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Observer design for a class of nonlinear systems combining dissipativity with interconnection and damping assignment

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Abstract
A nonlinear observer design approach is proposed that exploits and combines port-Hamiltonian systems and dissipativity theory. First, a passivity-based observer design using interconnection and damping assignment for time variant state affine systems is presented by applying output injection to the system such that the observer error dynamics takes a port-Hamiltonian structure. The stability of the observer error system is assured by exploiting its passivity properties. Second, this setup is extended to develop an observer design approach for a class of systems with a time varying state affine forward and a nonlinear feedback contribution. For a class of nonlinear systems, the theory of dissipative observers is adapted and combined with the results for the passivity-based observer design using interconnection and damping assignment. The convergence of the compound observer design is determined by a linear matrix inequality. The performance of both observer approaches is analyzed in simulation examples.

KEYWORDS
dissipative system, interconnection and damping assignment, LMI, observer design, output injection, port-Hamiltonian system, state affine system

1 | INTRODUCTION

Nonlinear controllers generally rely on the availability of the full state information. However, only a subset of the needed state information is usually measurable so that the non-measurable states have to be reconstructed from the knowledge of the system equations, inputs and outputs.

Observer designs for linear time invariant (LTI) and linear time variant (LTV) systems are well studied. Typical examples are Luenberger observer designs or optimal observer designs like the Kalman filter. However, these concepts cannot be directly applied to nonlinear systems. Possible solutions are given by transformations of nonlinear systems into observer normal forms,[1-3] bilinear systems up to output injections,[4] or state affine systems.[5-8] There are also various developments of Kalman filtering for nonlinear systems, for example, the extended Kalman filter, which makes use of the local linearisation and the solution of a Riccati equation at each time step.

Based on dissipativity concepts,[9-14] a dissipative observer design[15-17] is presented providing a systematic approach to solve the observer design problem for certain nonlinear systems. The system is decomposed into a linear part and a nonlinear perturbation term in the feedback loop. The resulting error system is in a Lur'e system structure[18] so that the observer error dynamics is linear with a nonlinear feedback perturbation. Assuming that the nonlinear perturbation satisfies a quadratic dissipativity condition, the observer gain can be designed to assure the stability of the observer error dynamics by fulfilling certain linear matrix inequalites (LMIs). This observer approach covers several other observer designs like circle criterion design,[19] high gain observer[20] or Lipschitz observer.[21]

Port-Hamiltonian systems (PHSs) are a subclass of passive systems. They offer a framework to model various linear and nonlinear finite-dimensional and linear infinite-dimensional systems on the basis of an underlying Dirac structure.[22-24] The passivity property of PHSs in terms of an input-output pairing enables the development of powerful control methods for nonlinear systems. For example, the concept of interconnection and damping assignment passivity-based control (IDA-PBC) provides a controller design so that the interconnected system consisting of the nonlinear system and the controller is a PHS.[25] A full order observer for a class of PHSs is presented in Reference 26. Thereby, the observer is designed so that the augmented system, which consists of the system and the observer, preserves the PHS structure. Stability is then achieved using passivity theory in terms of a new input-output pairing. Moreover, an observer for linear PHSs is presented in Reference 27 to estimate the magnetic and temperature profiles in a fusion reactor. The observer gain is chosen so that the error dynamics is in a PHS form. Furthermore, the observer approach is extended by an integral part. Extensions to interval observers for linear PHS are available.[28] An observer design applying contraction analysis for PHSs is presented in Reference 29. However, these observer designs can only be applied to certain types of PHSs.

The state estimation for state affine systems by a passivity-based observer design using interconnection and damping assignment (IDA-PBO) was first presented by the authors in Reference 30. The system under consideration is divided into measurable and non-measurable states and output-input injection is applied. Based on this, the observer design adapts the underlying idea of IDA-PBC. The observer is set up so that the resulting observer error dynamics form a PHS. The stability conditions rely on the definiteness of the matrix, which describes the system's energy dissipation.

This work presents two main results. First, the IDA-PBO is extended, so that the considered system class may depend explicitly on time. Thus, this observer approach can be applied to general LTV systems. Secondly, a combination of the augmented IDA-PBO and the dissipative design is presented by considering a system with a state affine forward and a dissipative nonlinear feedback part. The convergence of the observers is verified by taking into account the properties of PHSs and dissipativity theory. It is shown that the convergence of the combined observer design is determined by an LMI emphasizing the synergy of the IDA-PBO and the theory of dissipative observers. The resulting observer design approach can be applied to a wider class of nonlinear systems, overcoming the limitations of each technique.

This article is organized as follows: Section 2 introduces the problem and the basic observer structure. In Section 3, the observer design for time varying state affine systems by interconnection and damping assignment is presented. Section 4 introduces an observer approach for a class of nonlinear systems by the combination of the enhanced IDA-PBO with a dissipative observer. Examples and simulation results are presented and evaluated for each proposed design. Some final remarks are provided in Section 5.

Notation

The set of real numbers larger or equal to t_0 is denoted by $\mathbb{R}^+_{t_0} = \{t \in \mathbb{R} | t \geq t_0\}$. To reduce notation overhead and if clear from the context, the arising tuple $(t, \boldsymbol{y}, \boldsymbol{u})$ with $t \in \mathbb{R}^+_{t_0}$, output $\boldsymbol{y} : \mathbb{R}^+_{t_0} \to \mathbb{R}^p$ and input $\boldsymbol{u} : \mathbb{R}^+_{t_0} \to \mathbb{R}^k$ is denoted by $\boldsymbol{\xi} \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$. Moreover, time dependencies are omitted if clear from the context. We write $\boldsymbol{\xi} \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ to refer to a condition that holds true for all $\boldsymbol{\xi}$, that is, for all $t \geq t_0$, $\boldsymbol{u}$ and $\boldsymbol{y}$. If not stated otherwise, $\|\boldsymbol{\alpha}\|$ denotes the $\mathbb{R}^n$-norm of a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$.

2 | PROBLEM FORMULATION

In the following, an observer design for nonlinear systems that can be decomposed into a time varying state affine part, a nonlinear feedback part and a perturbation term, that is,

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(t, \boldsymbol{y}, \boldsymbol{u})\boldsymbol{x} + G(t, \boldsymbol{y}, \boldsymbol{u})\boldsymbol{\psi}(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}) + \boldsymbol{\vartheta}(t, \boldsymbol{y}, \boldsymbol{u}), \quad t > t_0, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= C\boldsymbol{x}, \quad t \geq t_0 \\ \boldsymbol{\sigma} &= H\boldsymbol{x}, \quad t \geq t_0 \end{aligned} \tag{1}$$

is developed. This class of systems resembles the ones investigated in References 15-17. The main difference is that the forward path in terms of $A(t,\boldsymbol{y},\boldsymbol{u})\boldsymbol{x}$ is not restricted to an LTI structure. Moreover, LTV systems or linear PHSs denote subclasses of (1).

It is assumed that a unique solution of (1) exists for all $t \geq t_0$. The input is given by $\boldsymbol{u}(t) \in \mathbb{R}^k$ and the output is denoted as $\boldsymbol{y}(t) \in \mathbb{R}^p$. The output matrix is considered in the form $C = [C_1, 0]$ with the invertible matrix $C_1 \in \mathbb{R}^{p\times p}$. Without loss of generality, C_1 is taken as the $p \times p$ identity matrix $C_1 = I$ implying $\boldsymbol{y} = \boldsymbol{x}_1$. The state vector is divided into $\boldsymbol{x}(t) = [\boldsymbol{x}_1^\mathsf{T}(t), \boldsymbol{x}_2^\mathsf{T}(t)]^\mathsf{T} \in \mathbb{R}^n$, where $\boldsymbol{x}_1(t) \in \mathbb{R}^p$ denotes measurable and $\boldsymbol{x}_2(t) \in \mathbb{R}^{n-p}$ non-measurable states. The system matrix $A : \mathbb{R}^+_{t_0} \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^{n\times n}$ is partitioned as

$$A(t,\boldsymbol{y},\boldsymbol{u}) = \begin{bmatrix} A_{11}(t,\boldsymbol{y},\boldsymbol{u}) & A_{12}(t,\boldsymbol{y},\boldsymbol{u}) \\ A_{21}(t,\boldsymbol{y},\boldsymbol{u}) & A_{22}(t,\boldsymbol{y},\boldsymbol{u}) \end{bmatrix} \tag{2}$$

with $A_{11}(t,\boldsymbol{y},\boldsymbol{u}) \in \mathbb{R}^{p\times p}$, $A_{12}(t,\boldsymbol{y},\boldsymbol{u}) \in \mathbb{R}^{p\times(n-p)}$, $A_{21}(t,\boldsymbol{y},\boldsymbol{u}) \in \mathbb{R}^{(n-p)\times p}$, $A_{22}(t,\boldsymbol{y},\boldsymbol{u}) \in \mathbb{R}^{(n-p)\times(n-p)}$ and the elements of each matrix are assumed to be locally Lipschitz in $(\boldsymbol{u},\boldsymbol{y})$ and piecewise continuous in t. Moreover, $A(t,\boldsymbol{y},\boldsymbol{u})$ is assumed uniformly bounded. Furthermore, $\boldsymbol{\sigma} : \mathbb{R}^n \to \mathbb{R}^r$ is a linear function of the states with $H \in \mathbb{R}^{r\times n}$. The nonlinear functions $\boldsymbol{\vartheta} : \mathbb{R}^+_{t_0} \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^n$, $\boldsymbol{\psi} : \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^q$ and the elements of the matrix $G : \mathbb{R}^+_{t_0} \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^{n\times q}$ are assumed to be locally Lipschitz in $(\boldsymbol{u},\boldsymbol{y})$ and piecewise continuous in t.

A full order observer for (1) is proposed as

$$\begin{aligned} &\dot{\hat{\boldsymbol{x}}} = A(t,\boldsymbol{y},\boldsymbol{u})\hat{\boldsymbol{x}} - L(t,\boldsymbol{y},\boldsymbol{u})(\hat{\boldsymbol{y}} - \boldsymbol{y}) + \boldsymbol{\vartheta}(t,\boldsymbol{y},\boldsymbol{u}) + G(t,\boldsymbol{y},\boldsymbol{u})\boldsymbol{\psi}(\hat{\boldsymbol{\sigma}} + N(t,\boldsymbol{y},\boldsymbol{u})(\hat{\boldsymbol{y}} - \boldsymbol{y}),\boldsymbol{y},\boldsymbol{u}), \quad t > t_0, \quad \hat{\boldsymbol{x}}(t_0) = \hat{\boldsymbol{x}}_0 \\ &\hat{\boldsymbol{y}} = C\hat{\boldsymbol{x}}, \quad t \geq t_0 \\ &\hat{\boldsymbol{\sigma}} = H\hat{\boldsymbol{x}}, \quad t \geq t_0. \end{aligned} \tag{3}$$

The elements of the observer gain matrices $L : \mathbb{R}^+_{t_0} \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^{n\times p}$ and $N : \mathbb{R}^+_{t_0} \times \mathbb{R}^p \times \mathbb{R}^k \to \mathbb{R}^{r\times p}$ are assumed to be locally Lipschitz in $(\boldsymbol{u},\boldsymbol{y})$ and piecewise continuous in t. Subsequently, conditions for the matrices $L(t,\boldsymbol{y},\boldsymbol{u})$ and $N(t,\boldsymbol{y},\boldsymbol{u})$ are determined to achieve that $\hat{\boldsymbol{x}}$ converges to $\boldsymbol{x}$ with asymptotic or even exponential convergence rate. Thereby, two cases are distinguished:

(i) The matrix $G(t,\boldsymbol{y},\boldsymbol{u})$ is assumed to be zero, implying a time varying state affine system with a perturbation term to acquire conditions for $L(t,\boldsymbol{y},\boldsymbol{u})$. The obtained observer approach extends the theory of observers for state affine systems[5-7] and IDA-PBO[30] as the system is allowed to depend explicitly on time. Furthermore, the observer design is simple, since its stability properties depend on particular properties of the arising matrices.

(ii) The nonlinear feedback part is included, that is, $G(t,\boldsymbol{y},\boldsymbol{u}) \neq 0$. The observer (3) is developed by combining the results obtained for the first case with the theory of dissipative observers.[15-17] This constitutes a more general observer design approach, since dissipative observers are typically designed for systems with a LTI forward path. Conditions for the matrices $N(t,\boldsymbol{y},\boldsymbol{u})$ and $L(t,\boldsymbol{y},\boldsymbol{u})$ are determined by solving an LMI.

In the following, both cases will be introduced starting with the generalized IDA-PBO (case (i)) to motivate its combination with dissipative observers (case (ii)).

3 | IDA-PBO FOR TIME VARYING STATE AFFINE SYSTEMS

In this section, a generalized IDA-PBO is developed for systems that may explicitly depend on time. The observer gain matrix $L(t,\boldsymbol{y},\boldsymbol{u})$ is given by transforming the error system dynamics into a PHS structure.

3.1 | Observer design

For $G(t,\boldsymbol{y},\boldsymbol{u}) = 0$ system (1) reduces to

$$\begin{aligned} &\dot{\boldsymbol{x}} = A(t,\boldsymbol{y},\boldsymbol{u})\boldsymbol{x} + \boldsymbol{\vartheta}(t,\boldsymbol{y},\boldsymbol{u}), \quad t > t_0, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ &\boldsymbol{y} = C\boldsymbol{x}, \end{aligned} \tag{4}$$

which is a state affine system in the non-measurable states $\boldsymbol{x}_2$.

Assumption 1. System (4) is uniformly observable for all $t \geq t_0$.

In this case, the observer (3) is set up with $N(t, \boldsymbol{y}, \boldsymbol{u}) = 0$, that is,

$$
\begin{aligned}
\dot{\hat{\boldsymbol{x}}} &= A(t, \boldsymbol{y}, \boldsymbol{u})\hat{\boldsymbol{x}} - L(t, \boldsymbol{y}, \boldsymbol{u})(\hat{\boldsymbol{y}} - \boldsymbol{y}) + \boldsymbol{\vartheta}(t, \boldsymbol{y}, \boldsymbol{u}), \quad t > t_0, \quad \hat{\boldsymbol{x}}(t_0) = \hat{\boldsymbol{x}}_0 \\
\hat{\boldsymbol{y}} &= C\hat{\boldsymbol{x}}.
\end{aligned} \tag{5}
$$

By introducing the observer error $\tilde{\boldsymbol{x}} = \hat{\boldsymbol{x}} - \boldsymbol{x}$ and the output error $\tilde{\boldsymbol{y}} = \hat{\boldsymbol{y}} - \boldsymbol{y}$, the resulting error system reads as

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{x}}} &= A(t, \boldsymbol{y}, \boldsymbol{u})\tilde{\boldsymbol{x}} - L(t, \boldsymbol{y}, \boldsymbol{u})\tilde{\boldsymbol{y}}, \quad t > t_0, \quad \tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0 \\
\tilde{\boldsymbol{y}} &= C\tilde{\boldsymbol{x}}.
\end{aligned} \tag{6}
$$

The main objective is to transform the error dynamics (6) into a PHS structure given by

$$
\dot{\tilde{\boldsymbol{x}}} = (J(\boldsymbol{\xi}) - R(\boldsymbol{\xi}))\, P\tilde{\boldsymbol{x}}, \quad t > t_0, \quad \tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0 \tag{7}
$$

with bounded matrices $P = P^{\mathsf{T}} \succ 0$, $J(\boldsymbol{\xi}) = -J^{\mathsf{T}}(\boldsymbol{\xi})$ and $R(\boldsymbol{\xi}) = R^{\mathsf{T}}(\boldsymbol{\xi}) \succcurlyeq 0$ for all $\boldsymbol{\xi} \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$ to exploit its favorable stability properties. To achieve this, the matching equation

$$
A(\boldsymbol{\xi}) - L(\boldsymbol{\xi})C = (J(\boldsymbol{\xi}) - R(\boldsymbol{\xi}))P \tag{8}
$$

has to be fulfilled for all $\boldsymbol{\xi} \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$. In other words, the matrix $L(\boldsymbol{\xi})$ has to reshape the error dynamics by inserting interconnection and damping elements. Thereby, the matrix $R(\boldsymbol{\xi})$ is used to stabilize the observer error system and to determine its convergence behavior. Using the separation into measurable and non-measurable states and by partitioning the matrices according to (2) together with $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{x}}_1$, the error dynamics (6) reads

$$
\begin{bmatrix} \dot{\tilde{\boldsymbol{x}}}_1 \\ \dot{\tilde{\boldsymbol{x}}}_2 \end{bmatrix} = \begin{bmatrix} A_{11}(\boldsymbol{\xi}) & A_{12}(\boldsymbol{\xi}) \\ A_{21}(\boldsymbol{\xi}) & A_{22}(\boldsymbol{\xi}) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \tilde{\boldsymbol{x}}_2 \end{bmatrix} - \begin{bmatrix} L_{11}(\boldsymbol{\xi}) & 0 \\ L_{21}(\boldsymbol{\xi}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \tilde{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11}(\boldsymbol{\xi}) - L_{11}(\boldsymbol{\xi}) & A_{12}(\boldsymbol{\xi}) \\ A_{21}(\boldsymbol{\xi}) - L_{21}(\boldsymbol{\xi}) & A_{22}(\boldsymbol{\xi}) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \tilde{\boldsymbol{x}}_2 \end{bmatrix}. \tag{9}
$$

The matrix $L(\boldsymbol{\xi})$ has to be designed so that the error dynamics fulfills

$$
\begin{bmatrix} \dot{\tilde{\boldsymbol{x}}}_1 \\ \dot{\tilde{\boldsymbol{x}}}_2 \end{bmatrix} = \left(\begin{bmatrix} J_{11}(\boldsymbol{\xi}) & J_{12}(\boldsymbol{\xi}) \\ -J_{12}^{\mathsf{T}}(\boldsymbol{\xi}) & J_{22}(\boldsymbol{\xi}) \end{bmatrix} - \begin{bmatrix} R_{11}(\boldsymbol{\xi}) & R_{12}(\boldsymbol{\xi}) \\ R_{12}^{\mathsf{T}}(\boldsymbol{\xi}) & R_{22}(\boldsymbol{\xi}) \end{bmatrix} \right) \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathsf{T}} & P_{22} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_1 \\ \tilde{\boldsymbol{x}}_2 \end{bmatrix}, \tag{10}
$$

where $J_{ii}(\boldsymbol{\xi}) = -J_{ii}^{\mathsf{T}}(\boldsymbol{\xi}),\ i \in \{1, 2\}$ and $R_{ii}(\boldsymbol{\xi}) = R_{ii}^{\mathsf{T}}(\boldsymbol{\xi}),\ i \in \{1, 2\}$. In reference to the matching condition (8), the following matrix equalities have to be fulfilled

$$
L_{11}(\boldsymbol{\xi}) = A_{11}(\boldsymbol{\xi}) - (J_{11}(\boldsymbol{\xi}) - R_{11}(\boldsymbol{\xi}))\, P_{11} - (J_{12}(\boldsymbol{\xi}) - R_{12}(\boldsymbol{\xi}))\, P_{12}^{\mathsf{T}} \tag{11a}
$$

$$
L_{21}(\boldsymbol{\xi}) = A_{21}(\boldsymbol{\xi}) + \left(J_{12}^{\mathsf{T}}(\boldsymbol{\xi}) + R_{12}^{\mathsf{T}}(\boldsymbol{\xi})\right) P_{11} - (J_{22}(\boldsymbol{\xi}) - R_{22}(\boldsymbol{\xi}))\, P_{12}^{\mathsf{T}} \tag{11b}
$$

$$
0 = A_{12}(\boldsymbol{\xi}) - (J_{11}(\boldsymbol{\xi}) - R_{11}(\boldsymbol{\xi}))\, P_{12} - (J_{12}(\boldsymbol{\xi}) - R_{12}(\boldsymbol{\xi}))\, P_{22} \tag{11c}
$$

$$
0 = A_{22}(\boldsymbol{\xi}) + \left(J_{12}^{\mathsf{T}}(\boldsymbol{\xi}) + R_{12}^{\mathsf{T}}(\boldsymbol{\xi})\right) P_{12} + (J_{22}(\boldsymbol{\xi}) + R_{22}(\boldsymbol{\xi}))\, P_{22}. \tag{11d}
$$

Theorem 1. *Consider the observer error dynamics (6) with (11) which results in the desired dynamics (7). Let $P = P^{\mathsf{T}} \succ 0$, $R(\boldsymbol{\xi}) = R^{\mathsf{T}}(\boldsymbol{\xi}) \succcurlyeq 0$ and $J(\boldsymbol{\xi}) = -J^{\mathsf{T}}(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$. Then the equilibrium point $\mathbf{0}$ of the observer error dynamics (7):*

(i) is uniformly stable for all $\xi \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$;

(ii) is globally exponentially stable if the matrix $R(\xi)$ is positive definite for all $\xi \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$. Let $0 < \mu_{\min} \le \mu_{\max}$ denote the minimal and the maximal eigenvalue of P. Then there exists an $\epsilon > 0$ such that

$$\|\tilde{\boldsymbol{x}}\| \le \sqrt{\frac{\mu_{\max}}{\mu_{\min}}} \exp\left(-\frac{\epsilon}{\mu_{\max}} t\right) \|\tilde{\boldsymbol{x}}_0\| . \tag{12}$$

In addition, the following assertions hold true:

(iii) The observer error dynamics approaches the limit determined by $\lim_{t\to\infty} \tilde{\boldsymbol{x}}^{\mathsf{T}} PR(\xi)P\tilde{\boldsymbol{x}} = 0$ as $t \to \infty$, if for all $\xi \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ the derivative $\dot{R}(\xi)$ exists and $M(\xi) = \mathrm{sym}(\dot{R}(\xi) + 2R(\xi)[J(\xi) - R(\xi)])$ is bounded, where $\mathrm{sym}(\cdot)$ yields the symmetric part.

(iv) The equilibrium point $\mathbf{0}$ of (7) is asymptotically stable, if for all $\xi \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ the derivative $\dot{R}(\xi)$ exists, $M(\xi) = \mathrm{sym}(\dot{R}(\xi) + 2R(\xi)[J(\xi) - R(\xi)])$ is bounded, and $\lim_{t\to\infty} R(\xi) = R^{\infty} \succ 0$.

The proof of these results is provided in Appendix A1.

Remark 1 (Observer design for LTV systems). This observer design in addition provides a methodology for the determination of the observer gain matrix $L(\xi)$ for LTV systems. In this case, time derivatives with respect to the matrix $A(\xi)$ are not needed for the design compared to eigenvalue placement using the Ackermann formula,[31] which makes use of the transformation into observer canonical form and the compensation of the time-variance.

The matching conditions (8) or (11), respectively, are rather general but various solutions for different parameterizations of $J(\xi)$, $R(\xi)$ and P can be obtained. For this note that only (11c) and (11d) have to hold true since $L_{11}(\xi)$ and $L_{21}(\xi)$ are degrees of freedom so that (11a) and (11b) can always be fulfilled. One possible approach to relax (11c) and (11d) is to assign $J(\xi)$ and $R(\xi)$ to determine either P_{12} or P_{22} directly by one equation. Another possible procedure is to assign P to calculate $J(\xi)$ and $R(\xi)$. This is presented in the following for a special case of $A_{22}(\xi)$.

3.2 | Systematic solution approach

An explicit solution scheme for the matrix $L(\xi)$ to directly achieve stability for (7) is given if $A_{22}(\xi)$ is positive semidefinite for all $\xi \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$. Here, P in (10) is considered as a block diagonal matrix, that is,

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} = P^{\mathsf{T}} \succ 0, \quad P_{11} = P_{11}^{\mathsf{T}} \succ 0, \quad P_{22} = P_{22}^{\mathsf{T}} \succ 0. \tag{13}$$

By assigning $P_{12} = 0$, the matching conditions (11) are relaxed. This approach denotes the opposite of the procedure presented in the example in Section 3.3. It is noteworthy to highlight that the following equations may also be derived from (11) by simply inserting $P_{12} = 0$. However, the following procedure shows the strong dependency on the definiteness of A_{22}. To obtain $L(\xi)$, the system matrix is partitioned into three matrices

$$A(\xi) = \underbrace{\begin{bmatrix} A_{11}(\xi) & 0 \\ A_{21}(\xi) & 0 \end{bmatrix}}_{(\star)} + \underbrace{\begin{bmatrix} 0 & A_{12}(\xi) + R_{12}(\xi)P_{22} \\ 0 & J_{22}(\xi)P_{22} \end{bmatrix}}_{(\star\star)} - \underbrace{\begin{bmatrix} 0 & R_{12}(\xi)P_{22} \\ 0 & R_{22}(\xi)P_{22} \end{bmatrix}}_{(\star\star\star)}. \tag{14}$$

Recalling the matching equation (8), that is, $A(\xi) - L(\xi)C = (J(\xi) - R(\xi))P$, the first matrix $(\star)$ contains matrix elements to be compensated in the course of the observer design. This yields

$$L_{11}(\xi) = A_{11}(\xi) - (J_{11}(\xi) - R_{11}(\xi))\, P_{11} \tag{15a}$$

$$L_{21}(\xi) = A_{21}(\xi) + \left(J_{12}^{\mathsf{T}}(\xi) + R_{12}^{\mathsf{T}}(\xi)\right) P_{11} \tag{15b}$$

The second matrix (⋆⋆) defines entries of the interconnection matrix $J(\xi)$ by assigning

$$J_{12}(\xi) = A_{12}(\xi)P_{22}^{-1} + R_{12}(\xi) \tag{16a}$$

$$J_{22}(\xi)P_{22} = \frac{1}{2}\left(A_{22}(\xi) - A_{22}^{\mathsf{T}}(\xi)\right). \tag{16b}$$

The third matrix (⋆⋆⋆) defines the entry

$$R_{22}(\xi)P_{22} = -\frac{1}{2}\left(A_{22}(\xi) + A_{22}^{\mathsf{T}}(\xi)\right) \tag{17}$$

of the dissipation matrix $R(\xi)$. Note that the matrix $A_{22}(\xi)$ is divided into its skew-symmetric and symmetric part to define $J_{22}(\xi)$ and $R_{22}(\xi)$. Inserting (16a) in (15b) yields the observer gain matrix

$$L(\xi) = \begin{bmatrix} A_{11}(\xi) - (J_{11}(\xi) - R_{11}(\xi))\,P_{11} \\ A_{21}(\xi) + \left(A_{12}^{\mathsf{T}}(\xi)P_{22}^{-1} + 2R_{12}^{\mathsf{T}}(\xi)\right)P_{11} \end{bmatrix}. \tag{18}$$

The submatrices $J_{11}(\xi) = -J_{11}^{\mathsf{T}}(\xi)$, $R_{11}(\xi) = R_{11}^{\mathsf{T}}(\xi)$ and $R_{12}(\xi)$ are degrees of freedom to adjust the performance of the observer (5). To analyze the stability properties of (10) first the definiteness of $R(\xi)$ is assessed in the following Lemma.

Lemma 1. *Let $R_{11}(\xi) \succ 0$ for all $\xi \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$. The matrix $R(\xi)$ is positive definite, if*

$$R_{22}(\xi) - R_{12}^{\mathsf{T}}(\xi)R_{11}^{-1}(\xi)R_{12}(\xi) \succ 0 \quad \forall \xi \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}. \tag{19}$$

The matrix $R(\xi)$ is positive semidefinite, if

$$R_{22}(\xi) - R_{12}^{\mathsf{T}}(\xi)R_{11}^{-1}(\xi)R_{12}(\xi) \succcurlyeq 0 \quad \forall \xi \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}. \tag{20}$$

The proof of this result is obtained by taking into account the Schur complement.[32] The matrices $R_{11}(\xi)$ and $R_{12}(\xi)$ serve as degrees of freedom to ensure that these conditions hold true. In many cases, $R_{12}(\xi) = 0$ is sufficient if the submatrix $A_{22}(\xi)$ satisfies $A_{22}(\xi) + A_{22}^{\mathsf{T}}(\xi) \preccurlyeq 0$ for all $\xi \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$ so that $R_{22}(\xi) \succcurlyeq 0$ according to (17). However, $R_{12}(\xi) \neq 0$ can introduce additional coupling in the system in terms of (16a).

Taking into account Lemma 1, the stability properties of the observer error dynamics (6) with $L(\xi)$ assigned according to (18) or equivalently (7) follow directly from Theorem 1 depending on $R(\xi)$ being positive semidefinite or positive definite, respectively. In particular asymptotic stability can be verified taking account Theorem 1 (ii).

Corollary 1. *Consider $L(\xi)$ as defined in (18) so that the observer error dynamics (6) is equivalently given by (10) with $P_{12} = 0$ as determined by (13). Let (A-1) condition (20) be satisfied with $R_{12}(\xi) = 0$, (A-2) $A_{22}(\xi) + A_{22}^{\mathsf{T}}(\xi) \preccurlyeq 0$ so that $R_{22}(\xi) \succcurlyeq 0$ for all $\xi \in \mathbb{X}_{\boldsymbol{u}\boldsymbol{y}}^{t_0}$ and assume that (A-3) the derivatives $\dot{R}(\xi)$, $\dot{J}(\xi)$ exist and are uniformly bounded. Then (10) is asymptotically stable.*

The proof of this result is provided in Appendix A2.

3.3 | Example

The observer design is applied to estimate the states of a magnetic levitation system.[33] An iron ball levitates in a magnetic field created by an electromagnet or coil, respectively. Magnetic flux x_1 and ball position x_2 are assumed to be measurable so that the ball's momentum x_3 has to be estimated. An observer approach based on the port-Hamiltonian framework is presented in Reference 26. For simplicity, physical parameters are omitted. The dynamics are given by

$$\begin{aligned} \dot{x}_1 &= -(1 - x_2)x_1 + u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \frac{1}{2}x_1^2 - g \end{aligned}$$

with the external input voltage u and the gravitational acceleration $g = 9.81$. The electromagnet is located at $x_2 = 1$. The levitating ball is assumed to not touch the electromagnet so that $x_2 < 1$ for all $t \geq 0$. The state affine system representation according to (4) reads

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} y_1(y_2 - 1) + u \\ 0 \\ \frac{1}{2} y_1^2 - g \end{bmatrix}, \quad t > 0, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}. \tag{21}$$

The submatrix A_{22} reduces to the scalar $a_{22} = 0$ and is positive semidefinite. Thus, an asymptotic stable observer error system is given according to Corollary 1 by assigning the PHS matrices and the observer gain matrix in the form

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = I \quad \Rightarrow \quad L = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (\Sigma_a),$$

where I is the identity matrix. Exponential stability can be proven for a different setup by utilizing Theorem 1. To reduce the degrees of freedom of the matching equations (11c) and (11d), the matrices $R \in \mathbb{R}^{3\times3}$, $J \in \mathbb{R}^{3\times3}$ and $P \in \mathbb{R}^{3\times3}$ are assigned as

$$R = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & \alpha \end{bmatrix}, \quad J = \begin{bmatrix} 0 & \beta & 1 \\ -\beta & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & \boldsymbol{p}_{12} \\ \boldsymbol{p}_{12}^{\mathsf{T}} & p_{22} \end{bmatrix}$$

with $\alpha, \beta > 0$, $P_{11} \in \mathbb{R}^{2\times2}$, $\boldsymbol{p}_{12} \in \mathbb{R}^2$, and $p_{22} \in \mathbb{R}$. The matching conditions (11c) and (11d) read

$$\boldsymbol{0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix} \boldsymbol{p}_{12} \tag{22a}$$

$$0 = \begin{bmatrix} 2 & 0 \end{bmatrix} \boldsymbol{p}_{12} + \alpha p_{22}. \tag{22b}$$

The vector $\boldsymbol{p}_{12}$ is directly determined by

$$\boldsymbol{p}_{12} = \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

according to (22a). The scalar p_{22} follows by (22b), that is,

$$p_{22} = -\frac{1}{\alpha} \begin{bmatrix} 2 & 0 \end{bmatrix} \boldsymbol{p}_{12} = \frac{2\beta}{\alpha(\alpha^2 + \beta^2)}.$$

The matrix P_{11} denotes a degree of freedom to be chosen so that P is positive definite. For example, if $\alpha = 2$ and $\beta = 4$, then a solution is given by

$$P_{11} = \frac{12}{20} I \quad \Rightarrow \quad P = \frac{1}{20} \begin{bmatrix} 12 & 0 & -4 \\ 0 & 12 & -2 \\ -4 & -2 & 4 \end{bmatrix} \quad \Rightarrow \quad L = \frac{1}{5} \begin{bmatrix} 6 & -12 \\ 12 & 6 \\ 4 & -1 \end{bmatrix} \quad (\Sigma_e).$$

The eigenvalues of P read $\lambda_1 = 2/20$, $\lambda_2 = 12/20$ and $\lambda_3 = 14/20$. Thus, exponential stability of the observer error system is given according to Theorem 1 (ii).

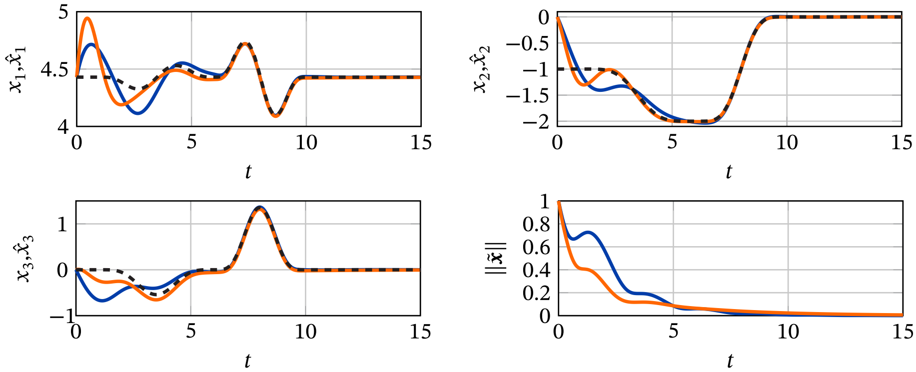

FIGURE 1 Evolution of the states $\boldsymbol{x}$ of (21) (dashed black), estimated states $\hat{\boldsymbol{x}}$ of the observer (5) with observer gain Σ_a (asymptotically stable) (blue) and Σ_e (exponentially stable) (orange). The error norms $\|\tilde{\boldsymbol{x}}\|$ is colored accordingly. The initial conditions read $\boldsymbol{x}_0 = [\sqrt{2g}, -1, 0]^{\mathsf{T}}$ and $\hat{\boldsymbol{x}}_0 = [\sqrt{2g}, 0, 0]^{\mathsf{T}}$ for both observers [Colour figure can be viewed at wileyonlinelibrary.com]

Simulation results can be seen in Figure 1. The plants initial state reads $\boldsymbol{x}_0 = [\sqrt{2g}, -1, 0]^{\mathsf{T}}$. The initial states of the two observers are given by $\hat{\boldsymbol{x}}_0 = [\sqrt{2g}, 0, 0]^{\mathsf{T}}$. Due to the instability of the system, the evaluation of the observer performance is done in closed-loop by taking into account a state feedback control based on exact input-state linearization with the ball position x_2 serving as linearizing output. The task is to balance the ball at the setpoint $x_2 = 0$. Due to the initial deviation both observers given by (5) with observer gain Σ_a (asymptotically stable) and Σ_e (exponentially stable), respectively, show some transients before converging to the real states. The analysis of the observer error norm reveals that Σ_e yields a slightly faster convergence to zero than Σ_a. Concluding, this example shows that different parameterizations with different stability conditions can be achieved. The first approach is more straightforward but the second one yields the stronger stability condition.

4 | DISSIPATIVE IDA-PBO

For $G(\boldsymbol{\xi}) \neq 0$, the system (1) involves a nonlinear feedback. A nonlinear observer (3) is obtained by adapting the results of Section 3 for a time varying state affine system in combination with the theory of dissipative observers[15-17] to deal with the nonlinear feedback. Conditions for $L(\boldsymbol{\xi})$ and $N(\boldsymbol{\xi})$ are developed in terms of an LMI.

4.1 | Preliminaries

This section summarizes some fundamental results for the observer design in Section 4.2. Dissipativity theory[12-14,18] is in the following adapted to systems of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= (J(\boldsymbol{\xi}) - R(\boldsymbol{\xi}))P\boldsymbol{x} + B(\boldsymbol{\xi})\boldsymbol{v}, \quad t > t_0, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{\eta} &= \Lambda(\boldsymbol{\xi})\boldsymbol{x}, \qquad t \geq t_0 \\ \boldsymbol{v} &= -\boldsymbol{\psi}(t, \boldsymbol{\eta}), \quad t \geq t_0 \end{aligned} \tag{23}$$

with $J(\boldsymbol{\xi}) = -J^{\mathsf{T}}(\boldsymbol{\xi})$, $R(\boldsymbol{\xi}) = R^{\mathsf{T}}(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{X}_{\boldsymbol{u},\boldsymbol{y}}^{t_0}$, $P = P^{\mathsf{T}} > 0$, the state affine time variant forward path in terms of $\boldsymbol{\eta}$ with $\Lambda : \mathbb{R}_{t_0}^{+} \times \mathbb{R}^{p} \times \mathbb{R}^{k} \to \mathbb{R}^{g \times n}$ locally Lipschitz in $(\boldsymbol{u}, \boldsymbol{y})$ and piecewise continuous in t, and the nonlinear feedback in terms of the function $\boldsymbol{\psi} : \mathbb{R}_{t_0}^{+} \times \mathbb{R}^{g} \to \mathbb{R}^{k}$. Let $\Omega \subseteq \mathbb{R}^{g}$ be a subset containing the origin. The nonlinearity $\boldsymbol{\psi}(t, \boldsymbol{\eta})$ is assumed piecewise continuous in t and locally Lipschitz in $\boldsymbol{\eta} \in \Omega$ with $\boldsymbol{\psi}(t, \boldsymbol{0}) = \boldsymbol{0}$. Furthermore, consider a quadratic supply rate

$$\omega(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\alpha}^{\mathsf{T}} Q \boldsymbol{\alpha} + 2\boldsymbol{\alpha}^{\mathsf{T}} S \boldsymbol{\beta} + \boldsymbol{\beta}^{\mathsf{T}} K \boldsymbol{\beta}, \tag{24}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^{a}$, $\boldsymbol{\beta} \in \mathbb{R}^{b}$, $Q = Q^{\mathsf{T}} \in \mathbb{R}^{a \times a}$, $S \in \mathbb{R}^{a \times b}$, $K = K^{\mathsf{T}} \in \mathbb{R}^{b \times b}$. In the following, the stability of (23) is determined by Lyapunov arguments. Consider the positive definite Lyapunov function candidate $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathsf{T}} P \boldsymbol{x}$. Its rate of change is given by

$$\dot{V} = 2\boldsymbol{x}^{\mathsf{T}} P \dot{\boldsymbol{x}} = -2\boldsymbol{x}^{\mathsf{T}} P R(\boldsymbol{\xi}) P \boldsymbol{x} + 2\boldsymbol{x}^{\mathsf{T}} P B(\boldsymbol{\xi}) \boldsymbol{v}, \tag{25}$$

since $J(\xi) = -J^{\mathsf{T}}(\xi)$ is skew-symmetric. Taking into account the supply rate (24) and introducing an $\epsilon > 0$, the inequality

$$\dot{V} = -2\boldsymbol{x}^{\mathsf{T}} PR(\xi)P\boldsymbol{x} + 2\boldsymbol{x}^{\mathsf{T}} PB(\xi)\boldsymbol{v} \leq \omega(\boldsymbol{\eta}, \boldsymbol{v}) - \epsilon V(\boldsymbol{x}) \tag{26}$$

is imposed. Using $\boldsymbol{\eta} = \Lambda(\xi)\boldsymbol{x}$ and (24), (26) can be rewritten as

$$\dot{V} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -2PR(\xi)P & PB(\xi) \\ B^{\mathsf{T}}(\xi)P & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \Lambda^{\mathsf{T}}(\xi)Q\Lambda(\xi) & \Lambda^{\mathsf{T}}(\xi)S \\ S^{\mathsf{T}}\Lambda(\xi) & K \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} - \epsilon V(\boldsymbol{x}). \tag{27}$$

To determine exponential stability of (23) in terms of ϵ, the following definitions[12-14] are recalled.

Definition 1. The state affine part $((J(\xi) - R(\xi))P, B(\xi), \Lambda(\xi))$ of (23) is called (Q, S, K)-state strictly dissipative (SSD) with respect to the supply rate (24), if there exists an $\epsilon > 0$ such that the LMI

$$\begin{bmatrix} -2PR(\xi)P + \epsilon P & PB(\xi) \\ B^{\mathsf{T}}(\xi)P & 0 \end{bmatrix} - \begin{bmatrix} \Lambda^{\mathsf{T}}(\xi)Q\Lambda(\xi) & \Lambda^{\mathsf{T}}(\xi)S \\ S^{\mathsf{T}}\Lambda(\xi) & K \end{bmatrix} \preccurlyeq 0 \tag{28}$$

holds true for all $t \geq t_0$.

Definition 2. The nonlinearity $\boldsymbol{\psi}(t, \boldsymbol{\eta})$ of (23) is said to be (Q, S, K)-dissipative with respect to $\boldsymbol{\eta}$, if the supply rate

$$\omega(\boldsymbol{\psi}(t, \boldsymbol{\eta}), \boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{\psi}(t, \boldsymbol{\eta}) \\ \boldsymbol{\eta} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & K \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}(t, \boldsymbol{\eta}) \\ \boldsymbol{\eta} \end{bmatrix}$$

is positive semidefinite for all $t \geq t_0$ and $\boldsymbol{\eta} \in \Omega$.

Conditions for exponential stability of (23) are obtained in the following result.

Lemma 2. *If there exist $Q \in \mathbb{R}^{k \times k}$, $S \in \mathbb{R}^{k \times g}$ and $K \in \mathbb{R}^{g \times g}$ such that (i) the state affine part $((J(\xi) - R(\xi))P, B(\xi), \Lambda(\xi))$ is $(-K, S^{\mathsf{T}}, -Q)$-SSD and (ii) the nonlinear feedback $\boldsymbol{\psi}(t, \boldsymbol{\eta})$ is (Q, S, K)-dissipative for all $t \geq t_0$, then the system (23) is locally exponentially stable for all $\boldsymbol{\eta} \in \Omega$.*

The exponential stability of (23) applies locally around the origin for $\boldsymbol{\eta} \in \Omega \subset \mathbb{R}^g$. If $\Omega = \mathbb{R}^g$, that is, $\boldsymbol{\eta} \in \mathbb{R}^g$, then the system is globally exponentially stable. The proof of Lemma 2 is summarized in Appendix A3.

Remark 2. Considering a vector-valued nonlinearity $\boldsymbol{\psi}(t, \boldsymbol{\eta})$ it may be useful to use several supply rates, for example, one for each vector element. Assuming that there exist M independent supply rates, a general supply rate is given by

$$\omega(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{m=0}^{M} \theta_m \omega_m(\boldsymbol{\alpha}, \boldsymbol{\beta}), \quad \theta_m \geq 0. \tag{29}$$

Thereby, the state affine part is said to be $(-K_\theta, S_\theta^{\mathsf{T}}, Q_\theta)$-SSD and the nonlinearity is said to be $(Q_\theta, S_\theta, K_\theta)$-dissipative.

4.2 | Observer design

Consider the observer (3) for system (1). Utilizing the observer error $\tilde{\boldsymbol{x}} = \hat{\boldsymbol{x}} - \boldsymbol{x}$ and the output error $\tilde{\boldsymbol{y}} = \hat{\boldsymbol{y}} - \boldsymbol{y}$, the observer error dynamics reads

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= A(\xi)\tilde{\boldsymbol{x}} - L(\xi)\tilde{\boldsymbol{y}} + G(\xi)\left(\boldsymbol{\psi}(\hat{\boldsymbol{\sigma}} + N(\xi)\tilde{\boldsymbol{y}}, \boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{\psi}(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})\right), \quad t > t_0, \ \tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0 \\ \tilde{\boldsymbol{\sigma}} &= H\tilde{\boldsymbol{x}}, \quad t \geq t_0 \\ \tilde{\boldsymbol{y}} &= C\tilde{\boldsymbol{x}}, \quad t \geq t_0. \end{aligned} \tag{30}$$

The observer design proceeds in two steps. First, the time varying state affine part of the error dynamics is transformed to a desired PHS structure by utilizing the matrix $L(\xi)$, that is,

$$A(\xi)\tilde{\boldsymbol{x}} - L(\xi)\tilde{\boldsymbol{y}} = (J(\xi) - R(\xi))P\tilde{\boldsymbol{x}}$$

as done to transform (6) into (7). This procedure illustrates the structural connection between the IDA-PBO and this setup. Second, the theory of dissipative observers is applied. Considering the linear function

$$\boldsymbol{z} = \tilde{\boldsymbol{\sigma}} + N(\xi)\tilde{\boldsymbol{y}} = \underbrace{(H + N(\xi)C)}_{=H_N(\xi)}\tilde{\boldsymbol{x}}$$

and the nonlinear function

$$\begin{aligned}\boldsymbol{\psi}(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{\psi}(\hat{\boldsymbol{\sigma}} + N(\xi)\tilde{\boldsymbol{y}}, \boldsymbol{y}, \boldsymbol{u}) &= \boldsymbol{\psi}(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{\psi}(\boldsymbol{\sigma} + \tilde{\boldsymbol{\sigma}} + N(\xi)\tilde{\boldsymbol{y}}, \boldsymbol{y}, \boldsymbol{u}) \\ &= \boldsymbol{\psi}(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{\psi}(\boldsymbol{\sigma} + \boldsymbol{z}, \boldsymbol{y}, \boldsymbol{u}) =: \boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})\end{aligned} \tag{31}$$

the error system (30) can be rewritten as

$$\begin{aligned}&\dot{\tilde{\boldsymbol{x}}} = (J(\xi) - R(\xi))\,P\tilde{\boldsymbol{x}} + G(\xi)\boldsymbol{v}, \quad t > t_0, \quad \tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0 \\ &\boldsymbol{z} = H_N(\xi)\tilde{\boldsymbol{x}} \\ &\boldsymbol{v} = -\boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}).\end{aligned} \tag{32}$$

In the following, the properties of $J(\xi)$, $R(\xi)$ and P will be used in combination with dissipativity theory to determine the stability of (32). Note that (32) structurally resembles (23) by taking into account the following relations:

$$\boldsymbol{x} \leftrightarrow \tilde{\boldsymbol{x}}, \boldsymbol{\eta} \leftrightarrow \boldsymbol{z}, \Lambda(\xi) \leftrightarrow H_N(\xi), B(\xi) \leftrightarrow G(\xi), \boldsymbol{\psi}(t, \boldsymbol{\eta}) \leftrightarrow \boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}).$$

Moreover, the nonlinearity fulfills $\boldsymbol{\phi}(\boldsymbol{0}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{0}$ for all $(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})$. Hence, the results of Section 4.1 can be applied to obtain the following result. In this case, the set $\Omega \subseteq \mathbb{R}^n$ contains all $\boldsymbol{z}$ such that Definition 2 with respect to $\boldsymbol{\phi}$ and $\boldsymbol{z}$ is fulfilled.

Theorem 2. *If there exist $Q = Q^\mathsf{T} \in \mathbb{R}^{q\times q}$, $S \in \mathbb{R}^{q\times r}$, $K = K^\mathsf{T} \in \mathbb{R}^{r\times r}$, $L(\xi)$, $N(\xi)$, $R(\xi) = R^\mathsf{T}(\xi)$ for all $\xi \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ and $P = P^\mathsf{T} \succ 0$ so that (A-1) the nonlinearity $\boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})$ introduced in (31) is (Q, S, K)-dissipative with respect to $\boldsymbol{z}$ uniformly in $(\boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})$ and (A-2) the state affine part $(J(\xi) - R(\xi))P, G(\xi), H_N(\xi))$ of (32) is $(-K, S^\mathsf{T}, -Q)$-SSD, that is, there exists an $\epsilon > 0$ so that*

$$\begin{bmatrix} -2PR(\xi)P + \epsilon P + H_N^\mathsf{T}(\xi)KH_N(\xi) & PG(\xi) - H_N^\mathsf{T}(\xi)S^\mathsf{T} \\ G^\mathsf{T}(\xi)P - SH_N(\xi) & Q \end{bmatrix} \preccurlyeq 0 \quad \forall \xi \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}, \quad \forall \boldsymbol{z} \in \Omega, \tag{33}$$

then (32) is locally exponentially stable for $\boldsymbol{z} \in \Omega$. The exponential convergence of the error norm is given by

$$\|\tilde{\boldsymbol{x}}\| \leq \sqrt{\frac{\mu_{\max}}{\mu_{\min}}}\,\|\tilde{\boldsymbol{x}}_0\| \exp\left(-\frac{\epsilon}{2}t\right), \tag{34}$$

where $0 < \mu_{\min} \leq \mu_{\max}$ denote the minimal and maximal eigenvalue of P.

The proof of Theorem 2 can be found in Appendix A4. There are several possibilities to calculate a suitable supply rate such that $\boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})$ is (Q, S, K)-dissipative with respect to $\boldsymbol{z}$. For example, using Lipschitz conditions, the mean value theorem or sector conditions.[15,16,18] Note that in Theorem 2, the symmetric matrix $R(\xi)$ does not have to be positive semidefinite. A solution for the LMI (33) can be obtained using numerical calculations, for example, in terms of semidefinite programming. Moreover, applying the theory of Section 3.2, then $R_{22}(\xi)P_{22} = -1/2\ (A_{22}(\xi) + A_{22}^\mathsf{T}(\xi))$ is fixed by the system to be observed.

4.3 Example

The performance of the proposed observer is evaluated for a modified version of the Lorenz system,[34] which may show chaotic behavior.[35] The system is augmented by an additional nonlinearity, that is,

$$\begin{aligned}\dot{x}_1 &= a(t)(x_2 - x_1) + \mu(t)\arctan(x_2)\\ \dot{x}_2 &= bx_1 - x_2 - x_1x_3\\ \dot{x}_3 &= x_1x_2 - cx_3\end{aligned} \tag{35}$$

with real positive coefficients $a(t) = 10 + \sin(t)$, $b = 28$, $c = 8/3$ and $\mu(t) \in \mathbb{R}$. Assuming that x_1 can be measured, (35) can be rewritten for $t_0 = 0$ in the form

$$\begin{aligned}\dot{\boldsymbol{x}} &= \underbrace{\begin{bmatrix}-a(t) & a(t) & 0\\ b & -1 & -y\\ 0 & y & -c\end{bmatrix}}_{=A(t,y)}\boldsymbol{x} + \underbrace{\begin{bmatrix}\mu(t)\\ 0\\ 0\end{bmatrix}}_{=\boldsymbol{g}(t)}\arctan(\sigma), \quad t > 0, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0\\ y &= \underbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}_{=C}\boldsymbol{x} = x_1, \quad t \geq 0\\ \sigma &= \underbrace{\begin{bmatrix}0 & 1 & 0\end{bmatrix}}_{=H}\boldsymbol{x} = x_2, \quad t \geq 0.\end{aligned} \tag{36}$$

With (3), the observer is given by

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}} &= A(t,y)\hat{\boldsymbol{x}} - \boldsymbol{l}(t)(\hat{y} - y) + \boldsymbol{g}(t)\arctan(\hat{\sigma} + n(t)(\hat{y} - y)), \quad t > 0, \quad \hat{\boldsymbol{x}}(0) = \hat{\boldsymbol{x}}_0\\ \hat{y} &= \hat{x}_1, \quad t \geq 0\\ \hat{\sigma} &= \hat{x}_2, \quad t \geq 0.\end{aligned} \tag{37}$$

The observer gain vector is chosen as

$$\boldsymbol{l}(t) = \begin{bmatrix}\alpha - a(t) & a(t) + b & 0\end{bmatrix}^{\mathsf{T}} \tag{38}$$

as proposed in Section 3.2. The resulting observer error system reads

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}} &= (J(t,y) - R)\boldsymbol{x} + \boldsymbol{g}(t)v, \quad t > 0, \quad \tilde{\boldsymbol{x}}(t_0) = \tilde{\boldsymbol{x}}_0\\ z &= H_n(t)\tilde{\boldsymbol{x}}\\ v &= -\phi(z, x_2) = \arctan(x_2 + z) - \arctan(x_2)\end{aligned}$$

with the corresponding matrices

$$J(t,y) = \begin{bmatrix}0 & a(t) & 0\\ -a(t) & 0 & -y\\ 0 & y & 0\end{bmatrix}, \quad R = \begin{bmatrix}\alpha & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & c\end{bmatrix}, \quad H_n(t) = \begin{bmatrix}n(t)\\ 1\\ 0\end{bmatrix}^{\mathsf{T}}, \quad P = I.$$

To apply the exponential stability condition from Theorem 2, the (Q, S, K)-dissipativity of $\phi(z, x_2)$ is investigated in view of Definition 2. Herein, matrices (Q, S, K) reduce to scalars (q, s, k) due to $\phi(z, x_2)$ being a scalar function.

4.3.1 | Stability analysis

Sector conditions for the nonlinearity are determined to specify the supply rate $\omega(\phi, z)$. Based on this, the LMI condition of Theorem 2 is analyzed to determine α and $n(t)$ ensuring the exponential stability of the observer error system. Given $\phi(z, x_2) = \arctan(x_2) - \arctan(x_2 + z)$ it follows with $\phi(0, x_2) = 0$ and

$$\frac{\partial}{\partial z}\phi(z, x_2) = -\frac{1}{1 + (x_2 + z)^2} \in [-1, 0]$$

that the nonlinearity is in the sector $[-1, 0]$ with respect to z uniformly in x_2, that is,

$$(\phi(z, x_2) + z)(\phi(z, x_2) - 0) = \phi^2(z, x_2) + \phi(z, x_2)z \leq 0.$$

Note that this is valid for all $z \in \mathbb{R}$ and thus $\Omega = \mathbb{R}$. As a result, a suitable supply rate is given by

$$\omega(\phi, z) = -\phi^2(z, x_2) - \phi(z, x_2)z \geq 0$$

and $\phi(z, x_2)$ is $(-1, -1/2, 0)$-dissipative with respect to z. Taking into account the LMI (33) for some $\epsilon > 0$ yields that

$$\begin{bmatrix} \epsilon - 2\alpha & 0 & 0 & \mu(t) + \frac{n(t)}{2} \\ 0 & \epsilon - 2 & 0 & \frac{1}{2} \\ 0 & 0 & \epsilon - 2c & 0 \\ \mu(t) + \frac{n(t)}{2} & \frac{1}{2} & 0 & -1 \end{bmatrix} \preccurlyeq 0 \quad \forall t > 0$$

has to hold true to achieve exponential stability of the error system (32). By assigning $n(t) = -2\mu(t)$, this simplifies to study the LMI

$$\Pi = \begin{bmatrix} \epsilon - 2\alpha & 0 & 0 & 0 \\ 0 & \epsilon - 2 & 0 & \frac{1}{2} \\ 0 & 0 & \epsilon - 2c & 0 \\ 0 & \frac{1}{2} & 0 & -1 \end{bmatrix} \preccurlyeq 0.$$

Using the Schur complement Π is negative semidefinite if $\epsilon \leq \min\{2\alpha, 7/4, 2c\}$. In view of the used system parameters and $\epsilon > 0$, this implies $0 < \epsilon \leq \min\{2\alpha, 7/4, 16/3\}$. If $\alpha \geq 7/8$ is assigned, then the maximal decay rate is $\epsilon_{\max} = 7/4$. As a result, Theorem 2 implies the exponential stability of the observer error dynamics. As the set $\Omega = \mathbb{R}$, convergence is global, cf. Lemma 2. It should be pointed out that also a time-varying gain $\alpha(t)$ may be considered.

4.3.2 | Simulation results

In the following, the observer performance is analyzed in simulation scenarios. For this $\mu(t) = 4 + 2\cos(t)$ is assigned and a mismatch between the initial states $\boldsymbol{x}_0 = [10, 10, 10]^\top$ of the plant and $\hat{\boldsymbol{x}}_0 = [0, 0, 0]^\top$ of the observer is induced. Considering a fast exponential decay rate, the degrees of freedom are set as $\alpha = 5$ and $n(t) = -2\mu(t)$ as investigated in the previous section. Considering the stability analysis of the previous section, the degrees of freedom are set as $\alpha = 5 \geq 7/8$ and $n(t) = -2\mu(t)$ so that the observer error convergence is faster than the exponential decay rate. Note that if α is very large, then a high gain observer with $n = 0$ can achieved. Simulation results are shown in Figure 2. As predicted by the theoretical results the observer convergence is achieved. The time evolution of the error norm $\|\tilde{\boldsymbol{x}}\| = \|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$ decreases faster than the exponential decay rate (34) for $\epsilon = 7/4$ plotted by a black line for comparison purposes.

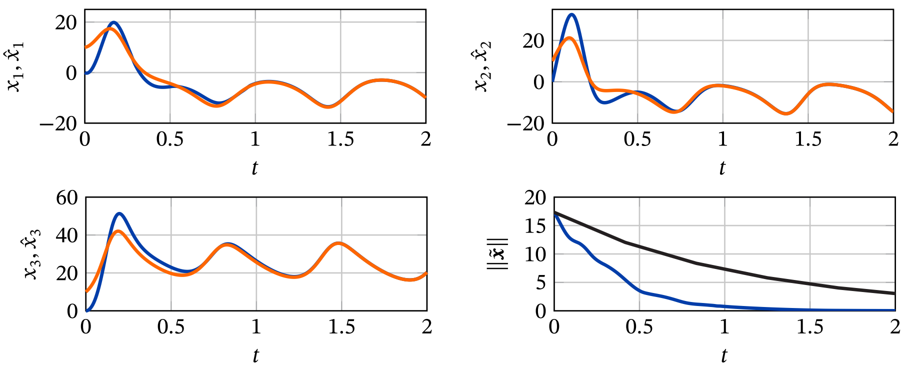

FIGURE 2 Evolution of the states $\boldsymbol{x}$ of (36) (orange), estimated states $\hat{\boldsymbol{x}}$ of (37) (blue) and error norm $\|\tilde{\boldsymbol{x}}\|$ in nominal case for $\alpha = 5$, $n(t) = -2\mu(t)$ with initial conditions $\boldsymbol{x}_0 = [10, 10, 10]^\mathsf{T}$ and $\hat{\boldsymbol{x}}_0 = [0, 0, 0]^\mathsf{T}$. The black line shows the exponential decay rate of the error norm in terms of (34) with $\epsilon = 7/4$ [Colour figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSIONS

Two observer design approaches are presented for a class of nonlinear systems by using the PHS framework and dissipativity theory. First, the application of an IDA-PBO is extended to general LTV systems by taking into account an explicit time dependency of the differential equations. Asymptotic and exponential stability is determined by solving an LMI. Secondly, an observer is developed by combining the enhanced IDA-PBO with the theory of dissipative observers. The obtained observer approach can be applied to a class of systems with an LTV forward and nonlinear feedback path, thus overcoming the limitations of an IDA-PBO or dissipative observer. Exponential stability can be ensured under certain assumptions given by matrix properties and LMIs. An example for the last observer design shows how to adjust the observer parameters and the performance is confirmed by numerical simulation results. The extension by an integral part[27,36] or parameter adaptation[37] to improve the robustness of the observers are open topics for future research.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

CONFLICT OF INTEREST

The authors declare no conflicts of interest.

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APPENDIX A. PROOFS

A.1 Proof of Theorem 1

The observer error dynamics (6) with $L(\boldsymbol{\xi})$ assigned according to (11) is equivalently described by (7) with $J(\boldsymbol{\xi}) = -J^{\mathsf{T}}(\boldsymbol{\xi})$, $R(\boldsymbol{\xi}) = R^{\mathsf{T}}(\boldsymbol{\xi}) \succcurlyeq 0$ for all $\boldsymbol{\xi} \in \mathbb{X}_{\boldsymbol{u},\boldsymbol{y}}^{t_0}$ and $P = P^{\mathsf{T}} \succ 0$. Consider the Hamiltonian $\mathcal{H}(\tilde{\boldsymbol{x}}) = 1/2\, \tilde{\boldsymbol{x}}^{\mathsf{T}} P \tilde{\boldsymbol{x}}$ as Lyapunov function candidate satisfying $1/2\, \mu_{\min} \|\tilde{\boldsymbol{x}}\|^2 \leq \mathcal{H}(\tilde{\boldsymbol{x}}) \leq 1/2\, \mu_{\max} \|\tilde{\boldsymbol{x}}\|^2$ for $0 < \mu_{\min} \leq \mu_{\max}$ the minimal and the maximal eigenvalue of P, respectively. The rate of change of $\mathcal{H}(\tilde{\boldsymbol{x}})$ along a solution trajectory $\tilde{\boldsymbol{x}}$ of (7) reads

$$\dot{\mathcal{H}} = \tilde{\boldsymbol{x}}^{\mathsf{T}} P \dot{\tilde{\boldsymbol{x}}} = \tilde{\boldsymbol{x}}^{\mathsf{T}} P \left(J(\boldsymbol{\xi}) - R(\boldsymbol{\xi}) \right) P \tilde{\boldsymbol{x}} = -\tilde{\boldsymbol{x}}^{\mathsf{T}} P R(\boldsymbol{\xi}) P \tilde{\boldsymbol{x}} \leq 0,$$

since $R(\boldsymbol{\xi}) \succcurlyeq 0$ and $J(\boldsymbol{\xi})$ is skew-symmetric. With this, the stability assertions for the observer error dynamics stated in Theorem 1 are verified.

(i) The property $\dot{\mathcal{H}} \leq 0$ together with the upper and lower bounds for $\mathcal{H}(\tilde{\boldsymbol{x}})$ implies uniform stability of (7) according to Theorem 4.8 in Khalil.[18] Moreover, the error norm satisfies

$$\frac{\mu_{\min}}{2}\|\tilde{\boldsymbol{x}}\|^2 \leq \mathcal{H}(\tilde{\boldsymbol{x}}) \leq \mathcal{H}(\tilde{\boldsymbol{x}}_0) \leq \frac{\mu_{\max}}{2}\|\tilde{\boldsymbol{x}}_0\|^2 \quad \Rightarrow \quad \|\tilde{\boldsymbol{x}}\| \leq \sqrt{\frac{\mu_{\max}}{\mu_{\min}}}\,\|\tilde{\boldsymbol{x}}_0\|\,. \tag{A1}$$

(ii) As $R(\boldsymbol{\xi}) \succ 0$ for all $\boldsymbol{\xi} \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ it follows that $PR(\boldsymbol{\xi})P \succ 0$ for all $\boldsymbol{\xi} \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$. For all $\boldsymbol{\xi} \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ let $\lambda(\boldsymbol{\xi}) > \epsilon > 0$ be the minimal eigenvalue of $PR(\boldsymbol{\xi})P$. Hence $\tilde{\boldsymbol{x}}^T PR(\boldsymbol{\xi})P\tilde{\boldsymbol{x}} \geq \lambda(\boldsymbol{\xi})\|\tilde{\boldsymbol{x}}\|^2 \geq \epsilon\|\tilde{\boldsymbol{x}}\|^2 \geq 0$ and hence $\dot{\mathcal{H}} \leq -\epsilon\|\tilde{\boldsymbol{x}}\|^2$. Thus, global exponential stability follows from Theorem 4.10 of Khalil.[18] The decay rate is calculated by utilizing the upper bound of the Hamiltonian, that is, $\dot{\mathcal{H}} \leq -\epsilon\|\tilde{\boldsymbol{x}}\|^2 \leq -2\epsilon/\mu_{\max}\mathcal{H}(\tilde{\boldsymbol{x}})$, which yields $\mathcal{H}(\tilde{\boldsymbol{x}}) \leq \mathcal{H}(\tilde{\boldsymbol{x}}_0)\exp(-\epsilon/\mu_{\max}t)$. Substituting the upper and lower bounds for $\mathcal{H}(\tilde{\boldsymbol{x}})$ results in (12).

(iii) Differentiation of $\dot{\mathcal{H}}$ with respect to t in view of (7) results in

$$\ddot{\mathcal{H}} = -\tilde{\boldsymbol{x}}^{\mathsf{T}}P\left[\dot{R}(\boldsymbol{\xi}) - 2R^2(\boldsymbol{\xi}) + R(\boldsymbol{\xi})J(\boldsymbol{\xi}) - J(\boldsymbol{\xi})R(\boldsymbol{\xi})\right]P\tilde{\boldsymbol{x}} = -\tilde{\boldsymbol{x}}^{\mathsf{T}}P\mathrm{sym}\left(\dot{R}(\boldsymbol{\xi}) + 2R(\boldsymbol{\xi})[J(\boldsymbol{\xi}) - R(\boldsymbol{\xi})]\right)P\tilde{\boldsymbol{x}},$$

where sym(M) denotes the symmetric part of matrix M. The second line follows as $R(\boldsymbol{\xi})J(\boldsymbol{\xi}) - J(\boldsymbol{\xi})R(\boldsymbol{\xi}) = R(\boldsymbol{\xi})J(\boldsymbol{\xi}) + (R(\boldsymbol{\xi})J(\boldsymbol{\xi}))^{\mathsf{T}} = 2\mathrm{sym}(R(\boldsymbol{\xi})J(\boldsymbol{\xi}))$. Hence, if $M(\boldsymbol{\xi}) = \mathrm{sym}(\dot{R}(\boldsymbol{\xi}) + 2R(\boldsymbol{\xi})[J(\boldsymbol{\xi}) - R(\boldsymbol{\xi})])$ is bounded, then $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi})$ is uniformly continuous in time, which together with the lower boundedness of $\mathcal{H}(\tilde{\boldsymbol{x}})$ and $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi}) \leq 0$ implies[38] that $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi}) = -\tilde{\boldsymbol{x}}^{\mathsf{T}}PR(\boldsymbol{\xi})P\tilde{\boldsymbol{x}} \to 0$ as $t \to \infty$.

(iv) The previous result can be strengthened to deduce asymptotic stability provided that $\lim_{t\to\infty} R(\boldsymbol{\xi}) = R^{\infty} \succ 0$. Recalling $P \succ 0$ in this case the limit of $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi}) = -\tilde{\boldsymbol{x}}^{\mathsf{T}}PR(\boldsymbol{\xi})P\tilde{\boldsymbol{x}} \to 0$ as $t \to \infty$ implies the convergence of $\tilde{\boldsymbol{x}}$ to the origin.

A.2 Proof of Corollary 1

Recall that the matrix $R_{12}(\boldsymbol{\xi})$ is a degree of freedom setting $R_{12}(\boldsymbol{\xi}) = 0$ preserves the positive semidefiniteness of $R(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{X}^{t_0}_{\boldsymbol{u},\boldsymbol{y}}$ according to (20) since $R_{11}(\boldsymbol{\xi}) \succ 0$ and $R_{22}(\boldsymbol{\xi}) \succcurlyeq 0$ by assumption (A-2). Proceeding now as in the proof of Theorem 1 (i) and (ii) we can conclude using the Lyapunov function candidate $\mathcal{H}(\tilde{\boldsymbol{x}}) = 1/2\,\tilde{\boldsymbol{x}}^{\mathsf{T}}P\tilde{\boldsymbol{x}}$ with $1/2\,\mu_{\min}\|\tilde{\boldsymbol{x}}\|^2 \leq \mathcal{H}(\tilde{\boldsymbol{x}}) \leq 1/2\,\mu_{\max}\|\tilde{\boldsymbol{x}}\|^2$, where $0 < \mu_{\min} \leq \mu_{\max}$ are the minimal and the maximal eigenvalue of P, that

$$\dot{\mathcal{H}} = -\tilde{\boldsymbol{x}}_1^{\mathsf{T}}P_{11}R_{11}(\boldsymbol{\xi})P_{11}\tilde{\boldsymbol{x}}_1 - \tilde{\boldsymbol{x}}_2^{\mathsf{T}}P_{22}R_{22}(\boldsymbol{\xi})P_{22}\tilde{\boldsymbol{x}}_2. \tag{A2}$$

In view of assumptions (iii) $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi})$ is uniformly continuous in time, which together with the lower boundedness of $\mathcal{H}(\tilde{\boldsymbol{x}})$ and $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi}) \leq 0$ implies[38] that $\dot{\mathcal{H}}(\tilde{\boldsymbol{x}}, \boldsymbol{\xi}) = -\tilde{\boldsymbol{x}}^{\mathsf{T}}PR(\boldsymbol{\xi})P\tilde{\boldsymbol{x}} \to 0$ as $t \to \infty$. Taking into account $P_{ii} \succ 0$, $R_{11}(\boldsymbol{\xi}) \succ 0$ and $R_{22}(\boldsymbol{\xi}) \succcurlyeq 0$ this implies $\tilde{\boldsymbol{x}}_1^{\infty} = \lim_{t\to\infty}\tilde{\boldsymbol{x}}_1 = \boldsymbol{0}$ so that the solution $\tilde{\boldsymbol{x}}_2$ converges to

$$\lim_{t\to\infty}\tilde{\boldsymbol{x}}_2^{\mathsf{T}}P_{22}R_{22}(\boldsymbol{\xi})P_{22}\tilde{\boldsymbol{x}}_2 \to 0. \tag{A3}$$

Noting that $\lim_{t\to\infty}\int_{t_0}^{t} \mathrm{d}/\mathrm{d}\tau\,\tilde{\boldsymbol{x}}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty}\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}(t_0) = -\tilde{\boldsymbol{x}}(t_0)$ is finite under the preliminary of bounded initial error Barbalat's lemma implies $\lim_{t\to\infty}\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{0}$ if $\dot{\tilde{\boldsymbol{x}}}$ is uniformly continuous. The latter is ensured by assumption (A-3), which implies that $\ddot{\tilde{\boldsymbol{x}}}$ exists and is bounded. Thus (10) with (13), (16), (17), and $R_{12}(\boldsymbol{\xi}) = 0$ yields

$$\boldsymbol{0} = \lim_{t\to\infty}\begin{bmatrix} A_{12}(\boldsymbol{\xi}) \\ A_{22}(\boldsymbol{\xi}) \end{bmatrix}\tilde{\boldsymbol{x}}_2 = \begin{bmatrix} A_{12}^{\infty} \\ A_{22}^{\infty} \end{bmatrix}\tilde{\boldsymbol{x}}_2^{\infty},$$

where $\tilde{\boldsymbol{x}}_2^{\infty} = \lim_{t\to\infty}\tilde{\boldsymbol{x}}_2$. In view of (16b) and (17), we obtain from (A3) that

$$(\tilde{\boldsymbol{x}}_2^{\infty})^{\mathsf{T}}P_{22}\left(A_{22}^{\infty} + (A_{22}^{\infty})^{\mathsf{T}}\right)\tilde{\boldsymbol{x}}_2^{\infty} = 0.$$

These equations in the limit as $t \to \infty$ imply

$$\tilde{\boldsymbol{x}}_2^{\infty} \in \ker\left(A_{12}^{\infty}\right) \cap \ker\left(A_{22}^{\infty}\right) \cap \ker\left((A_{22}^{\infty})^{\mathsf{T}}\right). \tag{A4}$$

Recalling that the output matrix $C = [I, 0]$ extracts the states $\boldsymbol{x}_1$ or $\tilde{\boldsymbol{x}}_1$, respectively, it follows that (A4) excludes non-zero $\tilde{\boldsymbol{x}}_2^\infty \neq \boldsymbol{0}$ as in this case $\tilde{\boldsymbol{x}}_2^\infty \in \ker(A_{12}^\infty) \cap \ker(A_{22}^\infty)$ denotes a direction that is not observable. To illustrate this let $\tilde{\boldsymbol{x}}_2^\infty = \boldsymbol{r} \neq \boldsymbol{0}$ satisfy (A4) and recall $\tilde{\boldsymbol{x}}_1^\infty = \boldsymbol{0}$. Then

$$\underbrace{\begin{bmatrix} A_{11}^\infty & A_{12}^\infty \\ A_{21}^\infty & A_{22}^\infty \end{bmatrix}}_{=A^\infty} \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{r} \end{bmatrix}}_{=\boldsymbol{v}} = \boldsymbol{0}$$

shows that $\boldsymbol{v}$ is an eigenvector of $A^\infty = \lim_{t\to\infty} A(\boldsymbol{\xi})$ to the eigenvalue $\lambda = 0$. As $C\boldsymbol{v} = [I, 0]\boldsymbol{v} = \boldsymbol{0}$ the Popov–Belevitch–Hautus eigenvector criteria yields that the pair (A^∞, C) is not observable, which contradicts Assumption 1 imposing uniform observability of (4). Hence $\tilde{\boldsymbol{x}}_2^\infty = \boldsymbol{r} = \boldsymbol{0}$, which together with $\tilde{\boldsymbol{x}}_1^\infty = \boldsymbol{0}$ proves asymptotic convergence of the observer error to the origin.

A.3 Proof of Lemma 2

System (23) is $(-K, S^\mathsf{T}, -Q)$-SSD by assumption. Thus, the rate of change of the positive definite Lyapunov function candidate $V(\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T} P \boldsymbol{x}$ with $P > 0$ is given by applying (27), that is,

$$\dot{V} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -2PR(\boldsymbol{\xi})P & PB(\boldsymbol{\xi}) \\ B^\mathsf{T}(\boldsymbol{\xi})P & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\Lambda^\mathsf{T}(\boldsymbol{\xi})K\Lambda(\boldsymbol{\xi}) & \Lambda^\mathsf{T}(\boldsymbol{\xi})S^\mathsf{T} \\ S\Lambda(\boldsymbol{\xi}) & -Q \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} - \epsilon V(\boldsymbol{x}).$$

Inserting the definition $\boldsymbol{v} = -\boldsymbol{\psi}(t, \boldsymbol{\eta})$ yields

$$\begin{aligned} \dot{V} &\leq \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\psi}(t,\boldsymbol{\eta}) \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\Lambda^\mathsf{T}(\boldsymbol{\xi})K\Lambda(\boldsymbol{\xi}) & -\Lambda^\mathsf{T}(\boldsymbol{\xi})S^\mathsf{T} \\ -S\Lambda(\boldsymbol{\xi}) & -Q \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\psi}(t,\boldsymbol{\eta}) \end{bmatrix} \\ -\epsilon V(\boldsymbol{x}) &= -\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\psi}(t,\boldsymbol{\eta}) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \Lambda^\mathsf{T}(\boldsymbol{\xi})K\Lambda(\boldsymbol{\xi}) & \Lambda^\mathsf{T}(\boldsymbol{\xi})S^\mathsf{T} \\ S\Lambda(\boldsymbol{\xi}) & Q \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{\psi}(t,\boldsymbol{\eta}) \end{bmatrix} - \epsilon V(\boldsymbol{x}). \end{aligned}$$

Inserting $\boldsymbol{\eta} = \Lambda(\boldsymbol{\xi})\boldsymbol{x}$ and using that the nonlinear function $\boldsymbol{\psi}(t, \boldsymbol{\eta})$ is (Q, S, K)-dissipative by assumption implies

$$\dot{V} \leq -\underbrace{\begin{bmatrix} \boldsymbol{\psi}(t,\boldsymbol{\eta}) \\ \boldsymbol{\eta} \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & K \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}(t,\boldsymbol{\eta}) \\ \boldsymbol{\eta} \end{bmatrix}}_{\geq 0} - \epsilon V(\boldsymbol{x}) \leq -\epsilon V(\boldsymbol{x}).$$

The exponential stability follows from $V(\boldsymbol{x}) \leq V(\boldsymbol{x}_0)\exp(-\epsilon t)$ for all $t \geq t_0$.

A.2 Proof of Theorem 2

Consider the positive definite Lyapunov function candidate $V(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^\mathsf{T} P \tilde{\boldsymbol{x}}$ with $P > 0$. The system (32) is $(-K, S^\mathsf{T}, -Q)$-SSD by assumption. Thus, the rate of change of $V(\tilde{\boldsymbol{x}})$ is given by adapting (27), that is,

$$\dot{V} = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{v} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -2PR(\boldsymbol{\xi})P & PG(\boldsymbol{\xi}) \\ G^\mathsf{T}(\boldsymbol{\xi})P & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{v} \end{bmatrix} \leq \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{\phi} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -H_N^\mathsf{T}(\boldsymbol{\xi})KH_N(\boldsymbol{\xi}) & -H_N^\mathsf{T}(\boldsymbol{\xi})S^\mathsf{T} \\ -SH_N(\boldsymbol{\xi}) & -Q \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{\phi} \end{bmatrix} - \epsilon V(\tilde{\boldsymbol{x}}).$$

Using $\boldsymbol{v} = -\boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})$ yields

$$\dot{V} = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{v} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -2PR(\boldsymbol{\xi})P & PG(\boldsymbol{\xi}) \\ G^\mathsf{T}(\boldsymbol{\xi})P & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{v} \end{bmatrix} \leq \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{v} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -H_N^\mathsf{T}(\boldsymbol{\xi})KH_N(\boldsymbol{\xi}) & H_N^\mathsf{T}(\boldsymbol{\xi})S^\mathsf{T} \\ SH_N(\boldsymbol{\xi}) & -Q \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{v} \end{bmatrix} - \epsilon V(\tilde{\boldsymbol{x}})$$

and the LMI (33) follows by rearranging the terms. The exponential stability of (32) is given by Lemma 2, since $\boldsymbol{\phi}(\boldsymbol{z}, \boldsymbol{\sigma}, \boldsymbol{y}, \boldsymbol{u})$ is (Q, S, K)-dissipative by assumption. The Lyapunov function can be bounded by $V(\tilde{\boldsymbol{x}}) \leq V(\tilde{\boldsymbol{x}}_0) \exp(-\epsilon t)$ and $\mu_{\min} \|\tilde{\boldsymbol{x}}\|^2 \leq V(\tilde{\boldsymbol{x}}) \leq \mu_{\max} \|\tilde{\boldsymbol{x}}\|^2$, where $0 < \mu_{\min} \leq \mu_{\max}$ are the minimal and maximal eigenvalue of P. Combining the upper and lower bounds for $V(\tilde{\boldsymbol{x}})$ yields the estimate

$$\|\tilde{\boldsymbol{x}}\| \leq \sqrt{\frac{\mu_{\max}}{\mu_{\min}}} \|\tilde{\boldsymbol{x}}_0\| \exp\left(-\frac{\epsilon}{2} t\right)$$

and thus proves the claim.