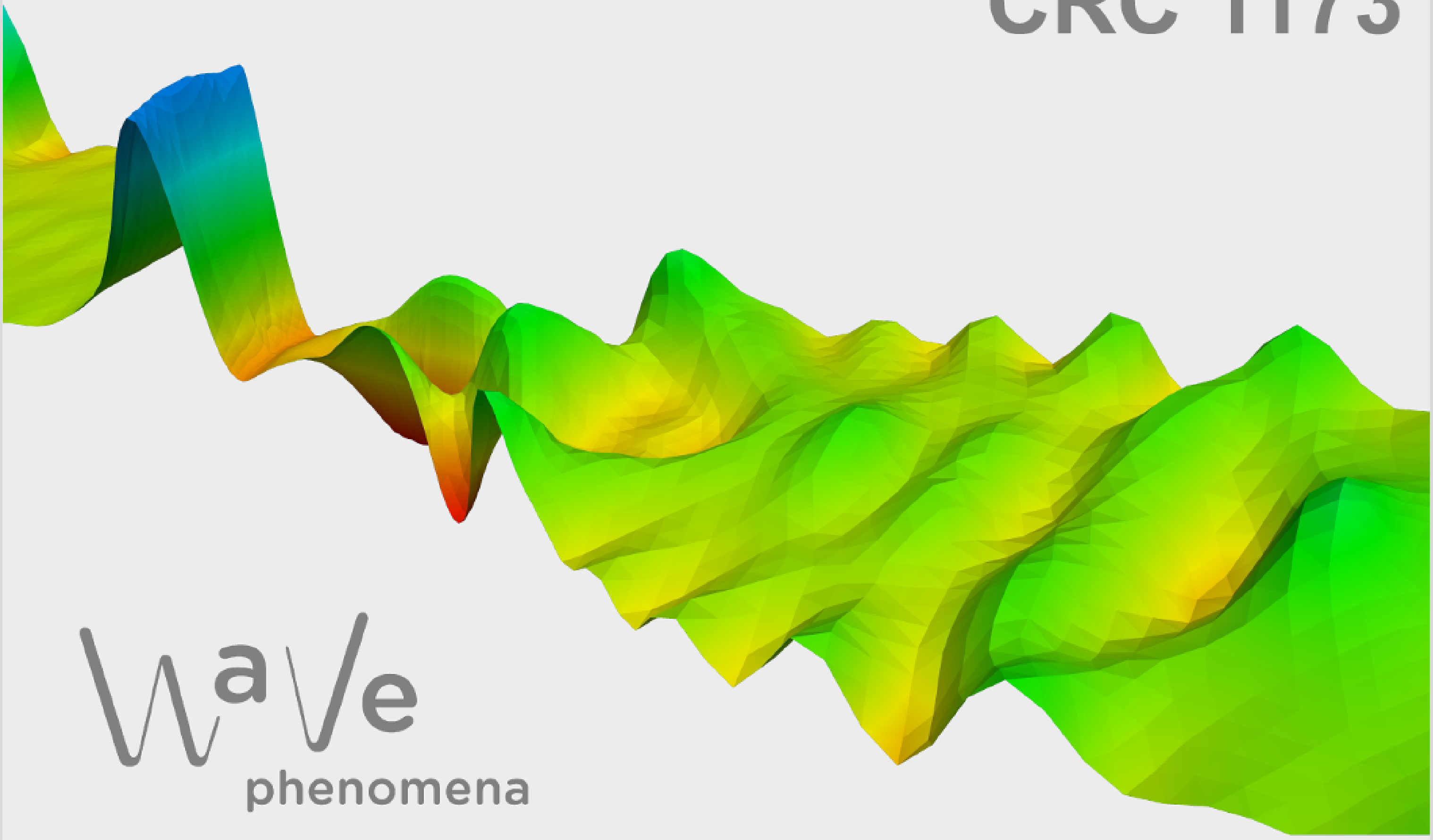
Error bounds for discrete minimizers of the Ginzburg–Landau energy in the high-κ regime

Benjamin Dörich, Patrick Henning

CRC Preprint 2023/11, March 2023



Participating universities

Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

TU
WIEN

Funded by

DFG

ISSN 2365-662X

ERROR BOUNDS FOR DISCRETE MINIMIZERS OF THE GINZBURG–LANDAU ENERGY IN THE HIGH-κ REGIME

BENJAMIN DÖRICH AND PATRICK HENNING

Abstract. In this work, we study discrete minimizers of the Ginzburg–Landau energy in finite element spaces. Special focus is given to the influence of the Ginzburg–Landau parameter κ. This parameter is of physical interest as large values can trigger the appearance of vortex lattices. Since the vortices have to be resolved on sufficiently fine computational meshes, it is important to translate the size of κ into a mesh resolution condition, which can be done through error estimates that are explicit with respect to κ and the spatial mesh width h. For that, we first work in an abstract framework for a general class of discrete spaces, where we present convergence results in a problem-adapted κ-weighted norm. Afterwards we apply our findings to Lagrangian finite elements and a particular generalized finite element construction. In numerical experiments we confirm that our derived L^2- and H^1-error estimates are indeed optimal in κ and h.

1. Introduction

Superconductors are materials that allow to conduct electricity without any electrical resistance. Letting $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, denote a bounded polyhedral Lipschitz domain occupied by a superconducting material, the superconductivity in Ω can be modeled by a complex-valued wave function $u : \Omega \to \mathbb{C}$ which is called the order parameter. The physical quantity of interest is $|u|^2$ which denotes the density of the superconducting electron pairs, where in the appropriate scaling, it holds $0 \leq |u|^2 \leq 1$. This means that the material is not superconducting (in normal state) in $x \in \Omega$ if $|u(x)|^2 = 0$ and behaves like a perfect superconductor if $|u(x)|^2 = 1$. In between, different degrees of superconductivity are possible. Of particular interest are mixed normal-superconducting states where both phases coexist in a lattice of quantized vortices [1].

Mathematically, the order parameter can be characterized as a minimizer of the Ginzburg–Landau energy (or Gibbs free energy) given by

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u + \mathrm{i}\kappa A u|^2 + \frac{\kappa^2}{2}\big(1 - |u|^2\big)^2 \, dx, \tag{1.1}$$

where $A\colon \Omega \to \mathbb{R}^d$ is a real-valued magnetic potential and κ is the so-called Ginzburg–Landau parameter, a material parameter that correlates with the temperature and determines the type of superconductor. By the necessary condition for local extrema, any minimizer $u \in H^1(\Omega)$ must fulfill the condition $E'(u) = 0$, which is known as the Ginzburg–Landau equation (GLE) and reads written out (cf. [18])

$$\operatorname{Re} \int_\Omega \big(\nabla u + \mathrm{i}\kappa A u\big) \cdot \big(\nabla \varphi + \mathrm{i}\kappa A \varphi\big)^* + \kappa^2 \big(|u|^2 - 1\big) u \varphi^* \, dx = 0 \qquad \text{for } \varphi \in H^1(\Omega). \tag{1.2}$$

The real-valued magnetic potential $A\colon \Omega \to \mathbb{R}^d$ in the GLE is typically unknown and can be inferred from an external magnetic field H through the condition $H = \operatorname{curl} A$ which is then added as a penalty term to the energy. In this work we consider the simplifying case that A is given, where the focus of our analysis is rather the influence of κ on the accuracy of numerical approximations.

1991 *Mathematics Subject Classification.* Primary: 65N12, 65N15. Secondary: 65N30, 35Q56.

Key words and phrases. Ginzburg–Landau equation, superconductivity, error analysis, finite element method.

The first author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173. The second authors also acknowledges the support by the Deutsche Forschungsgemeinschaft through the grant HE 2464/7-1. Parts of this paper were written while the authors enjoyed the kind hospitality of the Institute for Mathematical Science in Singapore in February 2023 during the program on "Multiscale Analysis and Methods for Quantum and Kinetic Problems".

In fact, the size of the parameter κ is crucial for the appearance of vortices [38–41]. On the one hand, if κ is too small, no vortices will appear. On the other hand, the larger the value of κ, the more vortices appear in the lattice and the more point-like they become [2, 38]. The so-called high-κ regime is hence the physically most interesting regime, but numerically it is also the most challenging one because it requires fine meshes to resolve all lattice structures. This raises an important practical question: how fine do we have to select the mesh size relative to the size of κ so that the numerical approximations capture the correct vortex pattern? Motivated by this question, the main goal of this work is to derive rigorous error bounds for the discrete minimizers with constants that are explicit and optimal in the spatial parameter h and the Ginzburg–Landau parameter κ.

To the best of our knowledge, the only work where the approximation properties of discrete solutions to the stationary GLE were analyzed is the seminal SIAM Review article by Du, Gunzburger and Peterson [18] (see also [19] for periodic boundary conditions). The paper considers H^1-error estimates in finite element spaces for both the order parameter u and the magnetic potential A. The proof technique considers fixed (compact) intervals of κ-values and does not trace all κ-dependencies that enter through the size of these intervals and through uniform bounds for certain operator norms (that are linked to the chosen interval). The proof also considers a modified setting where $E''(u)$ is assumed to have a trivial kernel. However, the solutions to the GLE (1.2) are known to be only locally unique up to gauge transformations [18]. In our case, these transformations are of the form $u \mapsto e^{\mathrm{i}\theta}u$ for any $\theta \in \mathbb{R}$. In fact, it is easily seen that $E(u) = E(e^{\mathrm{i}\theta}u)$ for all such θ, which hence leads to a cluster of (qualitatively equivalent) solutions $e^{\mathrm{i}\theta}u$. In turn, we have $\langle E''(u)\, e^{\mathrm{i}\pi/2}u, \cdot\rangle = 0$ which shows that $E''(u)$ can become singular. Hence, it makes sense to revisit the results [18] with new proof techniques that allow us to follow all κ-dependencies and which allow us to avoid an assumption of local uniqueness.

Our analysis is performed in a general framework of finite element methods, and we state our results under natural assumptions on the discrete spaces. We first establish bounds on the discrete minimizers which are explicit in κ and independent of h. This enables us to provide an abstract convergence result which identifies a suitable, continuous minimizer of (1.1). This a priori information is crucial in the derivation of the error bounds. In order to exploit the structure of the problem, we have to study the properties of the second Fréchet derivative of the energy E. In particular, we carry over the inf-sup stability to our discrete setting under a smallness condition related to the product κh. Let us emphasize that this is not a technical issue, but is indeed observed in our numerical experiments. We employ a problem adapted scalar product and its Ritz projection, which captures the one-dimensional kernel of E'', to extract optimal error bounds not only for the H^1-norm, but also new error bounds for the L^2-norm and the energy. Our numerical experiments confirm that the predicted scaling of the error in κ and h is asymptotically sharp.

It is worth to mention that, aside from stationary Ginzburg–Lindau equations, there has been a lot of work on the numerical analysis of the time-dependent problem that describes the dynamics of superconductors, where we exemplarily refer to [11, 12, 14–17, 20, 22, 23, 30–32] and the references therein. For works with a particular emphasis on tracing the influence of κ in the estimates, we refer to [7–9] for the case of vanishing vector potentials A. Due to the different nature of the time-dependent problem, we will not discuss the equation any further here.

The rest of the paper is organized as follows: In Section 2, we introduce the analytical framework and present some results on continuous minimizers of (1.1). In particular, we discuss the assumptions concerning uniqueness of minimizers. For an abstract finite element space discretization, we present in Section 3 our main results on the existence, boundedness, and approximation of discrete minimizers. An application to linear Lagrange finite elements is also given. Numerical experiments which illustrate our theoretical findings and confirm the convergence rates as well as the κ-dependency of our bounds are shown in Section 4. The proofs of our main results are given in Section 5. Finally, in Section 6 we present a nonstandard application of the abstract result to spaces based on the Localized Orthogonal Decomposition for which we prove that the resolution coupling between h and κ can be relaxed.

Notation. For a complex number $z \in \mathbb{C}$, we use z^* for the complex conjugate of z. In the whole paper we further denote by $L^2 := L^2(\Omega, \mathbb{C})$ the Hilbert space of L^2-integratable complex functions, but equipped with the *real* scalar product $m(u,v) := \operatorname{Re}\int_\Omega v\, w^*\, dx$ for $v, w \in L^2$. Hence, we interpret the space as a *real* Hilbert space. Analogously, we equip the space $H^1 := H^1(\Omega, \mathbb{C})$ with the scalar product $m(v,w) + m(\nabla v, \nabla w)$. This interpretation is crucial so that the Fréchet derivatives of E are meaningful and exist on H^1. For any space X, we denote its dual space by X'. Note that this implies, that the elements of the dual space of H^1 consist of real-linear functionals, which are not necessarily complex-linear. For example, if $F(v) := m(f, v)$ for some $f \in L^2$, then it holds $F(\alpha\, v) = \alpha\, F(v)$ if $\alpha \in \mathbb{R}$, but in general *not* if $\alpha \in \mathbb{C}$.

In the following C will denote a generic constant which is independent of κ and the spatial mesh parameter h, but might depend on numerical constants as well as Ω and A. In particular, we will write $\alpha \lesssim \beta$ if there is a constant C independent of κ and h such that $\alpha \leq C\, \beta$.

2. Analytical framework

In this section, we present several results concerning the continuous minimizers of (1.1).

From now on, we assume that the magnetic potential A satisfies

$$A \in L^\infty(\Omega, \mathbb{R}^d), \qquad \operatorname{div} A = 0 \text{ in } \Omega, \qquad A \cdot \nu = 0 \text{ on } \partial\Omega. \tag{2.1}$$

Further, we introduce the dual pairing $\langle u, \varphi \rangle := \langle u, \varphi \rangle_{(H^1)', H^1}$, and the bilinear forms given by

$$m(u,\varphi) := \operatorname{Re}\int_\Omega u \varphi^*\, dx, \qquad a_A(u,\varphi) := \operatorname{Re}\int_\Omega \big(\nabla u + \mathrm{i}\kappa A u\big) \cdot \big(\nabla \varphi + \mathrm{i}\kappa A \varphi\big)^*\, dx, \tag{2.2}$$

as well as the norm $||\varphi||^2_{H^1} := ||\nabla\varphi||^2_{L^2} + ||\varphi||^2_{L^2}$, the scaled norms

$$||\varphi||^2_{H^1_\kappa} := ||\nabla\varphi||^2_{L^2} + \kappa^2 ||\varphi||^2_{L^2}, \qquad ||\varphi||^2_{H^2_\kappa} := ||\varphi||^2_{H^2} + \kappa^2 ||\varphi||^2_{H^1_\kappa}, \tag{2.3}$$

and the induced norm $||f||_{(H^1_\kappa)'} = \sup_{\varphi \in H^1} \frac{f(\varphi)}{||\varphi||_{H^1_\kappa}}$. We abbreviate $A_\infty = ||A||_{L^\infty}$, and define the stabilized inner product on $H^1 = H^1(\Omega)$ for $u, \varphi \in H^1$ by

$$\hat{a}_\kappa(u,\varphi) := a_A(u,\varphi) + \beta^2 m(u,\varphi)_{L^2}, \quad \text{with } \beta^2 = \kappa^2(A^2_\infty + 1). \tag{2.4}$$

We call it stabilized since this modification enables us to show boundedness and coercicvity of $\hat{a}_\kappa(\cdot,\cdot)$ with respect to the H^1_κ-norm defined in (2.3).

Lemma 2.1. *There are κ-independent constants $C_{bnd}, C_{coe} > 0$ such that for all $v, \varphi \in H^1$*

$$\hat{a}_\kappa(v,\varphi) \leq C_{bnd}\, ||v||_{H^1_\kappa} ||\varphi||_{H^1_\kappa}, \qquad \textit{and} \qquad \hat{a}_\kappa(\varphi,\varphi) \geq C_{coe}\, ||\varphi||^2_{H^1_\kappa}.$$

Proof. The boundedness is a straightforward application of the Cauchy-Schwarz inequality. For the coercivity, we note that by Young's inequality it holds

$$|\nabla\varphi + \mathrm{i}\kappa A\varphi|^2 \geq |\nabla\varphi|^2 - 2|\nabla\varphi||\kappa A\varphi| + |\kappa A\varphi|^2 \geq \frac{1}{2}|\nabla\varphi|^2 - \kappa^2 A^2_\infty |\varphi|^2.$$

By the choice of β, we conclude the lower bound. □

A straightforward calculation shows that the energy is (real-)Fréchet differentiable and satisfies for all $\varphi \in H^1$

$$\langle E'(u), \varphi \rangle = \operatorname{Re}\int_\Omega \big(\nabla u + \mathrm{i}\kappa A u\big) \cdot \big(\nabla\varphi + \mathrm{i}\kappa A\varphi\big)^* + \kappa^2\big(|u|^2 - 1\big) u \varphi^*\, dx. \tag{2.5}$$

In particular any minimizer $u \in H^1$ satisfies $E'(u) = 0$. Our first result collects the existence of a minimizer u and its properties.

Theorem 2.2. *For every $\kappa \geq 0$ there exists a minimizer $u \in H^1$ of (1.1). Further, any minimizer fulfills*

$$|u(x)| \leq 1 \textit{ for all } x \in \Omega, \qquad ||u||_{H^1_\kappa} \lesssim \kappa, \quad \textit{and if } \Omega \textit{ is convex then } u \in H^2 \textit{ and } \quad ||u||_{H^2_\kappa} \lesssim \kappa^2,$$

where the hidden constants in the above estimates are independent of κ and u.

Proof. First note that the energy E is continuous in $H^1(\Omega)$, and further weakly lower semi-continuous, see e.g., [42, Thm. 1.6]. In addition, a simple calculation shows

$$E(u) = \hat{a}_\kappa(u,u) + \frac{\kappa^2}{2}\int_\Omega \big(1+\frac{2\beta^2}{\kappa^2} - |u|^2\big)^2 + 1 - \big(1+\frac{2\beta^2}{\kappa^2}\big)^2\,dx,$$

and hence $E(u) \to \infty$ as $||u||_{H^1_\kappa} \to \infty$. The standard arguments then imply the existence of a minimizer, see e.g., [42, Thm. 1.2]. For the pointwise bound, we refer to [18, Prop. 3.11], which implies a bound in L^2 independent of κ. We further have

$$||\nabla u||_{L^2} \leq ||\nabla u + \mathrm{i}\kappa A u||_{L^2} + \kappa A_\infty ||u||_{L^2} \lesssim E(0)^{1/2} + \kappa \lesssim \kappa.$$

Since $E'(u) = 0$, we rearrange to

$$a_A(u,\varphi) = -\kappa^2 \operatorname{Re}\int_\Omega \big(|u|^2-1\big)u\varphi^*\,dx = m(f,\varphi)$$

with $||f||_{L^2} \lesssim \kappa^2$, and obtain with (2.1)

$$\operatorname{Re}\int_\Omega \nabla u\cdot\nabla\varphi^*\,dx = m(f,\varphi) - \operatorname{Re}\int_\Omega \big(2\mathrm{i}\kappa A\cdot\nabla u + \kappa^2|A|^2 u\big)\varphi^*\,dx.$$

If Ω is convex, standard elliptic regularity theory (cf. [24]) gives us

$$||u||_{H^2} \lesssim ||f||_{L^2} + \kappa^2||u||_{L^2} + \kappa||\nabla u||_{L^2} \lesssim \kappa^2,$$

where we used the L^2- and H^1-bounds for u in the last step. □

Since u is a global minimizer of the energy E, it must not only hold $\langle E'(u),\varphi\rangle = 0$ but also $\langle E''(u)\varphi,\varphi\rangle \geq 0$ for all $\varphi\in H^1$. Later we will make use of these conditions. For that we require a corresponding representation of the second Fréchet derivative of E. This and its properties are summarized in the following lemma.

Lemma 2.3. (a) The energy is twice (real-)Fréchet differentiable and satisfies for $\varphi, v\in H^1$

$$\langle E''(u)v,\varphi\rangle = \operatorname{Re}\int_\Omega \big(\nabla v + \mathrm{i}\kappa A v\big)\cdot\big(\nabla\varphi + \mathrm{i}\kappa A\varphi\big)^* + \kappa^2\big((|u|^2-1)v\varphi^* + u^2 v^*\varphi^* + |u|^2 v\varphi^*\big)\,dx.$$

(b) For $\varphi, v\in H^1$ it holds

$$\langle E''(u)v,\varphi\rangle = \langle E''(u)\varphi, v\rangle \quad\text{and}\quad |\langle E''(u)v,\varphi\rangle| \lesssim ||v||_{H^1_\kappa}||\varphi||_{H^1_\kappa}.$$

Proof. The Fréchet derivative is computed in a straightforward manner, and the symmetry follows from the representation by noting the real part in front of the integral. For the bound, we employ Lemma 2.1 as well as $|u|\leq 1$. □

Let u be a minimizer of (1.1), then by the invariance under complex rotation, also $e^{\mathrm{i}\phi}u$ is a minimizer for any $\phi\in\mathbb{R}$. In particular, one easily shows that $\langle E''(u)\,\mathrm{i}u,\varphi\rangle = 0$ holds for all $\varphi\in H^1$. To tackle this indefiniteness, we define the $m(\cdot,\cdot)$-orthogonal complement of $\mathrm{i}u$ in H^1 by

$$H^1_{\mathrm{i}u} := H^1 \cap (\mathrm{i}u)^\perp := \{\varphi\in H^1 \,|\, m(\mathrm{i}u,\varphi) = 0\}.$$

In our error analysis we will restrict ourselves to this space. The choice of $H^1_{\mathrm{i}u}$ is further discussed in connection with Assumption 2.5 below.

Note that $H^1_{\mathrm{i}u}$ is a closed subspace of H^1. Since the variational problems in the following proofs are posed on this subspace, we show the following properties of their solutions.

Lemma 2.4. *For any $f\in L^2(\Omega)$, there is $z\in H^1_{\mathrm{i}u}\subset H^1(\Omega)$ such that*

$$\hat{a}_\kappa(z,\varphi) = m(f,\varphi), \qquad \text{for all } \varphi\in H^1_{\mathrm{i}u},$$

and there hold the bounds

$$||z||_{H^1_\kappa} \lesssim ||f||_{(H^1_\kappa)'} \lesssim \frac{1}{\kappa}||f||_{L^2} \quad \text{and, if } \Omega \text{ is convex, then } z\in H^2 \text{ and } ||z||_{H^2_\kappa} \lesssim ||f||_{L^2},$$

where the (hidden) constants in the bounds are independent of κ.

Proof. Since $\hat{a}_\kappa(\cdot,\cdot)$ is still coercive on $H^1_{\mathrm{i}u}$, we immediately obtain the unique solution, and also the bounds in H^1_κ. Furthermore, we have for any $f \in L^2$ that

$$||f||_{(H^1_\kappa)'} = \sup_{||\varphi||_{H^1_\kappa}=1} m(f,\varphi) \leq \sup_{||\varphi||_{H^1_\kappa}=1} \frac{1}{\kappa}||f||_{L^2}\kappa||\varphi||_{L^2} \leq \frac{1}{\kappa}||f||_{L^2}, \tag{2.6}$$

which yields the second inequality. For the bound in the H^2_κ-norm for convex domains, let $\varphi \in H^1$ and decompose as $\varphi = \widehat{\varphi} + \alpha(\mathrm{i}u)$ with $\widehat{\varphi} \in H^1_{\mathrm{i}u}$ and $\alpha = m(\varphi, \mathrm{i}u)||u||^{-2}_{L^2}$. Then,

$$\begin{aligned}\hat{a}_\kappa(z,\varphi) &= \hat{a}_\kappa(z,\widehat{\varphi}) + \alpha\,\hat{a}_\kappa(z,\mathrm{i}u) = m(f,\widehat{\varphi}) + \alpha\,\hat{a}_\kappa(z,\mathrm{i}u)\\ &= m(f,\varphi) - \alpha\, m(f,\mathrm{i}u) + \alpha\, a_A(z,\mathrm{i}u),\end{aligned}$$

where we used (2.4) in the last step. We first note

$$|m(f,\varphi) - \alpha\, m(f,\mathrm{i}u)| \leq 2||f||_{L^2}||\varphi||_{L^2}\,,$$

and then employ $E'(\mathrm{i}u) = 0$ to obtain

$$|a_A(z,\mathrm{i}u)| = |\langle E'(\mathrm{i}u), z\rangle - \kappa^2\,\mathrm{Re}\int_\Omega \big(|u|^2-1\big)\mathrm{i}uz^*\,dx| \leq \kappa^2||u||_{L^2}||z||_{L^2} \lesssim ||f||_{L^2},$$

where we exploited $\kappa^2||z||_{L^2} \lesssim ||f||_{L^2}$ in the last line. Altogether we have shown that there exists some $f_z \in L^2$ such that it holds for all $\varphi \in H^1$

$$\hat{a}_\kappa(z,\varphi) = m(f_z,\varphi), \quad ||f_z||_{L^2} \lesssim ||f||_{L^2}. \tag{2.7}$$

We conclude as in Theorem 2.2: We write

$$\hat{a}_\kappa(z,\varphi) = \mathrm{Re}\int_\Omega \nabla z\cdot\nabla\varphi^*\,dx + \mathrm{Re}\int_\Omega \big(\beta^2 z + 2\mathrm{i}\kappa A\cdot\nabla z + \kappa^2|A|^2 z\big)\varphi^*\,dx, \tag{2.8}$$

and since the second term is in L^2, we have $z \in H^2$ and

$$||z||_{H^2} \lesssim ||f_z||_{L^2} + \kappa^2||z||_{L^2} + \kappa||\nabla z||_{L^2} \lesssim ||f||_{L^2},$$

where we used the L^2- and H^1-bounds for z in the last step. □

We now turn to the key assumption in our analysis. As we have seen above, we cannot expect uniqueness of a minimizer due to the rotation invariance. However, we assume that apart from this, the minimizer is locally unique. For that we can restrict the energy to an appropriate subspace. To be precise, if u is a global minimizer of E, we know that $E'(u) = 0$ and that the spectrum of $E''(u)$ is non-negative. On the other hand, it is easily seen that $\mathrm{i}u$ is an eigenfunction of $E''(u)$ with eigenvalue 0. This eigenvalue corresponds to the aforementioned invariance of E under rotations of the form $e^{\mathrm{i}\phi}$. By assuming that the remaining spectrum of $E''(u)$ is strictly positive we can hence guarantee that the solution u is locally unique (up to rotations). A positive spectrum of $E''(u)$ on the $m(\cdot,\cdot)$-orthogonal complement of the eigenfunction $\mathrm{i}u$ (i.e. the space $H^1_{\mathrm{i}u}$) implies inf-sup stability of $E''(u)$ on $H^1_{\mathrm{i}u}$. This is precisely what the following assumption says.

Assumption 2.5. *Let u be a minimizer of (1.1). Then, there is a constant $C_{\mathrm{sol}}(u,\kappa) \gtrsim 1$ such that*

$$C^{-1}_{\mathrm{sol}}(u,\kappa) \leq \inf_{v\in H^1_{\mathrm{i}u}} \sup_{\varphi\in H^1_{\mathrm{i}u}} \frac{\langle E''(u)v,\varphi\rangle}{||v||_{H^1_\kappa}||\varphi||_{H^1_\kappa}}. \tag{2.9}$$

Let us note that the condition $C_{\mathrm{sol}}(u,\kappa) \gtrsim 1$ is not a restriction, since one can drop the condition replacing $C_{\mathrm{sol}}(u,\kappa)$ by $1 + C_{\mathrm{sol}}(u,\kappa)$ at every occurrence.

Remark 2.6. *From our numerical experiments, the precise growth of $C^{-1}_{\mathrm{sol}}(u,\kappa)$ with respect to κ does not become clearly visible. In fact, it turns out to be difficult to numerically compute the inf-sup constants on a space which contains information on the exact solution. In addition, we are not aware of any literature (neither in analysis nor numerics) addressing the (spectral) properties of $E''(u)$. We are convinced that this an interesting research question which might be pursued in the future, both analytically and numerically.*

From the above assumption, we can conclude solvability and a priori bounds which will play a crucial role in the presented error analysis below. Let us note that the inclusion $H^1_{\mathrm{iu}} \subset H^1$ implies for the dual spaces $(H^1)' \subset (H^1_{\mathrm{iu}})'$.

Corollary 2.7. *Let Assumption 2.5 hold.*

(a) For any $f \in (H^1_{\mathrm{iu}})'$, there is a unique $z \in H^1_{\mathrm{iu}}$ such that

$$\langle E''(u)z, \varphi\rangle = \langle f, \varphi\rangle, \quad \text{for all} \quad \varphi \in H^1_{\mathrm{iu}}, \tag{2.10}$$

which satisfies the estimate

$$||z||_{H^1_\kappa} \leq C_{\mathrm{sol}}(u,\kappa)||f||_{(H^1_\kappa)'}.$$

(b) Let $z \in H^1_{\mathrm{iu}}$ be the solution of (2.10) with $f \in L^2$. Then, it further holds

$$||z||_{H^1_\kappa} \leq \frac{C_{\mathrm{sol}}(u,\kappa)}{\kappa}||f||_{L^2} \quad \text{and, if } \Omega \text{ is convex, then } z \in H^2 \text{ and } ||z||_{H^2_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)||f||_{L^2}.$$

Proof. By standard theory for indefinite differential equations (cf. [5]), the inf-sup stability in Assumption 2.5 directly gives the well-posedness of (2.10) together with the stability estimate $||z||_{H^1_\kappa} \leq C_{\mathrm{sol}}(u,\kappa)||f||_{(H^1_\kappa)'}$, hence proving (a). The first estimate in (b) is obtained from (2.6). Using this observation, we conclude that $z \in H^1_{\mathrm{iu}}$ solves

$$\hat{a}_\kappa(z,\varphi) = m(\widetilde{f},\varphi), \quad \text{for all } \varphi \in H^1_{\mathrm{iu}}\,,$$

for some $\widetilde{f} \in L^2$ with $||\widetilde{f}||_{L^2} \lesssim C_{\mathrm{sol}}(u,\kappa)||f||_{L^2}$, and thus Lemma 2.4 gives the claim. □

If one considers domains with smooth boundaries, and uses magnetic vector potential in some higher order Sobolev spaces, higher regularity of the minimizer u can be derived. However, for our purposes the H^2-regularity is sufficient, and we hence turn to the spatial discretization.

3. SPACE DISCRETIZATION AND MAIN RESULTS

Let us consider some finite dimensional finite element space V_h which is a subspace of $H^1(\Omega)$ and where we recall that we assume Ω to be a polygonal (or polyhedral) Lipschitz domain. By h we denote a spatial parameter which tends to zero for a finer spatial resolution. We consider the closed subspace $V_h^\perp$ of V_h given by $V_h^\perp = V_h \cap (\mathrm{i}u)^\perp \subset H^1_{\mathrm{iu}}$ with orthogonality with respect to $m(\cdot,\cdot)$. Further, we denote by $\mathrm{R}^\perp_{\kappa,h} : H^1_{\mathrm{iu}} \to V_h^\perp$ the orthogonal projection satisfying

$$\hat{a}_\kappa(\mathrm{R}^\perp_{\kappa,h}w, \varphi_h) = \hat{a}_\kappa(w,\varphi_h), \quad \text{for all } \varphi_h \in V_h^\perp.$$

In the following assumption, we introduce some abstract conditions which are sufficient to carry out our error analysis and which are later verified for our examples.

Assumption 3.1. *The family of (non-empty) finite element spaces V_h has the following properties:*

(a) The family of spaces V_h is dense in $H^1(\Omega)$ in the sense that for each $\varphi_h \in H^1$ we have

$$\lim_{h\to 0} \inf_{\varphi_h \in V_h} ||\varphi - \varphi_h||_{H^1} = 0.$$

(b) Let $w \in H^1_{\mathrm{iu}}$ and $f \in L^2$ such that $\hat{a}_\kappa(w,\varphi) = m(f,\varphi)$ for all $\varphi \in H^1_{\mathrm{iu}}$. Then, it holds

$$||w - \mathrm{R}^\perp_{\kappa,h}w||_{H^1_\kappa} \lesssim h||f||_{L^2}, \tag{3.1}$$

where the constant is independent of h and κ.

The most prominent example that fulfills Assumption 3.1 are linear Lagrange finite element spaces which are discussed at the end of this section. Property (a) is obvious and to verify property (b), one first replaces $||f||_{L^2}$ by $||w||_{H^2_\kappa}$, and uses (for a convex domain Ω) H^2-regularity. We give the details below. Another, non-trivial example of generalized finite elements spaces is presented in Section 6.

Recall that we want to minimize the functional E from (1.1) over V_h, i.e.,

$$E(u_h) = \inf_{\varphi_h \in V_h} E(\varphi_h), \qquad E(\varphi_h) = \frac{1}{2} \int_\Omega |\nabla \varphi_h + \mathrm{i}\kappa A \varphi_h|^2 + \frac{\kappa^2}{2}\big(1 - |\varphi_h|^2\big)^2 \, dx. \tag{3.2}$$

Note that since V_h is finite dimensional, the existence of a minimizer u_h is always guaranteed. Our first result shows bounds on the discrete minimizer u_h in different norms and the corresponding energy, which are independent of h and behave in the parameter κ the same way as the exact minimizer u studied in Theorem 2.2. The proof is postponed to Section 5.

Lemma 3.2. *For all $h > 0$ let u_h be a minimizer of (3.2). Then there hold the bounds*

$$E(u_h) \lesssim \kappa^2 \qquad ||u_h||_{L^2} \lesssim 1, \qquad ||\nabla u_h||_{L^2} \lesssim \kappa, \qquad ||u_h||_{H^1_\kappa} \lesssim \kappa,$$

where the hidden constants are independent of h and κ.

Our main findings are collected in the following theorem. We provide error bounds for the discrete minimizers which are explicit in the parameter κ and the mesh width h. In addition, we show that the error behaves as the quasi-best approximation of $H^1_{\mathrm{i}u}$ in $V_h^\perp$.

Theorem 3.3. *Let Assumption 2.5 and 3.1 hold, and let $h \leq h_0$ be sufficiently small such that in particular $\kappa C_{\mathrm{sol}}(u,\kappa)h$ is small. Then, there is neighborhood $U \subset H^1(\Omega)$ of each discrete minimizer u_h of (3.2) such that there is a unique minimizer $u \in U$ of (1.1) with*

$$m(u_h, \mathrm{i}u) = 0,$$

and we have the error bounds

$$\begin{aligned} ||u - u_h||_{H^1_\kappa} &\lesssim (1 + \kappa C_{\mathrm{sol}}(u,\kappa)h) \inf_{\varphi_h \in V_h^\perp} ||u - \varphi_h||_{H^1_\kappa}, \\ ||u - u_h||_{L^2} &\lesssim h\, C_{\mathrm{sol}}(u,\kappa)\,(1 + \kappa C_{\mathrm{sol}}(u,\kappa)h) \inf_{\varphi_h \in V_h^\perp} ||u - \varphi_h||_{H^1_\kappa}, \end{aligned}$$

as well as the following estimate on the error in the energy

$$0 \leq E(u_h) - E(u) \lesssim ||u - u_h||^2_{H^1_\kappa} \big(1 + \kappa^{1/2}||u - u_h||_{H^1_\kappa} + \kappa ||u - u_h||^2_{H^1_\kappa}\big),$$

where the hidden constants are independent of κ, $C_{\mathrm{sol}}(u,\kappa)$, and h.

The proof is divided in several steps which are outlined in detail in Section 5. The first application of the results are Lagrangian finite elements. In the following, we denote by $\mathcal{T}_h$ a conforming family of partitions of the domain Ω consisting of simplical elements K. For the space $\mathcal{P}_1(K)$ of complex-valued polynomials of degree less than or equal to 1 on K, we consider the finite element space

$$V_h := \{\varphi_h \in H^1(\Omega) \mid \varphi_h|_K \in \mathcal{P}_1(K) \text{ for all } K \in \mathcal{T}_h\}. \tag{3.3}$$

We assume that the partition $\mathcal{T}_h$ is shape-regular and the L^2-projection, defined via $m(\pi_h v, \varphi_h) = m(v, \varphi_h)$ for all $\varphi_h \in V_h$, is H^1-stable, i.e.,

$$||\pi_h \varphi||_{H^1} \lesssim ||\varphi||_{H^1}, \tag{3.4}$$

with a constant independent of h. This condition is always fulfilled for quasi-uniform triangulations, but is also valid for certain adaptively refined meshes. For a detailed discussion on criteria when (3.4) holds, we refer to [6, 13]. In this setting, we obtain convergence rates which are explicit in the parameter κ and the mesh width h.

Corollary 3.4. *Let the conditions of Theorem 3.3 hold, and assume in addition that Ω is convex. For Lagrangian finite elements which satisfy (3.4), the error bounds in Theorem 3.3 can be further estimated by*

$$||u - u_h||_{H^1_\kappa} \lesssim \kappa^2 h \qquad \text{and} \qquad ||u - u_h||_{L^2(\Omega)} \lesssim C_{\mathrm{sol}}(u,\kappa)\kappa^2 h^2,$$

where the hidden constants are independent of κ, $C_{\mathrm{sol}}(u,\kappa)$, and h.

The proof is presented in Section 5.

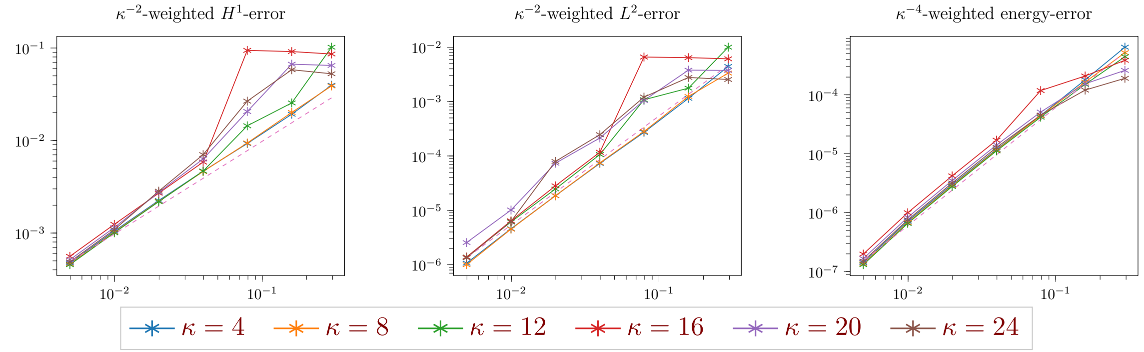


Figure 1. Convergence in the mesh size h for κ-weighted errors in the H^1- and L^2-norm and for the energy, for $\kappa = 4, 8, 12, 16, 20, 24$. The errors in L^2 and H^1 are scaled by κ^{-2} and the error in energy by κ^{-4}. The dashed lines indicate order $\mathcal{O}(h)$ in the left figure, and order $\mathcal{O}(h^2)$ in the center and right figure.

4. NUMERICAL EXPERIMENTS

Before we present the proof of our main result, we illustrate our theoretical findings with some numerical examples confirming the rates and the κ-dependence in our error bounds.

4.1. **Implementation.** For the discretization in space with linear Lagrange finite elements, we use the open source Python tool FEniCS [3, version 2018.1.0]. To compute a discrete minimizer, we applied a steepest descent approach using an implicit Euler method for the L^2 gradient flow. A direct application yields the following nonlinear iteration

$$m(u_h^{n+1}, \varphi_h) = m(u_h^n, \varphi_h) - \tau \, \langle E'(u_h^{n+1}), \varphi_h \rangle,$$

where $\tau > 0$ is some parameter. To avoid the solution of nonlinear systems several times, we replace $E'(u^{n+1})$ by the linearization

$$\langle E'(u_h^{n+1}), \varphi_h \rangle \to a_A(u_h^{n+1}, \varphi_h) + \kappa^2 \operatorname{Re} \int_\Omega (|u_h^n|^2 - 1) u_h^{n+1} \varphi_h^* \, dx,$$

and thus have to solve the following linear system for $u_h^{n+1} \in V_h$

$$m(u_h^{n+1}, \varphi_h) + \tau \, a_A(u_h^{n+1}, \varphi_h) + \tau \, \kappa^2 \operatorname{Re} \int_\Omega (|u_h^n|^2 - 1) u_h^{n+1} \varphi_h^* \, dx = m(u_h^n, \varphi_h)$$

for all $\varphi_h \in V_h$. In our experiments, we set $\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$, and use (on the coarsest mesh) the initial value $u_0 = 0.8 + 0.6\mathrm{i}$, For the finer grids, we use a "ramping" procedure, i.e., we use the minimizer on a coarser grid as the initial value on the next finer grid. The magnetic potential is chosen as

$$A(x, y) := \sqrt{2} \begin{pmatrix} \sin(\pi x) \cos(\pi y) \\ -\cos(\pi x) \sin(\pi y) \end{pmatrix},$$

and satisfies the assumptions in (2.1). Further, we set $\tau = 1$, and used the stopping criterion $|E(u_h^{n+1}) - E(u_h^n)| < \delta$ for a tolerance $\delta = 10^{-10}$. The code to reproduce the results presented in this paper is available on request.

4.2. **Numerical results.** We first illustrate the convergence in the spatial parameter h for different values of κ. To this end, we computed a reference solution on a finer grid using $h_{\max} \sim 2.5 \cdot 10^{-3}$. In order to compare the results for different values of κ, we divide the error in the H^1_κ- and L^2-norm by κ^2 and the energy by κ^4, see Figure 1. Here we recall that according to Corollary 3.4 we expect the H^1_κ-error to convergence with the rate $\kappa^2 h$, the L^2-error with the rate $\kappa^2 h^2$ and the energy-error with the rate $\kappa^4 h^4$. Indeed, we observe the predicted convergence in h and, in particular, the numerical experiments confirm the κ-scaling in our error estimates. The plot further indicates that the constants in front of the normalized errors are independent of κ.

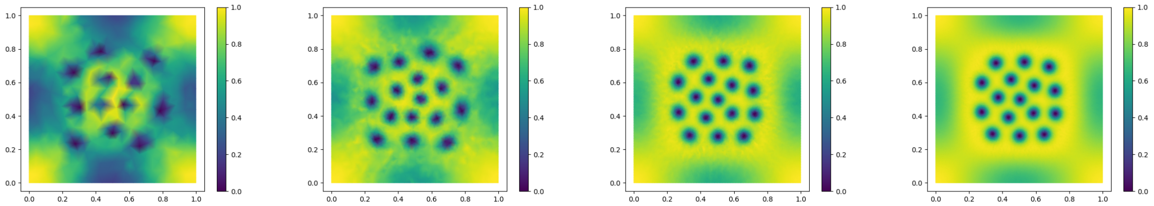


FIGURE 2. Minimizers for the Ginzburg–Landau parameter $\kappa = 48$ and different mesh widths $h \approx 8 \cdot 10^{-2}, 4 \cdot 10^{-2}, 2 \cdot 10^{-2}, 1 \cdot 10^{-2}$ (from left to right).

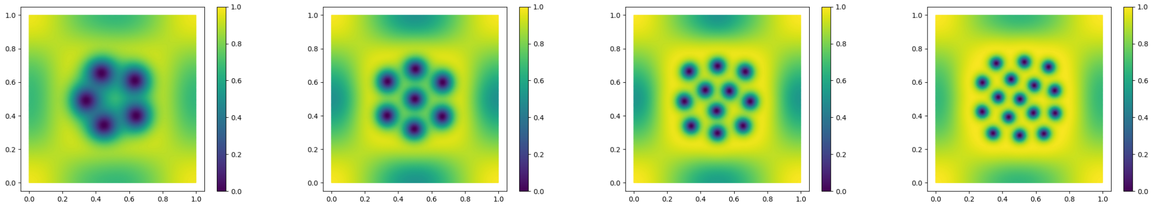


FIGURE 3. Different minimizers corresponding to the Ginzburg–Landau parameters $\kappa = 16, 24, 36, 48$ (from left to right) for $h \approx 2.5 \cdot 10^{-3}$.

Let us also note that for larger values of κ, we observe a preasymptotic behavior in h. We expect that this is related to the smallness condition for $\kappa C_{\mathrm{sol}}(u, \kappa)h$ stated in the theorem, which is required below in Lemma 5.4 for the discrete inf-sup stability. Since beyond the (numerically observed) threshold $\kappa h < 1$, the errors coincide for all values of κ, this is still in alignment with our theory.

In our second experiment, we first computed for $\kappa = 48$ the discrete minimizers for different values of $h \approx 8 \cdot 10^{-2}, 4 \cdot 10^{-2}, 2 \cdot 10^{-2}, 1 \cdot 10^{-2}$, see Figure 2. We observe that the number of vortices remains constant on the different discretization levels, but the precise pattern is only resolved on the finest grids. On the other hand, we plotted the minimizers for different values of $\kappa = 16, 24, 36, 48$, see Figure 3. We observe that the number of vortices increases with larger values of κ, which is in agreement with analytical results [2, 38].

5. PROOF OF THE MAIN RESULT

In this section, we provide the proof of our main results Theorem 3.3 and Corollary 3.4. We first show an abstract convergence result in order to identify possible limits of a sequence of discrete minimizers. Those are then used to establish convergence with rates, if we are sufficiently close to a continuous minimizer. Throughout this section, we let Assumptions 2.5 and 3.1 hold.

5.1. **Abstract convergence result.**

In order to deduce convergence, we first establish bounds on minimizes in the discrete space V_h which are independent of the spatial parameter h as formulated in Lemma 3.2.

Proof of Lemma 3.2. First note that for all $h > 0$ we have $0 \in V_h$, and thus by the minimizing property, we conclude the bound on the energy

$$E(u_h) \leq E(0) \leq \frac{\kappa^2}{2} \operatorname{vol}(\Omega).$$

This gives on the one hand

$$||\nabla u_h + \mathrm{i}\kappa A u_h||_{L^2} \leq E(u_h)^{1/2} \leq \kappa \operatorname{vol}(\Omega)^{1/2},$$

and on the other hand we estimate

$$\frac{\kappa^2}{2}\|1-|u_h|\,\|_{L^2}^2 \leq \frac{\kappa^2}{2}\int_\Omega \big(1-|u_h|\big)^2\big(1+|u_h|\big)^2\,dx \leq E(0) = \frac{\kappa^2}{2}\operatorname{vol}(\Omega)^{1/2},$$

and thus conclude

$$\|u_h\|_{L^2} \leq \|1-|u_h|\|_{L^2} + \operatorname{vol}(\Omega)^{1/2} \leq 2\operatorname{vol}(\Omega)^{1/2}.$$

Combining the estimates above, the bound on $\|\nabla u_h\|_{L^2}$ directly follows. ☐

With the uniform estimates on the discrete minimizers, following the approach in [10], we employ the Banach–Alaoglu theorem to obtain some limit which is an exact minimizer and by Assumption 2.5 locally unique up to complex rotation.

Proposition 5.1. *Denote by* $(u_h)_{h>0}$ *a family of minimizers of* (3.2)*. Then, there exists a minimizer* u_0 *of* (1.1) *such that there is a monotonically decreasing sequence* $(h_n)_{n\in\mathbb{N}}$ *with*

$$\lim_{n\to\infty}\|u_0-u_{h_n}\|_{H^1_\kappa} = 0.$$

In particular, we can define the twisted approximations

$$\widetilde{u}_{h_n} := e^{\mathrm{i}\phi_n}u_{h_n} \qquad \textit{where } \phi_n\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}] \textit{ is chosen such that } m(\widetilde{u}_{h_n},\mathrm{i}u_0)=0\,,$$

which also converge in H^1*, i.e.,*

$$\lim_{j\to\infty}\|\widetilde{u}_{h_n}-u_0\|_{H^1_\kappa} = 0.$$

Conversely, for any n*, the minimizer* u_{h_n} *is an approximation to* $e^{-\mathrm{i}\phi_n}u_0$*.*

Remark 5.2. *The assertion of Proposition 5.1 can be interpreted as follows. Assume that there exists a (sub-)sequence of discrete minimizers that keeps a positive distance to all exact minimizers, then this would be a contradiction to Proposition 5.1. Hence, for* h *sufficiently small, one always arrives at a neighborhood of some minimizer* u_0*, which is precisely the claim in Theorem 3.3.*

Proof of Proposition 5.1. The proof of convergence of a subsequence is along the lines of [10] if one takes into account the bounds provided in Lemma 3.2 together with the weak lower semi-continuity of E, see Theorem 2.2, and Assumption 3.1.

For the twisted approximations, we note that we can find some $\phi_n\in[-\frac{\pi}{2},\frac{\pi}{2}]$ such that real part of the inner product with $\mathrm{i}u_0$ vanishes if n is large enough. Thus, we obtain by the choice of ϕ_n

$$\sin\phi_n\; m(u_{h_n},u_{h_n}) = m(e^{\mathrm{i}\phi_n}u_{h_n},\mathrm{i}u_{h_n}) = m(e^{\mathrm{i}\phi_n}u_{h_n},\mathrm{i}u_{h_n}-\mathrm{i}u_0).$$

Since the right-hand side tends to zero, either $u_0=0$ or $\phi_n\to 0$ holds. In any case, we have

$$\|\widetilde{u}_{h_n}-u_0\|_{H^1_\kappa} \leq \|u_{h_n}-u_0\|_{H^1_\kappa} + |1-e^{\mathrm{i}\phi_n}|\,\|u_{h_n}\|_{H^1_\kappa} \to 0,$$

which yields the assertion. ☐

5.2. **Discrete inf-sup stability.** In order to derive the error estimates, we first establish a discrete version of the inf-sup condition in (2.9). In the proof, we need the following consequence of Assumption 3.1.

Corollary 5.3. *Let Assumption 3.1 hold, and let* $z\in H^1_{\mathrm{i}u}$ *be the solution of*

$$\langle E''(u)z,\varphi\rangle = \langle f,\varphi\rangle, \qquad \textit{for all } \varphi\in H^1_{\mathrm{i}u}.$$

Then, it holds the estimate

$$\|z-\mathrm{R}^\perp_{\kappa,h}z\|_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)h\|f\|_{L^2}. \tag{5.1}$$

Proof. We rewrite with Lemma 2.3

$$\hat{a}_\kappa(z,\varphi) = m(f,\varphi) - m(f_z,\varphi) \qquad \text{for all } \varphi\in H^1_{\mathrm{i}u}.$$

Here, f_z satisfies

$$\|f_z\|_{L^2} \lesssim \kappa^2\|z\|_{L^2} \lesssim C_{\mathrm{sol}}(u,\kappa)\|f\|_{L^2},$$

where we used part (b) in Corollary 2.7, and the approximation (3.1) in Assumption 3.1 gives the claim. ☐

The proof of the next lemma, which states the discrete inf-sup stability, is inspired by the thesis [37, Prop. 8.2.7], where this was done for the Helmholtz equation.

Lemma 5.4. (a) If $\kappa C_{\mathrm{sol}}(u,\kappa)h$ is sufficiently small, it holds for all $w_h \in V_h^{\perp}$

$$||w_h||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa) \sup_{\varphi_h \in V_h^{\perp}} \frac{\langle E''(u)w_h, \varphi_h\rangle}{||\varphi_h||_{H^1_\kappa}},$$

where the constant is independent of h and κ.

(b) For any $f \in (H^1_{\mathrm{iu}})'$, there is a unique $w_h \in V_h^{\perp}$ such that

$$\langle E''(u)w_h, \varphi_h\rangle = \langle f, \varphi_h\rangle, \qquad \text{for all } \varphi_h \in V_h^{\perp}$$

and it holds

$$||w_h||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)||f||_{(H^1_\kappa)'}.$$

Proof. Part (b) is a classical stability bound for inf-sup stable problems, cf. [5, Thm. 2.1]. Hence, claim (b) directly follows once we have shown (a). To do so, we fix $w_h \in V_h^{\perp}$ and observe for arbitrary $z \in H^1_{\mathrm{iu}}$

$$\langle E''(u)w_h, w_h + z\rangle = a_A(w_h, w_h) + \kappa^2 \operatorname{Re}\int_\Omega \big(2|u|^2 - 1\big)w_h w_h^* + u^2 w_h^* w_h^* \,dx + \langle E''(u)z, w_h\rangle. \tag{5.2}$$

Now let $z \in H^1_{\mathrm{iu}}$ be the solution to

$$\langle E''(u)z, \varphi\rangle = m(f, \varphi) \quad \text{ for all } \varphi \in H^1_{\mathrm{iu}} \qquad \text{with} \quad f = (\beta^2 + 2\kappa^2)w_h$$

and insert it into (5.2). Then, we obtain from (2.4) together with Lemma 2.1 and Lemma 2.3 that

$$\begin{aligned}||w_h||^2_{H^1_\kappa} &\lesssim \langle E''(u)w_h, w_h + z\rangle \lesssim \langle E''(u)w_h, w_h + \mathrm{R}^{\perp}_{\kappa,h}z\rangle + ||w_h||_{H^1_\kappa}||\mathrm{R}^{\perp}_{\kappa,h}z - z||_{H^1_\kappa}\\ &\lesssim \sup_{\varphi_h \in V_h^{\perp}} \frac{\langle E''(u)w_h, \varphi_h\rangle}{||\varphi_h||_{H^1_\kappa}}||w_h + \mathrm{R}^{\perp}_{\kappa,h}z||_{H^1_\kappa} + ||w_h||_{H^1_\kappa}||\mathrm{R}^{\perp}_{\kappa,h}z - z||_{H^1_\kappa}.\end{aligned}$$

It remains to study the terms with z. Here, we establish with Corollary 2.7 and (5.1) the bound

$$\kappa||z||_{H^1_\kappa} + h^{-1}||z - \mathrm{R}^{\perp}_{\kappa,h}z||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)||f||_{L^2} \lesssim \kappa C_{\mathrm{sol}}(u,\kappa)||w_h||_{H^1_\kappa}.$$

From this, we finally conclude

$$||w_h||^2_{H^1_\kappa} \lesssim \sup_{\varphi_h \in V_h^{\perp}} \frac{\langle E''(u)w_h, \varphi_h\rangle}{||\varphi_h||_{H^1_\kappa}} C_{\mathrm{sol}}(u,\kappa)||w_h||_{H^1_\kappa} + \kappa C_{\mathrm{sol}}(u,\kappa)h||w_h||^2_{H^1_\kappa},$$

and obtain the assertion (a) if $\kappa C_{\mathrm{sol}}(u,\kappa)h$ is sufficiently small by absorption. □

5.3. **Convergence with rates.** After these preparations, we can derive the error equation employing the second Fréchet derivative E''. To this end, we strive for a representation of the form

$$\langle E''(u)(\mathrm{R}^{\perp}_{\kappa,h}u - u_h), \varphi_h\rangle = \varepsilon_h(\varphi_h), \tag{5.3}$$

for $\varphi_h \in V_h^{\perp}$ and employ Lemma 5.4 to conclude a bound for $\mathrm{R}^{\perp}_{\kappa,h}u - u_h$. The right-hand side ε_h is studied in the following lemma.

Lemma 5.5. *Let u and u_h be minimizers of (1.1) and (3.2), respectively.*

(a) For $\varphi_h \in V_h^{\perp}$ it holds the representation (5.3) where $\varepsilon_h = \varepsilon_h^{\mathrm{lin}} + \varepsilon_h^{\mathrm{nonlin}}$ and

$$\begin{aligned}\varepsilon_h^{\mathrm{lin}}(\varphi_h) &= \kappa^2 \operatorname{Re}\int_\Omega \Big((|u|^2 - 1)(\mathrm{R}^{\perp}_{\kappa,h}u - u) + u^2(\mathrm{R}^{\perp}_{\kappa,h}u - u)^* + |u|^2(\mathrm{R}^{\perp}_{\kappa,h}u - u)\Big)\varphi_h^* \,dx\\ &\quad - \beta^2 \operatorname{Re}\int_\Omega (\mathrm{R}^{\perp}_{\kappa,h}u - u)\varphi_h^* \,dx,\\ \varepsilon_h^{\mathrm{nonlin}}(\varphi_h) &= 2\kappa^2 \operatorname{Re}\int_\Omega |u|^2 u \varphi_h^* \,dx + \kappa^2 \operatorname{Re}\int_\Omega |u_h|^2 u_h \varphi_h^* \,dx - \kappa^2 \operatorname{Re}\int_\Omega 2\big(|u|^2 u_h + u^2 u_h^*\big)\varphi_h^* \,dx.\end{aligned}$$

(b) The error terms are bounded by

$$\begin{aligned}||\varepsilon_h^{\mathrm{lin}}||_{(H^1_\kappa)'} &\lesssim \kappa||u-\mathrm{R}^\perp_{\kappa,h}u||_{L^2},\\ ||\varepsilon_h^{\mathrm{nonlin}}||_{(H^1_\kappa)'} &\lesssim \kappa\big(||u-u_h||^2_{L^4}+||u-u_h||^3_{L^6}\big),\end{aligned}$$

where the constants are independent of h and κ.

Proof. Inserting the exact solution u, we decompose ε_h as

$$\varepsilon_h^{\mathrm{lin}}(\varphi_h)=\langle E''(u)(\mathrm{R}^\perp_{\kappa,h}u-u),\varphi_h\rangle, \qquad \varepsilon_h^{\mathrm{nonlin}}(\varphi_h)=\langle E''(u)(u-u_h),\varphi_h\rangle,$$

and treat the two terms separately. We begin with the linear part and use Lemma 2.3, the definition of $\hat{a}_\kappa(\cdot,\cdot)$ in (2.4), and the orthogonality condition of $\mathrm{R}^\perp_{\kappa,h}$ to obtain

$$\begin{aligned}&\langle E''(u)(\mathrm{R}^\perp_{\kappa,h}u-u),\varphi_h\rangle\\ &\quad= a_A(\mathrm{R}^\perp_{\kappa,h}u-u,\varphi_h)+\kappa^2\,\mathrm{Re}\int_\Omega\Big((|u|^2-1)(\mathrm{R}^\perp_{\kappa,h}u-u)+u^2(\mathrm{R}^\perp_{\kappa,h}u-u)^*+|u|^2(\mathrm{R}^\perp_{\kappa,h}u-u)\Big)\varphi_h^*\,dx\\ &\quad= -\beta^2\,\mathrm{Re}\int_\Omega(\mathrm{R}^\perp_{\kappa,h}u-u)\varphi_h^*\,dx\\ &\qquad+\kappa^2\,\mathrm{Re}\int_\Omega\Big((|u|^2-1)(\mathrm{R}^\perp_{\kappa,h}u-u)+u^2(\mathrm{R}^\perp_{\kappa,h}u-u)^*+|u|^2(\mathrm{R}^\perp_{\kappa,h}u-u)\Big)\varphi_h^*\,dx.\end{aligned}$$

Using that $\kappa||\varphi_h||_{L^2}\le||\varphi_h||_{H^1_\kappa}$ gives the first estimate in part (b).

For the nonlinear part, we note with Lemma 2.3 the identity for $v,\varphi\in H^1$

$$\langle E''(v)v,\varphi\rangle=\langle E'(v),\varphi\rangle+2\kappa^2\,\mathrm{Re}\int_\Omega|v|^2v\varphi^*\,dx.$$

Since $\langle E'(u),\varphi_h\rangle=\langle E'(u_h),\varphi_h\rangle=0$, we expand

$$\begin{aligned}\langle E''(u)(u-u_h),\varphi_h\rangle&=\langle E''(u)u,\varphi_h\rangle-\langle E''(u_h)u_h,\varphi_h\rangle+\langle E''(u_h)u_h,\varphi_h\rangle-\langle E''(u)u_h,\varphi_h\rangle\\ &=2\kappa^2\,\mathrm{Re}\int_\Omega|u|^2u\varphi_h^*\,dx-2\kappa^2\,\mathrm{Re}\int_\Omega|u_h|^2u_h\varphi_h^*\,dx\\ &\quad+\kappa^2\,\mathrm{Re}\int_\Omega 2\big(|u_h|^2-|u|^2\big)u_h\varphi_h^*+\big(u_h^2-u^2\big)u_h^*\varphi_h^*\,dx\\ &=2\kappa^2\,\mathrm{Re}\int_\Omega|u|^2u\varphi_h^*\,dx+\kappa^2\,\mathrm{Re}\int_\Omega|u_h|^2u_h\varphi_h^*\,dx\\ &\quad-\kappa^2\,\mathrm{Re}\int_\Omega\big(2|u|^2u_h+u^2u_h^*\big)\varphi_h^*\,dx,\end{aligned}$$

where we collected terms in the last step. For the estimate, we write $u_h=u-e_h$ and compute

$$2|u|^2u+|u_h|^2u_h-\big(2|u|^2u_h+u^2u_h^*\big)=2u|e_h|^2+e_h^2u^*-|e_h|^2e_h, \tag{5.4}$$

which together with $|u|\le 1$ and the Hölder inequality gives the second bound. □

Now we have everything together to prove the first part of Theorem 3.3, i.e., the H^1_κ-estimates for the discrete minimizers.

Proposition 5.6. *Let u and u_h be minimizers of* (1.1) *and* (3.2)*, respectively, and assume the orthogonality $m(u_h,\mathrm{i}u)=0$.*

(a) We have for the fully discrete error

$$||u-u_h||_{H^1_\kappa}\lesssim||u-\mathrm{R}^\perp_{\kappa,h}u||_{H^1_\kappa}+\kappa\,C_{\mathrm{sol},h}(\kappa)||u-\mathrm{R}^\perp_{\kappa,h}u||_{L^2}+\kappa\,C_{\mathrm{sol},h}(\kappa)\big(||u-u_h||^2_{L^4}+||u-u_h||^3_{L^6}\big)$$

(b) For h sufficiently small, we have for the (unique) minimizer u in Proposition 5.1

$$||u-u_h||_{H^1_\kappa}\lesssim||u-\mathrm{R}^\perp_{\kappa,h}u||_{H^1_\kappa}+\kappa\,C_{\mathrm{sol},h}(\kappa)||u-\mathrm{R}^\perp_{\kappa,h}u||_{L^2}.$$

Let us point out that (a) holds for any minimizers u and u_h. But to ensure that the higher order terms are indeed negligible, we need the a priori information from the abstract convergence result in Proposition 5.1.

Proof of Proposition 5.6. (a) Using the triangle inequality, we obtain

$$\|u-u_h\|_{H^1_\kappa} \lesssim \|u-\mathrm{R}^\perp_{\kappa,h}u\|_{H^1_\kappa} + \|\mathrm{R}^\perp_{\kappa,h}u-u_h\|_{H^1_\kappa},$$

and are left to bound the second term. Lemmas 5.4 and 5.5 then give

$$\|\mathrm{R}^\perp_{\kappa,h}u-u_h\|_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)\|\varepsilon_h\|_{(H^1_\kappa)'} \lesssim \kappa C_{\mathrm{sol}}(u,\kappa)\big(\|u-\mathrm{R}^\perp_{\kappa,h}u\|_{L^2}+\|u-u_h\|^2_{L^4}+\|u-u_h\|^3_{L^6}\big),$$

and the bound is established.

(b) With the convergence shown in Proposition 5.1 for h sufficiently small, we can absorb the higher order terms, and obtain the claimed estimate for $h \le h_0$. ☐

We can further show quadratic convergence in the L^2-norm for the discrete minimizers using an Aubin–Nitsche argument.

Lemma 5.7. *Let u and u_h be a minimizers of (1.1) and (3.2), respectively, and assume the orthogonality $m(u_h, \mathrm{i}u) = 0$. We have for the fully discrete error*

$$\begin{aligned}\|u-u_h\|_{L^2} \lesssim{}& C_{\mathrm{sol}}(u,\kappa)h\|u-u_h\|_{H^1_\kappa}\\ &+ C_{\mathrm{sol}}(u,\kappa)\kappa\|u-u_h\|_{L^2}\big(\|u-u_h\|_{L^3}+\|u-u_h\|^2_{L^6}\big),\end{aligned}$$

and hence for h sufficiently small, it holds for the (unique) minimizer u in Proposition 5.1

$$\|u-u_h\|_{L^2} \lesssim C_{\mathrm{sol}}(u,\kappa)h\|u-u_h\|_{H^1_\kappa}.$$

Proof. Recall the abbreviation $e_h = u-u_h$, and let $z \in H^1_{\mathrm{i}u}$ be the solution of

$$m(E''(u)z,\varphi) = m(e_h,\varphi),$$

and note that Corollary 2.7 and (5.1) give the estimate

$$\kappa\|z\|_{H^1_\kappa} + h^{-1}\|z-\mathrm{R}^\perp_{\kappa,h}z\|_{H^1_\kappa} \le C_{\mathrm{sol}}(u,\kappa)\|e_h\|_{L^2}. \tag{5.5}$$

Using the symmetry of E'', we can decompose the error as

$$\|e_h\|^2_{L^2} = \langle E''(u)\,e_h, z-\mathrm{R}^\perp_{\kappa,h}z\rangle + \langle E''(u)\,e_h, \mathrm{R}^\perp_{\kappa,h}z\rangle = E^1 + \varepsilon_h^{\mathrm{nonlin}}(\mathrm{R}^\perp_{\kappa,h}z),$$

where $\varepsilon_h^{\mathrm{nonlin}}$ is defined in Lemma 5.5. We estimate the first term with Lemma 2.3 and (5.5)

$$E^1 \lesssim \|e_h\|_{H^1_\kappa}\|z-\mathrm{R}^\perp_{\kappa,h}z\|_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)h\|e\|_{H^1_\kappa}\|e_h\|_{L^2}.$$

For the second term, we use the representation of $\varepsilon_h^{\mathrm{nonlin}}$ in Lemma 5.5 and (5.4) together with (5.5) and the Hölder equation to obtain

$$\begin{aligned}|\varepsilon_h^{\mathrm{nonlin}}(\mathrm{R}^\perp_{\kappa,h}z)| &\lesssim \kappa^2\|u-u_h\|_{L^2}\big(\|u-u_h\|_{L^3}+\|u-u_h\|^2_{L^6}\big)\|\mathrm{R}^\perp_{\kappa,h}z\|_{H^1}\\ &\lesssim \kappa C_{\mathrm{sol}}(u,\kappa)\|u-u_h\|_{L^2}\big(\|u-u_h\|_{L^3}+\|u-u_h\|^2_{L^6}\big)\|e_h\|_{L^2},\end{aligned}$$

where we used $\kappa\|\mathrm{R}^\perp_{\kappa,h}z\|_{H^1} \lesssim \kappa\|z\|_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)\|e_h\|_{L^2}$ in the last step. Combining the two bounds and dividing by $\|e_h\|_{L^2}$ gives the desired estimate. ☐

A similar trick gives the improved convergence of $\mathrm{R}^\perp_{\kappa,h}$ in the L^2-norm.

Lemma 5.8. *For κh small enough, the following bound holds for all $w \in H^1_{\mathrm{i}u}$*

$$\|w-\mathrm{R}^\perp_{\kappa,h}(w)\|_{L^2} \lesssim h\|w-\mathrm{R}^\perp_{\kappa,h}(w)\|_{H^1_\kappa},$$

where the constant is independent of h and κ.

Proof. We use an Aubin–Nitsche argument and let $z \in H^1_{\mathrm{i}u}$ be the solution of

$$\hat{a}_\kappa(z,\varphi) = m(w-\mathrm{R}^\perp_{\kappa,h}w,\varphi), \qquad \text{for all } \varphi \in H^1_{\mathrm{i}u}.$$

Using orthogonality, we have by (3.1) that

$$\|w-\mathrm{R}^\perp_{\kappa,h}w\|^2_{L^2} = \hat{a}_\kappa(z-\mathrm{R}^\perp_{\kappa,h}z, w-\mathrm{R}^\perp_{\kappa,h}w) \lesssim h\|w-\mathrm{R}^\perp_{\kappa,h}w\|_{L^2}\|w-\mathrm{R}^\perp_{\kappa,h}w\|_{H^1_\kappa},$$

and the claim follows. ☐

Finally, we provide the error bounds for the energy which behaves in the lowest order as the square of the error in the H^1_κ-norm.

Lemma 5.9. *Let u and u_h be minimizers of (1.1) and (3.2), respectively. The error in the energies is bounded by*

$$0 \le E(u_h) - E(u) \lesssim \|u-u_h\|_{H^1_\kappa}^2 \left(1 + \kappa^{1/2}\|u-u_h\|_{H^1_\kappa} + \kappa\|u-u_h\|_{H^1_\kappa}^2\right).$$

We note that the powers of κ can be improved in the case $d=2$, but since the leading order term does not change, we will not give any details here.

Proof of Lemma 5.9. Since $V_h \subset H^1$, we have $E(u) \le E(u_h)$, and thus the lower bound. In the next step, we derive the representation

$$\begin{aligned} E(u_h) - E(u) &= \frac{1}{2} a_A(u-u_h, u-u_h) \\ &\quad + \frac{\kappa^2}{4} \operatorname{Re} \int_\Omega (1-|u_h|^2)^2 - (1-|u|^2)^2 + 4(|u|^2-1)u(u-u_h)^* \, dx, \end{aligned} \tag{5.6}$$

Let us first note the identity

$$\frac{1}{2} a_A(u-u_h, u-u_h) = \frac{1}{2} a_A(u_h,u_h) - \frac{1}{2} a_A(u,u) + a_A(u, u-u_h),$$

and rewrite the energies as

$$\begin{aligned} E(u_h) - E(u) &= \frac{1}{2} a_A(u_h,u_h) - \frac{1}{2} a_A(u,u) + \frac{\kappa^2}{4} \operatorname{Re} \int_\Omega (1-|u_h|^2)^2 - (1-|u|^2)^2 \, dx \\ &= \frac{1}{2} a_A(u-u_h, u-u_h) + \frac{\kappa^2}{4} \operatorname{Re} \int_\Omega (1-|u_h|^2)^2 - (1-|u|^2)^2 \, dx - a_A(u, u-u_h). \end{aligned}$$

Since u is a minimizer, we have $\langle E'(u), u-u_h\rangle = 0$ and thus by (2.5)

$$- a_A(u, u-u_h) = \kappa^2 \operatorname{Re} \int_\Omega (|u|^2-1)u(u-u_h)^* \, dx,$$

and hence (5.6) holds. The first term of the representation gives the H^1_κ-norm in the estimate, and it remains to study the nonlinear part. We first investigate the difference of the squares. As before, we write $u_h = u - e_h$ and obtain

$$\begin{aligned} \left(1 - |u-e_h|^2\right)^2 &= \left(|u|^2 + |e_h|^2 - 1 - 2\operatorname{Re}(ue_h^*)\right)^2 \\ &= |u|^4 + 1 - 2|u|^2 + 4\operatorname{Re}(ue_h^*) - 4|u|^2 \operatorname{Re}(ue_h^*) + \mathcal{O}(|e_h|^2 + |e_h|^3 + |e_h|^4), \end{aligned}$$

which gives

$$(1-|u_h|^2)^2 - (1-|u|^2)^2 = 4\operatorname{Re}(ue_h^*) - 4|u|^2\operatorname{Re}(ue_h^*) + \mathcal{O}(|e_h|^2 + |e_h|^3 + |e_h|^4).$$

We now show that the part, which is linear in e_h, is canceled by the last term in (5.6). In fact, since it holds

$$4\operatorname{Re}(|u|^2-1)u(u-u_h)^* = 4|u|^2\operatorname{Re}(ue_h^*) - 4\operatorname{Re}(ue_h^*),$$

we conclude from (5.6), the fact that $|u| \le 1$ and the Hölder inequality the bound

$$E(u_h) - E(u) \lesssim \|u-u_h\|_{H^1_\kappa}^2 + \kappa^2\left(\|u-u_h\|_{L^2}^2 + \|u-u_h\|_{L^3}^3 + \|u-u_h\|_{L^4}^4\right).$$

To show the final estimate, we use interpolation theory, see e.g., [33, Thm. 2.6], with $\frac{1}{3} = \frac{\theta}{2} + \frac{1-\theta}{6}$ for $\theta = \frac{1}{2}$ to obtain for $w \in H^1$

$$\kappa^2\|w\|_{L^3}^3 \lesssim \kappa^{1/2}\left(\kappa\|w\|_{L^2}\right)^{3/2}\|w\|_{L^6}^{3/2} \lesssim \kappa^{1/2}\|w\|_{H^1_\kappa}^3,$$

and similarly with $\frac{1}{4} = \frac{\theta}{2} + \frac{1-\theta}{6}$ for $\theta = \frac{1}{4}$

$$\kappa^2\|w\|_{L^4}^4 \lesssim \kappa\left(\kappa\|w\|_{L^2}\right)\|w\|_{L^6}^3 \lesssim \kappa\|w\|_{H^1_\kappa}^4,$$

and the second claim is established. □

We can finally give the proof of our main result.

Proof of Theorem 3.3. We mainly collect the results shown in Proposition 5.6, Lemma 5.7, together with the L^2-estimate in Lemma 5.8, and Lemma 5.9, and the claims are established. ☐

5.4. Application to Lagrange finite elements.

In this section, we consider the linear Lagrange finite element space V_h as defined (3.3). In order to derive the corresponding error estimates through verifying the assumptions of Theorem 3.3, we require the L^2-orthogonal projection onto the ansatz space V_h as an auxiliary projection. We recall the L^2-projection for $v \in L^2$ as

$$m(\pi_h v, \varphi_h) = m(v, \varphi_h) \quad \text{for all} \quad \varphi_h \in V_h.$$

In the following lemma, we provide corresponding estimates in the H^1_κ-norm which are the first step towards verifying part (b) in Assumption 3.1.

Lemma 5.10. (a) The L^2-projection π_h is stable in H^1_κ, i.e., there hold the bounds

$$||\pi_h\varphi||_{H^1_\kappa} \lesssim ||\varphi||_{H^1_\kappa}, \qquad \varphi \in H^1,$$

where the constant is independent of h and κ.

(b) For all $z \in H^2$ it holds

$$||z - \pi_h z||_{H^1_\kappa} \lesssim h||z||_{H^2_\kappa},$$

where the constant is independent of h and κ.

(c) If Ω is convex and $z \in H^1_{\mathrm{i}u}$ satisfies for $f \in L^2$ the equation $\hat{a}_\kappa(z, \varphi) = m(f, \varphi)$ for all $\varphi \in H^1_{\mathrm{i}u}$, then

$$||z - \pi_h z||_{H^1_\kappa} \lesssim h||f||_{L^2}.$$

Proof. Due to (3.4), standard arguments lead to the bounds on the L^2-projection in part (a) and (b). Part (c) is a direct consequence of part (b) and Lemma 2.4. ☐

In the next lemma, we relate the orthogonal projection, which takes into account the orthogonality to $\mathrm{i}u$ in $m(\cdot,\cdot)$, to the L^2-projection.

Lemma 5.11. *For κh small enough, it holds the bound*

$$||\varphi - \mathrm{R}^\perp_{\kappa,h}(\varphi)||_{H^1_\kappa} \lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa}, \quad \varphi \in H^1_{\mathrm{i}u}.$$

Proof. For $\varphi_h \in V_h$ we let $P^\perp_{\mathrm{i}u} : V_h \to V^\perp_h$ be the mapping that adjusts the angle to $\mathrm{i}u$ via

$$P^\perp_{\mathrm{i}u}(\varphi_h) := \varphi_h - \frac{m(\varphi_h, \mathrm{i}u)}{m(\pi_h(\mathrm{i}u), \mathrm{i}u)}\pi_h(\mathrm{i}u).$$

We this we obtain for any $\varphi \in H^1_{\mathrm{i}u}$

$$\begin{aligned}
||\varphi - \mathrm{R}^\perp_{\kappa,h}(\varphi)||_{H^1_\kappa} &\lesssim ||\varphi - (P^\perp_{\mathrm{i}u} \circ \pi_h)\varphi||_{H^1_\kappa} \leq ||\varphi - \pi_h\varphi||_{H^1_\kappa} + \frac{m(\pi_h\varphi - \varphi, \mathrm{i}u)}{m(\pi_h(\mathrm{i}u), \mathrm{i}u)}||\pi_h(\mathrm{i}u)||_{H^1_\kappa} \\
&\lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa} + ||\varphi - \pi_h\varphi||_{L^2}\frac{\|u\|_{L^2}}{m(\pi_h(\mathrm{i}u) - \mathrm{i}u, \mathrm{i}u) + ||u||^2_{L^2}}||u||_{H^1_\kappa} \\
&\lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa} + \frac{\kappa}{1 - c\kappa h}||\varphi - \pi_h\varphi||_{L^2} \lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa},
\end{aligned}$$

where we used in the last step that $||\pi_h(\mathrm{i}u) - \mathrm{i}u||_{L^2} \lesssim h||u||_{H^1} \lesssim \kappa h$ holds. ☐

These preparations lead to the error bounds for our first application.

Proof of Corollary 3.4. From Lemmas 5.10 and 5.11, we obtain that Assumption 3.1 holds, and thus we can use the bounds in Theorem 3.3. In addition, we recall that Ω is assumed to be convex, and, hence, the approximation estimates due to Lemmas 5.10 and 5.11 yield

$$||u - \mathrm{R}^\perp_{\kappa,h}u||_{H^1_\kappa} \lesssim h||u||_{H^2_\kappa} \lesssim \kappa^2 h,$$

where we used Theorem 2.2 for the last step. This establishes the claims. ☐

6. Relaxed κ-dependencies in LOD spaces

In this final section we present a nonstandard application of the abstract approximation result in Theorem 3.3. For that we consider spaces based on the so-called Localized Orthogonal Decomposition (LOD). LOD spaces were originally developed in the context of elliptic multiscale problems with rough coefficients to efficiently handle low regularity and unresolved scales [35]. An introduction to the methodology is given in the textbook by Målqvist and Peterseim [36] and the review article by Altmann et al. [4]. Recently, new applications of these spaces emerged in the field of quantum mechanics where they were used to boost the performance of traditional discretizations [27, 29, 43]. As we will see see, the Ginzburg-Landau equation could be yet another promising application of LOD spaces in the context of quantum physics.

To define suitable LOD spaces for the GLE and to characterize its approximation properties in an abstract way, we start from a linear Lagrange finite element space V_h as defined in (3.3) and assume that the underlying triangulation $\mathcal{T}_h$ is shape-regular and quasi-uniform. The LOD space is now constructed from V_h by applying the inverse of a differential operator to the functions of V_h. In our case, we use the differential operator associated with the bilinear form $\hat{a}_\kappa(\cdot,\cdot,)$. The construction is made precise in the following definition.

Definition 6.1 (LOD spaces)**.** *Let $\hat{a}_\kappa(\cdot,\cdot)$ denote the symmetric, continuous and coercive bilinear form on $H^1(\Omega,\mathbb{C})$ given by (2.2) and let $\hat{\mathcal{A}}_\kappa^{-1}$ denote the corresponding solution operator on L^2, i.e., for $f \in L^2(\Omega,\mathbb{C})$ the image $\hat{\mathcal{A}}_\kappa^{-1} f \in H^1$ is given by the solution to*

$$\hat{a}_\kappa(\, \hat{\mathcal{A}}_\kappa^{-1} f \,,\, \varphi\,) = m(f,\varphi) \qquad \text{for all } \varphi \in H^1.$$

With this definition, the LOD space based on $\hat{a}_\kappa(\cdot,\cdot)$ and V_h is given by

$$V_h^{\mathrm{LOD}} := \hat{\mathcal{A}}_\kappa^{-1} V_h.$$

We note that the above definition of LOD spaces formally differs from the construction given in the classical references [26, 28, 35]. However, the characterizations are indeed equivalent as can be extracted from e.g., [25] and [4].

From a practical perspective it is also important to note that the space V_h^{LOD} admits a quasi-local basis, i.e., basis functions that are (super-)exponentially decaying in distances of the mesh size h. Details on the practical computation/approximation of such basis functions are given in [21] and recent super-localization strategies are presented in [25]. Corresponding numerical errors that might arise from the approximation of basis functions are well understood [4] and will be for brevity disregarded in the following error analysis.

The approximation properties of the idealized space V_h^{LOD} are summarized in the following proposition.

Proposition 6.2 (Approximation properties of V_h^{LOD})**.** *Let V_h^{LOD} be the LOD-space from Definition 6.1 and let $f \in L^2$ be given. If $u \in H^1$ denotes the solution to*

$$\hat{a}_\kappa(u,\varphi) = m(f,\varphi) \qquad \text{for all } \varphi \in H^1$$

and if $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} u \in V_h^{\mathrm{LOD}}$ denotes the corresponding $\hat{a}_\kappa(\cdot,\cdot)$-Ritz-projection of u in V_h^{LOD}, then it holds

$$||u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} u||_{H^1_\kappa} \lesssim h\, ||f - \pi_h f||_{L^2}, \tag{6.1}$$

where we recall $\pi_h : L^2 \to V_h$ as the L^2-projection on V_h. The hidden constant in (6.1) is generic and depends on the coercivity and continuity constants of $\hat{a}_\kappa(\cdot,\cdot)$, as well as the mesh regularity, but it does not depend on h and κ.

Furthermore, for every $\phi \in H^1$ there exists a unique decomposition such that

$$\phi = \phi^{\mathrm{LOD}} + \phi_0, \qquad \text{where } \ \phi^{\mathrm{LOD}} \in V_h^{\mathrm{LOD}}, \quad \pi_h \phi_0 = 0 \quad \text{and} \quad \hat{a}_\kappa(\phi^{\mathrm{LOD}}, \phi_0) = 0. \tag{6.2}$$

The result is standard and can be for instance found in [29] for homogeneous Dirichlet boundary conditions. For generalizations to higher order FE spaces and to only piecewise smooth source terms f, we refer to [34].

Analogously, to standard Lagrange finite elements, it is also possible to quantify the approximation properties of $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}$ for general smooth functions. This is done in the following lemma.

Lemma 6.3. *Let V_h^{LOD} be the LOD-space from Definition 6.1 and let $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$ denote the corresponding Ritz-projection w.r.t. $\hat{a}_\kappa(\cdot,\cdot)$. Then, for every $w \in H^2$, there is a $f_w \in L^2$ such that*

$$\hat{a}_\kappa(w,\varphi) = m(f_w,\varphi) \quad \textit{and} \quad ||f_w||_{L^2} \lesssim ||w||_{H^2_\kappa}.$$

Consequently, for all $w \in H^2$ it holds

$$||w - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} w||_{H^1_\kappa} \lesssim h||w||_{H^2_\kappa}.$$

Proof. From (2.8), we obtain using integration by parts

$$\hat{a}_\kappa(w,\varphi) = \mathrm{Re}\int_\Omega \big(-\Delta w + \beta^2 w + 2\mathrm{i}\kappa A \nabla w + \kappa^2 |A|^2 w\big)\varphi^* \, dx =: m(f_w,\varphi)$$

and the bound for $||f_w||_{L^2}$ follows. Proposition 6.2 finishes the second part of the lemma. □

From this lemma, we can deduce property (a) in Assumption 3.1. For the second property, we need a variant of this result given in the next lemma.

Lemma 6.4. *Let V_h^{LOD} be the LOD-space from Definition 6.1 and let $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$ denote the corresponding Ritz-projection w.r.t. $\hat{a}_\kappa(\cdot,\cdot)$. For $f \in L^2$ let $w \in H^1_{\mathrm{i}u}$ be the solution of*

$$\hat{a}_\kappa(w,\varphi) = m(f,\varphi) \quad \textit{for all} \quad \varphi \in H^1_{\mathrm{i}u}.$$

Then, it holds

$$||w - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} w||_{H^1_\kappa} \lesssim h||f||_{L^2}.$$

Proof. As in the proof of Lemma 2.4 in (2.7), we know that w solves the variational problem also tested against all $\varphi \in H^1$ for some modification of f which is bounded in L^2 by $||f||_{L^2}$. Hence, the assertion follows from Proposition 6.2. □

To apply the general error estimates in Theorem 3.3, we need to verify Assumption 3.1 for the LOD space V_h^{LOD}. As H^2 is a dense subset of H^1, the property (a) follows from the second part of Lemma 6.3. For property (b), we require the following lemma.

Lemma 6.5. *Let again $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$ denote the Ritz-projection onto the LOD-space V_h^{LOD} and let*

$$\mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}} : H^1_{\mathrm{i}u} \to V_h^{\mathrm{LOD}} \cap (\mathrm{i}u)^\perp$$

denote the corresponding Ritz-projection onto $V_h^{\mathrm{LOD}} \cap (\mathrm{i}u)^\perp$. If h is small enough, in particular $h \lesssim \kappa^{-1}$, then it holds for all $\varphi \in H^1_{\mathrm{i}u}$

$$||\varphi - \mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}}\varphi||_{H^1_\kappa} \lesssim ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H^1_\kappa}.$$

Proof. To proceed as in the proof of Lemma 5.11, we note that by the LOD-decomposition (6.2) we have $\pi_h\left(\mathrm{i}u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u)\right) = 0$. Hence, with the approximation properties of π_h:

$$||\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u) - \mathrm{i}u||_{L^2} \lesssim h\,||\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u) - \mathrm{i}u||_{H^1_\kappa} \lesssim h\,||\mathrm{i}u||_{H^1_\kappa} \lesssim h\,\kappa. \tag{6.3}$$

This implies for all $\varphi \in H^1_{\mathrm{i}u}$

$$\begin{aligned}
||\varphi - \mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}}\varphi||_{H^1_\kappa} &\lesssim ||\varphi - \left(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi - \frac{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi,\mathrm{i}u)}{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u),\mathrm{i}u)}\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u)\right)||_{H^1_\kappa} \\
&\le ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H^1_\kappa} + \frac{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi - \varphi,\mathrm{i}u)}{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u),\mathrm{i}u)}||\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u)||_{H^1_\kappa} \\
&\overset{(6.3)}{\lesssim} ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H^1_\kappa} + ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{L^2}\frac{||u||_{L^2}}{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u) - \mathrm{i}u,\mathrm{i}u) + ||u||_{L^2}^2}||u||_{H^1_\kappa} \\
&\lesssim ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H^1_\kappa} + \frac{\kappa}{1 - c\kappa h}||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{L^2} \lesssim ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H^1_\kappa}.
\end{aligned}$$

□

Lemma 6.5 together with Theorem 3.3 guarantees that the H^1_κ-error between an exact solution u and a corresponding approximation in the LOD-space is bounded by $||u - \mathrm{R}^{\mathrm{LOD}}_{\kappa,h} u||_{H^1_\kappa}$. The next lemma quantifies this error.

Lemma 6.6. *Let u be a minimizer of (1.1) and let $\mathrm{R}^{\mathrm{LOD}}_{\kappa,h} : H^1 \to V_h^{\mathrm{LOD}}$ be the Ritz-projection onto V_h^{LOD}. Then it holds at least*

$$||u - \mathrm{R}^{\mathrm{LOD}}_{\kappa,h} u||_{H^1_\kappa} \lesssim \kappa^3 \, h^2$$

and, if Ω is convex, we have $u \in H^2$ and the estimate improves to

$$||u - \mathrm{R}^{\mathrm{LOD}}_{\kappa,h} u||_{H^1_\kappa} \lesssim \kappa^4 \, h^3.$$

Proof. We want to apply Proposition 6.2. By $E'(u) = 0$ we have for every $\phi \in H^1_\kappa$ that

$$\hat{a}_\kappa(u,\varphi) = \beta \,\mathrm{Re} \int_\Omega u\varphi^* \, dx - \kappa^2 \,\mathrm{Re} \int_\Omega \big(|u|^2 - 1\big) u \varphi^* \, dx = m(\beta^2 u - \kappa^2\big(|u|^2 - 1\big)u, \varphi).$$

Since $\beta^2 u - \kappa^2\big(|u|^2 - 1\big)u$ is at least in H^1 and even in H^2 for convex domains, one easily verifies that for $s = 0, 1, 2$

$$||\beta^2 u - \kappa^2\big(|u|^2 - 1\big)u||_{H^s} \lesssim \kappa^2\big(||u||^s_{H^1} + ||u||_{H^s}\big) \leq \kappa^{s+2},$$

where we used the bounds from Theorem 2.2 and in particular repeatedly $|u| \leq 1$. The estimate now follows with Proposition 6.2 and standard estimates for the L^2-projection π_h on $P1$ finite element spaces. ☐

By collecting the previous results we obtain our final main result which shows the superapproximation properties of the LOD space, even on nonconvex domains.

Theorem 6.7. *Let Assumption 2.5 hold and let h be sufficiently small in the sense of Theorem 3.3. If V_h^{LOD} denotes the LOD-space from Definition 6.1 and if $u_h^{\mathrm{LOD}} \in V_h^{\mathrm{LOD}}$ is a corresponding minimizer of the Ginzburg–Landau energy with*

$$E(u_h^{\mathrm{LOD}}) = \inf_{\varphi \in V_h^{\mathrm{LOD}}} E(\varphi),$$

then, there is neighborhood $U \subset H^1(\Omega)$ of u_h^{LOD} and a unique minimizer $u \in U$ of (1.1) with $m(u_h^{\mathrm{LOD}}, \mathrm{i}u) = 0$ and such that

$$C_{\mathrm{sol}}^{-1}(u,\kappa) \, ||u - u_h^{\mathrm{LOD}}||_{L^2(\Omega)} + h \, ||u - u_h^{\mathrm{LOD}}||_{H^1_\kappa} \lesssim \kappa^3 h^3,$$

and for convex domains Ω (and consequently H^2-solutions) it even holds

$$C_{\mathrm{sol}}^{-1}(u,\kappa) \, ||u - u_h^{\mathrm{LOD}}||_{L^2(\Omega)} + h \, ||u - u_h^{\mathrm{LOD}}||_{H^1_\kappa} \lesssim \kappa^4 h^4.$$

Proof. Proposition 6.2 and Lemmas 6.3, 6.4, and 6.5 guarantee that Assumption 3.1 is fulfilled for V_h^{LOD}. Hence, we can apply Theorem 3.3 together with Lemmas 6.5 and 6.6 to conclude that for all sufficiently small h and for $u \in H^s$ with $s \in \{1, 2\}$ it holds

$$||u - u_h^{\mathrm{LOD}}||_{H^1_\kappa} \lesssim ||u - \mathrm{R}^{\perp,\mathrm{LOD}}_{\kappa,h} u||_{H^1_\kappa} \lesssim ||u - \mathrm{R}^{\mathrm{LOD}}_{\kappa,h} u||_{H^1_\kappa} \lesssim \kappa^{s+2} \, h^{s+1},$$

and

$$||u - u_h^{\mathrm{LOD}}||_{L^2} \lesssim C_{\mathrm{sol}}(u,\kappa) h \, ||u - u_h^{\mathrm{LOD}}||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa) \, \kappa^{s+2} \, h^{s+2}.$$

☐

Remark 6.8. *It is worth to note that, in LOD-spaces, one can also improve the smallness condition on $\kappa C_{\mathrm{sol}}(u,\kappa)h$ required for the inf-sup condition in Lemma 5.4. In fact, a precise inspection of the proof leads to a smallness condition on $\kappa^2 C_{\mathrm{sol}}(u,\kappa)h^2$, which is in general weaker if $1 \lesssim C_{\mathrm{sol}}(u,\kappa)$. However, since the abstract result contains a term of the form $1 + \kappa C_{\mathrm{sol}}(u,\kappa)h$, one cannot exploit this any further in the error analysis, and we thus refrain from giving the proof here.*

References

[1] A. A. Abrikosov, *Nobel lecture: Type-II superconductors and the vortex lattice*, Rev. Mod. Phys. **76** (2004), no. 3,1, 975–979.

[2] A. Aftalion, *On the minimizers of the Ginzburg-Landau energy for high kappa: the axially symmetric case*, Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), no. 6, 747–772. MR1720515

[3] M. S. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes, and G. N. Wells, *The fenics project version 1.5*, Archive of Numerical Software **3** (2015), no. 100. `https://doi.org/10.11588/ans.2015.100.20553`.

[4] R. Altmann, P. Henning, and D. Peterseim, *Numerical homogenization beyond scale separation*, Acta Numer. **30** (2021), 1–86. MR4298217

[5] I. Babuška, *Error-bounds for finite element method*, Numer. Math. **16** (1971), 322–333. MR288971

[6] R. E. Bank and H. Yserentant, *On the H^1-stability of the L_2-projection onto finite element spaces*, Numer. Math. **126** (2014), no. 2, 361–381. MR3150226

[7] S. Bartels, *A posteriori error analysis for time-dependent Ginzburg-Landau type equations*, Numer. Math. **99** (2005), no. 4, 557–583. MR2121069

[8] ———, *Robust a priori error analysis for the approximation of degree-one Ginzburg-Landau vortices*, M2AN Math. Model. Numer. Anal. **39** (2005), no. 5, 863–882. `https://doi.org/10.1051/m2an:2005038`. MR2178565

[9] S. Bartels, R. Müller, and C. Ortner, *Robust a priori and a posteriori error analysis for the approximation of Allen-Cahn and Ginzburg-Landau equations past topological changes*, SIAM J. Numer. Anal. **49** (2011), no. 1, 110–134. MR2764423

[10] H. Chen, X. Gong, and A. Zhou, *Numerical approximations of a nonlinear eigenvalue problem and applications to a density functional model*, Math. Methods Appl. Sci. **33** (2010), no. 14, 1723–1742. `https://doi.org/10.1002/mma.1292`. MR2723492

[11] Z. Chen, *Mixed finite element methods for a dynamical Ginzburg-Landau model in superconductivity*, Numer. Math. **76** (1997), no. 3, 323–353. MR1452512

[12] Z. Chen and S. Dai, *Adaptive Galerkin methods with error control for a dynamical Ginzburg-Landau model in superconductivity*, SIAM J. Numer. Anal. **38** (2001), no. 6, 1961–1985. MR1856238

[13] L. Diening, J. Storn, and T. Tscherpel, *On the Sobolev and L^p-stability of the L^2-projection*, SIAM J. Numer. Anal. **59** (2021), no. 5, 2571–2607. MR4320894

[14] Q. Du, *Finite element methods for the time-dependent Ginzburg-Landau model of superconductivity*, Comput. Math. Appl. **27** (1994), no. 12, 119–133. MR1284135

[15] ———, *Global existence and uniqueness of solutions of the time-dependent Ginzburg-Landau model for superconductivity*, Appl. Anal. **53** (1994), no. 1-2, 1–17. `https://doi.org/10.1080/00036819408840240`. MR1379180

[16] ———, *Discrete gauge invariant approximations of a time dependent Ginzburg-Landau model of superconductivity*, Math. Comp. **67** (1998), no. 223, 965–986. MR1464143

[17] Q. Du and P. Gray, *High-kappa limits of the time-dependent Ginzburg-Landau model*, SIAM J. Appl. Math. **56** (1996), no. 4, 1060–1093. MR1398408

[18] Q. Du, M. D. Gunzburger, and J. S. Peterson, *Analysis and approximation of the Ginzburg-Landau model of superconductivity*, SIAM Rev. **34** (1992), no. 1, 54–81. MR1156289

[19] ———, *Modeling and analysis of a periodic Ginzburg-Landau model for type-*II *superconductors*, SIAM J. Appl. Math. **53** (1993), no. 3, 689–717. MR1218380

[20] H. Duan and Q. Zhang, *Residual-based a posteriori error estimates for the time-dependent Ginzburg-Landau equations of superconductivity*, J. Sci. Comput. **93** (2022), no. 3, Paper No. 79, 47. MR4507136

[21] C. Engwer, P. Henning, A. Målqvist, and D. Peterseim, *Efficient implementation of the localized orthogonal decomposition method*, Comput. Methods Appl. Mech. Engrg. **350** (2019), 123–153. MR3926249

[22] H. Gao, L. Ju, and W. Xie, *A stabilized semi-implicit Euler gauge-invariant method for the time-dependent Ginzburg-Landau equations*, J. Sci. Comput. **80** (2019), no. 2, 1083–1115. MR3977199

[23] H. Gao and W. Sun, *Analysis of linearized Galerkin-mixed FEMs for the time-dependent Ginzburg-Landau equations of superconductivity*, Adv. Comput. Math. **44** (2018), no. 3, 923–949. MR3814665

[24] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364

[25] M. Hauck and D. Peterseim, *Super-localization of elliptic multiscale problems*, Math. Comp. **92** (2023), no. 341, 981–1003. MR4550317

[26] P. Henning and A. Målqvist, *Localized orthogonal decomposition techniques for boundary value problems*, SIAM J. Sci. Comput. **36** (2014), no. 4, A1609–A1634. MR3240855

[27] P. Henning and A. Persson, *On optimal convergence rates for discrete minimizers of the Gross–Pitaevskii energy in LOD spaces*, 2021. ArXiv e-print 2112.08485.

[28] P. Henning and D. Peterseim, *Oversampling for the Multiscale Finite Element Method*, SIAM Multiscale Model. Simul. **11** (2013), no. 4, 1149–1175. MR3123820

[29] P. Henning and J. Wärnegård, *Superconvergence of time invariants for the Gross-Pitaevskii equation*, Math. Comp. **91** (2022), no. 334, 509–555. MR4379968

[30] B. Li, *Convergence of a decoupled mixed FEM for the dynamic Ginzburg-Landau equations in nonsmooth domains with incompatible initial data*, Calcolo **54** (2017), no. 4, 1441–1480. MR3735822

[31] B. Li and Z. Zhang, *A new approach for numerical simulation of the time-dependent Ginzburg-Landau equations*, J. Comput. Phys. **303** (2015), 238–250. MR3422711

[32] ———, *Mathematical and numerical analysis of the time-dependent Ginzburg-Landau equations in nonconvex polygons based on Hodge decomposition*, Math. Comp. **86** (2017), no. 306, 1579–1608. MR3626529

[33] A. Lunardi, *Interpolation theory*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 16, Edizioni della Normale, Pisa, 2018. Third edition [of MR2523200]. MR3753604

[34] R. Maier, *A high-order approach to elliptic multiscale problems with general unstructured coefficients*, SIAM J. Numer. Anal. **59** (2021), no. 2, 1067–1089. MR4246089

[35] A. Målqvist and D. Peterseim, *Localization of elliptic multiscale problems*, Math. Comp. **83** (2014), no. 290, 2583–2603. MR3246801

[36] ———, *Numerical homogenization by localized orthogonal decomposition*, SIAM Spotlights, vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, [2021] ©2021. MR4191211

[37] J. M. Melenk, *On generalized finite-element methods*, Ph.D. Thesis, 1995. MR2692949

[38] E. Sandier and S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Progress in Nonlinear Differential Equations and their Applications, vol. 70, Birkhäuser Boston, Inc., Boston, MA, 2007. MR2279839

[39] ———, *From the Ginzburg-Landau model to vortex lattice problems*, Comm. Math. Phys. **313** (2012), no. 3, 635–743. MR2945619

[40] S. Serfaty, *Stable configurations in superconductivity: uniqueness, multiplicity, and vortex-nucleation*, Arch. Ration. Mech. Anal. **149** (1999), no. 4, 329–365. MR1731999

[41] S. Serfaty and E. Sandier, *Vortex patterns in Ginzburg-Landau minimizers*, XVIth International Congress on Mathematical Physics, 2010, pp. 246–264. MR2730781

[42] M. Struwe, *Variational methods*, Fourth, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 34, Springer-Verlag, Berlin, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems. MR2431434

[43] Z. Wu and Z. Zhang, *Convergence analysis of the localized orthogonal decomposition method for the semiclassical Schrödinger equations with multiscale potentials*, J. Sci. Comput. **93** (2022), no. 3, Paper No. 73, 30. MR4505120

INSTITUTE FOR APPLIED AND NUMERICAL MATHEMATICS, KARLSRUHE INSTITUTE OF TECHNOLOGY, 76149 KARLSRUHE, GERMANY

Email address: `benjamin.doerich@kit.edu`

DEPARTMENT OF MATHEMATICS, RUHR-UNIVERSITY BOCHUM, 44801 BOCHUM, GERMANY

Email address: `patrick.henning@rub.de`