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On the scattering of a plane wave by a perturbed open periodic waveguide

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We consider the scattering of a plane wave by a locally perturbed periodic (with respect to x_1) medium. If there is no perturbation, it is usually assumed that the scattered wave is quasi-periodic with the same parameter as the incident plane wave. As it is well known, one can show existence under this condition but not necessarily uniqueness. Uniqueness fails for certain incident directions (if the wavenumber is kept fixed), and it is not clear which additional condition has to be assumed in this case. In this paper, we will analyze three concepts. For the limiting absorption principle (LAP), we replace the refractive index $n = n(x)$ by $n(x) + i\varepsilon$ in a layer of finite width and consider the limiting case $\varepsilon \rightarrow 0$. This will give an unsatisfactory condition. In a second approach, we require continuity of the field with respect to the incident direction. This will give the same satisfactory condition as the third approach where we approximate the incident plane wave by an incident point source and let the location of the source tend to infinity.

KEYWORDS

periodic structure, radiation condition, scattering problem

MSC CLASSIFICATION

35B27, 35J05

1 | INTRODUCTION

Let $k > 0$ be the wave number and $\hat{\theta} \in \mathbb{R}^2$ be a unit vector with $\hat{\theta}_2 < 0$ which are fixed. In polar coordinates, we express $\hat{\theta}$ as $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ for some $|\theta| < \frac{\pi}{2}$. Furthermore, let $n \in L^\infty(\mathbb{R}^2)$ be the real valued index of refraction, which is assumed to be 2π -periodic with respect to x_1 and equal to 1 for $|x_2| > h_0$ for some $h_0 > 0$. Let $q \in L^\infty(\mathbb{R}^2)$ have compact support in $Q := (0, 2\pi) \times (-h_0, h_0)$. We refer to Figure 1 for a sketch of the geometry. It is the aim to solve

$$\Delta u + k^2(n + q)u = 0 \text{ in } \mathbb{R}^2 \quad (1)$$

for the total field $u(x) = e^{ik\hat{\theta} \cdot x} + u^s(x)$ as the sum of the incident plane wave of direction $\hat{\theta}$ and the scattered field u^s . Furthermore, a suitable radiating condition for u^s has to be assumed.

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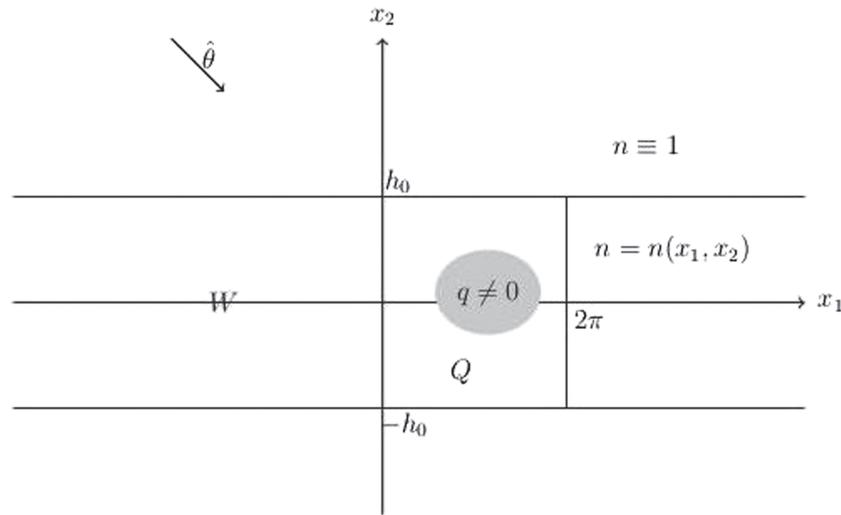


FIGURE 1 Sketch of the geometry. [Colour figure can be viewed at wileyonlinelibrary.com]

Scattering theory in periodic structures has many important applications in optics, micro-electronics, and photonic crystals. We refer to [1] for an introduction and overview till 1980. During the past decades, a huge number of publications appeared on this topic. Most of them deal with the scattering of periodic curves such as [2–7] or [8–11], if the problem is considered as a special case of a scattering problem by a rough surface. For the scattering by a layer with a space dependent refractive index, we refer to, for example, [12–15].

In the first part of the paper, we consider the unperturbed case; that is, $q = 0$. We note that the incident field $u^i(x) = e^{ik\hat{\theta}\cdot x}$ is α -quasi-periodic with respect to x_1 with parameter $\alpha = k\hat{\theta}_1 = k \sin \theta$. (Recall that a function $\phi = \phi(x_1)$ is α -quasi-periodic if $\phi(x_1 + 2\pi) = e^{2\pi i \alpha} \phi(x_1)$ for all $x_1 \in \mathbb{R}$.) Therefore, it is common (see, e.g., [2, 16–18]) to assume that also the scattered field has to be quasi-periodic with the same parameter α , and then the Rayleigh expansion provides a suitable radiation condition.

As we will recall below, for fixed $k > 0$, there exist angles θ of incident directions, for which no uniqueness holds under the Rayleigh expansion. For these particular angles (which correspond to so-called propagative wave numbers, see Definition 2.1, by $\alpha = k \sin \theta$), it is not clear which solution is—mathematically or physically—the correct one.

There are at least three ways to derive a correct radiation condition in this case where no uniqueness holds. A classical way is to apply the limiting absorption principle (LAP). Noting that the scattered field satisfies the inhomogeneous Helmholtz equation (for $q = 0$)

$$\Delta u^s + k^2 n u^s = -k^2(n-1)u^i \text{ in } \mathbb{R}^2 \quad (2)$$

with incident plane wave $u^i(x) = e^{ik\hat{\theta}\cdot x}$, we observe that the application of the LAP to the wave number k ; that is, replacing k by $k + i\varepsilon$ does not seem to work because in that case, the right hand side $f(x) = (k + i\varepsilon)^2(n(x) - 1)u^i(x) = (k + i\varepsilon)^2(n(x) - 1)e^{ik\hat{\theta}\cdot x}e^{-\varepsilon\hat{\theta}\cdot x}$ vanishes for $|x_2| > h_0$ but is not even bounded in the layer $W := \mathbb{R} \times (-h_0, h_0)$ (note that $-\hat{\theta} \cdot x$ is not bounded from above in W). An alternative is to apply the LAP to the refractive index n ; that is, replace $n(x)$ by $n(x) + i\varepsilon$ inside the waveguide. Since also in this case we do not expect a $H^1(\mathbb{R}^2)$ -solution for u^s , we have to add a radiation condition. The “upwards propagation radiation condition” gives uniqueness in $H^1_{loc}(\mathbb{R}^2)$ even in the case of general q , that is, with refractive index $n(x) + q(x) + i\varepsilon$. In the unperturbed case $q = 0$, this condition is equivalent to the Rayleigh expansion. In Section 3, we will study the question of convergence when ε tends to zero. It will turn out that this principle gives an unexpected and unsatisfactory answer in the case where no uniqueness holds.

The second approach demands continuity of the solution with respect to the angle of incidence. As we will see in Section 6, this will pick one particular solution and gives an additional condition on the field.

In Section 5, we will follow a third approach and consider first the scattering of an incident point source at $z \in \mathbb{R}^2$ with $z_2 > h_0$ and later let z tend to infinity. Therefore, the incident field is given by $u^i_z(x) = \Phi(x, z)$, $x \in \mathbb{R}^2 \setminus \{z\}$, where $\Phi(x, z) = \frac{i}{4}H_0^{(1)}(k|x - z|)$ denotes the fundamental solution. We recall the asymptotic behavior

$$\Phi(x, z) = \gamma \frac{e^{ik|z|}}{\sqrt{|z|}} e^{-ikx \cdot z/|z|} + \mathcal{O}(|z|^{-3/2}), \quad |z| \rightarrow \infty,$$

uniformly with respect to directions $z/|z|$ and x from bounded sets. Here, $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$. Therefore, if $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ with $|\theta| < \frac{\pi}{2}$ is the direction of the incident plane wave, we define the source to be $z = -t\hat{\theta}$ and note that

$$\frac{1}{\gamma} \lim_{t \rightarrow \infty} \left[\sqrt{t} e^{-ikt} \Phi(x, -t\hat{\theta}) \right] = e^{ik\hat{\theta} \cdot x} \quad (3)$$

uniformly for x from bounded sets. Therefore, we expect that the solution of the scattering problem of a point source at $z = -t\hat{\theta}$ (multiplied with the factor $\frac{1}{\gamma} \sqrt{t} e^{-ikt}$) converges to a solution of the scattering problem for the plane incident field of direction $\hat{\theta}$. We will prove this convergence result for the unperturbed case, that is, for $q = 0$, in Section 5.

Section 6 is devoted to the case where q is general, that is, where the refractive index is given by $n + q$.

All three approaches use the theory of quasi-periodic scattering problems (either because the problems themselves are quasi-periodic or via the Floquet-Bloch transform), which we repeat in Section 2. Also, the problems are singular in the sense that they involve invertible operators L_ε for $\varepsilon \neq 0$, which tend to an operator L_0 as $\varepsilon \rightarrow 0$, which is singular. For treating the convergence of the corresponding solutions of $L_\varepsilon u_\varepsilon = r_\varepsilon$, we apply an abstract singular perturbation result, which we learned from [19], section 1.4. We recall and extend it in Theorem 2.7 of Section 2.

Let us summarize some notations on sets and spaces. Let again $W := \mathbb{R} \times (-h_0, h_0)$ and $Q := (0, 2\pi) \times (-h_0, h_0)$ and, furthermore, $Q^\infty := (0, 2\pi) \times \mathbb{R}$ and $Q_+^{h_0} := (0, 2\pi) \times (h_0, \infty)$ and $Q_-^{h_0} := (0, 2\pi) \times (-\infty, -h_0)$ and $\Gamma_\pm := (0, 2\pi) \times \{\pm h_0\}$; that is, $Q^\infty = Q_-^{h_0} \cup \Gamma_- \cup Q \cup \Gamma_+ \cup Q_+^{h_0}$. We set $\Gamma := \Gamma_+ \cup \Gamma_-$.

Let $H_{loc}^1(\mathbb{R}^2) := \{u : \mathbb{R}^2 \rightarrow \mathbb{C} : u|_K \in H^1(K) \text{ for all open and bounded discs } K\}$ be the usual local Sobolev space and $H_{\alpha,loc}^1(\mathbb{R}^2) := \{u \in H_{loc}^1(\mathbb{R}^2) : u(\cdot, x_2) \text{ is } \alpha\text{-quasi-periodic}\}$ where a function u is α -quasi-periodic if $u(x_1 + 2\pi, x_2) = e^{i\alpha 2\pi} u(x_1, x_2)$ for all $x = (x_1, x_2)$. We identify $H_{\alpha,loc}^1(\mathbb{R}^2)$ sometimes with $H_{\alpha,loc}^1(Q^\infty) := \{u : Q^\infty \rightarrow \mathbb{C} : u|_{Q^H} \in H^1(Q^H) \text{ for all } H > 0 \text{ and } u(\cdot, x_2) \text{ is } \alpha\text{-quasi-periodic}\}$ where $Q^H := (0, 2\pi) \times (-H, H)$; that is, identify quasi-periodic functions on $(0, 2\pi)$ with those on \mathbb{R} – as we do also by identifying the space $\{f \in L^2(\mathbb{R}^2) : f \text{ vanishes outside of } Q^\infty\}$ with $L^2(Q^\infty)$. In the same way, $H_\alpha^1(Q)$ is defined. The space $H_{per}^1(Q)$ denotes the subspace of $H^1(Q)$ of 2π -periodic functions with respect to x_1 . Finally, the space $H_\alpha^{1/2}(\Gamma)$ is the trace space of $H_{\alpha,loc}^1(Q^\infty)$ on Γ and $H_\alpha^{-1/2}(\Gamma)$ the dual of $H_\alpha^{1/2}(\Gamma)$.

2 | QUASI-PERIODIC PROBLEMS AND A SINGULAR PERTURBATION RESULT

We first recall some notations.

Definition 2.1.

- (a) $\alpha \in \mathbb{R}$ is called a cut-off value if there exists $\ell \in \mathbb{Z}$ with $|\ell + \alpha| = k$.
- (b) $\alpha \in \mathbb{R}$ is called a propagative wave number (or quasi-momentum or Floquet spectral value) if there exists a non-trivial $\phi \in H_{\alpha,loc}^1(\mathbb{R}^2)$ such that

$$\Delta \phi + k^2 n \phi = 0 \text{ in } \mathbb{R}^2, \quad (4)$$

and ϕ satisfies the Rayleigh expansion

$$\phi(x) = \sum_{\ell \in \mathbb{Z}} \phi_\ell^\pm e^{i(\ell+\alpha)x_1 + i\sqrt{k^2 - (\ell+\alpha)^2}|x_2|} \text{ for all } \pm x_2 > h_0 \quad (5)$$

for some $\phi_\ell^\pm \in \mathbb{C}$ where the convergence is uniform with respect to $\{x \in (0, 2\pi) \times \mathbb{R} : |x_2| \geq h_0 + \delta\}$ for all $\delta > 0$. The functions ϕ are called propagating (or guided) modes.

If we decompose k into $k = \tilde{\ell} + \kappa$ with $\tilde{\ell} \in \mathbb{N} \cup \{0\}$ and $\kappa \in (-1/2, 1/2]$, we observe that the cut-off values are given by $\pm \kappa + \ell$ for any $\ell \in \mathbb{Z}$.

Since with α also $\alpha + \ell$ for every $\ell \in \mathbb{Z}$ is a propagative wave number, we can restrict ourselves to propagative wave numbers in $(-1/2, 1/2]$.

Under the following assumption, it can easily be seen that every propagating mode ϕ corresponding to some propagative wave number α is evanescent; that is, $\phi_\ell^\pm = 0$ for all $|\ell + \alpha| \leq k$; that is, there exist $c, \delta > 0$ with $|\phi(x)| \leq c e^{-\delta|x_2|}$ for all $|x_2| > h_0$.

Assumption 2.2. Let $|\ell + \alpha| \neq k$ for all propagative wave numbers α and all $\ell \in \mathbb{Z}$. In other words, we assume that the cut-off values are no propagative wave numbers.

Under Assumption 2.2, it can also be shown (see, e.g., [20]) that at most, a finite number of propagative wave numbers exist in $[-1/2, 1/2]$. Furthermore, if α is a propagative wave number with mode ϕ , then $-\alpha$ is a propagative wave number with mode $\bar{\phi}$. Therefore, we can numerate the propagative wave numbers in $[-1/2, 1/2]$ such they are given by $\{\tilde{\alpha}_j : j \in J\}$ where $J \subset \mathbb{Z}$ is symmetric with respect to 0 and $\tilde{\alpha}_{-j} = -\tilde{\alpha}_j$ for $j \in J$. Furthermore, it is known that every eigenspace

$$X_j := \left\{ \phi \in H_{\tilde{\alpha}_j, \text{loc}}^1(\mathbb{R}^2) : \phi \text{ satisfies (4) and (5)} \right\} \quad (6)$$

is finite dimensional with some dimension $m_j > 0$. We note that the elements of X_j are in $H^2(Q^\infty)$ and even analytic for $|x_2| > h_0$. We construct an orthonormal basis in X_j as follows. Let $j \in J$ be fixed. First, we choose an arbitrary inner product $(\cdot, \cdot)_{X_j}$ in X_j . Then we consider the following finite dimensional eigenvalue problem in X_j .

Determine $\lambda_{\ell, j} \in \mathbb{R}$, $\ell = 1, \dots, m_j$, and non-trivial $\phi_{\ell, j} \in X_j$ for $\ell = 1, \dots, m_j$ such that

$$-2i \int_{Q^\infty} \frac{\partial \phi_{\ell, j}}{\partial x_1} \bar{\psi} dx = \lambda_{\ell, j} (\phi_{\ell, j}, \psi)_{X_j} \text{ for all } \psi \in X_j. \quad (7)$$

This eigenvalue problem is self-adjoint because the left hand side defines a Hermitean sesqui-linear form on the finite dimensional space X_j . Let the eigenfunctions be normalized such that $(\phi_{\ell, j}, \phi_{\ell', j})_{X_j} = \delta_{\ell, \ell'}$ for $\ell, \ell' = 1, \dots, m_j$.

Remark 2.3. In [21], it is shown (for the case of the source problem $\Delta u + k^2 nu = -f$ in the half plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$ and additional Neumann boundary conditions for $x_2 = 0$) that the limiting absorption principle (LAP) with respect to k leads to the eigenvalue problem with inner product $(\phi, \psi)_{X_j} = 2k \int_{Q^\infty} n \phi \bar{\psi} dx$ while the LAP with respect to n in the layer W leads to the eigenvalue problem with inner product $(\phi, \psi)_{X_j} = k^2 \int_Q \phi \bar{\psi} dx$.

We make a further assumption, which is equivalent to the fact that the group velocities do not vanish (see [22]).

Assumption 2.4. Let $\lambda_{\ell, j} \neq 0$ for all $\ell = 1, \dots, m_j$ and $j \in J$; that is, there is no non-trivial $\phi \in X_j$ with $\int_{Q^\infty} \frac{\partial \phi}{\partial x_1} \bar{\psi} dx = 0$ for all $\psi \in X_j$.

In all of the paper, we make Assumptions 2.2 and 2.4 but mentioning this only in the formulations of the theorems.

After these preparations, we will now consider quasi-periodic source problems with source functions $f \in L^2(Q^\infty)$, which are not compactly supported.

Let $f \in L^2(Q^\infty)$ such there exist $c, \delta > 0$ with $|f(x)| \leq ce^{-\delta|x_2|}$ for all $|x_2| > h_0$. For any $\alpha \in \mathbb{R}$, consider the problem to determine $u \in H_{\alpha, \text{loc}}^1(Q^\infty)$ such that

$$\Delta u + k^2 nu = -f \text{ in } Q^\infty, \quad (8a)$$

and u satisfies the generalized Rayleigh condition

$$\sum_{\ell \in \mathbb{Z}} \left| (\text{sign } x_2) \frac{du_\ell(x_2)}{dx_2} - i\sqrt{k^2 - (\ell + \alpha)^2} u_\ell(x_2) \right|^2 \rightarrow 0, \quad |x_2| \rightarrow \infty. \quad (8b)$$

Here, $u_\ell(x_2) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x_1, x_2) e^{-i(\ell + \alpha)x_1} dx_1$ are the Fourier coefficients of $u(\cdot, x_2)$. The corresponding α -quasi-periodic Dirichlet-to-Neumann operator $\Lambda_\alpha : H_\alpha^{1/2}(\Gamma) \rightarrow H_\alpha^{-1/2}(\Gamma)$ is given by

$$(\Lambda_\alpha \phi)(x_1, \pm h_0) := \frac{i}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} \sqrt{k^2 - (\ell + \alpha)^2} \phi_\ell(\pm h_0) e^{i(\ell + \alpha)x_1}, \quad x_1 \in (0, 2\pi), \quad (9)$$

for $\phi \in H_\alpha^{1/2}(\Gamma)$.

The following theorem collects properties of the problem (8a), (8b). For a proof, we refer to [22], Theorems 4.1–4.3 and Remark 4.4.

Theorem 1. Let Assumptions 2.2 and 2.4 hold.

(a) For every $\alpha \in \mathbb{R}$, the problem (8a), (8b) is equivalent to the variational equation

$$\int_Q [\nabla u \cdot \nabla \bar{\psi} - k^2 n u \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u) \bar{\psi} ds = \int_Q f \bar{\psi} dx + \int_\Gamma \frac{\partial w}{\partial \nu} \bar{\psi} ds \quad (10)$$

for all $\psi \in H_\alpha^1(Q)$ where $\partial w / \partial \nu := \pm \partial w^\pm / \partial x_2$ on Γ_\pm . Here, $w^\pm \in H_{\alpha, \text{loc}}^1(Q_\pm^{h_0})$ are the (uniquely determined) solutions of $\Delta w^\pm + k^2 w^\pm = -f$ in $Q_\pm^{h_0}$, $w^\pm = 0$ on Γ_\pm , which satisfy the generalized Rayleigh condition (8b); that is,

$$\sum_{\ell \in \mathbb{Z}} \left| \left(\text{sign} x_2 \right) \frac{dw_\ell^\pm(x_2)}{dx_2} - i \sqrt{k^2 - (\ell + \alpha)^2} w_\ell^\pm(x_2) \right|^2 \rightarrow 0, \quad x_2 \rightarrow \pm\infty,$$

where $w_\ell^\pm(x_2)$ are the Fourier coefficients of $w^\pm(\cdot, x_2)$.

(b) For every $\alpha \in \mathbb{R}$, the variational equation (10) can be written as

$$L_\alpha u = r_\alpha \text{ in } H_\alpha^1(Q)$$

where $r_\alpha \in H_\alpha^1(Q)$ and $L_\alpha : H_\alpha^1(Q) \rightarrow H_\alpha^1(Q)$ are given by

$$\begin{aligned} (L_\alpha u, \psi)_{H^1(Q)} &= \int_Q [\nabla u \cdot \nabla \bar{\psi} - k^2 n u \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u) \bar{\psi} ds, \\ (r_\alpha, \psi)_{H^1(Q)} &= \int_Q f \bar{\psi} dx + \int_\Gamma \frac{\partial w}{\partial \nu} \bar{\psi} ds \end{aligned}$$

for $u, \psi \in H_\alpha^1(Q)$. The operator L_α is a Fredholm operator with index zero and Riesz number one (that is, the null spaces of L_α and L_α^* coincide). The operator L_α is invertible if, and only if, α is not a propagative wave number. If $\alpha = \tilde{\alpha}_j + \ell$ (for some $\ell \in \mathbb{Z}$) is a propagative wave number, then the null spaces of L_α and its adjoint L_α^* coincide and are given by the restrictions to Q of the corresponding modes in X_j .

- (c) If $\alpha = \tilde{\alpha}_j + \ell$ is a propagative wave number for some $\ell \in \mathbb{Z}$ and $j \in J$ then the problem (8a), (8b) is solvable if, and only if, $\int_{Q^\infty} f \phi dx = 0$ for all $\phi \in X_j$.
- (d) Define $J_\alpha : H_{\text{per}}^1(Q) \rightarrow H_\alpha^1(Q)$ by $(J_\alpha \phi)(x) := e^{i\alpha x_1} \phi(x)$ and $\tilde{r}_\alpha \in H_{\text{per}}^1(Q)$ and the operator \tilde{L}_α from $H_{\text{per}}^1(Q)$ into itself by $\tilde{r}_\alpha := J_\alpha^{-1} r_\alpha$ and $\tilde{L}_\alpha := J_\alpha^{-1} L_\alpha J_\alpha$, respectively. If $\tilde{\alpha} \in \mathbb{R}$ is not a cut-off value, then there exists a neighborhood $U \subset \mathbb{C}$ of $\tilde{\alpha}$ such that $\alpha \mapsto \tilde{r}_\alpha$ and $\alpha \mapsto \tilde{L}_\alpha$ are analytic as mappings from U into $H_{\text{per}}^1(Q)$ and $\mathcal{L}(H_{\text{per}}^1(Q))$, respectively.

We note that in the case where $f \in L^2(Q^\infty)$ has compact support in Q , the generalized Rayleigh condition (8b) can be replaced by the Rayleigh expansion (5), and the function w appearing in (10) vanishes. Application of this theorem yields existence of the following quasi-periodic scattering problem.

Theorem 2.6. Let Assumptions 2.2 and 2.4 hold. For a given wave number $k > 0$ and unit vector $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ with $|\theta| < \frac{\pi}{2}$; that is, $\hat{\theta}_2 = -\cos \theta < 0$, set $\alpha := k\hat{\theta}_1 = k \sin \theta$. Then there exists $u \in H_{\alpha, \text{loc}}^1(\mathbb{R}^2)$ such that $\Delta u + k^2 n u = 0$ in \mathbb{R}^2 , and $u^s(x) := u(x) - e^{ik\hat{\theta} \cdot x}$ satisfies the Rayleigh expansion (5).

Proof. The scattered field u^s satisfies (8a) with $f = k^2(n-1)u^i$ where $u^i(x) = e^{ik\hat{\theta} \cdot x}$ denotes the incident field. If α is not a propagative wave number, then there exists a unique solution u^s by parts (a) and (b) of Theorem 1. If $\alpha = \tilde{\alpha}_j + \ell$ is a propagative wave number for some $\ell \in \mathbb{Z}$ and $j \in J$, then we have to show that $\int_{Q^\infty} (n-1)u^i \bar{\phi} dx = 0$ for all $\phi \in X_j$.

From the differential equation for ϕ , we obtain

$$k^2 \int_{Q^\infty} (n-1)u^i \bar{\phi} dx = \int_{Q^\infty} u^i [\Delta \bar{\phi} + k^2 \bar{\phi}] dx = \int_{Q^\infty} [\Delta u^i + k^2 u^i] \bar{\phi} dx = 0$$

by Green's second theorem. We used that the product $u^i \bar{\phi}$ is 2π -periodic with respect to x_1 , that u^i is bounded, and that ϕ decays exponentially for $|x_2| \rightarrow \infty$. \square

The following theorem is a special case of a singular perturbation result in [19], Section 1.4. We add the characterization of the limiting solution and give a more direct proof for the convenience of the reader.

Theorem 2.7. *Let $\tilde{\alpha} \in I$ for some open interval $I \subset \mathbb{R}$, K_α compact operators from some Hilbert space X into itself and $r_\alpha \in \mathcal{R}(L_\alpha)$ for all $\alpha \in I$ where $L_\alpha := I - K_\alpha$, and $\mathcal{R}(L_\alpha)$ denotes the range of L_α . Furthermore, let L_α be one-to-one (thus invertible) for all $\alpha \neq \tilde{\alpha}$ and let $L_{\tilde{\alpha}} = I - K_{\tilde{\alpha}}$ have Riesz number one; that is, the null spaces of $L_{\tilde{\alpha}}$ and $L_{\tilde{\alpha}}^2$ coincide. Let $P : X \rightarrow \mathcal{N} := \mathcal{N}(L_{\tilde{\alpha}})$ be the projection onto the null space of $L_{\tilde{\alpha}}$ along the direct decomposition¹ $X = \mathcal{N} \oplus \mathcal{R}$ where $\mathcal{R} = \mathcal{R}(L_{\tilde{\alpha}})$. Finally, let $\alpha \mapsto r_\alpha$ and $\alpha \mapsto K_\alpha$ be analytic in a neighborhood $U \subset \mathbb{C}$ of $\tilde{\alpha}$ and let $PL'_\alpha|_{\mathcal{N}}$ be an isomorphism from \mathcal{N} onto itself where L'_α denotes the derivative of L_α with respect to α at $\alpha = \tilde{\alpha}$.*

Then the mapping $\alpha \mapsto u_\alpha := L_\alpha^{-1}r_\alpha$ has an extension to an analytic mapping from U into X . The limit $u_{\tilde{\alpha}} = \lim_{\alpha \rightarrow \tilde{\alpha}} u_\alpha$ is the unique solution of the system $L_{\tilde{\alpha}}u_{\tilde{\alpha}} = r_{\tilde{\alpha}}$ and $PL'_\alpha u_{\tilde{\alpha}} = Pr'_\alpha$ where r'_α denotes the derivative of r_α at $\alpha = \tilde{\alpha}$. Furthermore, there exists a closed interval $I_0 \subset I$ containing $\tilde{\alpha}$ in its interior and $c > 0$ with

$$\|u_\alpha\|_X \leq c \left[\sup_{\beta \in I_0} \|r_\beta\|_X + \sup_{\beta \in I_0} \|\partial r_\beta / \partial \beta\|_X \right] \text{ for all } \alpha \in I_0. \quad (11)$$

Proof. Without loss of generality we assume that $\tilde{\alpha} = 0$. First, we show uniqueness of the system $L_0 u_0 = r_0$ and $PL'_0 u_0 = Pr'_0$. Let $u_0^{(j)}$ for $j = 1, 2$ denote two solutions. Then $u_0 = u_0^{(1)} - u_0^{(2)}$ satisfies $L_0 u_0 = 0$ and $PL'_0 u_0 = 0$; that is, $u_0 \in \mathcal{N}$ and thus $u_0 = 0$ because PL'_0 is one-to-one on \mathcal{N} .

For $\alpha \neq 0$, we decompose u_α into $u_\alpha = u_\alpha^N + u_\alpha^R$ with $u_\alpha^N \in \mathcal{N}$ and $u_\alpha^R \in \mathcal{R}$ and project the equation $L_\alpha u_\alpha = r_\alpha$ onto \mathcal{N} and \mathcal{R} ; that is, $PL_\alpha(u_\alpha^N + u_\alpha^R) = Pr_\alpha$ and $QL_\alpha(u_\alpha^N + u_\alpha^R) = Qr_\alpha$ where $Q = I - P$ is the projection onto \mathcal{R} .

The operator $QL_\alpha|_{\mathcal{R}}$ is an isomorphism from \mathcal{R} onto itself as easily seen. Therefore, by a perturbation argument, there exist $A_\alpha := [QL_\alpha|_{\mathcal{R}}]^{-1}$ from \mathcal{R} onto itself for all α in a neighborhood $V \subset U$ of 0 and they depend analytically on $\alpha \in V$. Therefore, substituting $u_\alpha^R = A_\alpha(Qr_\alpha - QL_\alpha u_\alpha^N)$ into the first equation yields

$$PL_\alpha(I - A_\alpha QL_\alpha)u_\alpha^N = Pr_\alpha - PL_\alpha A_\alpha Qr_\alpha \text{ in } \mathcal{N},$$

which we write briefly as $C_\alpha u_\alpha^N = s_\alpha$. From $PL_0 = 0$ and $Pr_0 = 0$, we conclude that $C_0 = 0$ and $s_0 = 0$. Therefore, $C_\alpha u_\alpha^N = s_\alpha$ is equivalent to $\frac{1}{\alpha}(C_\alpha - C_0)u_\alpha^N = \frac{1}{\alpha}(s_\alpha - s_0)$. The operators $\frac{1}{\alpha}(C_\alpha - C_0)$ and the elements $\frac{1}{\alpha}(s_\alpha - s_0)$ depend analytically on α in the neighborhood V of $\alpha = 0$ with $\lim_{\alpha \rightarrow 0} \frac{1}{\alpha}(C_\alpha - C_0) = C'_0$ and $\lim_{\alpha \rightarrow 0} \frac{1}{\alpha}(s_\alpha - s_0) = s'_0$. By the chain rule, we compute $C'_0 = PL'_0|_{\mathcal{N}}$ and $s'_0 = Pr'_0 - PL'_0 A_0 r_0$. Since $C'_0 = PL'_0|_{\mathcal{N}}$ is invertible by assumption also $\frac{1}{\alpha}(C_\alpha - C_0)$ is invertible for α in some interval I_0 and its inverses are uniformly bounded with respect to $\alpha \in I_0$, thus $\|u_\alpha^N\|_X \leq c\|(s_\alpha - s_0)/\alpha\|_X \leq c' \sup_\beta \|s'_\beta\|_X$. Also, it is easily seen that u_α^N converges to the unique solution $u_0^N \in \mathcal{N}$ of $C'_0 u_0^N = s'_0$; that is, of $PL'_0 u_0^N = Pr'_0 - PL'_0 A_0 r_0$.

Finally, we observe from above that u_α^R converges to $u_0^R = A_0(Qr_0 - QL_0 u_0^N) = A_0 r_0$. Therefore, u_0^N satisfies $PL'_0 u_0^N = Pr'_0 - PL'_0 u_0^R$; that is, $PL'_0 u_0 = Pr'_0$, which ends the proof. \square

Remark 2.8. From the proof of this theorem, we observe that we can modify the assumptions on the mappings $\alpha \mapsto r_\alpha$ and $\alpha \mapsto K_\alpha$. If these mappings are only continuously differentiable in an open interval $J \subset I$ (as a subset of \mathbb{R}) which contains $\tilde{\alpha}$, then the solution maps $\alpha \mapsto u_\alpha$ is continuous from J into X , and the estimate (11) holds. Also, if the assumption on the injectivity of L_α holds only for $\alpha \in J$ with $\alpha > \tilde{\alpha}$, then the one-sided limit $u_{\tilde{\alpha}} = \lim_{\alpha \rightarrow \tilde{\alpha}, \alpha > \tilde{\alpha}} u_\alpha$ exists and solves the system $L_{\tilde{\alpha}} u_{\tilde{\alpha}} = r_{\tilde{\alpha}}$ and $PL'_\alpha u_{\tilde{\alpha}} = Pr'_\alpha$.

¹This is an immediate consequence of Riesz' third theorem, see, e.g., [23], Theorem 3.3.

3 | THE LIMITING ABSORPTION PRINCIPLE

In this section we consider the unperturbed case; that is, $q = 0$, and prove the limiting absorption principle (LAP) with respect to the refractive index; that is, we replace $n(x)$ in $W := \mathbb{R} \times (-h_0, h_0)$ by $n(x) + i\varepsilon p(x)$ for $\varepsilon > 0$ and let ε tend to zero. Here $p \in L^\infty(W)$ is any fixed non-negative function which is 2π -periodic with respect to x_1 and satisfies $p(x) \geq p_0$ on some open set $\Omega \subset Q$ for some $p_0 > 0$. As an example we can take the constant function $p = 1$. Therefore, let

$$n_\varepsilon(x) := \begin{cases} n(x) + i\varepsilon p(x) & \text{for } x \in W, \\ 1 & \text{for } x \in \mathbb{R}^2 \setminus W. \end{cases}$$

The incident plane wave is given by $u^i(x) = e^{ik\hat{\theta} \cdot x}$ where $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ for some fixed $|\theta| < \frac{\pi}{2}$. Then u^i is α -quasi-periodic with parameter $\alpha := k\hat{\theta}_1 = k \sin \theta$. Therefore, for $\varepsilon > 0$, we wish to determine $u_\varepsilon \in H_{\alpha, \text{loc}}^1(Q^\infty)$ such that

$$\Delta u_\varepsilon + k^2 n_\varepsilon u_\varepsilon = 0 \text{ in } \mathbb{R}^2 \quad (12)$$

and the scattered field $u_\varepsilon^s := u_\varepsilon - u^i$ satisfies the Rayleigh expansion (5). The scattered field satisfies $\Delta u_\varepsilon^s + k^2 n_\varepsilon u_\varepsilon^s = -k^2(n_\varepsilon - 1)u^i$, and by (10), its variational form is given by

$$\int_Q [\nabla u_\varepsilon^s \cdot \nabla \bar{\psi} - k^2 n_\varepsilon u_\varepsilon^s \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u_\varepsilon^s) \bar{\psi} ds = k^2 \int_Q (n_\varepsilon - 1) u^i \bar{\psi} dx$$

for all $\psi \in H_\alpha^1(Q)$. Green's theorem applied to u^i and ψ in Q yields

$$\int_Q [\nabla u^i \cdot \nabla \bar{\psi} - k^2 n_\varepsilon u^i \bar{\psi}] dx - \int_\Gamma \frac{\partial u^i}{\partial \nu} \bar{\psi} ds = -k^2 \int_Q (n_\varepsilon - 1) u^i \bar{\psi} dx$$

and thus by adding both equations

$$\begin{aligned} & \int_Q [\nabla u_\varepsilon \cdot \nabla \bar{\psi} - k^2 n_\varepsilon u_\varepsilon \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u_\varepsilon) \bar{\psi} ds \\ &= \int_\Gamma \left[\frac{\partial u^i}{\partial \nu} - \Lambda_\alpha u^i \right] \bar{\psi} ds = \int_{\Gamma_+} \left[\frac{\partial u^i}{\partial x_2} - \Lambda_\alpha u^i \right] \bar{\psi} ds \\ &= 2ik \hat{\theta}_2 e^{ik\hat{\theta}_2 h_0} \int_0^{2\pi} e^{i\alpha x_1} \overline{\psi(x_1, h_0)} dx_1 = -2ik \cos \theta e^{-ikh_0 \cos \theta} \int_0^{2\pi} e^{i\alpha x_1} \overline{\psi(x_1, h_0)} dx_1 \end{aligned} \quad (13)$$

for all $\psi \in H_\alpha^1(Q)$. Here, we used that for $x_2 < -h_0$, the incident field satisfies the Rayleigh condition; thus, $\frac{\partial u^i}{\partial \nu} = \Lambda_\alpha u^i$ on Γ_- . Furthermore, for $x_2 > h_0$, the α -quasi-periodic solution of the Dirichlet problem with boundary data u^i on Γ_+ is given by $e^{i\alpha x_1 + |\hat{\theta}_2|(x_2 - 2h_0)}$; thus, $\Lambda_\alpha u^i = ik|\hat{\theta}_2|e^{i\alpha x_1 + ik\hat{\theta}_2 h_0}$ on Γ_+ .

Lemma 3.1. *For all $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in H_{\alpha, \text{loc}}^1(Q^\infty)$ of (12), (5), or, equivalently, (13).*

Proof. Since by Theorem 1, this equation can be written as $L_\varepsilon u_\varepsilon = r$ in $H_\alpha^1(Q)$ where L_ε is a Fredholm operator of index zero it suffices to prove uniqueness. For $u^i = 0$, we substitute $\psi = u_\varepsilon$ into the variational equation and obtain

$$\begin{aligned} 0 &= \int_Q [|\nabla u_\varepsilon|^2 - k^2 n_\varepsilon |u_\varepsilon|^2] dx - \int_\Gamma (\Lambda_\alpha u_\varepsilon) \overline{u_\varepsilon} ds \\ &= \int_Q [|\nabla u_\varepsilon|^2 - k^2 n_\varepsilon |u_\varepsilon|^2] dx - i \sum_{\sigma \in \{+, -\}} \sum_{\ell \in \mathbb{Z}} \sqrt{k^2 - (\ell + \alpha)^2} |u_{\varepsilon, \ell}(\sigma h_0)|^2. \end{aligned}$$

Taking the imaginary part

$$0 = -\varepsilon k^2 \int_Q p |u_\varepsilon|^2 dx - \sum_{\sigma \in \{+, -\}} \sum_{|\ell + \alpha| < k} \sqrt{k^2 - (\ell + \alpha)^2} |u_{\varepsilon, \ell}(\sigma h_0)|^2$$

yields $u_\varepsilon = 0$ in Ω . Unique continuation implies that u_ε vanishes in all of Q . \square

Theorem 3.2. *Let Assumptions 2.2 and 2.4 hold, and let $u_\varepsilon \in H_{\alpha, \text{loc}}^1(Q^\infty)$ be the unique solution of the quasi-periodic scattering problem (12), (5) for the plane incident wave of direction $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ for some fixed $|\theta| < \frac{\pi}{2}$. Here, $\alpha := k\hat{\theta}_1 = k \sin \theta$. Then u_ε converges to some u_0 in $H^1(Q)$ which is a solution of (12), (5) for $\varepsilon = 0$. Furthermore, in the case that $\alpha = \tilde{\alpha}_j + \ell$ (for some $\ell \in \mathbb{Z}$ and $j \in J$) is a propagative wave number, u_0 is the only solution which satisfies in addition $\int_Q p u_0 \bar{\phi} dx = 0$ for all modes $\phi \in X_j$.*

Proof. We note that now α is fixed and ε takes the role of the parameter which tends to zero. We write (13) again in the form $L_\varepsilon u_\varepsilon = r$ where $L_\varepsilon : H_\alpha^1(Q) \rightarrow H_\alpha^1(Q)$ and $r \in H_\alpha^1(Q)$ are given by (compare with part (b) of Theorem 1)

$$(L_\varepsilon u, \psi)_{H_\alpha^1(Q)} := \int_Q [\nabla u \cdot \nabla \bar{\psi} - k^2 n_\varepsilon u \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u) \bar{\psi} ds,$$

$$(r, \psi)_{H_\alpha^1(Q)} := -2ik \cos \theta e^{-ikh_0 \cos \theta} \int_0^{2\pi} e^{i\alpha x_1} \overline{\psi(x_1, h_0)} dx_1$$

for $u, \psi \in H_\alpha^1(Q)$.

If α is not propagative wave number, then L_0 is invertible, and one has convergence of u_ε to the unique solution u_0 of $L_0 u_0 = r$ in $H^1(Q)$ as ε tends to zero.

Let now α be a propagative wave number. It is the aim to apply Theorem 2.7 in the modification of Remark 2.8 with $X = H_\alpha^1(Q)$. Then we know from Theorem 1 that the Riesz number of L_0 is one and the null spaces \mathcal{N} of L_0 and its adjoint L_0^* coincide and are given by the restrictions to Q of the space of corresponding propagating modes. Furthermore, L_ε depends obviously analytically on ε . It remains to show that r is in the range of L_0 and that $PL'_0|_{\mathcal{N}}$ is an isomorphism from \mathcal{N} onto itself (where L'_0 denotes the derivative with respect to ε at $\varepsilon = 0$). Since the null spaces of L_0 and its adjoint L_0^* coincide, we have to show that $(r, \phi)_{H_\alpha^1(Q)} = 0$ for all propagating modes ϕ corresponding to the propagative wave number α . We have

$$(r, \phi)_{H_\alpha^1(Q)} = -2ik \cos \theta e^{-ikh_0 \cos \theta} \int_0^{2\pi} e^{i\alpha x_1} \overline{\phi(x_1, h_0)} dx_1 = 0$$

because the Fourier coefficients of the propagating modes ϕ vanish for all $|\ell + \alpha| < k$, in particular for $\ell = 0$ because $|\alpha| = k |\sin \theta| < k$. Furthermore,

$$(L'_0 v, \psi)_{H^1(Q)} = -ik^2 \int_Q p v \bar{\psi} dx, \quad v, \psi \in H_\alpha^1(Q),$$

which shows that $PL'_0|_{\mathcal{N}}$ is an isomorphism from \mathcal{N} onto itself. Application of Theorem 2.7 yields convergence of u_ε to u_0 as ε tends to zero, and u_0 solves the $k \sin \theta$ -quasi-periodic scattering problem and, in addition, $\int_Q p u_0 \bar{\phi} dx = 0$ for all modes ϕ . \square

This result is quite unsatisfactory because the orthogonality condition $\int_Q p u_0 \bar{\phi} dx = 0$ depends on p . The scattering problem for the limiting case $\varepsilon = 0$, however, is independent of p . Therefore, also the extra condition in the case of a propagative wave number should be independent of p .

4 | CONTINUITY WITH RESPECT TO THE DIRECTION OF INCIDENCE

We continue with the unperturbed case; that is, $q = 0$, and the scattering of a plane wave $u_\varphi^i(x) = e^{ik\hat{\varphi}\cdot x}$ for some $\hat{\varphi} = \begin{pmatrix} \sin \varphi \\ -\cos \varphi \end{pmatrix}$ with $|\varphi| < \frac{\pi}{2}$ such that $\alpha := k\hat{\varphi}_1 = k \sin \varphi$ is not a propagative wave number in the sense of Definition 2.1. Then Theorem 1 yields uniqueness and existence of a α -quasi-periodic solution u_φ of $\Delta u_\varphi + k^2 n u_\varphi = 0$ such that $u_\varphi - u_\varphi^i$ satisfies the Rayleigh expansion (5). Let now $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ with $|\theta| < \frac{\pi}{2}$ such that $\tilde{\alpha} := k\hat{\theta}_1 = k \sin \theta$ is a propagative wave number and consider φ in a neighborhood of θ . It is the aim to prove that the unique solution u_φ converges to a solution u_θ of the problem for $\hat{\theta}$ and give a characterization.

We recall from (13) that the scattering problem for the incident direction $\hat{\varphi}$ is equivalent to the variational equation

$$\begin{aligned} & \int_Q [\nabla u_\varphi \cdot \nabla \bar{\psi} - k^2 n u_\varphi \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u_\varphi) \bar{\psi} ds \\ & = -2ik \cos \varphi e^{-ikh_0 \cos \varphi} \int_0^{2\pi} e^{i\alpha x_1} \overline{\psi(x_1, h_0)} dx_1 \text{ for all } \psi \in H_\alpha^1(Q) \end{aligned} \quad (14)$$

where $\alpha = k \sin \varphi$. With this variational formulation of the scattering problem we are able to prove the following convergence result.

Theorem 4.1. *Let Assumptions 2.2 and 2.4 hold and let $\tilde{\alpha} := k \sin \theta$ for some $|\theta| < \frac{\pi}{2}$ be a propagative wave number; that is, $\tilde{\alpha} = k \sin \theta = \tilde{\ell} + \tilde{\alpha}_j$ for some $\tilde{\ell} \in \mathbb{Z}$ and $j \in J$. Furthermore, let u_φ be the unique solution of the $k \sin \varphi$ -quasi-periodic scattering problem of the plane wave incidence of direction $\hat{\varphi} = \begin{pmatrix} \sin \varphi \\ -\cos \varphi \end{pmatrix}$ for φ in a neighborhood of θ . Then u_φ converges in $H^1(Q)$ to some u_θ as φ tends to θ , and u_θ is a $\tilde{\alpha}$ -quasi-periodic solution of the scattering problem corresponding to the incident field of direction $\hat{\theta}$ and the only solution which also satisfies $\int_{Q^\infty} \frac{\partial u_\theta}{\partial x_1} \bar{\phi} dx = 0$ for all propagating modes $\phi \in X_j$.*

Proof. We transform (14) into the 2π -periodic form by setting $\tilde{u}_\varphi(x) = e^{-ik \sin \varphi x_1} u_\varphi(x)$ and replacing $\psi(x)$ by $e^{ik \sin \varphi x_1} \psi(x)$ for some $\psi \in H_{per}^1(Q)$. Then we substitute the form of the Dirichlet-Neumann map and use partial integration $\int_Q \frac{\partial \bar{\psi}}{\partial x_1} \tilde{u}_\varphi dx = -\int_Q \frac{\partial \tilde{u}_\varphi}{\partial x_1} \bar{\psi} dx$. This yields

$$\begin{aligned} & \int_Q \left[\nabla \tilde{u}_\varphi \cdot \nabla \bar{\psi} - 2ik \sin \varphi \frac{\partial \tilde{u}_\varphi}{\partial x_1} \bar{\psi} - k^2 (n - \sin^2 \varphi) \tilde{u}_\varphi \bar{\psi} \right] dx \\ & - i \sum_{\sigma \in \{+, -\}} \sum_{\ell \in \mathbb{Z}} \sqrt{k^2 - (\ell + k \sin \varphi)^2} \tilde{u}_{\varphi, \ell}(\sigma h_0) \overline{\psi_\ell(\sigma h_0)} \\ & = -2ik \cos \varphi e^{-ikh_0 \cos \varphi} \int_0^{2\pi} \overline{\psi(x_1, h_0)} dx_1 \text{ for all } \psi \in H_{per}^1(Q). \end{aligned} \quad (15)$$

Here, $\tilde{u}_{\varphi, \ell}(\pm h_0)$ are the Fourier coefficients of $\tilde{u}_\varphi(\cdot, \pm h_0)$. We write this as $\tilde{L}_\varphi \tilde{u}_\varphi = \tilde{r}_\varphi$ in $H_{per}^1(Q)$ where $\tilde{L}_\varphi := J_\alpha^{-1} L_\alpha J_\alpha$ as in Theorem 1. Since $\tilde{\alpha} = k \sin \theta$ is a propagative wave number it is not a cut-off value by Assumption 2.2. Therefore, by Theorem 1 the operator \tilde{L}_φ satisfies the smoothness assumptions of Theorem 2.7 in a neighborhood of θ , and also the right hand side \tilde{r}_φ depends obviously analytically on φ . Furthermore, \tilde{L}_φ has Riesz number one and the null spaces \mathcal{N} of L_θ and its adjoint L_θ^* coincide and are given by the restrictions to Q of the space of corresponding propagating modes (transformed to the periodic case). The derivatives with respect to φ are given by

$$\begin{aligned} (\tilde{L}'_\varphi \tilde{v}, \tilde{\psi})_{H^1(Q)} & = -2ik \cos \varphi \int_Q \left[\frac{\partial \tilde{v}}{\partial x_1} + ik \sin \varphi \tilde{v} \right] \bar{\psi} dx \\ & + ik \cos \varphi \sum_{\sigma \in \{+, -\}} \sum_{\ell \in \mathbb{Z}} \frac{\ell + k \sin \varphi}{\sqrt{k^2 - (\ell + k \sin \varphi)^2}} \tilde{v}_\ell(\sigma h_0) \overline{\tilde{\psi}_\ell(\sigma h_0)}, \\ (\tilde{r}'_\varphi, \tilde{\psi})_{H^1(Q)} & = 2ik \sin \varphi e^{-ikh_0 \cos \varphi} \left[1 - ik h_0 \cos \varphi \right] \int_0^{2\pi} \overline{\tilde{\psi}(x_1, h_0)} dx_1 \end{aligned}$$

for $\tilde{v}, \tilde{\psi} \in H_{per}^1(Q)$. To show that $P\tilde{L}'_\theta$ is one-to-one on \mathcal{N} , we compute $(\tilde{L}'_\theta \tilde{v}, \tilde{\psi})_{H^1(Q)}$ for $\tilde{v}, \tilde{\psi} \in \mathcal{N}$. As mentioned above, $\tilde{v}(x) = e^{-ik \sin \theta x_1} v(x)$ and $\tilde{\psi}(x) = e^{-ik \sin \theta x_1} \psi(x)$ in Q with propagating modes $v, \psi \in X_j$ which have expansions outside of Q in the forms

$$v(x) = \frac{1}{\sqrt{2\pi}} \sum_{|\ell+k \sin \theta| > k} v_\ell(\pm h_0) e^{i(\ell+k \sin \theta)x_1 - \sqrt{(\ell+k \sin \theta)^2 - k^2}(|x_2| - h_0)}, \pm x_2 > h_0,$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \sum_{|\ell+k \sin \theta| > k} \psi_\ell(\pm h_0) e^{i(\ell+k \sin \theta)x_1 - \sqrt{(\ell+k \sin \theta)^2 - k^2}(|x_2| - h_0)}, \pm x_2 > h_0,$$

respectively. A direct computation yields that

$$\int_Q \frac{\partial v}{\partial x_1} \bar{\psi} dx = \int_Q \left[\frac{\partial \tilde{v}}{\partial x_1} + ik \sin \theta \tilde{v} \right] \bar{\tilde{\psi}} dx \text{ and} \quad (16a)$$

$$\int_{h_0}^{\infty} \int_0^{2\pi} \frac{\partial v}{\partial x_1} \bar{\psi} dx_1 dx_2 = \frac{i}{2} \sum_{|\ell+k \sin \theta| > k} \frac{\ell + k \sin \theta}{\sqrt{(\ell + k \sin \theta)^2 - k^2}} v_\ell(h_0) \overline{\psi_\ell(h_0)}$$

$$= -\frac{1}{2} \sum_{|\ell+k \sin \theta| > k} \frac{\ell + k \sin \theta}{\sqrt{k^2 - (\ell + k \sin \theta)^2}} v_\ell(h_0) \overline{\psi_\ell(h_0)} \quad (16b)$$

and analogously for the integral over $(0, 2\pi) \times (-\infty, -h_0)$. Therefore,

$$(\tilde{L}'_\theta \tilde{v}, \tilde{\psi})_{H^1(Q)} = -2ik \cos \theta \int_{Q^\infty} \frac{\partial v}{\partial x_1} \bar{\psi} dx$$

for $\tilde{v}, \tilde{\psi} \in \mathcal{N}$. Therefore, $P\tilde{L}'_\theta \tilde{v} = 0$ for some $\tilde{v} \in \mathcal{N}$ implies that $\int_{Q^\infty} \frac{\partial v}{\partial x_1} \bar{\psi} dx = 0$ for all $\psi \in X_j$ which implies that v vanishes identically by Assumption 2.4.

Application of Theorem 2.7 yields continuity of $\varphi \mapsto \tilde{u}_\varphi$ in $H^1(Q)$ and $P\tilde{L}'_\theta \tilde{u}_\varphi = P\tilde{r}'_\theta \varphi$; that is, $(\tilde{L}'_\theta \tilde{u}_\varphi, \tilde{\psi})_{H^1(Q)} = (\tilde{r}'_\theta \varphi, \tilde{\psi})_{H^1(Q)}$ for all $\tilde{\psi} \in \mathcal{N}$. As above we go back to the quasi-periodic fields u_θ and ψ . We observe that for $x_2 > h_0$ and $x_2 < -h_0$ the total field u_θ is given by

$$u_\theta(x) = e^{ik \sin \theta x_1} \left[e^{-ik \cos \theta x_2} - e^{ik \cos \theta (x_2 - 2h_0)} \right]$$

$$+ \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} u_{\theta, \ell}(h_0) e^{i(\ell+k \sin \theta)x_1 - \sqrt{(\ell+k \sin \theta)^2 - k^2}(x_2 - h_0)}, \quad x_2 > h_0,$$

$$u_\theta(x) = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} u_{\theta, \ell}(-h_0) e^{i(\ell+k \sin \theta)x_1 - \sqrt{(\ell+k \sin \theta)^2 - k^2}(-x_2 - h_0)}, \quad x_2 < -h_0,$$

where $u_{\theta, \ell}(\pm h_0) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u_\theta(x_1, \pm h_0) e^{-i(\ell+k \sin \theta)x_1} dx_1$. From this and the fact that

$\int_0^{2\pi} \psi(x_1, h_0) e^{-ik \sin \theta x_1} dx_1$ vanishes the propagating modes, we conclude as before that

$$(\tilde{L}'_\theta \tilde{u}_\varphi, \psi)_{H^1(Q)} = -2ik \cos \theta \int_{Q^\infty} \frac{\partial u_\theta}{\partial x_1} \bar{\psi} dx \text{ and } (\tilde{r}'_\theta \varphi, \psi)_{H^1(Q)} = 0$$

for all propagating modes $\psi \in X_j$ which proves $\int_{Q^\infty} \frac{\partial u_\theta}{\partial x_1} \bar{\psi} dx = 0$ for all modes. \square

We note that this condition on u_θ is independent of h_0 in contrast to the condition obtained by the LAP.

5 | APPROXIMATION BY POINT SOURCES

We begin with the scattering problem of a point source at $z \in \mathbb{R}^2$ with $z_2 > h_0$. The total field $u_z(x) = \Phi(x, z) + u_z^s(x)$ is required to satisfy

$$\Delta u_z + k^2(n+q)u_z = 0 \text{ in } \mathbb{R}^2 \setminus \{z\}, \quad (17)$$

where $\Phi(x, z) = \frac{i}{4}H_0^{(1)}(k|x-z|)$ denotes again the fundamental solution. Furthermore, the scattered part $u_z^s = u_z - \Phi(\cdot, z)$ is required to be smooth at $x = z$; that is, $u_z^s \in H_{loc}^1(\mathbb{R}^2)$, and u_z has to satisfy the following open waveguide radiation condition.

Definition 5.1. Let $\psi_+, \psi_- \in C^\infty(\mathbb{R})$ be any (fixed) functions with $\psi_\pm(x_1) = 1$ for $\pm x_1 \geq \sigma_0$ (for some $\sigma_0 > 2\pi + 1$) and $\psi_\pm(x_1) = 0$ for $\pm x_1 \leq \sigma_0 - 1$. Denote by D a disc centered at the origin which contains the source z and the support of q and by $W_H := \mathbb{R} \times (-H, H)$ the layer of width $2H$ for any $H > 0$.

A solution $u \in H_{loc}^1(\mathbb{R}^2 \setminus \bar{D})$ of $\Delta u + k^2nu = 0$ in $\mathbb{R}^2 \setminus \bar{D}$ satisfies the open waveguide radiation condition with respect to given inner products $(\cdot, \cdot)_{X_j}$ in X_j if

(a) u has a decomposition in the form $u = u_{rad} + u_{prop}$ where $u_{rad} \in H^1(W_H \setminus \bar{D})$ for all $H > h_0$ and

$$u_{prop}(x) = \sum_{j \in J} \left[\psi_+(x_1) \sum_{\lambda_{\ell,j} > 0} a_{\ell,j} \phi_{\ell,j}(x) + \psi_-(x_1) \sum_{\lambda_{\ell,j} < 0} a_{\ell,j} \phi_{\ell,j}(x) \right] \quad (18)$$

for $x \in \mathbb{R}^2 \setminus \bar{D}$ and some $a_{\ell,j} \in \mathbb{C}$. Here, $\lambda_{\ell,j} \in \mathbb{R}$ and $\phi_{\ell,j} \in X_j$ for $j \in J$ are the eigenvalues and corresponding eigenfunctions, respectively, of the eigenvalue problem (7) in X_j .

(b) The radiating part u_{rad} satisfies the following generalized angular spectrum radiation condition

$$\int_{-\infty}^{\infty} \left| (\text{sign } x_2) \frac{\partial(\mathcal{F}u_{rad})(\omega, x_2)}{\partial x_2} - i\sqrt{k^2 - \omega^2} (\mathcal{F}u_{rad})(\omega, x_2) \right|^2 d\omega \rightarrow 0 \quad (19)$$

as $|x_2| \rightarrow \infty$ where $(\mathcal{F}u_{rad})(\cdot, x_2)$ denotes the Fourier transform of $u_{rad}(\cdot, x_2)$ with respect to x_1 . We normalize the Fourier transform as $(\mathcal{F}\phi)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s) e^{-ist} ds$ for $t \in \mathbb{R}$.

Remark: We refer to [2, 8, 24] for the original angular spectrum representation condition (ASR) which can shortly be formulated as $(\text{sign } x_2) \frac{\partial(\mathcal{F}u_{rad})(\omega, \cdot)}{\partial x_2} - i\sqrt{k^2 - \omega^2} (\mathcal{F}u_{rad})(\omega, \cdot) = 0$ on \mathbb{R} for almost all ω .

We transform this scattering problem to a problem with a compactly supported source. Indeed, for some $\varepsilon > 0$ we choose a function $\eta \in C^\infty(\mathbb{R}^2)$ with $\eta(y) = 1$ for $|y| \leq \varepsilon/2$ and $\eta(y) = 0$ for $|y| \geq \varepsilon$. We decompose u_z as $u_z = \eta_z \Phi(\cdot, z) + \hat{u}_z^s$ with $\hat{u}_z^s := u_z - \eta_z \Phi(\cdot, z)$ where we have set $\eta_z(x) = \eta(x-z)$. Then \hat{u}_z^s satisfies (note that $(1-n+q)\eta_z$ vanishes identically if $z_2 > h_0 + \varepsilon$)

$$\Delta \hat{u}_z^s + k^2(n+q)\hat{u}_z^s = -f_z \text{ in } \mathbb{R}^2, \quad (20)$$

where the right hand side $f_z := 2\nabla\eta_z \cdot \nabla_x \Phi(\cdot, z) + \Delta\eta_z \Phi(\cdot, z)$ is supported in the annulus $\{x \in \mathbb{R}^2 : \varepsilon/2 < |x-z| < \varepsilon\}$ which we assume to be in D .

It has been shown in [20] for the case of a half plane problem that the radiation condition of Definition 5.1 for compactly supported source functions $f \in L^2(Q)$ is a consequence of the limiting absorption principle. In [22], it is shown that the source problem (20) for any source function $f \in L^2(Q)$ has a unique solution satisfying the open waveguide radiation condition. Furthermore, we note that the solution \hat{u}_z^s of (20) satisfies the open waveguide condition if, and only if, the solution u_z of (17) satisfies the radiation condition because $\hat{u}_z^s - u_z$ vanishes for $|x-z| > \varepsilon$ and can be subsumed into the radiating part.

From now on, we consider again the unperturbed case $q = 0$. In this case, the coefficients $a_{\ell,j} = a_{\ell,j}(z)$ are given explicitly by

$$a_{\ell,j}(z) := \frac{2\pi i}{|\lambda_{\ell,j}|} \int_{|x-z|<\varepsilon} f_z(x) \overline{\phi_{\ell,j}(x)} dx, \quad (21)$$

see again [22]. It is the aim to prove the following convergence result.

Theorem 5.2. *Let Assumptions 2.2 and 2.4 hold and let $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \in \mathbb{R}^2$ be a fixed unit vector with $|\theta| < \frac{\pi}{2}$; that is, $\hat{\theta}_2 < 0$. In addition, let $\tilde{\alpha} := k\hat{\theta}_1 = k \sin \theta$ not be a cut-off value in the sense of Definition 2.1. Let u_t be the unique solution of the unperturbed (that is, for $q = 0$) scattering problem of the point source at $z = -t\hat{\theta}$ for $t|\hat{\theta}_2| > 2h_0$ such that $u_t^s := u_t - \Phi(\cdot, -t\hat{\theta}) \in H_{loc}^1(\mathbb{R}^2)$ and u_t satisfies the open waveguide radiation condition of Definition 5.1. Then*

$$\frac{1}{\gamma} \lim_{t \rightarrow \infty} \left[\sqrt{t} e^{-ikt} u_t \right] = v_\theta \text{ in } H^1(Q_R) \quad (22)$$

for any $R > 0$ where $Q_R := (-R, R) \times (-h_0, h_0)$, and where $v_\theta \in H_{\tilde{\alpha},loc}^1(\mathbb{R}^2)$ solves the $\tilde{\alpha}$ -quasi-periodic scattering problem $\Delta v_\theta + k^2 n v_\theta = 0$ in \mathbb{R}^2 such that the scattered field $v_\theta^s(x) := v_\theta(x) - e^{ik\hat{\theta} \cdot x}$ satisfies the Rayleigh expansion (5) for $\alpha = \tilde{\alpha} = k \sin \theta$.

If $\tilde{\alpha} = k \sin \theta = \ell + \tilde{\alpha}_j$ is a propagative wave number (for some $\ell \in \mathbb{Z}$ and $j \in J$) with corresponding space X_j of propagating modes then the total field v_θ is the only solution which satisfies in addition

$$\int_{Q^\infty} \frac{\partial v_\theta}{\partial x_1} \bar{\phi} dx = 0 \text{ for all } \phi \in X_j. \quad (23)$$

We note that the convergence of the total fields in (22) corresponds exactly to the convergence of the incident fields in (3). Therefore, this theorem justifies rigorously the assumption that one searches right away for $k \sin \theta$ -quasi-periodic solutions of the scattering problem. We note however that this result holds also for the case that $k \sin \theta$ is a propagative wave number. In this case, there is no uniqueness of the scattering problem by the plane wave of direction $\hat{\theta}$ of incidence, and Theorem 5.2 formulates the extra orthogonality condition (23) which coincides with the condition of Theorem 4.1.

We were not able to prove Theorem 5.2 in the case that $k \sin \theta - \ell$ is one of the cut-off values $\pm \kappa$ for some $\ell \in \mathbb{Z}$.

Proof of Theorem 5.2²: For the moment, we consider any $z \in \mathbb{R}^2$ with $z_2 > h_0 + \varepsilon$. From (20) (for $q = 0$), we note that the radiating part $u_{z,rad}$ of u_z^s solves

$$\Delta u_{z,rad} + k^2 n u_{z,rad} = -f_z - g_z \text{ in } \mathbb{R}^2, \quad (24)$$

where

$$f_z = 2 \nabla \eta_z \cdot \nabla_x \Phi(\cdot, z) - \Delta \eta_z \Phi(\cdot, z) = (\Delta + k^2) [(\eta_z - 1) \Phi(\cdot, z)] \text{ and} \quad (25a)$$

$$g_z = (\Delta + k^2 n) u_{z,prop} = \sum_{j \in J} \sum_{\ell=1}^{m_j} a_{\ell,j}(z) \varphi_{\ell,j} \text{ with} \quad (25b)$$

$$\varphi_{\ell,j}(x) = \begin{cases} 2\psi'_+(x_1) \frac{\partial \phi_{\ell,j}(x)}{\partial x_1} + \psi'_+(x_1) \phi_{\ell,j}(x) & \text{if } \lambda_{\ell,j} > 0, \\ 2\psi'_-(x_1) \frac{\partial \phi_{\ell,j}(x)}{\partial x_1} + \psi'_-(x_1) \phi_{\ell,j}(x) & \text{if } \lambda_{\ell,j} < 0. \end{cases}$$

Now, we use the Floquet-Bloch transform F to transform (24) to a family of quasi-periodic problems. For functions $v \in C_0^\infty(\mathbb{R}^2)$, the transform is defined as

$$(Fv)(x, \alpha) := \sum_{\ell \in \mathbb{Z}} v(x_1 + 2\pi\ell, x_2) e^{-i\alpha 2\pi\ell}, \quad x \in \mathbb{R}^2.$$

²We note already here that we will interrupt the proof by four lemmas.

Then it is known (see, e.g., [25–27]) that F has an extension to an isomorphism from $H^1(W)$ onto

$$H_{QP}^1(Q \times I) := \left\{ u \in L^2(Q \times I) : \begin{array}{l} u(\cdot, \alpha) \in H_a^1(Q) \text{ for almost all } \alpha \text{ and} \\ \alpha \mapsto \|u(\cdot, \alpha)\|_{H^1(Q)} \text{ is in } L^2(I) \end{array} \right\}$$

where $I = (-1/2, 1/2)$ (or any other interval of length 1). The inverse is given by $u = \int_I (Fu)(\cdot, \alpha) d\alpha$ in W where $(Fu)(\cdot, \alpha)$ is extended α -quasi-periodically to W .

We know from [22] that the Floquet-Bloch transformed equation

$$\Delta u_{\alpha,z} + k^2 n u_{\alpha,z} = -(Ff_z)(\cdot, \alpha) - (Fg_z)(\cdot, \alpha) \text{ in } Q^\infty \quad (26)$$

for $u_{\alpha,z} = (Fu_{z,rad})(\cdot, \alpha)$ is solvable for all $\alpha \in \mathbb{R}$ (without exception) and that $\alpha \mapsto u_{\alpha,z}$ has an extension to a mapping in $W^{1,1}(I, H^1(Q))$ and is even analytic in neighborhoods of points $\tilde{\alpha}$ which are no cut-off values. By part (b) of Theorem 1 this equation can be written as a variational equation in the form

$$\int_Q [\nabla u_{\alpha,z} \cdot \nabla \bar{\psi} - k^2 n u_{\alpha,z} \bar{\psi}] dx - \int_\Gamma (\Lambda_\alpha u_{\alpha,z}) \bar{\psi} ds = \int_Q (Fg_z)(\cdot, \alpha) \bar{\psi} dx + \int_\Gamma \frac{\partial w_{\alpha,z}}{\partial \nu} \bar{\psi} ds \quad (27)$$

for all $\psi \in H_a^1(Q)$ or shortly as $L_\alpha u_{\alpha,z} = r_{\alpha,z}$ in $H_a^1(Q)$ where $r_{\alpha,z} \in H_a^1(Q)$ denotes the Riesz representation of the right hand side. Note that Ff_z vanishes in Q and therefore appears only implicitly in $w_{\alpha,z}^\pm$. The functions $w_{\alpha,z}^\pm \in H_{\alpha,loc}^1(Q_\pm^{h_0})$ are the α -quasi-periodic solutions of

$$\begin{aligned} \Delta w_{\alpha,z}^+ + k^2 w_{\alpha,z}^+ &= -(Ff_z)(\cdot, \alpha) - (Fg_z)(\cdot, \alpha) \\ &= -(\Delta + k^2)F((\eta_z - 1)\Phi(\cdot, z)) - (Fg_z)(\cdot, \alpha) \\ &= -(\Delta + k^2)F((\eta_z - 1)\Phi(\cdot, z) + \Phi(\cdot, z^*)) - (Fg_z)(\cdot, \alpha) \end{aligned} \quad (28a)$$

in $Q_+^{h_0}$ with $w_{\alpha,z}^+ = 0$ for $x_2 = h_0$ and

$$\Delta w_{\alpha,z}^- + k^2 w_{\alpha,z}^- = -(Fg_z)(\cdot, \alpha) \text{ in } Q_-^{h_0} \quad (28b)$$

with $w_{\alpha,z}^- = 0$ for $x_2 = -h_0$, satisfying the generalized Rayleigh condition (8b). Here, we used the definition of f_z and the fact that η_z vanishes in $Q_-^{h_0}$. The point $z^* = (z_1, 2h_0 - z_2)^\top$ is the reflection of z at the line $x_2 = h_0$.

Lemma 5.3. *Let $w_{\alpha,z}^\pm$ be the solutions of (28a) and (28b), respectively. Then $\partial w_{\alpha,z}^\pm / \partial x_2$ are given by*

$$\begin{aligned} \frac{\partial w_{\alpha,z}^+(x_1, h_0)}{\partial x_2} &= \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} e^{i\sqrt{k^2 - (\ell + \alpha)^2}(z_2 - h_0)} e^{i(\ell + \alpha)(x_1 - z_1)} \\ &\quad + \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} \int_{h_0}^{\infty} (Fg_z)_\ell(y_2, \alpha) e^{i\sqrt{k^2 - (\ell + \alpha)^2}(y_2 - h_0)} dy_2 e^{i(\ell + \alpha)x_1}, \\ \frac{\partial w_{\alpha,z}^-(x_1, -h_0)}{\partial x_2} &= -\frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} \int_{h_0}^{\infty} (Fg_z)_\ell(-y_2, \alpha) e^{i\sqrt{k^2 - (\ell + \alpha)^2}(y_2 - h_0)} dy_2 e^{i(\ell + \alpha)x_1} \end{aligned}$$

for $x_1 \in (0, 2\pi)$ where $(Fg_z)_\ell(y_2, \alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (Fg_z)(y, \alpha) e^{-i(\ell + \alpha)y_1} dy_1$ are the Fourier coefficients of $(Fg_z)(\cdot, y_2, \alpha)$.

Proof. We write $(\eta_z - 1)\Phi(\cdot, z) + \Phi(\cdot, z^*) = -G^+(\cdot, z) + \eta_z \Phi(\cdot, z)$ where $G^+(x, z) = \Phi(x, z) - \Phi(x, z^*)$ denotes the Green's function for the half space $\{x \in \mathbb{R}^2 : x_2 > h_0\}$. Furthermore, the Floquet-Bloch transform $(FG^+(\cdot, z))(x, \alpha)$ is just the α -quasi-periodic Green's function in $Q_+^{h_0}$, given by

$$(FG^+(\cdot, z))(x, \alpha) = \frac{i}{4\pi} \sum_{\ell \in \mathbb{Z}} \frac{1}{\sqrt{k^2 - (\ell + \alpha)^2}} \left[e^{i\sqrt{k^2 - (\ell + \alpha)^2}|x_2 - z_2|} - e^{i\sqrt{k^2 - (\ell + \alpha)^2}(x_2 + z_2 - 2h_0)} \right] e^{i(\ell + \alpha)(x_1 - z_1)}.$$

Indeed, this follows from the connection between the Fourier transform \mathcal{F} and the Floquet-Bloch transform F

$$(F\phi)(\ell + \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s) e^{-is(\ell + \alpha)} ds = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (F\phi)(t, \alpha) e^{-i(\ell + \alpha)t} dt$$

(just decompose the region of integration into $\bigcup_{\ell \in \mathbb{Z}} (2\pi\ell, 2\pi\ell + 2\pi)$), writing this as $(F\phi)(t, \alpha) = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} (F\phi)(\ell + \alpha) e^{i(\ell + \alpha)t}$, and using formulas 3. and 4. in [28], Section 6.677.

Therefore, $F(G^+(\cdot, z) - \eta_z \Phi(\cdot, z))$ is smooth near $x = z$ and vanishes for $x_2 = h_0$ and satisfies the Rayleigh expansion (5) because η_z vanishes near $x_2 = h_0$ and for $|x| > |z| + \varepsilon$. Therefore,

$$w_{\alpha, z}^+ = F(G^+(\cdot, z) - \eta_z \Phi(\cdot, z))(\cdot, \alpha) + v_{\alpha, z}^+ \text{ in } Q_+^{h_0}$$

where $v_{\alpha, z}^+$ is the radiating solution of $\Delta v_{\alpha, z}^+ + k^2 v_{\alpha, z}^+ = -(Fg_z)(\cdot, \alpha)$ in $Q_+^{h_0}$ with $v_{\alpha, z}^+ = 0$ for $x_2 = h_0$. Expanding $v_{\alpha, z}^+$ into a Fourier series and solving the one dimensional boundary value problem $\frac{d}{dx_2} v_{\ell, \alpha, z}^+(x_2) + (k^2 - (\ell + \alpha)^2) v_{\ell, \alpha, z}^+(x_2) = -(Fg_z)_{\ell}(x_2, \alpha)$ for $x_2 > h_0$ and $v_{\ell, \alpha, z}^+(h_0) = 0$ and the generalized Rayleigh condition (8b) for its Fourier coefficients gives

$$v_{\ell, \alpha, z}^+(x_2) = \frac{i}{2} \int_{h_0}^{\infty} (Fg_z)_{\ell}(y_2, \alpha) \frac{e^{i\sqrt{k^2 - (\ell + \alpha)^2}|x_2 - y_2|} - e^{i\sqrt{k^2 - (\ell + \alpha)^2}(x_2 + y_2 - 2h_0)}}{\sqrt{k^2 - (\ell + \alpha)^2}} dy_2.$$

This proves the form for $w_{\alpha, z}^+$. Since $w_{\alpha, z}^-$ plays the role of $v_{\alpha, z}^+$ in $Q_-^{h_0}$ the representation is shown analogously. \square

With this result we rewrite (27) as

$$\begin{aligned} (L_{\alpha} u_{\alpha, z}, \psi)_{H^1(Q)} &= \int_Q (Fg_z)(\cdot, \alpha) \overline{\psi} dx + \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} e^{i\sqrt{k^2 - (\ell + \alpha)^2}(z_2 - h_0)} e^{-i(\ell + \alpha)z_1} \overline{\psi}_{\ell}(h_0) \\ &+ \sum_{\sigma \in \{+, -\}} \sum_{\ell \in \mathbb{Z}} \overline{\psi}_{\ell}(\sigma h_0) \int_{h_0}^{\infty} (Fg_z)_{\ell}(\sigma y_2, \alpha) e^{i\sqrt{k^2 - (\ell + \alpha)^2}(y_2 - h_0)} dy_2, \end{aligned} \tag{29}$$

where the operator L_{α} from $H_{\alpha}^1(Q)$ into itself is again defined as

$$(L_{\alpha} v, \psi)_{H^1(Q)} := \int_Q [\nabla v \cdot \nabla \overline{\psi} - k^2 n v \overline{\psi}] dx - \int_{\Gamma} (\Lambda_{\alpha} v) \overline{\psi} ds, \quad v, \psi \in H_{\alpha}^1(Q).$$

At this point we define the sources z to be $z = z(t) = -t\hat{\theta}$ for $t > 0$ where $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$ for $|\theta| < \frac{\pi}{2}$ is the fixed direction of the incident plane wave with $\hat{\theta}_2 = -\cos \theta < 0$. We choose $t > 0$ such that $z_2(t) = -t\hat{\theta}_2 = t \cos \theta > 2h_0$. Then $z_2(t) \rightarrow \infty$ as $t \rightarrow \infty$. We change the symbols slightly and write $u_{\alpha, t}$ and g_t and $a_{\ell, j}(t)$ for $u_{\alpha, z(t)}$ and $g_{z(t)}$ and $a_{\ell, j}(z(t))$, respectively.

It is now the aim to study the inverse Floquet-Bloch transform $u_t(x) = \int_{-1/2}^{1/2} u_{\alpha, t}(x) d\alpha$ when t tends to infinity. We will decompose u_t into components and split the region into parts and discuss the contributions separately.

From the definitions (25b) and (21) of g_z and $a_{\ell, j}(z)$, respectively, the exponential decay of $\phi_{\ell, j}$, and the fact that the support of f_z is contained in the disc $\{x \in \mathbb{R}^2 : |x - z| \leq \varepsilon\}$ we first note that $|a_{\ell, j}(t)| \leq ce^{-\delta t}$, and thus,

$$\|u_{t, prop}\|_{H^1(Q_R)} \leq c_R \sum_{j \in J} \sum_{\ell=1}^{m_j} |a_{\ell, j}(t)| \leq ce^{-\delta t} \tag{30}$$

for some $c > 0$ and

$$|(Fg_t)(x, \alpha)| + |\partial(Fg_t)(x, \alpha)/\alpha| \leq ce^{-\delta(t+|x_2|)}, \quad x \in Q^\infty \setminus Q, \quad (31)$$

for all $t > 0$, $R > 0$, and $\alpha \in [-1/2, 1/2]$.

We split the first series on the right hand side of (29) into propagating and evanescent parts. Decompose k again into the form $k = \tilde{\ell} + \kappa$ with $\tilde{\ell} \in \mathbb{N}_0$ and $\kappa \in (-1/2, 1/2]$. Then $\pm\kappa$ are the cut-off values. We can always decompose $[-1/2, 1/2]$ in the form $[-1/2, 1/2] = I_1 \cup I_2 \cup I_3$ with closed intervals I_m such that their interiors are pairwise disjoint and find corresponding sets $\mathcal{L}_m \subset \{-\tilde{\ell}, \dots, \tilde{\ell}\}$ such that $|\ell + \alpha| \leq k$ for all $\alpha \in I_m$ and $\ell \in \mathcal{L}_m$ and $|\ell + \alpha| \geq k$ for all $\alpha \in I_m$ and $\ell \notin \mathcal{L}_m$ for $m = 1, 2, 3$. For example, if $\kappa \geq 0$, then $I_1 = [-\kappa, \kappa]$ with $\mathcal{L}_1 = \{-\tilde{\ell}, \dots, \tilde{\ell}\}$, $I_2 = [-1/2, -\kappa]$ with $\mathcal{L}_2 = \{-\tilde{\ell} + 1, \dots, \tilde{\ell}\}$, and $I_3 = [\kappa, 1/2]$ with $\mathcal{L}_3 = \{-\tilde{\ell}, \dots, \tilde{\ell} - 1\}$. Some of the intervals can degenerate into points (as I_3 in the preceding example if $\kappa = 1/2$ or I_1 if $\kappa = 0$) and some of the sets \mathcal{L}_m can be empty (as \mathcal{L}_2 and \mathcal{L}_3 in the preceding example if $\tilde{\ell} = 0$). The cut-off values are contained in the boundary points of I_m .

For $\alpha \in I_m$ (where $m \in \{1, 2, 3\}$ is kept fixed), we rewrite (29) in the form

$$\begin{aligned} (L_\alpha u_{\alpha,t}, \psi)_{H^1(Q)} &= \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathcal{L}_m} e^{it[(\ell+\alpha)\hat{\theta}_1 + \sqrt{k^2 - (\ell+\alpha)^2}|\hat{\theta}_2|]} e^{-i\sqrt{k^2 - (\ell+\alpha)^2}h_0} \overline{\psi_\ell(h_0)} \\ &\quad + \frac{1}{\sqrt{2\pi}} \sum_{\ell \notin \mathcal{L}_m} e^{it[i(\ell+\alpha)\hat{\theta}_1 - \sqrt{(\ell+\alpha)^2 - k^2}|\hat{\theta}_2|]} e^{\sqrt{(\ell+\alpha)^2 - k^2}h_0} \overline{\psi_\ell(h_0)} \\ &\quad + \int_Q (Fg_t)(\cdot, \alpha) \overline{\psi} dx + \sum_{\sigma \in \{+, -\}} \sum_{\ell \in \mathbb{Z}} \overline{\psi_\ell(\sigma h_0)} \int_{h_0}^{\infty} (Fg_t)_\ell(\sigma y_2, \alpha) e^{i\sqrt{k^2 - (\ell+\alpha)^2}(y_2 - h_0)} dy_2 \end{aligned} \quad (32)$$

for all $\psi \in H_\alpha^1(Q)$. We recall that if α is not a propagative wave number, then this equation is uniquely solvable. If $\alpha = \tilde{\alpha}_j$ is a propagative wave number in I_m , then by the choice of $a_{\ell,j}(t)$, this equation is also solvable because $r_{\tilde{\alpha}_j,t}$ is orthogonal to X_j ; that is, the right hand side of (32) vanishes for modes $\psi = \phi_j \in X_j$ corresponding to $\tilde{\alpha}_j$. This has been shown in [22].

The right hand side of (32) suggests to decompose $u_{\alpha,t}$ for $\alpha \in I_m$ into a sum of the form

$$u_{\alpha,t} = \frac{i}{4\pi} \sum_{\ell \in \mathcal{L}_m} e^{it[(\ell+\alpha)\hat{\theta}_1 + \sqrt{k^2 - (\ell+\alpha)^2}|\hat{\theta}_2|]} \frac{1}{\sqrt{k^2 - (\ell + \alpha)^2}} v_{\ell,\alpha} + u_{\alpha,t}^{(1)} \quad (33)$$

with functions $v_{\ell,\alpha} \in H_\alpha^1(Q)$ for $\ell \in \mathcal{L}_m$ which are independent of t and solutions of

$$\begin{aligned} (L_\alpha v_{\ell,\alpha}, \psi)_{H^1(Q)} &= -2i\sqrt{2\pi} \sqrt{k^2 - (\ell + \alpha)^2} e^{-i\sqrt{k^2 - (\ell+\alpha)^2}h_0} \overline{\psi_\ell(h_0)} \\ &= -2i\sqrt{k^2 - (\ell + \alpha)^2} e^{-i\sqrt{k^2 - (\ell+\alpha)^2}h_0} \int_0^{2\pi} \overline{\psi(x_1, h_0)} e^{i(\ell+\alpha)x_1} dx_1 \end{aligned} \quad (34)$$

for all $\psi \in H_\alpha^1(Q)$. The solutions exist for all $\alpha \in I_m$ because for every propagative wave number $\alpha = \tilde{\alpha}_j \in I_m$ the right hand side of (34) vanishes for every $\psi = \phi \in X_j$. Indeed, in this case ϕ is evanescent; that is, the Fourier coefficients $\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \phi(x_1, h_0, \tilde{\alpha}) e^{-i(\ell+\tilde{\alpha})x_1} dx_1$ vanish for $|\ell + \tilde{\alpha}| < k$; that is, for all $\ell \in \mathcal{L}_m$. This proves existence of a solution for all $\alpha \in I_m$. The functions $v_{\ell,\alpha}$ are solutions of α -quasi-periodic scattering problems for plane wave incidence as the next lemma shows.

Lemma 5.4. $v_{\ell,\alpha}$ is the restriction to Q of a solution of the α -quasi-periodic scattering problem of the incident plane wave of direction $\hat{\theta}_\ell = \frac{1}{k} \left(\ell + \alpha, -\sqrt{k^2 - (\ell + \alpha)^2} \right)^\top$ to determine the total field $v_{\ell,\alpha}$ as the sum $v_{\ell,\alpha}(x) = e^{i(\ell+\alpha)x_1 - i\sqrt{k^2 - (\ell+\alpha)^2}x_2} + v_{\ell,\alpha}^s(x)$ such that

$$\Delta v_{\ell,\alpha} + k^2 n v_{\ell,\alpha} = 0 \text{ in } \mathbb{R}^2, \quad (35)$$

and the scattered field $v_{\ell,\alpha}^s$ satisfies the Rayleigh expansion (5) outside of Q .

Proof. We consider the scattering problem and make an ansatz for the solution in the form

$$v_{\ell,\alpha}(x) = \tilde{v}_{\ell,\alpha}^s(x) + \begin{cases} e^{i(\ell+\alpha)x_1} \left[e^{-i\sqrt{k^2-(\ell+\alpha)^2}x_2} - e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-2h_0)} \right], & x_2 > h_0, \\ 0, & x_2 < -h_0, \end{cases}$$

where

$$\tilde{v}_{\ell,\alpha}^s(x) = \frac{1}{\sqrt{2\pi}} \sum_{\ell' \in \mathbb{Z}} a_{\ell'}(\pm h_0) e^{i(\ell'+\alpha)x_1 + i\sqrt{k^2-(\ell'+\alpha)^2}(|x_2|-h_0)} \text{ for } \pm x_2 > h_0. \quad (36)$$

Then $v_{\ell,\alpha} = \tilde{v}_{\ell,\alpha}^s$ on $\Gamma = \Gamma_+ \cup \Gamma_-$, and thus,

$$\begin{aligned} \frac{\partial v_{\ell,\alpha}(x_1, h_0)}{\partial x_2} &= (\Lambda_\alpha v_{\ell,\alpha})(x_1, h_0) - 2i\sqrt{k^2 - (\ell + \alpha)^2} e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0} e^{i(\ell+\alpha)x_1}, \\ \frac{\partial v_{\ell,\alpha}(x_1, -h_0)}{\partial x_2} &= (\Lambda_\alpha v_{\ell,\alpha})(x_1, -h_0). \end{aligned}$$

Therefore, the variational form of (35) is

$$\begin{aligned} & \int_Q [\nabla v_{\ell,\alpha} \cdot \nabla \bar{\psi} - k^2 n v_{\ell,\alpha} \bar{\psi}] dx - \int_\Gamma (\Lambda v_{\ell,\alpha}) \bar{\psi} ds \\ &= -2i\sqrt{k^2 - (\ell + \alpha)^2} e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0} \int_0^{2\pi} e^{i(\ell+\alpha)x_1} \overline{\psi(x_1, h_0)} dx_1 \text{ for all } \psi \in H_{per}^1(Q) \end{aligned}$$

which coincides with (34). \square

Lemma 5.5. *Let $\tilde{\alpha}$ be a fixed value in the interior of I_m and $v_{\ell,\alpha}$ as in the previous lemma for $\ell \in \mathcal{L}_m$ and $\alpha \in I_m$. Then the solution map $\alpha \mapsto v_{\ell,\alpha}$ can be extended to an analytic map from an open neighborhood $U \subset \mathbb{C}$ of $\tilde{\alpha}$ into $H^1(Q)$. Furthermore, if $\tilde{\alpha} = \tilde{\alpha}_j$ is a propagative wave number, then this extension into $\tilde{\alpha}_j$ satisfies*

$$\int_{Q^\infty} \frac{\partial v_{\ell,\tilde{\alpha}_j}}{\partial x_1} \bar{\phi} dx = 0 \quad (37)$$

for all corresponding modes $\phi \in X_j$.

We omit the proof because it follows from Theorem 4.1 if one writes $(\ell + \alpha)/k$ as $(\ell + \alpha)/k = k \sin \varphi$ in the incident plane wave of direction $\hat{\theta}_\ell$.

Next we consider the remaining term

$$u_{\alpha,t}^{(1)} := u_{\alpha,t} - \frac{i}{4\pi} \sum_{\ell \in \mathcal{L}_m} e^{it[(\ell+\alpha)\hat{\theta}_1 + \sqrt{k^2-(\ell+\alpha)^2}|\hat{\theta}_2]} \frac{1}{\sqrt{k^2 - (\ell + \alpha)^2}} v_{\ell,\alpha}$$

of (33) which satisfies

$$\begin{aligned} (L_\alpha u_{\alpha,t}^{(1)}, \psi)_{H_\alpha^1(Q)} &= \frac{1}{\sqrt{2\pi}} \sum_{\ell \notin \mathcal{L}_m} e^{t[i(\ell+\alpha)\hat{\theta}_1 - \sqrt{(\ell+\alpha)^2 - k^2}|\hat{\theta}_2]} e^{\sqrt{(\ell+\alpha)^2 - k^2}h_0} \overline{\psi_\ell(h_0)} \\ &+ \int_Q (Fg_t)(x, \alpha) \overline{\psi(x)} dx + \sum_{\sigma \in \{+, -\}} \sum_{\ell \in \mathbb{Z}} \overline{\psi_\ell(\sigma h_0)} \int_{h_0}^\infty (Fg_t)_\ell(\sigma y_2, \alpha) e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)} dy_2 \end{aligned}$$

for all $\psi \in H_{per}^1(Q)$ which we write briefly as $L_\alpha u_{\alpha,t}^{(1)} = r_{\alpha,t}^{(1)} + r_{\alpha,t}^{(2)} + r_{\alpha,t}^{(3)}$.

Lemma 5.6. *There exists $c > 0$ such that*

$$\int_{I_m} \|u_{\alpha,t}^{(1)}\|_{H^1(Q)} d\alpha \leq \frac{c}{t} \text{ for all } t|\hat{\theta}_2| \geq 2h_0.$$

Proof. We decompose I_m into a finite union of closed intervals $I \subset I_m$ with non-intersecting interiors where I is one of the following two types.

First case: Let $I \subset I_m$ does not contain any of the propagative wave numbers $\tilde{\alpha}_j$. Then \tilde{L}_α^{-1} is uniformly bounded with respect to $\alpha \in I$. We estimate the three terms $r_{\alpha,t}^{(j)}$ on the right hand side. The inequality of Cauchy-Schwarz and the trace theorem yields for every $\alpha \in I_m$

$$\|r_{\alpha,t}^{(1)}\|_{H^1(Q)} \leq c \left(\sum_{\ell \notin \mathcal{L}_m} e^{-2\sqrt{(\ell+\alpha)^2-k^2}(t|\hat{\theta}_2|-h_0)} \right)^{1/2}$$

where $c > 0$ is independent of α and t . Furthermore, (31) implies

$$\|r_{\alpha,t}^{(2)}\|_{H^1(Q)} \leq \|(Fg_t)(\cdot, \alpha)\|_{L^2(Q)} \leq ce^{-\delta t} \text{ for all } \alpha \in I_m.$$

For $r_{\alpha,t}^{(3)}$, we consider first $|\ell| \geq k+1$. Then $|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}| = e^{-\sqrt{(\ell+\alpha)^2-k^2}(y_2-h_0)}$, and thus,

$$\begin{aligned} & \left[\sum_{|\ell| \geq k+1} |\psi_\ell(h_0)| \int_{h_0}^{\infty} |(Fg_t)_\ell(y_2, \alpha)| |e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}| dy_2 \right]^2 \\ & \leq \sum_{|\ell| \geq k+1} |\psi_\ell(h_0)|^2 \sum_{|\ell| \geq k+1} \int_{h_0}^{\infty} |(Fg_t)_\ell(y_2, \alpha)|^2 dy_2 \int_{h_0}^{\infty} e^{-2\sqrt{(\ell+\alpha)^2-k^2}(y_2-h_0)} dy_2 \\ & \leq c_1 \|\psi\|_{H^1(Q)}^2 \|(Fg_t)(\cdot, \alpha)\|_{L^2(Q_+^{h_0})}^2 \leq c_2 e^{-2\delta t} \|\psi\|_{H_{per}^1(Q)}^2 \end{aligned}$$

for all $\alpha \in [-1/2, 1/2]$ by (31). The remaining finite sum is estimated as

$$\begin{aligned} & \sum_{|\ell| \leq k+1} |\psi_\ell(h_0)| \int_{h_0}^{\infty} |(Fg_t)_\ell(y_2, \alpha)| |e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}| dy_2 \\ & \leq \sum_{|\ell| \leq k+1} |\psi_\ell(h_0)| \int_{h_0}^{\infty} |(Fg_t)_\ell(y_2, \alpha)| dy_2 \leq \frac{\sqrt{2k+3}}{\sqrt{2\pi}} \|\psi\|_{H^1(Q)} \int_0^{2\pi} \int_{h_0}^{\infty} |(Fg_t)(y, \alpha)| dy_2 dy_1 \\ & \leq c \|\psi\|_{H^1(Q)} e^{-\delta t} \text{ for all } \alpha \in [-1/2, 1/2] \end{aligned}$$

where we used (31) again. The restrictions of these estimates to $\alpha \in I$ and the uniform boundedness of \tilde{L}_α^{-1} yield the existence of $c > 0$ with

$$\|u_{\alpha,t}^{(1)}\|_{H^1(Q)} \leq c \left(\sum_{\ell \notin \mathcal{L}_m} e^{-t\sqrt{(\ell+\alpha)^2-k^2}|\hat{\theta}_2|} \right)^{1/2} + ce^{-\delta t} \quad (38)$$

for all $\alpha \in I$ and $t|\hat{\theta}_2| \geq 2h_0$

Second case: Let $I \subset I_m$ contain no cut-off value (that is, $I \subset \text{int}I_m$; that is, $|\ell + \alpha| \neq k$ for all $\ell \in \mathbb{Z}$ and $\alpha \in I$). In this case we wish to apply Theorem 2.7 (in the modification of Remark 2.8) to the equation $\tilde{L}_\alpha \tilde{u}_{\alpha,t}^{(1)} = \tilde{r}_{\alpha,t}^{(1)} + \tilde{r}_{\alpha,t}^{(2)} + \tilde{r}_{\alpha,t}^{(3)}$ in the space $H_{per}^1(Q)$ of periodic functions. We have to show that $\tilde{r}_{\alpha,t}^{(j)}$ is differentiable with respect to α and have to bound

the derivative. First we note that in this case of I there exists $c_0 > 0$ with $\sqrt{(\ell + \alpha)^2 - k^2} \geq c_0(|\ell| + 1)$ for all $\alpha \in I$ and $\ell \notin \mathcal{L}_m$. We begin with $\tilde{r}_{\alpha,t}^{(1)}$. For $\alpha \in I$ and $\ell \notin \mathcal{L}_m$, we have

$$\left| \frac{\partial}{\partial \alpha} e^{it(\ell + \alpha)\hat{\theta}_1 - \sqrt{(\ell + \alpha)^2 - k^2}(t|\hat{\theta}_2| - h_0)} \right| = \left| it\hat{\theta}_1 - \frac{\ell + \alpha}{\sqrt{(\ell + \alpha)^2 - k^2}}(t|\hat{\theta}_2| - h_0) \right| e^{-\sqrt{(\ell + \alpha)^2 - k^2}(t|\hat{\theta}_2| - h_0)} \leq ct e^{-c_0(|\ell| + 1)(t|\hat{\theta}_2| - h_0)}.$$

This yields

$$\sum_{\ell \notin \mathcal{L}_m} \left| \frac{\partial}{\partial \alpha} e^{it(\ell + \alpha)\hat{\theta}_1 - \sqrt{(\ell + \alpha)^2 - k^2}(t|\hat{\theta}_2| - h_0)} \right| |\psi_\ell(h_0)| \leq ct \|\psi\|_{H_{per}^1(Q)} \left(\sum_{\ell \notin \mathcal{L}_m} e^{-2c_0(|\ell| + 1)(t|\hat{\theta}_2| - h_0)} \right)^{1/2} \leq ct e^{-c_0(t|\hat{\theta}_2| - h_0)} \|\psi\|_{H_{per}^1(Q)};$$

that is, $\|\partial \tilde{r}_{\alpha,t}^{(1)} / \partial \alpha\|_{H_{per}^1(Q)} \leq ct e^{-c_0(t|\hat{\theta}_2| - h_0)} \leq ct e^{-c_0 t |\hat{\theta}_2| / 2}$ for all $\alpha \in I$ and $t|\hat{\theta}_2| \geq 2h_0$.

The estimates of $\|\partial \tilde{r}_{\alpha,t}^{(j)} / \partial \alpha\|_{H_{per}^1(Q)}$ for $j = 2, 3$ follow the same arguments as for $\|r_{\alpha,t}^{(j)}\|_{H^1(Q)}$ using in addition that $|\ell + \alpha| / |\sqrt{k^2 - (\ell + \alpha)^2}|$ is uniformly bounded with respect to $\ell \in \mathbb{Z}$ and $\alpha \in I$.

Therefore, application of Remark 2.8 yields an estimate of the form (38) where the second term is replaced by $c_2 t e^{-c_3 t}$ for some $c_2, c_3 > 0$. Since we can decompose I_m as a finite union of closed intervals I of the first or second type with non-intersecting interiors,³ we have an estimate of the form

$$\|u_{\alpha,t}^{(1)}\|_{H^1(Q)} \leq c_1 \left(\sum_{\ell \notin \mathcal{L}_m} e^{-t\sqrt{(\ell + \alpha)^2 - k^2} |\hat{\theta}_2|} \right)^{1/2} + c_2 t e^{-c_3 t}$$

for all $\alpha \in I_m$ and $t|\hat{\theta}_2| \geq 2h_0$. Therefore, by the inequality of Cauchy-Schwarz,

$$\int_{I_m} \|u_{\alpha,t}^{(1)}\|_{H^1(Q)} d\alpha \leq c \left(\sum_{\ell \notin \mathcal{L}_m} \int_{I_m} e^{-t\sqrt{(\ell + \alpha)^2 - k^2} |\hat{\theta}_2|} d\alpha \right)^{1/2} + c_2 t e^{-c_3 t}.$$

For large values of $|\ell|$, say $|\ell| \geq k + 1$, we use the estimate $\sqrt{(\ell + \alpha)^2 - k^2} \geq c_0|\ell|$ which yields that the series over $|\ell| \geq k + 1$ decays exponentially to zero as t tends to infinity. For fixed $\ell \notin \mathcal{L}_m$ with $|\ell| \leq k + 1$, we make the substitution $\beta = \psi(\alpha) = \sqrt{(\ell + \alpha)^2 - k^2}$. Then

$$\int_{I_m} e^{-t\sqrt{(\ell + \alpha)^2 - k^2} |\hat{\theta}_2|} d\alpha = \int_{\psi(I_m)} e^{-t\beta|\hat{\theta}_2|} \frac{\beta}{\sqrt{\beta^2 + k^2}} d\beta \leq \frac{1}{k} \int_{\psi(I_m)} \beta e^{-t\beta|\hat{\theta}_2|} d\beta$$

which tends to zero as $1/t^2$. Indeed, if $\psi(I_m) = [a, b]$ with $b > a \geq 0$, then this follows from

$$\int_a^b \beta e^{-s\beta} d\beta = \frac{1}{s} (a e^{-sa} - b e^{-sb}) - \frac{1}{s^2} (e^{-sb} - e^{-sa}).$$

This ends the proof. □

³Note that the cut-off values are non-propagative wave numbers by assumption.

We go back to the decomposition (33) of $u_{\alpha,t}$ for $\alpha \in I_m$ and consider the integrals (for $\ell \in \mathcal{L}_m$)

$$\frac{i}{4\pi} \int_{I_m} e^{it[(\ell+\alpha)\hat{\theta}_1 + \sqrt{k^2 - (\ell+\alpha)^2}|\hat{\theta}_2|]} \frac{1}{\sqrt{k^2 - (\ell + \alpha)^2}} v_{\ell,\alpha} d\alpha$$

in $H^1(Q_R)$ (for some fixed $R > 0$) with the method of stationary phase. We recall from Lemmas 5.4 and 5.5 that $v_{\ell,\alpha}$ is the total α -quasi-periodic field corresponding to the incident plane wave $e^{i(\ell+\alpha)x_1 - i\sqrt{k^2 - (\ell+\alpha)^2}x_2}$, which is analytic with respect to α in the interior of I_m and is also in $W^{1,1}(I_m, H^1(Q_R))$.

We define $\psi(s) = s\hat{\theta}_1 + \sqrt{k^2 - s^2}|\hat{\theta}_2|$ for $|s| \leq k$. Then it easily seen that $\tilde{s} = k\hat{\theta}_1$ is the only critical point (that is, $\psi'(\tilde{s}) = 0$) and $\psi(\tilde{s}) = k$ and $\psi''(\tilde{s}) = -\frac{1}{k\hat{\theta}_2^2} < 0$. There is exactly one $\tilde{\ell} \in \mathbb{Z}$ and $\tilde{\alpha} \in (-1/2, 1/2]$ with $\tilde{s} = k\hat{\theta}_1 = \tilde{\ell} + \tilde{\alpha}$. We note that $\tilde{\alpha} \neq \pm\kappa$; that is, $\tilde{\alpha}$ is not a cut-off value by assumption on $k\hat{\theta}_1$.

Then there exists exactly one interval $I_{\tilde{m}}$ such that $\tilde{\alpha}$ is in the interior of $I_{\tilde{m}}$ and $\tilde{\ell} \in \mathcal{L}_{\tilde{m}}$.

Since $v_{\tilde{\ell},\alpha}$ is smooth in $I_{\tilde{m}}$, the method of stationary phase is applicable to the integral over $I_{\tilde{m}}$, which gives

$$\begin{aligned} & \frac{i}{4\pi} \int_{I_{\tilde{m}}} e^{it[(\tilde{\ell}+\alpha)\hat{\theta}_1 + \sqrt{k^2 - (\tilde{\ell}+\alpha)^2}|\hat{\theta}_2|]} \frac{1}{\sqrt{k^2 - (\tilde{\ell} + \alpha)^2}} v_{\tilde{\ell},\alpha} d\alpha \\ &= \frac{i}{4\pi \sqrt{k^2 - (\tilde{\ell} + \tilde{\alpha})^2}} e^{itk - i\pi/4} \sqrt{\frac{2\pi k \hat{\theta}_2^2}{t}} v_{\tilde{\ell},\tilde{\alpha}} + o(1/\sqrt{t}) \\ &= \gamma \frac{e^{ikt}}{\sqrt{t}} v_{\tilde{\ell},\tilde{\alpha}} + o(1/\sqrt{t}) \end{aligned}$$

as $t \rightarrow \infty$. For $\ell \in \mathcal{L}_{\tilde{m}} \setminus \{\tilde{\ell}\}$ the function $\alpha \mapsto (\ell + \alpha)\hat{\theta}_1 + \sqrt{k^2 - (\ell + \alpha)^2}|\hat{\theta}_2|$ is monotonous. Substituting $\beta = (\ell + \alpha)\hat{\theta}_1 + \sqrt{k^2 - (\ell + \alpha)^2}|\hat{\theta}_2|$ and using partial integration yields that these integrals decay as $\mathcal{O}(1/t)$. Therefore, by (33) and Lemma 5.6,

$$\int_{I_m} u_{\alpha,t} d\alpha = \gamma \frac{e^{ikt}}{\sqrt{t}} v_{\tilde{\ell},\tilde{\alpha}} + o(1/\sqrt{t})$$

as $t \rightarrow \infty$ in $H^1(Q_R)$. For the intervals I_m with $m \neq \tilde{m}$ and $\ell \in \mathcal{L}_m$, partial integration yields again that these integrals decay as $\mathcal{O}(1/t)$. Therefore, the integration can be done over all of $[-1/2, 1/2]$, and the inverse Floquet-Bloch transform gives

$$u_t = \int_{-1/2}^{1/2} u_{\alpha,t} d\alpha = \gamma \frac{e^{ikt}}{\sqrt{t}} v_{\tilde{\ell},\tilde{\alpha}} + o(1/\sqrt{t})$$

in $H^1(Q_R)$. From Lemma 5.4, we observe that $v_{\tilde{\ell},\tilde{\alpha}}$ is the solution of the $\tilde{\alpha}$ -quasi-periodic scattering problem for the incident plane wave $u^{inc}(x) = e^{i(\tilde{\ell}+\tilde{\alpha})x_1 - i\sqrt{k^2 - (\tilde{\ell}+\tilde{\alpha})^2}x_2} = e^{ik\hat{\theta} \cdot x}$; that is, $v_{\tilde{\ell},\tilde{\alpha}} = v_{\theta}$ with the field v_{θ} from Theorem 5.2. If $k\hat{\theta}_1$ is a propagative wave number $\tilde{\alpha}_j + \ell$ for some $\ell \in \mathbb{Z}$, then $\int_{Q^\infty} \frac{\partial v_{\theta}}{\partial x_1} \bar{\phi} dx = 0$ for all corresponding modes $\phi \in X_j$ by Lemma 5.5. Finally we note that the propagating part $u_{t,prop}$ tends to zero exponentially by (30) and $u_t = \tilde{u}_t^*$ on Q . This ends the proof of Theorem 5.2.

6 | THE CASE OF A LOCALLY PERTURBED PERIODIC INDEX

Now we consider the more general problem that the periodic refractive index n is perturbed by some function $q \in L^\infty(\mathbb{R}^2)$ with support in Q . The following result on uniqueness and existence has been shown in [22].

Theorem 6.1. *Let Assumptions 2.2 and 2.4 hold and, in the case $q \neq 0$, the additional assumption that no bound states exist; that is, no non-trivial $w \in H^1(\mathbb{R}^2)$ with $\Delta w + k^2(n + q)w = 0$ in \mathbb{R}^2 exist; that is, k^2 is not in the point spectrum of*

$-\frac{1}{n+q}\Delta$. Then for all $f \in L^2(Q)$ there exists a unique solution $u \in H_{loc}^1(\mathbb{R}^2)$ of $\Delta u + k^2(n+q)u = -f$ in \mathbb{R}^2 which satisfies the open waveguide radiation condition of Definition 5.1. Furthermore, the solution operator $f \mapsto u|_Q$ is bounded from $L^2(Q)$ into $H^1(Q)$.

It is the aim to prove the following extension of Theorem 5.2.

Theorem 6.2. *Let Assumptions 2.2 and 2.4 hold and let $\hat{\theta} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \in \mathbb{R}^2$ be a fixed unit vector with $|\theta| < \frac{\pi}{2}$; that is, $\hat{\theta}_2 < 0$. In addition, let $\tilde{\alpha} := k\hat{\theta}_1 = k \sin \theta$ not be a cut-off value in the sense of Definition 2.1 and assume that there exist no bound states. Let $w_t = \Phi(\cdot, -t\hat{\theta}) + w_t^s$ be the unique solution of the scattering problem*

$$\Delta w_t + k^2(n+q)w_t = 0 \text{ in } \mathbb{R}^2 \setminus \{-t\hat{\theta}\}, \quad (39)$$

of the point source at $z = -t\hat{\theta}$ for $t|\hat{\theta}_2| > 2h_0$ such that $w_t^s \in H_{loc}^2(\mathbb{R}^2)$ and w_t satisfies the open waveguide radiation condition of Definition 5.1. Then

$$\frac{1}{\gamma} \lim_{t \rightarrow \infty} \left[\sqrt{t} e^{-ikt} w_t \right] = v_\theta + w \text{ in } H^1(Q_R) \quad (40)$$

for any $R > 0$ where again $Q_R := (-R, R) \times (-h_0, h_0)$. Here, $v_\theta \in H_{\tilde{\alpha}, loc}^1(\mathbb{R}^2)$ is exactly the limit function as in Theorem 5.2; that is, v_θ solves the $\tilde{\alpha}$ -quasi-periodic scattering problem $\Delta v_\theta + k^2 n v_\theta = 0$ in \mathbb{R}^2 for such that the scattered field $v_\theta^s(x) := v_\theta(x) - e^{ik\hat{\theta} \cdot x}$ satisfies the Rayleigh expansion (5) for $|x_2| > h_0$. If $k\hat{\theta}_1 = k \sin \theta$ is a propagative wave number then v_θ satisfies in addition the orthogonality condition (37).

The function $w \in H_{loc}^1(\mathbb{R}^2)$ solves the source problem $\Delta w + k^2(n+q)w = -k^2 q v_\theta$ in \mathbb{R}^2 , and satisfies the open waveguide radiation condition of Definition 5.1.

Proof. We define $u_t = u_t^s + \Phi(\cdot, -t\hat{\theta})$ as in Theorem 5.2 to be the unique solution of the unperturbed scattering problem $\Delta u_t + k^2 n u_t = 0$ in $\mathbb{R}^2 \setminus \{-t\hat{\theta}\}$ for the point source incidence at $z = -t\hat{\theta}$ such that $u_t^s \in H_{loc}^1(\mathbb{R}^2)$ and u_t satisfies the open waveguide radiation condition of Definition 5.1. Then Theorem 5.2 implies that $\frac{1}{\gamma} e^{-ikt} \sqrt{t} u_t$ converges in $H^1(Q)$ to the solution v_θ of the $\tilde{\alpha}$ -quasi-periodic scattering problem for the plane wave of incidence $\hat{\theta}$. In the case that $k\hat{\theta}_1$ is a propagative wave number v_θ satisfies in addition the orthogonality condition (37). Then $\tilde{w}_t = w_t - u_t$ satisfies $\Delta \tilde{w}_t + k^2(n+q)\tilde{w}_t = -k^2 q u_t$ in \mathbb{R}^2 and the open waveguide radiation condition of Definition 5.1. The convergence of $\frac{1}{\gamma} e^{-ikt} \sqrt{t} u_t$ to v_θ in $H^1(Q)$ yields convergence of $\frac{1}{\gamma} e^{-ikt} \sqrt{t} \tilde{w}_t$ to w in $H^1(Q)$ because of the continuous dependence of the solution on the right hand side. This ends the proof. \square

7 | CONCLUDING REMARKS

In this paper, we have, first, justified the assumption that in a purely periodic structure the scattered field has the same quasi-periodicity property as the incident field. Second, in the case where no uniqueness holds, we have compared three additional properties to achieve uniqueness. While continuity with respect to the incident angle or the approximation of the incident plane wave by sources where the source point tends to infinity leads to the same additional property, the limiting absorption principle with respect to the refractive index leads to a different one which depends on the kind of absorption and is therefore not appropriate. Third, in the case where the periodic refractive index is locally perturbed, we have shown that the solution can be represented as the sum of the solution to the unperturbed problem and a correction term which satisfies the open waveguide radiation condition.

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CONFLICT OF INTEREST STATEMENT

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REFERENCES

1. R. Petit Ed., *Electromagnetic theory of gratings*. Springer, 1980.
2. G. Hu, W. Lu, and A. Rathsfeld, *Time-harmonic acoustic scattering from locally perturbed periodic curves*, SIAM J. Appl. Math. **81** (2021), no. 6, 2569–2595.
3. W. Lu and Y. Y. Lu, *High order integral equation method for diffraction gratings*, J. Opt. Soc. Am. **29** (2012), no. 5, 734–740.
4. A. Meier, T. Arens, S. N. Chandler-Wilde, and A. Kirsch, *A Nyström method for a class of integral equations on the real line with applications to scattering by diffraction gratings and rough surfaces*, J. Int. Equ. Appl. **12** (2000), 281–321.
5. J.-C. Nédélec and F. Starling, *Integral equation methods in a quasi-periodic diffraction problem for the time-harmonic Maxwell's equations*, SIAM J. Math. Anal. **22** (1991), no. 6, 1679–1701.
6. A. Rathsfeld, G. Schmidt, and B. H. Kleemann, *On a fast integral equation method for diffraction gratings*, Commun. Comput. Phys. **1** (2006), 984–1009.
7. Y. Wu and Y. Y. Lu, *Analyzing diffraction gratings by a boundary integral equation Neumann-t-Dirichlet map*, J. Opt. Soc. Am. **26** (2009), no. 11, 2444–2451.
8. S. N. Chandler-Wilde and P. Monk, *Existence, uniqueness, and variational methods for scattering by unbounded rough surfaces*, SIAM J. Math. Anal. **37** (2005), 598–618.
9. S. N. Chandler-Wilde and C. R. Ross, *Scattering by rough surfaces: The Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane*, Math. Meth. Appl. Sci. **19** (1996), 959–976.
10. S. N. Chandler-Wilde and B. Zhang, *A uniqueness result for scattering by infinite dimensional rough surfaces*, SIAM J. Appl. Math. **58** (1998), 1774–1790.
11. P. Li, H. Wu, and W. Zheng, *Electromagnetic scattering by unbounded rough surfaces*, SIAM J. Math. Anal. **43** (2011), no. 3, 1205–1231.
12. S. N. Chandler-Wilde, P. Monk, and M. Thomas, *The mathematics of scattering by unbounded, rough, inhomogeneous layers*, J. Comput. Appl. Math. **204** (2007), 549–559.
13. S. N. Chandler-Wilde and B. Zhang, *Electromagnetic scattering by an inhomogeneous conducting or dielectric layer on a perfectly conducting plate*, Proc. R. Soc. Lond. A **454** (1998), 519–542.
14. Z. Chen and H. Wu, *An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures*, SIAM J. Numer. Anal. **41** (2003), 799–826.
15. B. Zhang and S. N. Chandler-Wilde, *Acoustic scattering by an inhomogeneous layer on a rigid plate*, SIAM J. Appl. Math. **58** (1998), no. 6, 1931–1950.
16. G. Bao, D. C. Dobson, and J. A. Cox, *Mathematical studies in rigorous grating theory*, J. Opt. Soc. Am. A **12** (1995), no. 5, 1029–1042.
17. A.-S. Bonnet-Bendhia and F. Starling, *Guided waves by electromagnetic gratings and non-uniqueness examples for the diffraction problem*, Math. Methods Appl. Sci. **17** (1994), 305–338.
18. J. Elschner and G. Schmidt, *Diffraction of periodic structures and optimal design problems of binary gratings. Part I: Direct problems and gradient formulas*, Math. Meth. Appl. Sci. **21** (1998), 1297–1342.
19. D. L. Colton and R. Kress, *Integral equation methods in scattering theory*, 1992. Krieger, repr. ed., with corr. edition.
20. A. Kirsch and A. Lechleiter, *The limiting absorption principle and a radiation condition for the scattering by a periodic layer*, SIAM J. Math. Anal. **50** (2018), no. 3, 2536–2565.
21. A. Kirsch, *The trouble with the limiting absorption principle for periodic waveguides*, 2022.
22. A. Kirsch, *A scattering problem for a local perturbation of an open periodic waveguide*, Math. Meth. Appl. Sciences **45** (2022), no. 10, 5737–5773.
23. R. Kress, *Linear integral equations*, 3rd ed., Springer, 2014.
24. J. A. Desanto and P. A. Martin, *On angular-spectrum representations for scattering by infinite rough surfaces*, Wave Motion **24** (1996), 421–433.
25. S. Fliss and P. Joly, *Solutions of the time-harmonic wave equation in periodic waveguides Asymptotic behaviour and radiation condition*, Arch. Ration. Mech. Anal. **219** (2016), 349–386.
26. P. Kuchment, *Floquet Theory for Partial Differential Equations*, volume 60 of Operator Theory, Advances and Applications, Basel, Birkhäuser, 1993.
27. A. Lechleiter, *The floquet-Bloch transform and scattering from locally perturbed periodic surfaces*, J. Math. Anal. Appl. **446** (2017), no. 1, 605–627.
28. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 7th ed., Elsevier, Amsterdam, Boston, etc, 2007.

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