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The Karlsruhe physics course

Lecture notes

Mechanics

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Lecture notes

- **Mechanics**
- Thermodynamics
- Electromagnetism
- Optics

Der Karlsruher Physikkurs

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1

Substance-like quantities

There is a class of physical quantities which are particularly easy to handle: the substance-like quantities. These include:

- mass m
- energy E
- electric charge Q
- momentum $\vec{p}$
- entropy S
- amount of substance n

and others.

One may imagine each of these quantities as a kind of substance, and one can speak about them as one speaks about a substance. The physical reason for this is that for each such quantity a density (mass density, energy density, charge density ...) and a current (mass flow, energy flow, electric current ...) can be defined.

This fact leads to further properties of the substance-like quantities: They add up when combining two systems into one, Fig. 1.1. If the quantity X has the value X_1 in system S_1 and the value X_2 in system S_2, it has the value $X_1 + X_2$ in the composed system S. This rule does not apply to non-substance-like quantities, such as temperature, pressure or velocity.

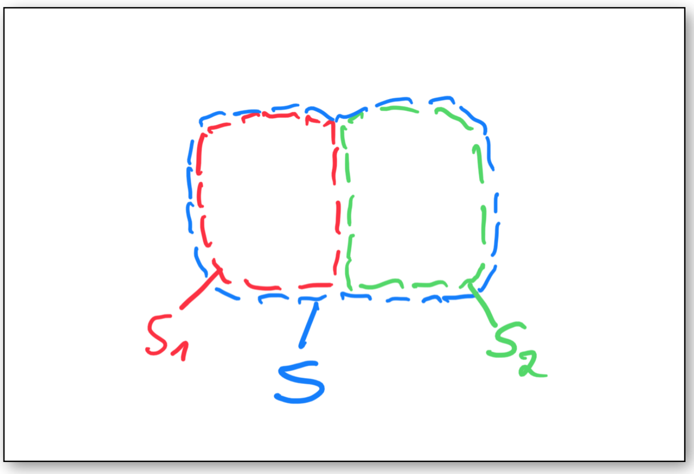

Fig. 1.1
Regarding the additivity of substance-like quantities

The question of whether or not a quantity is conserved is a meaningful question only for substance-like quantities. Energy and electric charge are conserved, entropy and amount of substance are not, because one can create entropy, and can create as well as destroy amount of substance. The question about the conservation of non-substance-like quantities, is meaningless, such as the question: “Is pressure a conserved quantity?”

For historical reasons, the strengths of some currents have their own names: The energy current strength is usually called power, and the momentum current strength is almost exclusively called force. Table 1.1 lists the most important substance-like quantities together with the corresponding currents.

Substance-like quantity		Current strength	
Name	Symbol (unit)	Name	Symbol (unit)
Mass	m (Kilogram, kg)	Mass current	– , (kg/s)
Energy	E (Joule, J)	Power	P (Watt, W = J/s)
Electric charge	Q (Coulomb, C)	Electric current	I (Ampere, A = C/s)
Momentum	$\vec{p}$ (Huygens, Hy)	Force	$\vec{F}$ (Newton, N = Hy/s)
Entropy	S (Carnot, Ct)	Entropy current	– , (Ct/s)
Amount of substance	n (Mol, mol)	Substance current	– , (mol/s)

Table 1.1
Some substance-like quantities and their currents

There are analogies existing between some areas of physics: From a relationship that is valid in one area of physics, one obtains a relationship that is valid in another by purely formal translation. In these analogies, substance-like quantities correspond to each other. In the following text we will often refer to the analogy between mechanics and electricity. In this case, momentum and electric charge, as well as force (= momentum current) and electric current, correspond to each other.

2

Momentum and momentum capacitance

2.1 Definition of momentum

A rolling carriage has momentum. The faster it rolls, and the heavier it is, the more momentum it has. The meaning of what is colloquially called momentum is very much in line with the meaning of the substance-like physical quantity momentum.

Huygens called the momentum “quantitas motus”, meaning quantity or amount of movement. A moving body contains a certain amount of momentum, just as an electrically charged body contains a certain amount of electricity.

At first we concentrate on the analysis of one-dimensional, rectilinear movements in the *x*-direction, and we define:

If a body moves in the positive *x*-direction, its momentum is positive. If the body moves in the negative *x*-direction, its momentum is negative. If the body is at rest, its momentum is zero.

Fig. 2.1 shows how the momentum of a body B can be measured. Unit bodies U i.e. bodies each of which carries one (negative) unit of momentum, are allowed to collide with B in such a way that they remain connected to B after the collision (“inelastic collision”). Momentum is thereby transferred from B to the unit bodies. One now lets unit bodies collide with B until B and all unit bodies already attached to it have come to rest. If *z* unit bodies are required for this, we know that B had *z* units of momentum at the beginning.

This measuring method assumes that no momentum is lost in the collision and no new momentum is generated. That this is the case can easily be proven in further experiments.

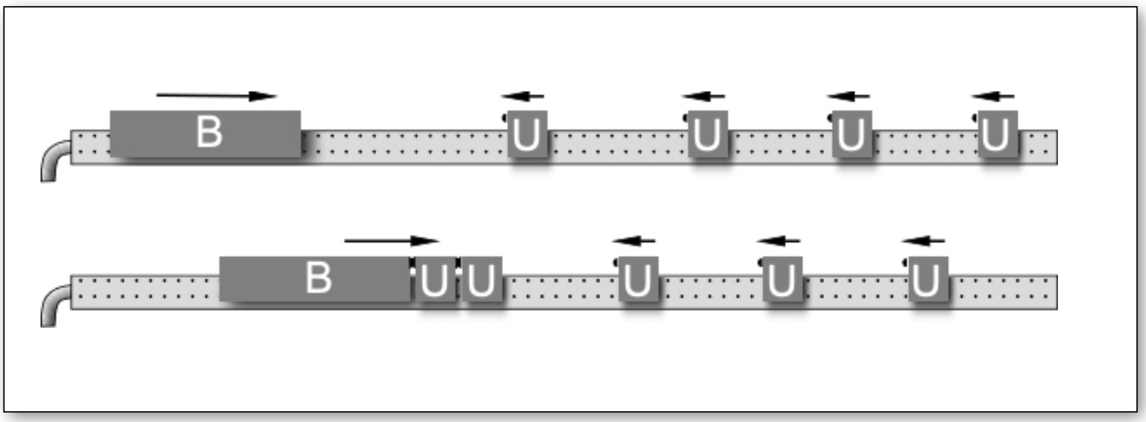

Fig. 2.1
Measuring momentum. B = Body whose momentum is to be measured. U = Body with one unit of momentum.

2.2 Momentum capacitance

We now ask what the momentum of a body depends on and ascertain:

The momentum of a body depends on

- the velocity of the body;
- the mass of the body.

It does not depend on e.g.

- the chemical composition of the body;
- the geometrical shape of the body.

The quantitative investigation shows that for not too high velocities ($v \ll c$) the following applies:

$$p \propto m \cdot v$$

The unit of measurement *Huygens* of the momentum is chosen in such a way that the proportionality becomes an equation:

$$p = m \cdot v \qquad (2.1)$$

Of course, this relationship only applies to that class of systems for which it has been experimentally verified: for bodies of not too high velocity. For other systems, e.g. electromagnetic fields, other relationships apply.

Equation (2.1) can also be read as follows: At a given velocity a body contains the more momentum the greater its mass is. Thus the mass is a measure for the *momentum capacity* of a body.

Table 2.1 lists some typical momentum values.

The relation of electricity which is analogous to equation (2.1) is

$$Q = C \cdot U$$

It tells us, that for a given voltage the plates of a capacitor carry the more electric charge Q, the higher the capacity C of the capacitor is.

It is experimentally established that momentum can neither be created nor destroyed:

Momentum is a conserved quantity.

Flying tennis ball	2 Hy
Flying soccer ball	12 Hy
Pedestrian	100 Hy
Moving passenger car	40 000 Hy
Earth (on its orbit around the sun)	$1.8 \cdot 10^{28}$ Hy
Photon of visible light	10^{-27} Hy

Table 2.1
Some typical momentum values

3

Force

3.1 Momentum currents

When electric charge is transferred from a place A to a place B, it is said that an electric current is flowing from A to B. In the same way, when momentum is transferred from a body A to a body B, one can say that a momentum current flows from A to B, Fig. 3.1 and Fig. 3.2.

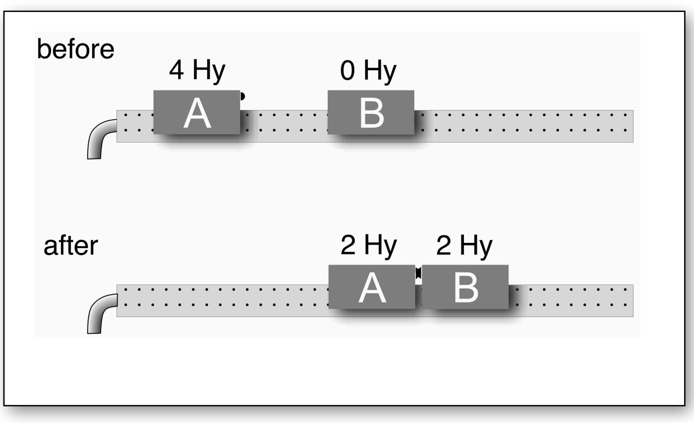

Fig. 3.1
Momentum is flowing from A to B. The momentum from body A distributes to both bodies A and B.

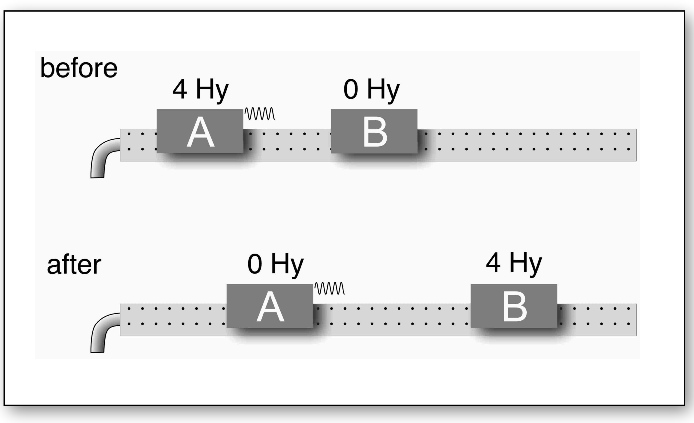

Fig. 3.2
Momentum is flowing from A to B. All of the momentum goes from A to B.

If the momentum of a body is allowed to flow into the earth, it is distributed in the earth; it is "diluted" so much that it can no longer be detected, Fig. 3.3. The analogous electrical situation is shown in Fig. 3.4.

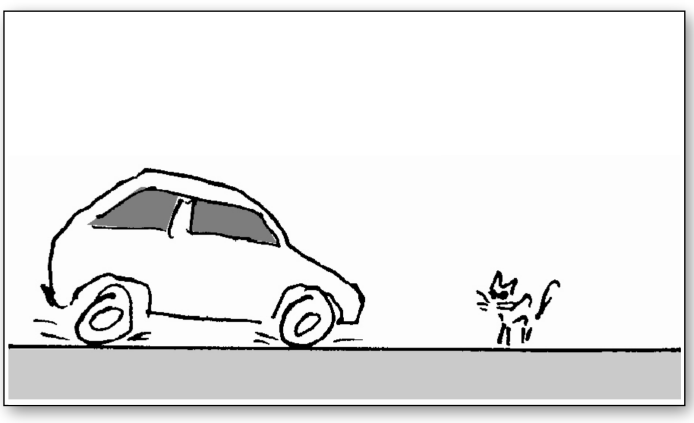

Fig. 3.3
The momentum flows from the car into the earth, where it dilutes so much that it can no longer be detected.

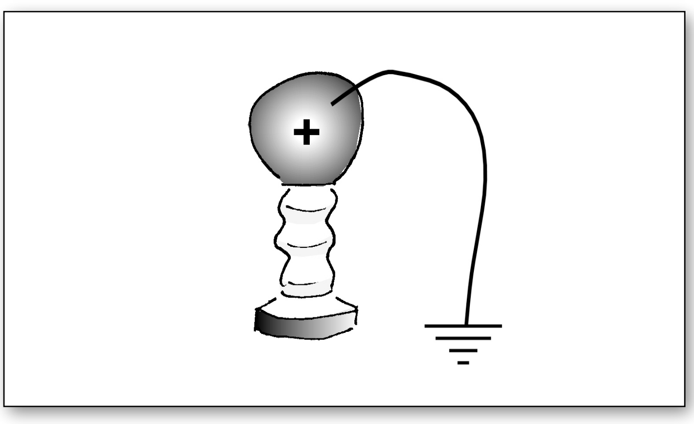

Fig. 3.4
The electric charge flows from the sphere into the earth, where it is diluted so much that it can no longer be detected.

If there is a connection, the momentum flows into the earth by itself. To make the momentum flow against its natural direction, a "momentum pump" is needed. In fig. 3.5 the person acts as a momentum pump. Fig. 3.6 shows the analog electrical situation.

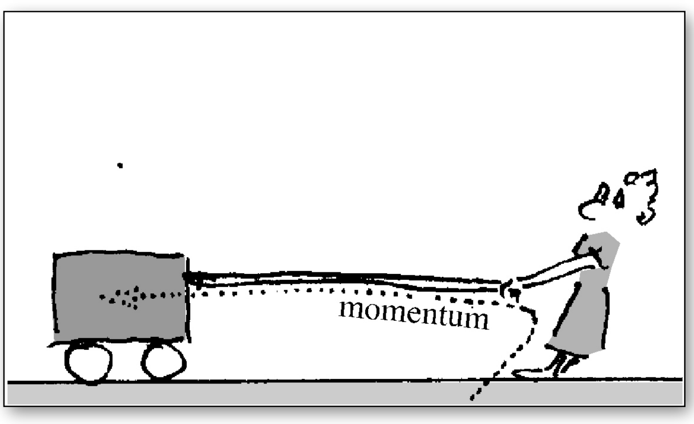

Fig. 3.5
The momentum of the carriage increases. The person "pumps" it out of the ground over the rope into the cart.

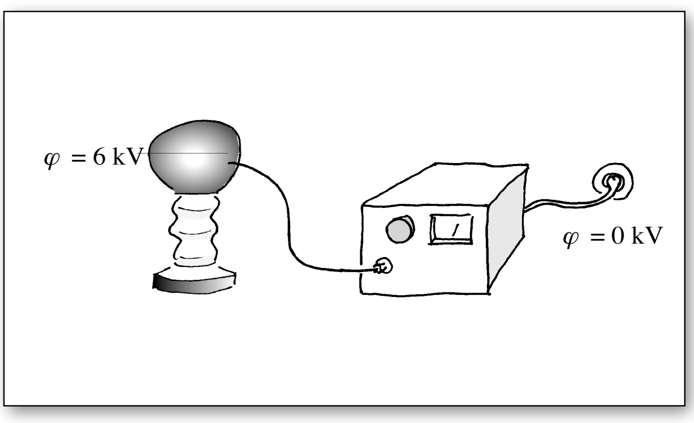

Fig. 3.6
The electric charge of the sphere increases. The power supply "pumps" it out of the ground via the cable onto the sphere.

It is easy to ascertain whether an object or other structure conducts the momentum well or poorly, Fig. 3.7:

- Solid objects conduct momentum currents;
- gases conduct momentum currents poorly;
- ropes conduct momentum currents in one direction only;
- wheels are often used for momentum insulation.

If two objects friction against each other, a momentum current flows between them. The lower the friction, the better the momentum insulation.

Non-material physical systems, the so-called fields, also conduct momentum. Fig. 3.8 shows how momentum flows through a magnetic field.

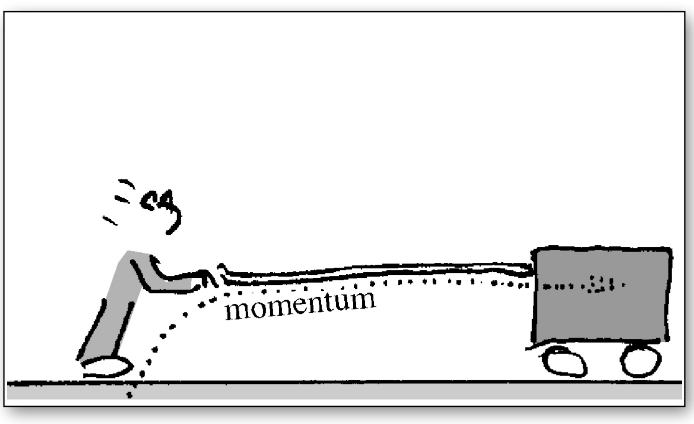

Fig. 3.7
The momentum of the carriage increases. The person "pumps" it out of the ground over the rod into the cart.

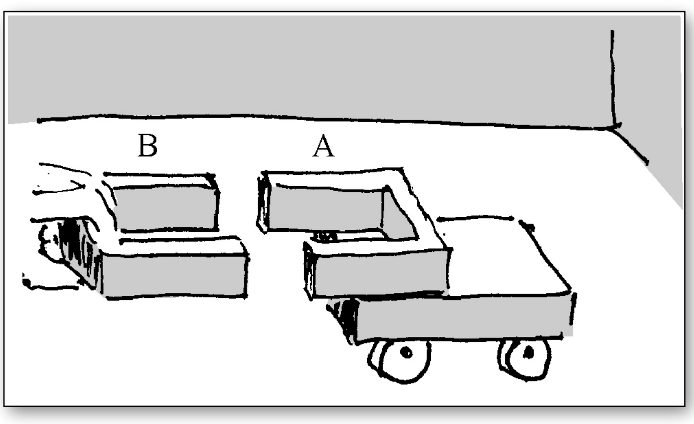

Fig. 3.8
The cart gets its momentum via the magnetic field between the two magnets.

3.2 Current direction

If an object (or other entity) is under tensile stress, a momentum current flows in it in the negative x-direction, Fig. 3.9. If an object is under compressive stress, momentum flows in the positive x-direction, Fig. 3.10. These statements are based on a convention: If the momentum of a body is counted positive when the body moves in the negative x-direction, the directions of the currents are reversed as well. (In electricity, too, one has made an arbitrary choice of sign: Electrons or rubbed sealing wax sticks are defined as negatively charged).

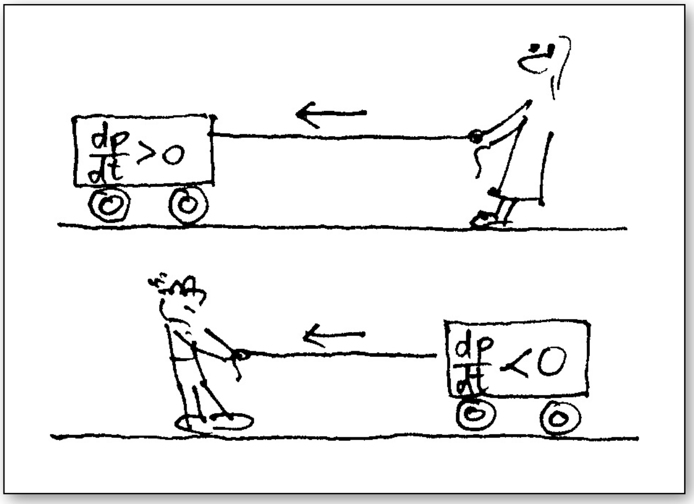

Fig. 3.9
Momentum flows in the negative x-direction.

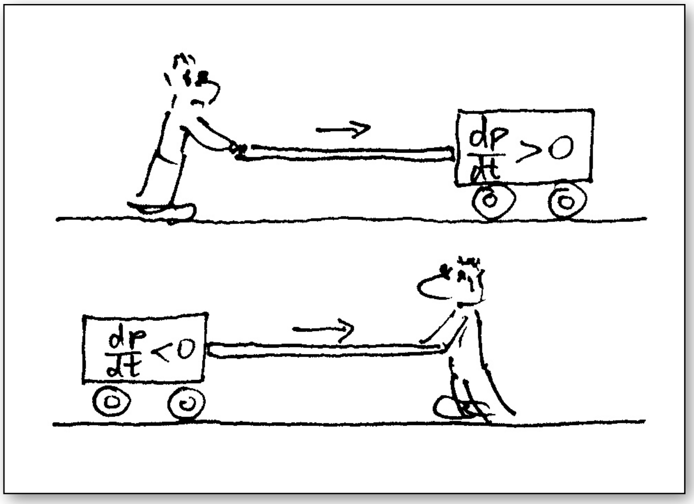

Fig. 3.10
Momentum flows in the positive x-direction.

3.3 Momentum current intensity

An object that a momentum current passes through is deformed. If the object resumes its old shape after the momentum flow has stopped, it is called *elastic*. Elastic bodies can be used to measure the momentum current strength or intensity. Such current meters are called dynamometers. They are handled in the same way as other current meters, e.g. ammeters, Fig. 3.11:

- The conductor in which flows the current to be measured is cut;
- the two new ends are connected to the two terminals of the ammeter or dynamometer, respectively.

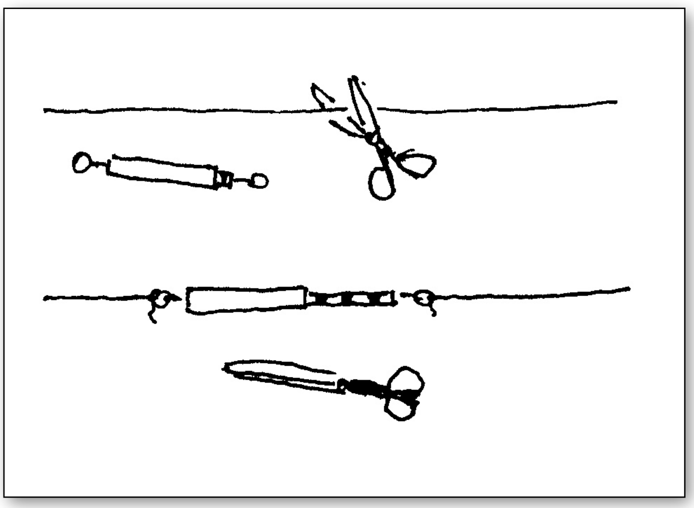

Fig. 3.11
Measuring the momentum current intensity

The unit of measurement of the momentum current strength is the Newton (N). We have

$$1\,\mathrm{N} = \frac{1\,\mathrm{Hy}}{\mathrm{s}}$$

In physics, the common name for momentum current strength is *force*. However, the verbal handling of the word force is somewhat different from that of the word current. In table 3.1 some translation rules are listed.

Momentum current	Force
A momentum current enters the body.	A force is acting on the body.
A momentum current flows from body A to body B.	Body A exerts a force on body B.
A momentum current flows through the rope.	Two forces that are equal and opposite are acting on the rope.

Table 3.1
The verbal handling of the words "force" and "momentum current"

3.4 Momentum current circuits

In figures 3.12 a to c the momentum change of body K is 10 Hy/s, because a momentum current of 10 N flows into the respective body. Due to the large mass of the body in Fig. 3.12 c, however, its velocity only changes very little. Nevertheless, a the momentum current flowing through the dynamometer into the body has the same intensity as that in Fig. 3.12 a. This shows that the flow of a momentum current has nothing to do with the movement of the momentum conductor.

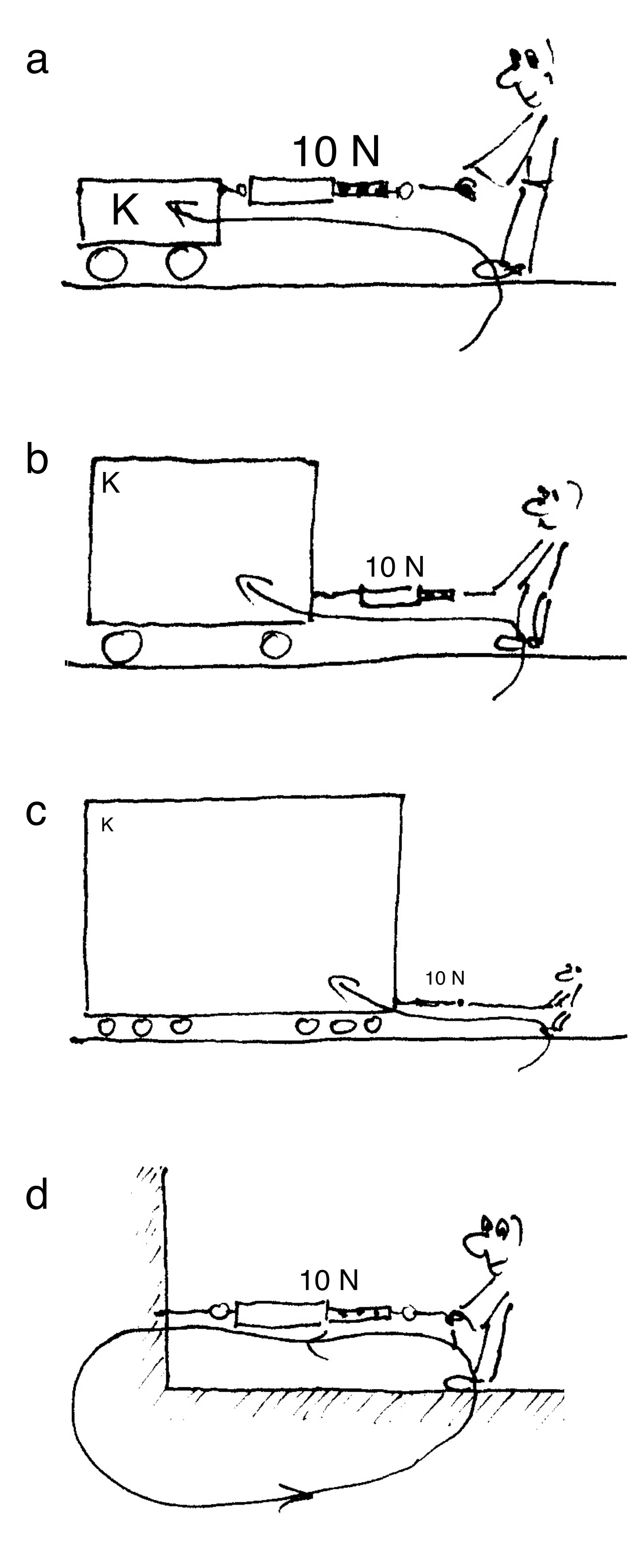

Fig. 3.12
The flow of a momentum current has nothing to do with the movement of the momentum conductor.

Finally, in fig. 3.12 d again a momentum current of 10 N is flowing. But this time there is no change in the momentum of any body. In this case the momentum flows in a closed "circuit". This situation is similar to the arrangement of Fig. 3.13: The upper spring is under tensile stress, i.e. the momentum current flows from right to left, the lower spring is under compressive stress, i.e. the momentum current flows from left to right.

Arrangements in which momentum does not accumulate anywhere are called static arrangements.

The momentum circuit of Fig. 3.13 illustrates a trivial experience: If there is compressive stress in any component of a static arrangement, there must be another component under tensile stress.

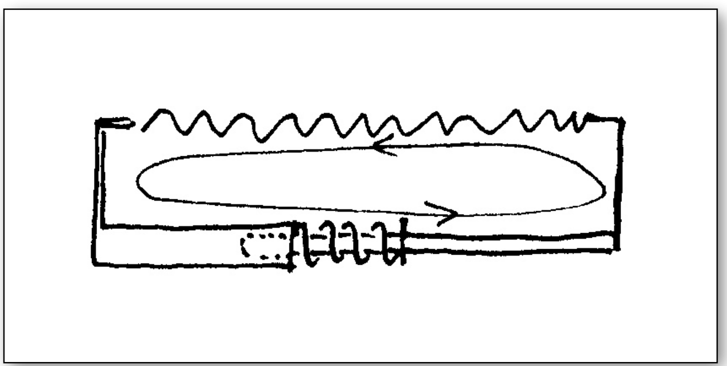

Fig. 3.13
Closed momentum circuit

It may seem strange to speak of currents in the context of static arrangements for two reasons:

1. Is it possible that a current flows continuously – without any drive, without any energy source? Apparently yes. After all, there are also electric "frictionless" currents, the super currents.

2. The arrangement of fig. 3.13 is right-left symmetrical. But why does the current distinguish one of the two directions? The answer is: The symmetry breaking is not caused by the current. We ourselves are responsible for this. Because we have defined that bodies moving to the right have positive momentum. So it is simply because our coordinate system is asymmetrical. *We* say: right and left are different, right is positive, left is negative. It is not nature who says it.

3.5 Momentum balance equation

Since momentum is a conserved quantity, its value within a region of space can only change if a momentum current flows into or out of the region. The flowing in and out can be realized in two different ways: Either, as in Fig. 3.14 a, by tension or pressure, or, as in Fig. 3.14 b, by moving momentum “convectively” into or out of the region.

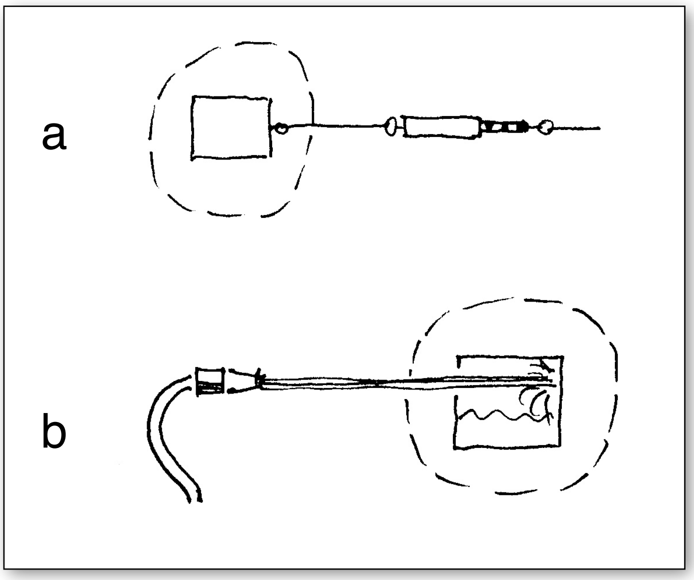

Fig. 3.14
Two kinds of momentum currents

Thus, the total momentum current I_p can have two contributions. Only the first one, the compression or tension term, is called force. The second one is the convective current F_{conv} . So we have

$$I_p = F + F_{conv}$$

The convective momentum current F_{conv}, for example in a water jet, can be expressed by the mass current I_m and the velocity v_j of the jet. For the change *dp* of the momentum in the region marked with a dashed line in Fig. 3.14 b the we get

$$dp = v_j \, dm.$$

and thus

$$\frac{dp_{conv}}{dt} = F_{conv} = v_j \frac{dm}{dt} = v_j \cdot I_m$$

The total intensity of the current can thus be written

$$I_p = F + v_j \, I_m$$

and for the momentum change in the region of space we can write

$$\frac{dp}{dt} = I_p \qquad (3.1)$$

Equation (3.1) is the *balance equation* for the momentum. Note that this relation is not a definition of the quantity I_p. It rather describes an experience: the experience that momentum is a conserved quantity. The quantities on the left and on the right side can be measured independently.

The balance equation for the electric charge is analogue to (3.1)

$$\frac{dQ}{dt} = I$$

just as the balance equation for the mass

$$\frac{dm}{dt} = I_m$$

4

Momentum and force as vector quantities

4.1 Momentum as a vector quantity

We now remove the restriction that bodies may only move in the positive or negative *x* direction. To specify the value of the momentum of a body of any direction of motion, we have to specify the amount of the momentum and the direction of motion, the "direction of the momentum". Thus, momentum is a vector quantity. Since momentum is substance-like, the additivity must be defined: The momentum is added according to the usual vector addition rule (parallelogram rule).

The momentums of the spatially separated systems A and B are $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$. Then the momentum of the system composed of A and B is

$$\vec{p} = \vec{p}_A + \vec{p}_B$$

(Velocity is not a substance-like quantity. The sum of two velocities corresponds to a change of the reference frame. Velocities can only be added as long as $|v| \ll c$).

It is often useful to decompose the momentum of a body into the *x*, *y* and *z* components:

$$\vec{p} = p_x\vec{e}_x + p_y\vec{e}_y + p_z\vec{e}_z$$

Momentum that points in the *x*-direction is called *x*-momentum for short. Accordingly there is also *y*- and *z*-momentum.

4.2 Force as a vector quantity

Electric charge is a scalar. Therefore also the electric current is a scalar. Momentum is a vector. Accordingly also the force, i.e. the momentum current intensity, must be a vector.

To specify a momentum current (or force), it is not enough to say, that e.g. 10 Hy/s are flowing in the corresponding conductor. One must also say, what kind of momentum is flowing: *x*-momentum, *y*-momentum, *z*-momentum or any combination of them. Therefore the momentum current intensity is a vector quantity.

The magnitude of this vector indicates the amount of momentum passing through a given surface per time. Its direction indicates the direction of the momentum flowing in the conductor.

Attention: The direction of the momentum current vector (= direction of the force) is in general not the same as that of the conductor through which the momentum flows (e.g. a rod).

Fig. 4.1 shows three examples of *x*-momentum flowing into a body. In all three cases it is said that “a force in *x*-direction acts on the body“. Notice that the direction of flow of the *x*-momentum is different in every partial image. In fig. 4.1(a) the *x*-momentum goes from right to left and enters the body from the right, in fig. 4.1(b) it goes from left to right and enters the body from the left and in fig. 4.1(c) it flows upwards and enters the body from below.

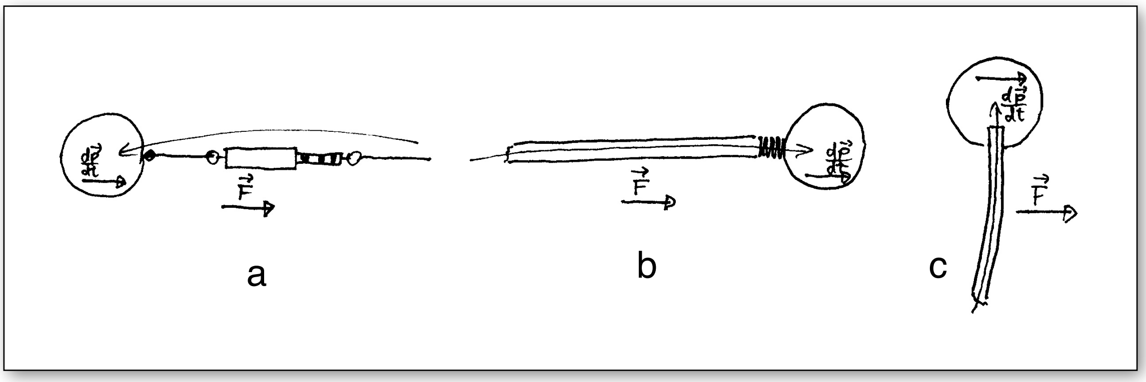

Fig. 4.1
(a) *x*-momentum flows from right to left. (b) *x*-momentum flows from left to right. (c) *x*-momentum flows from bottom to top.

Fig. 4.2 shows a body that is charged with *y*-momentum. A force of *y*-direction acts on the body. In fig. 4.2 a the momentum comes from the right; in fig. 4.2 b it also comes from the right, but it still has to pass through the loop.

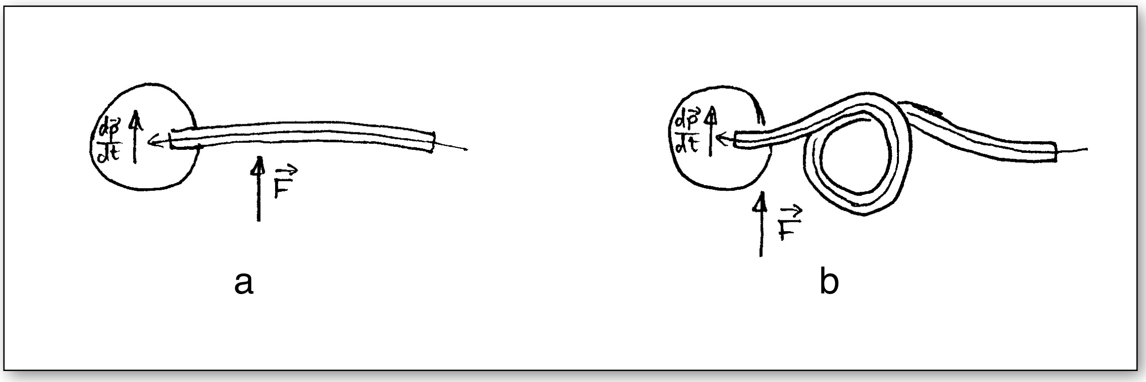

Fig. 4.2
(a) *y*-momentum flows from right to left. (b) *y*-momentum flows from the right through the loop to the left.

4.3. Newton's laws

We can now write the balance equation in vector form:

$$\frac{d\vec{p}}{dt} = \vec{I}_p \qquad \text{with} \qquad \vec{I}_p = \vec{F} + \vec{F}_{\text{conv}} \tag{4.1}$$

If convective momentum currents are excluded, the following remains

$$\frac{d\vec{p}}{dt} = \vec{F}$$

We now can formulate Newton's laws of motion.

1. First law

Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

Alternatively, equivalent to this

If no momentum current flows into or out of a body the momentum of the body does not change.

2. Second law

If a force $\vec{F}$ acts on a body, its momentum changes such that

$$\frac{d\vec{p}}{dt} = \vec{F}$$

Or

When a momentum current $\vec{F}$ flows into a body, its momentum changes such that

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{4.2}$$

3. Third law

When one body A exerts a force $\vec{F}$ on a second body B, the second body simultaneously exerts a force $-\vec{F}$ equal in magnitude and opposite in direction on the first body.

Or

If a momentum current of intensity $\vec{F}$ flows from a body A to a body B, then the momentum current leaving A and that entering B have the same intensity.

It is obvious that all three laws express the conservation of momentum. The first and the third one express the conservation in special situations: the first in the case where the momentum does not change; the third compares two sections through an unbranched momentum conductor.

From a modern point of view, one would formulate the content of Newton's laws more briefly in one single statement:

Momentum cannot be created or annihilated.

Or

The momentum of a body can change only by inflow or outflow.

If for a body or a particle the relation

$$\vec{p} = m \cdot \vec{v}$$

is valid, it follows from the Newton's (second) law

$$\vec{F} = \frac{d(m\vec{v})}{dt}$$

If, in addition, m is independent of the velocity, we get

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

With the acceleration $\vec{a} = d\vec{v}/dt$ we finally obtain

$$\vec{F} = m \cdot \vec{a}\,.$$

Often, this relation is called Newton's 2nd law. In fact, it contains Newton's momentum balance law, which is always valid, but it also contains the linear relationship between $\vec{p}$ and $\vec{v}$, which is not always valid.

4.4 The force of gravity

The *z*-axis is perpendicular to the surface of the earth, and it is oriented downward.

For a falling body, momentum and velocity increase. There is a flow of *z*-momentum into the body. This momentum comes from the earth via the gravitational field. If there is air friction, momentum also flows out of the body into the air.

The momentum current flowing from the earth into a body is used to define the gravitational mass. Observation shows that the total momentum current flowing into two completely similar bodies is twice that flowing into one of them. The momentum current is thus proportional to the magnitude of the body, as measured by some arbitrary extensive physical quantity. One then defines that two different bodies have the same gravitational mass m_S if the momentum currents flowing into the bodies from the earth are equal. The unit is defined by specifying that a certain reference body has the mass of 1 kg.

Thus, the gravitational mass is defined by the equation:

$$\vec{F} = m_S \cdot \vec{g}$$

$\vec{F}$ is called gravitational force or weight.

The magnitude of the vector $\vec{g}$ depends on the location. In central Europe, its value is 9.81 N/kg near the earth's surface, 9.83 N/kg at the north and south poles, and 9.78 N/kg at the equator. Later, we will get to know a more general meaning of $\vec{g}$.

Experience has shown that the inertial mass m and the gravitational mass m_S are exactly proportional to each other. Since one uses the same unit for both, they are even equal:

$$m = m_S$$

In the framework of classical mechanics, this equation expresses the observable equality of two different physical quantities. According to the general theory of relativity, inertia and gravity are the same phenomenon. Accordingly, there is also no difference between an inertial mass and a gravitational mass.

If the body is prevented from falling by fixing it somehow, Fig. 4.3, a closed momentum circuit is obtained: The momentum flows from the earth via the gravitational field into the body and from there via the suspension back into the earth.

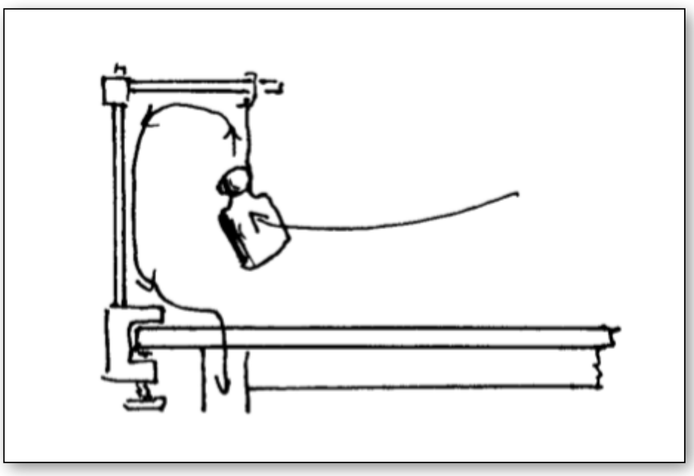

Fig. 4.3
Closed momentum circuit

To describe the state of the body in Fig. 4.3, one traditionally distinguishes between four different forces. Although all of them are of the same magnitude, they have to be distinguished from each other conceptually:

1. The force $\vec{F}_{sb}$, that the string exerts on the body;
2. the force $\vec{F}_{bs}$, that the body exerts on the string;
3. the force $\vec{F}_{eb}$, that the earth exerts on the body;
4. the force $\vec{F}_{be}$, that the body exerts on the earth.

The following relationships apply between these forces:

$$\vec{F}_{sb} = -\vec{F}_{bs} \quad (4.3a)$$

$$\vec{F}_{eb} = -\vec{F}_{be} \quad (4.3b)$$

$$\vec{F}_{sb} = -\vec{F}_{eb} \quad (4.3c)$$

Equations (4.3a) and (4.3b) are expressions of Newton's 3rd law. Equation (4.3c) states that the sum of the forces acting on the body is zero. This sum must be equal to zero, because the momentum of the body does not change.

Looking at the arrangement in the momentum current picture, we see that the four forces represent nothing more than the current strength of the same momentum current at four different locations: when it leaves the rope, when it enters the body, when it leaves the body, and when it enters the earth.

4.5 Momentum balance for rotational movements

The centrifugal force

A body of mass m is supposed to move frictionless with constant angular velocity

$$\vec{\omega} = \frac{d\vec{\alpha}}{dt}$$

on a circular path. ($d\vec{\alpha}$ = Angle in radians. $d\vec{\alpha}$ is a vector. If the curved fingers of the right hand point in the direction of rotation, the thumb indicates the direction of the $d\vec{\alpha}$ vector. It follows that the angular velocity is also a vector, Fig. 4.4).

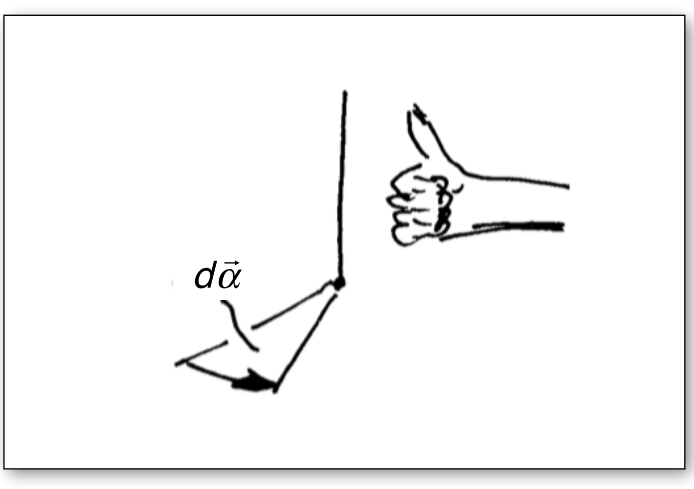

Fig. 4.4
The definition of the angular velocity vector

The fact that the body moves on a circular path means that its momentum is constantly changing, Fig. 4.5.

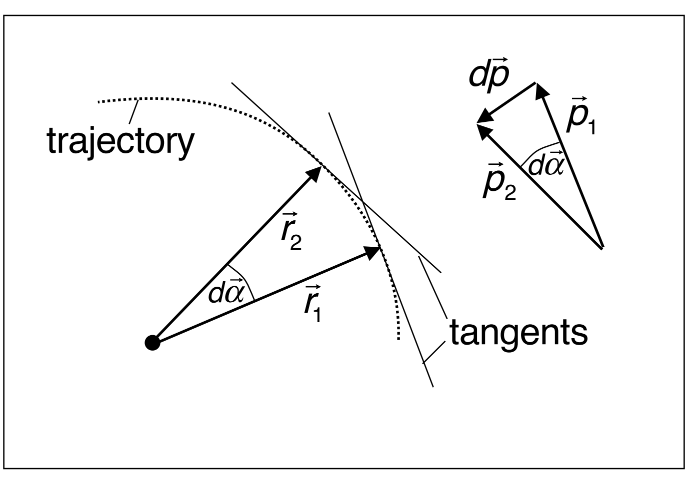

Fig. 4.5
For calculating the change of the momentum of a body making a circular motion

The change of momentum in the time interval dt is

$$d\vec{p} = \vec{p}_2 - \vec{p}_1 = d\vec{\alpha} \times \vec{p}$$

With $\vec{\omega} = d\vec{\alpha}/dt$ we get

$$\frac{d\vec{p}}{dt} = \vec{\omega} \times \vec{p}$$

and with $\vec{p} = m \times \vec{v}$ and $\vec{v} = \vec{\omega} \times \vec{r}$ finally

$$\frac{d\vec{p}}{dt} = m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]$$

For the momentum of the body to change, a force must act on it. (A momentum current must enter it.) This force is called *centripetal force*.

With $\vec{F} = d\vec{p}/dt$ we get for the centripetal force

$$\vec{F} = m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]$$

Obviously, this force can be measured. If, for example, we are dealing with a body that is being flung around with a string, we only need to install a force gauge in the string.

With this, we have established the momentum balance for the body.

It becomes more difficult when we sit on the body and make the balance in this new reference frame. On the force gauge we see that a momentum current still flows into the body. However, we do not notice any change of the momentum; in our new frame of reference the body is at rest. We therefore conclude that the momentum flows out again through an invisible conductor. Thus a current of the strength

$$\vec{F}_C = -m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]$$

leaves the body.

We say: On the body acts a force which keeps the balance to the centripetal force, the *centrifugal force*.

We already got to know the system through which this current is flowing away. It is the gravitational field.

The Coriolis force

We consider a vehicle of mass m moving on a disk that rotates with angular velocity $\vec{\omega}$, Fig. 4.6.

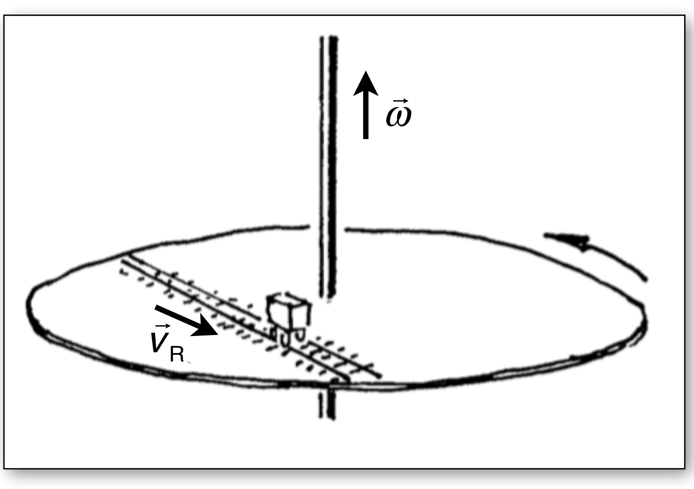

Fig. 4.6
In addition to the centrifugal force, another force acts on the vehicle transverse to the direction of the rails.

Let the velocity $\vec{v}_R$ of the vehicle relative to the disk be constant, as well as the angular velocity $\vec{\omega}$. It is possible in this case to express the change of the momentum of the vehicle by the quantities m, $\vec{v}_R$ and $\vec{\omega}$ alone. In fact it is

$$\frac{d\vec{p}}{dt} = m\{[\vec{\omega} \times (\vec{\omega} \times \vec{r})] + 2(\vec{\omega} \times \vec{v}_R)\} \qquad (4.4)$$

The first summand on the right side is equal to the centripetal force. In addition, there is a contribution to the change of momentum which is perpendicular to the velocity $\vec{v}_R$. To get an idea of this second summand, let us consider a special case: The vehicle moves radially outward on a rail. We consider the vehicle when it is so close to the center of the disk that we can neglect the first term on the right side of (4.4). (The second term is independent of the radius.) We can now think of the change of the second term as being composed of two components, Fig. 4.7.

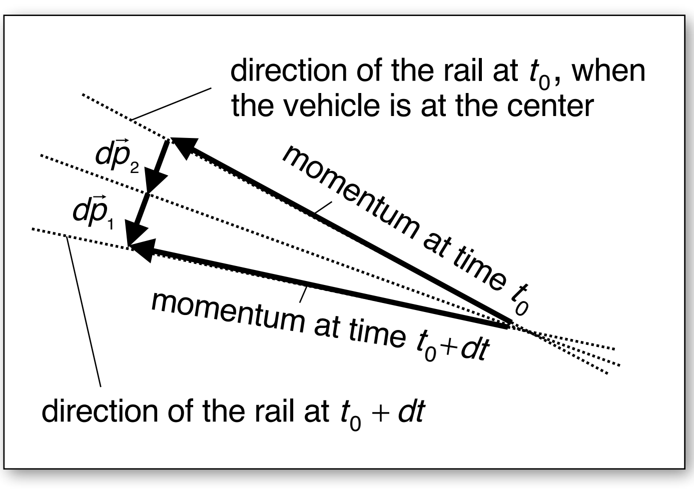

Fig. 4.7
Decomposition of the momentum change in the time interval dt into two parts. The third part is zero.

The first contribution comes from the fact that the momentum vector $m\vec{v}_R$ is rotated with the angular velocity $\vec{\omega}$. It amounts to

$$d\vec{p}_1 = m(d\vec{\alpha} \times \vec{v}_R) = m(\vec{\omega} \times \vec{v}_R)dt$$

The second contribution is due to the fact that, because of the radial motion, after the time dt has elapsed, the vehicle is located on a different circumference, i.e. at a point on the disk moving at a different tangential velocity:

$$d\vec{p}_2 = m \cdot d(\vec{\omega} \times \vec{r}) = m(\vec{\omega} \times \frac{d\vec{r}}{dt})dt = m(\vec{\omega} \times \vec{v}_R)dt$$

The sum of the two contributions is $2m(\vec{\omega} \times \vec{v}_R)dt$.

If the vehicle moves at a greater distance from the center, the first term in (4.4) has to be added

$$d\vec{p}_0 = m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]dt$$

so that in total we get

$$d\vec{p} = m\{[\vec{\omega} \times (\vec{\omega} \times \vec{r})] + 2(\vec{\omega} \times \vec{v}_R)\}dt$$

Also the force corresponding to the term $2m(\vec{\omega} \times \vec{v}_R)$ can be measured. It would manifest itself in a rail vehicle by a pressure on the rails transverse to their direction.

Again we establish the momentum balance in the rotating reference frame. Here, the momentum change is zero, although the forces following from equation (4.4) act on the body. So besides the centrifugal force

$$\vec{F}_C = -m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]$$

there must be yet another force

$$\vec{F}_{Cor} = -2m(\vec{\omega} \times \vec{v}_R)$$

that counterbalances the second summand on the right side of equation (4.4). This force is called *Coriolis force*. Also $\vec{F}_{Cor}$ describes a momentum current which flows out of the body into the gravitational field, and also this current exists only in the rotating reference frame.

4.6 Parallel momentum current circuits

We have seen that the force is a vector. What is the physical meaning of vector addition of forces? In the arrangement of Fig. 4.8, three ropes meet at a point, a node or *junction*.

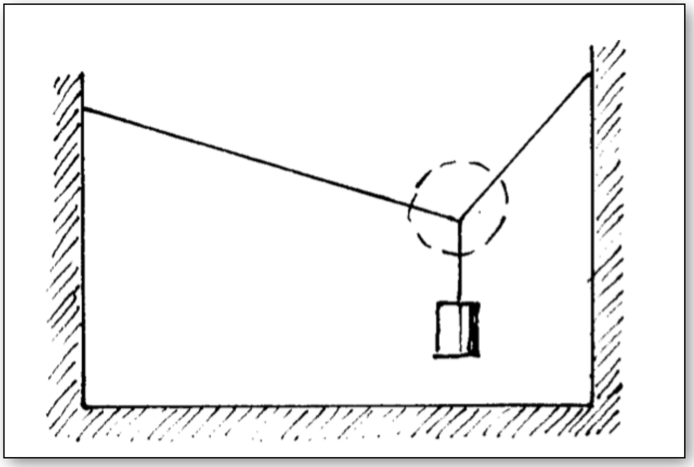

Fig. 4.8
The sum of the forces acting on the node is zero.

We apply the balance equation (4.2) to the dashed framed area:

$$\frac{d\vec{p}}{dt} = \sum_i \vec{F}_i \qquad i = 1, 2, 3$$

The total current $\vec{F}$ is the sum of the three partial currents $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$. The momentum in the considered range does not change, i.e. $d\vec{p}/dt = 0$. With the balance equation it follows:

$$\sum_i \vec{F}_i = 0 \qquad i = 1, 2, 3$$

The sum of the forces acting on the junction is equal to zero. (The total current flowing to the junction is equal to zero.)

The electrical analog of this rule is Kirchhoff's junction rule.

5

Momentum currents and energy currents

5.1 The relation between energy current strength, momentum current strength and velocity

A crate is pulled over the ground with constant velocity, Fig. 5.1. The rope is under tension, i.e. a momentum current flows through the rope to the left. From the fact that neither the momentum of the crate nor that of the person changes, we can conclude that the momentum flows back through the ground to the person. We are dealing with a closed momentum circuit.

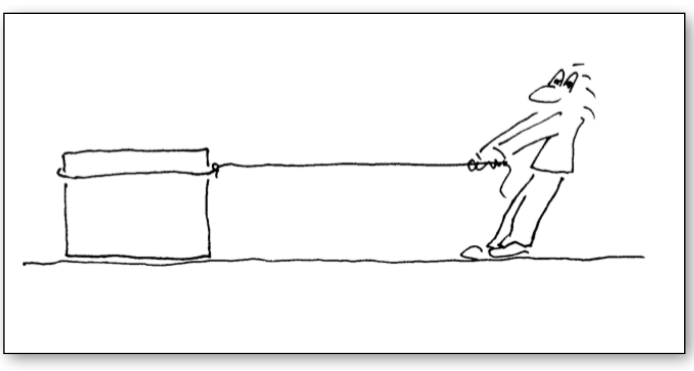

Fig. 5.1 The momentum circuit is closed via the ground.

The flow of the momentum current can be seen even better if the experiment is modified as shown in Fig. 5.2. The momentum circuit is isolated from the ground by the wheels. The flowing of the momentum current can be seen by means of the two springs: The upper spring is stretched, i.e. the momentum current flows to the left, the lower spring is compressed, i.e. the momentum current flows to the right.

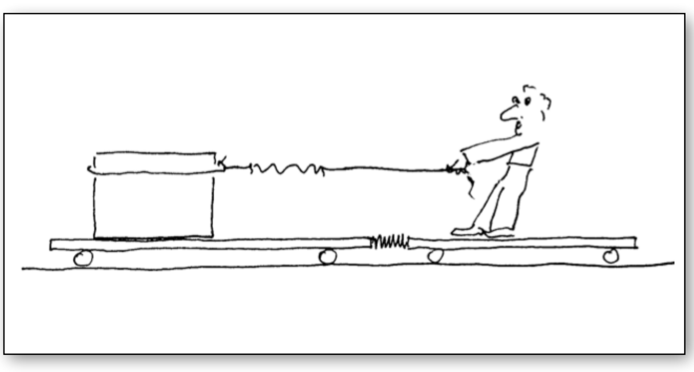

Fig. 5.2 The closed momentum circuit is isolated from the ground.

In the process, the person has to make an effort, and the underside of the crate heats up. This shows that energy is flowing from the person to the crate. It comes from the person's muscles and flows through the rope and the crate to the bottom of the crate. The energy flows only as long as the crate is moving and as long as the rope is under tensional stress. This means that the energy current strength P depends on the velocity v of the crate and on the force F acting on the crate (i.e. the strength of the momentum current flowing through the rope to the crate). We are looking for this relationship between P, v and F.

It is easy to see that at constant velocity $P \propto F$ if the person is pulling two crates side by side instead of one, Fig. 5.3.

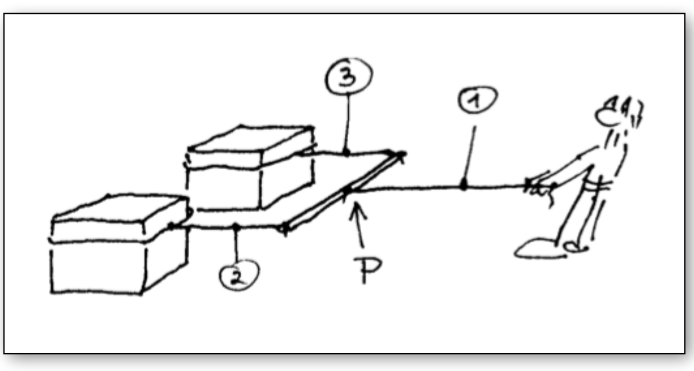

Fig. 5.3 In rope 1, both the momentum current intensity and the energy current intensity are twice as large as in rope 2 or in rope 3.

Because of the junction rule (applied to the point P), the following applies to the energy flow:

$$P_1 + P_2 + P_3 = 0$$

and to the momentum flow

$$F_1 + F_2 + F_3 = 0.$$

With $P_2 = P_3$ and $F_2 = F_3$ we get

$$P_1 = 2|P_2|$$

and

$$F_1 = 2|F_2|.$$

At constant velocity, a doubling of the force results in a doubling of the energy current strength.

It can be seen from Fig. 5.4 that for a constant energy current strength we obtain $F \propto 1/v$.

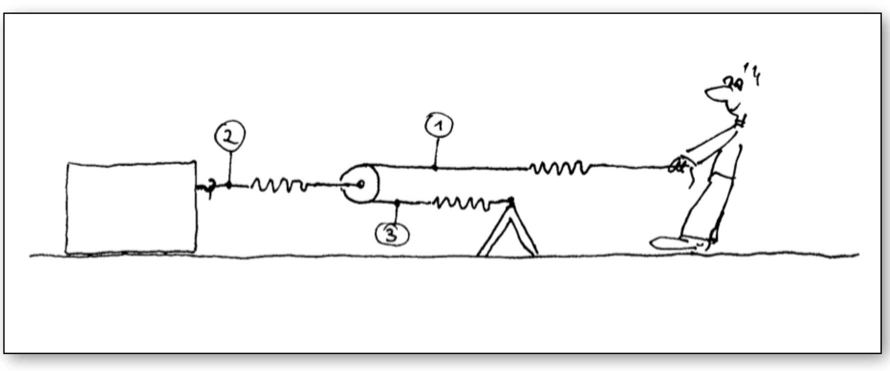

Fig. 5.4 The product of force and velocity has the same value for rope 1 as for rope 2.

For geometrical reasons

$$v_1 = 2v_2 .$$

Since there is no energy flow through rope 3 we have

$$P_1 = P_2$$

By applying the junction rule to the pulley, we get

$$F_1 + F_2 + F_3 = 0,$$

and taking into account the symmetry of the pulley,

$$F_2 = 2|F_1|.$$

Thus, at constant energy current strength, the force is proportional to the reciprocal of the velocity.

From

$$P \propto F \quad \text{for } v = \text{const}$$

and

$$F \propto 1/v \quad \text{for } P = \text{const}$$

we obtain

$$P \propto v \cdot F.$$

The unit of measurement of the energy is defined in such a way that we have

$$P = v \cdot F.$$

Measuring v in m/s and F in N, we obtain P in Watt (W).

If the whole arrangement is allowed to move with the velocity v', Fig. 5.5 (which is the same as if it were described in a different reference frame), then we not only have energy flow

$$P_S = (v' + \Delta v) F,$$

in the rope, where Δv is the velocity of the crate relative to the board. There is also an energy flow in the "return line", i.e. within the board:

$$P_R = -v' F.$$

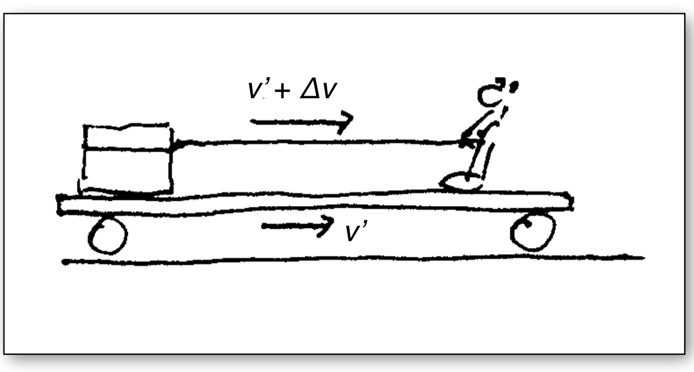

Fig. 5.5 Besides in the rope, an energy current also flows in the board.

Thus, the net energy flow from the person to the crate has the strength

$$P = P_S + P_R$$

or

$$P = \Delta v \cdot F$$

5.2 The analogy with electricity

The momentum circuit of Fig. 5.2 is analogous to a simple electric circuit, Fig. 5.6.

Fig. 5.6
An electrical circuit that is analogous to the momentum circuit of Fig. 5.2

The battery corresponds to the person, the filament of the lamp corresponds to the bottom of the crate. The energy current strength is

$$P = \Delta\Phi \cdot I\,.$$

We see that the electric potential Φ is the analog of the velocity.

The two equations

$$P = \Delta v \cdot F \quad \text{and} \quad P = \Delta\Phi \cdot I$$

express a general rule:

> *Every energy flow is accompanied by the flow of another substance-like quantity.*

We call the quantity flowing simultaneously with the energy the *energy carrier*. In the example with the moving crate the energy carrier is momentum, in the electric circuit it is electric charge.

Fig. 5.7 shows two technical devices that are analogous to each other: a pulley block and a transformer.

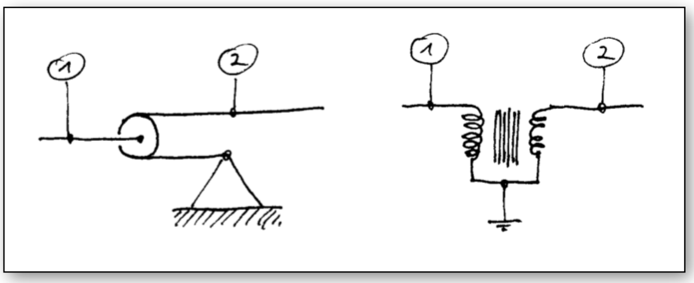

Fig. 5.7
Pulley block and transformer

Provided that they have no losses, the strength of the energy current flowing in is equal to that flowing out:

$$P_1 = P_2\,.$$

In both cases, what changes is the carrier current.

5.3 Generalization to three dimensions

The equation $P = \Delta v \cdot F$ is valid only for the case that $\Delta\vec{v}$ and $\vec{F}$ have the same direction. Δv and F are the magnitudes of these vectors. The relation can be easily generalized. It then reads

$$P = \Delta\vec{v} \cdot \vec{F}$$

Fig. 5.8 shows that we have to use the dot product of the two vetros. The force $\vec{F}$ is decomposed into a component parallel to the motion and a component perpendicular to it.

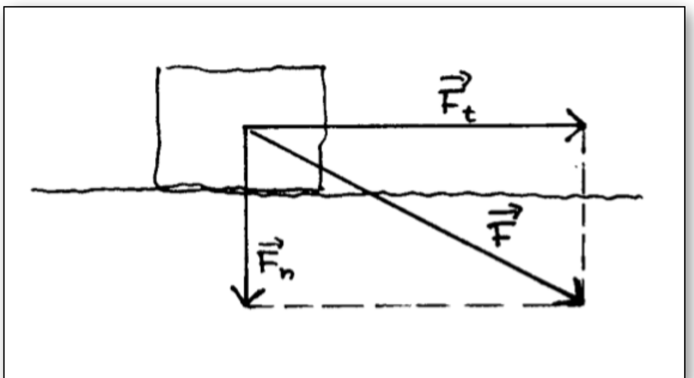

Fig. 5.8
Only the component $\vec{F}_t$ of $\vec{F}$ contributes to the energy current.

There is no energy flow connected with the perpendicular component. The energy flow is therefore the same as if only the projection $\vec{F}_t$ of $\vec{F}$ were present.

6

Energy stores

6.1 A moving body as an energy store – kinetic energy

In this chapter, we will examine mechanical energy storage devices. A mechanical energy store can be charged with energy using a momentum current, and the energy can be retrieved at a later time using the momentum current. It is characteristic for mechanical energy storage that the charging and the discharging process can be described by the relation

$$P = \vec{v} \cdot \vec{F} \ .$$

There are many different systems for which this is true. During charging – apart from energy – some other variables of the system always change their value. The energy content can be read from the value of these variables. We will calculate the relationship between the energy content and such other variables for several examples.

Let us begin with a carriage that is accelerated, Fig. 6.1. An energy current of strength $P = \vec{v} \cdot \vec{F}$ flows through the rope into the carriage. Not only the energy but also the momentum accumulates in the carriage.

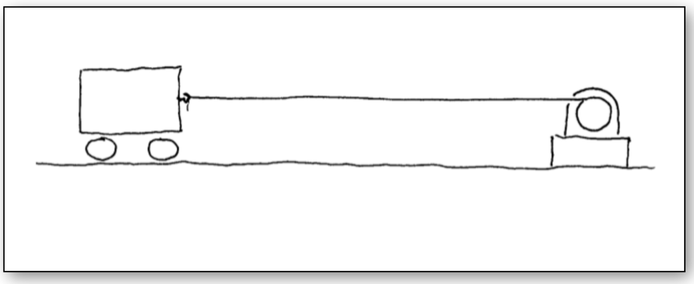

Fig. 6.1
A body is charged with momentum. Thereby its energy content increases.

With

$$P = \frac{dE}{dt}$$

and

$$\vec{F} = \frac{d\vec{p}}{dt}$$

the relation between the time rates of change of E and $\vec{p}$ of the carriage becomes

$$\frac{dE}{dt} = \vec{v}\frac{d\vec{p}}{dt}$$

Using $\vec{p} = m \cdot \vec{v}$ and integrating

$$E = \frac{1}{m}\int \vec{p}\, d\vec{p}$$

we obtain

$$E(\vec{p}) = E_0 + \frac{\vec{p}^2}{2m}$$

Here E_0 is the energy of the carriage for $\vec{p} = 0$. With $\vec{p} = m \cdot \vec{v}$ we get:

$$E(\vec{v}) = E_0 + \frac{m}{2}\vec{v}^2$$

Of course, these relations are valid only as long as $\vec{p} = m \cdot \vec{v}$, i.e. as long as $|\vec{v}| \ll c$. The term $\vec{p}^2/(2m) = (m/2)\vec{v}^2$ is called the *kinetic energy* of the body. Note that this does not mean that there are energies of different nature. With the adjective “kinetic” one only characterizes the system, in which the energy is stored.

6.2 The spring as an energy store

An elastic spring, Fig. 6.2, is stretched. An energy current of strength $P = \vec{v} \cdot \vec{F}$ flows through the rope to the spring. While the energy is stored within the spring, the momentum flows through the spring. Since the velocity of the left end of the spring is zero, P is zero as well.

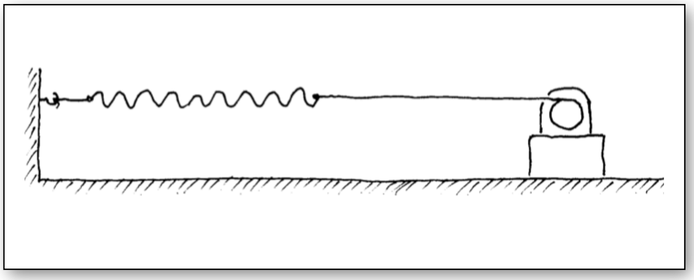

Fig. 6.2
A spring is stretched. Thereby, its energy content increases.

With

$$P = \frac{dE}{dt}$$

and

$$\vec{v} = \frac{d\vec{s}}{dt}$$

we get

$$\frac{dE}{dt} = \vec{F}\frac{d\vec{s}}{dt}$$

For a normal steel spring, the relationship between force and extension is linear, i.e. it holds that

$$\vec{F} = D\vec{s}$$

(Hooke's law, D is the spring constant). We thus get

$$E = D\int \vec{s}\, d\vec{s}$$

$$E(\vec{s}) = E_0 + \frac{D}{2}\vec{s}^2$$

or with $\vec{F} = D\vec{s}$

$$E(\vec{F}) = E_0 + \frac{\vec{F}^2}{2D}$$

6.3 The electric field as an energy store

The plates of a capacitor are pulled apart, while the electric charge is kept constant, Fig. 6.3. An energy current of strength $P = \vec{v} \cdot \vec{F}$ flows through the rope to the capacitor. The energy is deposited in the electric field located between the capacitor's plates.

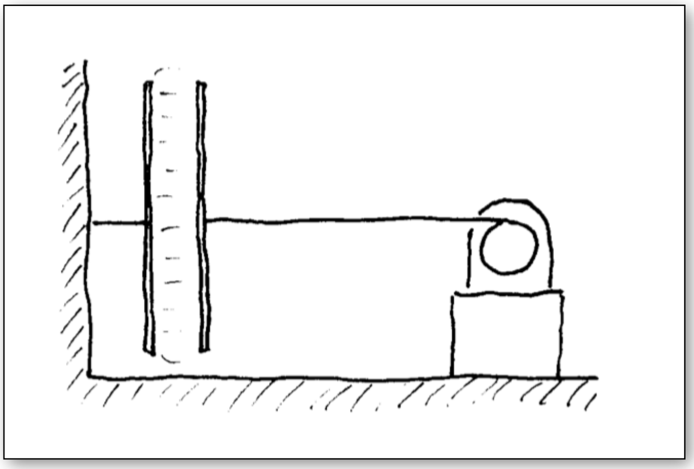

Fig. 6.3
The plates of a charged capacitor are pulled apart. Thereby the energy content of the field between the plates increases.

With

$$P = \frac{dE}{dt}$$

$$v = \frac{dx}{dt}$$

and

$$F = \frac{Q^2}{2\varepsilon_0 A}$$

(ε_0 = vacuum permittivity, A = plate area) we obtain by integration:

$$E(Q) = E_0 + \frac{Q^2 x}{2\varepsilon_0 A}$$

and with $C = \varepsilon_0 A/x$ (C = capacitance, x = plate distance):

$$E(Q) = E_0 + \frac{Q^2}{2C}$$

or with $Q = CU$

$$E(U) = E_0 + \frac{C}{2}U^2$$

6.4 The gravitational field as an energy store

A body is pulled upwards, Fig. 6.4. An energy flux of strength $P = \vec{v} \cdot \vec{F}$ flows through the rope to the body and further into the gravitational field of body and earth.

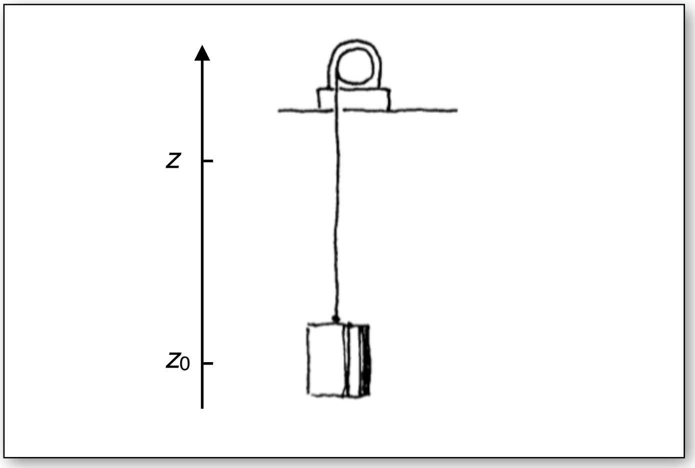

Fig. 6.4
A body is lifted. Thereby, energy is stored in the gravitational field.

With

$$P = \frac{dE}{dt}$$

$$v = \frac{dz}{dt}$$

and

$$F = mg$$

we obtain by integration:

$$E(z) - E(z_0) = mg(z - z_0)$$

$E(z)$ is called the *potential energy* of the body. It is said that when the body is lifted from z to z_0, its potential energy increases.

Note that the potential energy is not located within the body.

7

Collision processes

7. Collision processes

A collision is a transition from an initial state to a final state, in which at least two bodies or particles (e.g. also photons) are involved. In a collision, momentum and energy are redistributed among the colliding partners. There can also occur a reaction, i.e. particles are created or destroyed. In the initial and in the final state, the particles are completely decoupled: neither momentum nor energy flows between them. Only during the time of the transition, momentum and energy transfer takes place.

Energy and momentum conservation require that

$$\sum_i E_{\text{initial},\,i} = \sum_j E_{\text{final},\,j}$$

and

$$\sum_i \vec{p}_{\text{initial},\,i} = \sum_j \vec{p}_{\text{final},\,j}$$

The center of mass $\vec{r}_{\text{CM}}$ of a system of n bodies is defined by

$$\vec{r}_{\text{CM}} \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \vec{r}_i m_i$$

Deriving this equation with respect to time yields:

$$\dot{\vec{r}}_{\text{CM}} \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \dot{\vec{r}}_i m_i$$

Since the total momentum of the system (right side of the equation) remains constant during the collision, the velocity of the center of mass

$$\vec{v}_{\text{CM}} = \dot{\vec{r}}_{\text{CM}}$$

is also the same before and after the collision:

$$\vec{v}_{\text{CM, initial}} = \vec{v}_{\text{CM, final}}$$

The reference frame in which the center of mass of a set of bodies is at rest is called the center-of-mass frame.

We can write the energy of a set of n bodies as

$$E = \sum_{i=1}^{n} E_{0,i} + \sum_{i=1}^{n} E_{\text{kin},i}$$

$E_{0,i}$ is the internal energy, $E_{\text{kin},i}$ the kinetic energy of body i. The second sum can be decomposed once more

$$\sum_{i=1}^{n} E_{\text{kin},i} = E_{\text{kin,CM}} + \sum_{i=1}^{n} E_{\text{kin},i}^{(\text{CM})}$$

where

$$E_{\text{kin,CM}} = \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) v_{\text{CM}}^2$$

is the "kinetic energy of the center of mass", and $E_{\text{kin},i}^{(\text{CM})}$ is the kinetic energy of body i in the centre of mass frame.

Thus, the total energy becomes

$$E = \sum_{i=1}^{n} E_{0,i} + \sum_{i=1}^{n} E_{\text{kin},i}^{(\text{CM})} + E_{\text{kin,CM}}$$

If we consider the ensemble of bodies as a whole, the sum

$$\sum_{i=1}^{n} E_{\text{kin},i}^{(\text{CM})}$$

appears as a part of its internal energy. We therefore combine the two sums into E_0, the internal energy of the whole ensemble:

$$E = E_0 + E_{\text{kin,CM}}$$

where

$$E_0 = \sum_{i=1}^{n} E_{0,i} + \sum_{i=1}^{n} E_{\text{kin},i}^{(\text{CM})}$$

and

$$E_{\text{kin,CM}} = \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right) v_{\text{CM}}^2$$

Since the velocity of the center of mass $\vec{v}_{\text{CM}}$ remains constant in the collision, it follows that

$$E_{0,\text{ initial}} = E_{0,\text{ final}}$$

and

$$E_{\text{kin, initial}}^{(\text{CM})} = E_{\text{kin, final}}^{(\text{CM})}$$

Quantities that remain constant during the collision are called *collision invariants*. Thus, E, $\vec{p}$, $\vec{v}_{\text{CM}}$, E_0 and $E_{\text{kin,CM}}$ are collision invariants.

If in addition

$$\sum_{i=1}^{n} E_{\text{kin},i}^{(\text{CM})}$$

is a collision invariant, the collision is called *elastic*, if not it is called *inelastic*. The kinetic energy missing after an inelastic collision can either be stored or used for heat production. In the first case the collision is reversible, in the second it is not, Fig. 7.1.

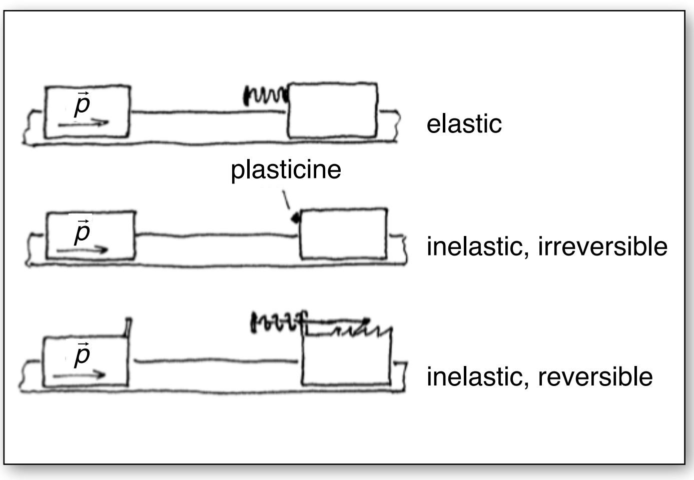

Fig. 7.1
Several types of collisions

8

Dissipative momentum currents – friction and viscosity

8. Dissipative momentum currents – friction and viscosity

In the three situations shown in Fig. 8.1, a momentum current flows from one body moving at constant velocity to another body at rest. The momentum current flows across a velocity gradient, much like the electric current in an electrical resistor flows across a potential gradient. Since in each case an energy current of strength $P = \Delta\vec{v} \cdot \vec{F}$ is dissipated, i.e., used to produce heat, we call these momentum currents dissipative momentum currents. Mechanical energy dissipation is called friction.

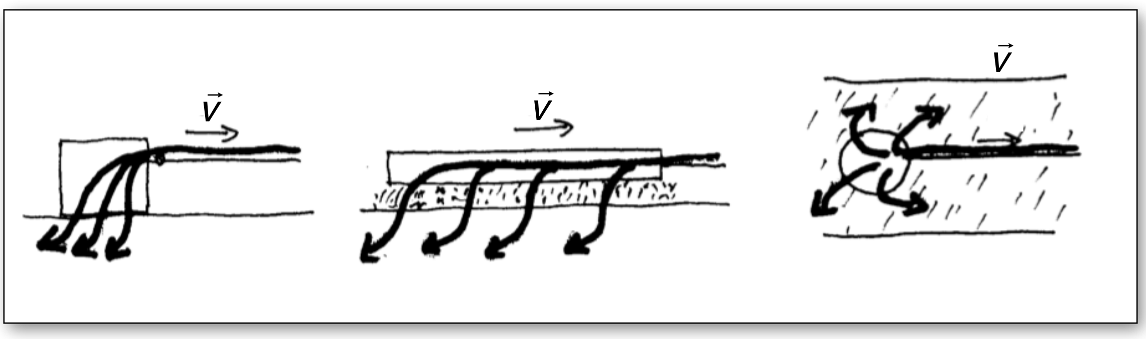

Fig. 8.1
In a friction process, momentum flows from the body of higher velocity to the body of lower velocity.

We recognize the validity of the following rule:

A dissipative momentum current always flows from the body of higher to the body of lower velocity.

The electrical analog of this rule is:

A dissipative electric current always flows from the body of higher to the body of lower electric potential.

The relationship between $\vec{F}$ and $\Delta\vec{v}$ is different in each of the three cases of Fig. 8.1, Fig. 8.2.

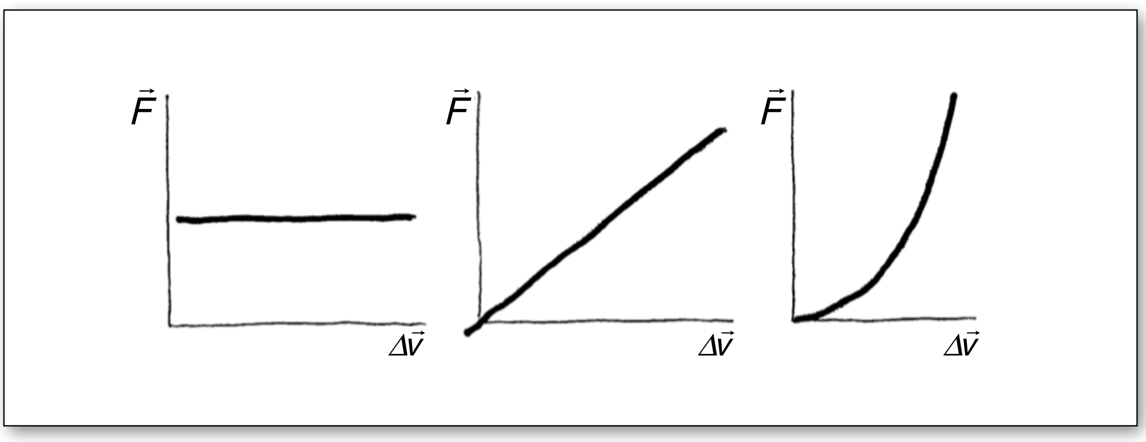

Fig. 8.2
The characteristic curves belonging to the processes shown in Fig. 8.1

The electrical analog of the $\Delta\vec{v} - \vec{F}$ characteristic is the $\Delta\Phi - I$ characteristic.

In the first case of Fig. 8.2, when two solid bodies slide over each other, the force is independent of the velocity difference. This case is realized with the brake and clutch of a car.

The second characteristic curve of Fig. 8.2 is obtained when the interfaces of two solid bodies sliding over each other are separated by a liquid layer (a lubricant). Here a kind of Ohm's law applies, and one can define a mechanical resistance R_p in analogy to the electrical resistance $R = \Delta\Phi/I$:

$$R_p = \frac{\Delta v}{F}$$

This case is realized in the shock absorber of a car: the force on the shock absorber is proportional to the velocity difference between the two attachments of the shock absorber.

The resistance R_p is related in a simple way to the area A of the bodies sliding over each other and their distance s, Fig. 8.3:

$$R_p = \frac{s}{\eta A}$$

This relationship is analogous to the well-known equation the electrical resistance:

$$R = \frac{s}{\sigma A}$$

η, the viscosity, is a material constant of the liquid that is conducting the momentum. It is the analog of the electrical conductivity σ. Therefore, η can also be called the momentum conductivity.

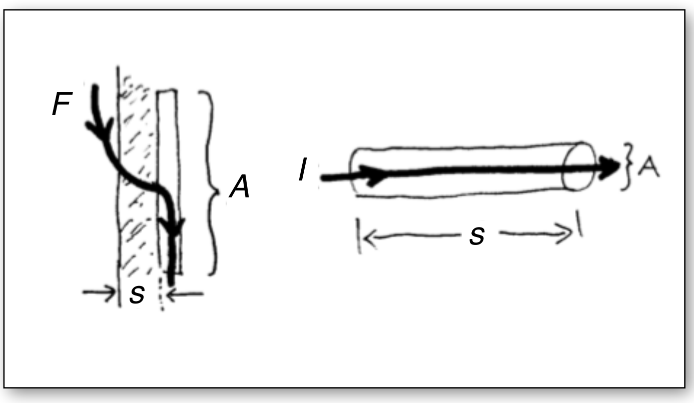

Fig. 8.3
Mechanical and electrical resistance law

The third characteristic curve in Fig. 8.2, where the force depends quadratically on the velocity, is obtained when a body is pulled through a liquid or a gas. It describes, for example, the air resistance of a car:

$$F = \frac{1}{2} c_W A \rho (\Delta v)^2$$

Here ρ is the density of the fluid, A is the cross-sectional area of the body perpendicular to the direction of motion, and c_W is the so-called drag coefficient. c_W is dimensionless and of the order of 1.

9

The analogy between mechanics and electricity – the dualism within mechanics and electricity

9.1 The analogy

There is an analogy between two fields of physics, if quantities of one field can be mapped on quantities of the other in such a way that the relations between the quantities of one field translate into correct relations between the quantities of the other. There are several such analogies in physics. Here we are concerned with an analogy between mechanics and electromagnetism. The quantities corresponding to each other are listed in Table 9.1.

mechanics	electricity
momentum $\vec{p}$	electric charge Q
force (momentum current strength) $\vec{F}$	electric current strength I
velocity $\vec{v}$	electric potential Φ
velocity difference $\Delta\vec{v}$	voltage $\Delta\Phi = U$
energy E	energy E
energy current strength P	energy current strength P

Table 9.1 Analogous quantities from mechanics and electromagnetism

The conserved quantity $\vec{p}$ is mapped onto the conserved quantity Q, and the energy is mapped onto itself.

It can be seen that from

$$P = \Delta\vec{v} \cdot \vec{F}$$

by purely formal translation

$$P = U \cdot I$$

is obtained.

If one looks at an object from a mechanical point of view, often only three properties are of interest:

- its inertia
- its elasticity
- its dissipative behavior.

The engineer likes to realize these three properties by spatially separated components, or he decomposes a given system in his mind into components, namely into those which are

- inert, but rigid and frictionless,
- elastic, but mass- and frictionless and
- dissipative, but rigid and massless.

Each of these three components is idealized in the sense that the variables describing the component are interrelated in a very simple way. These idealized components are the mass point, the elastic spring and the shock absorber. The relationships characterizing these components are:

- mass point: $\vec{p} = m \cdot \vec{v}$
- elastic spring: $\vec{F} = D \cdot \Delta\vec{s}$
- shock absorber: $\Delta\vec{v} = R_p \cdot \vec{F}$

In electricity, the situation is analogous. Here, too, one likes to break down a structure into components, three of which play a special role: the capacitor, the coil and the resistor. Also these components are approximately characterized by three simple relations:

- capacitor: $Q = C \cdot U$
- coil: $I = (N/L)\Phi$
- resistor: $U = R \cdot I$

If we look at the translation table 9.1, we see that these three components are analogous to the three mechanical components mentioned before, and we can thus extend our translation table, see table 9.2. The list of quantities analogous to each other is far from being complete. A particularly interesting pair of analogous quantities are the angular momentum and the electric dipole moment.

mechanics	electricity
mass point	capacitor
mass (momentum capacitance) m	capacitance C
spring	coil
reciprocal spring constant $1/D$	inductance L
shock absorber	resistor
mechanical resistance R_p	electric resistance R
viscosity (momentum conductivity) η	electric conductivity σ

Table 9.2 Analogous quantities and components from mechanics and electromagnetism

9.2 The dualism

Furthermore, within mechanics exists a structure which we will call dualism. Because of the analogy between mechanics and electromagnetism, electromagnetism also has this dual structure. What is it about? One transforms an arbitrary arrangement of the previously described components into another arrangement according to certain rules. Moreover, one maps quantities to other quantities according to certain rules. Then the mathematical structure of the old problem with the old quantities is the same as that of the new problem with the new quantities. If one applies the same translation rules twice in succession, one returns to the original problem. Table 9.3 lists the corresponding components, physical quantities and "topological rules".

	mechanics	**electricity**
components	mass point ⇔ spring shock absorber ⇔ shock absorber	capacitor ⇔ coil resistor ⇔ resistor
quantities	$\vec{p} \Leftrightarrow \Delta\vec{r}$ $\vec{F} \Leftrightarrow \Delta\vec{v}$ $m \Leftrightarrow 1/D$ $R_p \Leftrightarrow 1/R_p$ $E \Leftrightarrow E$ $P \Leftrightarrow P$	$Q \Leftrightarrow N\Phi$ $I \Leftrightarrow \Delta\varphi = U$ $C \Leftrightarrow L$ $R \Leftrightarrow 1/R = G$ $E \Leftrightarrow E$ $P \Leftrightarrow P$
topological rules	parallel circuit ⇔ series circuit junction rule ⇔ loop rule	

Table 9.3
The dualism within mechanics and within electricity

Also with the dualism energy plays a special role: It is self-dual.

Although the component resistor is dual to the component resistor, its reciprocal value, the conductance G, corresponds to the quantity resistance.

9.3 Example

We solve a mechanical problem, together with its electrical analogue, Fig. 9.1.

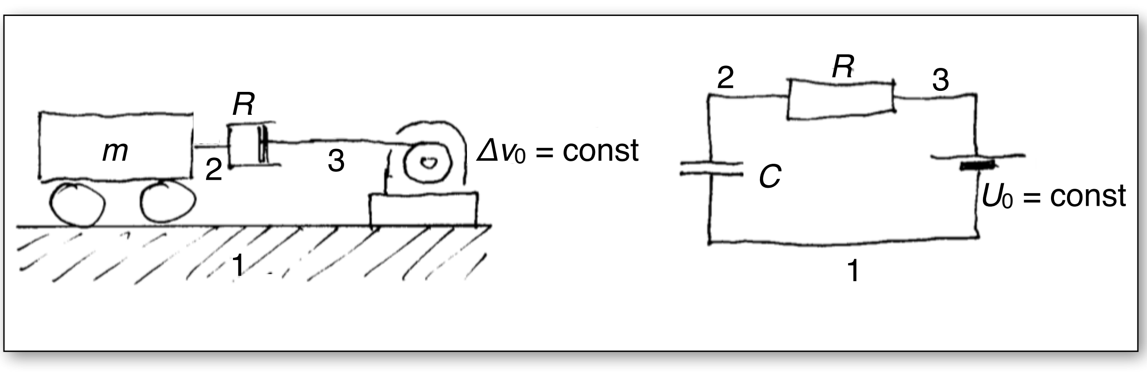

Fig. 9.1
Two systems that are analogous to each other

The mechanical version is on the left, the electrical one on the right. Thereafter, we solve the versions that are dual to both problems. Velocity differences and voltages are counted clockwise (e.g. $\Delta v_R = v_2 - v_3$ or $U_R = \Phi_2 - \Phi_3$). The subscript p on the mechanical resistance is omitted for clarity. We apply the *loop rule* to the circuit:

$$\Delta v_0 + \Delta v_m + \Delta v_R = 0 \qquad U_0 + U_C + U_R = 0$$

With

$$\Delta v_R = R \cdot F \qquad U_R = R \cdot I$$

and

$$p = m \cdot \Delta v_m \Rightarrow F = m \cdot \Delta \dot{v}_m \qquad Q = C \cdot U_C \Rightarrow I = C \cdot \dot{U}_C$$

we obtain

$$\Delta v_0 + \Delta v_m + Rm\Delta \dot{v}_m = 0 \qquad U_0 + U_C + RC\dot{U}_C = 0$$

The solutions of these differential equations are

$$\Delta v_m(t) = -\Delta v_0 \left(1 - e^{-\frac{t}{Rm}}\right) \qquad U_C(t) = -U_0 \left(1 - e^{-\frac{t}{RC}}\right)$$

From this, the time dependence of other quantities of the circuits can be calculated:

$$\Delta v_R(t) = -\Delta v_0 - \Delta v_m(t) \qquad U_R(t) = -U_0 - U_C(t)$$

$$= -\Delta v_0 e^{-\frac{t}{Rm}} \qquad = -U_0 e^{-\frac{t}{RC}}$$

$$F(t) = -\frac{\Delta v_0}{R} e^{-\frac{t}{Rm}} \qquad I(t) = -\frac{U_0}{R} e^{-\frac{t}{RC}}$$

We call

P_{total}	total strength of the energy current from the energy source (motor resp. battery) to the mass point resp. capacitor and the shock absorber resp. resistor
P_m and P_C	strength of the energy current to the mass point resp. capacitor
P_R	strength of the energy current to the shock absorber resp. resistor

We obtain

$$P_{\text{total}} = (\Delta v_m + \Delta v_R)F(t) = -\Delta v_0 F(t) \qquad P_{\text{total}} = (U_C + U_R)I(t) = -U_0 I(t)$$

$$= \frac{\Delta v_0^2}{R} e^{-\frac{t}{Rm}} \qquad = \frac{U_0^2}{R} e^{-\frac{t}{RC}}$$

$$P_m = \Delta v_m(t)F(t) \qquad P_C = U_C(t)I(t)$$

$$= \frac{\Delta v_0^2}{R} e^{-\frac{t}{Rm}} \left(1 - e^{-\frac{t}{Rm}}\right) \qquad = \frac{U_0^2}{R} e^{-\frac{t}{RC}} \left(1 - e^{-\frac{t}{RC}}\right)$$

$$P_R = \Delta v_R(t)F(t) = \frac{\Delta v_0^2}{R} \left(1 - e^{-\frac{t}{Rm}}\right)^2 \qquad P_R = U_R(t)I(t) = \frac{U_0^2}{R} \left(e^{-\frac{t}{RC}}\right)^2$$

Fig. 9.2 shows P_{total}, P_R and P_m (or P_C) as a function of time.

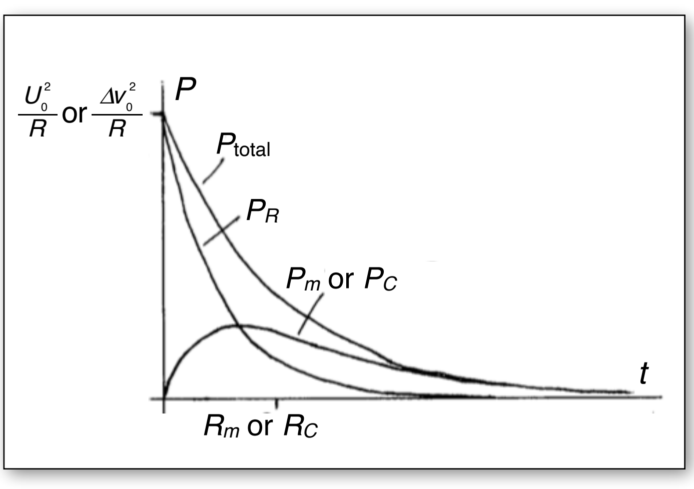

Fig. 9.2
Energy current strengths as a function of time

The comparison of the left side of our calculation with the right side shows that one of the two calculations could have been spared: One obtains it by purely formal translation from the other side.

Using the translation rules of the dualism, we now transform the problem into a new one, Fig. 9.3.

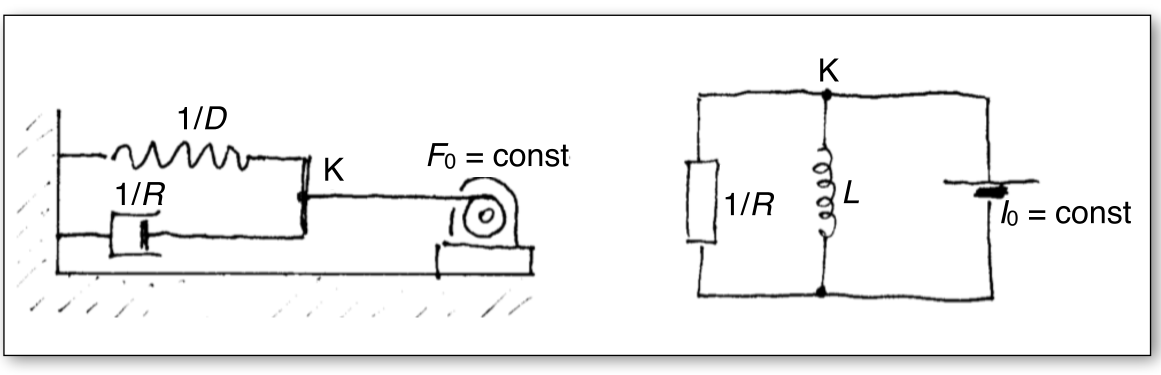

Fig. 9.3
The systems are analogous to each other, and are dual to those in Fig. 9.1.

Current intensities (also momentum current intensities) are counted positive toward the junction K.

We apply the junction rule to K:

$$F_0 + F_D + F_R = 0 \qquad I_0 + I_L + I_R = 0$$

With

$$\Delta v = F_R \cdot R = \frac{1}{D}\dot{F}_D \qquad U = I_R \cdot R = L\dot{I}_L$$

we obtain

$$F_0 + F_D + \frac{1}{RD}\dot{F}_D = 0 \qquad I_0 + I_L + \frac{L}{R}\dot{I}_L = 0$$

The solutions of these differential equations are

$$F_D(t) = -F_0\left(1 - e^{-RDt}\right) \qquad I_L(t) = -I_0\left(1 - e^{-\frac{R}{L}t}\right)$$

From this follows again the time dependence of other quantities. We do not continue the calculation, because one can see already how things work: One obtains the equations in this example line by line from those of the previous example by applying the translation rules of dualism.

Finally, Fig. 9.4 shows another problem including its electrical analogue and its two dual versions, which is very similar to the previous one. We leave the corresponding calculation to the reader.

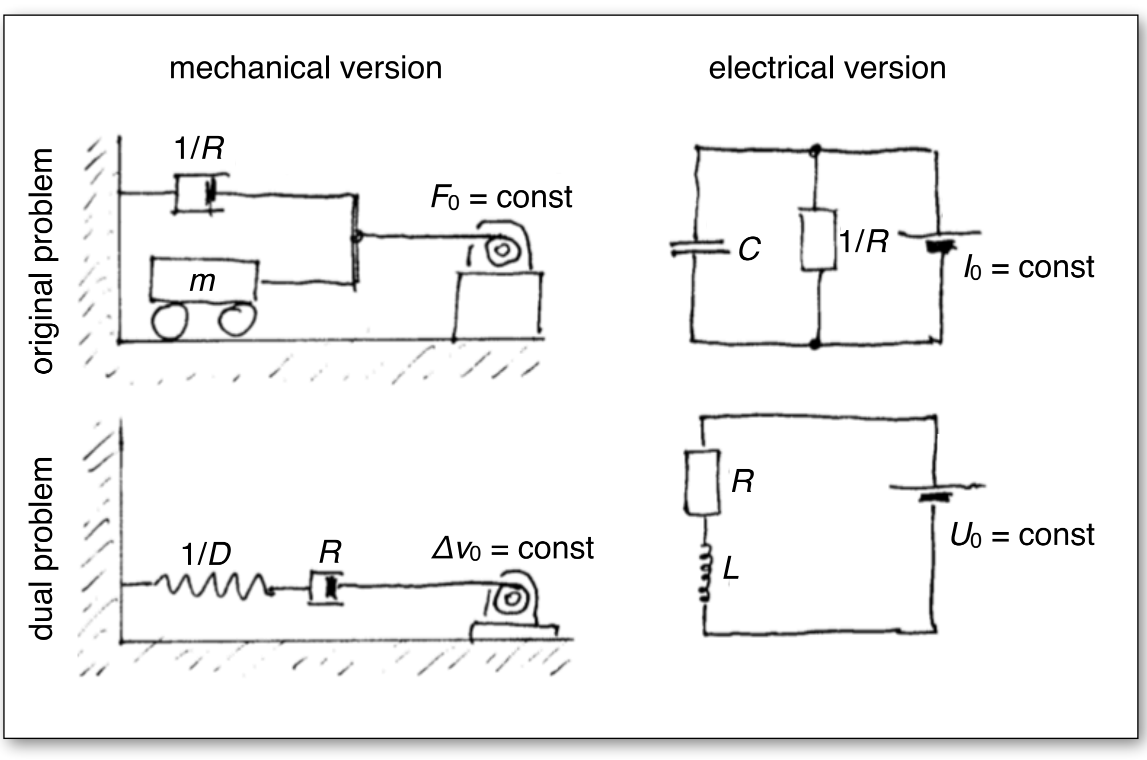

Fig. 9.4
A mechanical system with its electrical analog, as well as the two dual systems

9.4 Mechanical properties of materials

We had identified three different mechanical properties of bodies: inertia, described by the physical quantity mass m, elasticity, described by the spring constant D, and viscosity, described by a frictional resistance R_p.

The three quantities m, D and R_p. refer to an extended structure. However, each of these quantities also expresses a local property of a material. In addition to the local material quantities, only geometric quantities are included in the global quantities m, D and R_p.

Mass density

The local quantity describing the inertia is the mass density ρ. One divides the mass m contained in a region of space by the volume V of this region and obtains the average density. If the volume of the region is small against the total volume of the system under consideration, one omits the adjective “average” and simply speaks of the density “at the location” of the chosen space region. It is thus

$$\rho = \frac{m}{V}$$

Elastic modulus

A momentum current of strength F flows through an elastic rod of length s and cross-sectional area A. As long as the momentum current flows, the rod is shortened or lengthened by Δs relative to its normal length s. The relationship between F and Δs is described by Hooke’s law:

$$F = D \cdot \Delta s$$

The value of the global quantity D, the “spring constant”, depends on the dimensions of the bar. D is proportional to the cross-sectional area and inversely proportional to the length s:

$$D = E \cdot \frac{A}{s}$$

The factor of proportionality E is called the elastic modulus of the material. It depends only on the material of the bar. It is therefore

$$E = \frac{s}{A} \cdot D$$

Viscosity

We have already learned about the local quantity describing the dissipative behavior of matter: It is the viscosity η. It is related to the global quantity R_p via

$$\eta = \frac{s}{AR_p}$$

The description of the elastic and dissipative behavior of matter given here is highly simplified. In fact, neither can be described by a single number. A complete representation would show that both elastic modulus and viscosity are so-called tensors. Tensors are mathematical entities that require more than one numerical value to define them. For example, the elasticity tensor is determined by 21 independent numbers. If the material is isotropic, however, this number is reduced to 2. One of these is the elastic modulus just discussed, the other expresses how strongly the material deforms in the direction transverse to the applied force.

10

Oscillations

10.1 Kinematics and dynamics

Kinematics is concerned with the shape of the path of a body or particle, and it is concerned with how this path is traversed in time. It is therefore concerned with the function $\vec{r}(t)$. Mechanical processes are often classified according to kinematic criteria. Thus one speaks of

- rectilinear uniform motions;
- uniformly accelerated motions;
- uniform circular motions;
- harmonic motions;
- exponentially decaying motions;
- chaotic motions;
- etc.

However, such a classification by no means determines the dynamics of a process. One and the same kinematic type of motion can come about in quite different ways. Thus, a rectilinear uniform motion is present with a car driving at constant velocity on a straight stretch of the highway, but also with the famous force-free body of Newton's first law. The two processes have the same kinematics, but different dynamics.

10.2 Qualitative discussion of some examples of oscillations

We say there is an oscillation when the value of a physical quantity changes periodically, e.g. the momentum of a pendulum, the electric current in a resonant circuit or the reflectivity of a deciduous forest at λ = 500 nm. Here, of course, we restrict ourselves to mechanical oscillations, i.e., oscillations of quantities that play a role in mechanics. However, we do not define the term "oscillation" very narrowly. For example, we still speak of an oscillation when the periodic variation is modulated by an exponential function. Such an oscillation is called "damped".

Oscillations or oscillating systems can be characterized according to different criteria:

- What is the shape of the oscillation? (harmonic, sawtooth etc.)
- How many energy stores are involved in the oscillation?
- What proportion of the energy is dissipated per period?
- What energy and momentum currents flow into and out of the system?
- Does the system have characteristic frequencies?

We will first discuss some examples qualitatively from these points of view. We will see that kinematically identical oscillations can come about in quite different ways, i.e. have different dynamics.

One-dimensional elastic pendulum, Fig. 10.1

Oscillation shape: harmonic

The energy flows periodically from the bodies into the spring and back again. The momentum flows back and forth between the two bodies. Ideally, no energy is dissipated. The system has a single natural frequency.

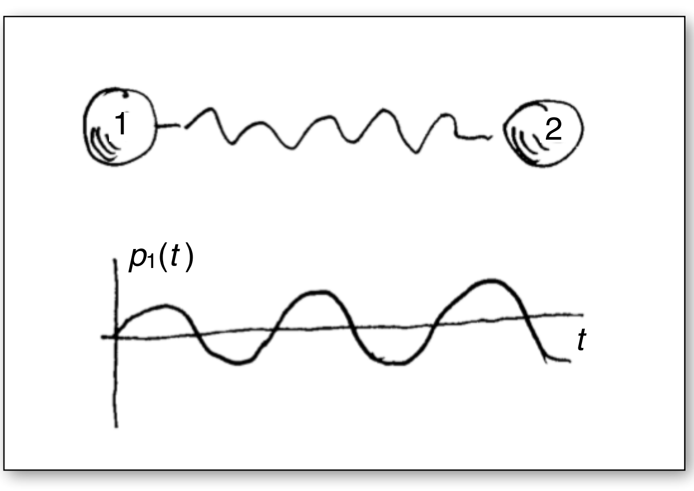

Fig. 10.1
One-dimensional elastic pendulum

Elastic ball between two hard walls, Fig. 10.2

Oscillation shape: rectangular

The energy remains all the time within the ball. The momentum is constant most of the time; only during the reversals momentum flows from the ball into the wall, or from the wall into the ball. Ideally, no energy is dissipated. The system does not have a specific oscillation period.

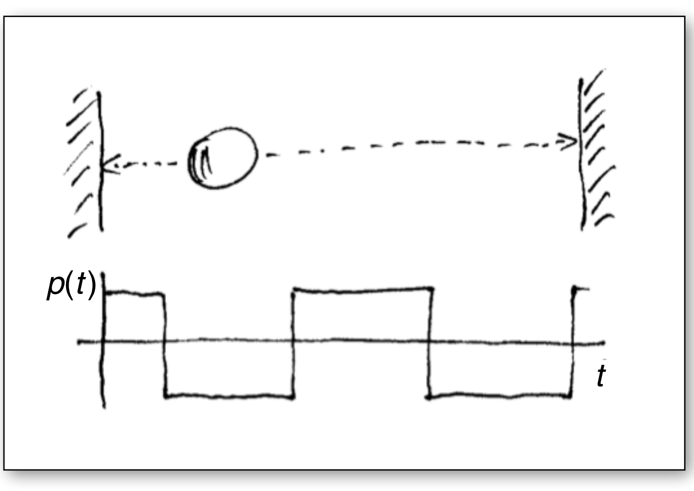

Fig. 10.2
An elastic ball is reflected back and forth between two hard walls.

Motor + oscillating body, Fig. 10.3

Oscillation shape: (almost) harmonic

Energy flows back and forth between the body and the motor. Momentum flows periodically from the body into the earth and back again. Ideally, no energy is dissipated. The system has no specific oscillation time.

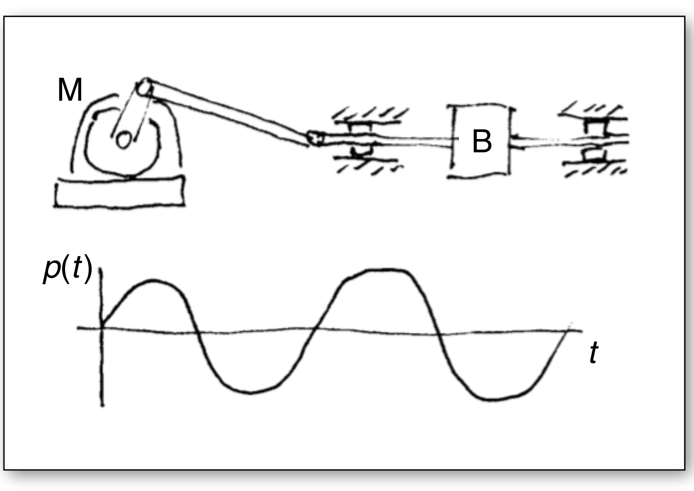

Fig. 10.3
The body B is moved back and forth by a motor M.

Relaxation oscillation, Fig. 10.4

Oscillation shape (of the amount of water in the upper container): sawtooth

A single energy store is periodically filled and emptied. A weak energy dissipation is necessary for the functioning. The oscillation time depends on the water flow.

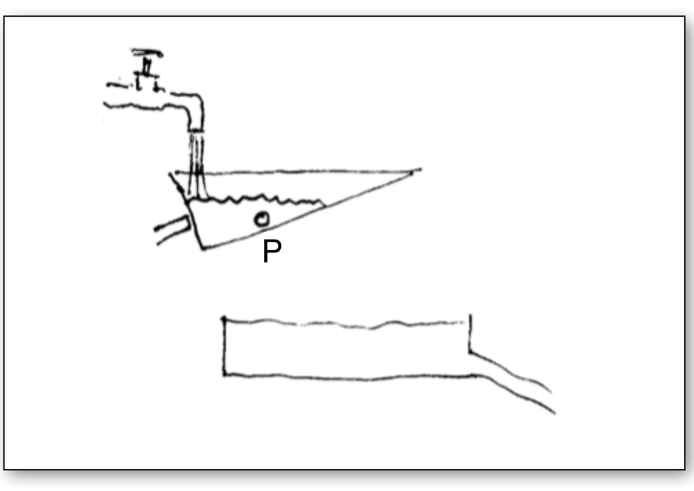

Fig. 10.4
The water tank with triangular profile tilts as soon as its center of mass has moved beyond the pivot point P to the right.

Harmonic relaxation oscillation, Fig. 10.5

Oscillation shape: harmonic

A single energy store is periodically filled and emptied. A strong energy dissipation is necessary for the functioning. The system has one single natural frequency.

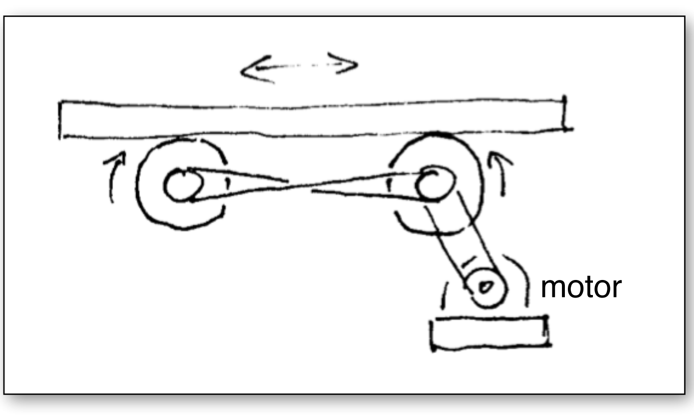

Fig. 10.5
The rod lies loosely on the rollers. It slides harmonically back and forth.

Damped oscillation, Fig. 10.6

Oscillation shape: harmonic with an exponentially decaying amplitude

Most of the energy flows back and forth between the spring and the sphere. A small amount is dissipated in the shock absorber for each oscillation. The system has a natural frequency.

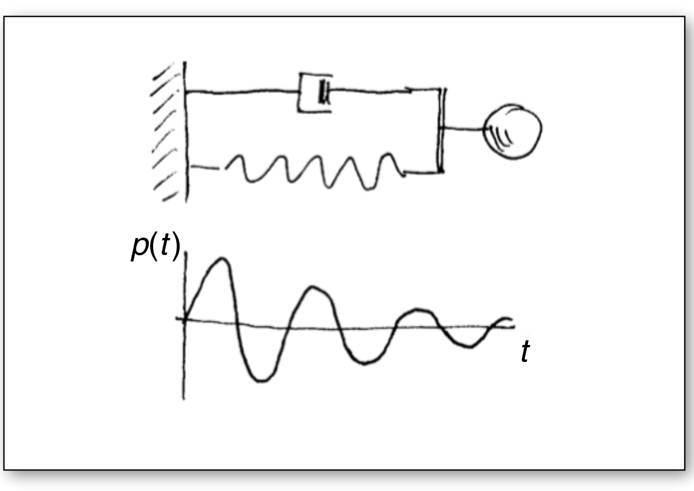

Fig. 10.6
Damped oscillation

Self-oscillation

They take place in most clocks (exceptions: sundial, hourglass, water clock). One uses a system, which executes oscillations as weakly damped as possible. The energy lost by dissipation is replaced by an energy flow from outside. The strength of this energy flow is controlled by the oscillator itself. As the damping increases, this type of oscillation steadily transitions to a relaxation oscillation. (Examples of vibrations that lie between these two types: string and wind instruments, squeaky door).

Driven oscillation, Fig. 10.7

Combination of the first (spring pendulum) and the third (motor + body)

Oscillation shape: harmonic

Depending on the frequency, the energy flows different ways. The system oscillates with any frequency.

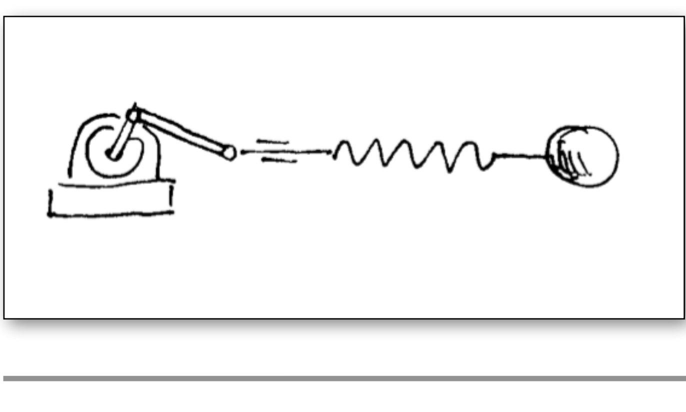

Fig. 10.7
Driven oscillation

10.3 The undamped elastic pendulum

The system, Fig. 10.8, consists of

- a spring (spring constant D);
- a body (mass m);
- the earth (mass infinite).

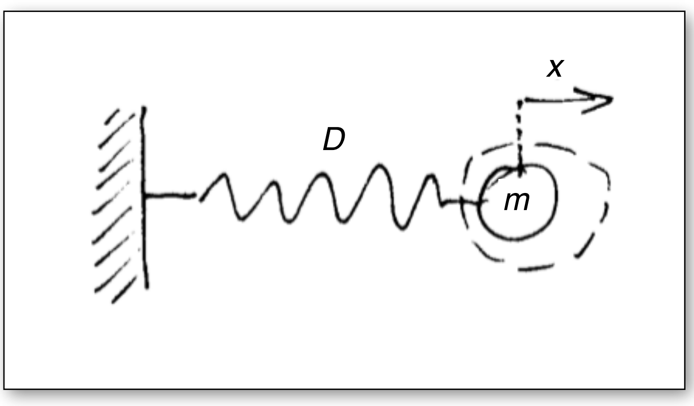

Fig. 10.8
Elastic pendulum

We apply Newton's second law (the balance equation for momentum) to the area surrounded by the dashed line,

$$\frac{dp}{dt} = F$$

insert

$$p = m\dot{x} \quad \text{and} \quad F = -Dx$$

and obtain

$$m\ddot{x} + Dx = 0$$

The solution of this differential equation is

$$x(t) = x_0 \sin(\omega t + \varphi) \quad \text{with} \quad \omega = \sqrt{\frac{D}{m}}$$

x_0 and φ define the initial conditions. x_0 is the amplitude, φ determines the position of the sine curve on the time axis.

We choose $\varphi = 0$, such that $x\,(t = 0) = 0$. From the solution $x(t)$ the values of the other variables can be calculated as a function of time. With

$$p(t) = m\dot{x}(t)$$

we get

$$p(t) = m\omega x_0 \cos(\omega t)$$

or

$$p(t) = p_0 \cos(\omega t) \quad \text{with} \quad p_0 = m\omega x_0$$

The force is obtained from $F\,(t) = -\,Dx\,(t)$

$$F(t) = -Dx_0 \sin(\omega t)$$

or

$$F(t) = F_0 \sin(\omega t) \quad \text{with} \quad F_0 = -Dx_0$$

It can be seen that the magnitude of the momentum current is maximum when the momentum itself is zero. The momentum flows periodically back and forth between the body and the earth.

The energy of the body is:

$$E_B = E_{B0} + \frac{p^2}{2m} = E_{B0} + \frac{p_0^{\,2}}{2m}\cos^2(\omega t)$$

or

$$E_B = E_{B0} + \frac{p_0^{\,2}}{2m}\frac{1}{2}(1 + \cos(2\omega t))$$

The energy of the spring is

$$E_S = E_{S0} + \frac{D}{2}x^2 = E_{S0} + \frac{D}{2}x_0^2 \sin^2(\omega t)$$

or

$$E_S = E_{S0} + \frac{Dx_0^{\,2}}{2}\frac{1}{2}(1 - \cos(2\omega t))$$

The amplitudes $p_0^2/(4m)$ and $Dx_0^2/4$ are equal to each other. The sum $E_B + E_S$ is therefore constant in time, i.e. energy flows back and forth between the body and the spring with the frequency 2ω.

We still calculate the strength of the energy flow between body and spring:

$$P = vF = \left(\omega x_0 \cos(\omega t)\right)\left(-Dx_0 \sin(\omega t)\right)$$
$$= -\frac{D\omega x_0^{\,2}}{2}\sin(2\omega t)$$

or

$$P = P_0 \sin(2\omega t) \quad \text{with} \quad P = -\frac{D\omega x_0^{\,2}}{2}$$

Fig. 10.9 shows various quantities of the spring pendulum as a function of time.

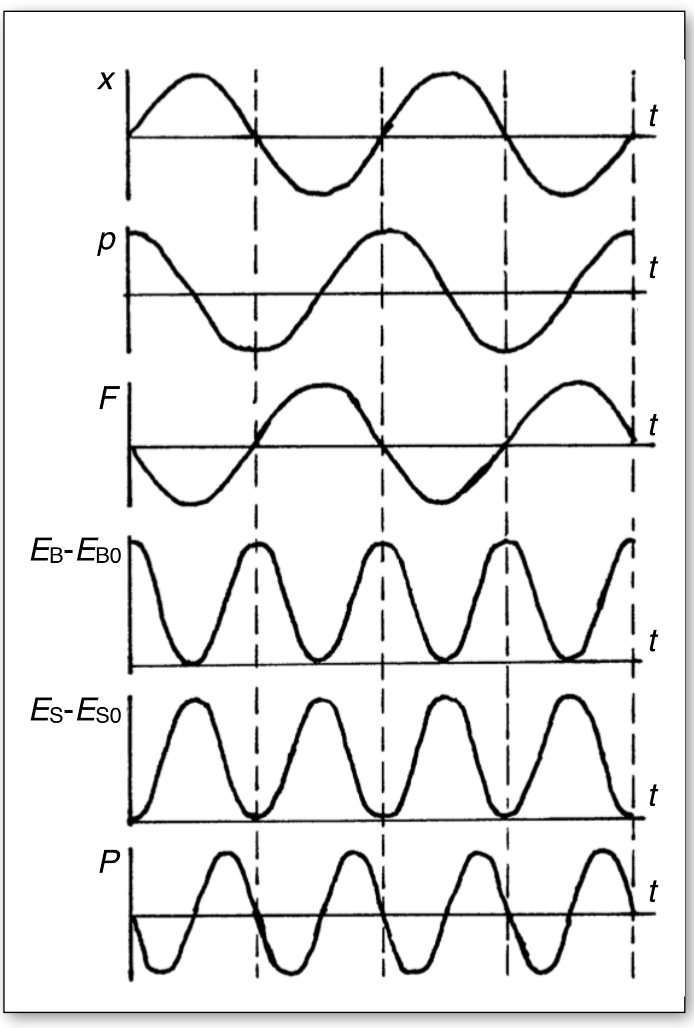

Fig. 10.9
Various quantities of the spring pendulum as a function of time

10.4 The damped elastic pendulum

Figure 10.10 schematically shows a spring pendulum with damping. The friction is represented by a shock absorber with

$$F_{\mathrm{R}} = -\frac{\dot{x}}{R} = -k\dot{x}$$

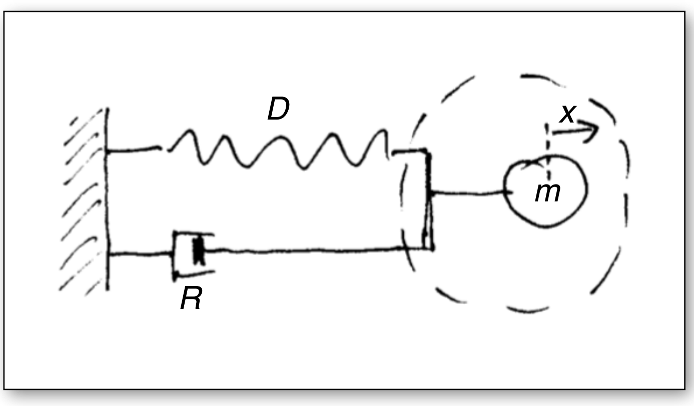

Fig. 10.10
Damped elastic pendulum

We apply Newton's second law to the area surrounded by the dashed line

$$\frac{dp}{dt} = F_{\mathrm{D}} + F_{\mathrm{R}}$$

With

$$p = m\dot{x}\ , \quad F_{\mathrm{D}} = -Dx \quad \text{and} \quad F_{\mathrm{R}} = -k\dot{x}$$

we obtain the differential equation

$$m\ddot{x} + k\dot{x} + Dx = 0$$

We insert the ansatz

$$x(t) = \mathrm{e}^{-\delta t}\left(x_1 \cos(\omega t) + x_2 \sin(\omega t)\right)$$

and obtain

$$\delta = \frac{k}{2m} \quad \text{and} \quad \omega = \pm\sqrt{\frac{D}{m} - \frac{k^2}{4m^2}}$$

If

$$\frac{D}{m} - \frac{k^2}{4m^2} > 0$$

this solution represents a harmonic oscillation with exponentially decreasing amplitude, Fig. 11. For

$$\frac{D}{m} - \frac{k^2}{4m^2} < 0$$

the harmonic oscillation becomes a sum of two exponential functions. To exclude solutions with positive exponents, it is best to make a new solution ansatz for this case

$$x(t) = x_1 \mathrm{e}^{-\alpha_1 t} + x_2 \mathrm{e}^{-\alpha_2 t}$$

Substituting into the differential equation yields

$$\alpha_{1,2} = \frac{k}{2m} \pm \sqrt{\frac{k^2}{4m^2} - \frac{D}{m}}$$

The decay process is fastest when it is *critically damped*, i.e. when

$$\frac{k^2}{4m^2} = \frac{D}{m}$$

If

$$\frac{k^2}{4m^2} < \frac{D}{m}$$

energy flows back and forth between the spring and the body. However, part of the energy flows to the shock absorber and is dissipated. The energy loss per period is obtained by comparing the energy content of the spring in two successive maxima.

We set $x_2 = 0$ and calculate the energy content of the spring in the maxima, i.e. when $\cos \omega t = 1$

$$E_{\mathrm{S,max}} = E_{\mathrm{S,0}} + \frac{D}{2} x_1^{\,2} \mathrm{e}^{-2\delta t}$$

It follows

$$\frac{dE_{\mathrm{S,max}}}{dt} = -\delta D x_1^{\,2} \mathrm{e}^{-2\delta t}$$

The Q factor of the system is defined as

$$Q = 2\pi \cdot \frac{\text{energy which per cycle flows back and forth}}{\text{energy dissipated per cycle}}$$

We obtain

$$Q = 2\pi \frac{E_{\mathrm{S,max}} - E_{\mathrm{S,0}}}{-\dfrac{dE_{\mathrm{S,max}}}{dt} T} = \frac{2\pi}{T} \frac{(D/2) x_1^{\,2} \mathrm{e}^{-2\delta t}}{\delta D x_1^{\,2} \mathrm{e}^{-2\delta t}} = \frac{\omega}{2\delta}$$

Here we have used $\omega = 2\pi T$.

With

$$\delta = \frac{k}{2m} \quad \text{and} \quad \omega = \sqrt{\frac{D}{m}}$$

we finally get

$$Q = \frac{\sqrt{Dm}}{k}$$

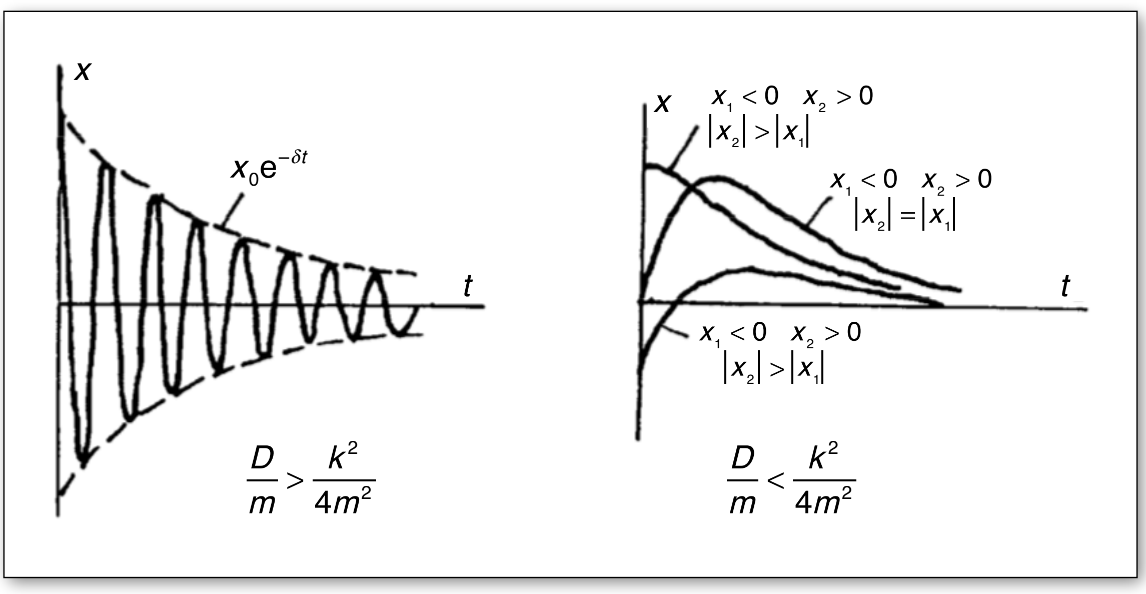

Fig. 10.11
Oscillation curve of a damped spring pendulum for different damping and for different initial conditions

10.5 Driven oscillations

We consider two arrangements, each consisting of a mass point, a spring, a shock absorber, and an energy source Q, Fig. 10.12. In one of them the energy source causes a harmonic displacement (and hence a harmonic velocity), in the other a harmonic momentum current. (The electrical analogues of these energy sources are the voltage-stabilized and current-stabilized ac power supplies).

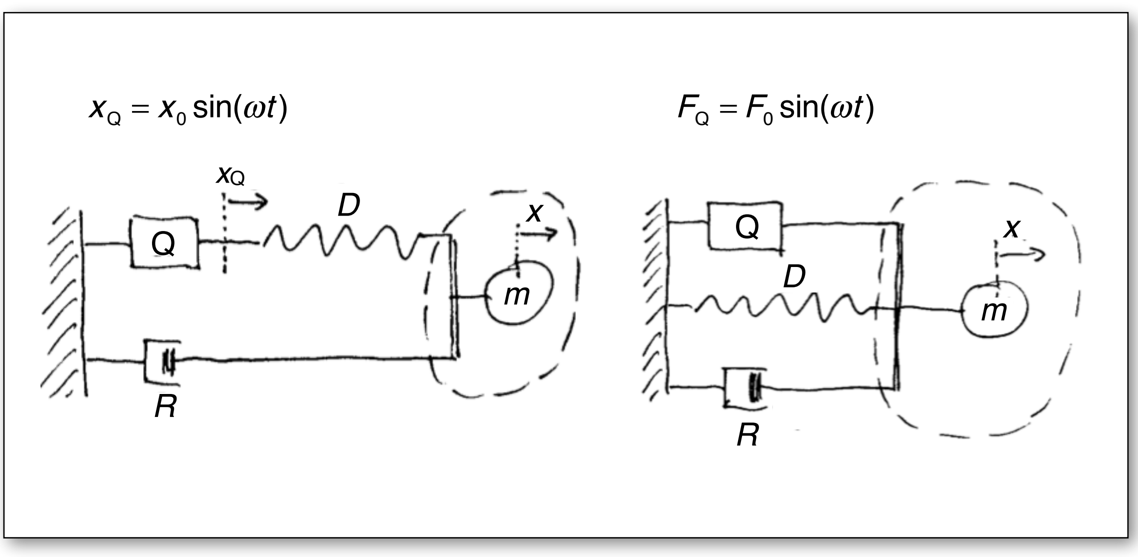

Fig. 10.12
The body of mass *m* performs driven oscillations. Applying the balance equation for the momentum to the dashed area leads to the same differential equation in both cases.

Applying the balance equation for momentum (Newton's second law) to the dashed area, the same differential equation is obtained for both arrangements.

$$dp/dt = F_D + F_R \qquad\qquad dp/dt = F_Q + F_D + F_R$$

$$p = m\dot{x} \qquad\qquad p = m\dot{x}$$

$$F_Q = F_0 \sin(\omega t)$$

$$F_D = -D(x - x_0) \qquad\qquad F_D = -Dx$$

$$F_R = -k\dot{x} \qquad\qquad F_R = -k\dot{x}$$

$$x_Q = x_{Q0}\sin(\omega t)$$

We call $Dx_{Q0} = F_0$ and obtain

$$m\ddot{x} + k\dot{x} + Dx = F_0 \sin(\omega t) \qquad\qquad m\ddot{x} + k\dot{x} + Dx = F_0 \sin(\omega t)$$

The general solution of this inhomogeneous differential equation is obtained as the sum of the general solution of the homogeneous equation plus a special solution of the inhomogeneous one. The solution of the homogeneous differential equation decays with time. It describes a transient process. We ask here only for that part of the solution which remains after the transient process has ended. We make the ansatz:

$$x(t) = x_0 \sin(\omega t - \varphi)$$

Substituting into the differential equation results in conditions for x_0 and φ:

$$\tan\varphi = \frac{\omega k}{m(\omega_0^2 - \omega^2)} \quad \text{where} \quad \omega_0 = \sqrt{\frac{D}{m}}$$

$$x_0 = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + k^2\omega^2}}$$

With

$$v(t) = \dot{x}(t)$$

we also get the velocity as a function of time

$$v(t) = v_0 \cos(\omega t - \varphi)$$

where

$$v_0 = \frac{\omega F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + k^2\omega^2}}$$

This allows to calculate the energy dissipated in the shock absorber:

$$P = v \cdot F_R = kv^2 = kv_0^2 \cos^2(\omega t - \varphi)$$

The time average of the energy dissipated in the shock absorber can be considered a measure of the intensity of the oscillation. Since the time average of $\cos^2(\omega t - \varphi)$ is equal to 1/2, we obtain for the time average $\bar{P}$ of the energy flow

$$\bar{P} = \frac{1}{2}kv_0^2$$

We substitute for v_0 the expression calculated earlier and obtain:

$$\bar{P}(\omega) = \frac{F_0^2}{2}\frac{k\omega^2}{m^2(\omega_0^2 - \omega^2)^2 + k^2\omega^2}$$

As could be expected, the dissipated energy depends on the frequency. For $\omega = \omega_0$ it has its maximum; the oscillation is most intense. The oscillator is said to be in *resonance* with the exciter. Fig. 10.13 shows the relationship graphically, for four different *k* values, i.e., for different dampings. The curves are called resonance curves.

Often, instead of the energy current, the position amplitude is represented as a function of the frequency. Such a curve also shows the resonance, but only roughly, because in such a representation the position of the maximum changes with the frequency. One could have represented several other quantities as a function of the frequency, for example the acceleration. In this case, the maximum would again have been at a different position. The most reasonable representation is the energy flow – or alternatively the velocity, because its maximum is also fixed at ω_0.

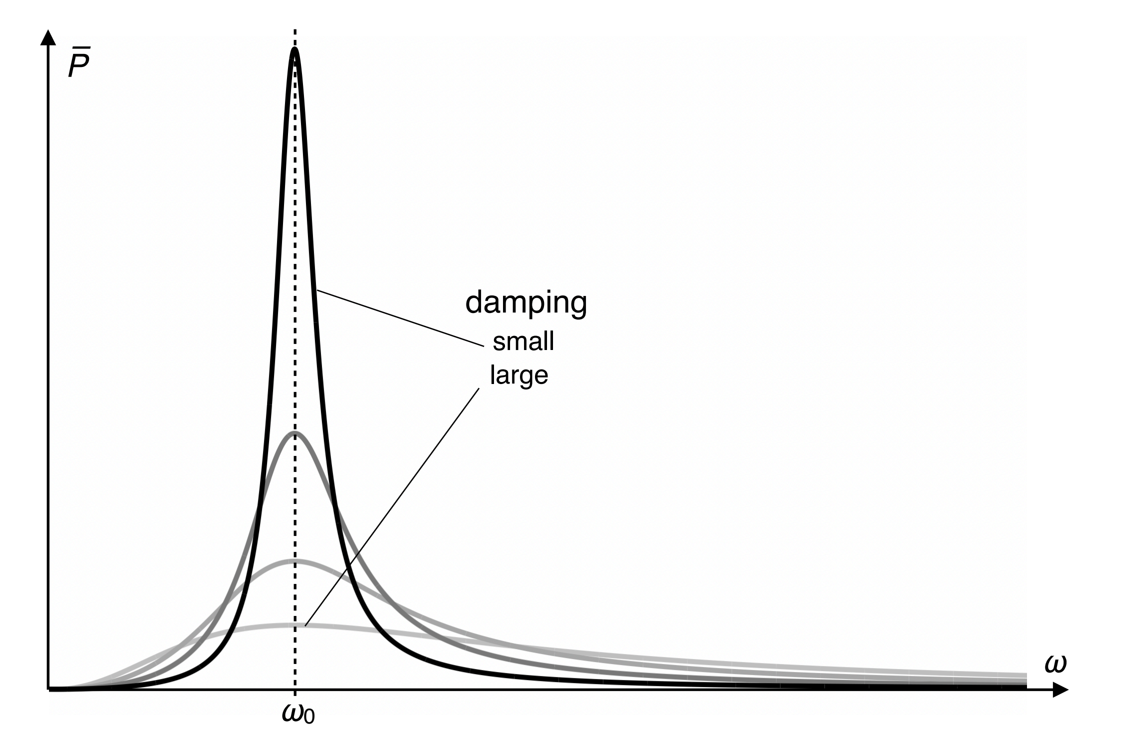

Fig. 10.13
The smaller the damping *k*, the higher and the narrower the peak of the resonance curve.

Let us finally consider the phase, or more precisely: the phase difference between the exciting force and the velocity of the oscillator, i.e. between the two factors on the right side of the equation

$P = v \cdot F_R$.

For the resonance case, i.e. for $\omega = \omega_0$ it is equal to zero.

Force and velocity being in phase means optimum energy transfer to the shock absorber. In the analogous electrical case, one would say that the “reactive power” is zero.

10.6 Electrical analogs and dual arrangements

Fig. 10.14 shows the electrical analogs as well as their dual arrangements of the systems discussed in Sections 10.3 through 10.5.

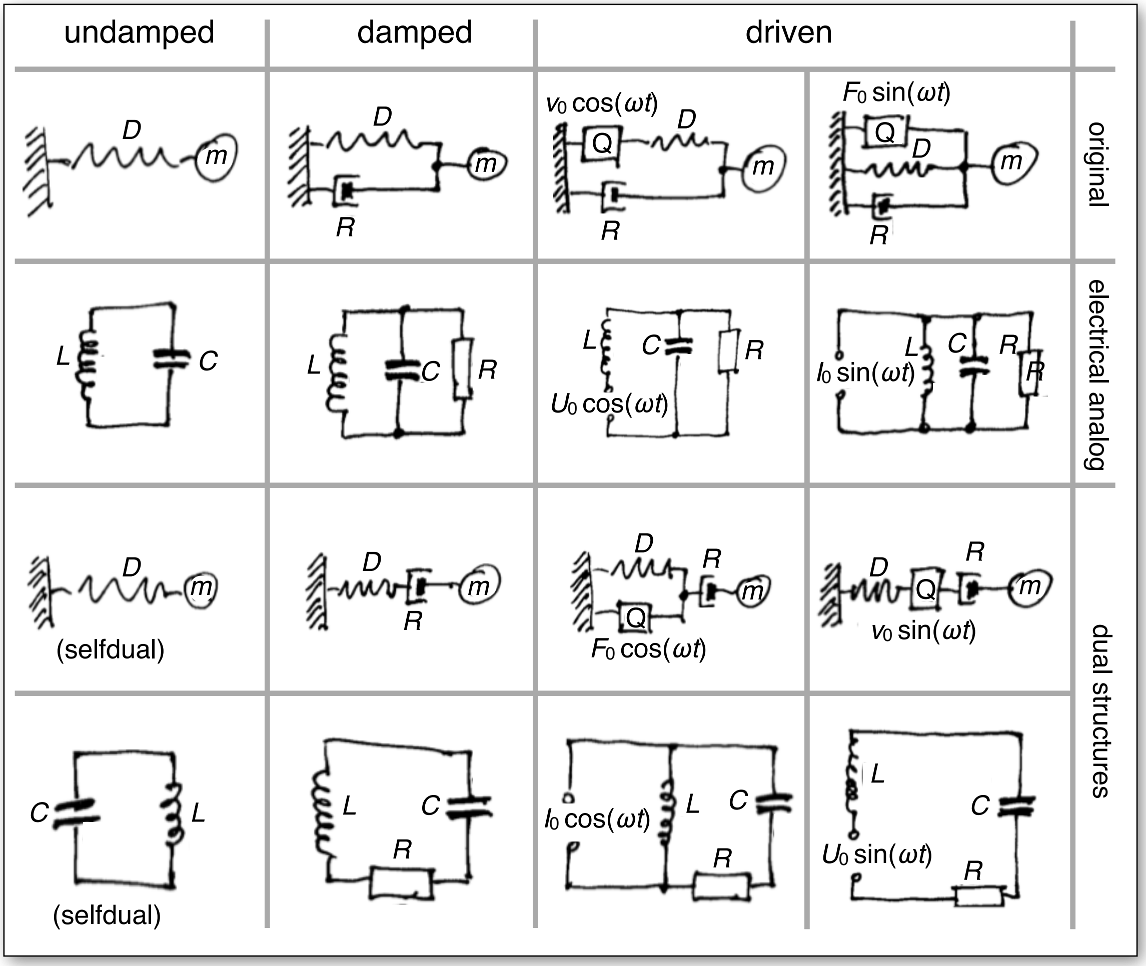

Fig. 10.14
Undamped, damped and forced oscillating systems as well as electrical analogs and dual systems

10.7 Two coupled spring pendulums

The arrangement of Fig. 10.15, which we will discuss next, provides an important basis for problems in atomic, molecular, and solid-state physics. The notations can be seen in the figure.

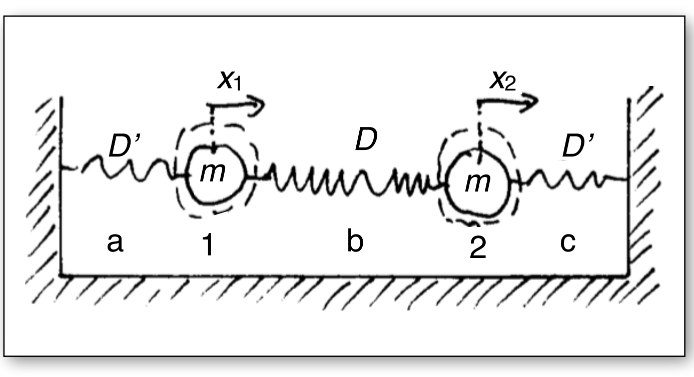

Fig. 10.15
Oscillating system with two degrees of freedom

In general, the two bodies perform a confusing motion. The mathematical solution of the problem shows that the motion, as well as the energy and momentum flows, are simpler than they appear. We apply the balance equation for momentum to the two regions surrounded by dashed lines in Fig. 10.15:

$$\frac{dp_1}{dt} = \sum_i F_{1,i} \qquad \frac{dp_2}{dt} = \sum_i F_{2,i}$$

With

$$p_1 = m\dot{x}_1 \quad p_2 = m\dot{x}_2$$

$$F_{1,a} = -D'x_1 \quad F_{1,b} = -D(x_1 - x_2) \quad F_{2,b} = -D(x_2 - x_1) \quad F_{2,c} = -D'x_2$$

we obtain the two coupled differential equations:

$$\text{(I)} \quad m\ddot{x}_1 + D(x_1 - x_2) + D'x_1 = 0$$

$$\text{(II)} \quad m\ddot{x}_2 + D(x_2 - x_1) + D'x_2 = 0$$

By addition (I) + (II) and subtraction (I) - (II) two new differential equations are obtained:

$$m(\ddot{x}_1 + \ddot{x}_2) + D'(x_1 + x_2) = 0$$

$$m(\ddot{x}_1 - \ddot{x}_2) + 2D(x_1 - x_2) + D'(x_1 - x_2) = 0$$

We introduce the new coordinates

$$q_1 = x_1 + x_2$$

and

$$q_2 = x_1 - x_2$$

The old coordinates depend on the new ones according to

$$x_1 = (1/2)(q_1 + q_2)$$

bzw.

$$x_2 = (1/2)(q_1 - q_2).$$

Using the new coordinates, the differential equations are

$$m\ddot{q}_1 + D'q_1 = 0$$

$$m\ddot{q}_2 + (2D + D')q_2 = 0$$

They are decoupled and can therefore be solved independently. We take the solution from section 10.3:

$$q_1(t) = q_{0,1}\sin(\omega_1 t + \varphi_1) \quad \text{with} \quad \omega_1 = \sqrt{\frac{D'}{m}}$$

$$q_2(t) = q_{0,2}\sin(\omega_2 t + \varphi_2) \quad \text{with} \quad \omega_2 = \sqrt{\frac{2D + D'}{m}}$$

From this the old coordinates can be calculated:

$$x_1(t) = \frac{1}{2}\left[q_{0,1}\sin(\omega_1 t + \varphi_1) + q_{0,2}\sin(\omega_2 t + \varphi_2)\right]$$

$$x_2(t) = \frac{1}{2}\left[q_{0,1}\sin(\omega_1 t + \varphi_1) - q_{0,2}\sin(\omega_2 t + \varphi_2)\right]$$

Thus, the oscillation of each of the coordinates is a superposition of two harmonic oscillations with frequencies ω_1 and ω_2.

Discussion

We set the initial conditions $\varphi_1 = 0$ and $\varphi_2 = 0$.

Normal modes

If $q_{10} \neq 0$ and $q_{20} = 0$, then x_1 and x_2 oscillate harmonically with the frequency ω_1 with the same amplitude and phase. If $q_{1,0} = 0$ and $q_{2,0} \neq 0$, then x_1 and x_2 oscillate harmonically with the frequency ω_2, with the same amplitude but in opposite phase. In these two cases, the system is said to perform *natural oscillations* or *normal mode oscillations*. Therefore q_1 and q_2 are also called the *normal coordinates* of the system.

Any state can be described as a superposition of the natural oscillations of the system.

Figure 10.16 shows the path of momentum and energy for both natural oscillations in two different oscillation states each.

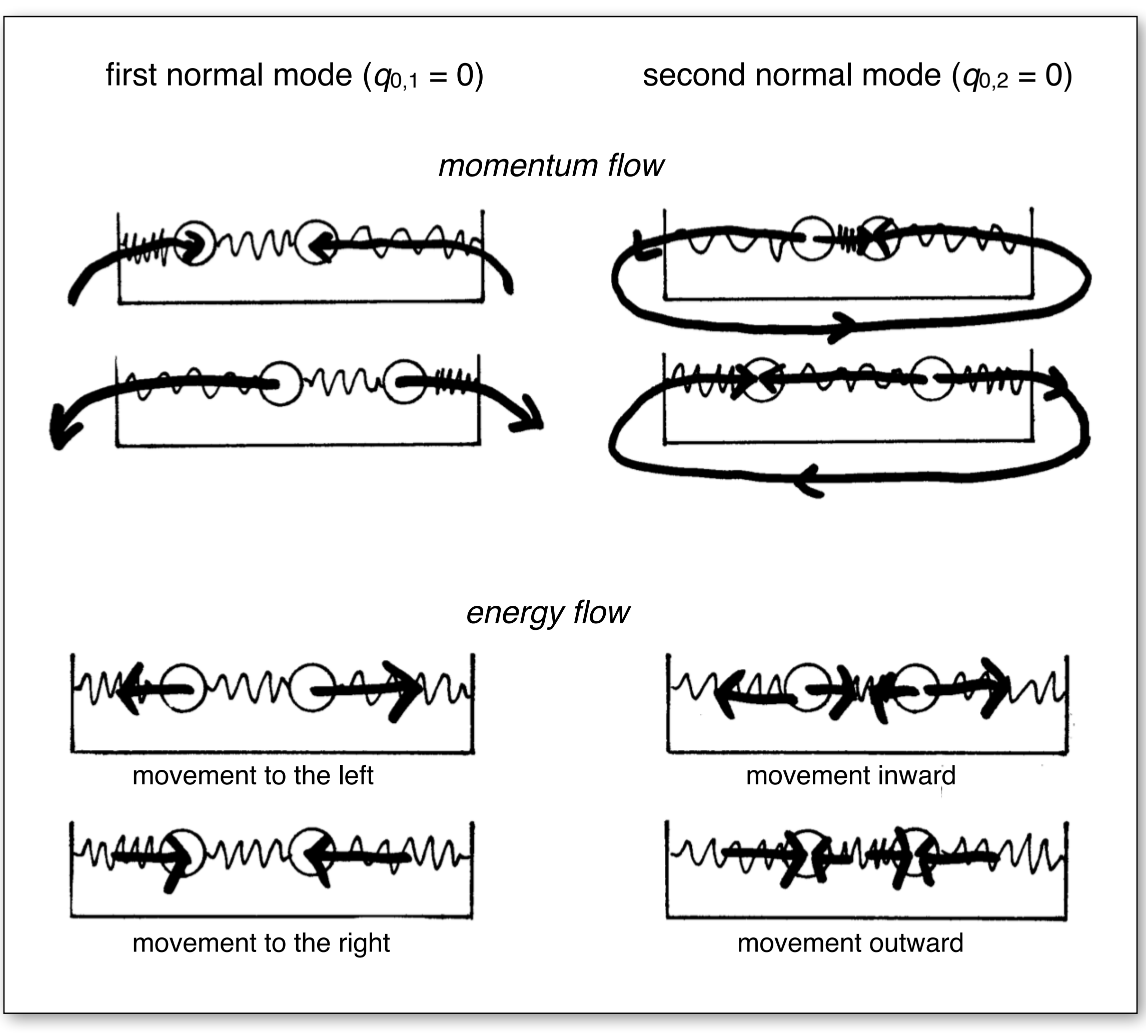

Fig. 10.16
Momentum and energy currents for the two normal modes of the system

Simple special case $D \ll D'$

If $D \ll D'$, i.e. if the middle spring is weak against the outer springs, the system can be considered as two spring pendulums (one body and one spring with D' each), which are weakly coupled to each other by the spring D. If $q_{1,0} = q_{2,0} = q_0$ is chosen as initial condition, a simple motion pattern is obtained:

$$x_1(t) = \frac{1}{2}q_0\left[\sin(\omega_1 t) + \sin(\omega_2 t)\right]$$

$$x_2(t) = \frac{1}{2}q_0\left[\sin(\omega_1 t) - \sin(\omega_2 t)\right]$$

After renaming $\omega_1 = \omega - \Delta\omega$ and $\omega_2 = \omega + \Delta\omega$ and using the well-known trigonometric formulas, we get:

$$x_1(t) = q_0\sin(\omega t)\cdot\cos(\Delta\omega t)$$

$$x_2(t) = -q_0\cos(\omega t)\cdot\sin(\Delta\omega t)$$

Because of $D \ll D'$, we have $\Delta\omega \ll \omega$. Therefore, $x_1(t)$ and $x_2(t)$ are harmonic oscillations of frequency ω *modulated* with the frequency $\Delta\omega$, Fig. 10.17. The energy flows back and forth between the right and the left spring pendulums with frequency $2\Delta\omega$. Within each pendulum, it flows back and forth at the high frequency 2ω between the body and the corresponding spring. Such an oscillation process is called a *beat*.

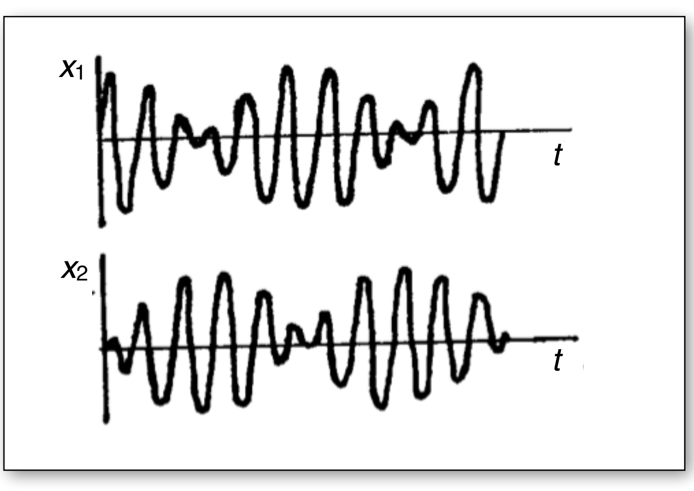

Fig. 10.17
The special case $D \ll D'$

Simple special case $D' \ll D$

From $D' \ll D$ follows $\omega_1 \ll \omega_2$. The system can be considered as a spring pendulum consisting of the two bodies and the central spring, which is weakly coupled (via D') to the earth, Fig. 10.18. For $D' \to 0$, the low frequency approaches zero, and the first natural oscillation turns into a translation.

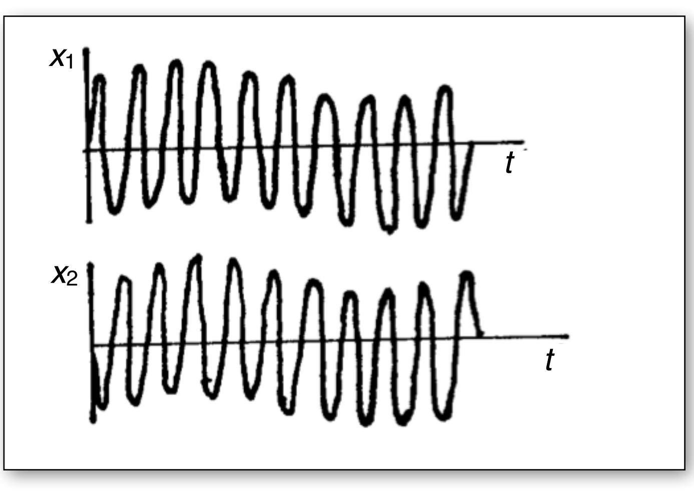

Fig. 10.18
The special case $D' \ll D$

10.8 Driven oscillations of two coupled pendulums

Fig. 10.19 shows the arrangement. One end of the pendulums is driven according to

$$x_Q = x_{Q0} \sin(\omega t)$$

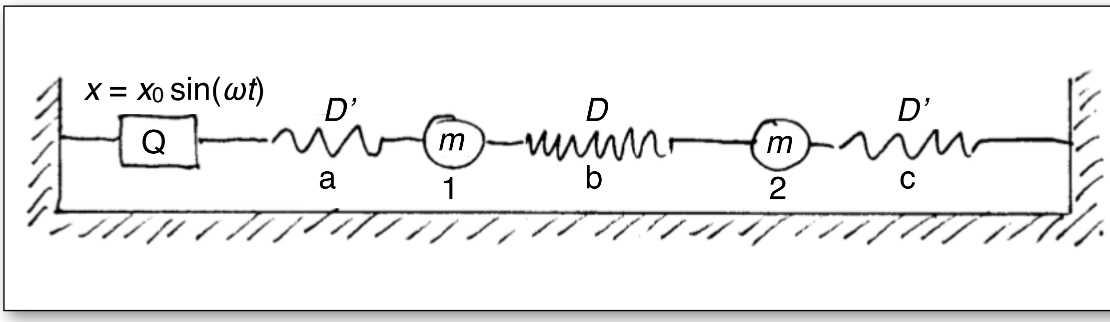

Fig. 10.19
Arrangement for generating driven oscillations of two coupled spring pendulums

Thereby we get

$$F_{1a} = -D'\left[x_1 - x_{Q0} \sin(\omega t)\right]$$

(instead of $F_{1a} = - D'x_1$). Thus the differential equations become:

$$\text{(I)}\quad m\ddot{x}_1 + D(x_1 - x_2) + D'x_1 = D'x_{Q0} \sin(\omega t)$$

$$\text{(II)}\quad m\ddot{x}_2 + D(x_2 - x_1) + D'x_2 = 0$$

Addition (I) + (II) and subtraction (I) - (II) yield the decoupled differential equations for q_1 and q_2:

$$m\ddot{q}_1 + D'q_1 = D'x_{Q0} \sin(\omega t)$$

$$m\ddot{q}_2 + (2D + D')q_2 = D'x_{Q0} \sin(\omega t)$$

These are two differential equations for common driven oscillations, like those in section 10.5 (however, we have assumed that there is no damping). If we draw the resonance curve, i.e. vary ω and observe the oscillation state, we find the following: If $\omega = \omega_1$, i.e. if ω *is* equal to the frequency of the first normal mode, this first normal mode is strongly excited. It is “in resonance with the exciter”. The strength of the energy flow from the exciter to the pendulum has a maximum. The analog is true if the excitation frequency is $\omega = \omega_2$.

10.9 Degrees of freedom

The number f of degrees of freedom indicates by how many independent time functions a system is described. In the example of section 10.7 $f = 2$, because the system is completely described by $x_1(t)$ and $x_2(t)$ or by $q_1(t)$ and $q_2(t)$ or by $p_1(t)$ and $p_2(t)$. Sometimes f is also called the *degree of freedom*.

Often, with a suitable choice of coordinates, each degree of freedom can be assigned to a simple motion process: a harmonic oscillation (vibrational degree of freedom), a rectilinear uniform motion (translational degree of freedom), a uniform rotation (rotational degree of freedom) …

Examples

- two coupled pendulums, see section 10.7: $q_1(t)$, $q_2(t)$;
- free masspoint $x(t)$, $y(t)$, $z(t)$;
- further examples, see Fig. 10.20.

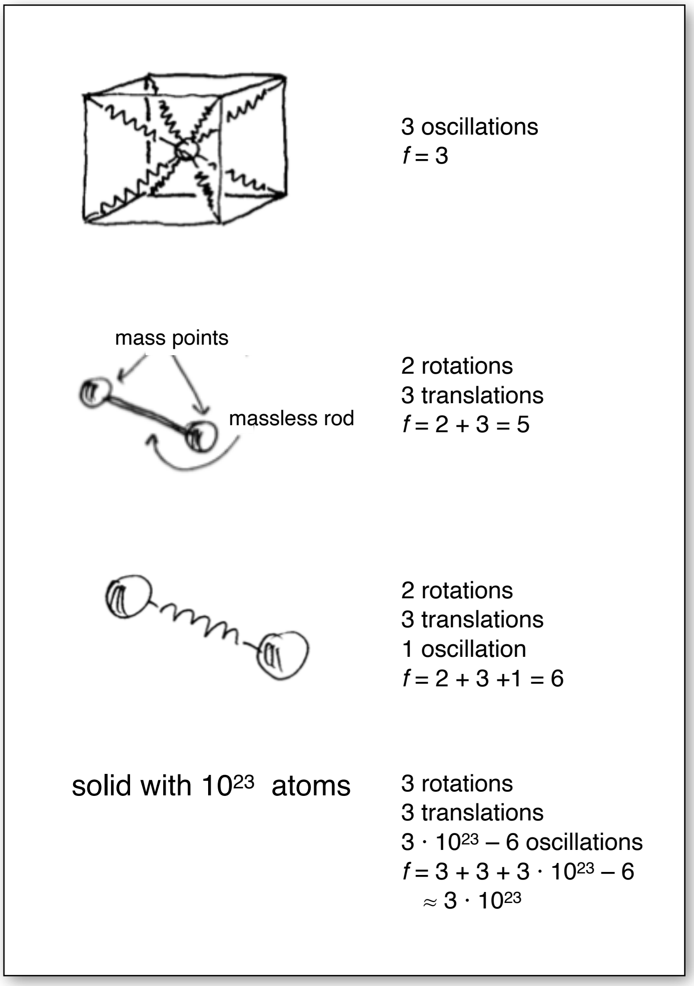

Fig. 10.20
Examples of some systems with degrees of freedom of different nature

10.10 Twelve coupled pendulums

The arrangement of Fig. 10.21a represents a one-dimensional model of a crystal. The system has 12 degrees of freedom and thus twelve normal modes of oscillation. Fig. 10.21b shows the shape of the first, second, third and twelfth normal mode. The arrows represent the length of the traversed paths between two reversal points. For the sake of clarity, this paths are shown transverse to the extension of the chain.

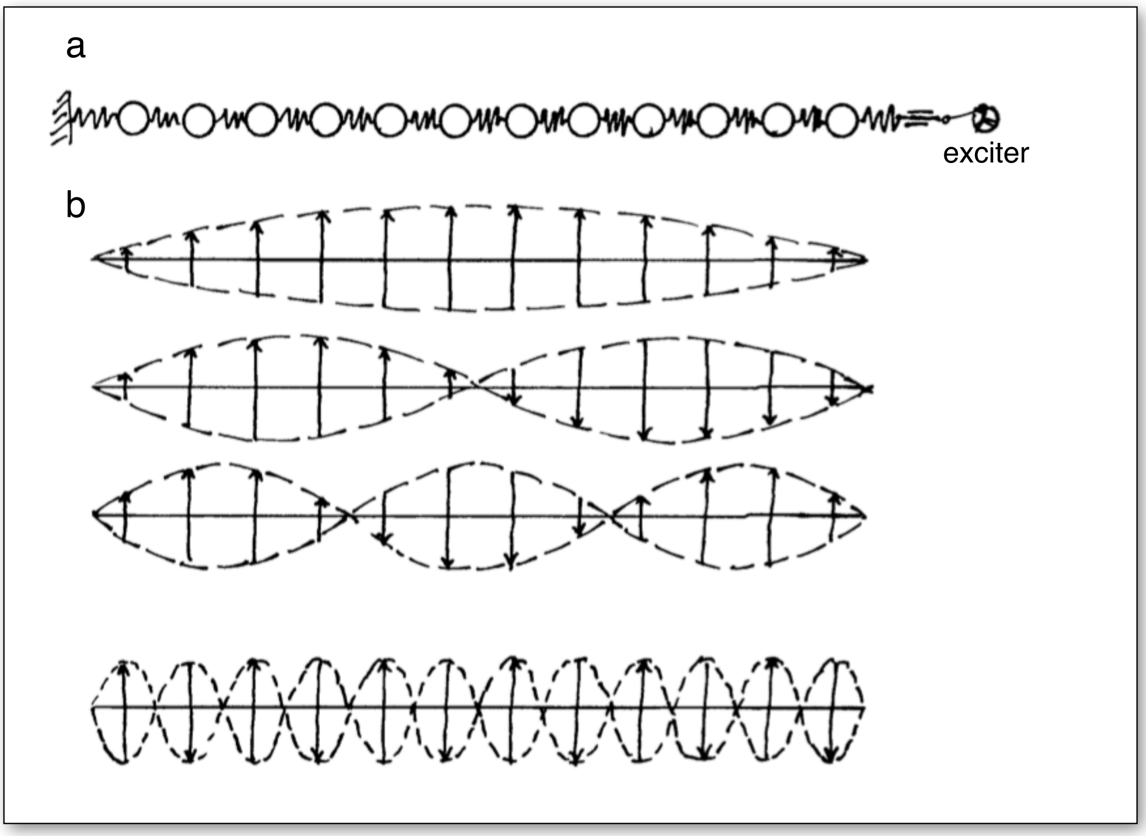

Fig. 10.21
(a) One-dimensional model of a crystal. (b) 1st, 2nd, 3rd and 12th normal mode

The 12 vibrational states look similar to standing waves, and each can be assigned a wavelength λ. One calls

$$k = 2\pi/\lambda$$

the *wave number*. The function $\omega = \omega(k)$ is called the *dispersion relation* of the arrangement, Fig. 10.22. The curve that can be laid through the points breaks off at $k = \pi/a$, where a is the distance between adjacent mass points. The dispersion relation of an arrangement with many more mass points looks almost the same, only the points are much closer together and the curve is practically continuous. Also in real crystals one measures dispersion relations which have this form.

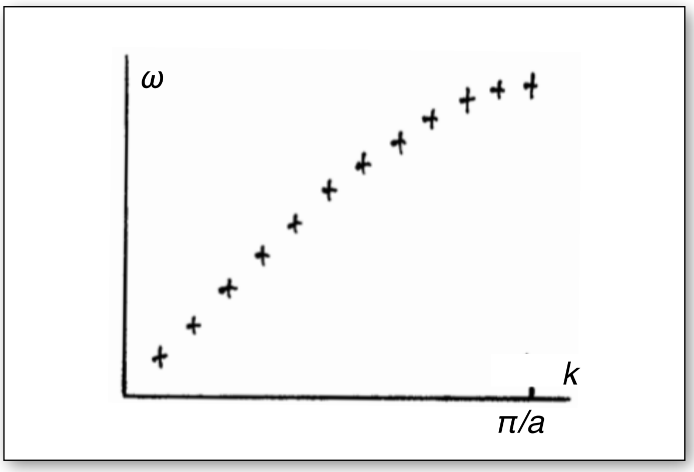

Fig. 10.22
Graphical representation of the dispersion relation of a system of 12 coupled oscillators

11

Chaotic processes

11. Chaotic processes

The processes we considered in Section 9.3 and in Chapter 10 are, from a certain point of view, unrealistic special cases. In fact, they have the following peculiarity: if the initial state of the system is given, one can use the corresponding differential equations to calculate the state of the system at a time arbitrarily far in the future and at a time arbitrarily far in the past. If one knows the initial state with some imprecision, one can also calculate the final state with some degree of inaccuracy. It is typical of this type of process that a small variation in the values of the variables characterizing the initial state results in a small variation in the final state values.

Actually, real systems almost always behave differently. If one asks for a final state which is too far in the future or for an earlier state which is far in the past, one finds that a small variation of the values of the initial state results in a very large variation of the values of the state to be calculated. The magnitude of this variation generally grows exponentially with the time interval between the initial state and the state to be calculated. Since one can characterize a state in principle only with a limited accuracy, it follows that one cannot in principle calculate the state at a time far in the future or far in the past.

Mathematically, such behavior manifests itself in the fact that the differential equation describing the system is non-linear. Such nonlinear systems are said to behave chaotically. In fact, the behavior of these systems looks chaotic – in the colloquial sense. But beware. Not every process that looks chaotic is also chaotic in the previously explained sense. A system of 20 linearly coupled oscillators, for example, can perform movements which look completely disordered, i.e. chaotic. Nevertheless, the system is not chaotic in the physical sense: its later states and its past states can be calculated from a given initial state.

Let us consider an example. Fig. 11.1a shows two gliders on an air track coupled by a spring. The system is excited so that the gliders oscillate against each other, but their center of mass remains at rest. Each glider moves sinusoidally, Fig. 11.1b. If an initial state is given, the states at any other times can be calculated. The system does not behave chaotically.

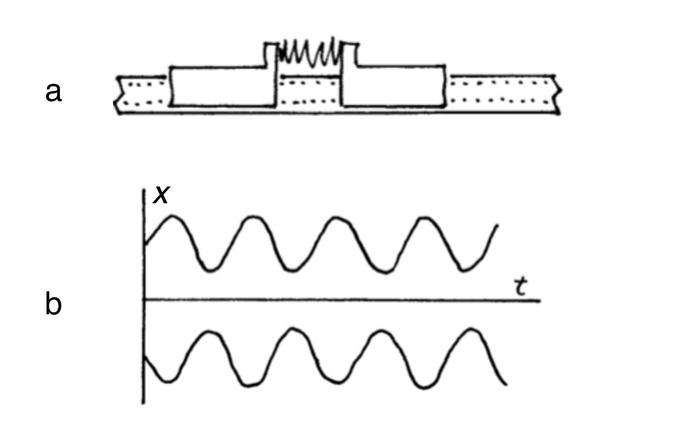

Fig. 11.1
(a) Two gliders on the air track swing about their common center of mass.
(b) Displacement-time graph of the two gliders

Fig. 11.2a shows a glider on an inclined air track. At the lower end of the track is a spring buffer which is quite hard. The glider moves accelerated to the left, is "reflected", moves again to the right, reverses again, and so on. The displacement-time graph consists of downward open parabolas, Fig. 11.2b. Also this system does not behave chaotically.

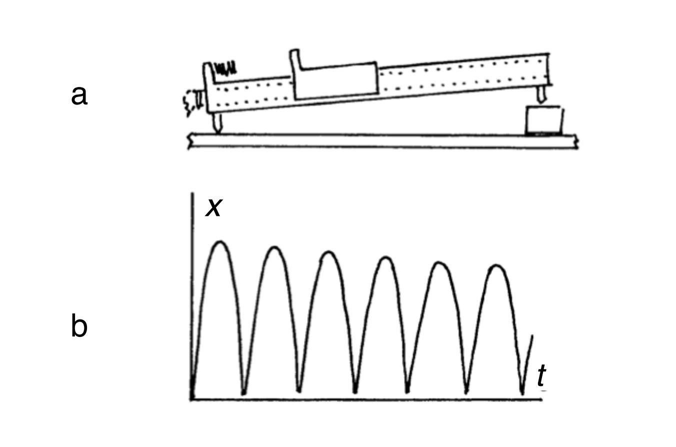

Fig. 11.2
(a) A single glider moves like a bouncing ball.
(b) Displacement-time graph of the glider

We now combine the two systems considered before, Fig. 11.3: Two coupled gliders as in Fig. 11.1 can move on an inclined air track as in Fig. 11.2.

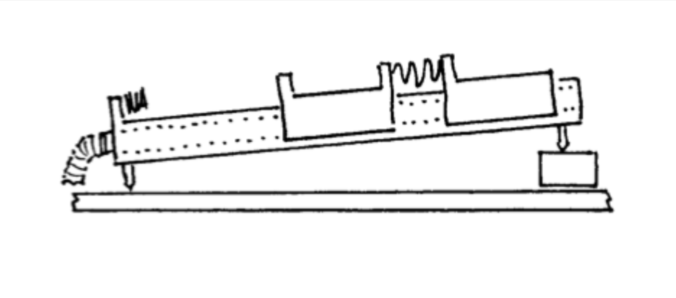

Fig. 11.3
Combination of the systems of Fig. 11.1 and Fig. 11.2

It turns out that the motion of the system is chaotic. Fig. 11.4 shows the displacement-time graph of one of the two gliders.

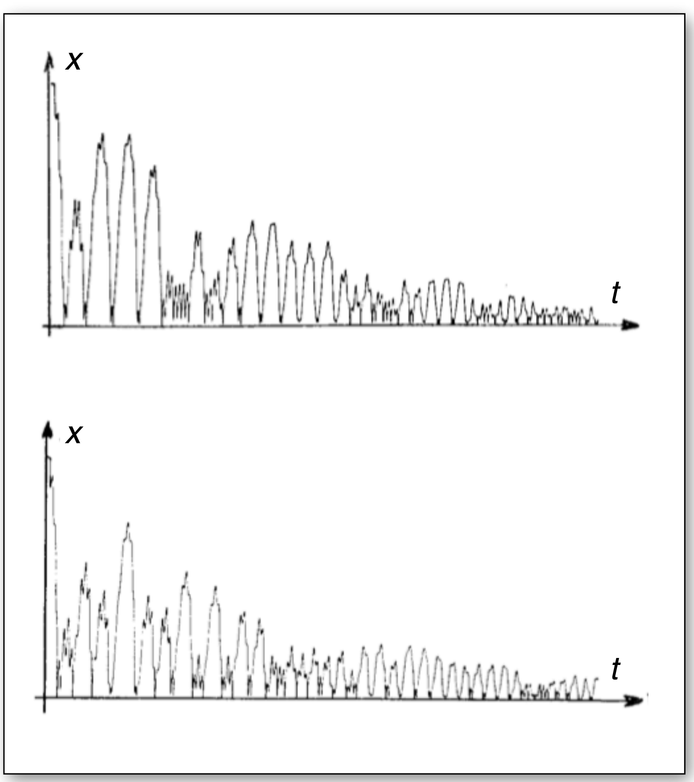

Fig. 11.4
Displacement-time graph of one of the two gliders of Fig. 11.3. The graph was recorded twice. The initial conditions were chosen the same way, as good as possible.

Fig. 11.4a suggests that this is a chaotic process. However, it might be that the process only looks chaotic, but is not chaotic in the sense of physics. We therefore repeat the experiment, setting up the initial conditions as best we can, exactly as in the experiment of 11.4a. We obtain the plot of Fig. 11.4b: despite almost the same initial conditions, a completely different behavior than in the first experiment results.

It is not hard to simulate the experiment on the computer. One enters initial conditions and runs the simulated process. One repeats the simulation with initial conditions that differ very slightly from those of the first run. The second time, the behavior is completely different. The computer thus confirms that the behavior of the system is chaotic.

A clear representation of the motion simulated on the computer is shown in Figure 11.5, where the motion of the center of mass and the relative motion of the two gliders against each other are plotted separately over time. Here, the system has been decomposed into two subsystems. The position variable of one of them is that of the center of mass, the other one is the distance between the two gliders. At each bouncing, the energy is redistributed between the two subsystems.

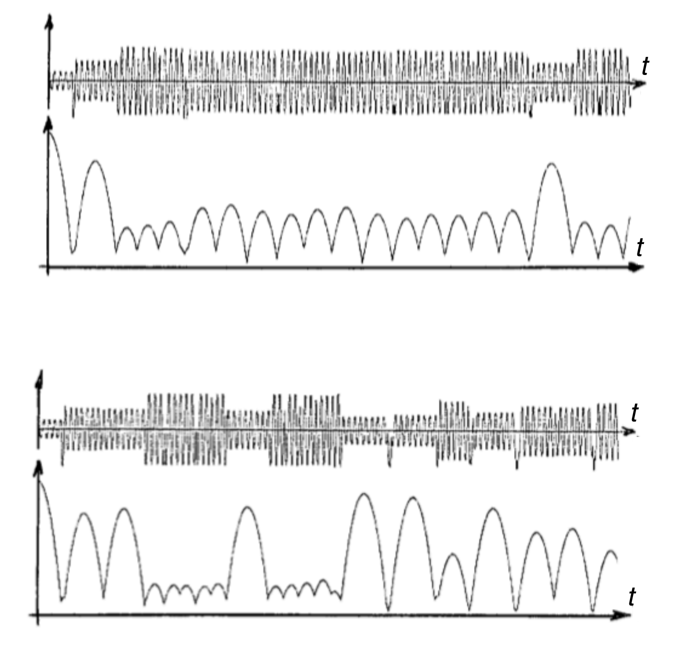

Fig. 11.5
The experiment of Fig. 11.3 was simulated twice on the computer. The initial conditions differed by about 5 %. The center of mass motion and the relative motion of the two gliders are shown separately.

12

Angular momentum and torque

12.1 Angular momentum as a substance-like quantity

A rotating flywheel is said to have momentum. However, what is colloquially called momentum in this context cannot be identified with the momentum of physics, because the total momentum of the flywheel is zero. Only its parts have momentum.

As a representative of the flywheel we consider a "dumbbell": two mass points 1 and 2, connected by a massless, rigid rod, Fig. 12.1. The masses of the two mass points are equal to each other. The dumbbell rotates about an axis which is perpendicular to the straight line connecting the two mass points, and which passes through the center of mass. The momentums of the mass points are $\vec{p}_1$ and $\vec{p}_2$. In the center of mass system, $\vec{p}_1 + \vec{p}_2 = 0$, i.e. $\vec{p}_1 = -\vec{p}_2$.

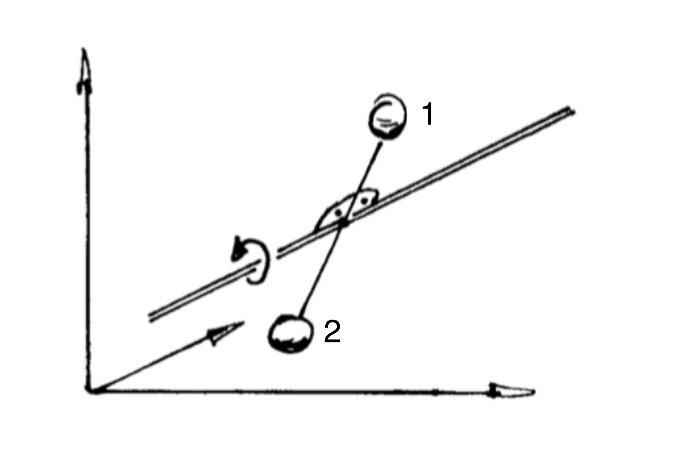

Fig. 12.1
The dumbbell rotates around an axis perpendicular to the straight line connecting the two mass points.

Is it possible to bring this dumbbell to a rest by letting the momentum of mass point 1 flow through the rod to mass point 2? Or by moving the two mass points to the center so that their momentums compensate each other? Experience shows that this is not possible. (In the case of a "dumbbell" consisting of two opposite electric charges, i.e. in the case of an electric dipole, this would certainly be possible). One can bring the dumbbell to a state of rest only by letting momentum currents flow between the system "dumbbell" and another system. This fact is an indication that we have to do here with a new conserved quantity.

This quantity is called *angular momentum* (or spin) and is abbreviated $\vec{L}$. It corresponds to what is colloquially called the momentum of the flywheel.

Let's get to know some of the properties of this quantity.

The flywheel can be carried around in space. This means that one can also move the angular momentum around in space.

Angular momentum can be transferred from one system to another, it can flow from one system to another, Fig. 12.2.

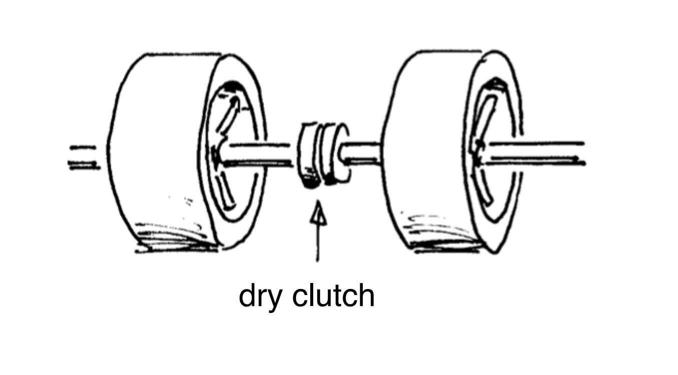

Fig. 12.2
The angular momentum flows through the shaft and the dry clutch from the left to the right flywheel.

The angular momentum has a direction, it is a vector. For a flywheel rotating around its axis of symmetry, the direction of the $\vec{L}$ vector is identical to the direction of the angular velocity vector $\vec{\omega}$.

If the angular momentum $-\vec{L}_0$ is transferred to a system which has the angular momentum $\vec{L}_0$, the resulting angular momentum is zero.

We consider a system consisting of many flywheels. We have

$$\vec{L} = \sum_i \vec{L}_i$$

where $\vec{L}_i$, is the angular momentum of the *i*-th flywheel.

Sometimes an angular momentum density can be given, Fig. 12.3.

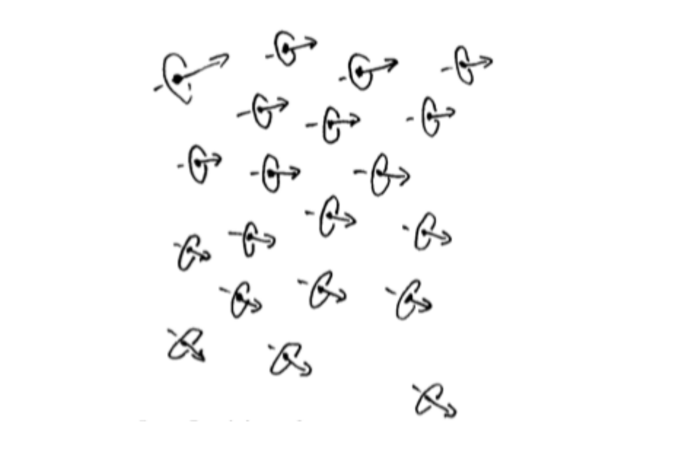

Fig. 12.3
Sometimes an angular momentum density can be defined.

We summarize:

The angular momentum

- is additive when systems are composed;
- can flow from one system to another;
- can have a density.

Thus, angular momentum is a substance-like quantity.

One finds experimentally that angular momentum is a conserved quantity.

One can use a procedure for measuring angular momentum values that is analogous to the procedure for measuring momentum described in Section 2.1.

We will see in the following section that in certain cases one can calculate the angular momentum of a system from the momentum distribution in the system.

12.2 The relationship between the angular momentum of a system of mass points and the momentum of the mass points

We consider a particular system: a “swarm” of mass points. The mass points can exchange momentum with each other, but no momentum should be exchanged with the outside environment. The distances between the mass points do not have to be fixed, and they can fly around arbitrarily. We start with the simplest case: with two mass points. Let the masses be m_1 and m_2, the position vectors $\vec{r}_1$ and $\vec{r}_2$.

We now claim that the quantity

$$\sum_{i=1}^{2} \vec{r}_i \times \vec{p}_i$$

is constant in time, as long as there is no momentum exchange of any of the mass points with the outside world. For the proof we consider the time derivative

$$\frac{d\sum \vec{r}_i \times \vec{p}_i}{dt} = \sum \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \sum \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

Since

$$\frac{d\vec{r}_i}{dt} = \vec{v}_i$$

is parallel to $\vec{p}_i$ the first vector product on the right-hand is zero, and we are left with

$$\frac{d\sum \vec{r}_i \times \vec{p}_i}{dt} = \vec{r}_1 \times \frac{d\vec{p}_1}{dt} + \vec{r}_2 \times \frac{d\vec{p}_2}{dt},$$

With

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i$$

we obtain

$$\frac{d\sum \vec{r}_i \times \vec{p}_i}{dt} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2,$$

If, as assumed, no momentum is exchanged with the environment, then momentum currents only flow *between* mass point 1 and mass point 2, and we have $\vec{F}_1 = -\vec{F}_2 = \vec{F}$. Thus

$$\frac{d\sum \vec{r}_i \times \vec{p}_i}{dt} = (\vec{r}_1 - \vec{r}_2) \times \vec{F},$$

Experience shows that the force vector $\vec{F}$ is parallel to the line connecting the mass points, i.e. to $(\vec{r}_1 - \vec{r}_2)$. So the right side of the last equation is zero, and thus the expression

$$\sum \vec{r}_i \times \vec{p}_i,$$

is constant in time.

The expression is constant in time as long as no momentum currents come from the outside or flow to the outside. We could therefore identify it with $\vec{L}$. But we do not, because the expression can be different from zero, although the considered system does not rotate at all. Instead, we define the angular momentum of the system consisting of two mass points:

$$\vec{L} = \vec{r}_1 \times \vec{p}_1^{\,(cm)} + \vec{r}_2 \times \vec{p}_2^{\,(cm)}$$

$\vec{p}_i^{\,(cm)}$ are the momentums of the mass points in the center of mass system. (In the center of mass system the sum of the momentums is equal to zero).

We show that an analogous equation is also valid for three mass points. Then the generalization for *n* mass points is obvious.

$$\frac{d\sum \vec{r}_i \times \vec{p}_i^{\,(cm)}}{dt} = \text{proof steps as before} = \sum \vec{r}_i \times \vec{F}_i,$$

Since again no momentum comes from outside, $\vec{F}_i$ represents the total strength of the momentum currents between mass point *i* and the other two mass points:

$$\vec{F}_i = \sum_{i \neq k} \vec{F}_{i,k},$$

and we get

$$\frac{d\sum \vec{r}_i \times \vec{p}_i^{\,(cm)}}{dt} = \vec{r}_1 \times (\vec{F}_{1,2} + \vec{F}_{1,3}) + \vec{r}_2 \times (\vec{F}_{2,3} + \vec{F}_{2,1}) + \vec{r}_3 \times (\vec{F}_{3,1} + \vec{F}_{3,2}),$$

Since momentum is conserved we have $\vec{F}_{i,k} = -\vec{F}_{k,i}$, and thus

$$\frac{d\sum \vec{r}_i \times \vec{p}_i^{\,(cm)}}{dt} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{1,2} + (\vec{r}_2 - \vec{r}_3) \times \vec{F}_{2,3} + (\vec{r}_3 - \vec{r}_1) \times \vec{F}_{3,1},$$

Due to the experience that $\vec{F}_{i,k}$ is parallel to the connection vector $(\vec{r}_i - \vec{r}_k)$, the right side becomes zero, i.e.

$$\sum \vec{r}_i \times \vec{p}_i^{\,(cm)},$$

is constant in time and we have for a system of mass points

$$\vec{L} = \sum \vec{r}_i \times \vec{p}_i^{\,(cm)}.$$

For the measuring unit of the angular momentum Euler (E) results:

1 E = 1 Hy · m = 1 N · m · s = 1 J · s.

Table 12.1 shows some typical values of the angular momentum.

the system earth - sun	$3 \cdot 10^{40}$ E
earth	$7 \cdot 10^{33}$ E
flywheel of a big steam engine	$10^4 \cdot 10^5$ E
flywheel of a toy steam engine	$5 \cdot 10^{-3}$ E
electron	$0.53 \cdot 10^{-34}$ E

Table 12.1
Values of the angular momentum

Finally a warning: The calculation carried out in this section is not a proof of the conservation of angular momentum. Nor is it the derivation of the conservation of angular momentum from that of momentum. The conservation of angular momentum was inserted into the calculation in the form of the following statement: “Experience shows that the force vector $\vec{F}$ is parallel to the line connecting the mass points, i.e. to $(\vec{r}_1 - \vec{r}_2)$.” This statement tells us that it is impossible to let the two momentums of a dumbbell simply flow to the center so that they compensate each other. So this statement is a way of formulating the conservation of angular momentum.

12.3 Balance equation for the angular momentum

A motor sets a flywheel in rotation, Fig. 12.4. As the angular momentum of the flywheel increases, angular momentum must flow through the drive shaft. An *angular momentum current* flows from the ground through the shaft into the flywheel.

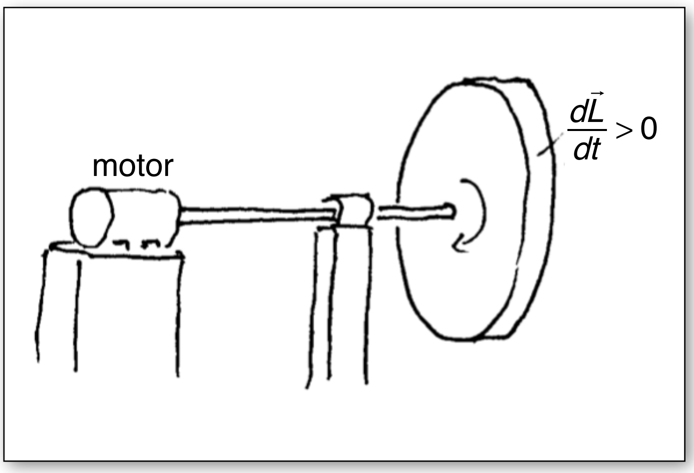

Fig. 12.4
An angular momentum current flows from the ground through the motor and shaft into the flywheel.

For the strength $\vec{M}$ of this current applies:

$$\vec{M} = \frac{d\vec{L}}{dt}$$

$\vec{M}$ is called *torque*. One says: “The motor exerts a torque on the flywheel”, or “a torque is transmitted to the flywheel”.

The function of some technical devices can be described like this:

- *Shaft:* conductor for angular momentum
- *Bearing:* insulator for angular momentum
- *Clutch:* switch for angular momentum current
- *Brake:* switch by which an angular momentum current can be directed to earth
- *Freewheel:* rectifier for angular momentum currents

If an angular momentum current flows through an elastic rod, the rod is twisted. The twist angle is a measure of the angular momentum current strength.

Since angular momentum is a conserved quantity, its value within a region of space can only change by an angular momentum current flowing into or out of the region. As in the case of momentum, the flow in and out can happen in two ways: either, as in the upper part of Fig. 12.5, via a shaft under torsional stress, or, as in the lower part of the figure, by angular momentum moving *convectively* into or out of the region.

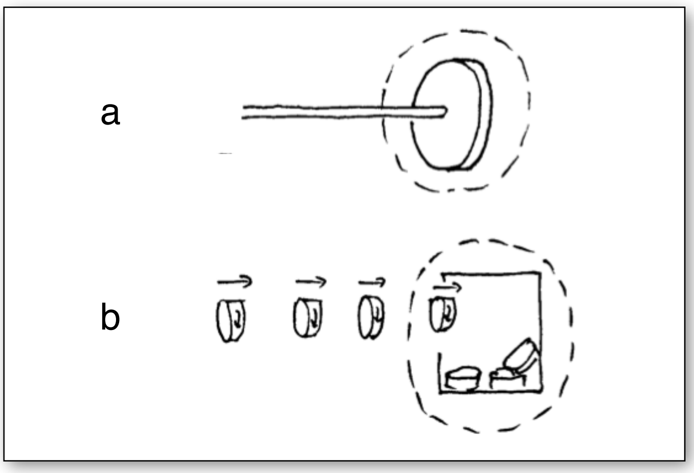

Fig. 12.5
Two types of angular momentum currents:(a) torque; (b) convective current.

Only the first of these two types of current is called torque $\vec{M}$. If the second is denoted by $\vec{M}_{conv}$, then the total current is

$$\vec{I}_L = \vec{M} + \vec{M}_{conv}$$

and we can write the balance equation for angular momentum:

$$\frac{d\vec{L}}{dt} = \vec{I}_L$$

12.4 The relationship between torque and forces

We calculate for a system of mass points

$$\frac{d\vec{L}}{dt} = \frac{d\sum \vec{r}_i \times \vec{p}_i^{\,(cm)}}{dt}.$$

The mathematical procedure is the same as in section 12.2. We now also want to allow forces to act on the system from the outside, but still with the restriction

$$\sum \vec{F}_i = 0.$$

Here $\vec{F}_i$ is the force acting from the outside on mass point *i*. If this condition is fulfilled, the center of mass momentum does not change. We get

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_i.$$

and because of

$$\frac{d\vec{L}}{dt} = \vec{M}.$$

we have

$$\vec{M} = \sum \vec{r}_i \times \vec{F}_i \quad \text{with} \quad \sum \vec{F}_i = 0 \qquad .$$

If the system consists of only two mass points, $\vec{F}_1 + \vec{F}_2 = 0$, i.e. the two force vectors are equal and opposite:

$$\vec{F}_1 = -\vec{F}_2 = \vec{F}$$

In this case the torque

$$\vec{M} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}$$

is called *force couple*.

Table 12.2 shows some typical torque values.

powerful ship engine	10^6 E/s
car engine	100 E/s
toy motor	10^{-2} E/s

Table 12.2
Some values of the torque

We now calculate the quantity $\vec{r} \times \vec{p}$ for a single mass point, where $\vec{r}$ is the position vector in an arbitrary coordinate system, and $\vec{p}$ is the momentum of the mass point. We denote the origin of the coordinate system by O. We calculate the time derivative

$$\frac{d(\vec{r} \times \vec{p})}{dt} = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{=0} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}$$

One often calls

$\vec{r} \times \vec{p}$ the angular momentum of the mass point with respect to O;

$\vec{r} \times \vec{F}$ the torque exerted on the mass point with respect to O.

However, these two quantities must be treated with caution, because their values depend not only on the choice of the velocity coordinate system, but also on that of the spatial coordinate system.

12.5 The moment of inertia

The faster a body rotates around a certain axis, the larger its angular momentum: the larger $\vec{\omega}$ the larger $\vec{L}$. The exact mathematical relationship between $\vec{\omega}$ and $\vec{L}$ can be quite complicated. It depends on the spatial distribution of the mass of the body under consideration, and it depends on the axis about which the body is rotating. We start the investigation of the $\vec{\omega}-\vec{L}$ relation with a simple special case: with a dumbbell rotating around its axis (through the center of mass and perpendicular to the straight line connecting the two mass points).

The angular momentum of the dumbbell is:

$$\vec{L} = \vec{r}_1 \times \vec{p}_1^{\,(cm)} + \vec{r}_2 \times \vec{p}_2^{\,(cm)}$$

Indices 1 and 2 refer to the two mass points.

With $\vec{p}_1^{\,(cm)} = -\vec{p}_2^{\,(cm)}$ we get

$$\vec{L} = (\vec{r}_1 - \vec{r}_2) \times \vec{p}_1^{\,(cm)}$$

Since $\vec{r}_1 - \vec{r}_2$ is perpendicular to $\vec{p}_1^{\,(cm)}$, we obtain for the magnitude of $\vec{L}$:

$$L = 2rp.$$

Here r is the magnitude of $(\vec{r}_1 - \vec{r}_2)/2$, i.e. the distance of the mass points from the axis, and p is the magnitude of $\vec{p}_1^{\,(cm)}$ and of $\vec{p}_2^{\,(cm)}$.

With $p = mv = m\omega r$ we get

$$L = 2r^2 m\omega,$$

or as a vector

$$\vec{L} = J\vec{\omega} \quad \text{with} \quad J = 2r^2 m \qquad (12.1)$$

We see that the angular momentum and angular velocity vectors are parallel and their magnitudes are proportional to each other. The proportionality factor is called moment of inertia. The moment of inertia tells us whether a body rotating at a given angular velocity contains much or little angular momentum. We can think of it as the angular momentum capacity of the system. It is thus a measure of the inertia of a body with respect to rotational motion. A large moment of inertia means that the body must be supplied with a large amount of angular momentum in order to increase its angular velocity.

Equation (12.1) has the same structure as the velocity-momentum relation

$$\vec{p} = m\vec{v}$$

or the relationship between voltage and electric charge

$$Q = CU.$$

For the moment of inertia of the dumbbell we have found

$$J = 2r^2 m$$

It depends not only on the mass of the dumbbell, but also on where the mass is located. The further away it is from the axis, i.e. the larger is r, the larger is also J, the more inert is the dumbbell (with regard to rotational movements).

We now investigate the ω-L relation for a more complicated body, a body which is central symmetric with respect to an axis. It is supposed to rotate around this symmetry axis. We decompose the body in our minds into small mass elements. The extension of each of these mass elements should be small against the radial extension of the body. Each mass element can be treated as a mass point. Because of the central symmetry of the body, the mass elements can be grouped into pairs, each of which forms a dumbbell whose axis coincides with the axis of rotation of the body, Fig. 12.6.

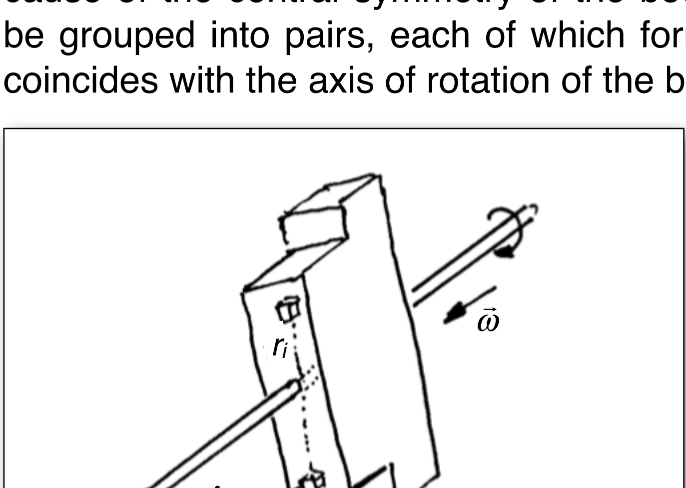

Fig. 12.6
The body can be decomposed into many dumbbells.

If the body rotates around its axis with the angular velocity ω, each dumbbell also rotates with this angular velocity and makes a contribution

$$2\omega r_i^2 \Delta m_i$$

to the total angular momentum. The dumbbells are numbered with the index i. Δm_i is the mass of each of the two mass elements of dumbbell i, and r_i is the distance of these mass elements from the axis of rotation.

We can also say that each half dumbbell, i.e. each mass element, provides the contribution

$$\Delta L_i = \omega r_i \Delta m_i$$

to the total angular momentum.

The total angular momentum is the sum of all these contributions:

$$L = \omega \sum_i r_i^2 \Delta m_i$$

or in vector notation

$$\vec{L} = J \cdot \vec{\omega} \quad \text{with} \quad J = \sum_i r_i^2 \Delta m_i$$

If one makes the decomposition into mass elements, that are smaller and smaller, one obtains as a limit value

$$J = \int r^2 \, dm \qquad (12.2)$$

We see: Again, $\vec{L}$ and $\vec{\omega}$ are proportional to each other.

We next realize that, in general, one cannot get along with just one number if one wants to characterize the inertia of a body with respect to rotational motions. We consider a body which is central symmetric with respect to several axes, e.g. the one in Fig. 12.7. We can use equation (12.2) to calculate three moments of inertia: one for the rotation about each of the three axes of symmetry. Of course, in general, a different value is obtained for each direction of rotation.

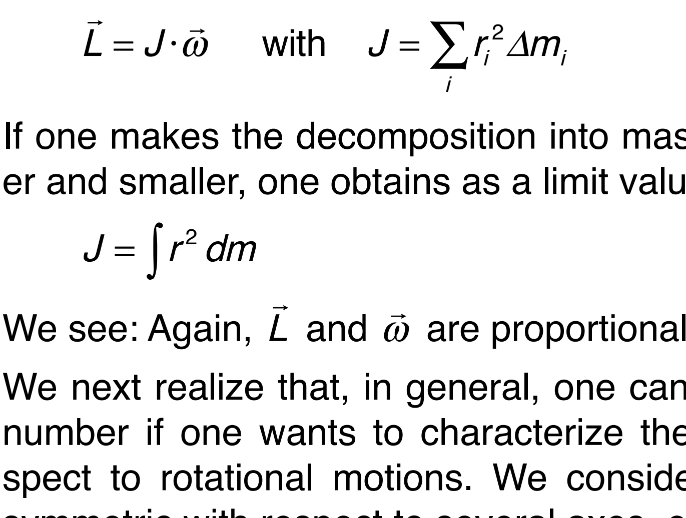

Fig. 12.7
The body is centrally symmetric with respect to three axes.

So we see that J is not simply a single number characteristic of the body. Rather, the moment of inertia has different values for different directions of rotation. But it is becoming even more complicated.

Again we let a barbell rotate, but this time around an axis that is no longer perpendicular to the line connecting the mass points, but forms an oblique angle to it. However, it should still pass through the center of mass, Fig. 12.8.

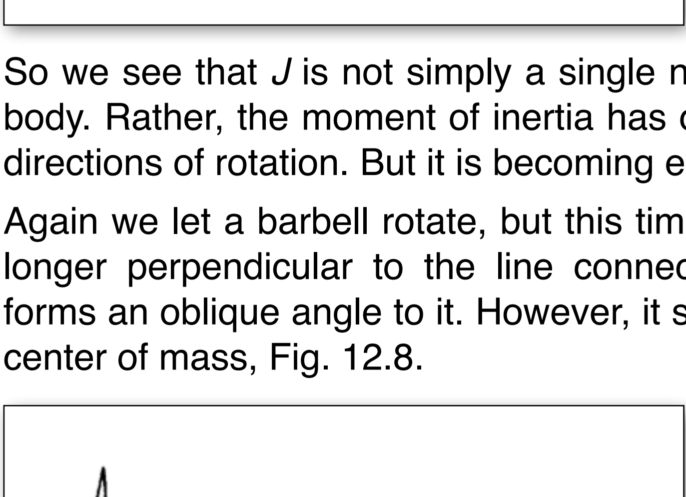

Fig. 12.8
The dumbbell rotates around an axis that is not perpendicular to the line connecting the mass points.

Again we look for the $\vec{\omega}-\vec{L}$ relation. We calculate:

$$\vec{L} = \vec{r}_1 \times \vec{p}_1^{\,(cm)} + \vec{r}_2 \times \vec{p}_2^{\,(cm)}$$

For this purpose we place the origin of the position vectors for the sake of convenience in the center of mass of the barbell, so that $\vec{r}_1 = -\vec{r}_2$. In addition, $\vec{p}_1^{\,(cm)} = -\vec{p}_2^{\,(cm)}$ anyway. Therefore it becomes

$$\vec{L} = 2\vec{r}_1 \times \vec{p}_1^{\,(cm)}$$

Thus, the angular momentum vector is perpendicular to the momentum vectors and perpendicular to the straight line connecting the mass points of the barbell, Fig. 12.9.

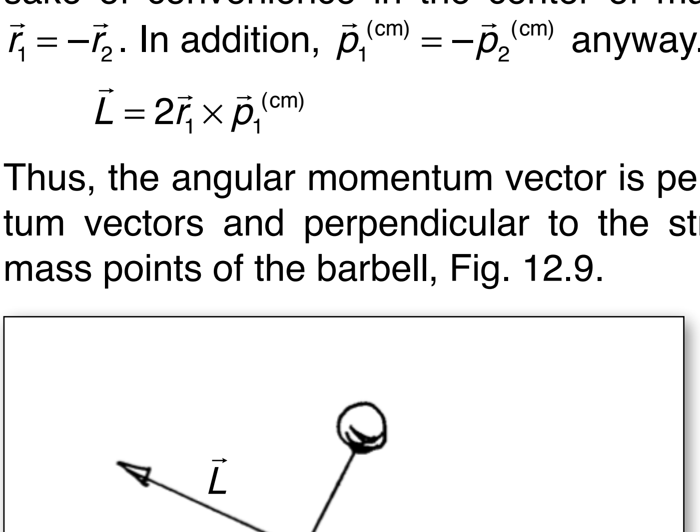

Fig. 12.9
The angular momentum vector is not parallel to the angular velocity vector.

Therefore, $\vec{L}$ does not have the same direction as $\vec{\omega}$. The $\vec{L}$ vector moves with angular velocity $\vec{\omega}$ on a cone surface about the $\vec{\omega}$-direction. We see that the relation $\vec{L} = J \cdot \vec{\omega}$ is no longer valid here.

Nevertheless, we can say that the relation between the vectors $\vec{\omega}$ and $\vec{L}$ is linear. If we change the magnitude of the angular velocity by a certain factor, but not its direction, the magnitude of the angular momentum changes by the same factor, while the direction of $\vec{L}$ does not change.

Mathematically, such a "linear transformation" of a vector into another is described by a *tensor*. One says that the moment of inertia is a tensor quantity and writes the relation between $\vec{\omega}$ and $\vec{L}$

$$\vec{L} = \mathbf{J} \cdot \vec{\omega}$$

The tensor **J** can be represented analytically by a 3×3 matrix containing the 9 "components" of the tensor. If one also represents the $\vec{\omega}$ vector by its components, one can multiply **J** and $\vec{\omega}$ and obtain $\vec{L}$ in component notation. Only 6 of the 9 components of the inertia tensor (= moment of inertia tensor) are independent of each other. This means that the inertial behavior of a body with respect to rotational motions is characterized uniquely by 6 numbers. The body can have an arbitrarily complicated distribution of its mass density – its rotational inertia is always determined by 6 numerical values.

These numbers, i.e. the components of the inertia tensor, behave similarly to the components of a vector. The components of one and the same vector have, depending on the coordinate system, different values. In the same way, the components of a tensor have different values, depending on the coordinate system.

Now there is a special choice of the coordinate system in which the components have a particularly visual meaning. It is the coordinate system in which the matrix representing the tensor has diagonal form. This means: If one lets the body rotate around one of the coordinate axes, i.e. if the angular velocity vector points in one of the three coordinate axis directions, the angular momentum vector also points in this direction: $\vec{\omega}$ and $\vec{L}$ are parallel to each other. This is true for all three coordinate axis directions. For each of these directions the following applies

$$\vec{L} = J_i \cdot \vec{\omega} \quad \text{with} \quad i = 1, 2, 3$$

To each of these directions belongs a value of the moment of inertia. The axes of rotation corresponding to these three directions are called *principal axes of inertia*. The corresponding values of the moment of inertia are the *principal moments of inertia*. It now becomes plausible why 6 numbers are needed to characterize the inertial behavior of a body: 3 numbers define the directions of the principal axes of inertia and 3 other numbers define the values of the 3 principal moments of inertia.

If a body rotates with constant angular velocity about an axis other than a principal axis of inertia, its angular momentum changes constantly. The components of the angular momentum which are perpendicular to $\vec{\omega}$ change sinusoidally. The corresponding inflow and outflow of angular momentum usually occurs through the bearings and is easy to observe or measure.

12.6 The decomposition of angular momentum – spin and orbital angular momentum

We show this decomposition for a simple case: for a system of 4 mass points, Fig. 12.10. The entire system A is decomposed into two spatially separated systems a and b.

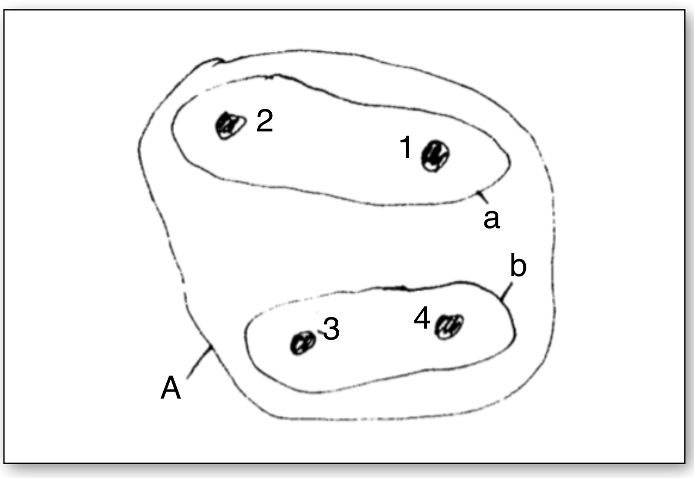

Fig. 12.10
The system A is decomposed into subsystems a and b.

We ask for the relation between the angular momentum $\vec{L}_A$ of the entire system and the angular momentums $\vec{L}_a$ and $\vec{L}_b$ of the subsystems. We first specify $\vec{L}_A$, $\vec{L}_a$ and $\vec{L}_b$ separately:

$$\vec{L}_A = \vec{r}_1 \times \vec{p}_1^{(A)} + \vec{r}_2 \times \vec{p}_2^{(A)} + \vec{r}_3 \times \vec{p}_3^{(A)} + \vec{r}_4 \times \vec{p}_4^{(A)}$$

$$\vec{L}_a = \vec{r}_1 \times \vec{p}_1^{(a)} + \vec{r}_2 \times \vec{p}_2^{(a)}$$

$$\vec{L}_b = \vec{r}_3 \times \vec{p}_3^{(b)} + \vec{r}_4 \times \vec{p}_4^{(b)}$$

Here $\vec{p}_i^{(x)}$ is the momentum of mass point i in the center of mass frame of system x, Fig. 12.11. A somewhat tedious calculation yields a simple and obvious result.

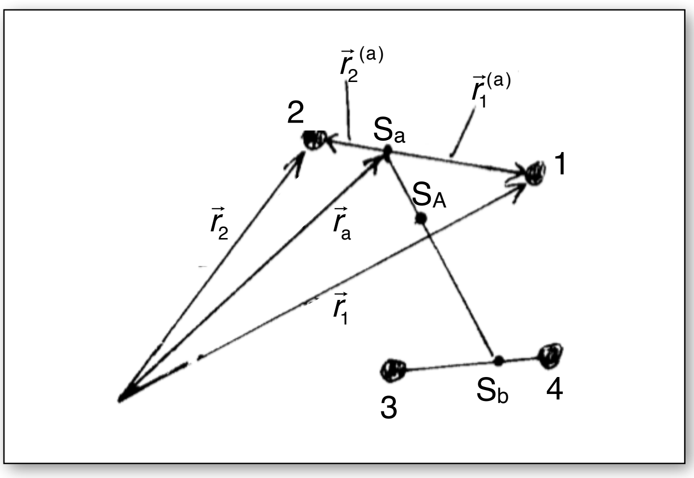

Fig. 12.11
The definition of the quantities used in the calculation

With

$$\vec{r}_1 = \vec{r}_a + \vec{r}_1^{(a)},\ \vec{r}_2 = \vec{r}_a + \vec{r}_2^{(a)},\ \vec{r}_3 = \vec{r}_b + \vec{r}_3^{(b)} \text{ and } \vec{r}_4 = \vec{r}_b + \vec{r}_4^{(b)}$$

we get

$$\vec{L}_A = (\vec{r}_a + \vec{r}_1^{(a)}) \times \vec{p}_1^{(A)} + (\vec{r}_a + \vec{r}_2^{(a)}) \times \vec{p}_2^{(A)} + (\vec{r}_b + \vec{r}_3^{(b)}) \times \vec{p}_3^{(A)} + (\vec{r}_b + \vec{r}_4^{(b)}) \times \vec{p}_4^{(A)}$$

We use

$$\vec{p}_1^{(A)} = m_1(v_a^{(A)} + v_1^{(a)})$$

and the corresponding relations for mass points 2 to 4, and we rearrange:

$$\begin{aligned}\vec{L}_A &= \vec{r}_a \times (\vec{p}_1^{(A)} + \vec{p}_2^{(A)}) + \vec{r}_b \times (\vec{p}_3^{(A)} + \vec{p}_4^{(A)}) \\ &+ m_1\vec{r}_1^{(a)} \times (v_a^{(A)} + v_1^{(a)}) + m_2\vec{r}_2^{(a)} \times (v_a^{(A)} + v_2^{(a)}) \\ &+ m_3\vec{r}_3^{(b)} \times (v_b^{(A)} + v_3^{(b)}) + m_4\vec{r}_4^{(b)} \times (v_b^{(A)} + v_4^{(b)})\end{aligned}$$

With $\vec{p}_1^{(A)} + \vec{p}_2^{(A)} = \vec{p}_a^{(A)}$ and $\vec{p}_3^{(A)} + \vec{p}_4^{(A)} = \vec{p}_b^{(A)}$ (center of mass momentums of systems a and b), and with $m_1\vec{r}_1^{(a)} + m_2\vec{r}_2^{(a)} = 0$ and $m_3\vec{r}_3^{(b)} + m_4\vec{r}_4^{(b)} = 0$ (definition of the centers of mass of S_a and S_b) we get

$$\vec{L}_A = (\vec{r}_a \times \vec{p}_a^{(A)}) + (\vec{r}_b \times \vec{p}_b^{(A)}) + (\vec{r}_1^{(a)} \times \vec{p}_1^{(a)}) + (\vec{r}_2^{(a)} \times \vec{p}_2^{(a)}) + (\vec{r}_3^{(b)} \times \vec{p}_3^{(b)}) + (\vec{r}_4^{(b)} \times \vec{p}_4^{(b)})$$

The last 4 summands are just equal to $\vec{L}_a + \vec{L}_b$. We thus obtain

$$\vec{L}_A = \vec{L}_a + \vec{L}_b + \vec{L}_{a\text{-}b} \quad \text{with} \quad \vec{L}_{a\text{-}b} = \sum_{a,b} \vec{r}_i \times \vec{p}_i^{(A)}$$

Thus, the total angular momentum of system A is equal to the sum of the angular momentums of the subsystems a and b plus the angular momentum $\vec{L}_{a\text{-}b}$ of a third system a-b. a-b is the system which is created when the mass of system a is united in the center of mass S_a and that of system b in S_b. So we have decomposed A into the three subsystems a, b and a-b.

Such a decomposition is often made when a and b are rigid bodies, but also with the atom, where for example a is an electron and b is the atomic nucleus. In these cases one calls $\vec{L}_a$ and $\vec{L}_b$ the intrinsic angular momentum or spin of system a and b, respectively.

If one of the systems a and b has a much larger mass than the other, e.g. $m_b \gg m_a$, the distance $\vec{r}_b - \vec{r}_a$ of the center of mass S_b from the total center of mass S_A is much smaller than $\vec{r}_a - \vec{r}_A$. If we place the origin of the position vectors in S_A, in the sum

$$\vec{L}_{a\text{-}b} = \sum_{a,b} \vec{r}_i \times \vec{p}_i^{(A)}$$

the contribution of system b is much smaller than that of system a.

Unfortunately, it has therefore become customary to call $\vec{L}_{a\text{-}b}$ the *orbital angular momentum* of system a in this case.

12.7 Angular momentum and energy

Angular momentum current and energy current

An energy current can flow through a shaft. For an energy current to flow, it is necessary that

- the shaft is rotating, i.e. $\vec{\omega} \neq 0$;
- an angular momentum current is flowing, i.e. $\vec{M} \neq 0$.

The energy current strength P is therefore a function of $\vec{\omega}$ and $\vec{M}$. We ask for the relation $P(\vec{M}, \vec{\omega})$. Let a momentum current flow through each of the two ropes in Fig. 12.12.

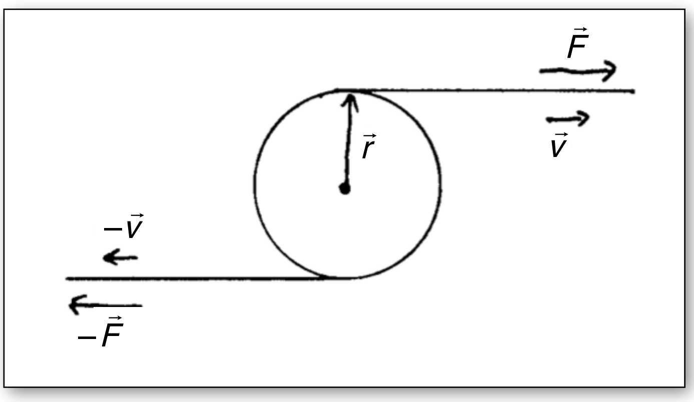

Fig. 12.12
Both energy and momentum flow through each of the two ropes.

The force vectors are equal and opposite to each other. So the force in both ropes together is zero. But there is an angular momentum current of the strength

$$\vec{M} = 2\vec{r} \times \vec{F}\ .$$

The energy current flowing through both ropes together to the wheel has a strength of

$$P = 2\vec{v}\vec{F}\ .$$

With $\vec{v} = \vec{\omega} \times \vec{r}$ we get

$$P = 2(\vec{\omega} \times \vec{r})\vec{F}\ .$$

According to the rules of vector calculus we obtain

$$P = 2\vec{\omega}(\vec{r} \times \vec{F})$$

and with $\vec{r} \times \vec{F} = \vec{M}/2$

$$P = \vec{\omega}\vec{M} \qquad (12.3)$$

Energy transmission by means of shafts

In Fig. 12.13, the angular momentum flows in a closed circuit between the energy source (turbine) and the energy receiver (generator): one part of its path is through the shaft, the other through the machine housings and the foundations. The energy accompanies the angular momentum only on a part of its path: only through the conductors for which $\vec{\omega} \neq 0$.

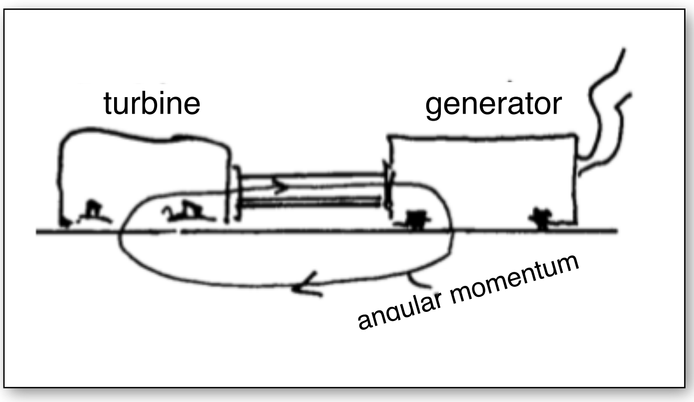

Fig. 12.13
Energy transfer through a shaft. Energy as well as angular momentum are flowing through the shaft.

Energy storage in a flywheel

A rotating flywheel has more energy than one at rest. To calculate this energy difference, we charge it with energy and with angular momentum. Thereby, an energy current and an angular momentum current flow into the flywheel. The currents are related to each other via equation (12.3). We apply the balance equations of energy and angular momentum to the flywheel:

$$P = \frac{dE}{dt} \quad \text{and} \quad \vec{M} = \frac{d\vec{L}}{dt}$$

So we get

$$\frac{dE}{dt} = \vec{\omega}\frac{d\vec{L}}{dt}$$

or

$$\int dE = \vec{\omega}\int d\vec{L}$$

and with $\vec{L} = J\vec{\omega}$

$$E = \frac{1}{J}\int \vec{L}\, d\vec{L}$$

$$E(\vec{L}) = \frac{\vec{L}^2}{2J} + E_0$$

or

$$E(\vec{\omega}) = \frac{J}{2}\vec{\omega}^2 + E_0$$

12.8 Parallel axis theorem

Let a body a of mass m rotate about an axis A fixed to the earth, Fig. 12.14. Suppose that the following is known:

- the angular velocity ω;
- the moment of inertia J of the body.

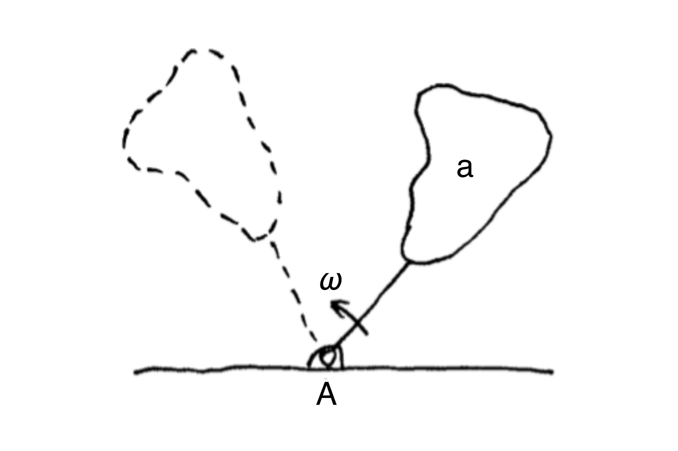

Fig. 12.14
Body a rotates around axis A.

How much angular momentum and how much energy contains

- the body a;
- the system body-earth?

Since the earth is rigid and heavy compared to the rotating body, we can imagine it to be replaced by a body b of infinite mass at the position of the axis of rotation, Fig. 12.15.

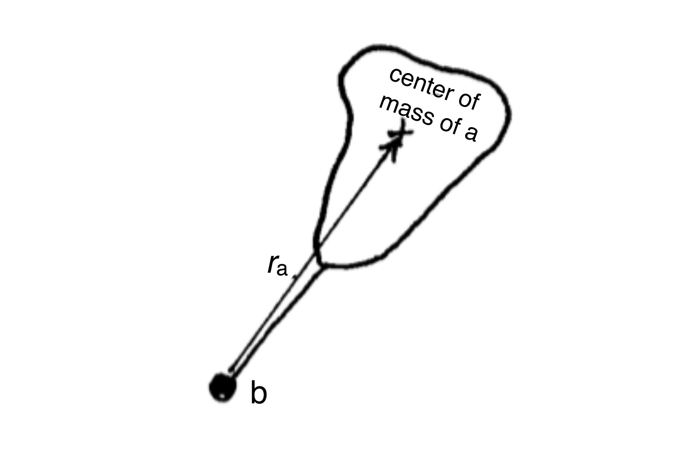

Fig. 12.15
The earth was replaced by an infinitely heavy body b at the location of the axis of rotation.

We now decompose the total system into subsystems as shown in Table 12.2. The total angular momentum L and the total kinetic energy E_{kin} are the sums of the corresponding values of the subsystems.

$$L = \omega J + \omega m r_a^2 = \omega(J + m r_a^2) = \omega J_A$$

$$E_{kin} = \frac{1}{2}(J + m r_a^2)\omega^2 = \frac{J_A}{2}\omega^2$$

We call

$$J_A = J + m r_a^2$$

the “moment of inertia of body a with respect to the axis of rotation A”. The relation between J and J_A is called *parallel axis theorem*. In fact, J_A is the sum of the moments of inertia of body a and the system a-b.

Table 12.2

subsystem	angular velocity	moment of inertia	angular momentum	kinetic energy
body a	ω	J	$J\omega$	$\frac{J}{2}\omega^2$
body b	0	not relevant	0	0
system a-b of the centers of mass of a and b	ω	$m r_a^2$	$\omega m r_a^2$	$\frac{1}{2} m r_a^2 \omega^2$

12.9 The analogy between electromagnetism, translational mechanics and rotational mechanics

The analogy is shown in table 12.3. It is more far-reaching than this table would suggest. In particular, a dualism exists also within rotational mechanics. Moreover, one could have introduced a second column referring to electricity, with the electric dipole moment at the top, the analog to the “momentum moment” $\vec{L}$. However, this column would be less interesting, because the electric dipole moment, in contrast to the angular momentum, is not a conserved quantity.

Table 12.3

electromagnetism	translational mechanics	rotational mechanics
Q	$\vec{p}$	$\vec{L}$
φ	$\vec{v}$	$\vec{\omega}$
U	$\Delta\vec{v}$	$\Delta\vec{\omega}$
C	m	J
L	$1/D$	$1/\tau$
$n\Phi$	$\vec{r}$	$\vec{\alpha}$
$Q = C \cdot U$	$\vec{p} = m \cdot \vec{v}$	$\vec{L} = J \cdot \vec{\omega}$
$P = U \cdot I$	$P = \Delta\vec{v} \cdot \vec{F}$	$P = \Delta\vec{\omega} \cdot \vec{M}$
$E = E_0 + \frac{Q^2}{2C}$	$E = E_0 + \frac{\vec{p}^2}{2m}$	$E = E_0 + \frac{\vec{L}^2}{2J}$

Let us consider two examples of the analogy which we need in the next section, Figs. 12.16 and 12.17. In the systems of Fig. 12.16, the substance-like quantity X (Q or $\vec{p}$ or $\vec{L}$) oscillates back and forth between system 1 and system 2.

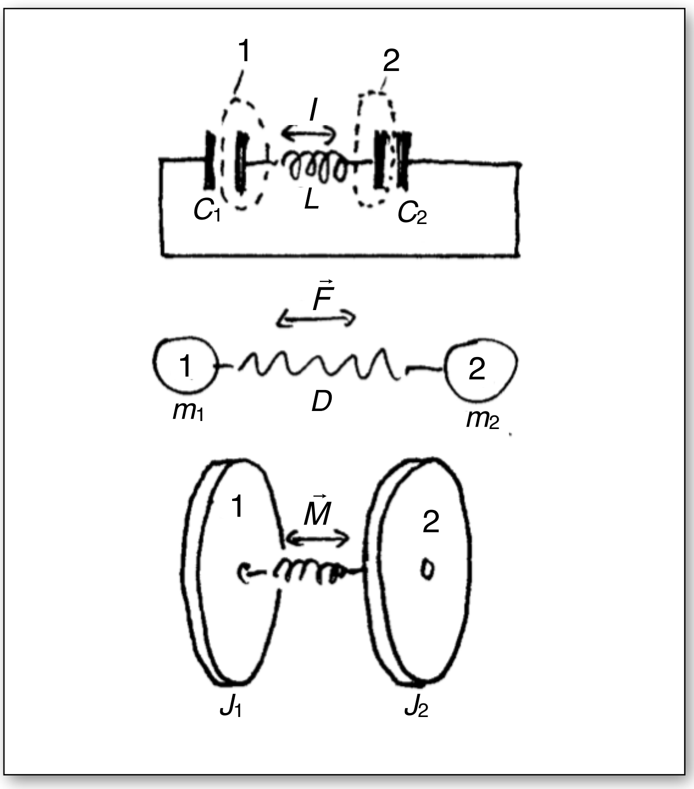

Fig. 12.16
The substance-like quantity (Q or $\vec{p}$ or $\vec{L}$) flows back and forth between subsystems 1 and 2.

In the systems of Fig. 12.17, X flows from the system where the intensive quantity ξ has the larger value into the system where ξ has the smaller value. The current stops flowing when the value of the intensive quantity has become equal in both systems. This state is called *equilibrium* with respect to the current of magnitude X.

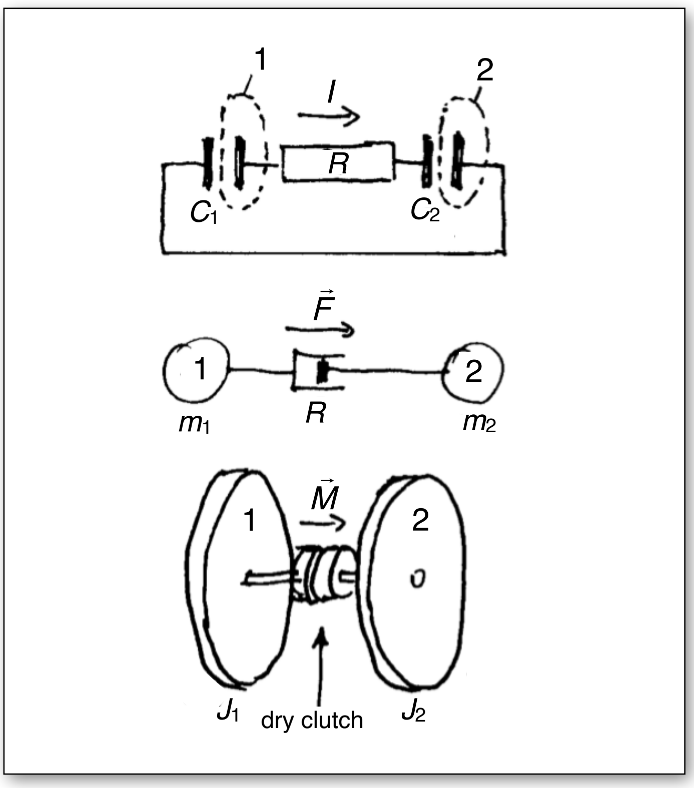

Fig. 12.17
The substance-like quantity flows from the subsystem in which the intensive variable has the higher value to the subsystem in which it has the lower value.

12.10 Appropriate decompositions into subsystems; the tides; spin-orbit coupling

We return to Section 12.6. One can often decompose a system into subsystems in different ways. The total angular momentum is in any case equal to the sum of the angular momentums of the subsystems. Sometimes angular momentum flows from one subsystem into another (in technical jargon: the subsystems are “coupled” to each other). Two extreme cases can be distinguished.

The angular momentum current between two subsystems can be dissipative. In this case, it flows until the ω-values of the two subsystems have become equal. In this case, there is equilibrium with respect to the angular momentum current or “rotational equilibrium”. An example of this is the angular momentum current flowing from the system “earth” into the system formed by the centers of mass of earth and moon. Such a current comes about because of the tides. (The systems moon and moon-earth are already in equilibrium, they already have the same angular velocity).

The angular momentum current between the subsystems can also be non-dissipative. Then it flows constantly back and forth between the subsystems. An example for this are the atoms. In the hydrogen atom, for example, angular momentum constantly flows back and forth between the subsystem electron and the subsystem formed by the centers of mass of the electron and the nucleus. This back and forth flow of angular momentum is called *spin-orbit coupling*.

Atoms with many electrons can be divided into subsystems in many ways. It is useful to do this in such a way that strong angular momentum currents occur only within the subsystems, but not between one subsystem and another. The angular momentum of each of the subsystems chosen in this way is then almost constant.

12.11 Torque equilibrium

We have seen in Chapter 4 how the junction rule for momentum currents can be used to solve a certain types of static problems. Fig. 12.18 shows a problem that cannot be solved in this way. The mass of the load and the geometry of the arrangement are given. The forces on the two supports P and Q are looked for.

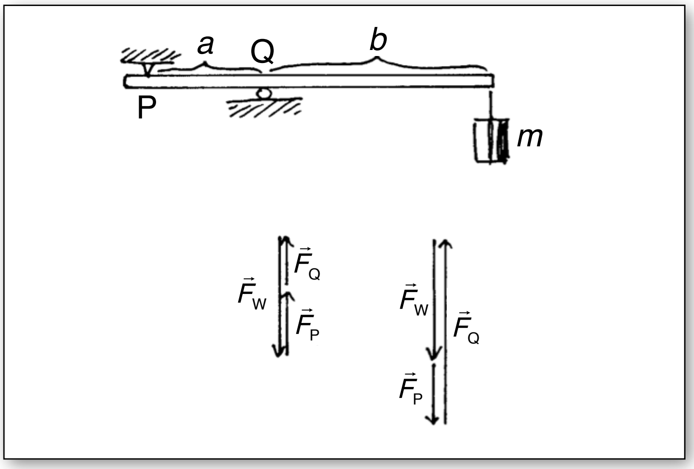

Fig. 12.18
The decomposition of $\vec{F}_W$ into two parallel forces is not unambiguous.

Since the three force vectors occurring in the problem are parallel, the decomposition of the weight force $\vec{F}_W$ into the forces on the supports is not unique. The lower drawing shows two of the infinitely many possibilities. The problem can be solved, however, if for the beam, in addition to the momentum balance

$$\sum \vec{F}_i = 0$$

the angular momentum balance for the beam is established:

$$\sum \vec{M}_i = \sum \vec{r}_i \times \vec{F}_i$$

We place the origin of the position vectors $\vec{r}_i$ in such a way that the calculation is as simple as possible. We place it in the point Q. Thus, the $\vec{r}_i$ are perpendicular to the force vectors and the vector products become products of the magnitudes of the vectors. Furthermore, the term $\vec{r}_Q \times \vec{F}_Q$ is omitted since $\vec{r}_Q = 0$. So there remains

$$-aF_P + bF_W = 0$$

With $F_W = -mg$ and $F_P + F_Q + F_W = 0$ we get

$$F_P = -\frac{b}{a} mg$$

$$F_Q = mg\left(1 + \frac{b}{a}\right)$$

13

Stress – momentum current density

13. Stress – momentum current density

We consider a piece of matter clamped between the jaws of a screw clamp so that a force acts in the *x* direction, Fig. 13.1. We place a vertical cross sectional area A anywhere through the piece of matter. The part on the left of the sectional area now exerts a force on the part on the right, or in other words, a momentum current flows through the cutting area.

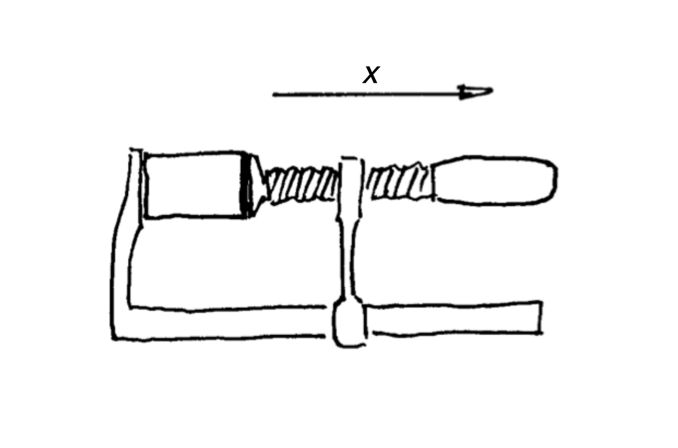

Fig. 13.1
The clamped body is under compressive stress in the *x* direction.

Often it is convenient to describe this situation locally. For this purpose one uses the quotient

$$\sigma = \frac{F}{A}$$

This quantity is called *mechanical stress* or *stress* for short or *momentum current density*. (A current per area is always called current density.) The quantity describes the local stress state of the matter. If σ is greater than zero at a point, a compressive stress prevails there, if σ is smaller than zero, there is tensile stress.

However, the stress state of a piece of matter is not yet unambiguously described by the specification of a single value of the stress. Independently of the force in the *x* direction considered first, there can also be a force in the *y* and in the *z* direction, Fig. 13.2. Matter can therefore be under three different stresses in three mutually perpendicular directions. In order to describe the local stress state completely, one must therefore specify:

- the direction of a right-angled tripod in which the stress components are independent of each other;
- the σ values belonging to the three independent directions.

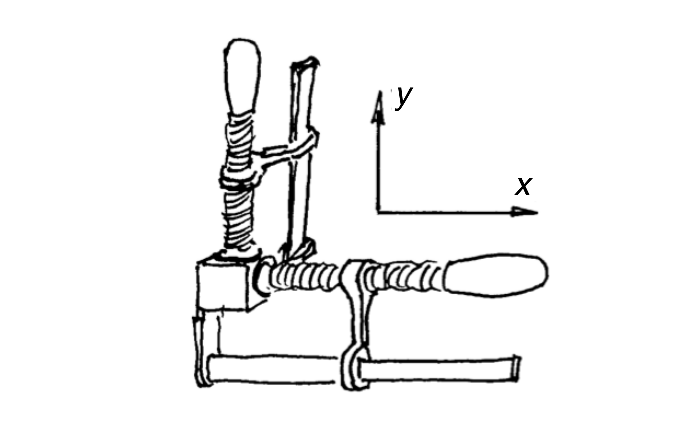

Fig. 13.2
The clamped body is under different compressive stresses in the *x* and *y* directions.

When discussing the moment of inertia, we had learned that a physical quantity which can be described in this way is a tensor. Thus, also mechanical stress is a tensor. By the way, the name “tensor” comes from this physical realization.

While the inertia tensor is associated with a whole body, the stress tensor is a local quantity, it refers to a single point. Its components can have different values at each point of a system.

14

Static fields

14.1 Fields and distributions

Fields are physical systems. There are several different kinds of fields. We consider here only the electromagnetic field and the gravitational field. So far, we have known the following properties of fields:

- in fields momentum currents are flowing (fields transmit forces);
- fields contain energy;
- in fields energy currents are flowing (however, we have observed these only as long as a field is changing).

So far a field was for us something similar to a spring in black box, from which only two “hooks” peek out: electric charges magnetic poles or masses. We had noticed the existence of the fields only by balancing: “Here ends the visible conductor of a momentum current, so here must begin an invisible conductor through which the momentum can continue to flow.” We now want to look into the box and answer different questions:

- How does the strength of the momentum current from one “hook” to the other, i.e. from one mass to another or from one charge to the other, depend on the values of the masses or charges and on their distances?
- What is the momentum current distribution within the field, what path do the momentum currents take in the field?
- How is the energy distributed in the field?

The air above the earth’s surface has different temperatures T, different pressures p, different mass and entropy densities ρ_m and ρ_S, and different chemical compositions at different positions. One can assign a T, p, ρ_m and ρ_S, value to each position, as well as concentration values of the different substances contained in the air.

It is said that the set of all T values $T(x,y,z)$ forms a temperature *field*, that of the p values p $p(x,y,z)$ a pressure *field* and so on. The word field is used here in a different meaning than we had explained above. To distinguish the two meanings we will avoid the term “field” in this second sense. Instead of temperature field we say temperature distribution, instead of pressure field we say pressure distribution, and if the quantity is the field strength, we speak of the field strength distribution.

We will see that a physical field can usually be described by such distributions. If the local physical quantities we use to describe the field have the same value at all points in space, the field is called *homogeneous*. The situation is the same as if we have air in a container and T, p, ρ, etc. have the same value everywhere in the container. Then it is enough to give a single T value, p value, etc. to completely describe the state of the air.

14.2 The physical quantity field strength

We consider an arbitrary distribution of small bodies B_1, B_2, B_3,… with masses m_1, m_2, m_3, …. We place another body B_0 of mass m_0 at a position $\vec{r}$ and find that a force is exerted on B_0, i.e. that a momentum current flows through the field to this body, Fig. 14.1.

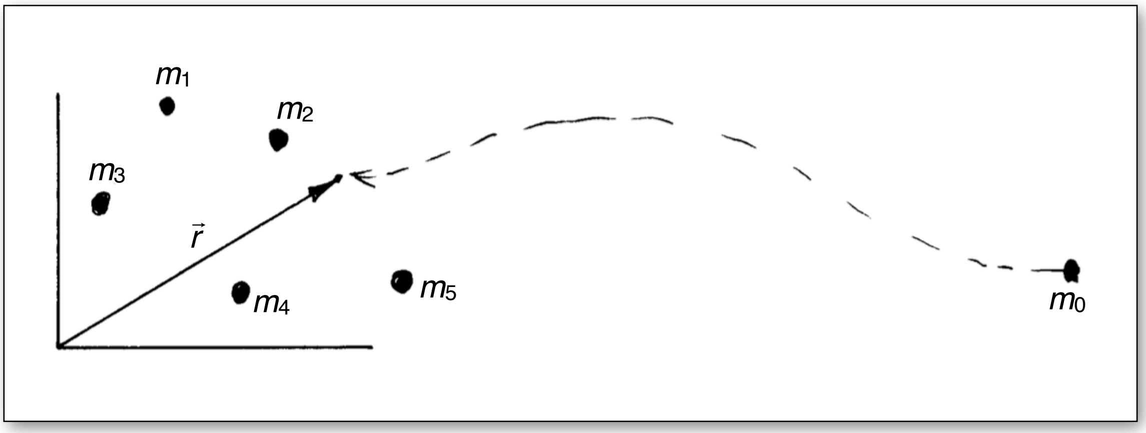

Fig. 14.1
If the mass is doubled from m_0 to $2m_0$, the force on the body with m_0 is also doubled.

We now double the mass of B_0 to $2m_0$ and find that the force on B_0 also doubles. It is as if we had simply placed two bodies with masses m_0 next to each other. The force of the field on the first one does not seem to disturb the force of the field on the second one. We conclude that $\vec{F}$ is proportional to m:

$$\vec{F} = \vec{g} \cdot m \tag{14.1}$$

The vectorial factor of proportionality $\vec{g}$ is independent of m. It characterizes the field which the bodies B_1, B_2, B_3,… generate at the place where body B_0 is located. We call it the strength of the gravitational field of the masses m_1, m_2, …m_n. Note that $\vec{g}(\vec{r})$ is the strength of the field in the state where body B_0 is not yet at the position $\vec{r}$.

All this applies analogously to static electric fields: The force exerted by an electric field on a small body with charge Q is proportional to Q. The proportionality factor $\vec{E}$ is called the strength of the electric field of the other charges, i.e. all charges except the “point charge” under consideration:

$$\vec{F} = \vec{E} \cdot Q$$

14.3 Newton's law of gravitation – Coulomb's law

Newton's law of gravitation (1687)

The position of two mass points m_1 and m_2 is given by the position vectors $\vec{r}_1$ and $\vec{r}_2$, Fig. 14.2.

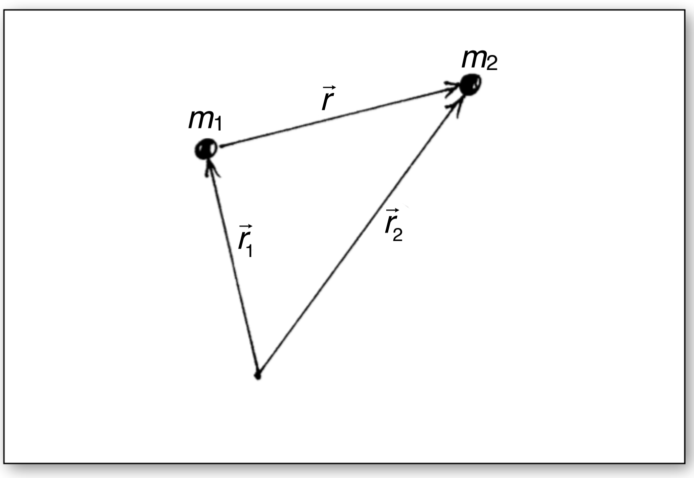

Fig. 14.2
The law of gravitation

Newton showed that the force $\vec{F}_2$ exerted by the mass point m_1 on the mass point m_2 is given by

$$\vec{F}_2 = -G\frac{m_1 \cdot m_2}{r^2}\left(\frac{\vec{r}}{r}\right)$$

Here is

$$G = 6.67 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$$

a universal constant, the *gravitational constant*, and $\vec{r} = \vec{r}_2 - \vec{r}_1$ is the distance vector between m_1 and m_2. From this, one obtains the field strength $\vec{g}_1$ of the field of m_1 alone:

$$\vec{g}_1(\vec{r}) = -G\frac{m_1}{r^2}\left(\frac{\vec{r}}{r}\right)$$

If we consider m_1 and the corresponding field alone, we can omit the index 1. We thus obtain the field strength distribution of the field of a mass point of mass m:

$$\vec{g}(\vec{r}) = -G\frac{m}{r^2}\left(\frac{\vec{r}}{r}\right)$$

Thus, a vector $\vec{g}(\vec{r})$ is assigned to each point of the field. The field strength vector arrows point to the mass point m, Fig. 14.3.

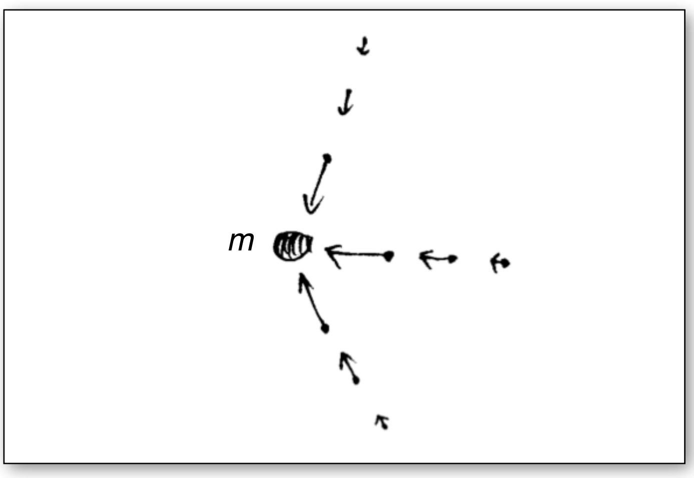

Fig. 14.3
The field strength arrows point to the mass point *m*.

Coulomb's laws (1785)

One hundred years after the publication of the law of gravitation, Coulomb showed that a similar relation holds for the force $\vec{F}_2$ exerted by a point charge Q_1 on a point charge Q_2, and exerted by a (point) magnetic pole of pole charge Q_{m1} on one of charge Q_{m2}. We start with the law for the electrostatic forces:

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 \cdot Q_2}{r^2}\left(\frac{\vec{r}}{r}\right)$$

Here

$$\varepsilon_0 = 8.85 \cdot 10^{-12} \frac{\mathrm{As}}{\mathrm{Vm}}$$

is the *electric constant*

For the electric field strength $\vec{E}_1$ of the field of Q_1 alone we get

$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}\left(\frac{\vec{r}}{r}\right)$$

or if we omit the index 1

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{\vec{r}}{r}\right)$$

If the charge is positive, the field strength vectors point away from the charge. If it is negative, they point towards the charge.

The corresponding magnetic laws are

$$\vec{F}_2 = \frac{1}{4\pi\mu_0}\frac{Q_{m1} \cdot Q_{m2}}{r^2}\left(\frac{\vec{r}}{r}\right)$$

and

$$\vec{H}(\vec{r}) = \frac{1}{4\pi\mu_0}\frac{Q_m}{r^2}\left(\frac{\vec{r}}{r}\right)$$

where

$$\mu_0 = 1.2566 \cdot 10^{-6} \frac{\mathrm{Vs}}{\mathrm{Am}}$$

is the *magnetic constant.*

14.4 Field line pictures – divergence-free fields

A particularly practical way to represent field strength distributions is the *field line diagram*. Instead of the field strength arrows one draws continuous lines, in such a way that the field strength vector arrows are tangents to the lines. In the case of the field of a mass point m_0, the field lines are straight lines running radially outward from m_0, as shown in Fig. 14.4.

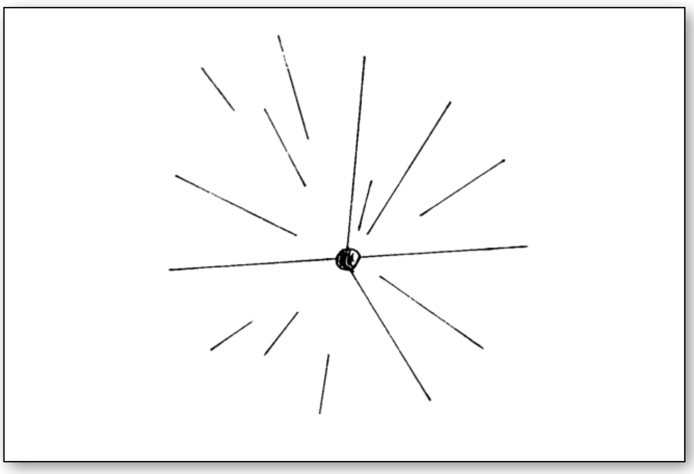

Fig. 14.4
The vector arrows have been replaced by lines to which they are tangents.

In this figure, however, another agreement has not yet been taken into account: The density of the lines, i.e. the number of lines Z per area A perpendicular to the lines, is chosen to be proportional to the field strength:

$$\frac{Z}{A} = k \cdot |\vec{g}|$$

Let us calculate how the number of lines depends on the distance r from the center. We calculate the number $Z(r)$ of lines that pierce a spherical surface with radius r.

$$Z(r) = k \cdot |\vec{g}| \cdot A = kG \cdot \frac{m_0}{r^2} \cdot 4\pi r^2 = kG \cdot 4\pi m_0$$

The number of lines is therefore independent of r. Each spherical surface is pierced by the same number of lines. This means that the field lines are continuous: They run from the mass point radially outwards. Now we assign a direction to each field line: the same direction as the field strength vectors which are tangential to the line. So the field lines come from outside and run radially towards the mass point, Fig. 14.5.

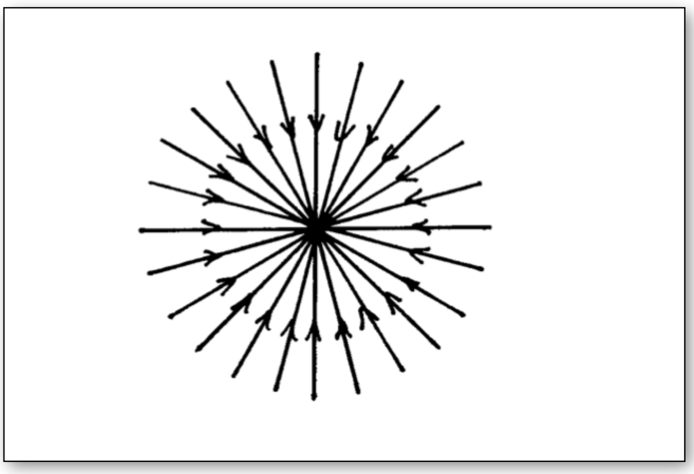

Fig. 14.5
Field lines only end on masses.

This statement is much more general than it seems from our derivation. Actually it holds true:

The field lines of static gravitational fields end on masses.

At places of the field, where the mass density is zero, no field lines begin or end. One also says: The field $\vec{g}(\vec{r})$ is divergence-free at places with $\rho_m(\vec{r}) = 0$. Mathematically this fact is expressed as follows

$$\operatorname{div} \vec{g}(\vec{r}) = 0 \quad \text{for} \quad \rho_m = 0$$

Thus, the field lines of the gravitational field have a property similar to that of the streamlines of a water flow. They suggest that something is flowing. But in fact they are no streamlines. In particular they represent neither the streamlines of the momentum nor the streamlines of the energy.

Everything said in this section applies *mutatis mutandis* to the electric field. In particular:

The electric field lines start at positive charges and end at negative charges.

14.5 Superposition of field strength distributions

The strength of the field in the vicinity of a mass point m_0 is

$$\vec{g}(\vec{r}) = -G\frac{m}{r^2}\left(\frac{\vec{r}}{r}\right) \quad (14.2)$$

If the mass of the mass point is doubled from m_0 to $2m_0$, the field strength is doubled at each position $\vec{r}$. But doubling the mass is the same as if one had put 2 mass points of mass m_0 at the same place. The strength of the field associated with the two mass points is therefore equal to the sum of the two field strengths, Fig. 14.6.

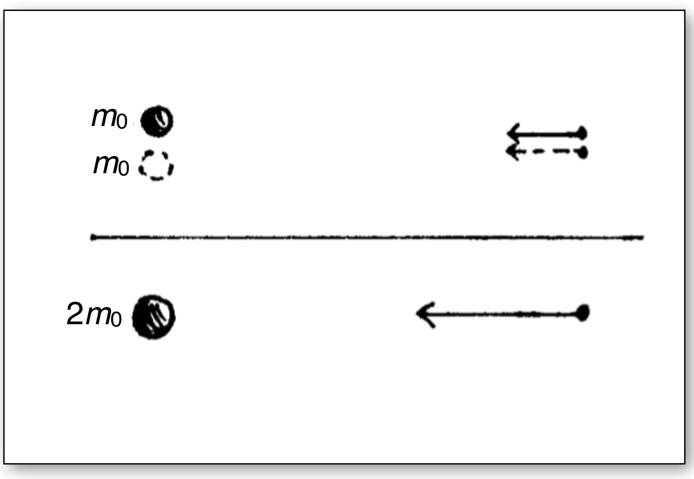

Fig. 14.6
The strength of the field produced by $2m_0$ at any point is twice that of the field produced by m_0.

Experience tells us that this is also true in a much more general way: the strength of the field at a point P of two mass points located at arbitrary positions is equal to the vector sum of the strengths of the fields that each mass point would produce at point P if it were present alone, Fig. 14.7.

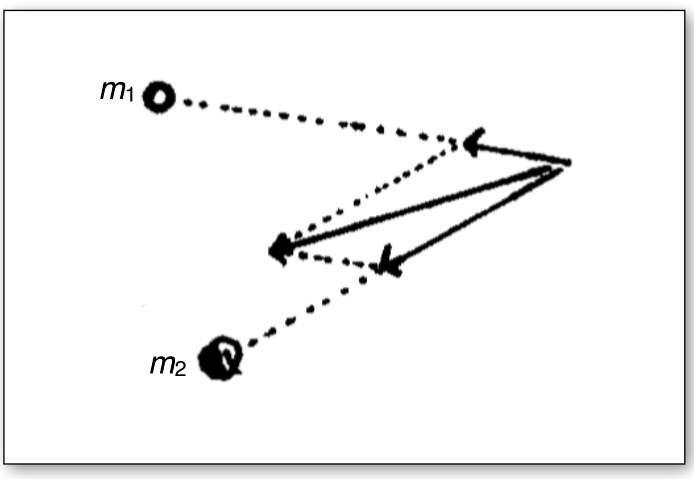

Fig. 14.7
The strength of the field of m_1 and m_2 is equal to the vector sum of the field strengths of the fields of m_1 and m_2 separately.

Therefore, one can construct the gravitational field strength of a mass distribution from the field strengths belonging to the individual masses. The analogous is valid for the electric field strength.

Attention: From the fact that the field strengths behave additively, it does not follow that all other quantities of the field also behave additively. In particular, the substance-like quantities of the fields do not behave additively.

Let us again look at the relationship

$$\vec{F} = \vec{g} \cdot m$$

We apply it to the body 0 in Figure 14.8.

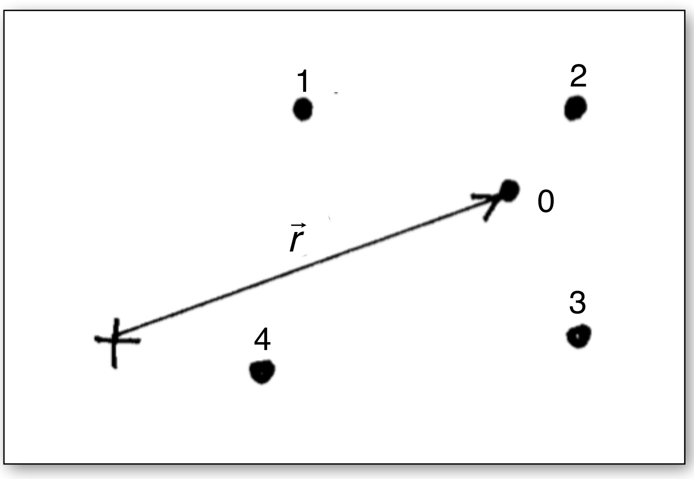

Fig. 14.8
Applying the relation $\vec{F} = \vec{g} \cdot m$ to body 0, $\vec{g}$ is the strength of the field of bodies 1 to 4.

$\vec{g}(\vec{r})$ is the strength of the field, which the bodies 1 to 4 create at the position $\vec{r}$. It is the strength which the gravitational field would have there if the body 0 were not present. $\vec{g}(\vec{r})$ is therefore not the strength of the actually existing field. The field that is actually present is quite different because of the presence of body 0. Figures 14.9a and 14.9b show this for the example where $\vec{g}(\vec{r})$ originates from a single mass point.

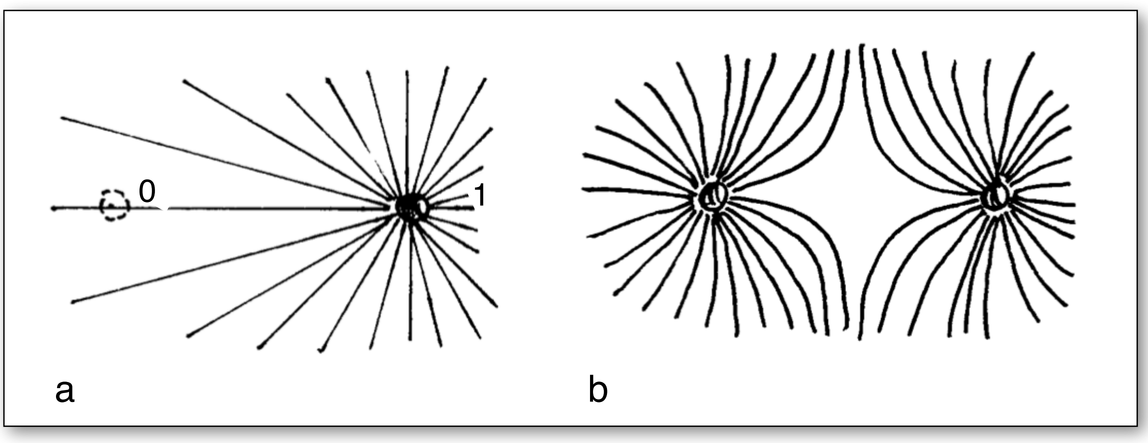

Fig. 14.9
(a) The field strength distribution to be substituted into $\vec{F} = \vec{g} \cdot m$. (b) The field strength distribution of the field actually present.

Fig. 14.9a shows the field line diagram of the field whose field strength must be substituted into $\vec{F}_0 = \vec{g} \cdot m_0$ if one wants to calculate the force $\vec{F}_0$ which mass point m_1 exerts on mass point m_0. Fig. 14.9b shows the field line diagram of the actually existing field.

The mass in $\vec{F} = \vec{g} \cdot m$ is often called test mass, the charge in $\vec{F} = \vec{E} \cdot Q$ correspondingly test charge, because one likes to imagine that one uses this mass or charge only to measure the field strength which would be present without it. Thus one takes it out again after the measurement, so that now the field strength actually has the value which one has determined with the help of the test mass or test charge.

14.6 Examples of field strength distributions

(a) The field strength distribution of an infinitely extended plate capacitor

The field lines of a uniformly electrically charged plate run perpendicular to the plate to both sides for reasons of symmetry, Fig. 14.10a and 14.10b. The field strength in a plate capacitor is obtained by adding the field strength of a positively charged plate and a negatively charged plate offset from the first, Fig. 14.10c. The resulting field has zero field strength everywhere outside the plates. Between them it is homogeneous.

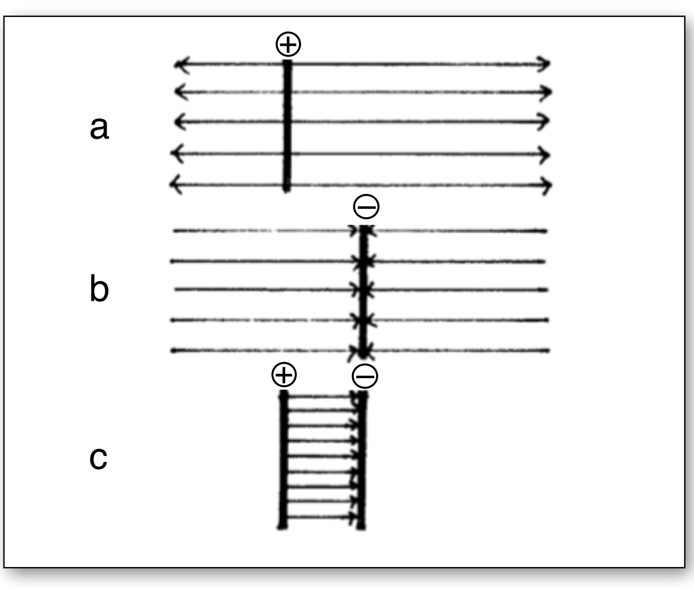

Fig. 14.10
The field inside the capacitor (c) results from the superposition of the fields of the plates (a and b).

In the case of two similarly charged plates or an arrangement of two flat parallel mass plates, the space between the plates is field-free, the field strength is zero here. Outside, the field is homogeneous, Fig. 14.11.

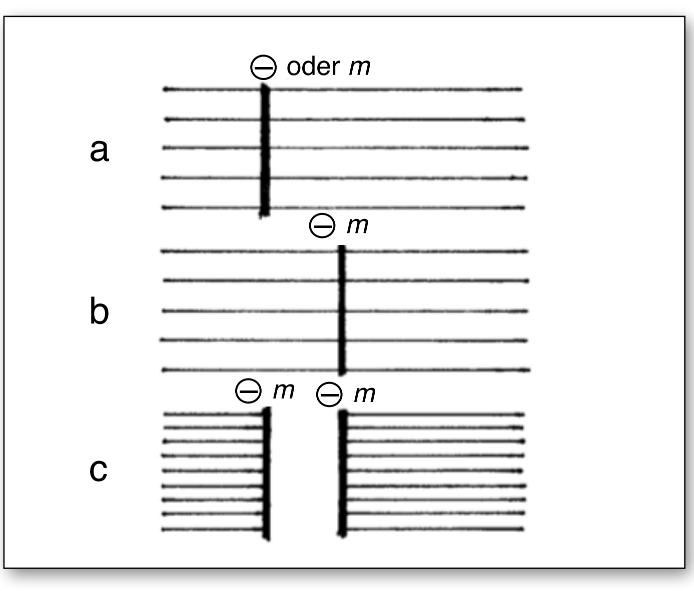

Fig. 14.11
The field between two similarly charged plates or between two mass plates (c) results from the superposition of the fields of the plates (a and b).

(b) The field strength distribution of the field of two point charges and of two mass points

The picture is obtained from that of the two separate point charges or masses by graphical addition of the field strength vectors. Fig. 14.12 shows the field of two point charges of the same magnitude but different sign.

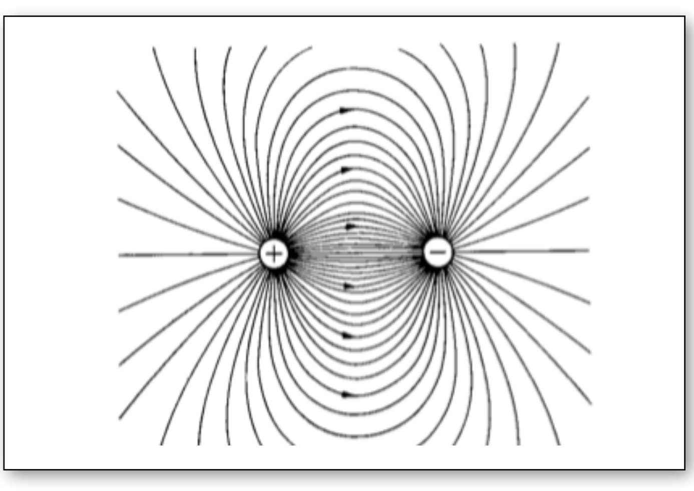

Fig. 14.12
$\vec{E}$ field lines of the field of two point charges of the same magnitude and opposite sign.

Fig. 14.13 shows the field strength distribution of two point charges of the same magnitude and the same sign. It is identical to that of two mass points of the same mass.

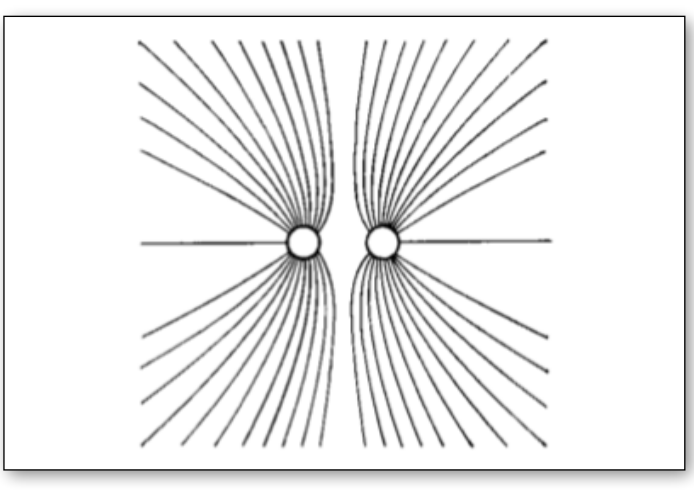

Fig. 14.13
$\vec{E}$ or $\vec{g}$ field lines of the field of two equal point charges or two equal mass points.

c) The field strength distribution of the field of a spherically symmetrical charge or mass distribution

For reasons of symmetry, the field lines must run radially outwards, Fig. 14.14. We place a spherical shell around the center in our mind, so that there are no sources (charges or masses) outside. The field line density here is determined by the total charge (resp. mass) inside the spherical surface. The field strength is therefore the same as if the whole charge resp. mass would be concentrated in the center of the sphere.

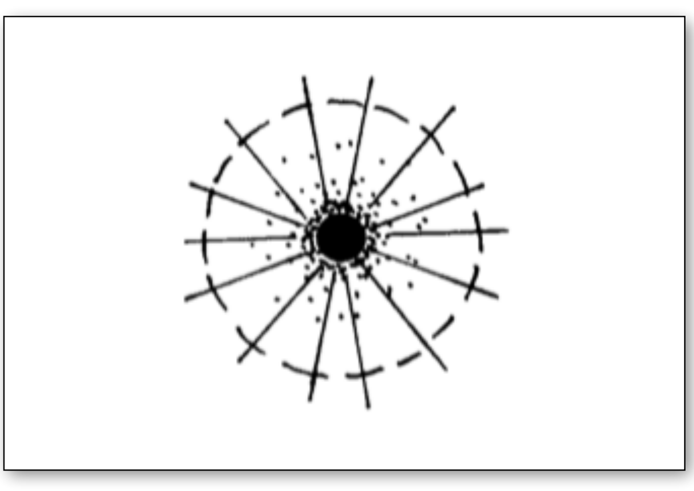

Fig. 14.14
Field lines of a spherically symmetric source distribution

From this follows e.g. that the field strength of the gravitational field at the earth's surface is the same, as if the whole mass of the earth would be concentrated in the earth's center. Thus it holds equation (14.2):

$$\vec{g}(\vec{r}) = -G\frac{m_0}{r^2}\left(\frac{\vec{r}}{r}\right)$$

where for m_0 the mass and for r the radius of the earth is to be inserted. With $m_0 = 5.977 \cdot 10^{24}$ kg and $r = 6.371 \cdot 10^6$ m we get

$$\vec{g}(\vec{r}) = -9.82\frac{\text{N}}{\text{kg}}\left(\frac{\vec{r}}{r}\right)$$

Substituting this value into equation (14.1), we obtain the weight at the earth's surface, compare section 4.4.

(d) The field strength distribution for a source distribution of the shape of a spherical shell

Outside of r_0 the field strength is the same as if the sources were concentrated in the center of the sphere. Inside the spherical shell, the field lines could only run radially for reasons of symmetry. If they would run like this, there would have to be a source in the center, which is not the case. Consequently, the field strength is zero everywhere inside the sphere, Fig. 14.15.

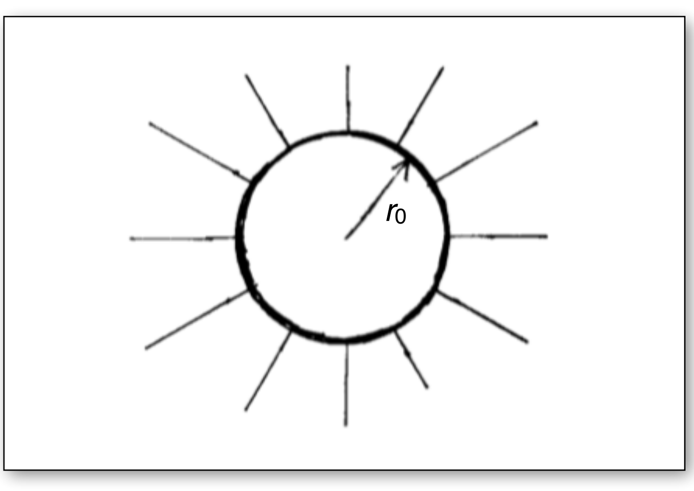

Fig. 14.15
Field lines of a spherical shell source distribution

If the electric charge is homogeneously distributed on a very thin spherical shell of radius r_0, the charge per area is

$$\frac{Q}{A} = \frac{Q}{4\pi r_0^2}$$

Since the field strength just outside the spherical surface is

$$\vec{E}(\vec{r}_0) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_0^2}\left(\frac{\vec{r}}{r}\right)$$

we get

$$\frac{|Q|}{A} = \varepsilon_0|\vec{E}|$$

This is a local statement about a point of the surface of the charged sphere. It is always valid if the field lines run away from a charged surface perpendicularly in only one direction.

In an analogous way we calculate the relation between the mass per area m/A and the field strength $\vec{g}$. With

$$\frac{m}{A} = \frac{m}{4\pi r_0^2}$$

and

$$\vec{g}(\vec{r}_0) = G\frac{m}{r_0^2}\left(\frac{\vec{r}}{r}\right)$$

we obtain

$$\frac{m}{A} = \frac{|\vec{g}|}{4\pi G}$$

14.7 Mechanical stress in static fields

We consider the charge of plate 2 in Fig. 14.16 as a test charge in the field of plate 1.

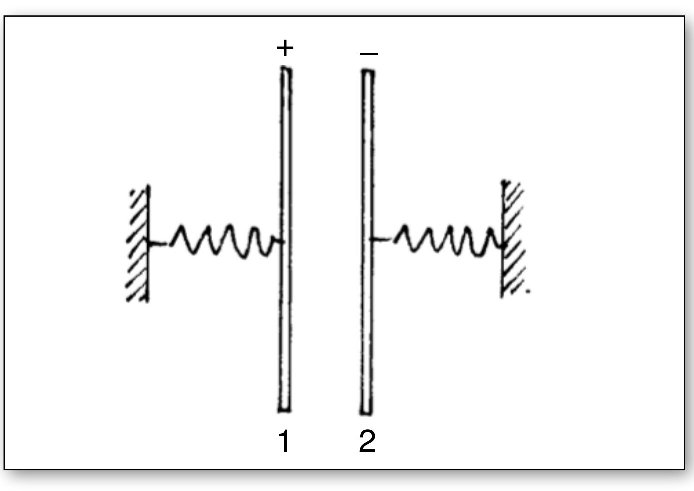

Fig. 14.16
The charge of plate 2 is considered as a test charge in the field of plate 1.

Since the validity of the equation $\vec{F}_2 = Q_2\vec{E}_1$ is limited to one point, we decompose plate 2 into many small segments of equal size, each carrying the charge Q_{2i}. Its total charge is

$$Q_2 = \sum_i Q_{2i}$$

On each of these segments Q_{2i} Plate 1 exerts the force

$$\vec{F}_{2i} = Q_{2i}\vec{E}_1$$

Since $\vec{E}_1$ has the same value at all the locations of all the charges Q_{2i}, and all Q_{2i} are equal to each other, the total force $\vec{F}_2$ on plate 2 becomes

$$\vec{F}_2 = \sum \vec{F}_{2i} = \vec{E}_1 \sum Q_{2i} = \vec{E}_1 Q_2$$

Because of the homogeneity of the field of plate 1, this formula is the same as if Q_2 were a point charge.

If $\vec{E}$ is the field strength in the complete capacitor, we get

$$\vec{E}_1 = \vec{E}/2$$

And thus

$$\vec{F}_2 = Q_2\frac{\vec{E}}{2}$$

According to section 14.7 (d) it is

$$\frac{|Q_2|}{A} = \frac{\varepsilon_0}{|\vec{E}|}$$

since to the right of plate 2 the field strength is zero. We therefore obtain

$$\vec{F}_2 = -\frac{\varepsilon_0}{2}\vec{E}^2 A$$

But this is not only the force that plate 1 exerts on plate 2. It is also the force that plate 1 exerts on the field immediately in front of plate 1, and it is the force that the field immediately in front of plate 2 exerts on plate 2, and it is also the force that the left half of the field exerts on the right half. In momentum current terms, F_x is the magnitude of the momentum current flowing from plate 1 to plate 2. It doesn't matter, of course, whether one considers the current at the location of plate 1 or plate 2, or anywhere in between: It has the same strength everywhere, Fig. 14.17.

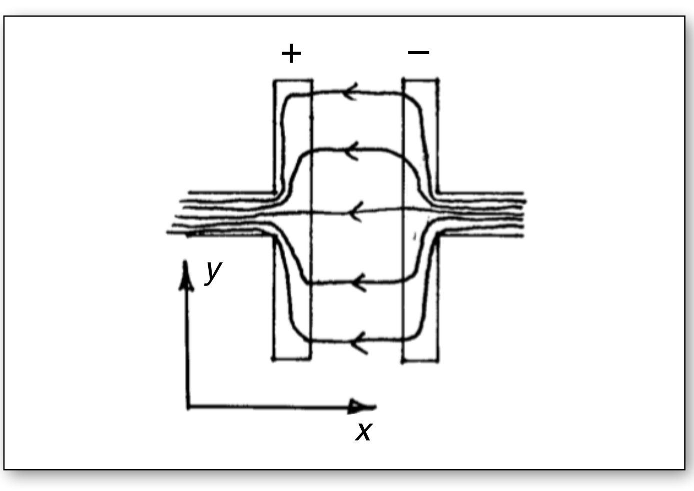

Fig. 14.17
The momentum current flows through the field from one plate to the other. The field is under tensile stress in the *x*-direction.

The fact that a momentum current flows through the field or that one part of the field exerts a force on another part is equivalent to the field itself being under mechanical tension. Since one part of the field pulls on the other, it is a tensile stress. The mechanical stress σ_x in the *x*-direction is obtained by dividing the force by the area:

$$\sigma_x = -\frac{\varepsilon_0}{2}\vec{E}^2$$

Therefore, there is tensile stress within the capacitor in the direction of the field lines, regardless of whether the left plate is positive and the right plate negative or vice versa. A small section of an electric field does not show by which arrangement the field is generated. It is therefore generally valid:

Within the electric field, there is tensile stress in the direction of the field lines.

We now consider the electric field of two equally charged plates, as well as the gravitational field of two parallel mass plates, Figs. 14.18a and 14.18b.

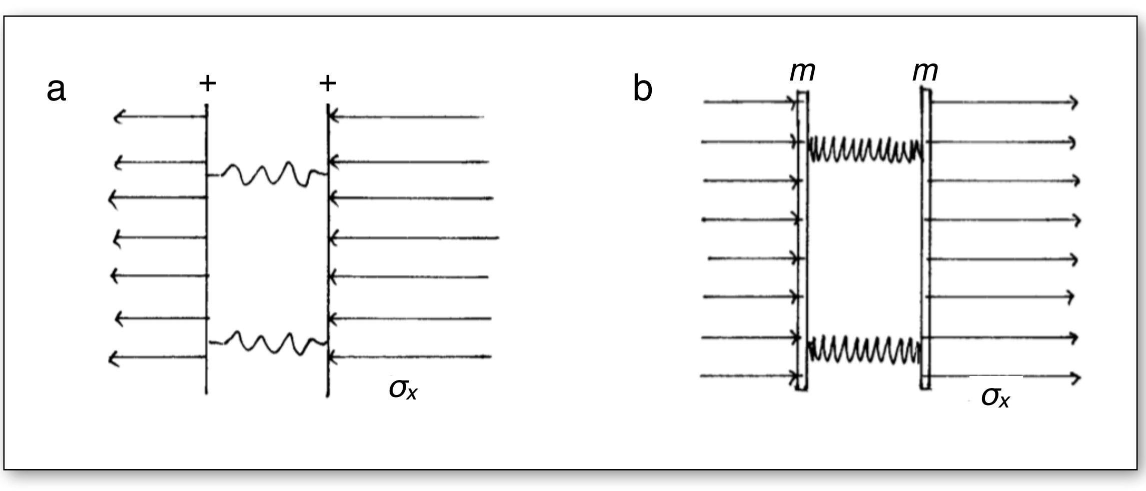

Fig. 14.18
(a) The like charged plates are pulled away from each other by the electric field. (b) The mass plates are pushed towards each other by the gravitational field.

We had already found the field lines in section 14.7. The space between the plates is field-free. Thus, no momentum current flows here. In the figure the momentum is conducted between the plates by springs. In the electric case, the springs and the field outside the plates are under tension. In the gravitational field, the springs are under compressive stress. Therefore, there must be compressive stress in the field as well:

Within the gravitational field, there is compressive stress in the direction of the field lines.

We calculate the value of the mechanical stress σ_x in the gravitational field like that of σ_x in the electric field.

$$\vec{F}_2 = m_2\vec{g}_1$$

If we denote the field strength outside the pair of plates by $\vec{g}$, then

$$\vec{g}_1 = \vec{g}/2$$

With this and with

$$\frac{m_2}{A} = \frac{|\vec{g}|}{4\pi G}$$

we get

$$\vec{F}_2 = \frac{1}{4\pi G}\vec{g}^2 A$$

and

$$\sigma_x = \frac{1}{8\pi G}\vec{g}^2$$

Therefore, equally charged bodies are not pushed away from each other by the field, but pulled away from one another. Likewise, two masses, for example the earth and the moon, are not pulled towards each other by the field, but are pushed towards each other from the outside.

In the electric field as well as in the gravitational field there is also a mechanical stress perpendicular to the field lines: in the electric field a compressive stress and in the gravitational field a tensile stress. It is easy to understand that in the electric field of a plate capacitor there is a compressive stress transverse to the field lines. The plates of the capacitor in Fig. 14.19 are under tensile stress in the *y*-direction. Thus, a *y*-momentum current flows within the plates from top to bottom. It can only flow back through the field. In the field, therefore, it flows from bottom to top. This means that there is a compressive stress in the field. The tensile and compressive stresses in electromagnetic fields were discovered by Faraday around 1840.

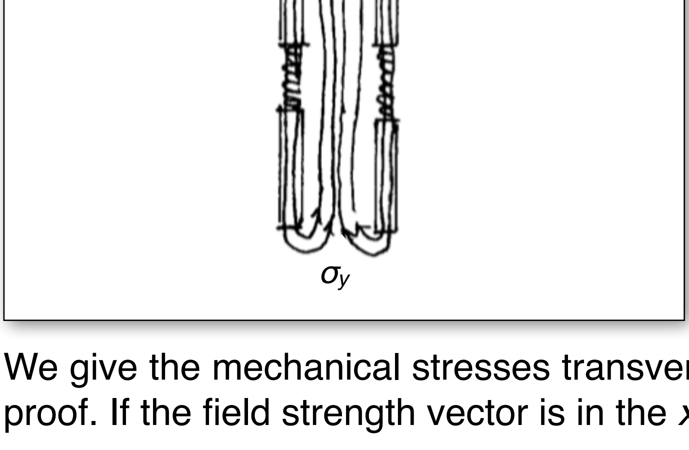

Fig. 14.19
The capacitor plates are under tensile stress. Therefore, the field between them must be under compressive stress in the vertical direction.

We give the mechanical stresses transverse to the field lines without proof. If the field strength vector is in the *x*-direction, then:

electric field	$\sigma_y = \frac{\varepsilon_0}{2}\vec{E}^2$	$\sigma_z = \frac{\varepsilon_0}{2}\vec{E}^2$
gravitational field	$\sigma_y = -\frac{1}{8\pi G}\vec{g}^2$	$\sigma_z = -\frac{1}{8\pi G}\vec{g}^2$

14.8 The energy distribution in the static electric field and in the static gravitational field

If one plate of a charged capacitor is moved against the other, as shown in Fig. 14.20, energy flows into the field of the capacitor according to

$$P = v \cdot F$$

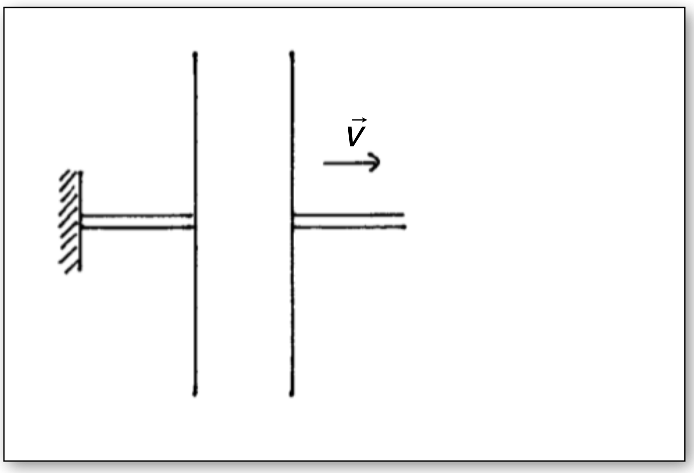

Fig. 14.20
If the right plate is moved to the right, energy flows into the field between the plates.

With $P = dE/dt$ (for the field between the plates) and $v = dx/dt$ we get

$$dE = Fdx$$

The total energy that is within the capacitor when the plate spacing is x_0 is obtained by integrating from $x = 0$ to $x = x_0$:

$$E = \int_0^{x_0} F\,dx$$

With

$$F = \frac{\varepsilon_0}{2}\vec{E}^2 A$$

(where A is the area of a capacitor plate) results in

$$E = \frac{\varepsilon_0}{2} A \vec{E}^2 x_0$$

The product of area and plate spacing is equal to the volume of the field. Therefore, the energy density $\rho_E = E/V$ becomes:

$$\rho_E = \frac{\varepsilon_0}{2}\vec{E}^2$$

To calculate the energy density of the gravitational field, we consider two parallel mass plates. The calculation is analogous to the previous one:

$$E = \int_0^{x_0} F\,dx$$

With

$$F = \frac{1}{8\pi G}\vec{g}^2 A$$

we obtain for the amount of energy flowing to the plates

$$E = \frac{1}{8\pi G}\vec{g}^2 A x_0$$

However, this amount of energy is not accompanied by the generation of field, but by the annihilation of the field between the plates. We can take this fact into account by saying that the energy density in the field is negative:

$$\rho_E = -\frac{1}{8\pi G}\vec{g}^2$$

So one can imagine that energy is needed to “build up” gravitational field free space.

14.9 The gravitational potential

The field strength distribution for a mass point is given by equation (14.2). The same equation is valid for any other spherically symmetric mass distributions, but only outside the region where the mass distribution is located.

We consider the field of a spherically symmetric mass distribution of total mass m_0. We bring a small body of mass m to the location $\vec{r}_0$. Let the origin of the position vectors lie in the center of symmetry O of the mass distribution m_0. We now move the body of mass m to another position $\vec{r}$, Fig. 14.21. If this position is farther away from O than $\vec{r}_0$, energy must be supplied for the displacement. This energy is transferred into the gravitational field common to both masses m_0 and m.

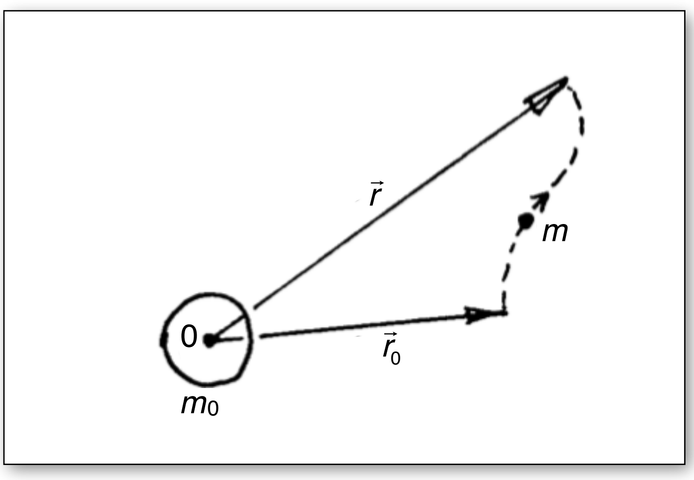

Fig. 14.21
For the displacement of the small body from $\vec{r}_0$ o to $\vec{r}$ energy must be invested.

We calculate the amount of energy to be supplied:

$$E(\vec{r})-E(\vec{r}_0)=\int_{\vec{r}_0}^{\vec{r}}\vec{F}\,d\vec{r}=G\cdot m\cdot m_0\int_{\vec{r}_0}^{\vec{r}}\frac{dr}{r^2}=-G\cdot m\cdot m_0\left(\frac{1}{r}-\frac{1}{r_0}\right) \qquad (14.3)$$

The greater the distance $\vec{r}$ to which m is brought, the more energy is required.

We now divide this energy by the mass m of the body we are moving. This energy per mass

$$\frac{E(\vec{r})-E(\vec{r}_0)}{m}=G\cdot m_0\left(\frac{1}{r}-\frac{1}{r_0}\right)$$

is independent of m. It only depends on the mass m_0 of the mass distribution. It is therefore a function of $\vec{r}$, describing the field of m_0 alone. We call the quantity

$$\frac{E(\vec{r})}{m}=V(\vec{r})$$

the *gravitational potential*.

Using (14.3), we can write the potential of the spherically symmetric mass distribution:

$$V(\vec{r})-V(\vec{r}_0)=-G\cdot m_0\left(\frac{1}{r}-\frac{1}{r_0}\right)$$

This equation defines the gravitational potential only up to an arbitrary additive constant. This means that the zero point of V can be chosen arbitrarily. One often defines

$$V(\vec{r}\to\infty)=0$$

With that we get

$$V(\vec{r})=-\frac{G\cdot m_0}{r}$$

The description of a field using the potential distribution $V(\vec{r})$ is equivalent to the description using the field strength distribution $\vec{g}(\vec{r})$. One can be calculated from the other. We have shown the connection between the two by the example of the field of the spherically symmetric mass distribution. In general we have

$$V(\vec{r})-V(\vec{r}_0)=-\int_{\vec{r}_0}^{\vec{r}}\vec{g}(\vec{r})\,d\vec{r}$$

$$\vec{g}(\vec{r})=-\nabla V(r)$$

The condition $V(\vec{r})=\text{const}$ defines a surface of constant potential. For different values V_i different *equipotential surfaces* result. The equipotential surfaces of the field of a spherically symmetric mass distribution are spherical surfaces, Fig. 14.22.

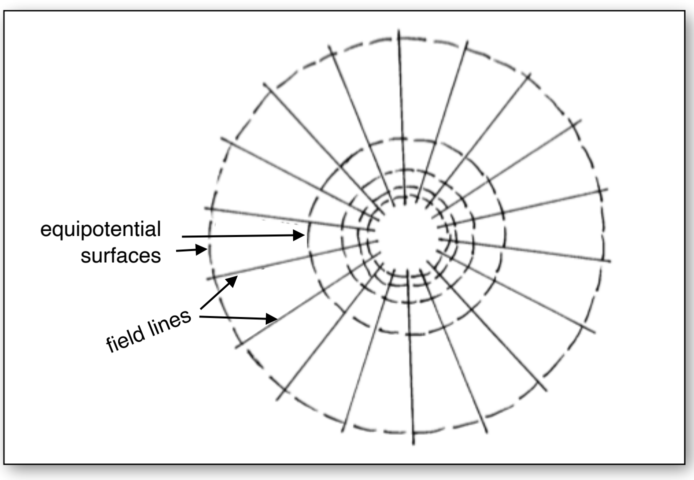

Fig. 14.22
Equipotential surfaces and field lines of a spherically symmetric mass distribution

The equipotential surfaces of a field are perpendicular to the field lines at every point. The graphical representation of the equipotential surfaces is therefore just as suggestive as that of the field lines.

Fig. 14.23 shows field lines and equipotential surfaces of two spherically symmetric bodies of different mass.

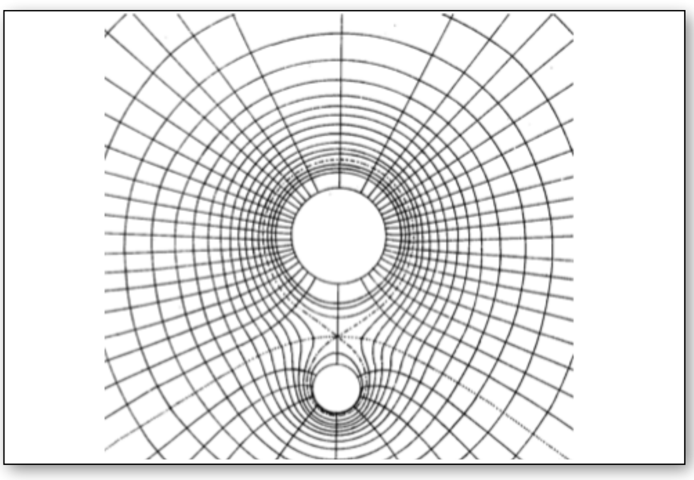

Fig. 14.23
Equipotential surfaces and field lines of two spherically symmetric bodies of different masses

14. 10 The two-body problem

We ask for the motion of two bodies attached to each other by a field, Fig. 14.24.

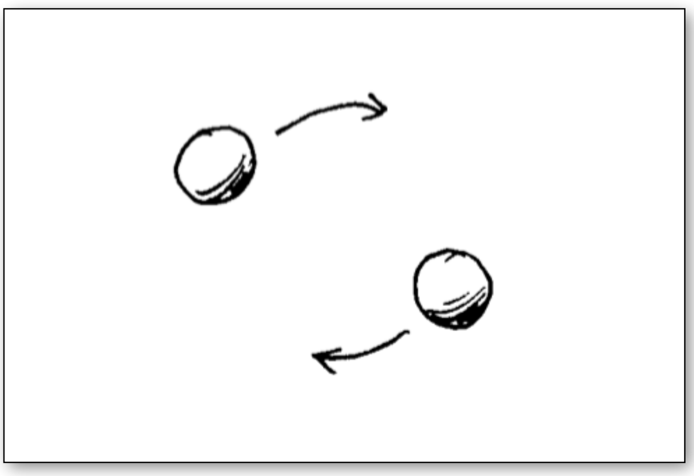

Fig. 14.24
Two-body problem

During the movement, momentum currents flow through the field between the two bodies. The field cannot absorb any momentum. Its momentum therefore does not change.

In addition, energy currents flow during the movement. Since the field can store energy, the energy of the field temporarily increases at the expense of the energy of the bodies and vice versa, Fig. 14.25a.

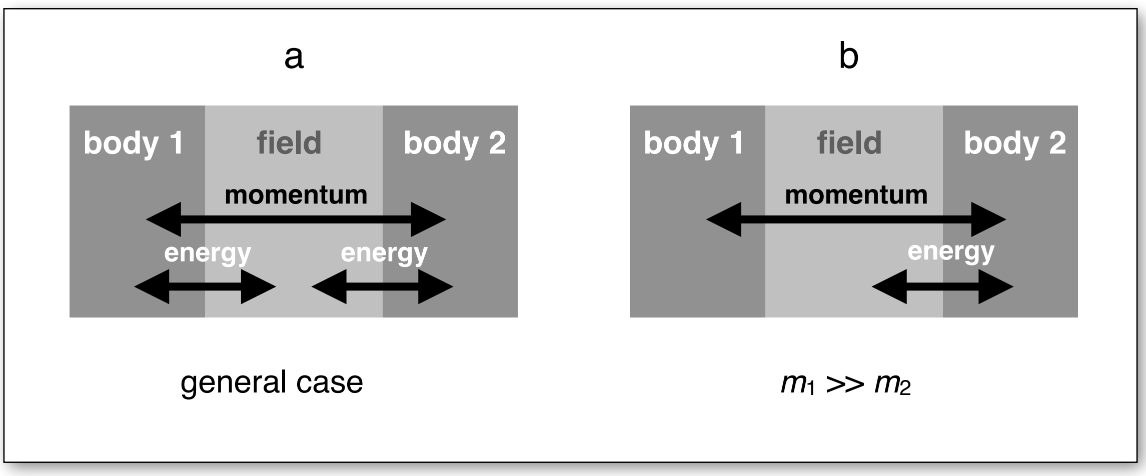

Fig. 14.25
Energy and momentum flow in the system body-field-body.
(a) The masses of the two bodies are of the same order of magnitude.
(b) The mass of one body is much larger than that of the other.

There is an important special case: the mass of one body is much larger than that of the other, i.e. $m_1 >> m_2$.

Examples:

- sun – earth: $m_{sun} >> m_{earth}$
- earth – moon: $m_{earth} >> m_{moon}$
- deflection coils of a cathode-ray tube – electron: $m_{tube} >> m_{electron}$
- atomic nucleus – electron: $m_{nucleus} >> m_{electron}$
- earth – apple: $m_{earth} >> m_{apple}$

In these cases, the energy practically only flows back and forth between the field and the light body, Fig. 14. 25b. Moreover, the motion, i.e., the velocity, of the heavy against that of the light is negligible. Why?

We consider the energy of the bodies in the center of mass system. From

$$E = \frac{\vec{p}^2}{2m} + E_0$$

follows

$$dE = \frac{\vec{p}}{m} d\vec{p}$$

$$d\vec{p}_1 = -d\vec{p}_2 = d\vec{p}$$

If momentum flows from one body to the other, it is $d\vec{p}_1 = -d\vec{p}_2 = d\vec{p}$. Furthermore, $\vec{p}_1 = -\vec{p}_2 = \vec{p}$. This gives

$$dE_1 = \frac{\vec{p}d\vec{p}}{m_1} \quad \text{and} \quad dE_2 = \frac{\vec{p}d\vec{p}}{m_2}$$

dE_1 and dE_2 have the same sign: If E_1 increases, then E_2 also increases. The energies dE_1 and dE_2 both come from the field. But because of $m_1 >> m_{2c}$ we have $dE_1 << dE_2$, i.e. dE_1 is negligible against dE_2. Thus, the energy practically only flows back and forth between the field and the light body 2. Because of

$$\vec{p}_2 = m_2\vec{v}_2 = -\vec{p}_1 = -m_1\vec{v}_1$$

with $m_1 >> m_2$ also $\vec{v}_1$ is negligible against $\vec{v}_2$. So in the center of mass system velocity and kinetic energy of the heavy body are practically zero.

Since body 1 does not move, the contribution of body 1 to the field is constant in time. Therefore, it is also said, “The field of body 1 is constant in time.” Obviously, here by “field” is meant the field strength distribution $\vec{g}_1(\vec{r})$.

14.11 The planetary motion

We apply the angular momentum conservation law to a two-body system. Let us imagine that one body is the sun (index ⊙), the other one a planet (index P).

$$\vec{L} = \vec{r}_\odot \times \vec{p}_\odot^{\,(cm)} + \vec{r}_P \times \vec{p}_P^{\,(cm)} = \text{const} \tag{14.4}$$

Here $\vec{p}_\odot^{\,(cm)}$ and $\vec{p}_P^{\,(cm)}$ are the momentums in the center of mass system. So we have

$$\vec{p}_\odot^{\,(cm)} = -\vec{p}_P^{\,(cm)}$$

Furthermore, let the zero point of the spatial coordinates $\vec{r}_\odot$ and $\vec{r}_P$ be the center of mass, so that the following applies

$$m_\odot \vec{r}_\odot + m_P \vec{r}_P = 0$$

or

$$\vec{r}_\odot = -\frac{m_P}{m_\odot}\vec{r}_P$$

Substituting into equation (14.4) results in

$$\frac{m_P}{m_\odot}\vec{r}_P \times \vec{p}_P^{\,(cm)} + \vec{r}_P \times \vec{p}_P^{\,(cm)} = \text{const}$$

Therefore

$$\vec{r}_P \times \vec{p}_P^{\,(cm)} = \text{const}$$

With $\vec{p} = m\vec{v} = m(d\vec{r}/dt)$ we get

$$m_P \vec{r}_P \times \vec{v}_P = m_P \vec{r}_P \times \frac{d\vec{r}_P}{dt} = \text{const}$$

or

$$\vec{r}_P \times \frac{d\vec{r}_P}{dt} = \text{const}$$

Now $(1/2)(\vec{r}_P \times d\vec{r}_P)$ is the area "swept" by the position vector $\vec{r}_P$ in the time interval dt. Therefore, because of $\vec{L} = \text{const}$, the area swept per time is constant in time. For the planet this means that its tangential velocity is small on that part of its orbit which is far from the sun, on the part which is close to the sun it is large, Fig. 14.26.

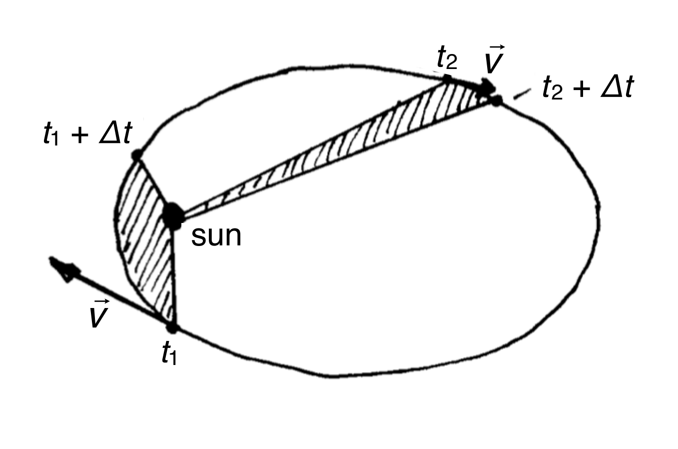

Fig. 14.26
The area swept per time by the planet's position vector is constant.

Kepler discovered this fact before the law of gravitation or the law of angular momentum conservation were known. It is called Kepler's second law.

The trajectories of a planet in the field of the sun or more generally, of a body in a $1/r^2$- field can be calculated analytically. But the calculation is quite tedious. We describe here simply the result: The planet moves on a conic section (ellipse, hyperbola, parabola, circle, straight line), with the sun in one of the two foci of the conic section. This was also discovered by Kepler (more precisely: that planets move on elliptical orbits). This is Kepler's first law.

The orbits of the planets of the sun are ellipses, whose eccentricity (with the exception of Mercury and Pluto) is very small, thus almost circular orbits. They lie approximately in a single plane, the revolution sense of all planets is the same.

For circular orbits in the $1/r^2$- field one can easily calculate a relation between orbital period and orbital radius. For a planet moving with angular velocity ω on a circular orbit with radius r, (see section 4.5) it is

$$\frac{d\vec{p}}{dt} = -m_P \omega^2 \vec{r}$$

This change of momentum is caused by the gravitational force

$$\vec{F}_P = -G\frac{m_\odot m_P}{r^2}\vec{e}_r$$

We thus have

$$m_P \omega^2 \vec{r} = G\frac{m_\odot m_P}{r^2}\vec{e}_r$$

The mass m_P of the planet drops out, and we are left with

$$\omega^2 r^3 = G m_0$$

The expression $\omega^2 r^3$, and also r^3/T^2 (T = orbital period), has therefore the same value for all planets of the sun

$$\frac{r^3}{T^2} = \text{const}$$

This relation is also valid for elliptic orbits, if one uses the great semi-axis of the ellipses for r. It was also discovered by Kepler and is called Kepler's third law.

We still calculate the kinetic energy of a body of mass m in the $1/r^2$-field of the sun for some special cases.

(a) The body is released from rest at a large distance r_0 from the sun. It approaches the sun and receives the energy ΔE from the gravitational field. Thus, its kinetic energy at the end of its falling motion is ΔE. According to equation (14.3) it is

$$\Delta E = G \cdot m_\odot \cdot m\left(\frac{1}{r} - \frac{1}{r_0}\right) = E_{kin}$$

If r_0 is very large compared to r, then

$$E_{kin} = G \cdot m_\odot \frac{m}{r}$$

(b) The body moves on a circular orbit with the radius r. In this case $d\vec{p}/dt = \vec{F}$ writes

$$m\frac{v^2}{r} = G\frac{m_\odot m}{r^2}$$

From this follows

$$E_{kin} = \frac{m}{2}v^2 = G \cdot m_\odot \frac{m}{2r}$$

Thus, the smaller the circular orbit, the greater the kinetic energy. However, it is always half as large as the kinetic energy it would have if it had fallen down freely from infinity to this radius.

If the kinetic energy of the body at $r = \infty$ is greater than zero, it cannot move on a closed trajectory.

14.12 Weightlessness

That the person in Fig. 14.27 feels heavy, means that he feels his own weight. For example, in the legs he feels the weight of his head, upper body and arms. So the person feels the momentum currents flowing through his body.

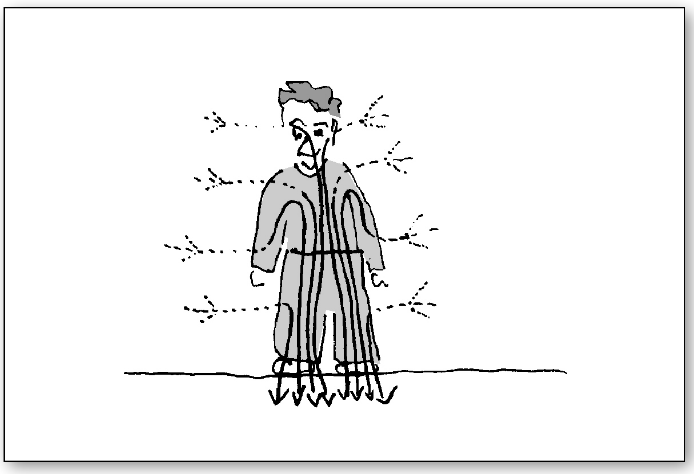

Fig. 14.27
The person feels heavy because momentum currents are flowing through him.

Fig. 14.28 shows a model person consisting of 2 blocks. Through the interface between the upper and lower block flows the momentum which has entered the upper block from the gravitational field. Through the boundary surface between the lower block and the earth additionally flows the momentum which has entered the lower block from the field.

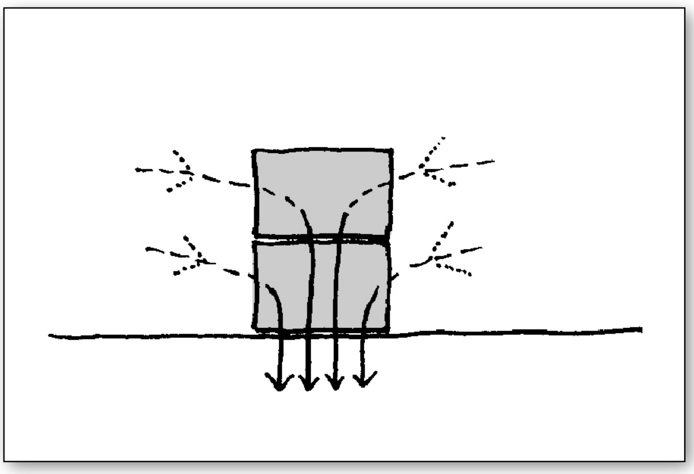

Fig. 14.28
Model person

To get rid of the feeling of heaviness, it is necessary to do something so that the momentum currents within the body disappear. There are two methods for this.

The one is to let no momentum flow in from the gravitational field. To achieve this, one must go to a place where there is no gravitational field of any planet or star.

The other method is much easier to realize: One simply prevents the outflow of the momentum that enters the various parts of the body from the gravitational field. For this it is enough to cut off the connection with the earth. In other words, one lets oneself fall freely. Fig. 14.29 shows the situation for our model person. Into each of the two blocks, and at each place of each block, momentum enters from the gravitational field. However, this momentum does not flow around within the matter of the blocks. In particular, it does not flow from the upper into the lower block. Therefore, the lower one no longer feels the weight of the upper one.

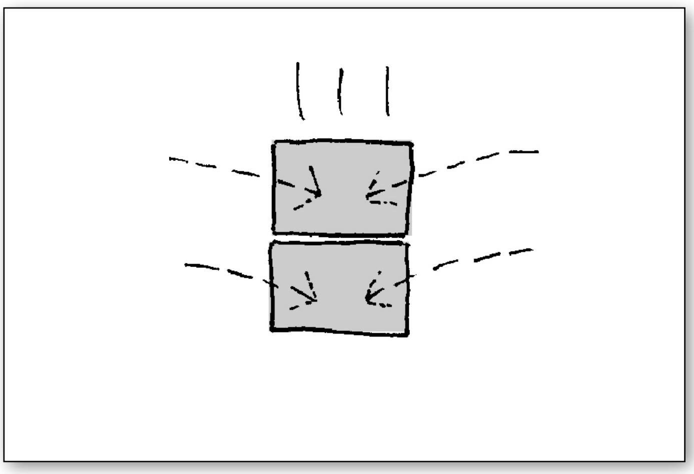

Fig. 14.29
The momentum coming from the gravitational field can no longer flow away.

While the blocks standing on the ground are under compressive stress, the free-falling blocks are stress-free.

By the way, the general theory of relativity shows that there is no difference in principle between the two realizations of weightlessness.

14.13 Tidal forces

We make a thought experiment. A kind of dumbbell, consisting of two bodies A and B of the same mass m and a light connecting rod, is in free fall in the inhomogeneous gravitational field, Fig. 14.30. Body A is located at a place of higher gravitational field strength than B. Therefore, a stronger momentum current flows from the gravitational field into A than into B.

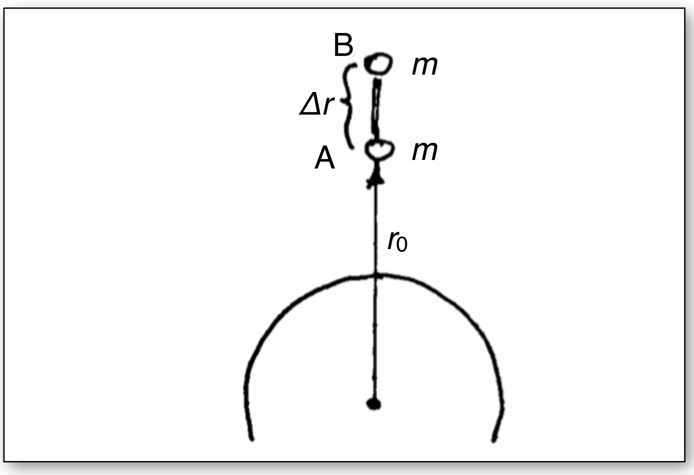

Fig. 14.30
From the gravitational field, a stronger momentum current flows into body A than into body B.

The corresponding current strengths are

$$F_{\text{field A}} = mg(r_0) \quad \text{and} \quad F_{\text{field B}} = m\left(g(r_0) + \frac{dg}{dr}\Delta r\right)$$

Since A and B are linked, the momentums of A and B can only change at the same rate:

$$\frac{dp_A}{dt} = \frac{dp_B}{dt}$$

In order for the momentum changes of A and B to be equal, momentum must constantly flow through the rod from A to B. We call the corresponding current F_T. The total momentum current flowing into body A must be just as strong as that flowing into B. Therefore the following must be valid:

$$F_A = F_B.$$

With

$$F_A = mg(r_0) - F_T \quad \text{and} \quad F_B = m\left(g(r_0) + \frac{dg}{dr}\Delta r\right) + F_T$$

we get

$$mg(r_0) - F_T = m\left(g(r_0) + \frac{dg}{dr}\Delta r\right) + F_T$$

From this we can calculate the strength of the momentum current in the rod:

$$F_T = -\frac{m}{2}\frac{dg}{dr}\Delta r$$

Since $dg/dr < 0$, F_T is positive: positive momentum flows from A to B. (The positive momentum direction is towards the earth.) This means that the rod is under tensile stress. A exerts a force on B via the rod. This force is called a *tidal force*, because it is also responsible for the tides on earth.

The tides are caused mainly by the inhomogeneity of the gravitational field of the moon in the region of space occupied by the earth. (Only to the smaller part the sun is also involved in the occurrence of the tides). So the earth is under tension in the direction of the connecting straight line to the moon. The water on the earth's surface can cede to this tension and forms "flood mountains" on opposite sides of the earth, Fig. 14.31.

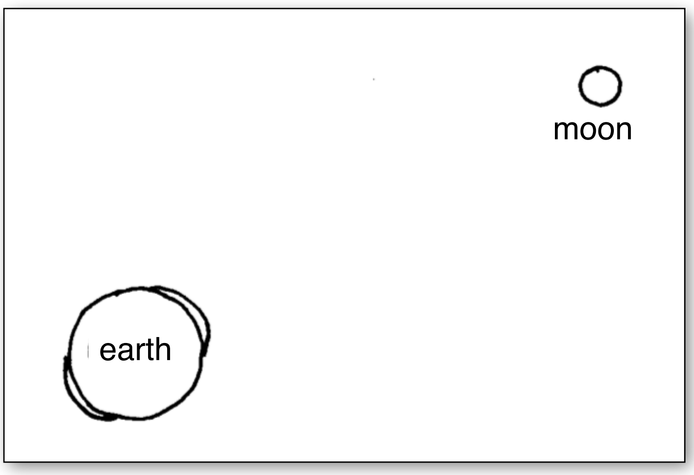

Fig. 14.31
The earth rotates beneath the flood mountains.

As the earth rotates, the flood mountains move relative to the earth's surface. This movement is associated with friction. Therefore, the inherent rotation of the earth is slowed down by the tides. Where does the angular momentum go?

15

Relativistic dynamics

15.1 Preliminary remarks

The previous chapters covered Newtonian mechanics. It is valid as long as all occurring velocities are small compared to 300 000 km/s. Relativistic mechanics abandons this restriction. One can develop relativistic mechanics from Newtonian mechanics by a single additional assumption. Historically, the development went like this: In an experiment (Michelson-Morley) it had been found that the velocity of light is independent of the reference frame. Einstein showed with the special theory of relativity that the resulting consequences go far beyond this special experiment. In particular, the whole Newtonian dynamics had to be corrected. The correctness of relativity was confirmed in the meantime in many experiments. We do not put at the beginning the invariance of the speed of light, but a fact, which has resulted historically as a consequence of it.

15.2 The identity of mass and energy

Newtonian mechanics knows four substance-like quantities, each of which satisfies a conservation law: momentum $\vec{p}$, angular momentum $\vec{L}$, mass m, and energy E. Relativity now claims that one of these quantities is superfluous, because it holds:

$$E = k \cdot m$$

where $k = 9.00 \cdot 10^{16}$ J/kg is a universal natural constant.

Sometimes it is said that this equation tells us that mass can be transformed into energy and likewise energy into mass. However, this does not really capture the point. Since k is a universal natural constant, the equation tells us that mass and energy are the same physical quantity. It says that systems which we have previously assumed to have only energy but no mass, e.g. the electrostatic field, also have mass, and it says that systems which Newtonian mechanics assumes to have only mass but no energy, e.g. a body at rest, also have energy.

The relation $E = k \cdot m$ also tells us that the properties we have known so far of the mass are also properties of the energy:

1. Mass is momentum capacitance. A body is inert, it changes its velocity only if momentum is added to it. So also energy is momentum capacitance, and we have:

$$\vec{p} = m\vec{v} = \frac{E}{k}\vec{v} \tag{15.1}$$

   This relation tells us that, for example, the energy in an electrostatic field is inert. But because of the large factor k this effect is not easy to measure.

2. Mass is the source of a gravitational field. So also energy, e.g. the energy in an electrostatic field, is the source of a gravitational field.

Actually, the relation $E = k \cdot m$ also tells us that mass has the properties which we knew so far from the energy. Thereby we realize that up to now we did not know such properties at all. Only now we learn which general properties are measured by the energy: Inertia and gravity.

Since the energy of a body increases with the velocity, it follows from (15.1) that a body becomes more and more inert as the velocity increases. It becomes more and more difficult to accelerate it. For $v \to 0$, relation (15.1) becomes that of Newtonian mechanics. The mass of Newtonian mechanics, which from now on we will denote by m_0, is thus the smallest value that the quantity $m = E/k$ can assume. m_0, is called the *rest mass* and $E_0 = k \cdot m_0$ the *rest energy* or *internal energy* of a system.

There is a field of physics where the designation of the quantities differs from those chosen here: particle physics. Here it is an important concern to characterize the particles by values which are independent of the state of motion: electric charge, spin, leptonic and baryonic charge. One such characteristic of a particle species is also the rest mass. In particle physics it is simply called mass.

In Newtonian mechanics, E and m are different quantities. In the light of relativity we should say that the energy consists of two parts: the internal energy and the remainder. Newtonian mechanics assumed that conservation laws apply separately to both parts. The theory of relativity teaches us that the two quantities individually do not satisfy a conservation law. There is only one conservation law for the sum of both. We call this sum energy.

15.3 The *E-p* relationship

In the theory of relativity the relation

$$P = \vec{v} \cdot \vec{F}$$

remains valid.

With the balance equations for energy and momentum we get:

$$dE = \vec{v}\, d\vec{p} \qquad (15.2)$$

In Newtonian mechanics, the relationship between energy and momentum is as follows

$$E = E_0 + \frac{\vec{p}^2}{2m_0}$$

Here m_0 is the mass and E_0 a constant term, whose value Newtonian mechanics does not specify. We now derive the relativistic energy-momentum relation. We substitute relation (15.1) into (15.2)

$$dE = \frac{k\,\vec{p}}{E} d\vec{p}$$

It follows

$$EdE = k\,\vec{p}\, d\vec{p}$$

and

$$d(E^2) = k\, d(p^2)$$

and further

$$d(E^2 - k\,p^2) = 0$$

Thus we have

$$E^2 - k\,p^2 = \text{const} = E_0{}^2$$

and

$$E = \sqrt{E_0{}^2 + k\vec{p}^2} \qquad (15.3)$$

As can be seen, E_0 is the energy of the system for $\vec{p} = 0$. Therefore, E_0 is the internal energy and thus equal to km_0. Fig. 15.1 shows the *E-p* relation for particles of different rest mass.

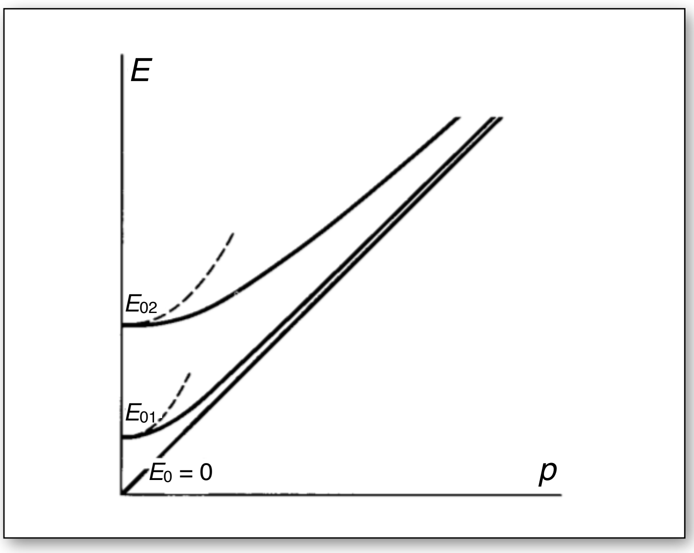

Fig. 15.1
Energy-momentum relationship for particles of various rest masses

The energy $E - E_0$, which the system has in addition to its internal energy, is called kinetic energy

$$E_{\text{kin}} = E - E_0 = \sqrt{E_0{}^2 + k\vec{p}^2} - E_0 \qquad (15.4)$$

Limiting cases

(a) The system is a body with little momentum, i.e. $k\vec{p}^2 \ll E_0{}^2$. With

$$E = E_0\sqrt{1 + \frac{k\vec{p}^2}{E_0{}^2}}$$

we obtain

$$E \approx E_0\left(1 + \frac{1}{2}\frac{k\vec{p}^2}{E_0{}^2}\right)$$

and with $E_0 = k \cdot m_0$

$$E \approx E_0 + \frac{\vec{p}^2}{2m_0}$$

(b) The system is light. For light $m_0 = 0$ and $E_0 = 0$. From this follows

$$E = c|\vec{p}|$$

(c) The system is a body with a very large momentum, i.e. $k\vec{p}^2 \gg E_0{}^2$. Thus

$$E = c|\vec{p}|$$

The energy only depends on the momentum, but no longer on the rest mass. The *E-p* relation is the same as for light.

15.4 The *v-p* relationship

From (15.1) and (15.3) follows

$$\vec{v}(\vec{p}) = \frac{k\vec{p}}{\sqrt{E_0^{\ 2} + k\vec{p}^2}}$$

and with

$$E_0 = k \cdot m_0$$

we get

$$\vec{v}(\vec{p}) = \frac{\sqrt{k} \cdot \vec{p}}{\sqrt{km_0^{\ 2} + \vec{p}^2}} \qquad (15.5)$$

Fig. 15.2. shows the graph of this function.

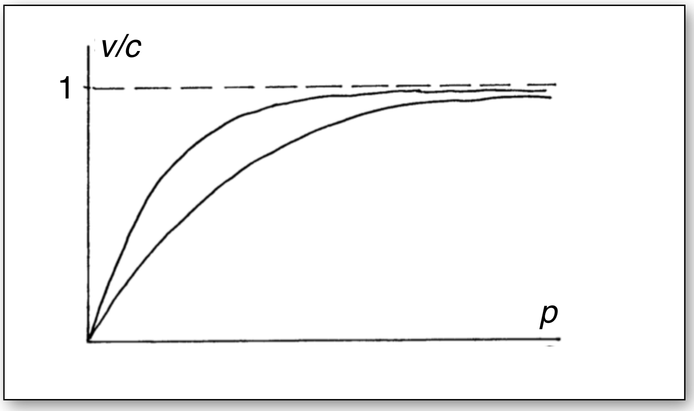

Fig. 15.2
Relationship between velocity and momentum for two particles of different rest masses

For increasing momentum, the velocity approaches a limit value.

$$\lim_{p \to \infty} v(p) = \sqrt{k} = 3 \cdot 10^8 \text{ m/s}$$

The absolute value of this velocity is called the *terminal speed c*. Just like *k*, it is a universal constant.

$$\sqrt{k} = c = 3 \cdot 10^8 \text{ m/s}$$

Particles whose rest mass is equal to zero, photons for instance, always move with the velocity *c*. That is why *c* is also called the speed of light.

For small values of the momentum, i.e. for

$$\vec{p}^2 \ll km_0^{\ 2}$$

we get the well known relation

$$\vec{v}(\vec{p}) = \frac{\vec{p}}{m_0}$$

or

$$\vec{p} = m_0\vec{v}$$

15.5 Examples

Charged particle in homogeneous electric field

Rest mass of the particle: m_0

electric charge of the particle: Q

Strength of the electric field without the particle: $\vec{E}$

The energy, momentum and velocity of the particle as a function of time are wanted.

$$\vec{F} = Q\vec{E}$$

$$\vec{p} = \vec{F} \cdot t = Q \cdot \vec{E} \cdot t$$

The momentum increases linearly with time, as in Newtonian mechanics, Fig. 15.3a. It grows without limit.

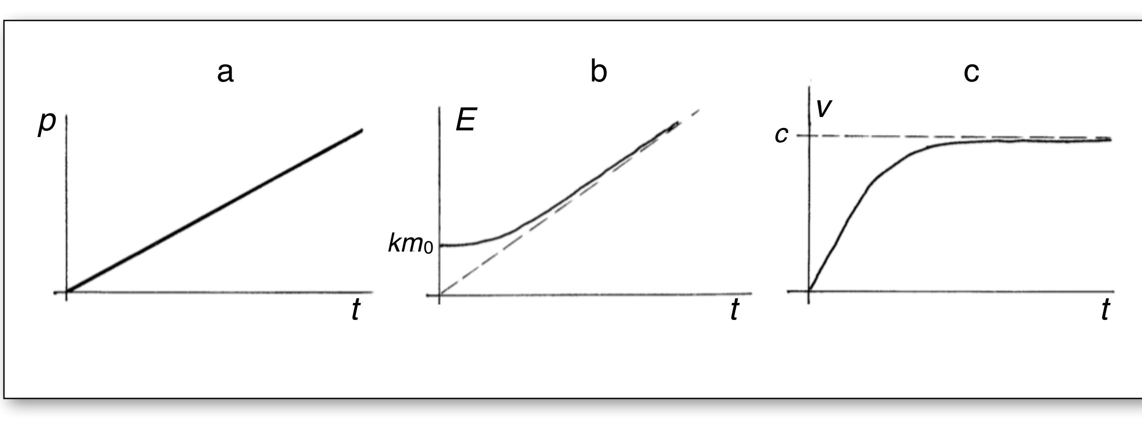

Fig. 15.3
Momentum (a), energy (b), and velocity (c) of a charged particle in a homogeneous electric field as a function of time.

$$E(t) = \sqrt{E_0^{\ 2} + k\vec{p}(t)^2}$$
$$= \sqrt{E_0^{\ 2} + \vec{F}^2 t^2}$$

For large t values, the energy increases linearly with time, Fig. 15.3b.

$$v(t) = \frac{k\,\vec{p}(t)}{E(t)}$$
$$= k\frac{\vec{F}t}{\sqrt{E_0^{\ 2} + k\vec{F}^2 t^2}}$$

For small t, the velocity grows linearly with time. For large t

$$|v(t)| = \frac{Ft}{m_0} \cdot \frac{\sqrt{k}m_0}{Ft} = \sqrt{k} = c$$

v approaches c asymptotically, Fig. 15.4c.

Thus, for large values of t, E and p increase linearly with t, while v remains constant.

Falling light

We consider a certain amount of light coming out of the lamp at the ceiling and moving downwards. Since the light has energy (= mass), a momentum current flows into the light from the gravitational field. The momentum of the light increases:

$$dp = Fdt$$

With $F = mg = (E/k)g$ we get

$$dp = \frac{E}{k}gdt$$

and with $dt = -\,dz/c$

$$dp = -\frac{E}{kc}gdz$$

Since gdz is equal to the change dV of the gravitational potential V, it follows

$$dp = -\frac{E}{kc}dV$$

With $E = cp$ we obtain the relative energy change dE/E of the light

$$\frac{dE}{E} = \frac{dV}{k} = \frac{g}{k}dz$$

The energy of the light increases, just like the energy of any other falling body or particle. With the light this manifests itself in the increase of the frequency, because energy and frequency ν are connected about

$$E = Zh\,\nu$$

(Z = number of photons, h = Planck constant)

On the earth the effect is very weak, but has been measured in the laboratory. For dz = 10 m results:

$$\frac{dE}{E} = \frac{10\ \text{m/s}^2 \cdot 10\ \text{m}}{9 \cdot 10^{16}\ \text{m}^2/\text{s}^2} \approx 10^{-15}$$

The fact that light coming from places of higher gravitational potential has a higher frequency at its place of arrival, i.e. at the place of lower gravitational potential, has a curious consequence. We can imagine that the light at the high gravitational potential is emitted by an oscillating system. Let the oscillation time of this system be $T = 1/\nu$, where ν is the frequency of the light. We can now conceive the oscillating system as a clock that emits signals at the time intervals T. We call this clock C_2. We now place at the place of the lower gravitational potential another clock C_1, which is constructed in exactly the same way as C_2.

An observer at the low gravitational potential now notices that the signals of C_2 arrive in shorter time intervals than those of his own clock C_1. He concludes that time runs faster on the high gravitational potential.

Accordingly, an observer at the high gravitational potential will notice that the clock C_1 runs slower than his own clock C_2, so that the time at the low gravitational potential runs slower than his own time.

The binding energy

During the reaction of 2 mol of atomic hydrogen to 1 mol of molecular hydrogen, energy is released, the binding energy. At the same temperature, the energy of atomic hydrogen is greater than that of molecular hydrogen. Accordingly, 1 mol H_2 weighs less than 2 mol H. However, this difference in weight is extremely small.

If, however, atomic nuclei react with each other, the binding energy per mole is five orders of magnitude greater and can be measured in the mass spectrometer. In the reaction

$$^{6}\text{Li} + {}^{2}\text{H} \rightarrow 2\ {}^{4}\text{He}$$

the energy released per mol is

$$\frac{\Delta E}{n} = 2.1 \cdot 10^{12}\ \text{J/mol}$$

Expressed in units of mass this is

$$\frac{\Delta m}{n} = \frac{\Delta E}{nk} = \frac{2.1 \cdot 10^{12}\ \text{J}}{9 \cdot 10^{16}\ (\text{m}^2/\text{s}^2)\text{mol}} = 0.23 \cdot 10^{-4}\ \text{kg/mol}$$

With

$$\frac{m}{n} = \left(\frac{m}{n}\right)_{\text{Li}} + \left(\frac{m}{n}\right)_{\text{H}} \approx 6\ \text{g/mol} + 2\ \text{g/mol} = 8\ \text{g/mol}$$

we obtain

$$\frac{\Delta m}{m} = \frac{0.023}{8} = 0.003$$

This mass difference is called *mass defect*.

The reason for building storage rings

High-energy physics deals with particle reactions that require a high amount of energy to take place. A given reaction between particle A and particle B takes place only when a minimum amount of energy is available, an amount that generally far exceeds the internal energy of the individual particles.

Since in the reaction between A and B the total momentum and thus the kinetic energy in the center of mass system is conserved, it is of no use to increase the kinetic energy of the system. Only the energy in the center of mass system A-B, i.e. the internal energy of the system A-B (see chapter 7), is available for the reaction. (One sees here the inappropriateness of the expression rest energy).

In some high-energy experiments, one only increases the energy of the particle species A. This is done by "accelerating" them. Particles B remain at rest, they form the "target". For the system A-B this means that not only its internal but also its kinetic energy is increased. This kinetic energy is lost for the reaction of A with B. We want to calculate what fraction of the energy introduced into A is available as internal energy in the center-of-mass system of A-B for the reaction. For simplicity, let A and B be particles of the same kind. So they have the same internal energy E_0.

From equation 15.4 follows

$$p = \frac{1}{c}\sqrt{E_{\text{kin}}^{\ \ 2} + 2E_{\text{kin}}E_0}$$

Since particles B are at rest before the reaction, the momentum of A before the reaction is equal to the momentum of A-B after the reaction:

$$\sqrt{E_{\text{kin A}}^{\ \ 2} + 2E_{\text{kin A}}E_0} = \sqrt{E_{\text{kin A-B}}^{\ \ 2} + 2E_{\text{kin A-B}}E_{0\ \text{A-B}}}$$

According to the law of energy conservation, we have

$$E_{\text{kin A}} + E_0 + E_0 = E_{\text{kin A-B}} + E_{0\ \text{A-B}}$$

and

$$E_{\text{kin A-B}} = E_{\text{kin A}} + 2E_0 - E_{0\ \text{A-B}}\ .$$

We enter $E_{\text{kin A-B}}$ into the momentum balance equation and resolve for $E_{0\ \text{A-B}}$

$$E_{0\ \text{A-B}} = \sqrt{2E_0(E_{\text{kin A}} + 2E_0)}$$

Thus, for high energies $E_{\text{kin A}} \gg E_0$, the energy $E_{0\ \text{A-B}}$ available for the reaction grows only with the square root of the expended energy $E_{\text{kin A}}$.

For example, if protons of 30 GeV are made to react with protons at rest, only 7 GeV are available for the reaction. Therefore it is better to charge both kinds of particles with momentum and energy in such a way that the momentum of the system A-B is zero. Then all the energy is available for the reaction.

This is realized by first charging the particles A with energy and momentum and "parking" them in a storage ring. Then the particles B are charged with E and p with the same accelerator and finally brought to the reaction with the particles A.

The light pressure of the sunlight

The energy current density j_E of the sunlight on the earth is about 1 kW/m² (= solar constant).

With $dE = cdp$ we get

$$P = cF$$

and

$$j_E = c\sigma$$

Therefore the pressure σ of the sunlight is

$$|\sigma| = \frac{1\ \text{kW/m}^2}{3 \cdot 10^8\ \text{m/s}} \approx 3 \cdot 10^{-6}\ \text{N/m}^2$$

If all the sunlight falling on the earth were absorbed, this would result in a momentum current of about $4.3 \cdot 10^8$ N to the earth. In fact, it is more, because the reflected light has momentum of the opposite sign than the incoming light.

15.6 Mass and inertia

In the framework of classical mechanics, we have seen mass being a measure of the inertia of a body. The mass had a fixed value and accordingly the inertia was a property characterizing the body. This changes when we turn to velocities which are no longer small compared to the terminal velocity.

First let us explain what we mean by inertia. A body is very inert if we have to supply it with a great amount of momentum in order for its velocity to change by a desired value. We therefore define inertia:

$$T := dp/dv$$

So the inertia is the slope of the function $p(v)$. This is just the inverse of the function $v(p)$, equation (15.5). We thus get

$$T(v) = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

see Fig. 15.4. Thus, T is given by the slope of the curve, see for example the red tangent to the curve. Obviously, this slope is velocity-dependent. Only at the beginning, in “classical approximation”, the slope dp/dv is equal to p/v, and thus equal to the rest mass, see the blue tangent.

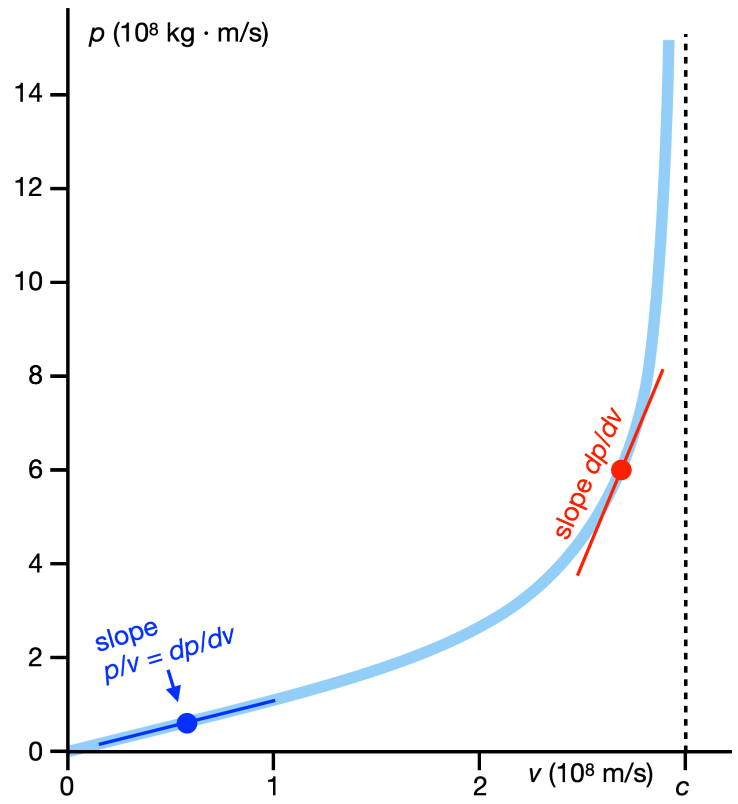

Fig. 15.4. The inertia of a body is given by the slope of the function $p(v)$. It depends on the velocity.

But isn’t it a pity about the beautiful interpretation of mass as a universal measure of inertia?

A pity perhaps – but why should mass be better off than other physical quantities? Let us remember:

- When we construct or invent a new theory, we are happy if the variables it contains measure simple properties known to us from our everyday experience. Most of the time, however, this does not quite work. Think of force, for example, or heat.
- The inertia behaves similar to some electrical quantities. The resistance characterizes an object: a resistor. If somebody says that the resistor has a resistance of 10 kΩ, then one is informed. However, this is only possible if the current is proportional to the voltage. But what if it is not? How do we characterize for example a semiconductor diode? In this case it is not enough to give one number. One has to give the *U*-*I* characteristic curve. The same applies to the capacitance. And we are in the same situation with the inertia. Inertia cannot be described by a single number; one needs a characteristic curve, Fig. 15.4.