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KPK

The Karlsruhe physics course

Lecture notes

Electromagnetism

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Lecture notes

- Mechanics
- Thermodynamics
- **Electromagnetism**
- Optics

Der Karlsruher Physikkurs

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Preface

These lecture notes belong to a lecture on experimental physics. What is the difference between experimental and theoretical physics? Instead of answering the question directly, we will give Maxwell the floor. In his famous *Treatise on Electricity & Magnetism* he makes some remarks about two physicists, one of whom, Faraday, can be described as a typical experimental physicist, the other, Ampère, as a typical theorist:

CHAPTER III.

ON THE INDUCTION OF ELECTRIC CURRENTS.

528.] The discovery by Örsted of the magnetic action of an electric current led by a direct process of reasoning to that of magnetization by electric currents, and of the mechanical action between electric currents. It was not, however, till 1831 that Faraday, who had been for some time endeavouring to produce electric currents by magnetic or electric action, discovered the conditions of magneto-electric induction. The method which Faraday employed in his researches consisted in a constant appeal to experiment as a means of testing the truth of his ideas, and a constant cultivation of ideas under the direct influence of experiment. In his published researches we find these ideas expressed in language which is all the better fitted for a nascent science, because it is somewhat alien from the style of physicists who have been accustomed to establish mathematical forms of thought.

The experimental investigation by which Ampère established the laws of the mechanical action between electric currents is one of the most brilliant achievements in science.

The whole, theory and experiment, seems as if it had leaped, full grown and full armed, from the brain of the 'Newton of electricity.' It is perfect in form, and unassailable in accuracy, and it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electro-dynamics.

The method of Ampère, however, though cast into an inductive form, does not allow us to trace the formation of the ideas which guided it. We can scarcely believe that Ampère really discovered the law of action by means of the experiments which he

176 MAGNETO-ELECTRIC INDUCTION. [529.

describes. We are led to suspect, what, indeed, he tells us himself*, that he discovered the law by some process which he has not shewn us, and that when he had afterwards built up a perfect demonstration he removed all traces of the scaffolding by which he had raised it.

Faraday, on the other hand, shews us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and is tempted to believe that, if he had the opportunity, he too would be a discoverer. Every student should therefore read Ampère's research as a splendid example of scientific style in the statement of a discovery, but he should also study Faraday for the cultivation of a scientific spirit, by means of the action and reaction which will take place between the newly discovered facts as introduced to him by Faraday and the nascent ideas in his own mind.

It was perhaps for the advantage of science that Faraday, though thoroughly conscious of the fundamental forms of space, time, and force, was not a professed mathematician. He was not tempted to enter into the many interesting researches in pure mathematics which his discoveries would have suggested if they had been exhibited in a mathematical form, and he did not feel called upon either to force his results into a shape acceptable to the mathematical taste of the time, or to express them in a form which mathematicians might attack. He was thus left at leisure to do his proper work, to coordinate his ideas with his facts, and to express them in natural, untechnical language.

The script is divided into two parts, **A** and **B**.

Part **A** could simply be called electricity. It deals with the electric charge and its current. The fact that the electric charge can never flow alone, that a charge current is always accompanied by currents of other physical quantities, plays an important role here.

The subject of Part **B** is a single physical system: the electromagnetic field. First, special classes of this system are treated: the electrostatic field and the magnetostatic field. It turns out that the structure of the theories of these two fields is identical. Next, the relationship between these fields is described, which is regulated by Amperes' Law and Faraday's Law (4th and 3rd Maxwell's equations). The treatment of the electromagnetic field is closely related to Maxwell's own account of his theory. Consequently, the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ are convenient tools for expressing other physically interesting quantities of the field: energy, energy current, momentum and momentum current (mechanical stress). All force laws can be derived from Maxwell's expression for the momentum current density.

Electrodynamics is full of structures and symmetries. The more of these structures one knows, the better one understands electrodynamics. However, beginners will not try to understand all structures at the same time. This lecture makes use of the symmetry in which the electric field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{H}$ correspond to one another. It allows an almost blind transformation of many laws of electrodynamics into an analogue. Another symmetry, which we call dualism, we had already experienced in mechanics: in it, electric current and voltage, inductance and capacitance, junctions and meshes, etc. correspond to each other. We meet it again in this script. We leave the treatment of a third symmetry, where the charge density and the electric current density correspond, to the lecture on theoretical physics.

A

THE ELECTRIC CHARGE

1

Electric charge and electric current

1.1 Balance equation of the electric charge

Just like momentum $\boldsymbol{p}$ for mechanics, electric charge is characteristic for the science of electricity.

The symbol of the electric charge is Q, the unit of measurement is Coulomb (C).

Like $\boldsymbol{p}$, also Q is a substance-like quantity, i.e.

- the value of the electric charge refers to a region of space;
- there is a charge density ρ_Q;
- there is a charge current intensity I_Q (= “electric current”);
- there is a charge current density $\boldsymbol{j}_Q$ (= “electric current density”).

If there is no risk of confusion, we omit the index Q, and simply write ρ, I or $\boldsymbol{j}$. In technical jargon, the electric current intensity is simply called “electric current” or, even shorter, “current”. The unit of measurement of the electric current is the ampere (A).

1 A = 1 C/s.

The electric current is measured with an ammeter. The measuring procedure is the same as for any other current measurement:

- Cut the cable in which the current flows;
- connect the newly created ends to the connections of the meter. The current now flows through the meter.

Like momentum, electric charge is a conserved quantity. For any given region of space applies

$$\frac{dQ}{dt} + I_Q = 0 \qquad (1.1)$$

This is the balance equation for the electric charge. It refers to a region of space, Fig. 1.1. dQ/dt is the rate of change of the electric charge within the region, I_Q is the electric current through the outwardly oriented boundary surface of the area.

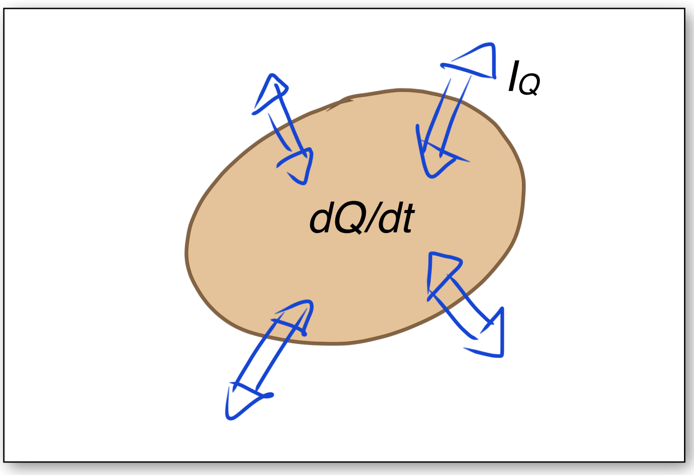

Fig. 1.1
The electric charge inside the region can only change if a current flows through the surface of the region.

(The signs in equation (1.1) are based on a convention. Closed surfaces are oriented outwards. In mechanics we had closed surfaces oriented inwards.)

In the next sections we will get to know a “local” version of the balance equation, the so-called continuity equation.

In the circuit of Fig. 1.2 dQ/dt is equal to zero everywhere (in the energy sources $dE/dt \neq 0$, but $dQ/dt = 0$). This simplifies equation (1.1) to

$$I_Q = \sum_i I_{Q,i} = 0$$

for any closed region of space for which $dQ/dt = 0$. Equation (1.2) is called the “Kirchhoff’s junction rule”.

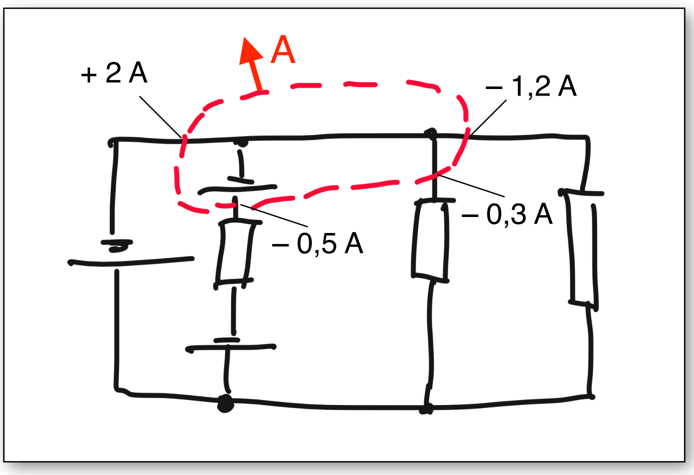

Fig. 1.2
The total current through the dotted area is zero.

The physical quantity Q can assume positive and negative values. Unfortunately, it is sometimes said that there are “two types of electric charge”, positive and negative. (Are there also 2 types of velocity or two types of momentum?)

1.2 Flux of a vector field – current density

A vector field $\boldsymbol{V}(\boldsymbol{r})$ is given. The flux of the vector field through the area S is the integral

$$\int_S \boldsymbol{V}(\mathrm{r})\, d\boldsymbol{A} \tag{1.3}$$

One can easily get an idea of this quantity if the vector field is a current density field. The flux of a current density field is simply equal to the current(= current intensity). In the case of an electric current, the following applies:

$$I = \int_S \boldsymbol{j}(\boldsymbol{r})\, d\boldsymbol{A} \tag{1.4}$$

This equation allows to calculate the current I flowing through the area S from the current density distribution $\boldsymbol{j}(\boldsymbol{r})$.

In Fig.1.3 the same current flows through the surfaces S_1 and S_2. As expected, the integral (1.4) provides the same value for both surfaces, because only the component of $d\boldsymbol{A}$ parallel to $\boldsymbol{j}$ contributes to the integral.

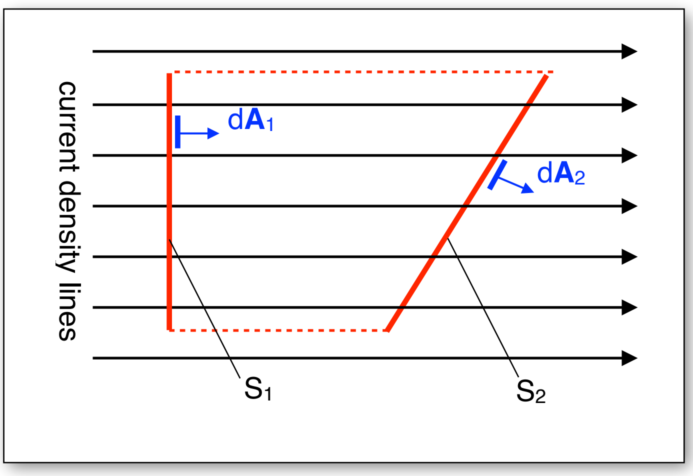

Fig. 1.3
The currents through surfaces S_1 and S_2 are equal.

The calculation of the integral (1.3) in Cartesian coordinates is done according to the following formula

$$\iint_S \boldsymbol{V}\, d\boldsymbol{A} = \iint_{S_{yz}} V_x\, dy\, dz + \iint_{S_{zx}} V_y\, dz\, dx + \iint_{S_{xy}} V_z\, dx\, dy$$

Here S_{yz} is the projection of the surface S onto the *y-z* coordinate plane, S_{zx} is the projection onto the *z-x* plane and S_{xy} is the projection onto the *x-y* plane.

For some currents, one can imagine that the flowing quantity moves at a well-defined velocity at every point of the flow field. Thus, the water of a river has a definite velocity at every point of the river, and we also say that the mass of the water moves at that velocity. There are cases where it is reasonable to assign more than one velocity to the flowing quantity at a given position. Thus, in the case of an electric current in a metallic conductor, a distinction is made between the velocity of the so-called mobile and immobile charge carriers. (In the reference system of the conductor, the velocity of the immobile charge carriers is zero). Finally, there are currents in which it is pointless to speak of a flow velocity, although the current density is a clearly defined quantity.

If a flow velocity $\boldsymbol{v}$ can be defined, there is a simple relationship between $\boldsymbol{v}$ and the current density:

$$\boldsymbol{j}_X = \rho_X\, \boldsymbol{v}$$

Here ρ_X is the density of the flowing quantity X and $\boldsymbol{j}_X$ is the current density.

We explain this relation with the help of Fig. 1.4. The quantity $dX = \rho_X\, Avdt$ contained in the space region of the volume $Adx = Avdt$ flows through the small area A in time dt. The current is therefore

$$I_X = \frac{dX}{dt} = \rho_X A v$$

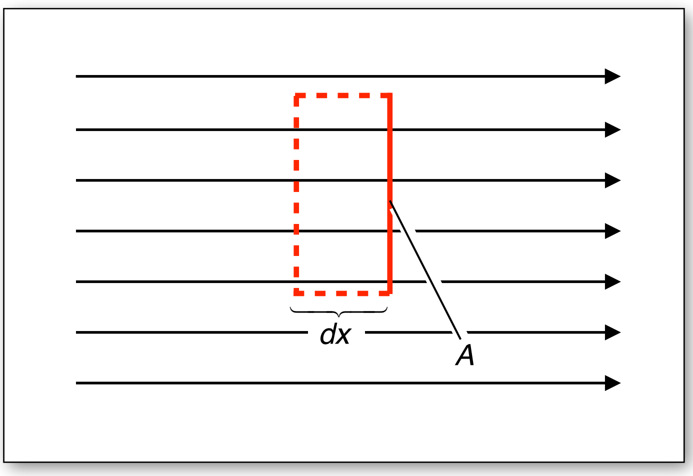

Fig. 1.4
In the time interval *dt*, the quantity contained in the region with the volume *Adx* flows through the surface *A*.

The magnitude of the current density is $j_X = I_X/A = \rho_X v$. Since the current density vector and the flow velocity vector are parallel, the following is obtained

$$\boldsymbol{j}_X = \rho_X \cdot \boldsymbol{v} \tag{1.5}$$

1.3. The divergence of a vector field – Gauss's theorem

The following derivation applies to all vector fields. It is particularly easy to understand if one imagines the current density field of an electric current: $\boldsymbol{j}(x,y,z)$.

The electric current I through the closed surface S in Figure 1.5 is

$$I = \oint\!\!\oint_S \boldsymbol{j}\, d\boldsymbol{A}$$

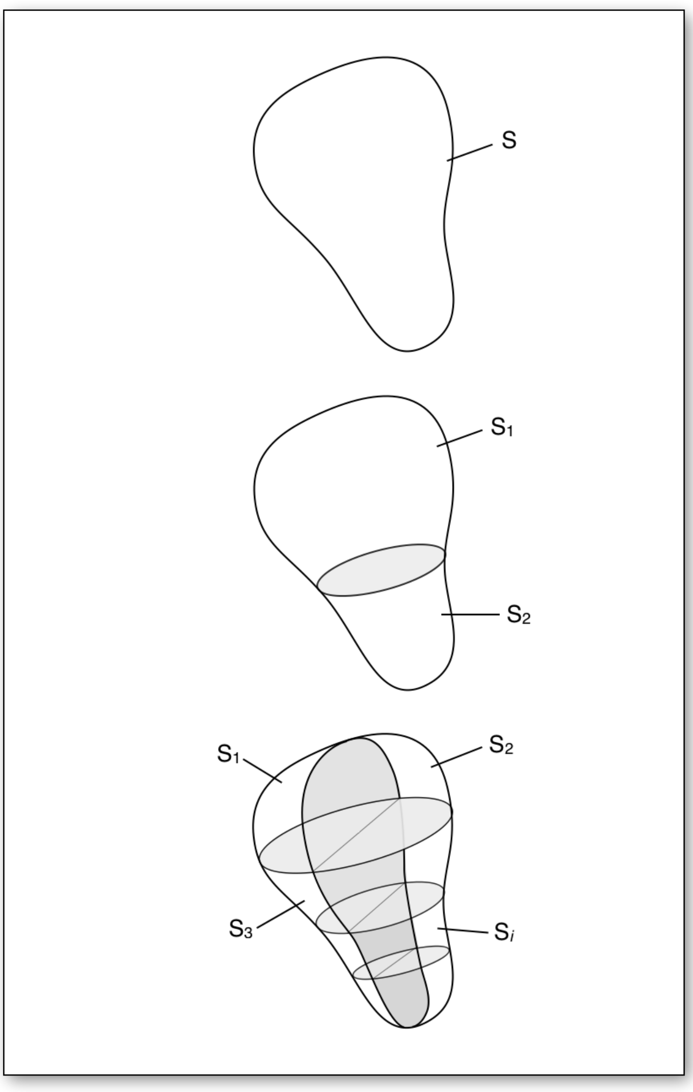

Fig. 1.5
The space enclosed by S is subdivided.

We divide the region enclosed by S into two regions with the boundary surfaces S1 and S2. A part of S1 coincides with a part of S2. It is

$$I = I_1 + I_2 = \oint\!\!\oint_{S_1} \boldsymbol{j}\, d\boldsymbol{A} + \oint\!\!\oint_{S_2} \boldsymbol{j}\, d\boldsymbol{A}$$

because the part of I_1, which flows through the common surface of S_1 and S_2, compensates the corresponding part of I_2. We now further divide the space into smaller and smaller sub-spaces and obtain

$$I = \sum I_i = \sum_i \oint\!\!\oint_{S_i} \boldsymbol{j}\, d\boldsymbol{A}$$

where I_i is the the current through the surface S_i of the i th subspace. We call the volume of the i th subspace V_i. For smaller and smaller divisions the I_i become smaller and smaller. The quotient I_i/V_i, on the other hand, goes against a limit value:

$$\frac{dI}{dV} = \lim_{V_i \to 0} \left[\frac{1}{V_i} \oint\!\!\oint_{S_i} \boldsymbol{j}\, d\boldsymbol{A} \right] = \text{div}\, \boldsymbol{j}$$

This limit is called the *divergence* of the field $\boldsymbol{j}$ at the point to which the volume has shrunk.

$$I = \oint\!\!\oint_S \boldsymbol{j}\, d\boldsymbol{A}$$

is a measure of how much of the flowing quantity – here the electric charge – flows out of or into the region bounded by S; it is a measure of the "yield" of the flow. Therefore, div $\boldsymbol{j}$ is also called the source density of the $\boldsymbol{j}$ field, even if the $\boldsymbol{j}$ does not refer to a flowing substance. We now write the sum:

$$I = \sum_i I_i = \sum_i V_i \frac{\oint\!\!\oint_{S_i} \boldsymbol{j}\, d\boldsymbol{A}}{V_i}$$

For $V_i \to 0$ the right side becomes

$$\iiint_{V_S} \text{div}\, \boldsymbol{j}\, dV$$

where V_S is the volume of the space enclosed by S. Thus

$$\oint\!\!\oint_S \boldsymbol{j}\, d\boldsymbol{A} = \iiint_{V_S} \text{div}\, \boldsymbol{j}\, dV \qquad (1.6)$$

This is *Gauss's theorem.* In words: The current through the surface of a region of space is equal to the volume integral over the source density within the region. If the field $\boldsymbol{j}$ is given in cartesian coordinates, the scalar field div $\boldsymbol{j}$ can easily be calculated. It is

$$\text{div}\, \boldsymbol{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$$

To prove this, we calculate the current I through the walls of a parallelepiped with the edge lengths dx, dy and dz (Fig. 1.6). Through the two surfaces perpendicular to the z axis flows (Attention – orientation of the surfaces outwards):

$$\begin{aligned} dI_{z,\,\text{below}} - dI_{z,\,\text{above}} &= -\left[j_z(x,y,z) + \frac{\partial j_z}{\partial x}\frac{dx}{2} + \frac{\partial j_z}{\partial y}\frac{dy}{2} \right] dx\,dy \\ &\quad + \left[j_z(x,y,z+dz) + \frac{\partial j_z}{\partial x}\frac{dx}{2} + \frac{\partial j_z}{\partial y}\frac{dy}{2} \right] dx\,dy \\ &= \left[-j_z(x,y,z) + j_z(x,y,z) + \frac{\partial j_z}{\partial z} dz \right] dx\,dy = \frac{\partial j_z}{\partial z} dV \end{aligned}$$

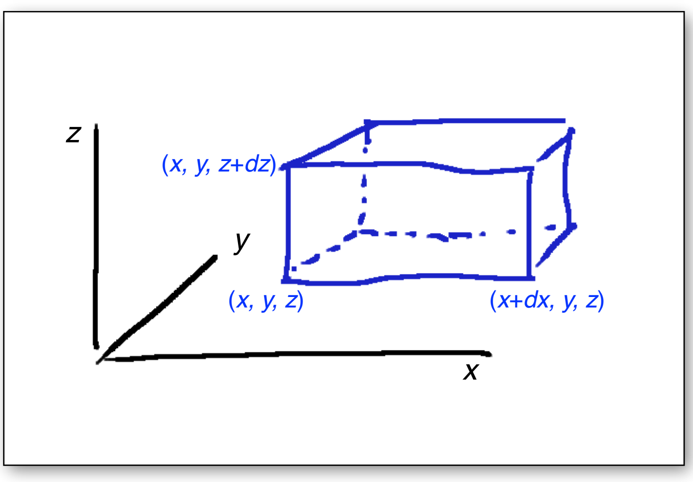

Fig. 1.6
For calculating the current through the walls of a parallelepiped

For the two surfaces perpendicular to the x axis and those perpendicular to the y axis, corresponding expressions are obtained so that the total current through all 6 surfaces results:

$$dI = \left[\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} \right] dV$$

With the equation defining the divergence it follows:

$$\text{div}\, \mathbf{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$$

1.4 The continuity equation for electric charge

The balance equation for the electric charge reads

$$\frac{dQ}{dt} + I = 0$$

We replace Q by

$$\iiint \rho \, dV$$

and, with Gauss's theorem,

$$I = \iint \boldsymbol{j} \, d\boldsymbol{A}$$

by

$$\iiint \mathrm{div} \boldsymbol{j} \, dV$$

and obtain

$$\iiint \frac{\partial \rho}{\partial t} dV + \iiint \mathrm{div}\, \boldsymbol{j} \, dV = 0$$

Since this equation applies to any region of space, we get for the integrands:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \boldsymbol{j} = 0 \qquad (1.7)$$

This is the balance equation in local form. It is also called the *continuity equation*. It says: The sources of the current density field are located where the charge density changes over time. A corresponding equation applies to any other conserved quantity.

Fig. 1.7 shows a field whose divergence is everywhere different from zero; on the left in vector arrow representation and on the right in field line representation.

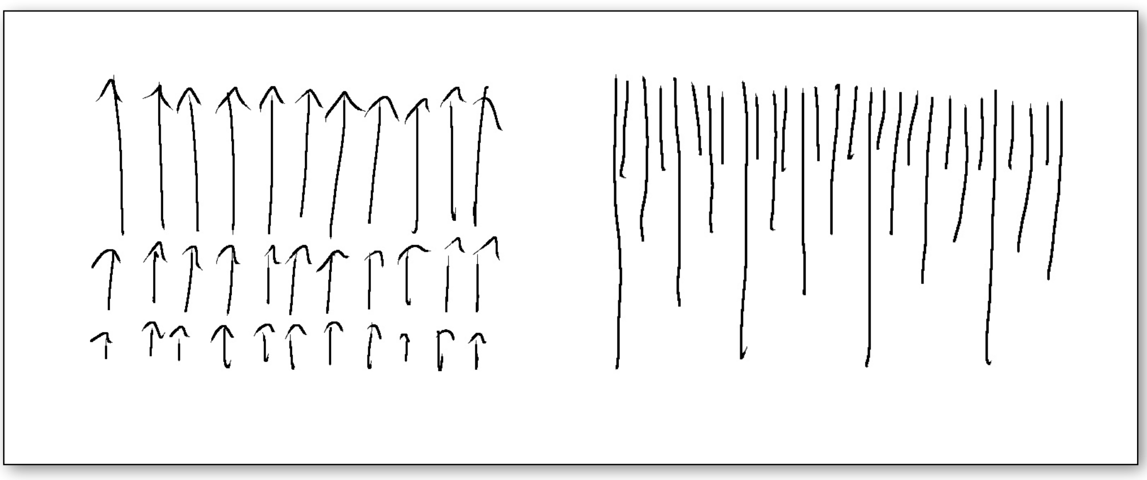

Fig. 1.7
Vector arrow and field line representation of a field whose divergence is different from zero everywhere

The divergence of the field of Fig. 1.8 is equal to zero everywhere.

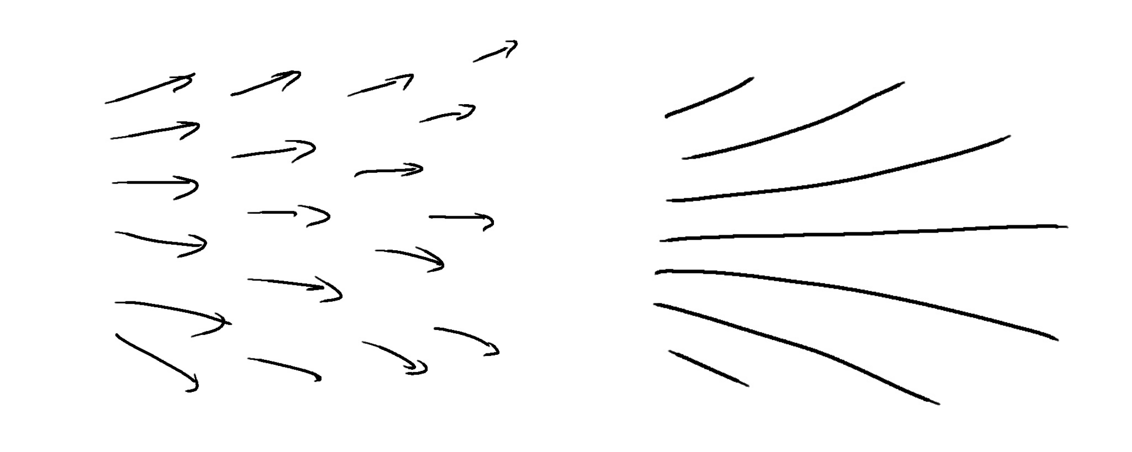

Fig. 1.8
Vector arrow and field line representation of a field whose divergence is zero everywhere

Finally, we apply the continuity equation to a mass flow. One can easily imagine the corresponding experiment.

Air is enclosed in a cylinder, Fig. 1.9. The piston (position coordinate x_K) is moved with the velocity v_K. (v_K must be small against the speed of sound so that the density in the whole cylinder is the same). The position coordinate of the air is x, its velocity $v(x)$.

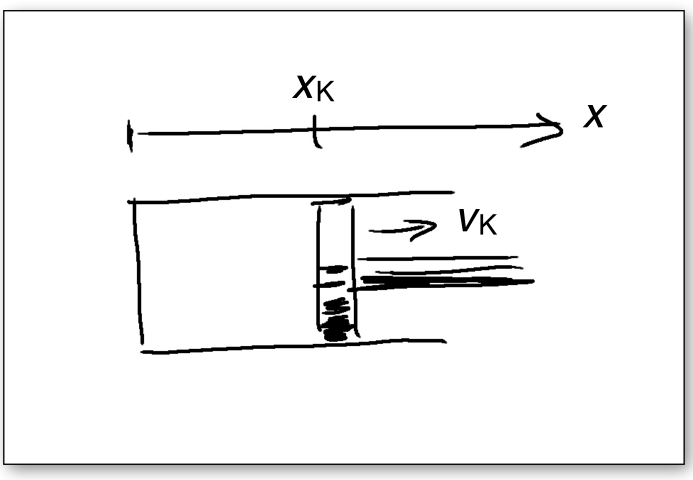

Fig. 1.9
Applying the continuity equation to the mass of air inside a cylinder

From

$$\frac{v(x)}{v_K} = \frac{x}{x_K}$$

we obtain the velocity field:

$$v(x) = \frac{x}{x_K} v_K$$

The mass density is

$$\rho_m = \frac{m}{A x_K}$$

With (1.5) we can calculate the mass current density:

$$j_m = \rho_m v = \frac{m}{A x_K} \frac{x}{x_K} v_K = \frac{m}{A} \frac{v_K}{x_K^2} x$$

This results in the divergence:

$$\mathrm{div}\, \mathbf{j}_m = \frac{\partial j_{mx}}{\partial x} = \frac{\partial j_m}{\partial x} = \frac{m}{A} \frac{v_K}{x_K^2}$$

On the other hand

$$\frac{\partial \rho_m}{\partial t} = \frac{m}{A} \frac{\partial (1/x_K)}{\partial t} = -\frac{m}{A} \frac{1}{x_K^2} \frac{\partial x_K}{\partial t} = -\frac{m}{A} \frac{v_K}{x_K^2}$$

Thus, div $\boldsymbol{j}_m$ is equal to $-\,\partial \rho_m / \partial t$, as could have been expected.

2

Electric current and energy current

2.1 The relationship between electric current and energy current

An energy current flows from the battery to the lamp in Fig. 2.1. In addition, an electric current flows in each of the two wires, from left to right in the upper wire and from right to left in the lower wire.

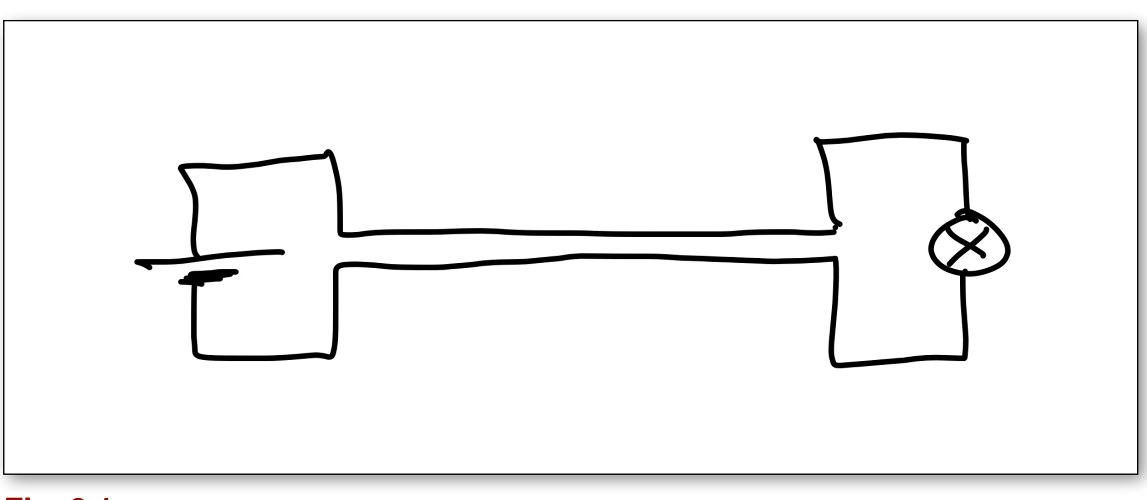

Fig. 2.1
From left to right flows an energy current, and in each wire flows an electric current.

If two circuits are superimposed so that the wires coincide, both the energy current and the electric current in each wire are doubled, Fig. 2.2. The following therefore applies

$P \sim I$

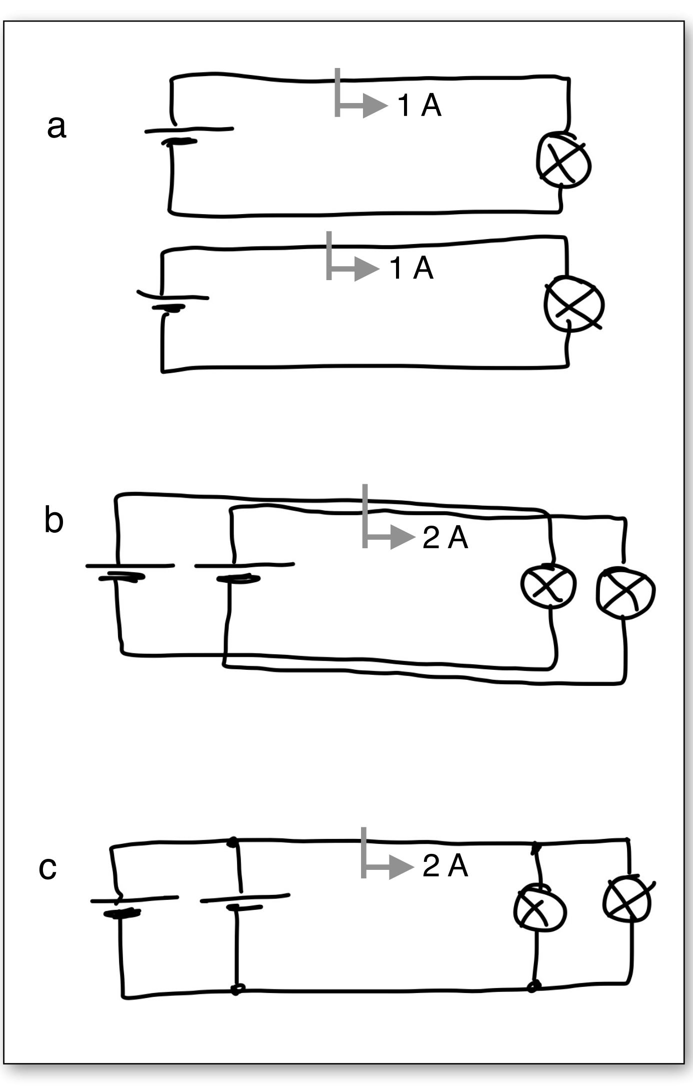

Fig. 2.2
Two identical circuits (a) are superimposed (b) and the pieces of wire lying on top of each other are replaced by a single wire (c).

Since the total electric current (forward and return lines taken together) has the value zero, the relationship between P and I must have the following form:

$P = -\phi_1 I + \phi_2 I$

ϕ is a quantity that has a certain value for one conductor. For the two lines in Fig. 2.1, ϕ must have different values. Otherwise we would obtain

$P = 0.$

Thus we have

$$P = (\phi_2 - \phi_1) I \,. \qquad (2.1)$$

ϕ is the *electric potential*. Only potential differences are defined by equation (2.1). The zero point of the potential can be defined arbitrarily. Usually the potential of the earth is set to zero. Its unit of measurement is Volt = Watt/Ampere = Joule/Coulomb. The difference $U = \phi_2 - \phi_1$ is called voltage.

The voltage between the two wires of an electric cable thus indicates how large is the energy current that is transmitted with a given electric current. This fact can also be expressed metaphorically: The electric charge “carries” the energy. The voltage indicates how much the carrier is “charged” with energy.

Equation (2.1) has the same structure as the equation

$$P = (\mathbf{v}_2 - \mathbf{v}_1)\mathbf{F} \,. \qquad (2.2)$$

That the equations (2.1) and (2.2) have the same structure is no coincidence. Every energy transport can be described by an equation of this type:

$$P = \xi \cdot I_X \,. \qquad (2.3)$$

Equation (2.3) states that an energy flow is always accompanied by the flow of another extensive or substance-like quantity X, Fig. 2.3a. We call the quantity X the *energy carrier*. The proportionality factor is a so-called *intensive quantity*.

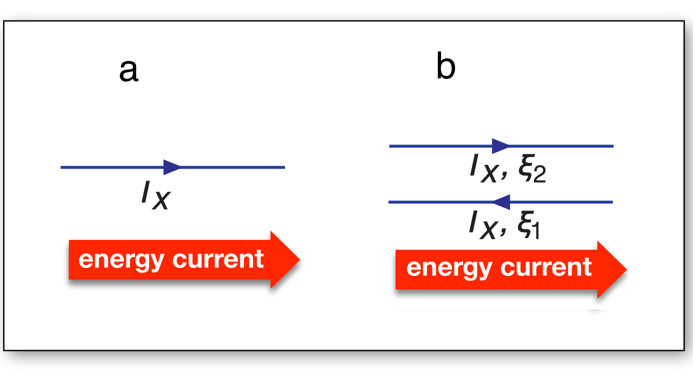

Fig. 2.3
(a) In addition to the energy flow, there is a flow of an energy carrier. (b) The carrier current has a return line.

Many energy transports are such that the carrier quantity X flows in a closed circuit, so that there is a forward and a return line for X, Fig. 2.3b. For a net energy flow to result, the intensive quantity in the forward and return lines must have different values. The net energy flow is then:

$$P = (\xi_2 - \xi_1) I_X \,. \qquad (2.4)$$

An example of this is the energy transport with a two-wire electrical cable that we are currently discussing.

We will consider two more examples of an energy transport, i.e. two more examples of equations of the type (2.3) or (2.4).

If energy is transferred “in the form of heat” (for example through the wall of a poorly insulated house), *entropy* flows in addition to the energy. The energy carrier X in equation (2.3) is thus the entropy S. The corresponding intensive quantity is the *absolute temperature* T. It is therefore

$$P = T \cdot I_S \,. \qquad (2.5)$$

The unit of measure of entropy is the Carnot (Ct), that of the absolute temperature the Kelvin (K). From equation (2.5) follows therefore

K · Ct = J

Finally, we consider a system in which a stationary chemical reaction is running, Fig. 2.4.

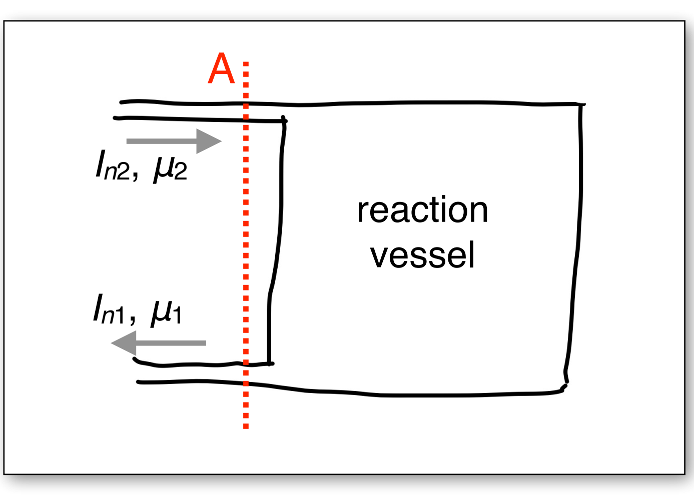

Fig. 2.4
Reaction vessel with feed line for the starting materials and return line for the reaction products

The starting materials for the reaction are fed to the reaction vessel through one pipe, the reaction products are discharged through another pipe. An energy current flows through surface A. It can again be described by an equation of the structure of equation (2.4):

$$P = (\mu_2 - \mu_1) I_n \,. \qquad (2.6)$$

Here μ is the *chemical potential.* The chemical potential is a quantity that is assigned to a substance or a mixture of substances. n is the *reaction conversion*. The unit of measurement of n is the mole. The unit of measurement of the *conversion rate* I_n is therefore mol/s. The unit of measurement of μ results from equation (2.6) to Joule/mol. This unit is sometimes abbreviated by Gibbs (G). So it is

G · mol = J.

The chemical potential of a substance depends on its pressure, temperature and state of aggregation. If the substance is dissolved, it also depends on the concentration and the nature of the solvent.

Back to electricity.

We once again put two circuits on top of each other. This time, however, so that the currents in two of the lines add up to zero, Fig. 2.5. From equation (2.1) it follows that the voltage in the last partial picture must be equal to the sum of the voltages in the first picture.

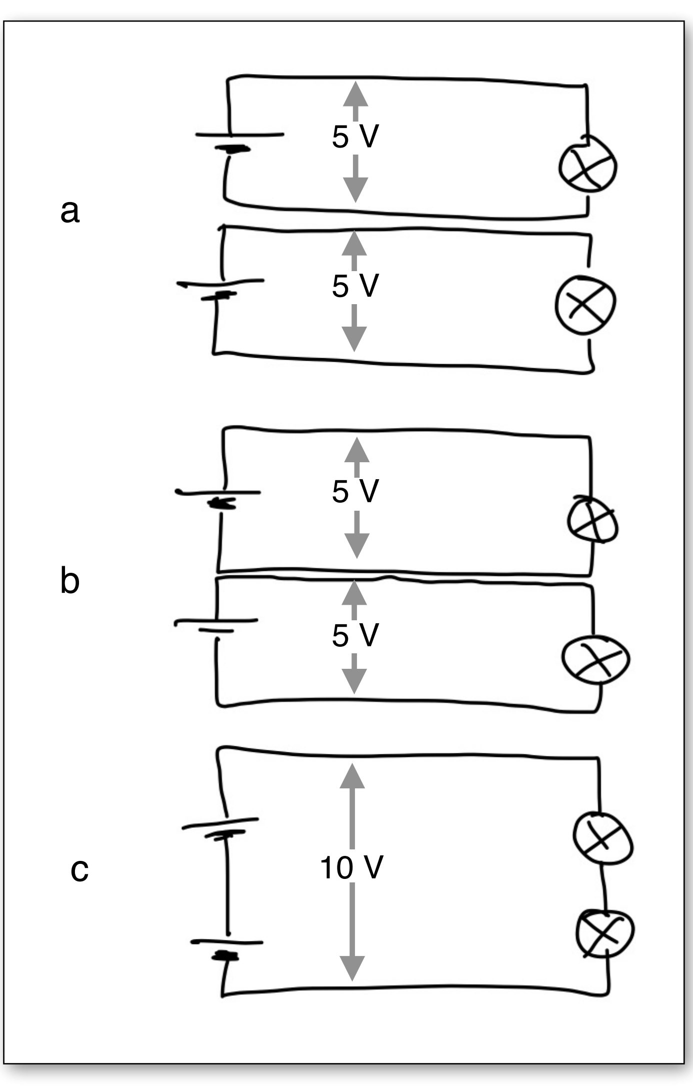

Fig. 2.5
Two identical circuits are combined in such a way that the currents in two lines add up to zero.

Since the potential has a certain value in every point of the circuit, we can formulate the “mesh rule”

$$\sum_i U_i = 0 \qquad (2.7)$$

The sum of all voltages in a “mesh” is zero. All voltages within a mesh must be counted in the same direction, Fig. 2.6.

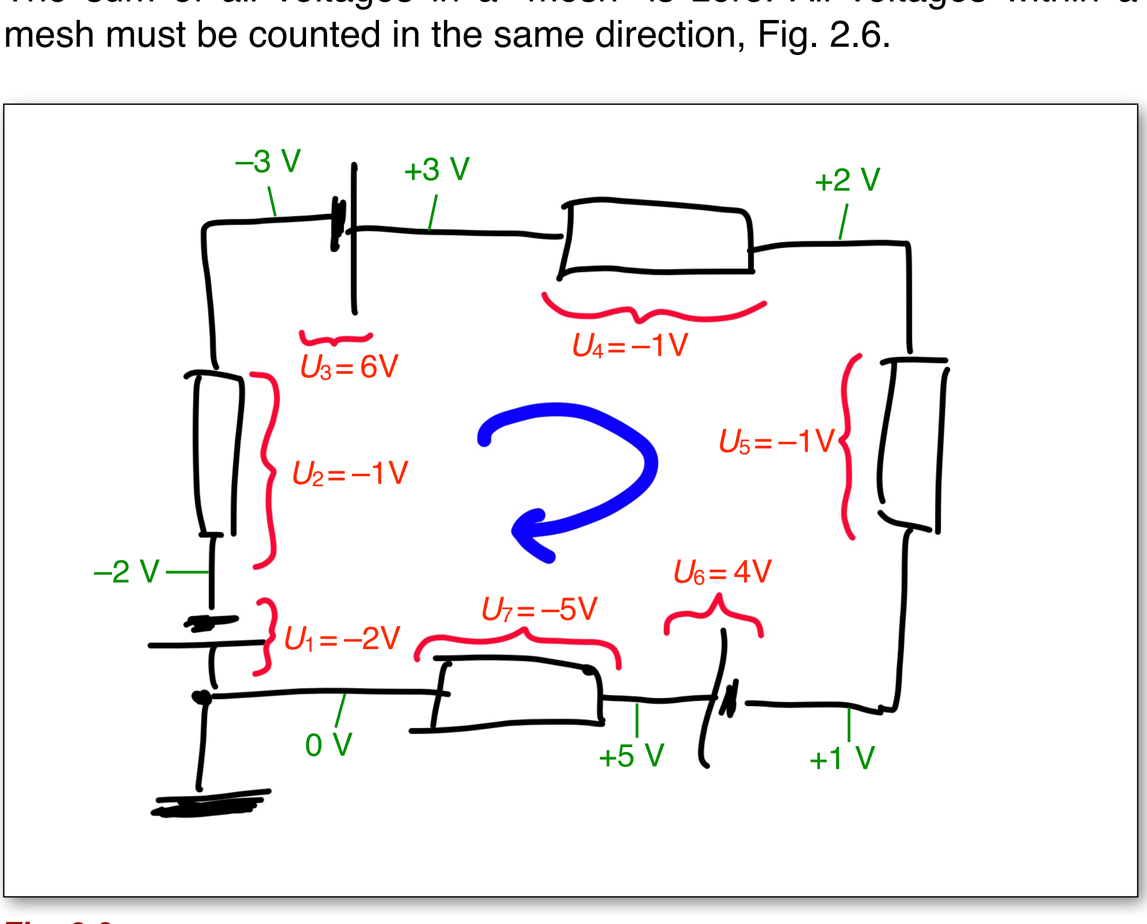

Fig. 2.6
The voltages within a mesh must be counted in the same direction.

Voltages are measured with the voltmeter. The two connections of the voltmeter are connected to the two points between which the voltage is to be measured. The question of the absolute value of the potential is meaningless, just as meaningless as the question of the absolute value of a velocity. (A speedometer, like a voltmeter, has two connections; it measures the speed difference between car and earth). Just as one must select a reference frame, i.e. a velocity zero point, to specify a velocity, one must define the “electrical reference frame”, i.e. the potential zero point, to specify an electrical potential.

2.2 Energy dissipation

Energy flows electrically into the device shown in Fig. 2.7. This energy is completely dissipated in the device. Dissipating energy means producing entropy with the help of energy.

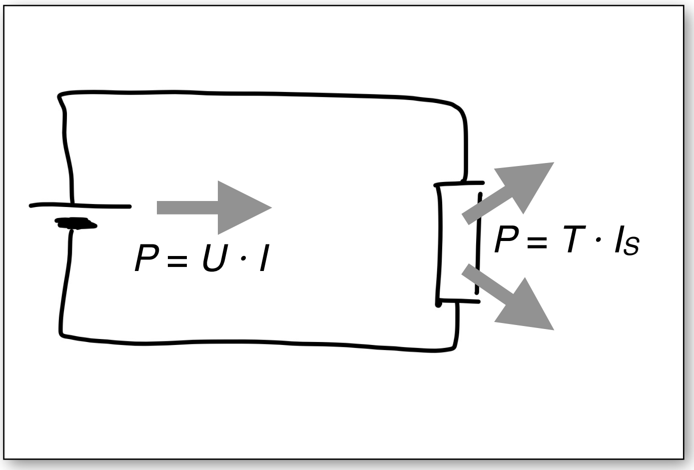

Fig. 2.7
All of the incoming energy is dissipated in the resistor.

The device could be a light bulb, the heating coil of an iron or a technical resistor, but not an electric motor or a battery that is being charged. Since all the incoming energy $U \cdot I$ is dissipated in the device, the following applies:

$$U \cdot I = T \cdot I_S$$

I_S is the entropy current leaving the device and T is its absolute temperature. Figure 2.8 shows the flow diagram of the process.

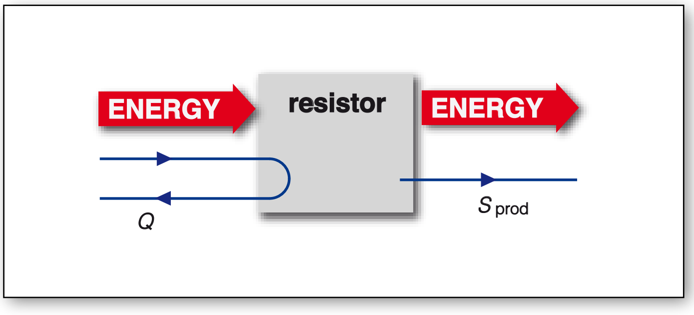

Fig. 2.8
Flow diagram of the electric resistor

It is a matter of experience that entropy can be created but not destroyed. The process of Fig. 2.8 can therefore not run backwards, it is *irreversible*, Fig. 2.9.

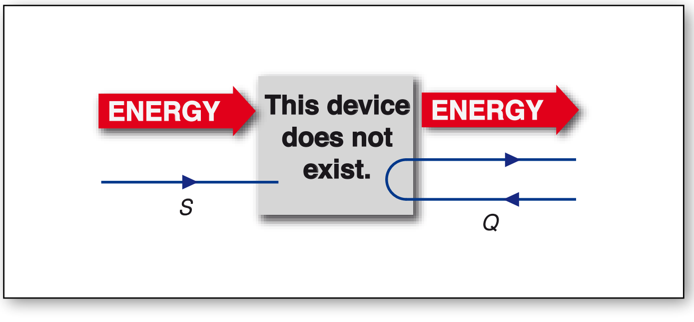

Fig. 2.9
The inverse of the process, that takes place in an electric resistor, is impossible.

2.3 Voltage as the drive of an electric current

A system in which energy is dissipated is said to have a resistance. The word resistance belongs to a pictorial representation of the process. According to it, the flow of the current is hindered by the resistor. That the current flows despite the obstruction is due to the voltage. It appears as a “drive” or the “cause” of the current. Although this image is very useful, it is just a human invention. One could just as well say that the current is the cause of the voltage (one often even says that the current causes a voltage “drop”).

2.4 Characteristics – Ohm's law

We look at objects with two electrical connections: Resistors, pieces of wire, diodes and other things. If we graphically illustrate for such an object the relationship between the electric current flowing through it and the voltage between its terminals, we get its *characteristic*, Fig. 2.10.

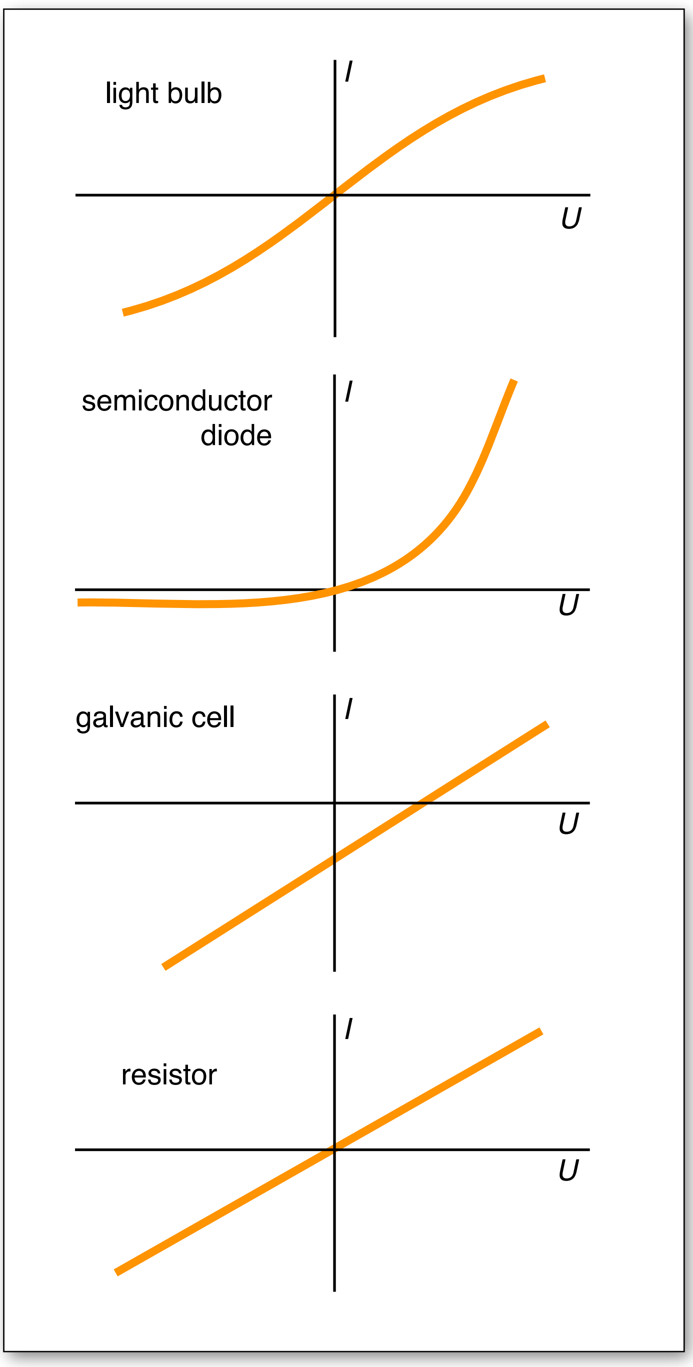

Fig. 2.10
Examples for current-voltage characteristic curves

For some objects, under certain conditions – constant temperature, current density not too high – a particularly simple relationship applies:

$U \sim I$.

It is said that Ohm's law applies to the object. It applies, for example, to metal wires (at a fixed temperature). In this case one calls the quotient

$R = U/I$

the *resistance* of the object. The unit of measurement of the resistance is the ohm, abbreviated Ω (1 Ω = 1 V/A).

For an "ohmic conductor" of length l with constant cross-section A, the following applies

$$R = \rho \frac{l}{A} = \frac{1}{\sigma} \frac{l}{A}$$

ρ is called *electric resistivity* (Attention: the same SI symbol as for mass density and charge density). The reciprocal value σ of the resistivity is called *electric conductivity*.

Ohm's law makes a global statement about a conductor. We want to derive from it a local relationship between the current density and the electric potential gradient.

We consider a rectangular volume element in a flow of the length dr (in the direction of the flow) and the cross-sectional area dA, Fig. 2.11.

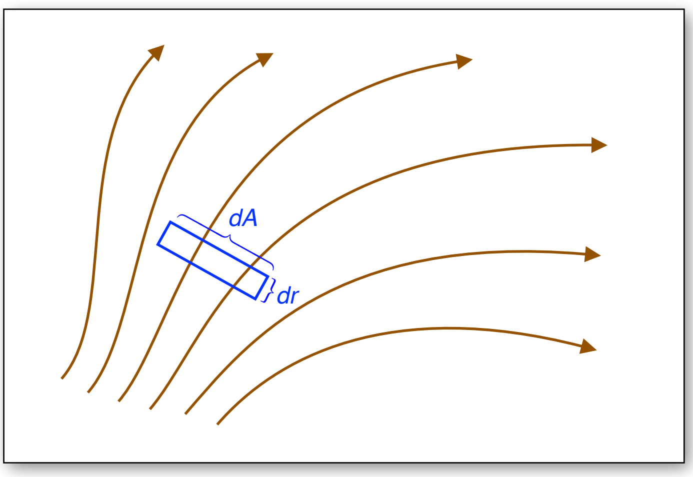

Fig. 2.11
As to the relationship between current density and potential gradient

The current $dI = |\boldsymbol{j}|dA$ through this cross-section is given by Ohm's law:

$$|\boldsymbol{j}|dA = \frac{d\Phi}{dR} = \frac{d\Phi}{dr}\sigma dA$$

where $d\Phi$ is the potential difference between the two surfaces separated by dr. From this follows $|\boldsymbol{j}| = \sigma\, d\phi/dr$. Since the current flows in the direction in which Φ decreases the most, we can write:

$$\boldsymbol{j} = -\sigma \operatorname{grad} \Phi \,. \tag{2.8}$$

2.5 The difference of the values of the intensive quantity as a drive of the current of the extensive quantity

For an electrical current to flow through an electrical resistor, it needs a drive: an electric voltage, Fig. 2.12. Sometimes $I \sim U$ (or $\boldsymbol{j} \sim$ grad ϕ) (Ohm's law). Entropy is produced in the resistor:

$T\, I_{S,\text{prod}} = \Delta\phi \cdot I$

Name of the process: *Joule heating*

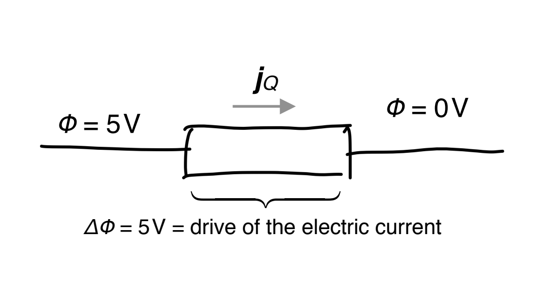

Fig. 2.12
Electric potential difference as a drive of an electric current

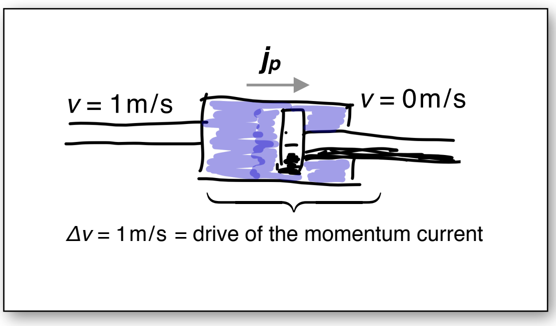

Fig. 2.13
Velocity difference as a drive of a momentum current

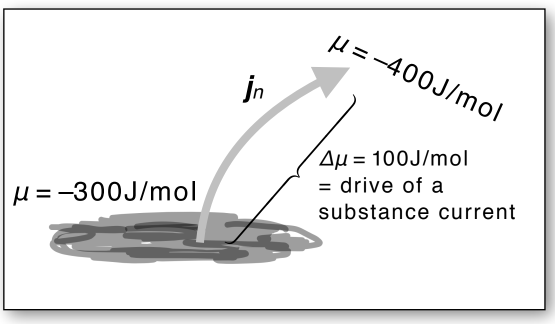

Fig. 2.14
Chemical potential difference as a drive of a current of the amount of substance

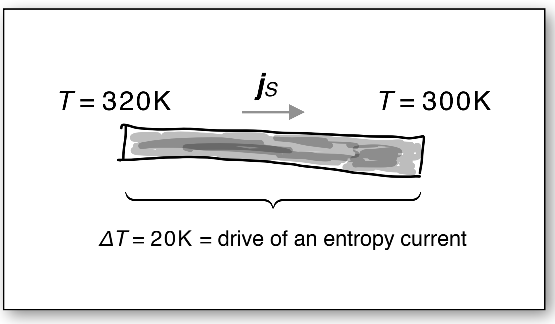

Fig. 2.15
Temperature difference as a drive of the entropy current

For a momentum current F to flow through a mechanical resistor (shock absorber, viscous medium), it needs a drive: a velocity difference, Fig. 2.13. Sometimes $F = \Delta v / R_p$. Entropy is produced in the mechanical resistor:

$T\, I_{S,\text{prod}} = \Delta v \cdot F$

Name of the process: *friction*

For a substance current to flow through a "chemical resistor" (e.g. gaseous water from one point of a room through the air to another), it needs a drive: a difference in the chemical potential μ, Fig. 2.14. Sometimes $j_n \sim$ grad μ (Fick's first law). During this process entropy is produced:

$T\, I_{S,\text{prod}} = \Delta\mu \cdot I_n$

Name of the process: *diffusion*

For an entropy current to flow through a thermal resistor (e.g. the wall of a house, a copper rod), it needs a drive: a temperature difference, Fig. 2.15. Sometimes $I_S \sim$ grad T. During this process entropy is produced:

$T\, I_{S,\text{prod}} = \Delta T \cdot I_S$

Name of the process: *thermal conduction*

We summarize: A difference of the values of the intensive quantity represents a drive for a current of the corresponding extensive quantity.

The currents of the extensive quantities flow only as long as the drive, i.e. the difference of the intensive quantities, is different from zero. If this difference equals zero, it is said that there is equilibrium with respect to the corresponding extensive quantity, Figs. 2.16 - 2.19.

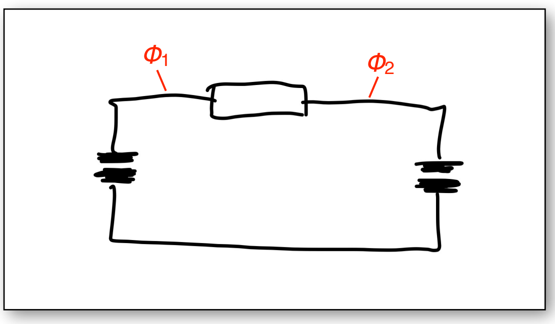

Fig. 2.16
Electric equilibrium: There is no more Q flowing if $\Phi_1 = \Phi_2$.

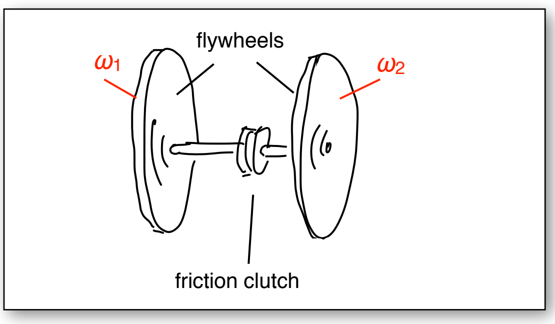

Fig. 2.17
Rotational equilibrium: There is no more L flowing if $\omega_1 = \omega_2$.

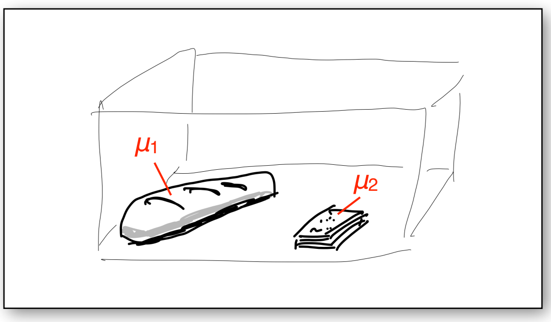

Fig. 2.18
Chemical equilibrium: There is no more substance flowing if $\mu_{\text{bread}} = \mu_{\text{crispbread}}$.

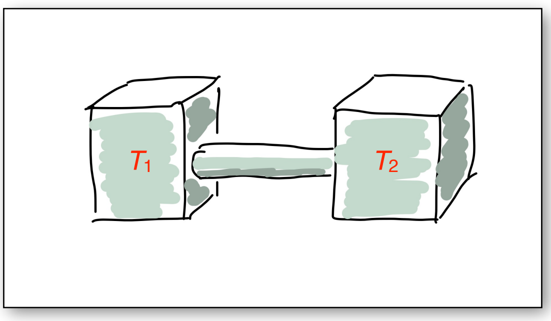

Fig. 2.19
Thermal equilibrium: There is no more S flowing if $T_1 = T_2$.

3

Electric current and substance current

3.1 The chemical potential

Just as an electrical potential difference represents a drive for a *Q* current, a chemical potential difference $\Delta\mu$ represents a drive for an *n* current. The value of the chemical potential always refers to a certain substance. Therefore, in case of doubt, the substance name is written in brackets behind the μ. In order to get a qualitative idea of the values of the chemical potential and the relationship between μ and other physical quantities, one only has to look from where to where substances flow.

A gas or liquid flows in a pipe from places of high pressure to places of low pressure. So the chemical potential depends on pressure, it grows with increasing pressure.

Water vapor mixed with air "diffuses" from places of high to places of low concentration. Likewise, salt dissolved in water diffuses from places of high to places of low concentration. μ therefore increases with the concentration.

In dry air, a puddle of water evaporates. The chemical potential of the water in the puddle must therefore be higher than in the air.

If silica gel is used to dry the air, the chemical potential of the water in the silica gel is lower than in the air. The chemical potential of a substance therefore also depends on the medium in which the substance is located.

3.2 Substances and particles

If the amount of a system doesn't matter, it is called a substance. 1 g air is the same substance as 1 kg air. In order to characterize a substance, however, the relationship between the values of the substance-like quantities is important.

So we have, for example

water: m/n = 18 g/mol

free electrons: m/n = 0.55 mg/mol and $Q/m = 1.76 \cdot 10^{11}$ C/kg

light: $E/p = 3 \times 10^8$ m/s.

Some of the substance-like quantities are quantized. What does this mean? If a system is isolated, i.e. if it cannot release or absorb anything of a quantity, the value of this quantity is an integer multiple of an elementary quantum, for example:

$Q = k_1 \cdot e \qquad e = 1.60 \cdot 10^{-19}$ C (elementary charge)

$L = k_2 \cdot \frac{\hbar}{2} \qquad \hbar = 1.05 \cdot 10^{-34}$ Js (quantum of angular momentum, Planck constant)

$n = k_3 \cdot \tau \qquad \tau = 1.66 \cdot 10^{-24}$ mol (elementary amount,1/Avogadro constant)

$k_1, k_2, k_3 =$ integer numbers

The quotient $F = e/\tau = 0.965 \cdot 10^5$ C/mol is called the *Faraday constant*.

A system for which the amount of substance has the value $n = 1\tau$ is called a particle. Under certain circumstances one can imagine it as a small localizable individual, but often this idea fails.

The particle electron e.g. is a system with

$n = 1\tau,\ Q = 1e,\ L = \hbar/2,\ E = \ldots,$ etc.

Particles with $Q \neq 0$ are called *charge carriers*.

Examples of charge carriers:

free electron
mobile electron in a semiconductor
electron hole in a semiconductor
free positron
Cu^{++} ion in aqueous solution
myon

Just as an ensemble of the values of all substance-like quantities belongs to a certain amount of a substance, an ensemble of flows of the corresponding substance-like quantities belongs to the flow of a substance. For example, an electron flow includes an electric current I, a mass flow I_m, a flow of the amount of substance I_n, an entropie flow I_S... Some of the currents are connected again in a characteristic way for a certain substance. Thus, for a current of free electrons we have

$I/I_m = 1.76 \cdot 10^{11}$C/kg.

The substance-like quantities are more or less "coupled to each other". Thus, electric charge is firmly coupled to the amount of substance and to mass. There is no electric current without a mass flow and without a flow of the amount of substance. So there is no such thing as a purely electric current. It follows from this that a substance or particle flow can be driven in different ways.

So one can drive an electron current:

- by creating an electric potential gradient; this potential gradient "pulls" at the charge of the electrons;
- by creating a gradient of the chemical potential; this pulls at the amount of substance of the electrons;
- by creating a T gradient; this pulls at the entropy of the electrons.

3.3 The electrochemical potential

We consider the current of any kind of charge carriers between the points a and b of a conductor, Fig. 3.1.

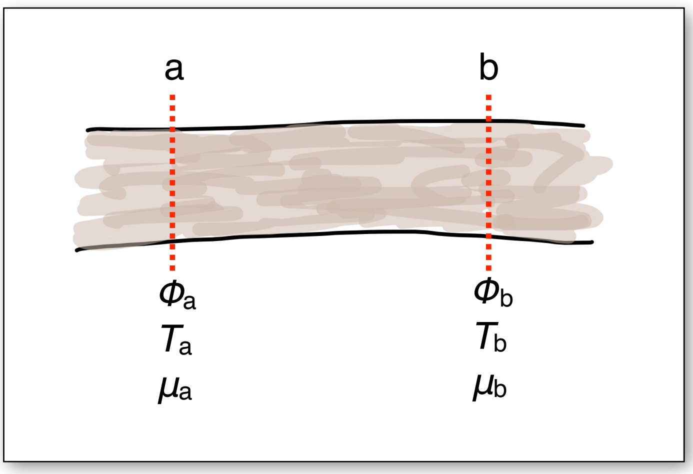

Fig. 3.1
A substance current can have various drives.

If all intensive variables have the same value at a and b, except for the electrical potential, i.e. if $T_a = T_b$, $\mu_a = \mu_b$,… and $\Phi_a \neq \Phi_b$, the particle flow is driven by the electric potential difference $\Delta\Phi = \Phi_a–\Phi_b$. In the conductor, energy is dissipated according to

$$P = T \cdot I_{S,prod} = (\Phi_a–\Phi_b)I.$$

If all intensive variables except μ have the same value at a and b, i.e. if $T_a = T_b$, $\Phi_a = \Phi_b$ etc. and $\mu_a \neq \mu_b$, then the particle flow is driven by the chemical potential difference $\Delta\mu = \mu_a - \mu_b$, and energy is dissipated in the conductor in accordance with

$$P = T \cdot I_{S,prod} = (\mu_a - \mu_b)\ I_n.$$

This equation can be regarded as a definition of the chemical potential. The unit of measurement of μ is J/mol, for which the abbreviation Gibbs (G) is also used:

1 G = 1 J/mol

If both Φ and μ have different values at a and b, the current has two drives: $\Delta\Phi$ and $\Delta\mu$. These can “pull” at the electrons in the same direction or in opposite directions.

The dissipated energy is then

$$P = T \cdot I_{S,prod} = (\Phi_a–\Phi_b)I + (\mu_a - \mu_b)\ I_n.$$

Now I and I_n are coupled to each other. A particle (n =1 τ) carries an integer z of elementary charges:

$$Q = ze.$$

For electrons, for example, $z = –1$.

The electric current and the amount-of-substance current are thus connected according to

$$\frac{I}{I_n} = \frac{ze}{\tau}$$

and with $e/\tau = F$ (= Faraday constant) follows

$$I = zFI_n\ . \qquad (3.1)$$

So the dissipated energy becomes

$$P = [(\Phi_a–\Phi_b)zF + (\mu_a - \mu_b)]I_n$$

The quantity

$$\eta = \mu + zF\Phi \qquad (3.2)$$

is called the *electrochemical potential* of the charge carriers in the respective environment. We thus obtain

$$P = (\eta_a - \eta_b)\ I_n \qquad (3.3)$$

We see that the total drive of the substance flow is given by the *electrochemical potential difference*

$$\Delta\eta = \eta_a - \eta_b$$

There is no particle or substance flow if $\Delta\eta = 0$, or if $\eta_a = \eta_b$. This means that

$$\mu_a - \mu_b = - zF\,(\Phi_a–\Phi_b)$$

Thus, the state of “no current” is not obtained if the electrical potential is the same everywhere, but if the electrochemical potential is the same.

3.4 The contact potential difference

The chemical potential of the electrons in different conductors is not the same. Therefore there is a chemical potential difference for the electrons in two metals. If it is defined that the chemical potential of free electrons in vacuum has the value 0 G, the values in Table 3.1 apply.

Table 3.1
Chemical potential of the electrons in several metals

substance	μ (in kG)
Ag	–460
Cs	–170
Cu	–430
Ni	–445
Pt	–515
W	–435

Between copper and platinum, for example, there is a chemical potential difference for electrons μ(Cu) – μ(Pt) = 85 kG. The chemical potential of the electrons is higher in Cu than in Pt. When a body of copper and a body of platinum are brought into contact with each other, initially electrons will flow from the copper into the platinum, following the chemical potential gradient. This causes the two metals to charge electrically, i.e. the copper positively and the platinum negatively. Thereby, the electric potential of copper increases and that of platinum decreases. The result is an electrical drive in the opposite direction to the chemical drive. If

$$F\Delta\Phi = \Delta\mu \ ,$$

or $\Delta\eta = 0$, i.e. if the electrical and the chemical drives are the same in the opposite directions, no particle current flows any more. There is *electrochemical equilibrium*.

Between two bodies made of different metals, which are in contact with each other, there is an electric potential difference, the *contact voltage* or Volta potential difference. We calculate the contact voltage between copper and platinum.

From

$$\eta = 0$$

follows

$$\Delta\Phi = (1/F)\Delta\mu.$$

With $F = 0.965 \cdot 10^5$ C/mol and $\Delta\mu = 85$ kG we obtain

$$\Delta\Phi = \Phi(\text{Cu}) - \Phi(\text{Pt}) = 0.88 \text{ V}.$$

Despite (or better: because of) this voltage, no electric current flows.

If a closed circuit is built from different metals, no current flows, Fig. 3.2.

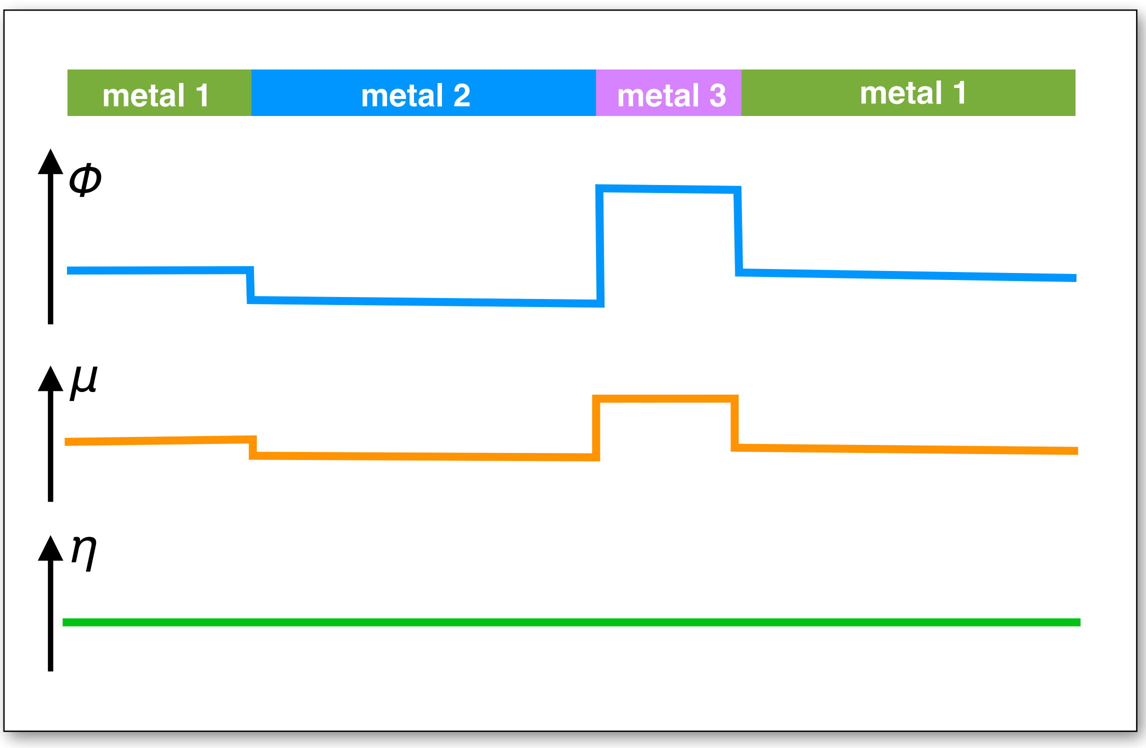

Fig. 3.2
Electrical, chemical and electrochemical potential in a closed “circuit”, that consists only of three conductors of different metals

The contact voltage cannot simply be measured with a voltmeter. Figure 3.3 shows why. A voltmeter always indicates the electrochemical potential difference. Only in the case that the chemical potential is the same in the two points between which one measures the electrochemical and the electric potential difference are equal (except for the factor zF).

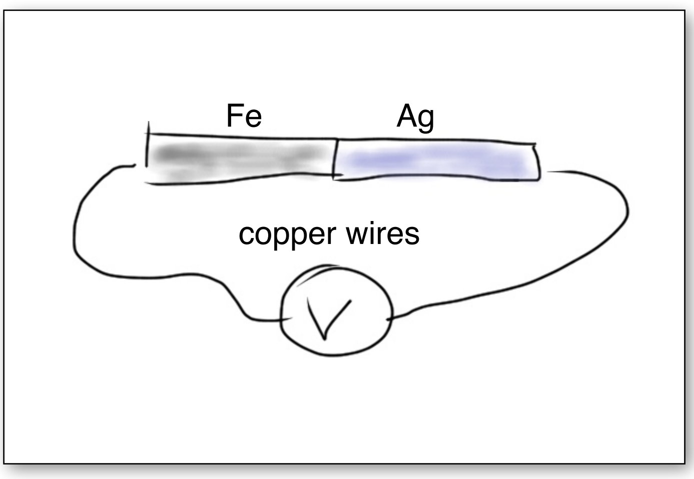

Fig. 3.3
The voltmeter does not measure the electrical potential difference between iron and silver, but the electrochemical potential difference.

If one nevertheless pretends that the voltmeter indicates the electric potential difference, there is usually no misfortune, because in many cases where one believes that one needs the electric potential difference, one actually needs the electrochemical one; for example, to calculate the electric current according to Ohm’s law.

Measuring the electric potential difference between two materials is quite difficult. The values in Table 3.1 are therefore subject to uncertainties.

3.5 The functioning of the galvanic cell

Even if the devices or “cells” are constructed differently, the essential features of their functioning are the same in all cases. In the following we try to understand it by means of a system, which is difficult to realize for technical reasons, but which is very transparent, so that it is easy to understand how it works.

The cell exploits the chemical potential difference that a gas – in our case hydrogen – passes through when it expands, Fig. 3.4.

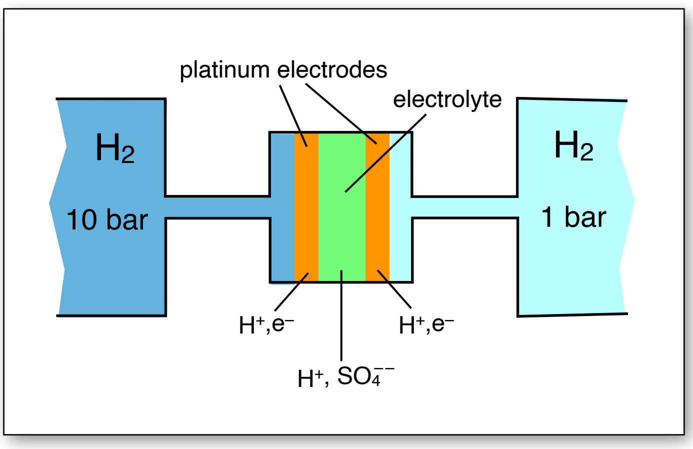

Fig. 3.4
Electrochemical cell. The chemical potential of hydrogen is higher in the left reservoir than in the right.

In the left reservoir there is hydrogen at high pressure, say 10 bar. In the right reservoir, the hydrogen pressure is 1 bar.

The left side of the cell has an inlet for the hydrogen at high pressure, and the right side has an outlet for the hydrogen at low pressure. Because of the pressure difference, and thus the chemical potential difference, the hydrogen “wants” to flow from left to right. However, this is not easy.

Behind the inlet is a platinum wall, a so-called *electrode*. There is another one on the outlet-side.

Platinum has the property of being able to absorb hydrogen. Since there is not enough space between the platinum atoms for the rather large hydrogen molecules, the hydrogen molecules disintegrate into electrons and protons, and these particles can move relatively freely in platinum.

Between the platinum electrodes there is an acid, e.g. sulfuric acid. Acids have the property that they are conductors for protons, but not for electrons.

Now, couldn’t at least the protons follow the chemical drive and flow to the right through the acid? In fact, at the very beginning a small number of protons flows through the acid from the left to the right platinum electrode. However, this causes an electric potential difference that represents a drive for the protons in the opposite direction. After a very short time, the two drives cancel each other and the proton current ceases to flow. The protons are in a state of electrochemical equilibrium. This means that the electric potential of the right platinum electrode is higher than that of the right.

It is now easy to open a path from left to right for the electrons as well: The two platinum electrodes are connected by a copper wire. Copper, like all metals, is a conductor for electrons and a non-conductor for protons. (We had just learned about platinum as an exception, it conducts both electrons and protons.)

This electron current through the copper can now be routed through an electrical energy receiver, Fig. 3.5.

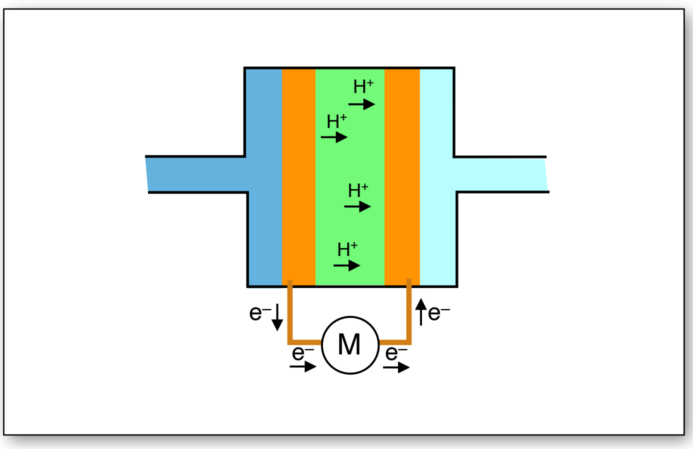

Fig. 3.5
The electric circuit is closed.

In practice, this cell works very poorly because the platinum is for the protons not as good a conductor as we assumed.

The really good cells use more complicated chemical reactions.

A chemical reaction is also driven by a chemical potential difference. These cells are set up in such a way that the reactants are spatially separated from each other. They can only come together if one of the substances is broken down into electrons plus ions in or on an electrode. Again, the ions pass through the electrolyte, the electrons through the outer part of the circuit.

It is not difficult to calculate the voltage of such a cell if the chemical potential difference of the reaction taking place in the cell is known. The chemical potentials can be taken from tables.

3.6 Fuel cell and electrolytic cell

If the chemical reaction

$$A + B \leftrightarrow C + D$$

is in equilibrium, the sum of the chemical potentials of the left side is equal to that of the right side:

$$\mu(A) + \mu(B) = \mu(C) + \mu(D) .$$

For a reaction A + B → C in equilibrium we have

$$\mu(A) + \mu(B) = \mu(C)$$

and for a reaction A + B → 2C it is

$$\mu(A) + \mu(B) = 2\mu(C) .$$

We consider the reaction

$$2H_2 + O_2 \leftrightarrow 2H_2O$$

From a table it can be seen that the chemical potential of the right side at normal pressure and room temperature is 474 kG lower than that of the left side. So there is a drive of

$$\Delta\mu = (2\mu(H_2) + \mu(O_2)) - 2\mu(H_2O) = 474 \text{ kG}$$

This difference is used in the hydrogen-oxygen fuel cell to drive an electric current. During electrolysis, i.e. the electrical decomposition of water into hydrogen and oxygen, this drive must be overcome.

Figure 3.6 shows the structure of a hydrogen-oxygen cell. The gas can enter each of the porous electrodes from one side, and the electrolyte from the other, but neither the gas nor the electrolyte can leave the electrode on the other side. The electrolyte is a conductor for H^+ ions, but not for electrons and oxygen ions.

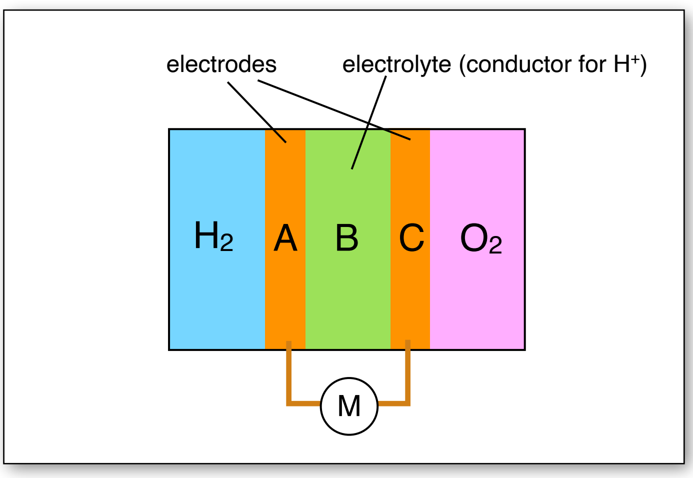

Fig. 3.6
Hydrogen-oxygen cell

Within the electrodes there are chemical equilibria, which are described by the following reaction equations:

electrode A	*electrode C*
$2H_2 \leftrightarrow 4H^+ + 4e^-$	$2H_2O \leftrightarrow 4H^+ + 4e^- + O_2$
$2\mu(H_2) = 4\mu_A(H^+) + 4\mu(e^-)$	$2\mu(H_2O) = 4\mu_C(H^+) + 4\mu(e^-) + \mu(O_2)$

The chemical potential of the electrons is the same in A and C, since the electrodes consist of the same material. The chemical potential of H^+, on the other hand, is very different because the oxidation reaction in C keeps its concentration in C low. We are looking for the electric potential difference between A and C. We subtract the right equation from the left equation:

$$(2\mu(H_2) + \mu(O_2)) - 2\mu(H_2O) = 4\mu_A(H^+) - 4\mu_C(H^+)$$

$$= 4(\eta_A(H^+) - F\Phi_A) - 4(\eta_C(H^+) - F\Phi_C)$$

In the last step of the equation $\eta = \mu + zF\Phi$ was used. Because the H^+ ions can flow freely back and forth between A and C through the electrolyte, there is electrochemical equilibrium for H^+ between A and C: $\eta_A(H^+) = \eta_C(H^+)$. One thus obtains

$$\Delta\mu = (2\mu(H_2) + \mu(O_2)) - 2\mu(H_2O) = 4F(\Phi_C - \Phi_A) = 4FU$$

Thus, the voltage between A and C becomes:

$$U = \frac{1}{4F}\Delta\mu \qquad (3.4)$$

where $\Delta\mu$ is the chemical potential difference of the total reaction of the cell.

With $\Delta\mu$ = 474 kG and F = 96500 C/mol one obtains U = 1.23 V.

This value applies as long as there is electrochemical equilibrium between the electrodes for H^+. There is no particle flow and no electric current.

If the circuit is closed via a load, the chemical drive for the H^+ ions is larger than the electrical drive, and an H^+ current flows from A to C. In the electrode C water is produced. The cell works as a fuel cell.

If, on the other hand, an external energy source ensures that the electrical drive of the H^+ ions from C to A is larger than the chemical drive from A to C, then H^+ flows from C to A and water is decomposed in the electrode C. This is the case when the H^+ ions are driven from C to A by an external energy source. The cell now works as an electrolytic cell.

In technical galvanic elements (lead accumulator, Leclanché element, Daniell element, Weston element), the electrodes simultaneously represent the “fuel reservoir”. The electrode material dissolves in the electrolyte. As part of the electrode, it has a different chemical potential than in the solution. This chemical potential difference is used to drive the electric current.

3.7 Coupled currents – the Onsager relations

We want to know the mathematical formalism using a simple example: the coupling between an electric current and a current of the amount of substance that has already been discussed.

The current of the amount of substance flowing in an appropriate conductor can be driven in two ways, Fig. 3.7:

- by a gradient grad μ of the chemical potential;
- by a gradient grad Φ of the electric potential pulling at the electric charge, which is firmly coupled to the amount of substance.

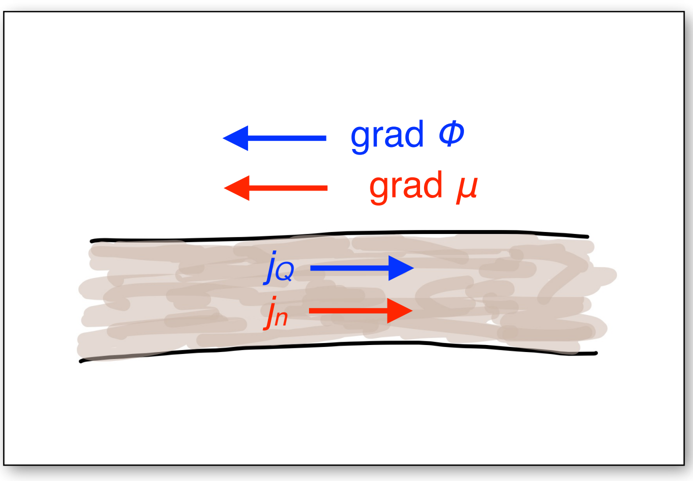

Fig. 3.7
Since n is coupled to Q, both the n current and the Q current can be driven by a Φ or a μ gradient.

Accordingly, a charge current can be driven:

- by a Φ gradient;
- by a μ gradient.

This can be expressed mathematically:

$$\mathbf{j}_n = L_{11}\ \text{grad}\ \mu + L_{12}\ \text{grad}\ \Phi \qquad (3.5a)$$

$$\mathbf{j}_Q = L_{21}\ \text{grad}\ \mu + L_{22}\ \text{grad}\ \Phi \qquad (3.5b)$$

$\mathbf{j}_n$ and $\mathbf{j}_Q$ are the current densities of the amount o substance and the electric charge. In the following we will consider the case where the gradients of μ and Φ are parallel to the x-direction. Equations (3.5a) and (3.5b) then simplify:

$$j_n = L_{11}\frac{d\mu}{dx} + L_{12}\frac{d\varphi}{dx} \qquad (3.6a)$$

$$j_Q = L_{21}\frac{d\mu}{dx} + L_{22}\frac{d\varphi}{dx} \qquad (3.6b)$$

Since Q is firmly coupled to n, the equations are linearly dependent: $j_Q = zFj_n$ (F = Faraday constant, z = integer). The analog will no longer be true if we select other currents, for example if we consider the entropy current density j_S instead of j_n (Section 3.8).

First we interpret the coefficients L_{ik}.

L_{11} is a measure of the n current caused by a given μ gradient in the event that no further drive is present ($d\Phi/dx = 0$). It has the meaning of a substance conductivity.

Accordingly, L_{22} is a measure for the electric current caused by a given Φ gradient, as long as no μ gradient is present. For $d\mu/dx = 0$ we have $j_Q = L_{22}\ d\Phi/dx$. The comparison with $j_Q = -\ \sigma\ d\Phi/dx$ (see equation (2.8)) shows that $L_{22} = -\ \sigma$, i.e. equal to the electrical conductivity.

L_{12} and L_{21} express that there is a coupling between j_n and j_Q. L_{12} indicates how strongly an n-current is driven by a Φ gradient and L_{21} how strongly a Q-current is driven by a μ gradient. It is obvious that if L_{12} is large, L_{21} must also be large, and vice versa. There is a general theorem that claims that basically

$$L_{12} = L_{21} \qquad (3.7)$$

always applies if two currents can be written in the form of equations (3.6a) and (3.6b). This relation is called *Onsager relation* after its discoverer. It can easily be proved in our concrete case.

The drive of a particle current is given by $d\eta/dx$:

$$j_n = L_1\frac{d\eta}{dx} = L_1\frac{d(\mu + zF\varphi)}{dx} = L_1\frac{d\mu}{dx} + L_1 zF\frac{d\varphi}{dx}$$

Comparing the coefficients with those in equation (3.6a) we obtain:

$L_{11} = L_1$ and $L_{12} = L_1 zF$

and thus

$L_{12} = zFL_{11}$.

If the case with $d\Phi/dx = 0$ is considered, with $j_Q = zFj_n$, equations (3.6a) and (3.6b) become

$$j_n = L_{11}\frac{d\mu}{dx}$$

$$j_Q = zFj_n = L_{21}\frac{d\mu}{dx}$$

Dividing one equation by the other we get:

$L_{21} = zFL_{11}$.

Thus we see that $L_{12} = L_{21}$ q. e. d.

If we divide equation (3.6b) by equation (3.6a), after setting $d\mu/dx$ = 0, we get

$L_{22} = zFL_{12}$

We can now express all 4 coefficients L_{ik} by the electrical conductivity σ and the Faraday constant F:

$$L_{22} = -\sigma \qquad L_{12} = L_{21} = -\frac{\sigma}{zF} \qquad L_{11} = -\frac{\sigma}{z^2F^2}$$

We now describe the coupling strength between the Q and the n current by a dimensionless constant m:

$$m = \left(\frac{j_n}{j_Q}\right)_{d\mu/dx=0} \cdot \left(\frac{j_Q}{j_n}\right)_{d\Phi/dx=0}$$

The first factor expresses how strongly n is dragged by Q if there is no proper drive for n ($d\mu/dx = 0$), the second factor is correspondingly a measure for how strongly Q is dragged by n. From equations (3.6a) and (3.6b) follows

$$m = \frac{L_{12}}{L_{22}} \cdot \frac{L_{21}}{L_{11}} = \frac{L_{12}^{\ 2}}{L_{11}L_{22}}$$

We use the terms for L_{21}, L_{12} and L_{22}:

$$m = \frac{\sigma^2/(zF)^2}{\left[-\sigma/(z^2F^2)\right](-\sigma)} = 1$$

In our case with a firm coupling, m = 1. For other currents, we expect smaller values for m.

3.8 Coupling between electric current and entropy current

The coupling between n and Q is a trivial special case. The equations (3.6a) and (3.6b) in the previous section can be replaced by a single equation:

$$j_n = -\frac{\sigma}{z^2F^2}\frac{d\eta}{dx}$$

j_Q is calculated from j_n simply by multiplying by zF.

If there is a temperature gradient in addition to the η gradient, we can no longer do without equations of the type of equations (3.6a) and (3.6b):

$$j_n = L_{11}\frac{d\eta}{dx} + L_{12}\frac{dT}{dx} \tag{3.8a}$$

$$j_S = L_{21}\frac{d\eta}{dx} + L_{22}\frac{dT}{dx} \tag{3.8b}$$

Of course, the coefficients L_{ik} now have different meanings than in the previous section. To interpret the equations, we will look at some special cases:

(1) $dT/dx = 0$, $d\eta/dx \neq 0$

$dT/dx = 0$ means: The temperature of the conductor is the same everywhere. Equation (3.8a) says what we already knew: An η gradient results in a substance flow, e.g. an electron current. Equation (3.8b) now says that this substance current drags along an entropy current.

(2) $d\eta/dx = 0$, $dT/dx \neq 0$

Equation (3.8b) tells what we already knew: A T gradient results in an S current (see section 2.5). Equation (3.8a) claims that a T gradient drives an electron current even though there is no electrochemical (and no electrical) potential difference.

(3) $j_n = 0$

We prevent the flow of a particle current by simply not installing the conductor in a circuit. From equation (3.8a) follows:

$$\frac{d\eta/dx}{dT/dx} = -\frac{L_{12}}{L_{11}}$$

A T gradient thus causes a gradient of the electrochemical potential.

We now look for the relation between the L_{ik} and the material coefficients, which can be found in tables.

We compare equation (3.8a) for $dT/dx = 0$ with the equation valid for $dT/dx = 0$

$$j_n = -\frac{\sigma_Q}{z^2F^2}\frac{d\eta}{dx}$$

and obtain

$$L_{11} = -\frac{\sigma_Q}{z^2F^2}$$

(We mark the electric conductivity with the index Q to distinguish it from the entropy conductivity σ_S).

Experimentally one finds the relationship:

$$j_S = -\sigma_S\frac{dT}{dx}$$

(See also section 2.5).

σ_S is the entropy conductivity. Tables usually give the “thermal conductivity” $\lambda = T\,\sigma_S$. Comparison with equation (3.8b) for $d\eta/dx = 0$ results in

$$L_{22} = -\sigma_S$$

The quantity

$$\alpha = -\frac{1}{zF}\left(\frac{d\eta/dx}{dT/dx}\right)_{j_n=0} \tag{3.9}$$

is called thermovoltage or thermoelectric emf. Its values can be found in tables. It indicates the electrochemical potential difference per degree K (unit V/K) between two points in the case that the current is zero. From equation (3.8a) follows:

$$\frac{L_{12}}{L_{11}} = -\alpha zF$$

We also calculate the coupling strength m:

$$m = \left(\frac{j_n}{j_S}\right)_{d\eta/dx=0}\cdot\left(\frac{j_S}{j_n}\right)_{dT/dx=0} = \frac{L_{12}}{L_{22}}\cdot\frac{L_{21}}{L_{11}} = \frac{L_{12}{}^2}{L_{11}L_{22}}$$

$$m = \alpha^2\frac{\sigma_Q}{\sigma_S}$$

Table 3.2 shows the values of σ_Q, σ_S and α for some metals at ambient temperature (≈ 300 K).

What does it mean that some α are positive and some are negative? In equation (3.9) the factor

$$\left(\frac{d\eta/dx}{dT/dx}\right)_{j_n=0}$$

is always negative because the temperature gradient builds up an opposite electrochemical potential gradient. $\alpha < 0$ therefore means $z < 0$, i.e. the charge carriers are negative – they are electrons. If $\alpha > 0$ we have $z > 0$. The charge carriers are positive. They are called “defect electrons”.

The quotient σ_Q/σ_S (last column of Table 3.2) is almost temperature-independent, although σ_Q and σ_S separately depend strongly on T. In addition, σ_Q/σ_S is almost the same for all metals. This fact is called *Wiedemann-Franz law*. One can conclude that the conduction of Q and S is realized by the same carriers.

For the order of magnitude of m we obtain:

$$m \approx 2{,}5 \cdot 10^{-12} \cdot 4{,}5 \cdot 10^{7} \approx 10^{-4}$$

Thus, the coupling between entropy and mass flow is very weak.

Metal	$10^{-7}\sigma_Q$ (Ω^{-1}m^{-1})	σ_S (JK^{-2}m^{-1}s^{-1})	$10^6\alpha$ (VK^{-1})	$10^{-7}\sigma_Q/\sigma_S$ (K^{2}V^{-2})
Ag	6.29	1.43	+1.5	4.4
Al	3.77	0.79	−1.7	4.8
Cs	0.5	0.12	+0.1	4.2
Cu	6.0	1.34	+1.86	4.5
Fe	1.03	0.27	+16.6	3.8
Hg	0.10	0.028	+8.6	3.6
Mg	2.25	0.52	+4.3	4.3
Na	2.38	0.47	−8.7	5.0
Ni	1.46	0.303	−20.0	4.8
Pb	0.48	0.118	−1.26	4.1
Pt	0.94	0.239	−5.13	3.9

Table 3.2
Electric conductivity, entropy conductivity and thermovoltage for some metals

3.9 Thermocouple and Peltier heatpump

According to equation (3.8a), there is an electrochemical potential difference between the ends of a copper wire which are at different temperatures T_1 and T_2 (Fig. 3.8).

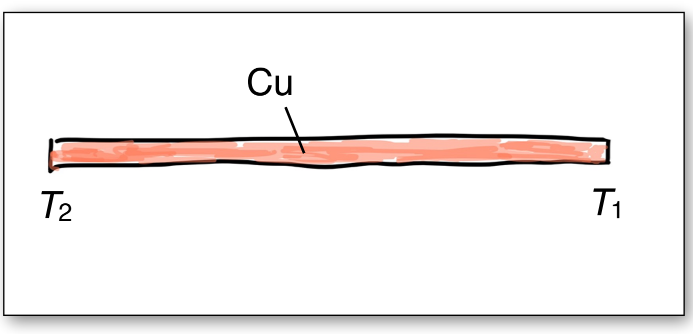

Fig. 3.8
Copper wire whose ends are at different temperatures

If the temperature difference is not too large, so that one can neglect the T dependence of σ, we have

$$\frac{\Delta\eta}{\Delta T} = \frac{d\eta/dx}{dT/dx} = -zF\alpha$$

and it follows

$$\Delta\eta = -zF\alpha\Delta T$$

We try to measure $\Delta\eta$ with the voltmeter. But the result is $\Delta\eta = 0$. Figure 3.9 shows why.

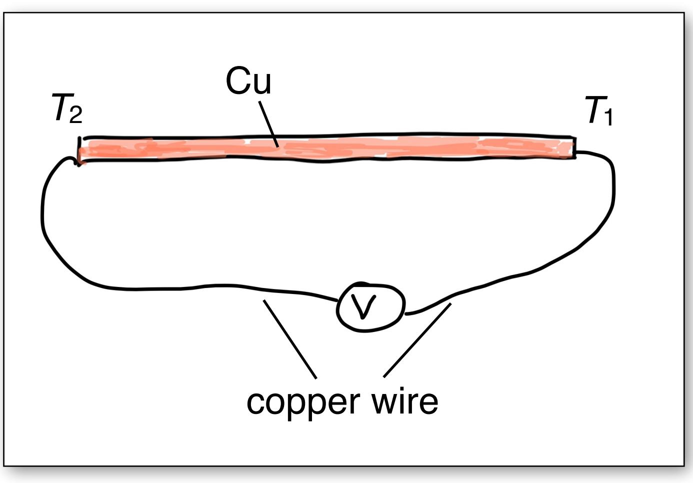

Fig. 3.9
The difference of the electrochemical potential between the terminals of the voltmeter is zero.

We now replace the second connection between the high and the low temperature with another metal, Fig. 3.10.

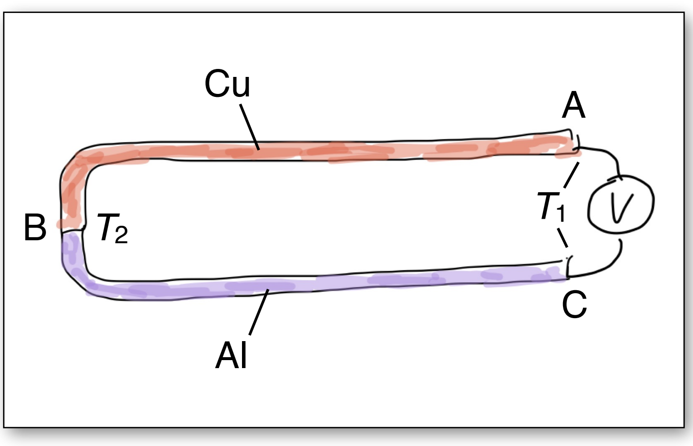

Fig. 3.10
Between A and C there is a measurable difference of the electrochemical potential.

We then have

$$\eta_A - \eta_B = -zF\alpha_{Cu}(T_1 - T_2)$$

and

$$\eta_B - \eta_C = -zF\alpha_{Al}(T_2 - T_1)$$

It follows

$$\eta_A - \eta_C = (\eta_A - \eta_B) + (\eta_B - \eta_C) = -zF(\alpha_{Cu} - \alpha_{Al})(T_1 - T_2)$$

So between points A and C there is a measurable electrochemical potential difference. Such an arrangement of two conductors made of different materials is called a thermocouple. It is used, among other things, to measure temperatures.

The same voltage is found when the circuit is interrupted at any other point, Fig. 3.11. The decisive factor is that the contact points between the two metals have different temperatures T_1 and T_2.

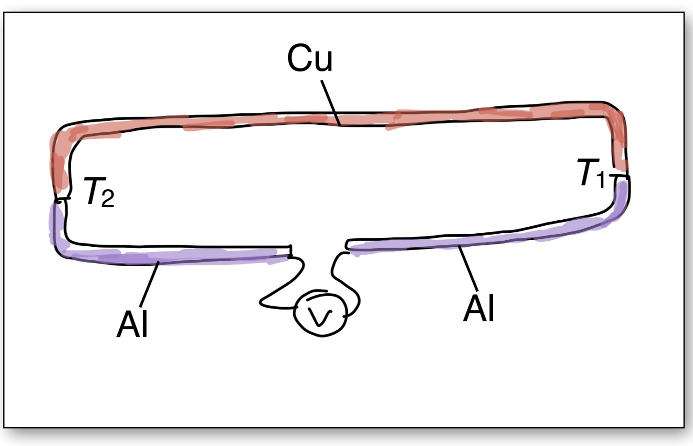

Fig. 3.11
One always finds the same potential difference, no matter where one interrupts the circuit.

If the two metals are joined to form a closed circuit, Fig. 3.12, a current flows. Its strength depends on the resistance, i.e. the cross-section and length of the conductors.

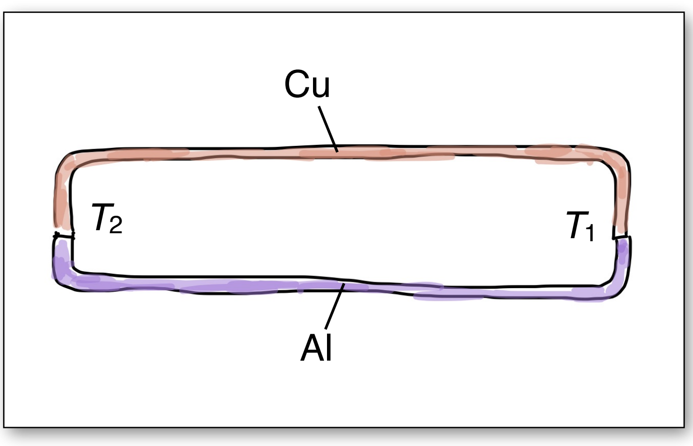

Fig. 3.12
Short-circuited thermocouple

If an electric energy consumer is connected instead of the voltmeter, the arrangement works as an "energy converter". An energy current of the strength $T_2 I_{S2}$ flows into the contact of the high temperature T_2. At the low temperature contact an energy current $T_1 I_{S1} < T_2 I_{S2}$ flows out. The difference flows out through the wires "in the form of electric energy". Because of the weak coupling between S and n, entropy essentially "slips down" the T mountain, thereby producing new entropy, instead of driving the electric current. Thermocouples are therefore highly irreversible energy converters. Their efficiency is much lower than that of a steam turbine plus generator, for example.

The thermocouple can also be operated in reverse: One "pumps" with an electrical energy source a particle current by the two contacts, Fig. 3.13.

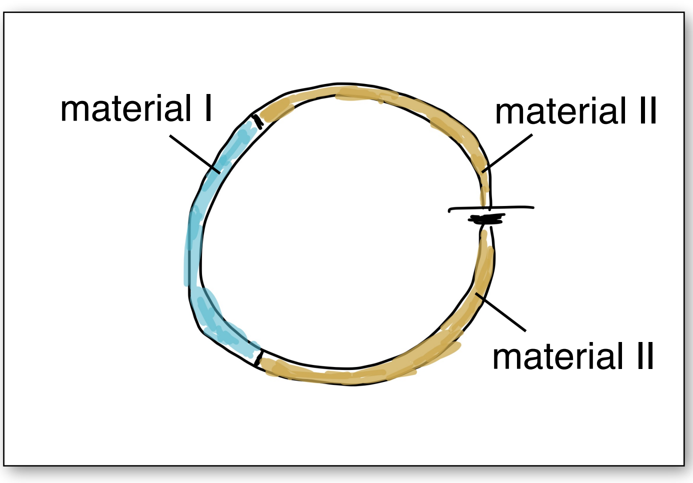

Fig. 3.13
Peltier heatpump

Since the particle flow in the two materials drags the entropy differently well, a net entropy flow occurs between the two contacts. If the contacts are thermally insulated against the environment, one of them heats up while the other cools down. This process is called the *Peltier effect*. Although such a heat pump is simple and robust, its efficiency is poor.

B

THE ELECTROMAGNETIC FIELD

4

Charge and polarization as sources of the electric field

4.1 The relationship between electric field strength and electric charge

There is always an electric field attached to electrically charged particles. The system electric field is recognizable by

- the forces it exerts on electrically charged bodies;
- the energy contained in it.

The electric field is a subsystem of the electromagnetic field: its states form a partial manifold of the states of the electromagnetic field.

If one brings into a given field at a certain position a very small electrically charged body, then a force acts on this "point charge" (one recognizes it by the fact that the momentum of the charged body changes). If the value of the charge Q is doubled, the value of the force $\boldsymbol{F}$ is also doubled:

$$\boldsymbol{F} \sim Q\,.$$

The vectorial factor of proportionality is thus characteristic for the field in the absence of the additional point charge. It is called the strength of the electric field. The SI symbol of the electric field strength is $\boldsymbol{E}$, the unit is N/C = V/m:

$$\boldsymbol{F} = \boldsymbol{E}Q \qquad (4.1)$$

The $\boldsymbol{E}$ vector field describes the electric field unambiguously. From $\boldsymbol{E}$, the mechanical stress (momentum current density) and the energy density can be calculated. The energy current density in the purely electric field is zero.

The $\boldsymbol{E}$ field distribution of a point charge Q follows from Coulomb's law

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\frac{\boldsymbol{r}}{r} \qquad (4.2)$$

$\boldsymbol{r}$ is the distance vector from the point charge, and

$$\varepsilon_0 = 8.854 \cdot 10^{-12}\ \text{C/(Vm)}$$

is the electric field constant. If another charge distribution $\rho_2(\boldsymbol{r})$ is added to a charge distribution $\rho_1(\boldsymbol{r})$, the associated field strength distributions add up vectorially:

$$\rho(\boldsymbol{r}) = \rho_1(\boldsymbol{r}) + \rho_2(\boldsymbol{r}) \quad \Rightarrow \quad \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_1(\boldsymbol{r}) + \boldsymbol{E}_2(\boldsymbol{r})\quad .$$

The flux of an arbitrary vector field $\boldsymbol{V}(\boldsymbol{r})$ through the area S is the integral

$$\iint_S \boldsymbol{V}(\boldsymbol{r})\,d\boldsymbol{A}$$

If $\boldsymbol{V}(\boldsymbol{r})$ is a current density, the flux is the corresponding current intensity.

We calculate the flux of the $\boldsymbol{E}$ field of a point charge through a closed spherical surface whose centre lies at the position of the point charge, Fig. 4.1:

$$\oint\!\!\oint_S \boldsymbol{E}(\boldsymbol{r})\,d\boldsymbol{A} = \frac{Q}{4\pi\varepsilon_0}\oint\!\!\oint_{\substack{\text{surface}\\ \text{of sphere}}}\frac{d\boldsymbol{A}}{r^2} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\oint\!\!\oint_{\substack{\text{surface}\\ \text{of sphere}}} d\boldsymbol{A} = \frac{Q}{\varepsilon_0}$$

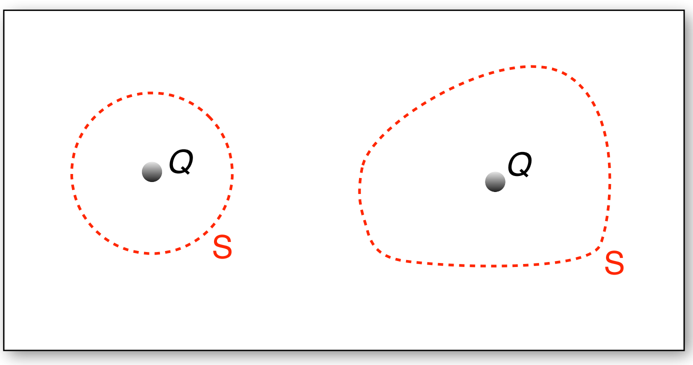

Fig. 4.1
Surfaces that enclose a point charge

For the calculation it was assumed that $d\boldsymbol{A}$ is parallel to $\boldsymbol{r}$ everywhere.

The flow through another arbitrarily shaped closed surface, which is placed around the point charge, has the same value, since field lines do not begin or end anywhere outside the charge.

We now place another charge distribution inside the closed surface and approximate it by a set of point charges. Since the field strengths add up when the charges are added, the following applies

$$\oint\!\!\oint_S \boldsymbol{E}(\boldsymbol{r})\,d\boldsymbol{A} = \frac{1}{\varepsilon_0}\sum_i Q_i$$

$\sum Q_i$ is the total charge that is located inside the area S. If we now describe the charge inside the surface by the charge density distribution $\rho(\boldsymbol{r})$, then

$$\oint\!\!\oint_S \boldsymbol{E}(\boldsymbol{r})\,d\boldsymbol{A} = \frac{1}{\varepsilon_0}\iiint \rho(\boldsymbol{r})\,dV \qquad (4.3)$$

With the Gauss's theorem

$$\oint\!\!\oint \boldsymbol{E}(\boldsymbol{r})\,d\boldsymbol{A} = \iiint \operatorname{div}\boldsymbol{E}\,dV$$

(4.3) becomes

$$\iiint \operatorname{div}\boldsymbol{E}\,dV = \frac{1}{\varepsilon_0}\iiint \rho\,dV$$

Since this relationship is correct for any area of space, the following must apply

$$\operatorname{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad (4.4)$$

Equations (4.3) and (4.4) express the fact that the electric charge is the place where the electric field is "attached" to matter.

4.2 The relation between the electric potential and the electric field strength

Let there be a point charge. We want to show that the value of the integral

$$\int_A^B \boldsymbol{E}\, d\boldsymbol{r}$$

only depends on the start and end points, but not on the integration path.

The contribution to the integral on the path between the radii r and r', Fig. 4.2, is

$$\boldsymbol{E}\, d\boldsymbol{r} = |\boldsymbol{E}|\, |d\boldsymbol{r}| \cos\alpha .$$

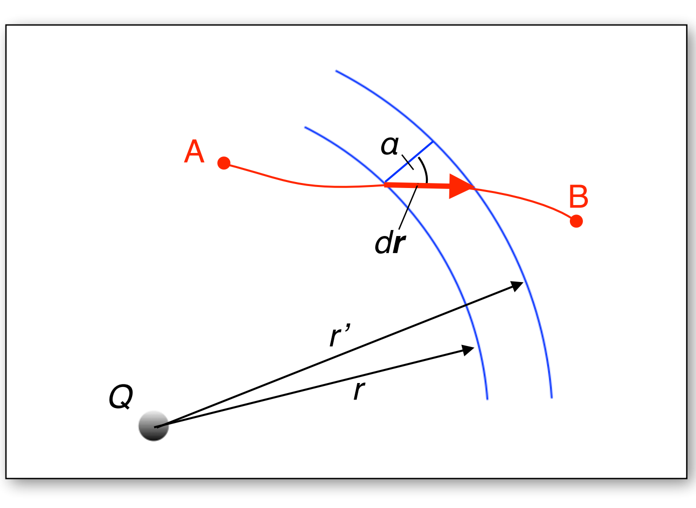

Fig. 4.2
The contribution to the integral on the path section between radii r and r' does not depend on the direction of this path section.

We also have

$$|d\boldsymbol{r}| = \frac{r' - r}{\cos\alpha} = \frac{dr}{\cos\alpha}$$

$|d\boldsymbol{r}|$ is the magnitude of the vector $d\boldsymbol{r}$, dr is the difference $r' - r$ of the radii. The magnitude of the field strength vector $\boldsymbol{E}$ is denoted by $|\boldsymbol{E}|$ in order to avoid confusion with the energy E.

Thus

$$\boldsymbol{E}\, d\boldsymbol{r} = |\boldsymbol{E}|\, dr .$$

The contribution therefore depends only on the two radii r and r', but not on the direction of the integration path. The entire integral consists of such contributions. Its value thus depends only on the distance of the points A and B from the point charge. In particular, the integral over the path S' in Fig. 4.3 has the same value as that over the path S.

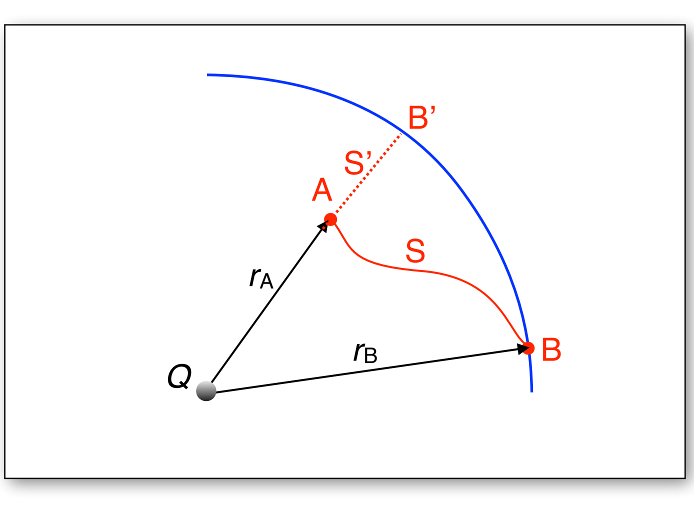

Fig. 4.3
The path integral over the electric field strength has the same value on path S as on path S'.

Thus we get

$$\int_A^B \boldsymbol{E}\, d\boldsymbol{r} = \int_A^{B'} \boldsymbol{E}\, d\boldsymbol{r} = \int_A^{B'} |\boldsymbol{E}|\, dr = \frac{Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B} \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

One can now imagine any charge distribution replaced by a collection of point charges. Since the field strengths of these point charges add up to the total field strength, the general rule applies

$$\int_A^B \boldsymbol{E}\, d\boldsymbol{r} \quad \text{is independent of the integration path}$$

We can thus define a potential function $\Phi(\boldsymbol{r})$:

$$\int_A^B \boldsymbol{E}\, d\boldsymbol{r} = -\left[\Phi(\boldsymbol{r}_B) - \Phi(\boldsymbol{r}_A)\right] \tag{4.5}$$

However, hereby only Φ differences are defined. The zero point of Φ can still be set arbitrarily.

For a point charge we get

$$\Phi(\boldsymbol{r}_A) - \Phi(\boldsymbol{r}_B) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

If we set $\Phi(r = \infty) = 0$ we obtain for the point charge

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \tag{4.6}$$

As a result of the definition equation (4.5) we get

$$\boldsymbol{E}(\boldsymbol{r}) = -\,\text{grad}\ \Phi(\boldsymbol{r}) \tag{4.7}$$

$\Phi(\boldsymbol{r})$ = const defines a set of surfaces. Each **E**-field line runs orthogonally to the corresponding surface at each point. We call these surfaces *field surfaces*. The field surfaces are thus surfaces of constant electric potential. Therefore they are also called *equipotential surfaces*.

(We will later look at fields for which no potential can be defined, but which can still be graphically represented with the help of field lines and field surfaces. So the term field surface is more general than the term equipotential surface.)

Fig. 4.4 shows the **E** field lines and the sections of the field surfaces with the drawing plane for two bodies with a spherically symmetrical charge distribution.

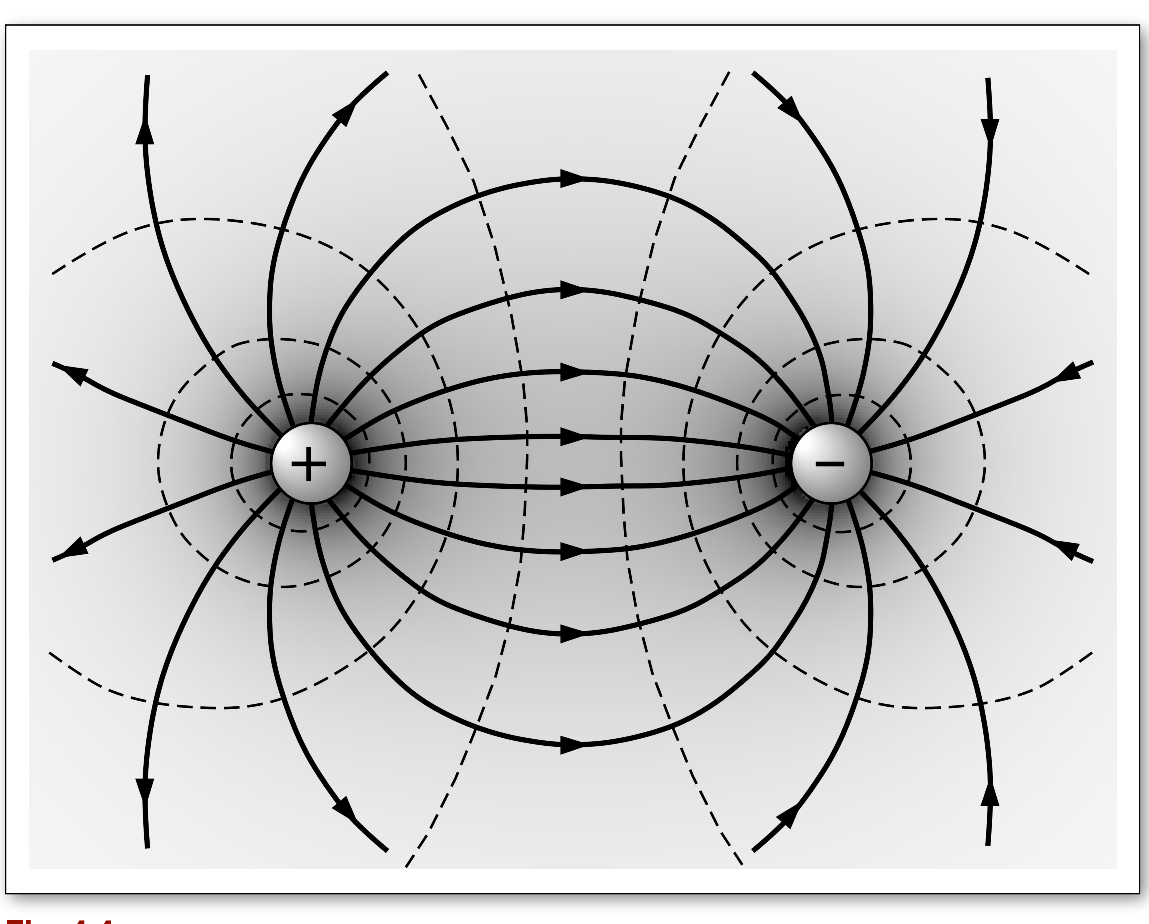

Fig. 4.4
Electric field lines (solid) and field surfaces (dashed) of the field of two spherical bodies with equal and opposite charges

On a charged particle (charge Q), which is in the electric field of the strength $\boldsymbol{E}$ of other charges, the field exerts a force $\boldsymbol{F} = Q\boldsymbol{E}$. If the particle is moving in the field, the energy flow into the particle is

$$P = \boldsymbol{vF} = Q\boldsymbol{vE}$$

This energy can be extracted during the movement. If the particle moves dissipatively, i.e. in an electrical conductor with resistance, the energy is used to produce entropy. If the particle moves from a point A to a point B, this energy amounts to

$$\Delta E = \int P\, dt = \int_{r_A}^{r_B} \boldsymbol{F}\, d\boldsymbol{r} = Q\int_{r_A}^{r_B} \boldsymbol{E}\, d\boldsymbol{r} = Q\left[\phi(\boldsymbol{r}_A) - \phi(\boldsymbol{r}_B)\right]$$

If an entire particle current, and thus an electric current, is flowing from A to B, then the energy flow

$$P = I\,(\Phi(\boldsymbol{r}_A) - \Phi(\boldsymbol{r}_B))$$

is dissipated.

The electric potential that is defined here is therefore identical to that defined by equation (2.1). We see, however, that not only an electrical conductor can be assigned a potential, but that every point of a static electric field has a potential.

With $\boldsymbol{E} = -\,\text{grad}\ \Phi$ we can simplify the equation $\boldsymbol{j} = -\,\sigma\, \text{grad}\ \Phi$ (Ohm's law, equation (2.8)):

$$\boldsymbol{j} = \sigma\boldsymbol{E} \tag{4.8}$$

Ohm's law applies in this form:

- only for isotropic media; in general σ is a tensor;
- only if the electrochemical potential can be replaced by the electric potential, i.e. for conductors with a constant chemical potential.

From div $\boldsymbol{E} = \rho/\varepsilon_0$ and $\boldsymbol{E} = -\,\text{grad}\ \Phi$ follows a relationship between ρ and Φ:

$$\text{div grad}\ \Phi = -\rho/\varepsilon_0 .$$

The operator div grad is abbreviated by Δ; it is called the Laplace operator. So we can write

$$\Delta\Phi = -\rho/\varepsilon_0 .$$

This equation is called Poisson's equation. In Cartesian coordinates the Laplace operator is:

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Often the charge distribution $\rho(\boldsymbol{r})$ is known and the field strength distribution $\boldsymbol{E}(\boldsymbol{r})$ is wanted.

Example: Spherically symmetric charge distribution

The **E**-field is radially symmetrical, Fig. 4.5.

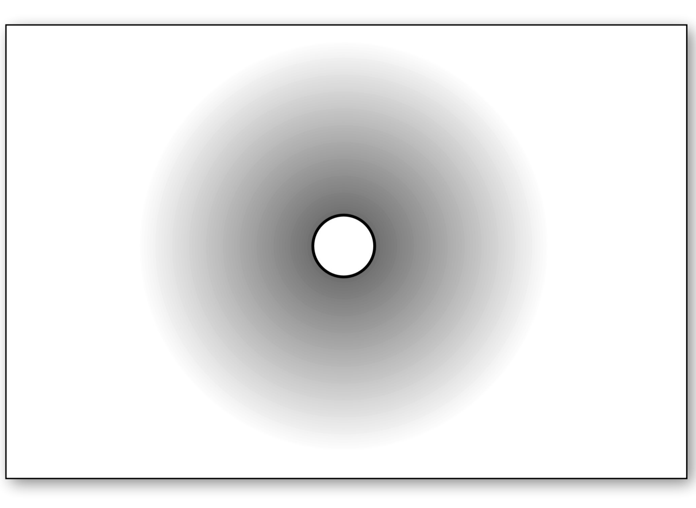

Fig. 4.5
Spherically symmetric charge distribution

The field strength can be calculated by applying equation (4.3) to a sphere with radius r. The left side is

$$\oint\!\!\oint \boldsymbol{E}\, d\boldsymbol{A} = |\boldsymbol{E}(\boldsymbol{r})|\, 4\pi r^2$$

The right side is the total charge $Q(r)$ within the sphere of radius r. So it is

$$|\boldsymbol{E}| = \frac{Q(r)}{\varepsilon_0 4\pi r^2}$$

The field strength at the distance r from the center of the distribution thus depends only on the total charge within the sphere of radius r, but not on how the charge density depends on r. Even if the entire charge inside the sphere were concentrated in the center of the sphere, the field strength would be the same. The field strength is therefore the same as that of a point charge located at the center. If the charge density inside the charge distribution between $r = 0$ and $r = r_0$ is equal to zero, then this entire interior space is field-free.

Example: Field between two infinitely extended parallel metal plates

The charge per surface area ρ_A is the same everywhere. However, the charge of one plate has the opposite sign of the charge of the other. For the dotted area in Figure 4.6, the following applies

$$\oint\!\!\oint \boldsymbol{E}\, d\boldsymbol{A} = |\boldsymbol{E}| \cdot |\boldsymbol{A}|$$

and

$$\iiint \rho\, dV = \rho_A A$$

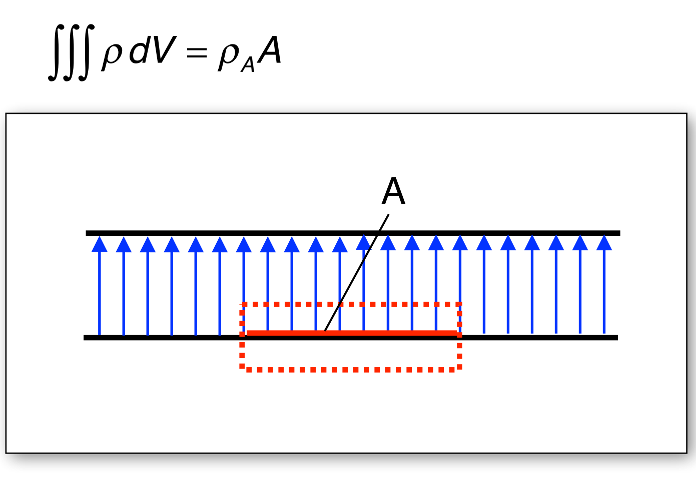

Fig. 4.6
Detail of two infinitely extended metal plates

With (4.3) we get

$$|\boldsymbol{E}| \cdot A = \frac{1}{\varepsilon_0}\rho_A A$$

and

$$|\boldsymbol{E}| = \frac{1}{\varepsilon_0}\rho_A \tag{4.9}$$

The **E** field between two oppositely charged parallel metal plates of finite extension is approximately homogeneous. The smaller the plate spacing against the lateral extension of the plates, the more homogeneous it is.

Example: Field of a homogeneous metal body

If an electric conductor that is electrically insulated from its surroundings is charged, the charge displaces until electrochemical equilibrium is achieved, i.e. until η has the same value everywhere.

If the material composition of the object is homogeneous, e.g. if it consists of a single metal, the chemical potential has the same value everywhere and it follows from η = const that also Φ = const.

So all points of the object are on the same electric potential. In particular, its surface is an equipotential surface, Fig. 4.7.

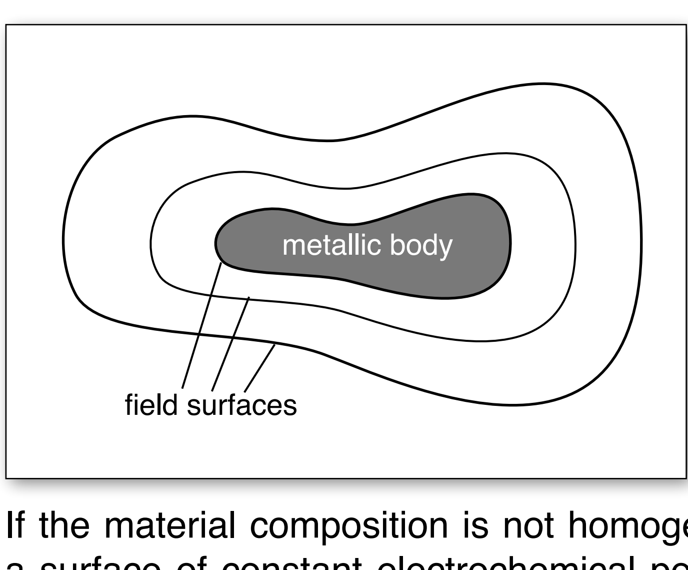

Fig. 4.7
The surface of a metallic body is an equipotential surface.

If the material composition is not homogeneous, the surface is only a surface of constant electrochemical potential. Since the deviation of the surface from an equipotential surface is only a few volts, the surface can practically be identified with an equipotential surface when dealing with high voltages.

4.3 Capacitance

We're looking at a charged, electrically conductive object. Its potential is Φ_1. There should be no other charged objects in its environment. For large distances from the object, the field strength goes towards zero, the potential towards a constant value: $\Phi(r \to \infty) = \Phi_\infty$.

If the charge density everywhere on the object is changed by the factor k from ρ_0 to $\rho = k\,\rho_0$, the following changes occur

- the total charge from Q_0 to $Q = kQ_0$;
- the field strength in each point of the field by the same factor k;
- the voltage between any two points by the same factor k.

The new charge distribution is again an equilibrium distribution, because for the object there is still Φ = const.

The voltage $U = \Phi_1 - \Phi_\infty$ has also changed by the factor k. So we have $Q/Q_0 = U/U_0$, or $U \sim Q$, or $Q = CU$.

The factor of proportionality C, which is independent of Q, is called the *capacitance* of the object. It expresses how much charge sits on the object for a given voltage between the body and a point infinitely far away.

We now consider two electrically conducting objects whose total charge is always zero, Fig. 4.8.

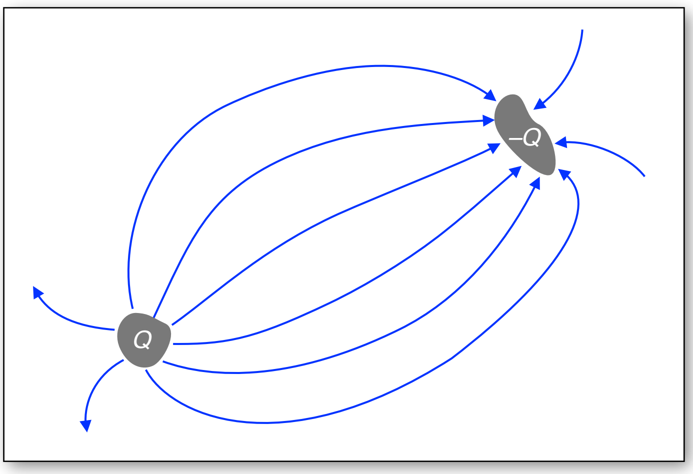

Fig. 4.8
Field of two bodies of equal and opposite charge

If we change the charge of each individual object by the factor k, the field strength changes everywhere by the factor k and also the voltage between the objects:

$$Q = CU \qquad (4.10)$$

C is the capacitance of the arrangement. If this capacitance is large compared to that of the two objects individually, the arrangement is called a *capacitor*.

Fig. 4.9 shows a plate capacitor.

Since its $\boldsymbol{E}$ field is approximately homogeneous, the following applies to it

$$\int_{\text{plate 1}}^{\text{plate 2}} \boldsymbol{E}\,d\boldsymbol{r} = |\boldsymbol{E}| \cdot d = U$$

Here, d is the distance between the plates. With equation (4.9) we obtain

$$Q = \varepsilon_0 AU/d,$$

and with (4.10) finally

$$C = \frac{\varepsilon_0 A}{d}$$

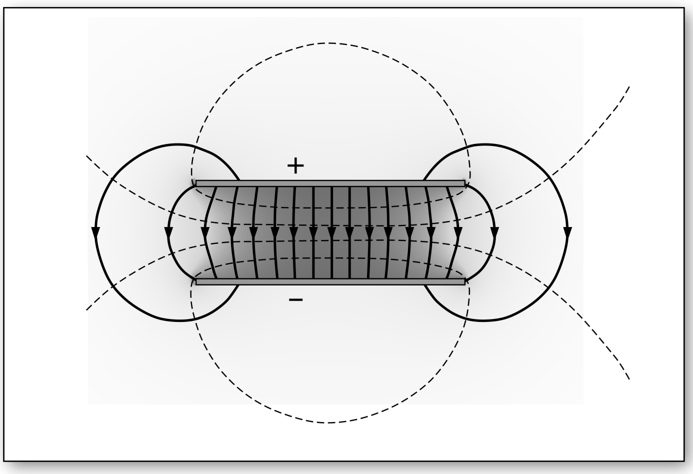

Fig. 4.9
Field of the plate capacitor

4.4 Dipoles, dipole density and polarization

An arrangement as in Fig. 4.10 of two point-like bodies with electric charges of equal magnitude but opposite sign is called a dipole.

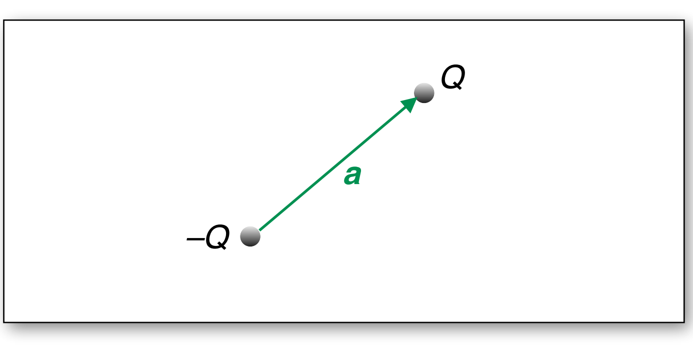

Fig. 4.10
Electric dipole

We call

$$\boldsymbol{p} = Q\boldsymbol{a} \qquad (4.12)$$

the *electric dipole moment*. (Compare: $\boldsymbol{L} = \sum \boldsymbol{r}_i \times \boldsymbol{p}_i$ is the angular momentum). Attention: The symbols of the momentum and the electric dipole moment are identical!

Just as it is mathematically particularly easy to deal with point charges, it is also easy to operate with point dipoles. One obtains a point dipole from a "real" dumbbell-shaped dipole by approaching $\boldsymbol{a} \to 0$ and simultaneously increasing Q so that $\boldsymbol{p}$ remains constant. For distances that are large compared to the distance between the dumbbell's point charges, the fields of the dumbbell and the point dipoles are identical.

We calculate the field of a point dipole by superposing the fields of two point charges and making the transition $\boldsymbol{a} \to 0$ with $\boldsymbol{p}$ = const.

For $r \to \infty$ the potential is defined to be zero. With equation (4.6) for the potential in point A (Fig. 4.11) the following results

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{-Q}{r_2} = \frac{1}{4\pi\varepsilon_0}\frac{Q(r_2 - r_1)}{r_1 r_2}$$

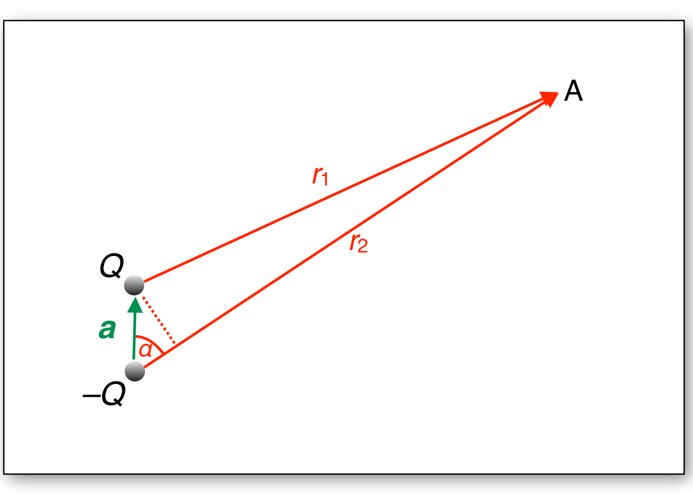

Fig. 4.11
Calculation of the potential distribution of the field of an electric dipole

For $\boldsymbol{a} \to 0$ we have $r_2 - r_1 = a \cos \alpha$ and $r_1 \cdot r_2 \approx r^2$, thus

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{Qa\cos\alpha}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{pr}}{r^3}$$

The equipotential surfaces (field surfaces) are shown in Fig. 4.12.

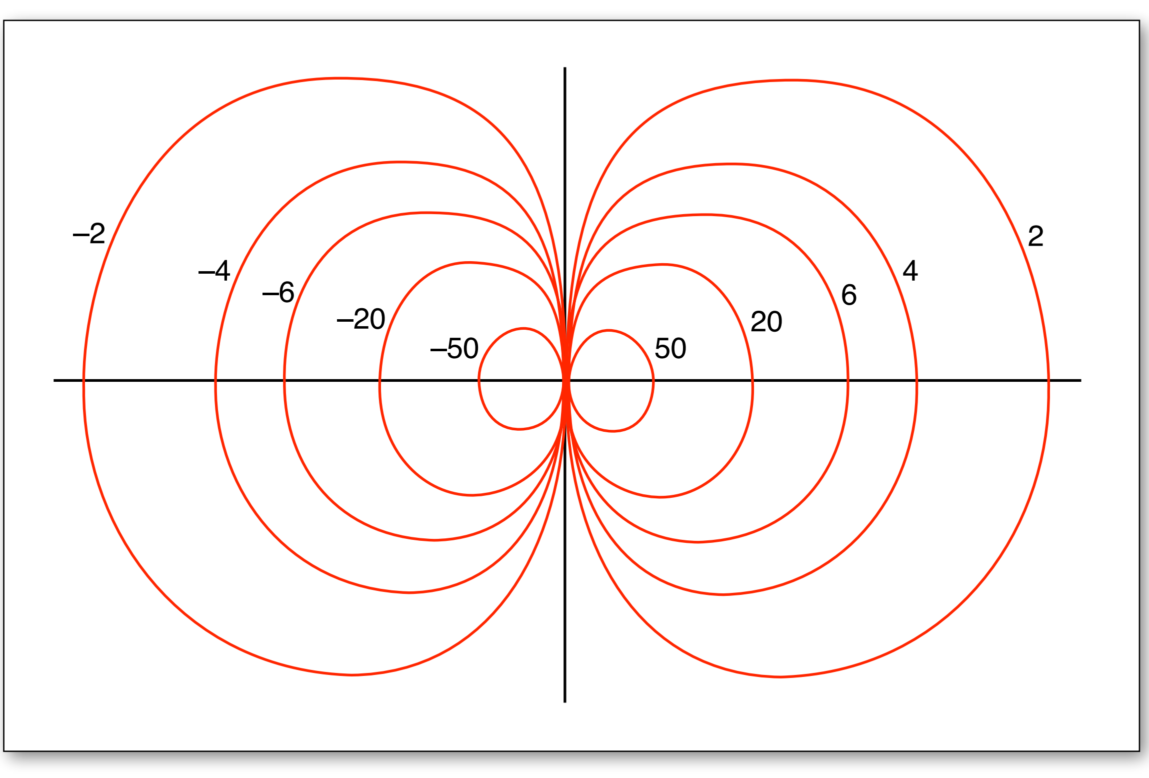

Fig. 4.12
Section through the equipotential surfaces of a point dipole

Just as in a field of field strength $\boldsymbol{E}$ a momentum current $\boldsymbol{F} = Q\boldsymbol{E}$ flows into a point-like body of charge Q, an angular momentum current

$$\boldsymbol{M} = \boldsymbol{p} \times \boldsymbol{E}$$

enters a point-like body with the electric dipole moment $\boldsymbol{p}$. (Attention $\boldsymbol{p}$ is the electric dipole moment, not momentum.)

Proof: First we imagine the dipole to be extended, Fig. 4.13.

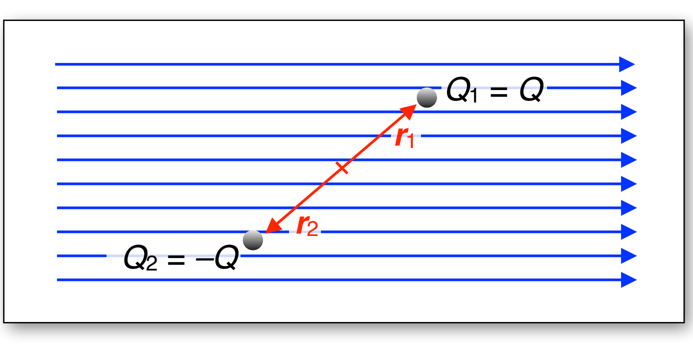

Fig. 4.13
The field exerts a torque on the dipole.

We have

$$\boldsymbol{M} = \sum_{1,2} \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_{1,2} \boldsymbol{r}_i \times Q_i\boldsymbol{E} = 2Q(\boldsymbol{r}_1 \times \boldsymbol{E})$$

and with $2\boldsymbol{r}_1 Q = \boldsymbol{p}$

$$\boldsymbol{M} = \boldsymbol{p} \times \boldsymbol{E} \qquad (4.13)$$

Many substances consist of molecules whose dipole moments are different from zero. If a body consisting of such molecules is brought into an "external" field, its dipoles are partially aligned. In some substances, the dipole moment vectors are parallel by nature.

We investigate the field of a piece of such polarized matter. Each molecule has the dipole moment $\boldsymbol{p}_i$. The field of a small volume element is obtained by superposing the fields of the n elementary dipoles located in it. At a sufficiently large distance, it is the same as a single dipole with the dipole moment $d\boldsymbol{p} = n\boldsymbol{p}_i$. We define the dipole moment density $\boldsymbol{P}$ (or dipole density for short):

$$d\boldsymbol{p} = \boldsymbol{P}\, dV \qquad (4.14)$$

$\boldsymbol{P}$ is also called *electric polarization*. $\boldsymbol{P}$ corresponds to a point of a body. $\boldsymbol{P}(\boldsymbol{r})$ is therefore a vector field.

We can replace the polarized matter in the volume element dV by non-polarized matter, which is oppositely charged at the two front faces, Figure 4.14.

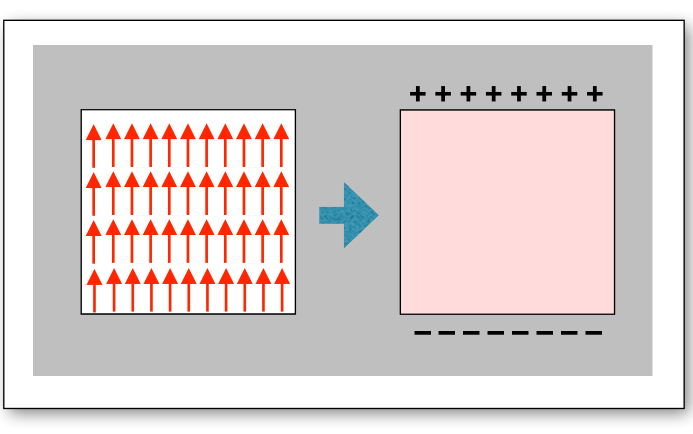

Fig. 4.14
The volume element polarized in its interior is equivalent to a volume element charged at the end faces.

If one places electric charges $|\boldsymbol{P}d\boldsymbol{A}|$ and $-|\boldsymbol{P}d\boldsymbol{A}|$ respectively, on the surfaces, the new matter element has the same field as the original polarized matter element. Now we build a macroscopic body out of many volume elements. The result is a body which is neutral inside, but whose base and top surface is charged, Fig. 4.15.

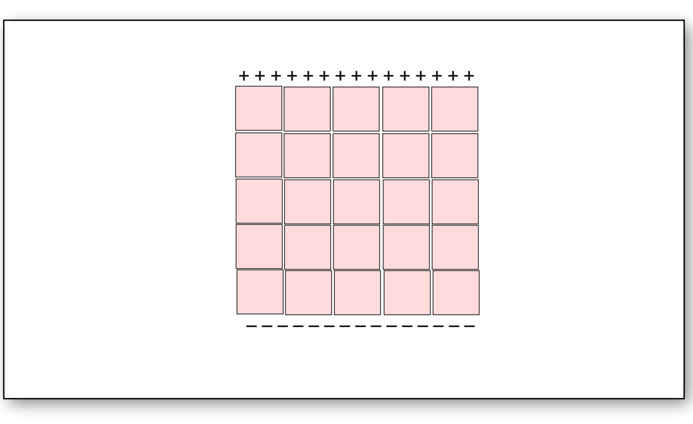

Fig. 4.15
A polarized body is neutral inside and charged at the surface.

A uniformly polarized body thus carries charges at its base and at its top surface, where

$$|Q_P| = |\boldsymbol{P}|A\,.$$

It is common to distinguish this polarization charge Q_P from the ordinary charge Q. Since Q_P cannot move freely, it is called *bound charge*.

The last equation can be generalized to (Fig. 4.16):

$$Q_P = -\oiint \boldsymbol{P}\, d\boldsymbol{A} \qquad (4.15)$$

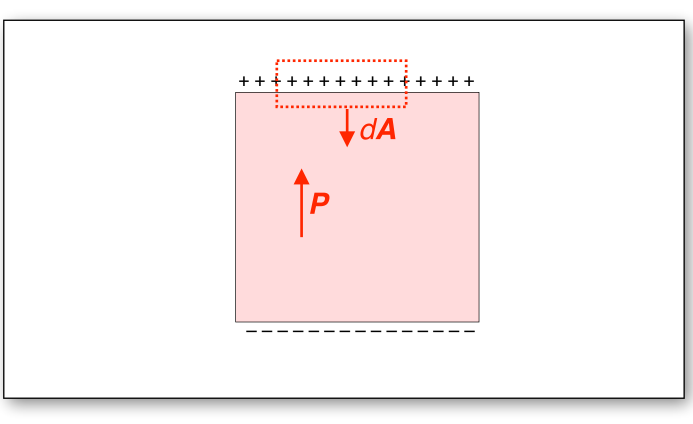

Fig. 4.16
The relationship between polarization and surface charge

In this form it also applies when the dipole density is no longer uniform. The relationship $|Q_P| = |\boldsymbol{P}|A$ results from this as a special case.

We want to transform equation (4.15) into a relation that is locally valid.

With

$$Q_P = \iiint \rho_P\, dV$$

(ρ_P = density of polarization charge) equation (4.15) becomes:

$$\iiint \rho_P\, dV = -\oiint \boldsymbol{P}\, d\boldsymbol{A}$$

We transform the right side with Gauss's theorem:

$$\iiint \rho_P\, dV = -\iiint \operatorname{div} \boldsymbol{P}\, dV$$

Since this equation applies to every region of space, we have:

$$\rho_P = -\operatorname{div} \boldsymbol{P} \qquad (4.16)$$

We can now generalize the equation div $\boldsymbol{E} = \rho/\varepsilon_0$ (Equation 4.4). Here, ρ is only the density of the free charge, i.e. the polarization charge is not contained in ρ. Therefore, in the case of polarized matter the term ρ_P/ε_0 has to be added to the right side of equation (4.4):

$$\operatorname{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0} + \frac{\rho_P}{\varepsilon_0}$$

We thus obtain

$$\operatorname{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0}\operatorname{div}\boldsymbol{P} \qquad (4.17)$$

If there is no free charge, then there is

$$\operatorname{div} \boldsymbol{E} = -(1/\varepsilon_0) \operatorname{div} \boldsymbol{P}.$$

This means that at a position where the $\boldsymbol{E}$ field has sources, the $\boldsymbol{P}$ field has sinks and vice versa. It is common to give equation (4.17) another form. First we write

$$\operatorname{div} (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho$$

We now abbreviate the sum after the div sign:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}\,. \qquad (4.18)$$

The quantity $\boldsymbol{D}$ is called the *electric displacement*. This makes equation (4.17) shorter and more memorable:

$$\operatorname{div} \boldsymbol{D} = \rho \qquad (4.19)$$

Integration and application of the Gauss's theorem yields

$$\oiint \boldsymbol{D}\, d\boldsymbol{A} = Q$$

Using

$$Q = \iiint \rho\, dV$$

we obtain

$$\oiint \boldsymbol{D}\, d\boldsymbol{A} = \iiint \rho\, dV \qquad (4.20)$$

This equation is *Maxwell's 1st equation*.

4.5 Distortion polarization, orientation polarization, electrostatic induction

In the following we investigate the behavior of a piece of matter that is brought into an electric field originating from other bodies. In order to obtain a situation that is as transparent as possible, we always select the arrangements in such a way that homogeneous fields are created: The field-producing bodies are two infinitely extended, oppositely equally charged plates, whose charge per surface has a value fixed once and for all. The piece of matter to be examined is an infinitely extended plate, which is located between the capacitor plates, Fig. 4.17.

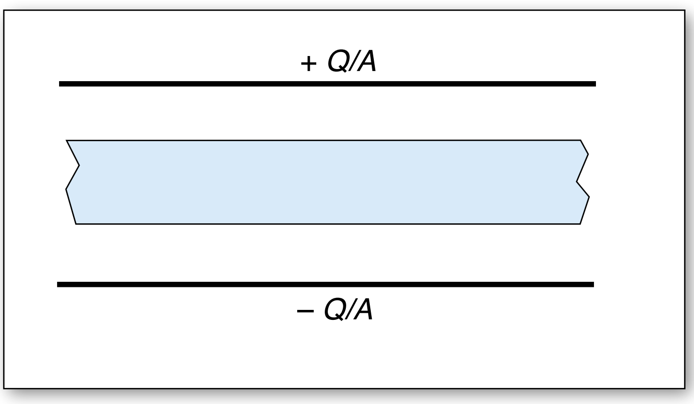

Fig. 4.17
A plate of the material to be examined is located between the plates of a capacitor.

Between the capacitor plates there is first a plate of polarized matter, Fig. 4.18. We are looking for the electric field strength within the matter.

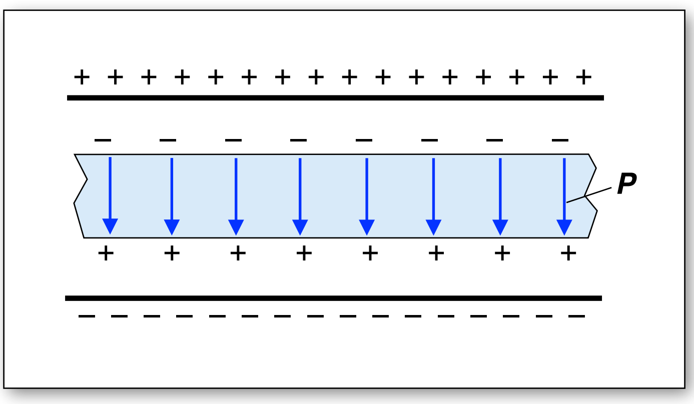

Fig. 4.18
Polarized matter between the plates of a capacitor

The electric field strength in the matter free space between the capacitor plates is (equation (4.9)):

$$|\boldsymbol{E}| = \frac{1}{\varepsilon_0}\frac{Q}{A}$$

Sources of the $\boldsymbol{P}$ field sit at the surface of the piece of matter, but no free charges. Therefore div $(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = 0$. It follows that the expression $\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ has the same value on both sides of the surface of the matter, i.e. inside (i) and outside (o). Since outside $\boldsymbol{P} = \boldsymbol{0}$, we have

$$\boldsymbol{E}_\text{o} = \boldsymbol{E}_\text{i} + \frac{1}{\varepsilon_0}\boldsymbol{P}$$

and

$$\boldsymbol{E}_\text{i} = \boldsymbol{E}_\text{o} - \frac{1}{\varepsilon_0}\boldsymbol{P} \qquad (4.21)$$

We are now dealing with the question of how the polarization of matter comes about. Polarized matter, like the one we have talked about so far, is normally not encountered at all: the polarization charges at the end surfaces are compensated by the ubiquitous free charges. On the other hand, it is possible to polarize matter by placing it in an electric field and inducing polarization. Depending on the material, this happens in different ways.

Distortion polarization: The positive charge within matter is shifted slightly against the negative charge under the influence of the $\boldsymbol{E}$ field. As long as $\boldsymbol{E}$ is not too large, the displacement, and thus the polarization, is proportional to the field strength in the matter.

$$\boldsymbol{P} = \varepsilon_0\, \chi_e \boldsymbol{E}\,. \qquad (4.22)$$

The dimensionless factor χ_e is called *electric susceptibility*. The larger χ_e is, the stronger the polarization is for a given $\boldsymbol{E}$ field strength.

Orientation polarization: Some substances consist of molecules that have a dipole moment different from zero but are generally disordered, e.g. water. When an electric field is applied, these dipoles are partially aligned. Here, too, equation (4.22) applies approximately.

Displacement and orientation polarization can only be found in non-conductors. Here the electric field strength is not zero, the field lines run through the matter. Non-conductors are therefore also called *Dielectrics* (from δια = through).

The values of χ_e are situated between 2 and 10 for most non-conductors. The value for water is unusually high, it is 80. This value is caused by the large dipole moment of the water molecules.

Electrostatic induction: If electrically conductive matter, i.e. a substance containing free charge carriers, is brought between the capacitor plates, Fig. 4.19, the charge shifts until the field strength within the matter equals zero.

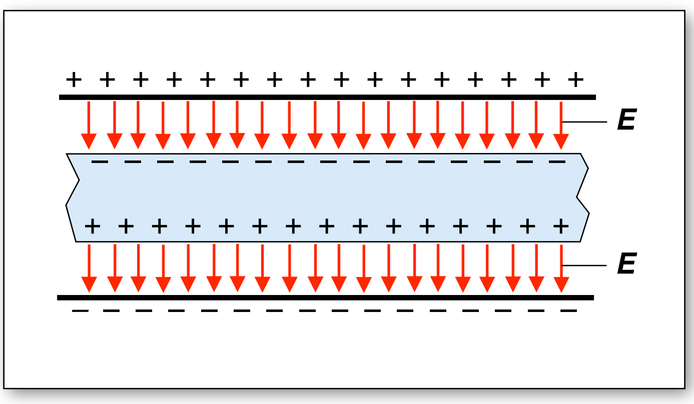

Fig. 4.19
Electrically conductive matter between the plates of a capacitor

Although this process is similar to the polarization of a non-conductor, it has a different name. The process is called induction and is not described by a $\boldsymbol{P}$-field. Rather, it is said that there are true charges at the boundary surfaces of the material.

4.6 The capacitance of a capacitor filled with matter

We now make the plate of polarizable material in Fig. 4.18 thicker and thicker, so that it finally fills the whole space between the capacitor plates, Fig. 4.20.

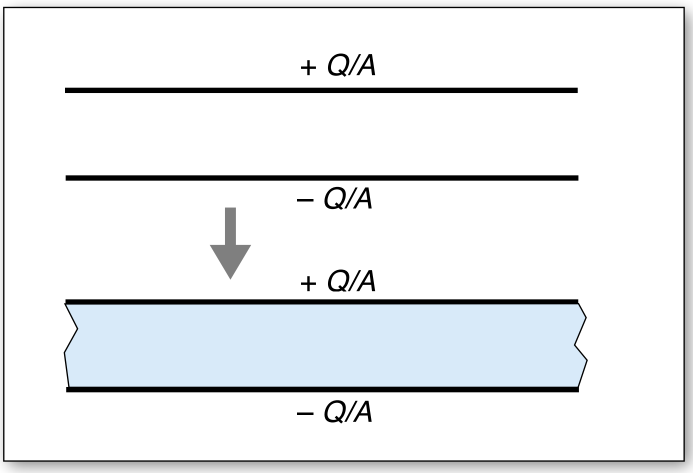

Fig. 4.20
The field strength is reduced by inserting the dielectric.

The field strength in the space not yet filled with matter remains constant during this process. According to equation (4.9) there is always

$$|\boldsymbol{E}| = \frac{1}{\varepsilon_0}\frac{Q}{A}$$

After the region has been filled with matter, the field strength there has decreased to the value given by equation (4.21). If we designate the field strength at a point before the insertion of matter with $\boldsymbol{E}_b$ and afterwards with $\boldsymbol{E}_a$, then we have:

$$\begin{aligned}\boldsymbol{E}_a &= \boldsymbol{E}_b - (1/\varepsilon_0)\,\boldsymbol{P} \\ &= \boldsymbol{E}_b - \chi_e\,\boldsymbol{E}_a\end{aligned}$$

It follows

$$\boldsymbol{E}_b = \boldsymbol{E}_a + \chi_e\,\boldsymbol{E}_a = (1 + \chi_e)\,\boldsymbol{E}_a$$

The factor

$$\varepsilon = 1 + \chi_e\,,$$

by which the field strength decreases is called the *dielectric constant*. (The material constant increased by 1 has its own name and its own symbol! Fortunately this is not usual in physics).

Because of $U = \int \boldsymbol{E}d\boldsymbol{r} = |\boldsymbol{E}|d$ the voltage decreases by the same factor:

$$U_b = \varepsilon U_a\,.$$

Since $Q = CU$, and since Q remains constant, the capacitance increases by ε:

$$C_a = \varepsilon\,C_b\quad.$$

We can thus generalize equation (4.11), which only applied to the matter-free capacitor:

$$C = \varepsilon\varepsilon_0\frac{A}{d} \tag{4.23}$$

In addition, equation (4.9) becomes

$$\rho_A = \varepsilon\varepsilon_0\,|\boldsymbol{E}| \tag{4.24}$$

If the matter is placed into a capacitor whose voltage is kept constant, its charge increases by the factor ε, Fig. 4.21, due to $Q = CU$:

$$Q_a = \varepsilon Q_b\,.$$

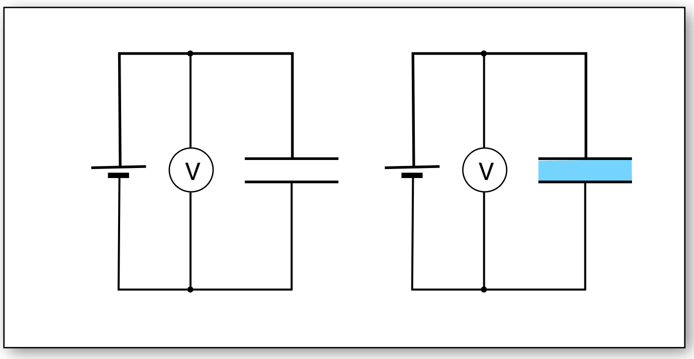

Fig. 4.21
If a dielectric is placed in a capacitor at a constant voltage, its charge increases.

4.7 Piezo and pyroelectric effect

Materials whose crystal structure is sufficiently asymmetrical are polarized when they are deformed. Deformation creates an electrical voltage between two opposite faces. This effect is called *piezoelectric effect*. Conversely, a deformation of the crystal results when a voltage is applied between the corresponding surfaces.

Crystals with even lower symmetry show the *pyroelectric effect*: When the temperature changes, the polarization changes; an electrical voltage is generated between two opposite faces.

4.8 The force exerted by one capacitor plate on the other

We want to calculate the force that one capacitor plate exerts on the other using equation (4.1):

$$\boldsymbol{F} = Q\boldsymbol{E}$$

We consider one of the plates, plate A, in the field of the other, plate B. Plate A carries the charge Q_A, the strength of the field of plate B is $\boldsymbol{E}_B$. However, is this equation applicable here at all? The condition for using equation (4.1) is that we are dealing with a single field strength value $\boldsymbol{E}_B$. And this means that the field strength at all positions over which the charge Q_A extends had the same value before the charge was brought there.

Now this is the case with our capacitor. The field of the left plate alone is shown in Fig. 4.22.

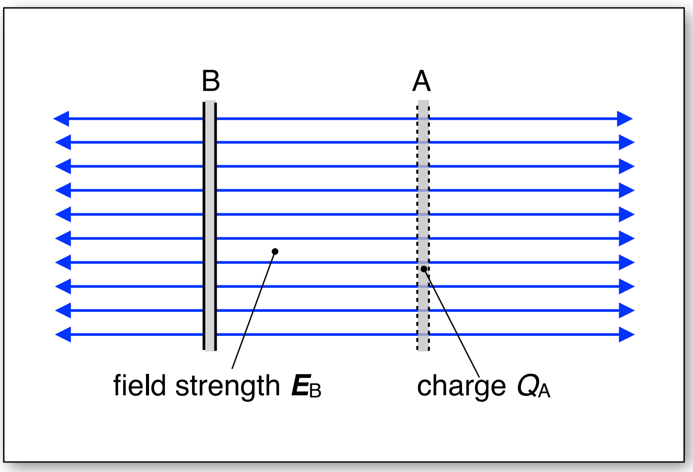

Fig. 4.22
Contribution of one of the two plates of a capacitor to the total field

Everywhere to the right of this plate, the field has the strength $\boldsymbol{E}_B$, where $\boldsymbol{E}_B$ is half as large as the field strength $\boldsymbol{E}$ in the complete capacitor:

$$\boldsymbol{E}_B = \frac{\boldsymbol{E}}{2}$$

This can be found by using equation (4.3). One then obtains for the force:

$$\boldsymbol{F} = \frac{\boldsymbol{E} \cdot Q}{2}$$

With equation (4.24) and $Q = \rho_A A$ it becomes

$$\boldsymbol{F} = \frac{Q^2}{2\varepsilon\varepsilon_0 A} \qquad (4.25)$$

4.9 The energy within the field of the capacitor

The energy is obtained by moving one plate against the other perpendicularly to the plane of the plate with the charge held constant, Fig. 4.23.

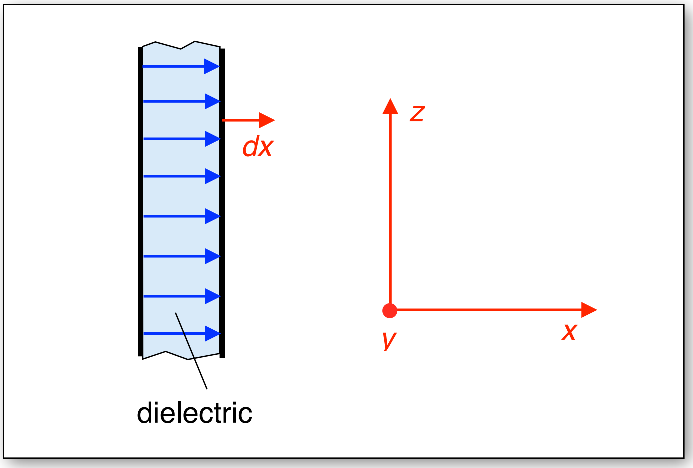

Fig. 4.23
Energy is needed to move the right plate to the right.

The space between the plates should remain completely filled with matter. Therefore, the dielectric is best imagined to be a liquid in which the whole arrangement is immersed.

From

$$dE = F\,dx.$$

with (4.25) we get

$$dE = \frac{Q^2}{2\varepsilon\varepsilon_0 A}dx$$

Integrated from $x = 0$ to $x = d$ the total energy in the field becomes

$$E = \frac{Q^2}{2\varepsilon\varepsilon_0 A}d \tag{4.26}$$

With (4.23) we finally obtain

$$E = \frac{Q^2}{2C} \tag{4.27}$$

5

Energy density and mechanical stress in the electric field

5.1 The energy density

According to equation (4.26), the energy in the homogeneous field of a capacitor is determined by the plate spacing d and the plate surface A:

$$E = \frac{Q^2}{2\varepsilon\varepsilon_0 A} d$$

We replace the charge Q with equation (4.24) and with $\rho_A = Q/A$:

$$E = \frac{\varepsilon\varepsilon_0}{2} \boldsymbol{E}^2 \cdot A \cdot d$$

Since $V = A \cdot d$ is the volume of the field, the energy density is

$$\rho_E = \frac{\varepsilon\varepsilon_0}{2} \boldsymbol{E}^2 \tag{5.1}$$

5.2 Tensile stress in the direction of the field lines

According to equation (4.25), the force exerted by one capacitor plate on the the other is:

$$\boldsymbol{F} = \frac{Q^2}{2\varepsilon\varepsilon_0 A}$$

But this force is not only exerted by one of the plates on the other, but also by one plate on the field immediately in front of it, and this region of field exerts it on the next, and so on. This means that there is a mechanical tensile stress within the field in the direction of the field lines. Its value can be obtained by dividing the force by the plate area A

$$\sigma_{\parallel} = \frac{Q^2}{2\varepsilon\varepsilon_0 A^2} = \frac{1}{2\varepsilon\varepsilon_0}\rho_A{}^2$$

The positive sign means that it is a tensile stress.

With eq. (4.24) we finally get

$$\sigma_{\parallel} = \frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 \qquad (5.2)$$

From the fact that electric field lines always begin or end at charges, we conclude that the electric field is pulling at charged matter.

If a soap bubble is electrically charged, it becomes larger. The electric field pulls the liquid lamella outwards, Fig. 5.1.

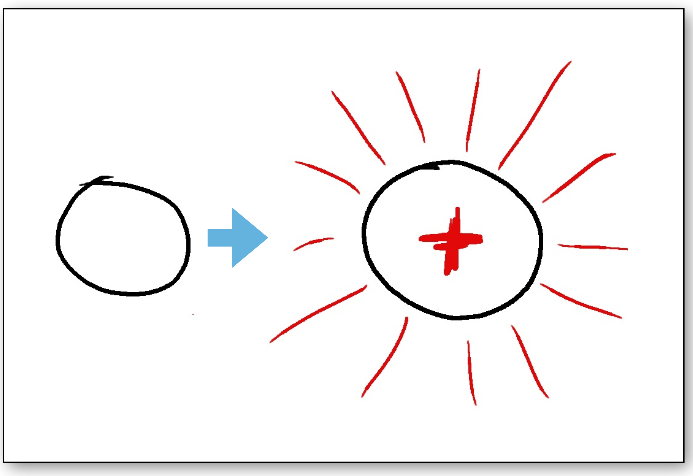

Fig. 5.1
The electric field pulls at the surface of the soap bubble.

But how can the repulsion between two bodies with charges of the same sign be explained? Fig. 5.2 shows the answer to this question. We look at the left body: the field pulls at all parts of its surface. Since the field strength is greater on the left than on the right, it pulls more strongly to the left than to the right, resulting in a net force to the left. Correspondingly, the same applies to the right body.

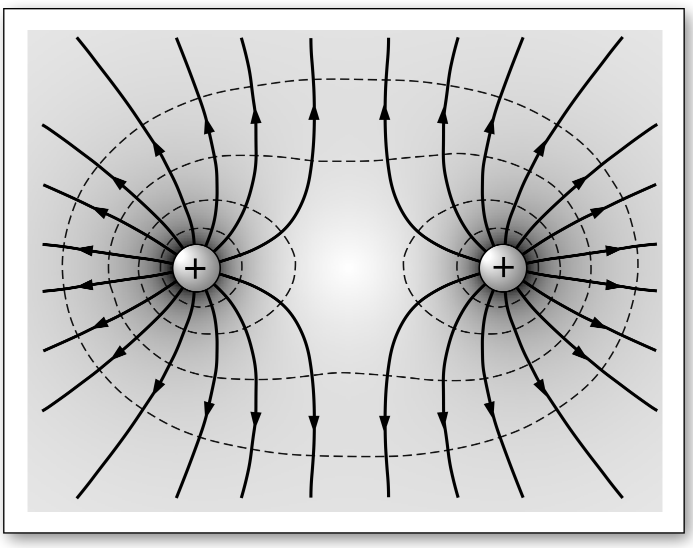

Fig. 5.2
Two bodies with charges of the same sign are pulled away from each other by the electric field.

Instead of “Bodies with like charges repel each other”, it would be more correct to say:

Bodies with like charges are pulled away from each other by the electric field.

We interpret Fig. 5.3 in a similar way. Again, the field pulls on the left body in all directions. This time, however, the field strength to the right of the body is greater than to the left, resulting in a net force to the right. So instead of “Bodies with opposite charges repel each other” we say more correctly:

Bodies with opposite charges are pulled towards each other by the electric field.

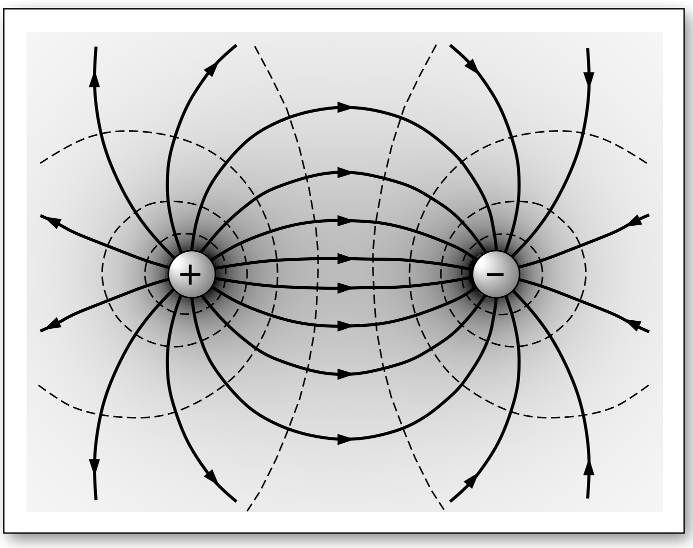

Fig. 5.3
Two bodies of opposite charges are attracted to each other by the electric field.

5.3 Compressive stress perpendicular to the field lines

We consider two small bodies that carry equal and opposite charges, Fig. 5.4.

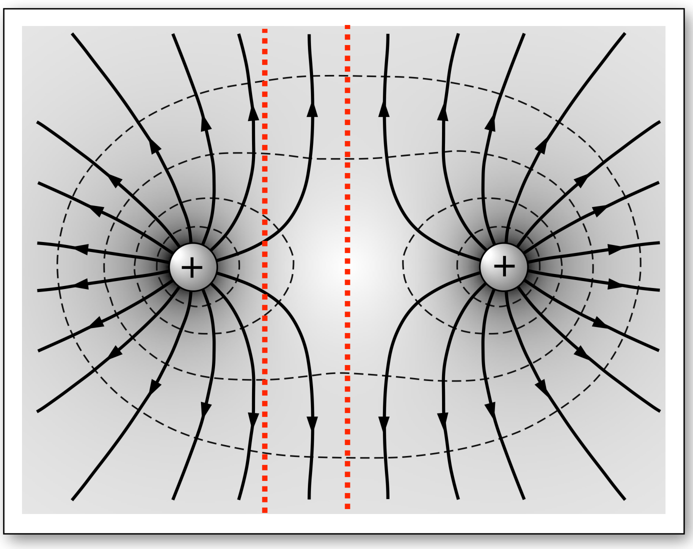

Fig. 5.4
In the symmetry plane there is pure compressive stress.

We now think of an infinitely extended surface, which is situated in such a way that one of our charged bodies lies on one side of it, the other body on the other side (left dashed line in Fig. 5.4). The net force that the field on one side exerts on the field on the other side of the surface must be a compressive force. However, it will generally be composed in a complicated way of the compressive and tensile stress contributions of the various surface elements of our interface.

Now there is one separating plane surface where things are simpler: the plane of symmetry between the two bodies (right dashed line). Here, all field strength vectors lie within the surface. They are perpendicular to the straight line that connects the two bodies. From the fact that there is a net compressive force in this surface, we can conclude that in the electric field there is compressive stress transverse to the field lines. We want to calculate this stress as a function of the field strength.

The plates of the capacitor in Fig. 5.5 can be pulled apart so that their area $y_0 z_0$ increases to $y_0(z_0 + dz_0)$. Hereby the charge is kept constant. When they are pulled apart, the energy of the field decreases.

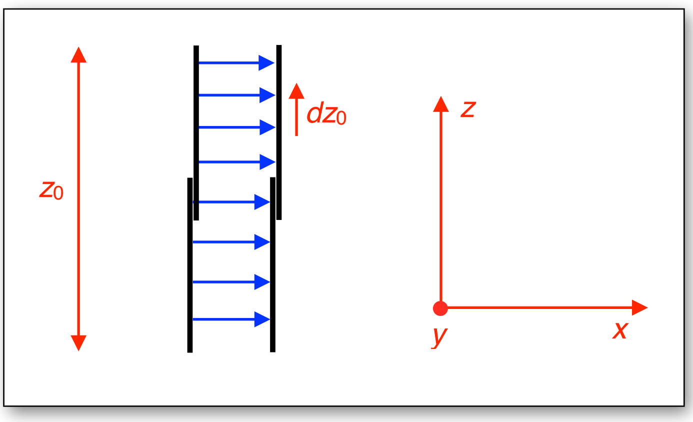

Fig. 5.5
When increasing the plate area, the field releases energy.

From equation (5.1) the energy of the field is

$$E = \frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 x_0 y_0 z_0 \tag{5.3}$$

(Attention: E stands for energy, $\boldsymbol{E}$ for the electric field strength). We now want to express the field strength by means of the charge, because the charge remains constant during the process under consideration. To do this we use (4.24)

$$\frac{Q}{y_0 z_0} = \rho_A = \varepsilon\varepsilon_0 |\boldsymbol{E}|$$

With (5.3) we get

$$E = \frac{Q^2}{2\varepsilon\varepsilon_0}\frac{x_0}{y_0 z_0}$$

With a displacement dz_0 the energy changes by

$$dE = -\frac{Q^2}{2\varepsilon\varepsilon_0}\frac{x_0}{y_0 {z_0}^2}dz_0$$

Comparing with $dE = F_z\, dz_0$ yields the force that the field exerts on its suspension, i.e. the plates

$$F_z = -\frac{Q^2}{2\varepsilon\varepsilon_0}\frac{x_0}{y_0}\frac{1}{{z_0}^2}$$

We again replace Q with $|\boldsymbol{E}|$ using equation (4.24)

$$F_z = -\frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 x_0 y_0$$

The mechanical stress $\sigma_\perp = F_z/(x_0 y_0)$ becomes

$$\sigma_\perp = -\frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 \tag{5.4}$$

Thus there is a compressive stress of the magnitude $\varepsilon\varepsilon_0|\boldsymbol{E}|^2/2$ perpendicular to the electric field lines.

6

The sources of the magnetic field

6.1 The magnetic field strength

The phenomena of magnetostatics are formally very similar to those of electrostatics. However, apart from the structural similarity, there seems to be no connection between these two areas if one limits oneself to static phenomena. However, a difference between electricity and magnetism follows from the fact that no isolated magnetic charges have been observed so far. Thus, it is only possible to operate with magnetic polarization charge, i.e. the analogue to the electric bound charge. However, handling bound magnetic charge is much more convenient than handling bound electric charge. Permanent electric dipoles are difficult to preserve because they are quickly neutralized by free charges. Since there is no free magnetic charge, permanent magnets can be stored very well.

Magnetic or magnetized matter consists of magnetic dipoles that cannot be decomposed. Here even the point-shaped dipole is realized: Electrons, for example, have a magnetic dipole moment (= magnetic moment) and sometimes behave like point-like particles. Electrons also have an angular momentum different from zero without having a spatially extended momentum distribution. The angular moment vector and the magnetic moment vector are antiparallel.

We designate the magnetic charge by Q_m (no SI symbol; the quantity is also called pole strength). Its SI unit is Weber (Wb), with 1 Wb = 1 Vs. The unit of the magnetic moment $\boldsymbol{m}$ is therefore Wb · m.

One notices that there appear electrical units of measurement. We can justify this fact only later when the relationship between electricity and magnetism is dealt with. We anticipate that magnetic units are obtained by replacing volts with amperes in the unit of the analog electrical quantity and vice versa. The unit of electric charge being As (abbreviated Coulomb), we obtain the unit of the magnetic charge to be Vs (abbreviated Weber).

That Part of a magnet which carries positive magnetic charge ($Q_m > 0$) is called the north pole, the negatively charged region ($Q_m < 0$) south pole. A typical value of the (bound) magnetic charge at one pole of a permanent magnet is 10^{-4} Wb. The magnetic moment of an electron is $1.166766 \cdot 10^{-29}$ Wb · m.

If one brings a "magnetic point charge Q_m" (e.g. the end of a very long and thin bar magnet) at a certain point in a given magnetic field, a force acts on it, Fig. 6.1. If the value of the charge is doubled, the value of the force is also doubled:

$$\boldsymbol{F} \sim Q_m .$$

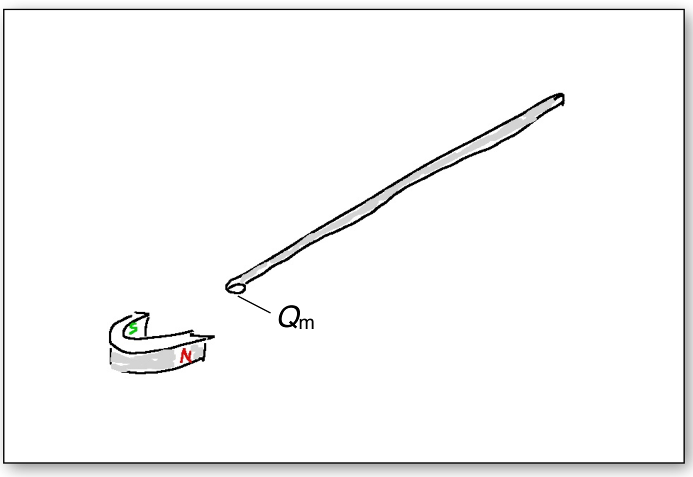

Fig. 6.1
To determine the strength of the field, one measures the force on the magnetic pole of charge Q_m.

The vectorial factor of proportionality is thus characteristic for the field without the point charge. It is called the magnetic field strength $\boldsymbol{H}$:

$$\boldsymbol{F} = \boldsymbol{H}\, Q_m \qquad (6.1)$$

The measuring unit of $\boldsymbol{H}$ is A/m.

Coulomb discovered not only the law of the force of electric point charges on each other, named after him, but also the corresponding magnetic law:

$$\boldsymbol{F} = \frac{1}{4\pi\mu_0} \frac{Q_{m1} Q_{m2}}{r^2} \frac{\mathbf{r}}{r} \qquad (6.2)$$

μ_0 is the *magnetic constant*:

$$\mu_0 = 4\pi \cdot 10^{-7}\ \text{Vs/Am} = 1.2566 \cdot 10^{-6}\ \text{Vs/Am}$$

From (6.2) follows the $\boldsymbol{H}$ field distribution of a magnetic point charge:

$$\boldsymbol{H} = \frac{1}{4\pi\mu_0} \frac{Q_m}{r^2} \frac{\boldsymbol{r}}{r}$$

In analogy to the electric dipole moment (equation (4.12)) we define the magnetic dipole moment

$$\boldsymbol{m} = Q_m \boldsymbol{a}.$$

Attention: In the literature the definition of $\boldsymbol{m}$ is not standardized! As an exception, we do not follow the SI convention.

Just as a force acts on a single magnetic pole, a torque acts on a magnetic dipole:

$$\boldsymbol{M} = \boldsymbol{m} \times \boldsymbol{H}$$

Compare with equation (4.13).

6.2 Magnetization

The atomic constituents of some materials have a magnetic moment different from zero. Sometimes the magnetic dipoles are aligned so that their dipole moment vectors are parallel. Then a macroscopic body also has a dipole moment different from zero. Such a body is called a magnet. We define the density $\boldsymbol{M}$ of the magnetic dipole moment (compare with equation (4.14)):

$$d\boldsymbol{m} = \boldsymbol{M}\,dV \tag{6.3}$$

The quantity $\boldsymbol{M}$ is called magnetization. $\boldsymbol{M}(\boldsymbol{r})$ is a vector field. The measuring unit of $\boldsymbol{M}$ is the Tesla (T). One finds 1 T = 1 Wb/m².

(Our $\boldsymbol{M}$ is μ_0 times the magnetization defined by the SI, just as our $\boldsymbol{m}$ is μ_0 times the magnetic dipole moment defined by the SI; different books use different names for the same quantity, give the same name to different quantities and use different symbols for the same quantity; this disagreement exists only for the magnetic pole strength, the magnetic dipole moment and its density; for the electric field quantities, as well as for $\boldsymbol{H}$ and the quantity $\boldsymbol{B}$ still to be introduced, there is agreement.)

Just as the bound electric charge at the ends of an electrically polarized body is the source of an $\boldsymbol{E}$ field, the magnetic charges at the ends of a magnet are the source of the $\boldsymbol{H}$ field. The mathematical formalism describing the relationship between sources and field is identical in both cases, and we can directly adopt the equations from Section 4.4, applying the following translation rules:

$\boldsymbol{E} \Rightarrow \boldsymbol{H}$

$\varepsilon_0 \Rightarrow \mu_0$

$\boldsymbol{P} \Rightarrow \boldsymbol{M}$

$\rho \Rightarrow$ zero (because there is no free magnetic charge.).

From equation (4.17) we get

$$\operatorname{div}\boldsymbol{H} = -\frac{1}{\mu_0}\operatorname{div}\boldsymbol{M} \tag{6.4}$$

or

$$\operatorname{div}(\mu_0\boldsymbol{H} + \boldsymbol{M}) = 0,$$

and with the abbreviation

$$\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M} \tag{6.5}$$

(analogous to (4.18)) we obtain the relationship analogous to (4.19)

$$\operatorname{div}\boldsymbol{B} = 0 \tag{6.6}$$

The quantity $\boldsymbol{B}$ is called *magnetic flux density*. Its measuring unit is the same as that of the magnetization, namely Tesla.

Equations (6.4) and (6.6) state that the $\boldsymbol{H}$ field has its sources where the $\boldsymbol{M}$ field has its sinks, and vice versa. Using Gauss's theorem, we transform the local equation (6.4) into an integral relation:

$$\oint\!\!\oint \boldsymbol{H}\,d\boldsymbol{A} = -\frac{1}{\mu_0}\oint\!\!\oint \boldsymbol{M}\,d\boldsymbol{A}$$

Correspondingly, (6.6) becomes

$$\oint\!\!\oint \boldsymbol{B}\,d\boldsymbol{A} = 0 \tag{6.7}$$

This equation is called *Maxwell's 2nd equation*.

Figure 6.2a shows the magnetization of a bar magnet. Bar magnets are manufactured in such a way that the magnetization lines run parallel to the longitudinal direction of the magnet, so that the poles are located at the end faces.

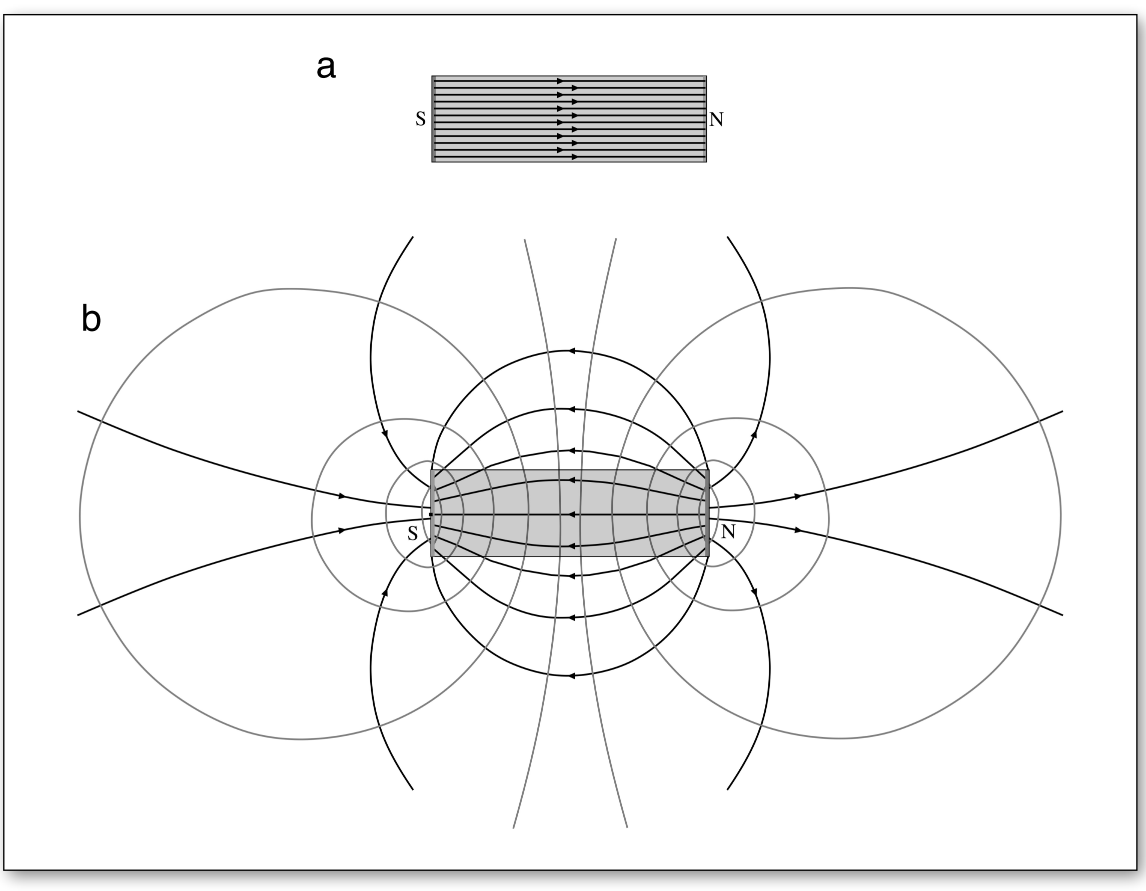

Fig. 6.2
Bar magnet; (a) Magnetization lines; (b) $\boldsymbol{H}$ field lines and magnetic equipotentials

Figure 6.2b shows the $\boldsymbol{H}$ field line image. (The magnetic equipotentials are also shown, see Section 6.3). If the magnetic charge at the end faces is replaced by electric charge, an electric field is obtained whose field lines are the same as in the figure.

Figure 6.3 shows the magnetization lines and magnetic field strength lines for a horseshoe magnet.

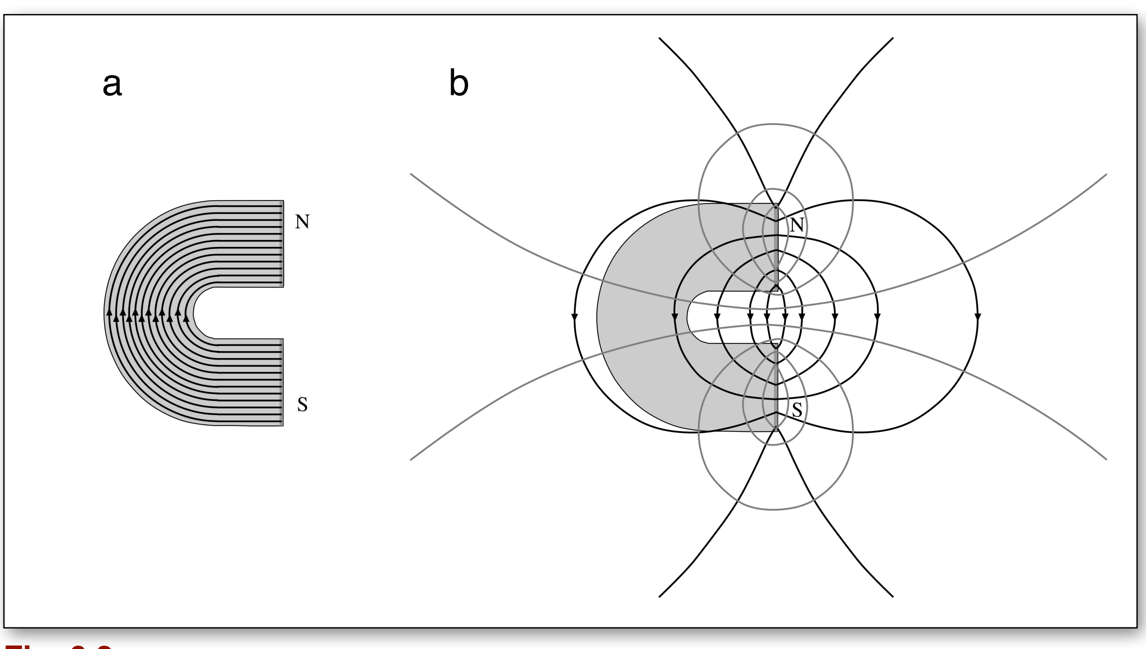

Fig. 6.3
Horseshoe magnet; (a) Magnetization lines; (b) $\boldsymbol{H}$ field lines and magnetic equipotentials

Figure 6.4 shows a ring magnet with a gap. The left partial image shows how it has been magnetized by the manufacturer. In the right partial picture one can see that the $\boldsymbol{H}$ field is practically concentrated on the region of the gap.

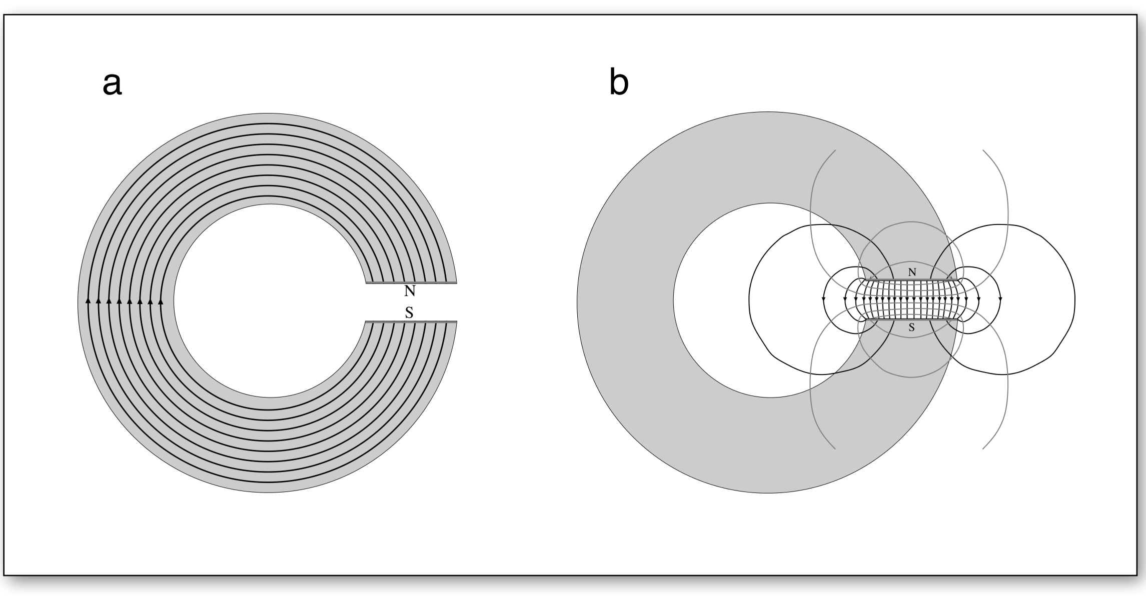

Fig. 6.4
Ring magnet; (a) Magnetization lines; (b) $\boldsymbol{H}$ field lines and magnetic equipotentials

Figure 6.5a shows a ring similar to that in Figure 6.4a, but this ring has no gap. What can be said about the $\boldsymbol{H}$ field and the $\boldsymbol{B}$ field? Figure 6.5b shows a magnetized spherical shell; on the inner surface there is a south pole (negative) charge, on the outer surface there is north pole (positive) charge. What can be said about the $\boldsymbol{H}$ field and the $\boldsymbol{B}$ field?

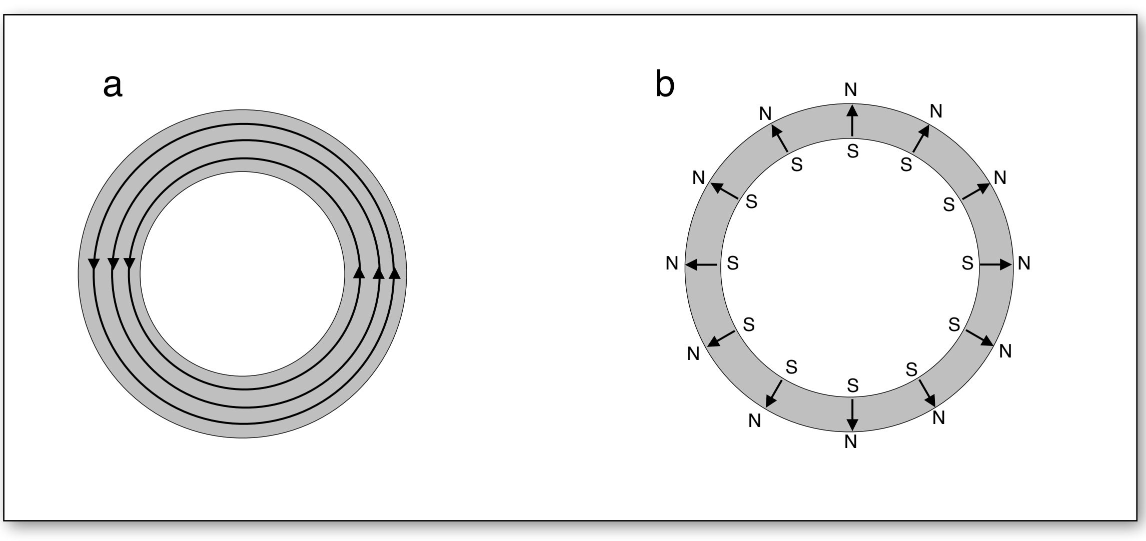

Fig. 6.5
(a) Ring magnet without gap; (b) Magnetized spherical shell. What can be said about the magnetic field strength and the magnetic flux density?

6.3 The magnetic potential

Just like

$$\int_A^B \boldsymbol{E}\, d\boldsymbol{r}$$

in the electrostatic field, the integral

$$\int_A^B \boldsymbol{H}\, d\boldsymbol{r}$$

in the magnetostatic field is independent of the integration path.

Therefore a magnetic potential $\Phi_m(\boldsymbol{r})$ can be defined (compare with equation (4.5)):

$$\int_A^B \boldsymbol{H}\, d\boldsymbol{r} = -\left[\Phi_m(\boldsymbol{r}_B) - \Phi_m(\boldsymbol{r}_A)\right] = U_m \qquad (6.8)$$

U_m is the magnetic potential difference between A and B. From the last equation further follows (compare with equation (4.7)):

$$\boldsymbol{H}(\boldsymbol{r}) = -\operatorname{grad} \Phi_m(\boldsymbol{r})$$

The measuring unit of the magnetic potential is Ampere. U_m is an SI symbol; for the magnetic potential there is no standardized symbol.

6.4 Induced magnetization

We're investigating how matter behaves in a magnetic field. We suppose the field to be homogeneous and to have a constant value outside of the matter. Such a field is realized, for example, by bringing a plate-shaped piece of matter into the narrow gap between the wide poles of a homogeneously magnetized magnet, Fig. 6.6.

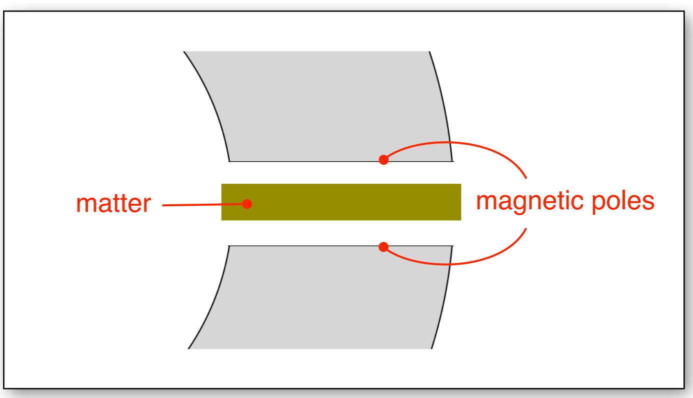

Fig. 6.6
Magnetizable matter between the poles of a magnet

Because of div ($\mu_0\boldsymbol{H} + \boldsymbol{M}$) = 0, the expression $\mu_0\boldsymbol{H} + \boldsymbol{M}$ has the same value on both sides of each surface of the piece of matter, i.e. inside (in) and outside (out). Since outside $\boldsymbol{M} = 0$ we have

$$\boldsymbol{H}_{\text{out}} = \boldsymbol{H}_{\text{in}} + \frac{1}{\mu_0}\boldsymbol{M}$$

and thus

$$\boldsymbol{H}_{\text{in}} = \boldsymbol{H}_{\text{out}} - \frac{1}{\mu_0}\boldsymbol{M} \qquad (6.9)$$

If the matter is not a magnet itself, i.e. if it is not magnetically polarized (= magnetized) from the outset, it is found that it is magnetized under the influence of an $\boldsymbol{H}$ field. Approximately we have

$$\boldsymbol{M} = \mu_0\, \chi_m\, \boldsymbol{H} \qquad (6.10)$$

χ_m is the *magnetic susceptibility*. With (6.9) we obtain

$$\boldsymbol{H}_{\text{in}} = \boldsymbol{H}_{\text{out}} - \chi_m \boldsymbol{H}_{\text{in}}$$

and

$$\boldsymbol{H}_{\text{out}} = (1 + \chi_m)\boldsymbol{H}_{\text{in}}$$

Now the matter is to fill the whole space between the magnetic poles, Fig. 6.7.

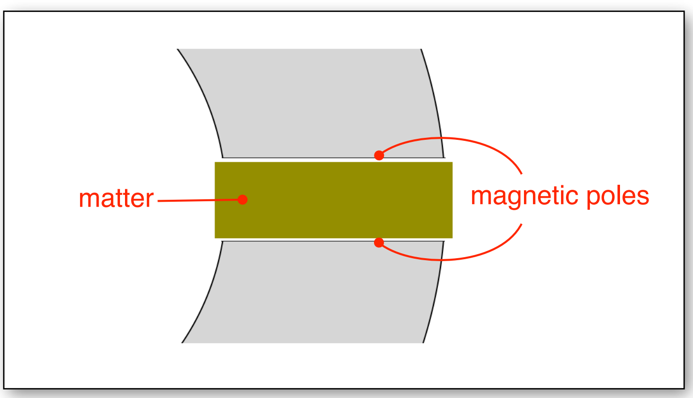

Fig. 6.7
By inserting magnetizable matter, the field strength is reduced.

If one compares the $\boldsymbol{H}$ field strength before the insertion of matter (b) with that after (a), then

$$\boldsymbol{H}_b = (1 + \chi_m)\boldsymbol{H}_a \ .$$

The field strength is thus changed by a factor of 1 + χ_m by the insertion of matter. This factor μ is called the *permeability* of the material:

$$\mu = 1 + \chi_m.$$

Because of $U_m = \int \boldsymbol{H} d\boldsymbol{r}$, the magnetic potential difference between the two magnetic poles also changes by this factor: $U_{m,b} = \mu U_{m,a}$. The magnetic potential difference is less commonly used than the electrical potential difference because it is harder to measure. All voltmeters make use of the fact that there is freely displaceable electric charge. Thanks to the displaceability of Q, a potential value can be easily transmitted by means of a wire. On the other hand, the magnetic field strength $\boldsymbol{H}$ is easier to measure than the electric field strength (e.g. with a Hall probe).

As in the case of the electric polarization, there are also different mechanisms for the magnetic polarization by an $\boldsymbol{H}$ field.

The electron has a magnetic moment. In many materials, the electrons of each atom or molecule adjust in such a way that the atom or molecule as a whole has a magnetic moment of zero. In other materials the molecules have a resulting magnetic moment. These are called *paramagnetic* substances. If they are brought into a magnetic field, the magnetic moments are oriented in the direction of the field (the paramagnetic polarization in the $\boldsymbol{H}$ field corresponds to the orientation polarization in the $\boldsymbol{E}$ field). Table 6.1 lists the susceptibility values of some paramagnetic substances.

Table 6.1
Examples of paramagnetic materials

Substance	χ_m
Oxigen (under normal conditions)	$0.14 \cdot 10^{-6}$
Liquid oxigen	$360 \cdot 10^{-6}$
Aluminum	$1.7 \cdot 10^{-6}$
Manganese	$80 \cdot 10^{-6}$
Sodium	$0.5 \cdot 10^{-6}$
Platinum	$19.3 \cdot 10^{-6}$

χ_m increases with decreasing temperature. The orientation of the magnetic moments at normal temperature is far from complete.

But even the substances whose molecules normally have no resulting magnetic moment become magnetic in the $\boldsymbol{H}$ field, namely in such a way that $\boldsymbol{M}$ and $\boldsymbol{H}$ point in opposite directions, so χ_m is negative. This process is called *diamagnetic* polarization. It also occurs in paramagnetic materials, but is overcompensated by the paramagnetic polarization. Table 6.2 contains the χ_m values of some diamagnetic substances.

Table 6.2
Examples of diamagnetic materials

Substance	χ_m
Lead	$-0.12 \cdot 10^{-6}$
Gold	$-3.1 \cdot 10^{-6}$
Copper	$-0.8 \cdot 10^{-6}$
Silver	$-1.5 \cdot 10^{-6}$
NaCl	$-1 \cdot 10^{-6}$
Water	$-0.72 \cdot 10^{-6}$
Bismuth	$-14 \cdot 10^{-6}$

6.5 Ferromagnetism

There are substances whose atoms have a resulting magnetic moment, and in which the magnetic dipoles of all atoms adjust themselves parallel. In these cases $\boldsymbol{M} \neq 0$, even though $\boldsymbol{H} = 0$. Since the orientation of the elementary magnetic moments is complete (in contrast to paramagnetic materials), $\boldsymbol{M}$ is very large.

These substances are called *ferromagnetic*. They include Fe, Ni, Co and alloys of these and other substances.

The reason that an ordinary piece of iron has no magnetic moment is the fact that the atomic dipoles are aligned in small differently oriented regions. These regions are called *magnetic domains*. Since they are oriented differently no magnetization can be detected macroscopically.

Fig. 6.8 shows the magnetization as a function of the field strength.

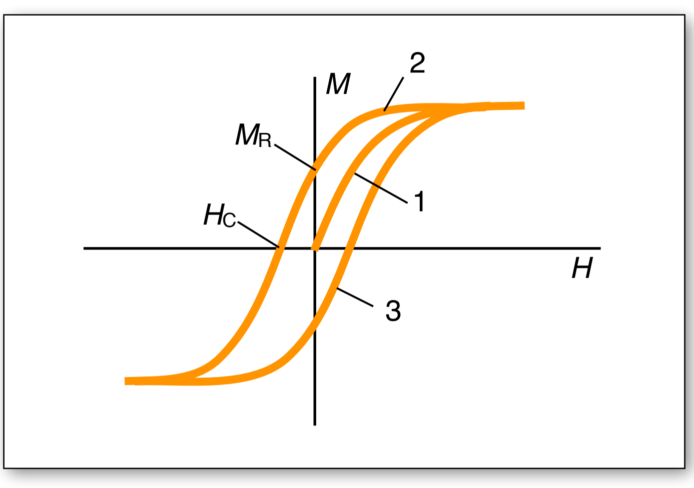

Fig. 6.8
M-H relation with hysteresis

If one starts with a non-magnetic piece of iron and lets the field strength increase, one moves on segment 1 of the curve. If one then lets $\boldsymbol{H}$ decrease up to negative values, one follows segment 2. When $\boldsymbol{H}$ increases again, curve 3 is followed. The magnetization M_R remaining at $\boldsymbol{H} = 0$ is called *remanence*, the field strength $\boldsymbol{H}_C$, which is necessary to bring the material back to zero magnetization, is the *coercivity*.

The magnetization at a given moment thus depends on the field strength at earlier times. The material thus has a "memory". Such a phenomenon is called "hysteresis". The effect is used for data storage.

The $\boldsymbol{M}$-$\boldsymbol{H}$ relationship is different for different ferromagnetic materials. Figure 6.9 and 6.10 show two examples.

Fig. 6.9a corresponds to a *hard magnetic* material. It is suitable for manufacturing permanent magnets. A permanent magnet should not change its magnetization when exposed to a magnetic field. Fig. 6.9b shows only the section marked by the blue box. It can be seen that in the corresponding $\boldsymbol{H}$ range the magnetization is independent of the field strength. (Of course you can change the magnetization "by force", as can be seen in fig. 6.9a.)

Today, values of M_R up to over 1 T and of H_K up to more than 150 kA/m can be achieved.

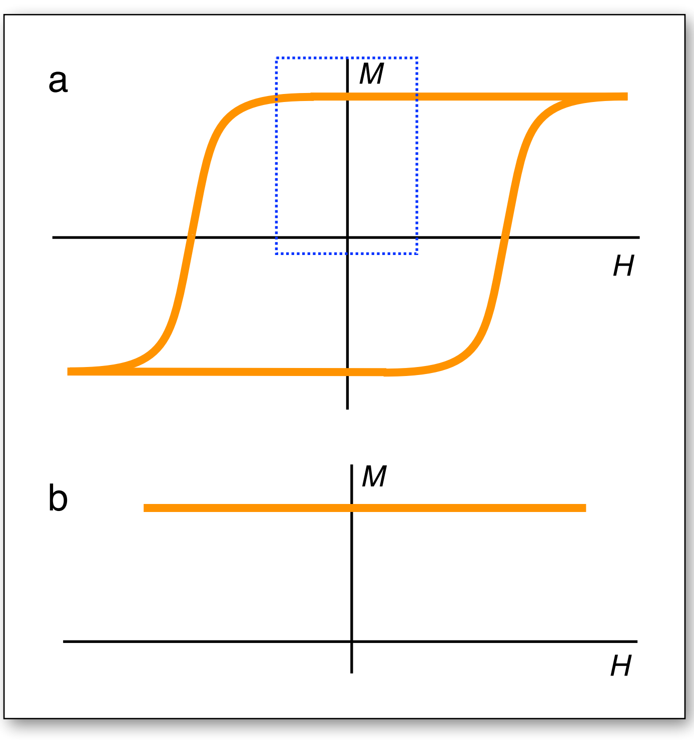

Fig. 6.9
(a) *M-H* relationship for a hard magnetic material.
(b) The blue box in (a) corresponds to the *H* range in which the material is normally used.

Figure 6.10 shows the $\boldsymbol{M}$-$\boldsymbol{H}$ relationship for a soft magnetic material (soft iron, Mu-metal). As long as $\boldsymbol{M}$ is small compared to the saturation magnetization, $\boldsymbol{M}$ is proportional to $\boldsymbol{H}$, i.e. $\boldsymbol{M} = \mu_0 \chi_m \boldsymbol{H}$, just as for diamagnetic and paramagnetic materials. However, χ_m is much larger here, namely a few tens of thousands. If such a material is brought into a magnetic field that originally had the value $\boldsymbol{H}_b$, then

$$\boldsymbol{H}_b = (1 + \chi_m)\boldsymbol{H}_a \quad \Rightarrow \quad \boldsymbol{H}_a \approx 1/10^4\, \boldsymbol{H}_b.$$

The field strength in the material thus becomes practically zero. Thus, a body made of a soft magnetic material expels the magnetic field from its interior.

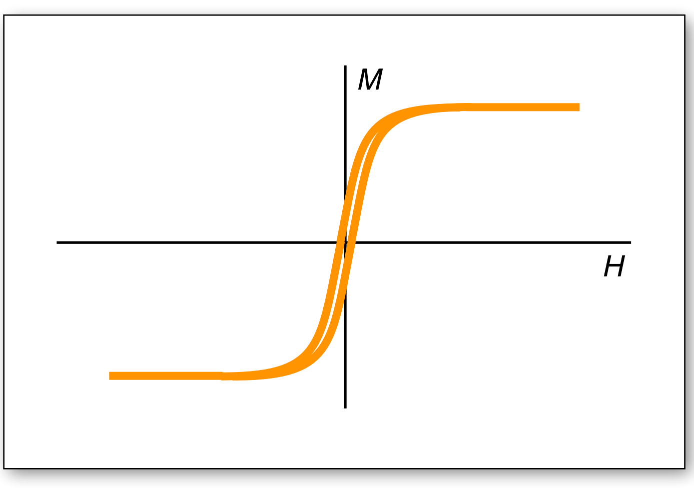

Fig. 6.10
M-H relationship for a soft magnetic material. As long as the material is not saturated, it expels the magnetic field from its interior, i.e. the field strength is very small.

Thus, a soft magnetic material behaves in the $\boldsymbol{H}$ field analogously to electrical conductors in the electric field. Just as the surface of an electrical conductor constitutes an electrical equipotential surface, the surface of soft iron is almost a surface of constant magnetic potential (see sections 4.2 and 4.5).

At a temperature characteristic of each ferromagnetic substance, the *Curie temperature*, ferromagnetism disappears. Above this temperature (for iron 1047 K), the material is an ordinary paramagnet.

With the help of components made of magnetically soft material, the field of a permanent magnet can be changed in a comfortable way. We suppose that the permanent magnet and the soft magnet are made of an ideal material. Figure 6.11a shows an example, Figure 6.11b shows the electrical analogue.

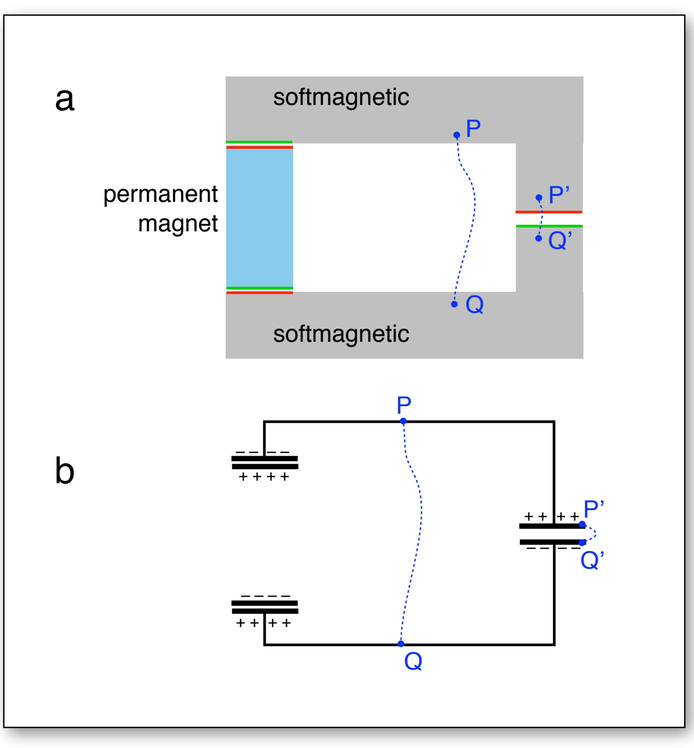

Fig. 6.11
Magnetic arrangements and its electric analogue.
Red and green: Positive and negative magnetic charge.

As in Figure 6.11b the wires are at constant electrical potential, the soft iron parts in Figure 6.11a are at constant magnetic potential. The electric field strength in the wires in Fig. 6.11b is zero and the magnetic field strength in the soft iron parts in Fig. 6.11a is nearly zero.

In Fig. 6.11b it is

$$\int_P^Q \boldsymbol{E}\, d\boldsymbol{r} = \int_{P'}^{Q'} \boldsymbol{E}\, d\boldsymbol{r}$$

and in Fig. 6.11a

$$\int_P^Q \boldsymbol{H}\, d\boldsymbol{r} = \int_{P'}^{Q'} \boldsymbol{H}\, d\boldsymbol{r}$$

Since the path PQ is much longer than the path P'Q', the field strength $\boldsymbol{E}$ or $\boldsymbol{H}$ respectively, on the path P'Q' is much greater than on the path PQ. Therefore the largest part of the induced electric charge or the magnetic polarization charge is located on the surfaces to the right. The field in the slit on the right side is almost homogeneous.

With the help of the soft iron pieces we have turned the originally inhomogeneous field of the rod magnet into a strong homogeneous field.

6.6 Magnetic capacitance

In analogy to $C = Q/U$ one could define a magnetic capacitance: $C_m = Q_m/U_m$. The unit of measurement of the magnetic capacitance is Wb/A = Henry (H). This quantity is not commonly used, although the example of the ring magnet in the previous section shows that it is a meaningful quantity. We will later learn about a quantity of the same dimension (measuring unit), the *inductance*. In order to obtain a well-defined space of high magnetic field strength, a magnetic capacitor can be built and connected to a permanent magnet (or electromagnet) via magnetic lines. The capacitance of the lines must be small compared to that of the "capacitor". Since $C_m = \mu_0 A/d$ (analogous to $C = \varepsilon_0 A/d$), the lines are laid in a large arc to the capacitor, i.e. in such a way that d is as large as possible for the lines. Fig. 6.12 shows how not to do it: the (magnetic) line capacitance is too large.

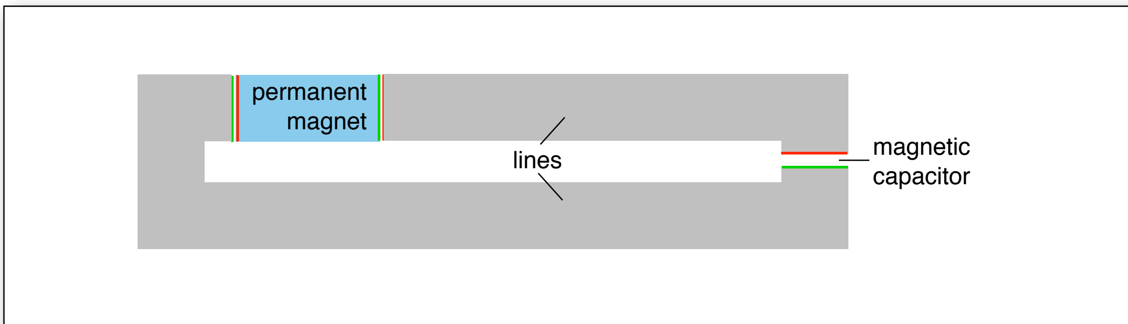

Fig. 6.12
The capacitance of the supply lines for the magnetic capacitor is too high.

7

Energy density and mechanical stress in the magnetostatic field

7. Energy density and mechanical stress in the magnetostatic field

We can translate the arguments of chapter 5 almost literally. We just have to replace:

$\boldsymbol{E} \rightarrow \boldsymbol{H}$

$\boldsymbol{P} \rightarrow \boldsymbol{M}$

$\chi_e \rightarrow \chi_m$

$\varepsilon_0 \rightarrow \mu_0$

In analogy to equation (5.1) we get the energy density

$$\rho_E = \frac{\mu\mu_0}{2}\boldsymbol{H}^2 \tag{7.1}$$

Parallel to the field lines of the ***H*** field there is a tensile stress (compare with (5.2))

$$\sigma_{||} = \frac{\mu\mu_0}{2}\boldsymbol{H}^2 \tag{7.2}$$

and perpendicular to the field lines a compressive stress (see also (5.4))

$$\sigma_{\perp} = -\frac{\mu\mu_0}{2}\boldsymbol{H}^2 \tag{7.3}$$

Since magnetic field lines always begin or end at magnetic poles, we conclude that the magnetic field always pulls at magnetic poles.

Magnetic poles of equal sign are pulled away from each other by the magnetic field, poles of opposite sign are pulled towards each other.

8

Maxwell’s equations

8.1 Ampère's law

In 1820, Oersted discovered that an electric current exerts a torque on a magnetic dipole. Ampère described the mechanical interaction between wires in which an electric current is flowing by means of an elegant theory. Faraday recognized that these phenomena could be used to infer the existence of a field. Maxwell finally generalized Ampères and Faraday's thoughts. In modern language, Oersted's discovery can be described as follows: A conductor in which electric charge is flowing is surrounded by a magnetic field which has no sources or sinks, i.e. div $\boldsymbol{H} = 0$, Fig. 8.1. The $\boldsymbol{H}$ field lines have neither a beginning nor an end.

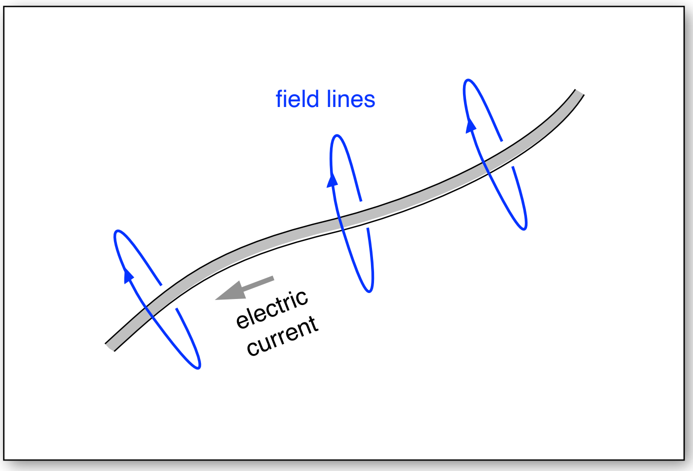

Fig. 8.1
Magnetic field of a conductor traversed by an electric current

From Ampère's theory follows the relationship between current and $\boldsymbol{H}$ field:

$$\oint \boldsymbol{H}\, d\boldsymbol{r} = I \quad \text{Ampère's law} \tag{8.1}$$

In words: The integral $\int \boldsymbol{H} d\boldsymbol{r}$ over a closed path is equal to the electric current flowing through this surface. This equation is based on the convention that the closed integration path together with the surface to which the current refers defines a right-hand screw.

This can also be formulated with the "right-hand rule": If the thumb of the right hand points in the direction of the electric current (= direction of the current density vectors), the curved fingers indicate the direction of the magnetic field lines, Fig. 8.2.

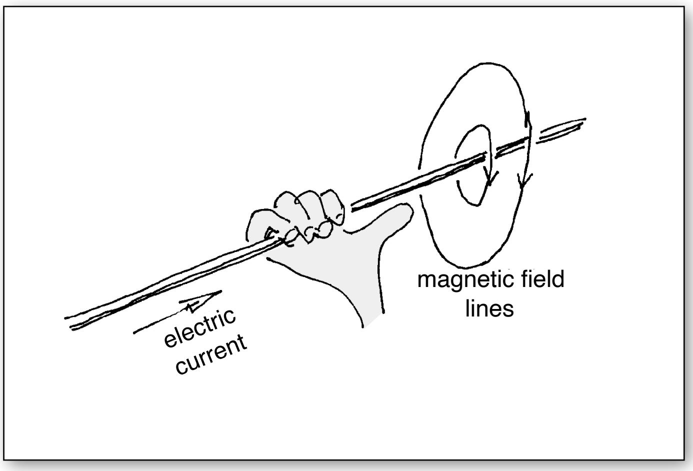

Fig. 8.2
If the thumb of the right hand points in the direction of the electric current, the curved fingers indicate the direction of the magnetic field strength.

With $I = \iint \boldsymbol{j} d\boldsymbol{A}$ Ampère's law becomes:

$$\oint \boldsymbol{H}\, d\boldsymbol{r} = \iint \boldsymbol{j}\, d\boldsymbol{A}$$

Attention: In an $\boldsymbol{H}$ field caused by an electric current, $\int \boldsymbol{H} d\boldsymbol{r}$ is *not* independent of the integration path. Therefore, a potential can only be defined in limited regions of space.

8.2 Calculation of magnetic field strengths

The straight conductor

The integration path is chosen to be a circle of radius r perpendicular to the direction of the wire, with its center at the center of the wire, Fig. 8.3.

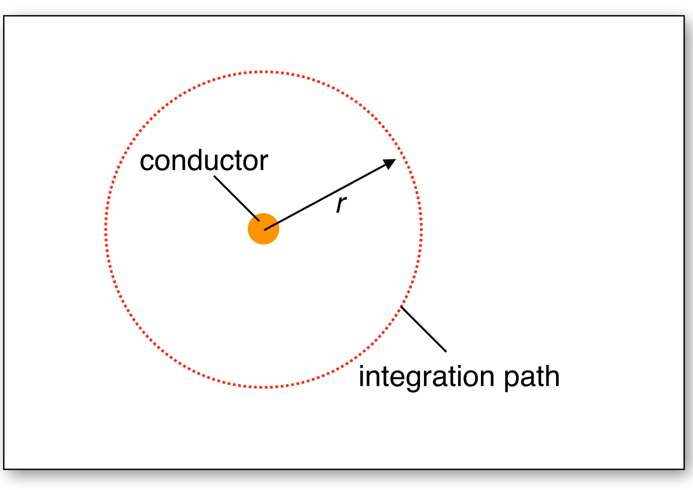

Fig. 8.3
Integration path for the calculation of the magnetic field strength around a straight wire

Because of the symmetry, the absolute value of $\boldsymbol{H}$ is constant along the entire integration path. The direction of $\boldsymbol{H}$ must be tangential to the circle, otherwise the $\boldsymbol{H}$ field would have sources or sinks on the wire, which is not the case. So it is

$$\oint \boldsymbol{H}\,d\boldsymbol{r} = |\boldsymbol{H}|2\pi r = I \quad \Rightarrow \quad |\boldsymbol{H}| = \frac{I}{2\pi r}$$

The long solenoid

As an integration path, we choose the path shown dashed in Fig. 8.4: Inside the coil, we follow a field line; outside the coil, the field strength is very small, we neglect its contribution to the integral:

$$\oint \boldsymbol{H}\,d\boldsymbol{r} = H \cdot l = N \cdot I$$

N = number or turns that cross the integration path.

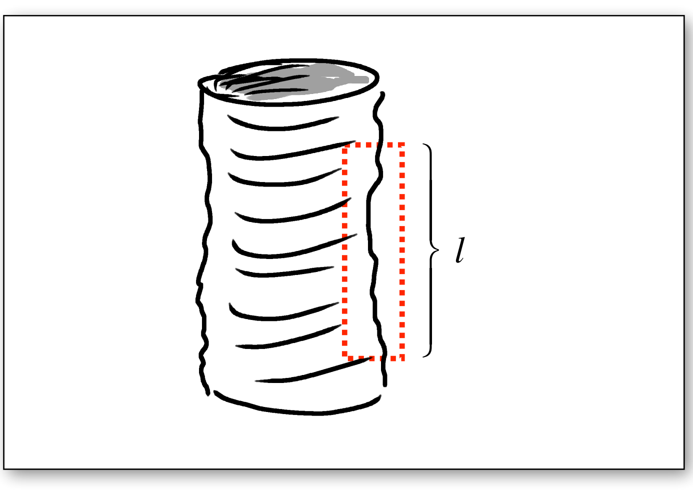

Fig. 8.4
Integration path for calculating the magnetic field strength in a long coil

It follows

$$H = \frac{N \cdot I}{l}$$

$\boldsymbol{H}$ does not depend on the position within the coil. In particular it does not depend on the position within a given coil cross-section. The $\boldsymbol{H}$ field is therefore homogeneous. In addition, $\boldsymbol{H}$ does not depend on the cross-sectional area of the coil, and given a certain number of turns per length, it does not depend on the length of the coil.

The toroidal coil

The radius R of the ring is supposed to be large against the "tube radius" r, Fig. 8.5. Then the field is almost homogeneous. The field strength outside the coil is zero. For the inside, one obtains by integration over a circular path of the length $l = 2\pi R$:

$$\oint \boldsymbol{H}\,d\boldsymbol{r} = |\boldsymbol{H}| \cdot l = N \cdot I \quad \Rightarrow \quad H = \frac{N \cdot I}{l}$$

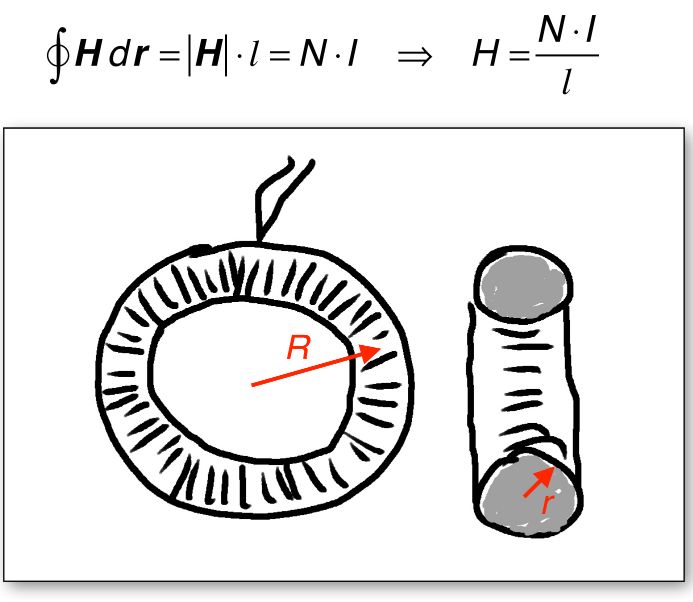

Fig. 8.5
Toroidal coil

The ring-shaped electromagnet

The distance between the plane pole surfaces is small in comparison to the lateral expansion of these surfaces. The $\boldsymbol{H}$ field between the poles is therefore almost homogeneous, Figure 8.6. We calculate the field strength H_{out} between the poles and the field strength H_{in} in the soft iron core.

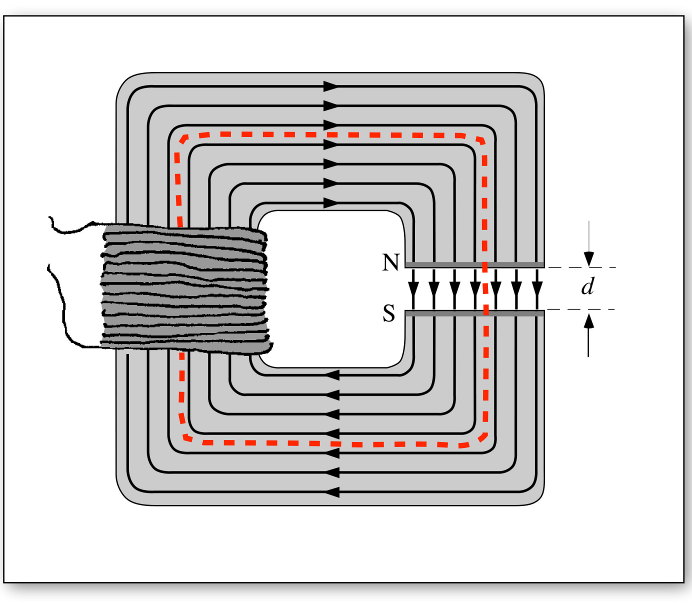

Fig. 8.6
Electromagnet with narrow gap. The field lines correspond to the magnetic flux density $\boldsymbol{B}$.

With

$\quad \operatorname{div} \boldsymbol{B} = 0$

we obtain

$\quad \mu_0 H_{out} = M + \mu_0 H_{in}$

With $\boldsymbol{M} = \mu_0\, \chi_m\, \boldsymbol{H}$ (equation (6.10)) we get

$\quad \mu_0 H_{out} = \mu_0 \chi_m H_{in} + \mu_0 H_{in} = \mu_0(1 + \chi_m) H_{in}$

Since $\chi_m >> 1$ we have

$\quad \mu_0 H_{out} \approx \mu_0 \chi_m H_{in}$

and thus

$$H_{in} = \frac{H_{out}}{\chi_m}$$

We integrate over the red dashed path. The path section inside the iron has the length l_{in}, the section outside has length l_{out}.

$\quad H_{in}\, l_{in} + H_{out}\, l_{out} = N \cdot I$

Here we do not neglect H_{in} against H_{out}, because of the factor l_{in} in front of H_{in}, which is big against l_{out}.

Replacing H_{in} with the help of the penultimate equation results in:

$$H_{out} = \frac{NI}{\dfrac{l_{in}}{\chi_m} + l_{out}}$$

With $H_{in} = H_{out}/\chi_m$ we get the field strength within the iron core

$$H_{in} = \frac{NI}{l_{in} + \chi_m l_{out}}$$

We consider two extreme cases.

$$\frac{l_{in}}{\chi_m} \ll l_{out} \quad \Rightarrow \quad H_{out} = \frac{NI}{l_{out}}, \quad H_{in} = \frac{NI}{\chi_m l_{out}}$$

If the gap is not too narrow, the field in the gap is just as strong as if the entire coil were wound over the short length l_{out}. In the iron the field strength is very small.

$$\frac{l_{in}}{\chi_m} \gg l_{out} \quad \Rightarrow \quad H_{out} = \frac{NI}{l_{in}} \chi_m, \quad H_{in} = \frac{NI}{l_{in}}$$

The field strength within the iron is the same as in a toroidal coil with the same number of turns. In the gap, $\boldsymbol{H}$ is greater by a factor of χ_m.

The long electromagnet (coil with iron core)

Since the field is inhomogeneous, it is harder to calculate. The effect of the iron core can be described qualitatively as follows, Fig. 8.7: With an empty coil, the largest contribution to $\int \boldsymbol{H} d\boldsymbol{r}$ comes from inside the coil. If the coil is filled with a soft magnetic material, $\boldsymbol{H}$ becomes very small inside. Since $\int \boldsymbol{H} d\boldsymbol{r} = NI$ has the same value as before, the external field must now make a large contribution. The outer field is thus strengthened by the iron core.

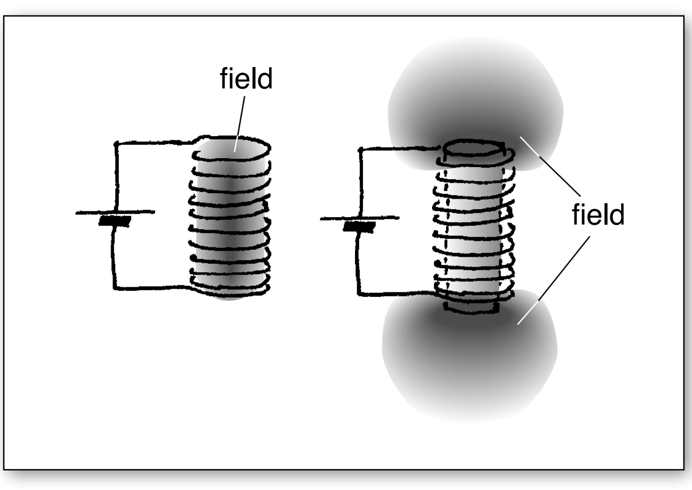

Fig. 8.7
The iron core expels the magnetic field from the coil.

8.3 Maxwell's forth equation

Ampère's law is only a provisional solution, valid only as long as the circuit is closed. In the case of Figure 8.8, where a line is interrupted by the empty space between the plates of a capacitor, it leads to an inconsistency. A current of constant strength I flows through the conductor so that the capacitor is charged.

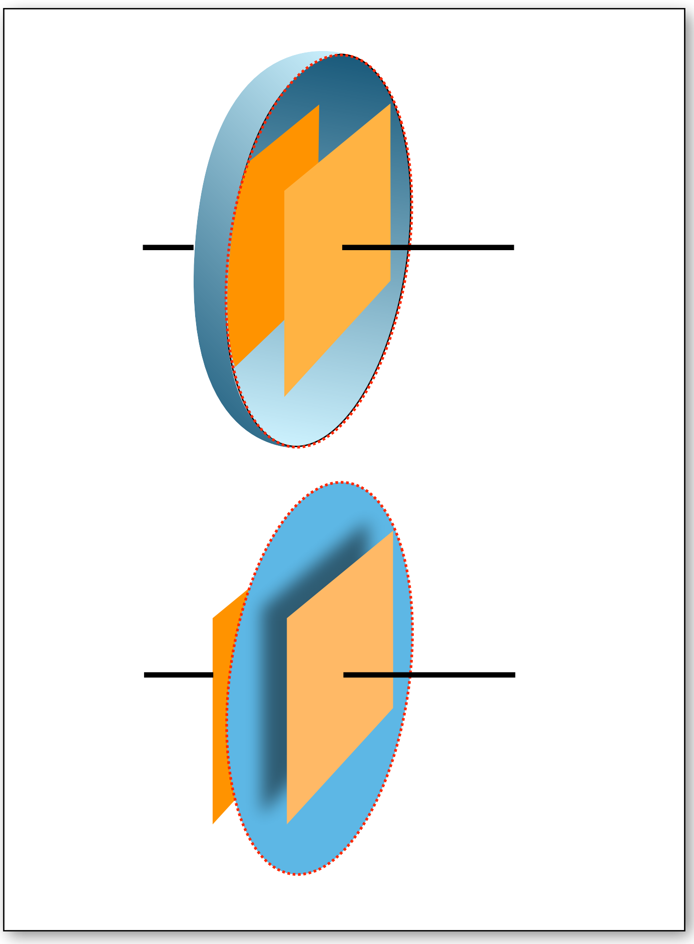

Fig. 8.8
Two possibilities for the choice of the integration surface with fixed boundary line

While this current is flowing, the integral $\int \boldsymbol{H}d\boldsymbol{r}$ has a well-defined, non-zero value on the dashed red line. But the right side of Ampère's law, $\iint \boldsymbol{j}d\boldsymbol{A}$, has different values depending on how the surface whose edge is the dashed line is chosen. If the surface cuts the wire (upper partial image) we have $\iint \boldsymbol{j}d\boldsymbol{A} = I$. If it runs through the space between the capacitor plates (lower partial image) we get $\iint \boldsymbol{j}d\boldsymbol{A} = 0$. In this second case $\int \boldsymbol{H}d\boldsymbol{r}$ cannot be equal to $\iint \boldsymbol{j}d\boldsymbol{A}$. This problem occurs only if the circuit is not closed, i.e. if $\boldsymbol{j}$ has sources or sinks. Maxwell therefore generalized Ampère's law by adding two terms to $\iint \boldsymbol{j}d\boldsymbol{A}$ on the right:

$$\oint_{\substack{\text{border}\\\text{line of S}}} \boldsymbol{H}\,d\boldsymbol{r} = \iint_S \boldsymbol{j}\,d\boldsymbol{A} + \iint_S \dot{\boldsymbol{P}}\,d\boldsymbol{A} + \varepsilon_0 \iint_S \dot{\boldsymbol{E}}\,d\boldsymbol{A} \tag{8.3}$$

With equation (4.18) the last two summands can be combined:

$$\oint_{\substack{\text{border}\\\text{line of S}}} \boldsymbol{H}\,d\boldsymbol{r} = \iint_S \boldsymbol{j}\,d\boldsymbol{A} + \iint_S \dot{\boldsymbol{D}}\,d\boldsymbol{A} \quad \text{Maxwell's 4th equation} \tag{8.4}$$

The ordinary electric current $\iint \boldsymbol{j}d\boldsymbol{A}$ has been supplemented by two further terms that contribute to the magnetic field:

- $I_p = dQ_p/dt = \iint (\partial \boldsymbol{P}/\partial t) d\boldsymbol{A}$ is the current caused by the displacement of polarization charges.
- Even if no polarizable matter is present, the space between the capacitor plates contributes to the magnetic field. This term behaves like an electric current as far as the magnetic field is concerned. It is, apart from the factor ε_0, the time derivative of the flow of the $\boldsymbol{E}$ field through the surface S.

Does this remove the inconsistency mentioned above? We calculate the right side of Maxwell's 4th equation for the case that the integration area lies between the capacitor plates. Since $\boldsymbol{j} = 0$ and $\partial \boldsymbol{P}/\partial t = 0$, only the term $\varepsilon_0 \iint (\partial \boldsymbol{E}/\partial t) d\boldsymbol{A}$ remains.

With $|\boldsymbol{E}| = Q/(\varepsilon_0 A)$ and $Q = I \cdot t$ we get

$$\varepsilon_0 \iint \dot{E}\, dA = \varepsilon_0 \iint \frac{\dot{Q}}{\varepsilon_0 A} dA = I$$

So it is $\int \boldsymbol{H}d\boldsymbol{r} = I$, in agreement with the result one gets when the integration surface intersects the wire.

Maxwell also interpreted the contribution $\iint (\partial \boldsymbol{D}/\partial t) d\boldsymbol{A}$ as an electric current and called it *displacement current*.

He called the sum $I + \int (\partial \boldsymbol{D}/\partial t) d\boldsymbol{A}$ the *true current*. According to this idea, there are only closed circuits at all. If the current density of the true current is referred to as C, the 4th Maxwell equation simplifies:

$$\oint \boldsymbol{H}\,d\boldsymbol{r} = \iint \boldsymbol{C}\,d\boldsymbol{A}$$

According to this view, the cause for $\int \boldsymbol{H}d\boldsymbol{r}$ is always an electric current.

The right-hand rule applies to all contributions to the current in equation (8.3).

8.4 Maxwell's third equation

Faraday recognized the structural similarity of electrical and magnetic phenomena. This insight led him to discover the law of induction. He did not find this law by chance, but consciously searched for it. The search took more than 10 years. If he had available that part of Maxwell's formalism that we have dealt with so far, his search would have gone faster. He could have found the law of induction simply by formally translating Maxwell's 4th equation (8.3):

$$\oint \boldsymbol{H}\, d\boldsymbol{r} = \iint \boldsymbol{j}\, d\boldsymbol{A} + \iint \dot{\boldsymbol{P}}\, d\boldsymbol{A} + \varepsilon_0 \iint \dot{\boldsymbol{E}}\, d\boldsymbol{A} \qquad \text{Maxwell's 4th equation}$$

$$\oint \boldsymbol{E}\, d\boldsymbol{r} = \iint \boldsymbol{j}_{\mathrm{m}}\, d\boldsymbol{A} + \iint \dot{\boldsymbol{M}}\, d\boldsymbol{A} + \mu_0 \iint \dot{\boldsymbol{H}}\, d\boldsymbol{A} \quad ?? \quad \left\{ \begin{array}{l} \text{not quite correct} \\ \text{version of} \\ \text{Maxwell's 3rd equation} \end{array} \right.$$

The only mistake in this formal translation is a sign error: the equation turns out to be correct if a minus sign is placed in front of $\int \boldsymbol{E} d\boldsymbol{r}$. (If the sign of the electric or magnetic charge had been defined the other way round, the minus sign would appear in Maxwell's 4th equation. However, it must be in one of the two, otherwise there is a conflict with the law of energy conservation.)

Since there are no isolated magnetic charges, there is no magnetic conduction current, so we can omit the term $\iint \boldsymbol{j}_{\mathrm{m}} d\boldsymbol{A}$. (As soon as someone discovers a magnetic monopole particle, we add it again).

So it remains

$$-\oint_{\substack{\text{border}\\ \text{line of S}}} \boldsymbol{E}\, d\boldsymbol{r} = \iint_{S} \dot{\boldsymbol{M}}\, d\boldsymbol{A} + \mu_0 \iint_{S} \dot{\boldsymbol{H}}\, d\boldsymbol{A} \qquad (8.5)$$

With equation (6.5) the two integrals on the right side can be summarized:

$$-\oint_{\substack{\text{border}\\ \text{line of S}}} \boldsymbol{E}\, d\boldsymbol{r} = \iint_{S} \dot{\boldsymbol{B}}\, d\boldsymbol{A} \quad \left\{ \begin{array}{l} \text{Maxwell's 3rd equation} \\ \text{(Faraday's law of induction)} \end{array} \right. \qquad (8.6)$$

Like Maxwell's 4th equation, the 3rd also makes a statement about a surface. More precisely: about a relationship between the edge and the interior of the surface. The integral $\int \boldsymbol{E} d\boldsymbol{r}$ over the boundary comes about through two contributions:

- the time rate of change of the flow of $\boldsymbol{M}$ through S
- the time rate of change of the flow of $\boldsymbol{H}$ (times μ_0) through S.

Here, too, the right side can be interpreted as a displacement current: a *magnetic displacement current*. We can thus say that around a magnetic current an electric field with is created, whose field lines surround the magentic current.

Attention: In an $\boldsymbol{E}$ field caused by induction, $\int \boldsymbol{E} d\boldsymbol{r}$ is *not* independent of the integration path. A potential can therefore only be defined in limited regions of space.

In order to obtain the direction of the electric field strength of the induced field, a hand rule can be applied again; but because of the minus sign in the equations (8.5) and (8.6) it is not a right-hand but a left-hand rule: If the thumb of the left hand points in the direction of the magnetic current (i.e. in the direction of $\partial \boldsymbol{B}/\partial t$), the curved fingers indicate the direction of the electric field lines.

8.5 Electric conductors in the induced electric field

To verify the electric field described by Maxwell's 3rd equation, a "test charge" could be used. However, such a demonstration is difficult because $\boldsymbol{E}$, and consequently $\boldsymbol{F} = \boldsymbol{E}Q$, is very small in a typical experimental setup. We therefore place an electric conductor in the region to be investigated. We then recognize the $\boldsymbol{E}$ field indirectly by an "induced voltage" appearing between the ends of the conductor, or by an "induced current" flowing in the conductor.

In order to obtain a mathematically manageable situation, we consider a magnetic field $\boldsymbol{H}_0$ (Fig. 8.9),

- which is homogeneous;
- whose field strength increases at a constant rate (e.g. in an electromagnet whose electric current strength increases at a constant rate): $\partial \boldsymbol{H}_0/\partial t$ = const

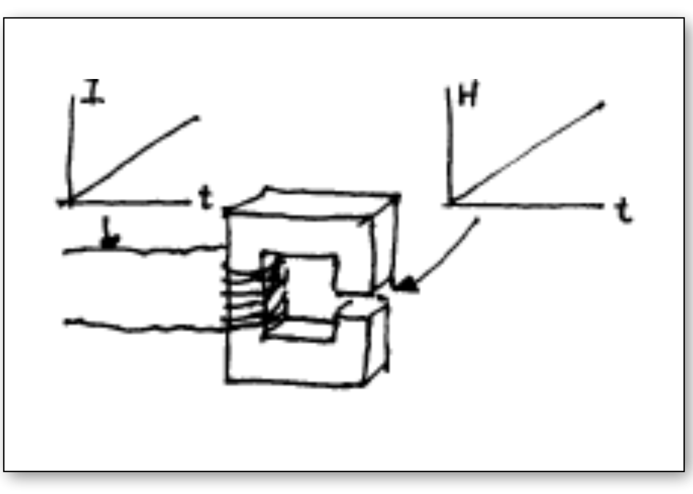

Fig. 8.9
The field in the gap of the electromagnet is homogeneous and increases at a constant rate.

We consider two special cases.

Short circuit

The conductor is a closed ring, Fig. 8.10. Because of the $\boldsymbol{E}$ field, an electric current flows in the conductor, which causes a magnetic field of strength $\boldsymbol{H}_i$. We apply Maxwell's 3rd equation:

$$\oint \boldsymbol{E}\, d\boldsymbol{r} = -\mu_0 \iint \dot{\boldsymbol{H}}\, d\boldsymbol{A} = -\mu_0 \iint (\dot{\boldsymbol{H}}_o + \dot{\boldsymbol{H}}_i)\, d\boldsymbol{A}$$

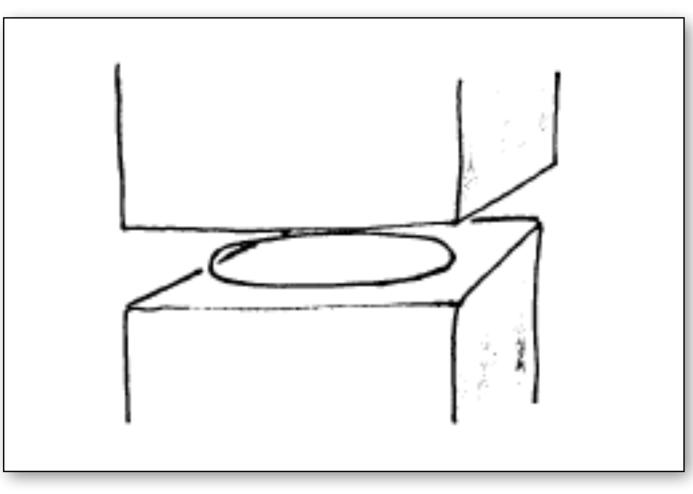

Fig. 8.10
Short-circuited conductor loop in a magnetic field whose strength increases at a constant rate

Now $\boldsymbol{H}_i$ is still unknown. But it is easy to find a consistent solution. We assume that $\partial \boldsymbol{H}_i/\partial t = 0$, i.e. $\boldsymbol{H}_i$ is constant in time. Then $\boldsymbol{E}$ is also constant in time everywhere and with $\boldsymbol{j} = \sigma\boldsymbol{E}$ also the electric current in the wire and consequently also $\boldsymbol{H}_i$, as we had assumed. Thus

$$\oint \boldsymbol{E}\, d\boldsymbol{r} = -\mu_0 \iint \dot{\boldsymbol{H}}_o\, d\boldsymbol{A}$$

So the equation has the same structure as if there were no conductor in the magnetic field. The conductor causes an additional magnetic field which is constant in time but has no influence on the induction process.

Open circuit

The conductor loop is interrupted, no electric current can flow, Fig. 8.11. The fact that no current flows means that the electric field strength in the wire is zero, since $\boldsymbol{j} = \sigma\boldsymbol{E}$. How can this be reconciled with Maxwell's 3rd equation? It cannot originate from the right side of the equation, i.e. the $\boldsymbol{H}$ field, because no current is flowing that could modify the $\boldsymbol{H}$ field.

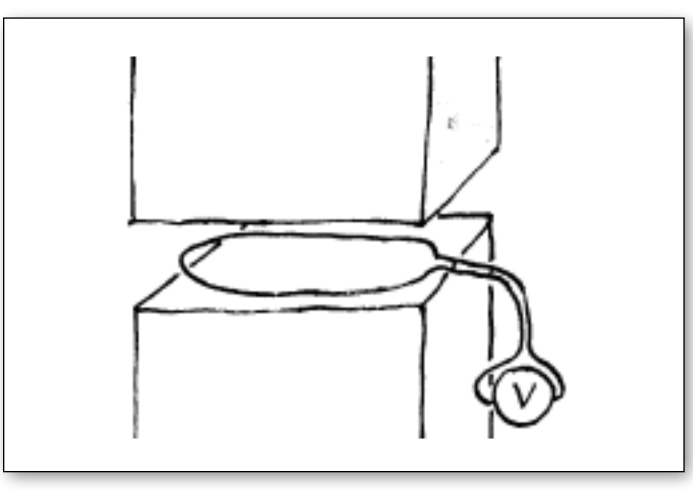

Fig. 8.11
Open conductor loop in a magnetic field whose strength increases at a constant rate

As soon as the experiment begins, charges displace in such a way that the $\boldsymbol{E}$ field strength within the conductor material becomes zero everywhere. Instead, an $\boldsymbol{E}$ field appears between the open ends of the loop, Fig. 8.12. This is just so strong that the integral over the path S in the gap between A and B equals

$\mu_0 \iint (\partial \boldsymbol{H}/\partial t) d\boldsymbol{A}$

If one calculates $\int \boldsymbol{E} d\boldsymbol{r}$ over the closed, dashed path, then only this section S makes a contribution. This integral is called induced voltage U_{ind}.

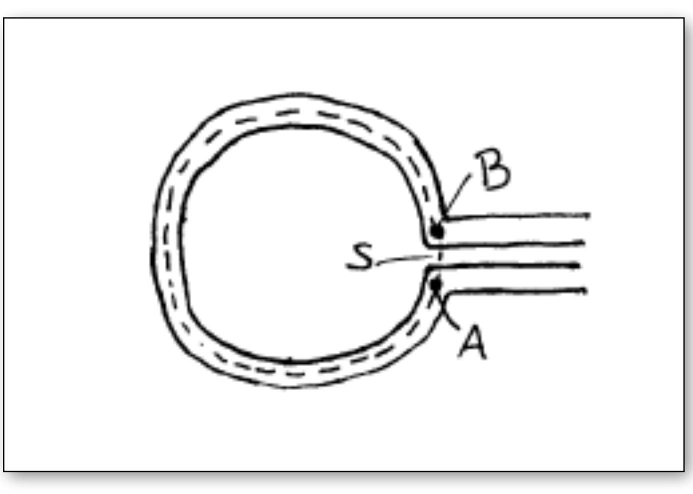

Fig. 8.12
Inside the conductor the electric field strength is zero.

The induced voltage can be measured between the leads with a voltmeter. In this case the 3rd Maxwell equation can be formulated as follows

$$U_{ind} = -\mu_0 \iint \dot{\boldsymbol{H}}\, d\boldsymbol{A}$$

or if $\chi_m \neq 0$, more generally

$$U_{ind} = -\iint \dot{\boldsymbol{B}}\, d\boldsymbol{A}$$

The integral

$$\phi = \iint \mathbf{B}\, d\mathbf{A}$$

is called *magnetic flux*. We thus obtain

$$U_{ind} = -\dot{\Phi}$$

Figures 8.13 to 8.17 show some induction experiments.

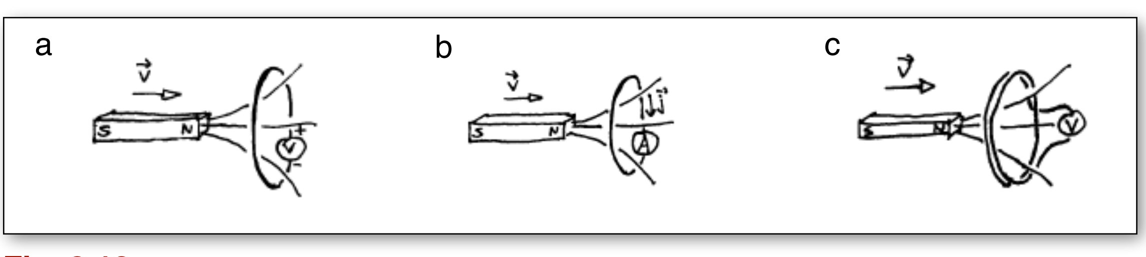

Fig. 8.13
(a) The permanent magnet is moved, the circuit is open.
(b) The permanent magnet is moved, the circuit is closed.
(c) The induced voltage is proportional to the number of turns.

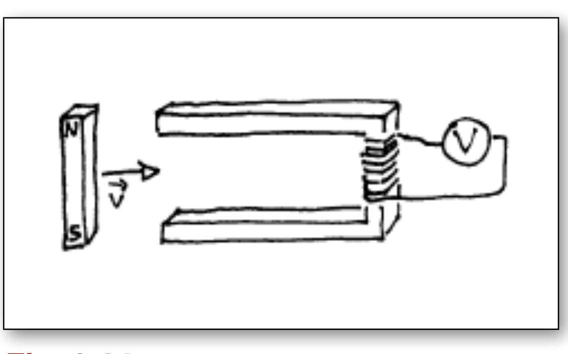

Fig. 8.14
The induced voltage is caused by a change of the magnetization.

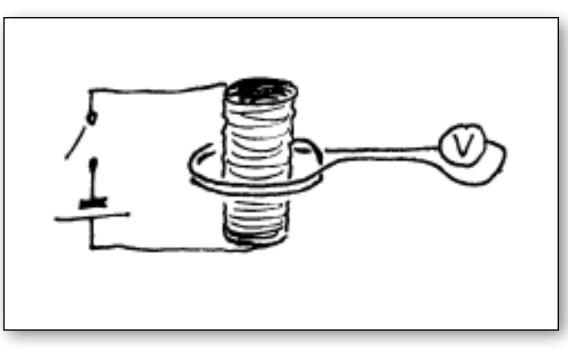

Fig. 8.15
The $\boldsymbol{H}$ field change is achieved by closing an electric circuit.

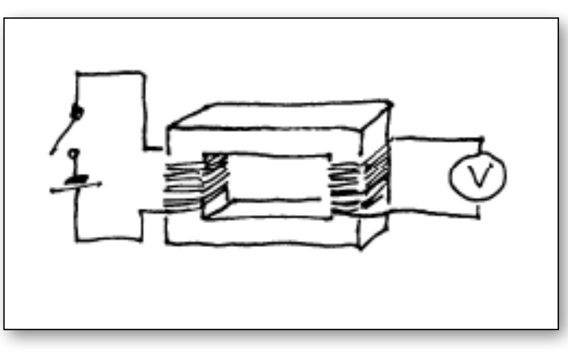

Fig. 8.16
The principle of operation of the transformer

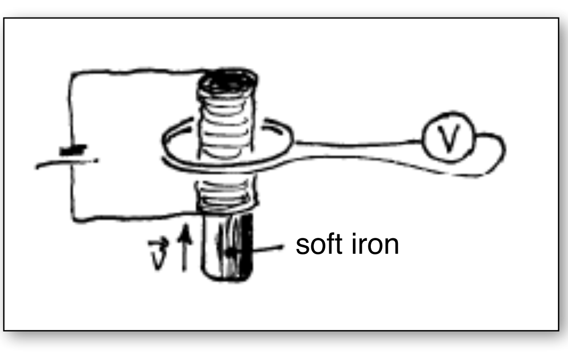

Fig. 8.17
The change of the $\boldsymbol{B}$ field distribution is caused by the displacement of the soft magnetic core of the coil.

In the arrangement shown in Fig. 8.18b, the conductor loop is moved instead of the magnet. The voltmeter reading is of course the same as in Fig. 8.18a, because the only difference between right and left is that a different reference frame has been chosen for the description. While the 3rd Maxwell equation is used to describe the left figure, the 4th Maxwell equation is required to describe the right one, where the $\boldsymbol{H}$ field of the permanent magnet is constant. However, the description using the 3rd Maxwell equation is more convenient; one will therefore choose the reference frame in such a way that the conductor loop is at rest.

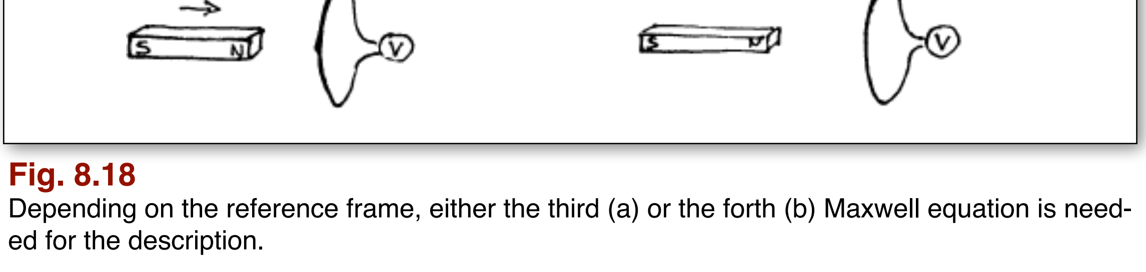

Fig. 8.18
Depending on the reference frame, either the third (a) or the forth (b) Maxwell equation is needed for the description.

Also in the arrangement shown in Fig. 8.19 the $\boldsymbol{H}$ field is constant in time. Therefore, the induced voltage is calculated in the reference frame in which the conductor loop is at rest and the magnet rotates.

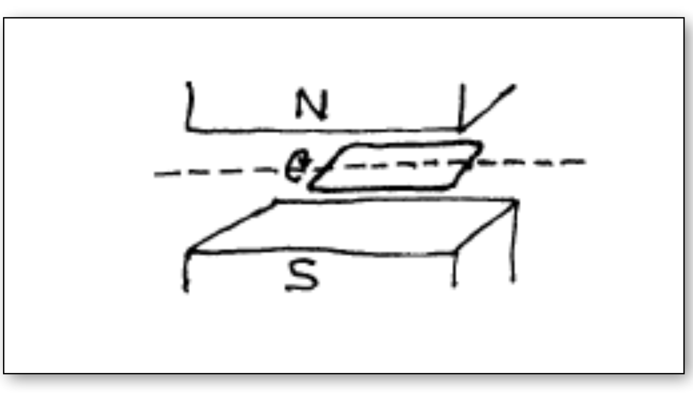

Fig. 8.19
The $\boldsymbol{H}$ field is constant in time. To calculate the induced voltage, one better goes into the reference frame in which the conductor loop is at rest.

8.6 Inductance

Inductance is a quantity of great technical importance. It characterizes a single-loop circuit in which the electric current has no sources or sinks. The circuit defines a surface S through which the flux passes.

$\iint \boldsymbol{B} d\boldsymbol{A} = \iint (\mu_0 \boldsymbol{H} + \boldsymbol{M}) d\boldsymbol{A}$

passes.

According to Maxwell's 4th equation, the field strength at each location is proportional to the electric current I in the circuit. As long as $\boldsymbol{M} \sim \boldsymbol{H}$, also $\boldsymbol{M}$ is proportional to I everywhere. It follows that both the flux $\iint \boldsymbol{H} d\boldsymbol{A}$ and $\iint \boldsymbol{M} d\boldsymbol{A}$ are proportional to I:

$$\Phi = \iint_S B\, dA = \iint_S (\mu_0 H + M)\, dA \propto I$$

We write this relationship

$$\Phi = L \cdot I \qquad (8.8)$$

The factor of proportionality L is called the *inductance* of the circuit. The unit of measurement is Vs/A = H (Henry). Since there are no real magnetic currents, there is also no electric analog to L.

With Maxwell's 3rd equation (8.6), we get

$$\oint \boldsymbol{E}\, d\boldsymbol{r} = -L\dot{I} \qquad (8.9)$$

If there is a resistor in the circuit, i.e. a short section whose resistance is large compared to that of the rest of the circuit, Fig. 8.20, only this section contributes to $\int \boldsymbol{E} d\boldsymbol{r}$, and the voltage over this resistor is

$$U = -L\dot{I}$$

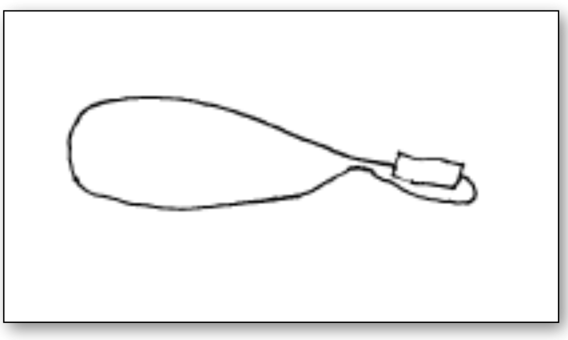

Fig. 8.20
The resistance of a small section of the circuit is large against that of the rest.

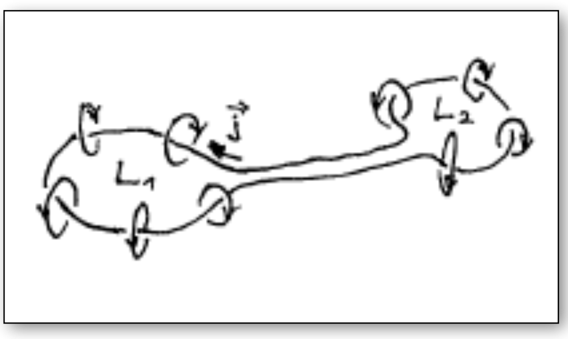

Fig. 8.21
The inductance of the circuit can be decomposed into two parts.

Often a circuit can be decomposed into parts as shown in Figure 8.21. There are two nearly closed surfaces and the flux caused by one part does not pass through the other. In this case, the inductance can also be decomposed and each part can be assigned its own inductance.

In the case of a coil with N turns, each $\boldsymbol{B}$ field line passes N times through the surface of the circuit. It thus contributes N times to the integral $\iint \boldsymbol{B} d\boldsymbol{A}$. Figure 8.22 shows that the surface of the circuit is penetrated 8 times by 4 field lines: All four subpictures are topologically equivalent.

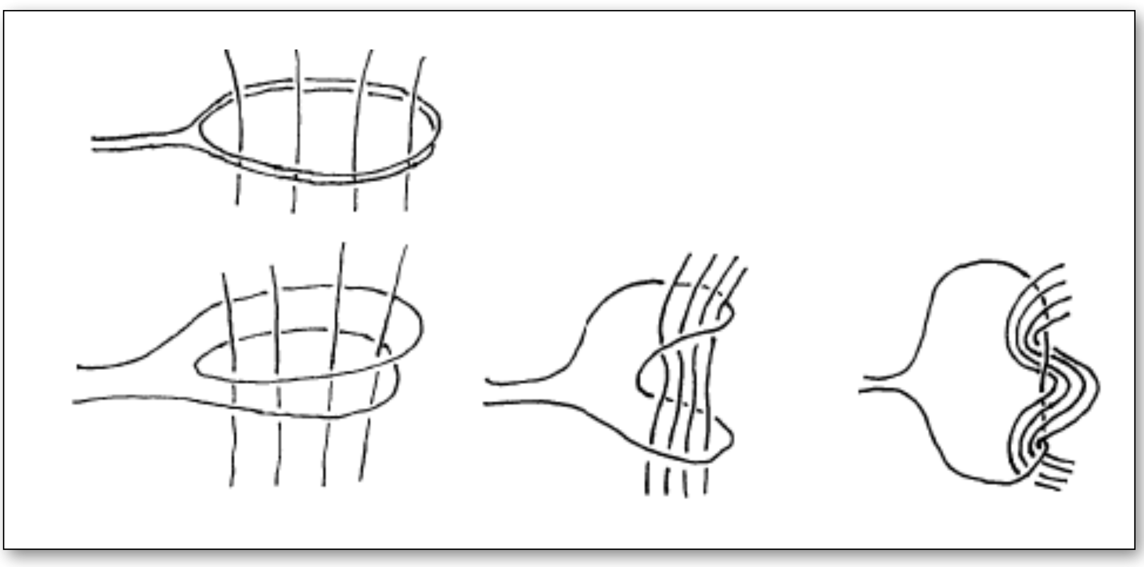

Fig. 8.22
The area defined by the circuit is crossed twice by each field line.

Let us calculate the inductance of a long coil. Inside the coil we have $H = (N/l)I$. With $B = \mu_0 H$ we find for the B flux which crosses the total area of the coil

$$\Phi = \mu_0 \cdot N \cdot A \cdot H = \mu_0 \frac{N^2 A}{l} I$$

By comparison with $\Phi = LI$ we get

$$L = \mu_0 \frac{N^2 A}{l}$$

If one designates the flux through a single cross-sectional area A with Φ', then $\Phi' = \Phi/N$, and from (8.8) we get:

$$N\Phi' = LI$$

If the coil is winded on a closed iron core, then (see section 8.2) $H = (N/l)I$, $M = \mu_0 \chi_m H$ and consequently

$$B = \mu_0 H + M = \mu_0 (1 + \chi_m) \frac{N}{l} I = \mu_0 \mu \frac{N}{l} I$$

We thus get

$$L = \mu\mu_0 \frac{N^2 A}{l} \qquad (8.10)$$

Finally, we calculate the energy content of the field of a coil with a closed iron core. The field is located within the iron core, and thus also the energy. With equation (7.1) and with $H = (N/l)I$ and $V = A \cdot l$ (where V is the volume) we obtain

$$E = \rho_E \cdot V = \frac{\mu\mu_0}{2} \frac{N^2}{l^2} I^2 A l = \frac{\mu\mu_0}{2} \frac{N^2 A}{l} I^2$$

Using (8.11) we get

$$E = \frac{L}{2} I^2 \qquad (8.11)$$

The range of validity of this equation is greater than it would appear from our derivation. It always applies when we are dealing with the magnetic field of a single-loop circuit.

8.7 Reference frames

This issue will only briefly be addressed here. A detailed treatment would lead directly to the theory of relativity.

1. We consider a uniformly electrically charged wire which is at rest in the reference frame S, Fig. 8.23. In this reference frame an electric field different from zero is observed, while ***H*** is equal to zero everywhere. In the reference frame S’, which moves relative to S in the direction of the wire, the charge is moving. Thus, an electric current is flowing and we find ***H*** ≠ 0.

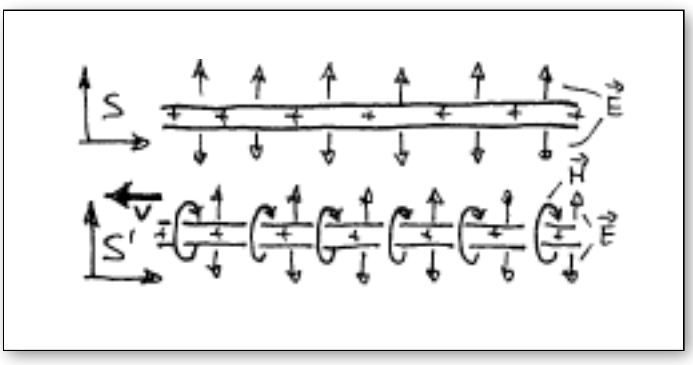

Fig. 8.23
In reference frame S only the electric field strength is different from zero, in reference frame S’ also the magnetic field strength.

2. Consider a bar magnet resting in the reference frame S, Fig. 8.24. In S, ***H*** ≠ 0, but ***E*** is zero everywhere. In the reference frame S’ the ***H*** flux through the surface σ changes with time, so ∫***E**dr* over the edge of this surface is not equal to zero. Thus in S’ the electric field strength is different from zero.

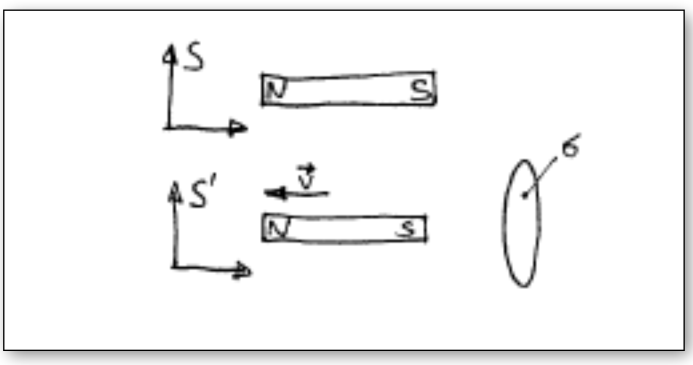

Fig. 8.24
In reference frame S only the magnetic, in reference frame S’ also the electric field strength is different from zero.

These two examples show that ***E*** and ***H*** are not only linked by some physical law, such as ***F*** and ***Q*** in Coulomb's Law. ***E*** and ***H*** rather transform into each other upon a change of the reference frame. The electric field and the magnetic field do not represent two different physical systems, but only one: the *electromagnetic field*. The values of the field strengths depend on the reference frame.

3. Finally, we consider a third situation in two different reference frames: A sphere that carries positive electric charge and that is at rest in a homogeneous magnetic field between the poles of a magnet, Figure 8.25a. Now, the sphere is to move relative to the magnet and perpendicular to the image plane. We consider the process in two different reference frames.

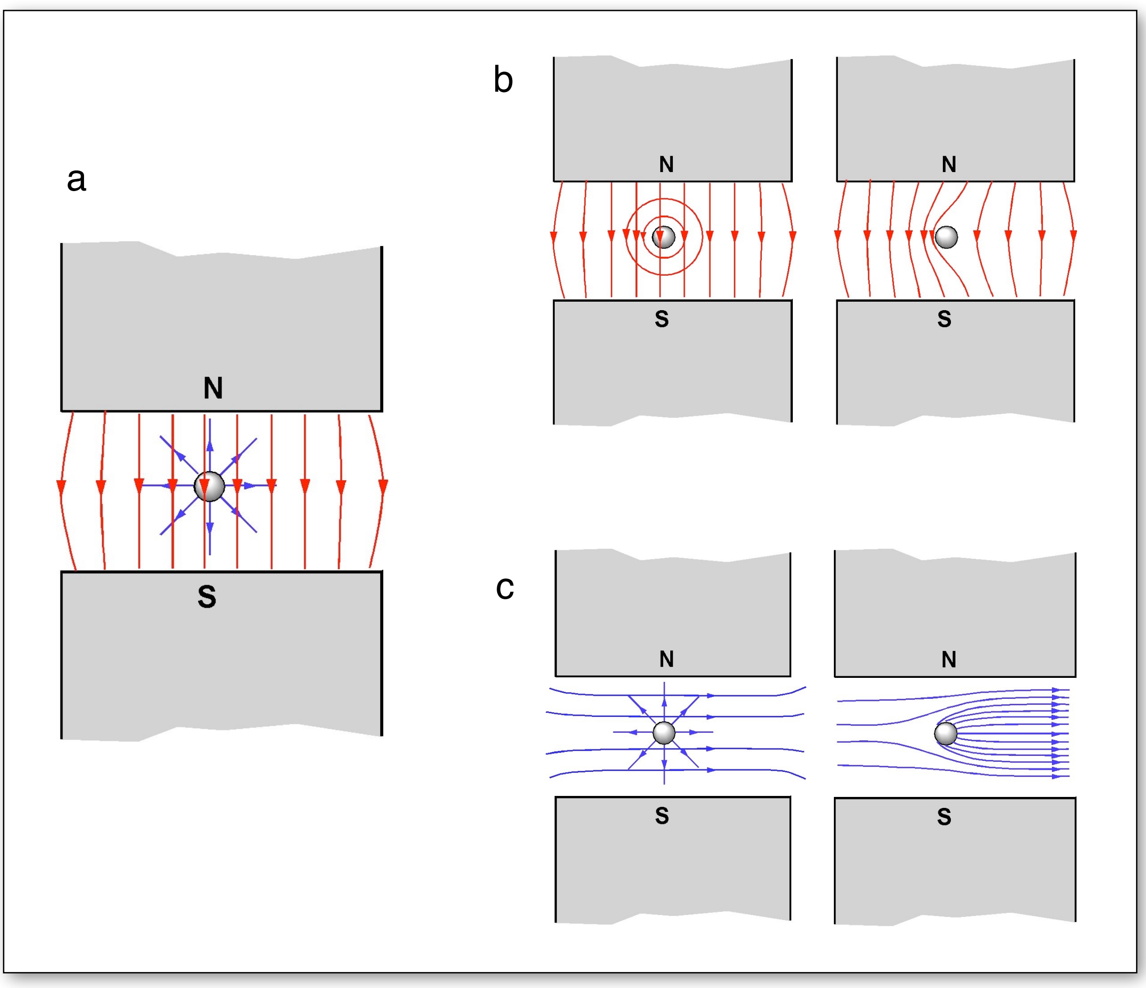

Fig. 8.25
(a) A charged sphere is at rest between the poles of a magnet.
(b) The magnet is at rest; the sphere moves out of the image plane.
(c) The sphere is at rest; the magnet moves into the image plane.

Once the magnet is at rest, and the sphere moves out of the image plane. The moving electric charge corresponds to a current and to this belongs a magnetic field. The left partial image of Fig. 8.25b shows the two magnetic fields that have not yet been combined: the field of the magnet and that of the moving sphere. In the right partial picture the field strengths were summed. The resulting field is stronger at the left side of the sphere than at the right. Since there is a compressive stress perpendicular to the field lines, the sphere is *pushed to the right*. For this interpretation we have used Maxwell’s 4th equation.

We now go into the reference frame of the sphere. The sphere is at rest and the magnet moves into the image plane. Thereby an electric field is induced which is almost homogeneous in the area between the poles. The left partial image of Fig. 8.25c shows the two electric fields that have not yet been combined: the field of the charged sphere and the electric field caused by the movement of the magnet. In the right partial picture the field strengths were added. The resulting electric field is stronger to the right of the sphere than to the left. Since there is tensile stress in the field line direction, the sphere is *pulled to the right*. We used Maxwell’s 3rd equation to interpret the experiment.

Thus, depending on the reference frame, the same phenomenon is explained once by the fourth and once by the third Maxwell equation. If one interprets such an experiment with the help of Maxwell’s 4th equation, the force that appears is called *Lorentz force*.

8.8 Summary of the equations

We want to compile the most important equations of electrodynamics once again. First there are the equations (4.20) and (6.7), which we already knew as Maxwell's 1st and 2nd equation. In addition we have equations (8.4) and (8.6), i.e. Maxwell's 4th and 3rd equation. The four Maxwell equations control the interaction of electric and magnetic fields and their sources.

$$\oint\!\!\!\oint \boldsymbol{D}\, d\boldsymbol{A} = \iiint \rho\, dV \qquad \text{Maxwell's 1st equation}$$

$$\oint\!\!\!\oint \boldsymbol{B}\, d\boldsymbol{A} = 0 \qquad \text{Maxwell's 2nd equation}$$

$$-\oint \boldsymbol{E}\, d\mathbf{r} = \iint \dot{\boldsymbol{B}}\, d\boldsymbol{A} \qquad \text{Maxwell's 3rd equation}$$

$$\oint \boldsymbol{H}\, d\boldsymbol{r} = \iint \boldsymbol{j}\, d\boldsymbol{A} + \iint \dot{\boldsymbol{D}}\, d\boldsymbol{A} \qquad \text{Maxwell's 4th equation}$$

Here we have written the equations in their integral form. For Maxwell's 1st and 2nd equation, we had already previously given a local or "differential" formulation, namely the equations (4.19) and (6.6). The 3rd and the 4th equations can also be formulated locally using the curl operator. We thus have

$$\mathrm{div}\boldsymbol{D} = \rho \qquad \text{Maxwell's 1st equation}$$

$$\mathrm{div}\boldsymbol{B} = 0 \qquad \text{Maxwell's 2nd equation}$$

$$-\mathrm{curl}\boldsymbol{E} = \dot{\boldsymbol{B}} \qquad \text{Maxwell's 3rd equation}$$

$$-\mathrm{curl}\boldsymbol{H} = \boldsymbol{j} + \dot{\boldsymbol{D}} \qquad \text{Maxwell's 4th equation}$$

9

Forces on moving charge carriers

9.1 The pressure of the magnetic field on an electric conductor

A field pushes or pulls on matter whenever the field "is attached to the matter"; in other words: when the sources of the field reside within the matter. In chapters 4 and 6 we got to know the electric and magnetic charge as sources of the electric or magnetic field and in chapters 5 and 7 we found that the fields on electrically or magnetically charged matter always pull.

In the previous chapter we got to know currents as further sources of fields: the electric current $\boldsymbol{j} + \partial \boldsymbol{D}/\partial t$ and the magnetic current $\partial \boldsymbol{B}/\partial t$. So fields must also exert forces on matter through which a current flows.

The situation is particularly simple when the electric current flows in a cylindrical waveguide in the direction of the cylinder axis (Fig. 9.1a). The magnetic field inside is zero. Outside it is unequal to zero. The field lines are parallel to the cylinder surface, the field surfaces are perpendicular to it, Fig. 9.1b. Because of equation (7.3) the field is pushing on the conductor.

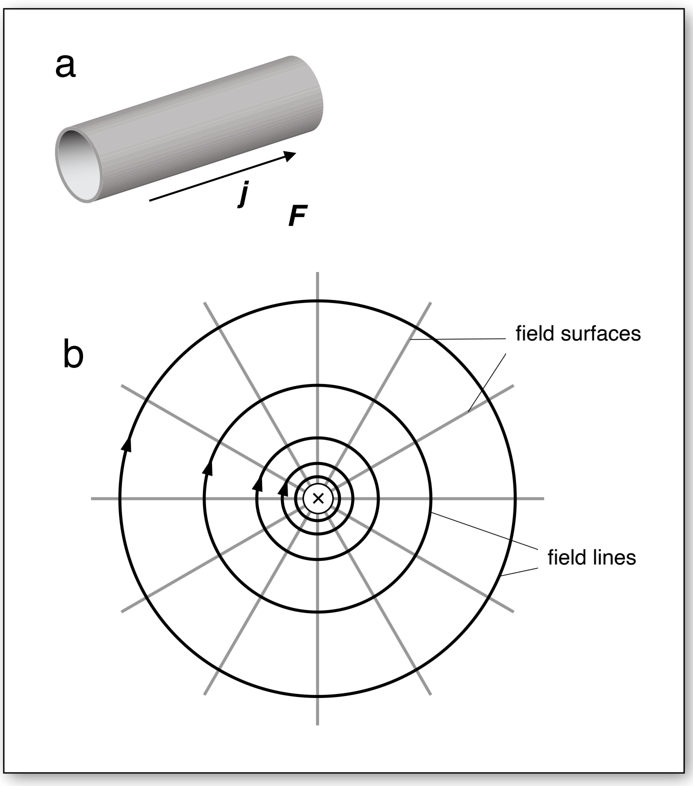

Fig. 9.1
(a) Hollow cylinder through which an electric current is flowing.
(b) Cross-section with field lines and field surfaces

We conclude that the magnetic field always pushes on an electric conductor.

The two wires shown in Fig. 9.2a in cross-section "attract" each other. We can now explain this fact: Let us look at the left wire. The magnetic field pushes on the wire from all sides. But to the left of it the field surfaces are denser than to the right. This results in a net force to the right. Correspondingly, the right wire. In the same way we explain why the wires of Fig.9.2b repel each other. Again we look at the left wire. The magnetic field pushes on it from all sides, but stronger from the right than from the left. This results in a net force to the left.

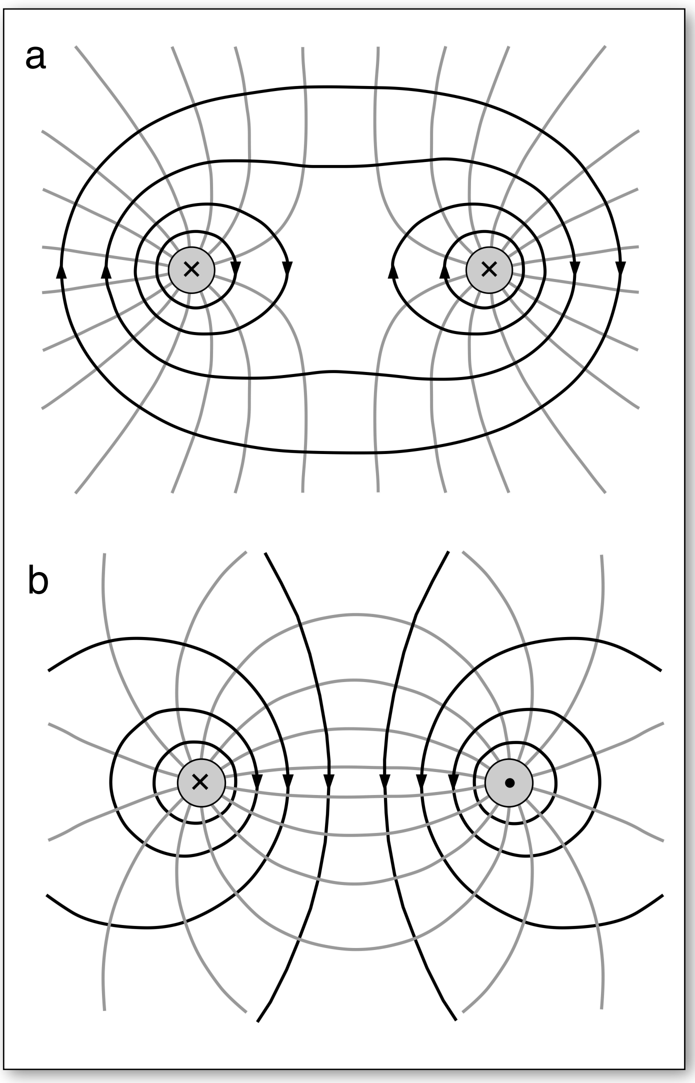

Fig. 9.2
The two wires are pushed towards each other (a) or away from each other (b) by the magnetic field.

We can thus formulate the following rules.

> Two wires in which an electric current flows in the same direction are pushed towards each other by the magnetic field. If the currents flow in opposite directions, the wires are pushed away from each other by the field.

9.2 The Lorentz force

Equation (7.3) is only suitable for calculating the force on a conductor if the field distribution is geometrically simple, as in the case of Fig. 9.1. In many other cases it is easier with another formula to achieve the result. We want to derive this other formula.

We choose again an arrangement that is geometrically simple, Fig. 9.3: an infinitely extended plate through which an electric current with a homogeneous current density distribution is flowing (current density in *z* direction, current intensity by width = *I/b*).

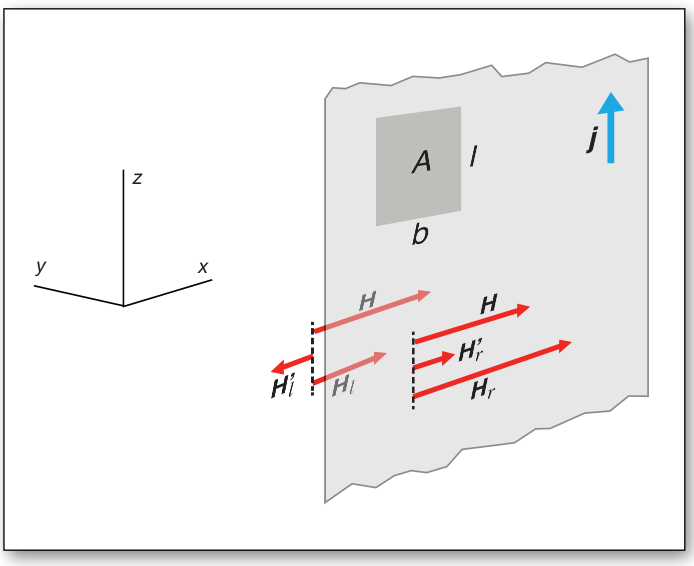

Fig. 9.3
In an external magnetic field a force is exerted on a flat conductor through which an electric current is flowing.

We bring the plate into a magnetic field of strength $\boldsymbol{H}$, the direction of which is parallel to the plate and perpendicular to the current direction. The susceptibility of the medium, and thus μ, is the same on both sides of the plate. In this configuration, only the *x* component of $\boldsymbol{H}$, the *y* component of the force $\boldsymbol{F}$ and the *z* component of the current density $\boldsymbol{j}$ are different from zero. We therefore limit the calculation to these components and omit the indices *x*, *y*, and *z*.

We calculate the force that acts on the plate:

$$F = (\sigma_r - \sigma_l)A$$

(“l” means left, “r” right.)

To calculate $\sigma = -(\mu\mu_0/2)\boldsymbol{H}^2$ we need the actual field strength on the left and right. This is composed of the field strength *H* of the “external” field and the strength *H’* of the field whose source is the electric current in the plate.

We calculate the field strength *H’* caused by the plate using Maxwell’s 4th equation, Fig. 9.4:

$$H'_r b_r + H'_l b_l = (I/b)\, b$$

and with $b_r = b$ and $b_l = -b$ we get $H'_r - H'_l = I/b$. Because of the symmetry we also have $H'_r = -H'_l$. So we obtain

$$H'_r = I/(2b) \text{ and } H'_l = -(I/2)b.$$

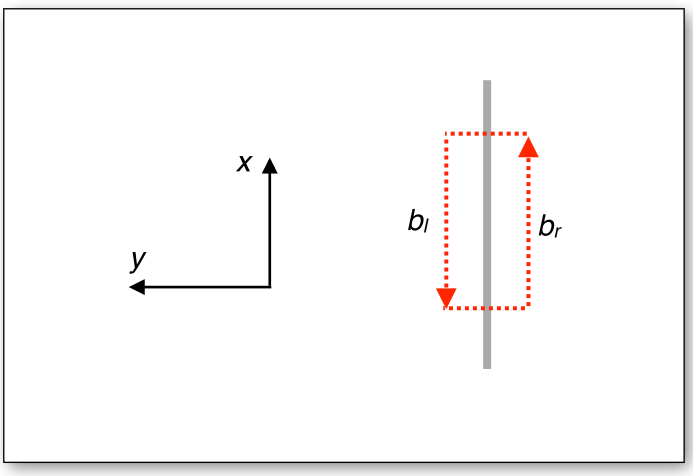

Fig. 9.4
Calculating the path integral over the magnetic field strength

For the total field strength we get

left $H_l = H - I/(2b)$ and right $H_r = H + I/(2b)$.

With this the difference of the mechanical stresses is obtained

$$\sigma_r - \sigma_l = -\frac{\mu\mu_0}{2}(H_r^2 - H_l^2) = -\frac{\mu\mu_0}{2}\left[4\frac{HI}{2b}\right] = -\mu\mu_0 H\frac{I}{b}$$

and with $A = l \cdot b$

$$F = -(\sigma_r - \sigma_l)A = \mu\mu_0 lIH$$

This equation is a special case of the vector relationship

$$\boldsymbol{F} = \mu\mu_0 I(\boldsymbol{l} \times \boldsymbol{H}) = I(\boldsymbol{l} \times \boldsymbol{B}) \qquad (9.1)$$

The vector $\boldsymbol{l}$ has the same direction as the electric current density. Equation (9.1) does not only apply under the simple conditions of our calculation. It always describes the force acting on a conductor in which an electric current is flowing and which is located in a magnetic field. The field strength $\boldsymbol{H}$ to be inserted into the equation is the strength of the field that would be present in the absence of the conductor. The force calculated with equation (9.1) is called the *Lorentz force*.

9.3 Examples of the Lorentz force

Force on a conductor with an electric current (Fig. 9.5)

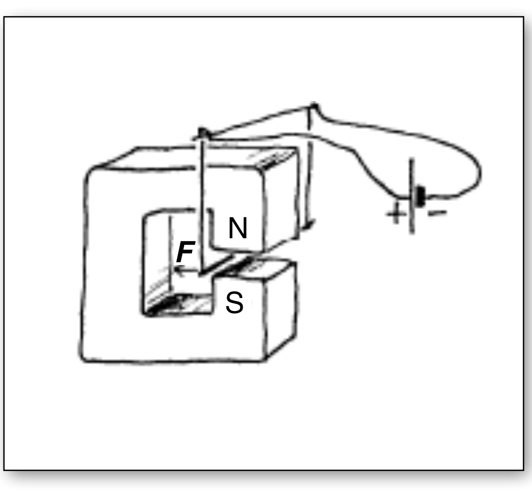

Fig. 9.5
The magnetic field pushes the wire to the left.

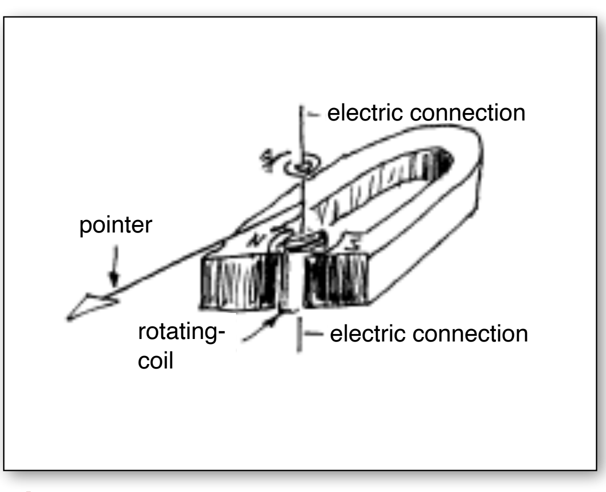

Fig. 9.6
Galvanometer

The *galvanometer*, Fig. 9.6, is based on this principle. It was used in analog current, voltage and resistance meters.

Force between parallel conductors in which an electric current is flowing

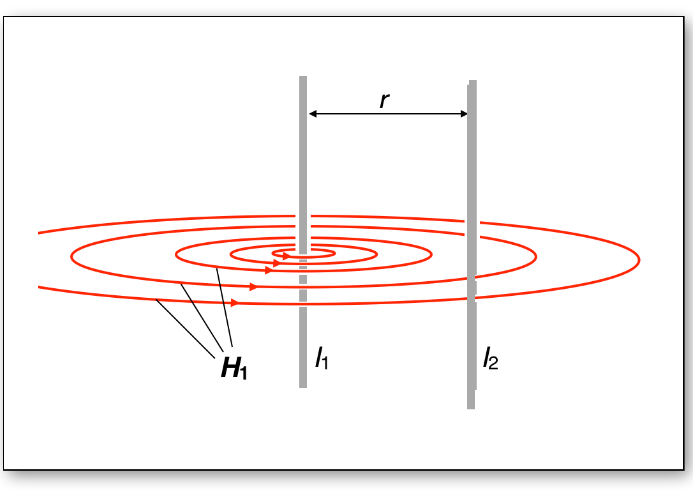

Fig. 9.7
Wire 2 in the field of wire 1

We consider wire 2 in the field of wire 1, Fig. 9.7, and apply equation (9.1). In our case we have $F = \mu_0 I_2 l H_1$. With $H_1 = I_1/(2\pi r)$, we get

$$\frac{F}{l} = \frac{\mu_0}{2\pi}\frac{I_1 \cdot I_2}{r} \qquad (9.2)$$

If the direction of the electric current in the two wires is the same, the wires attract each other, otherwise they repel each other.

Force on moving charged particles

We assume, the electric current within a conductor is due to the motion of electrically charged particles with the uniform velocity v :

$$I = \rho v A = \frac{Q}{V} v A = \frac{Q}{l} v$$

Then the Lorentz force will be

$$\boldsymbol{F} = \frac{Q}{l} v(\boldsymbol{l} \times \boldsymbol{B})$$

Since the vector $\boldsymbol{v}$ has the same direction as $\boldsymbol{l}$, we can write

$$\boldsymbol{F} = Q(\boldsymbol{v} \times \boldsymbol{B}) \qquad (9.3)$$

In this equation there is no longer any geometric variable of the conductor. It therefore also applies to individual charge packages, and thus also to particles. If the particle is an electron, then

$$\boldsymbol{F} = -e(\boldsymbol{v} \times \boldsymbol{B}) \qquad (9.4)$$

Electrons in a homogeneous magnetic field

Let an electron move at velocity $\boldsymbol{v}$ in a magnetic field which would be homogeneous without the presence of the electron and would have the field strength $\boldsymbol{H}$. Let $\boldsymbol{v}$ be perpendicular to $\boldsymbol{H}$. On the electron acts the force $\boldsymbol{F} = -e(\boldsymbol{v} \times \boldsymbol{B})$. This results in a change of the momentum. Since $\boldsymbol{F}$ is perpendicular to $\boldsymbol{v}$, it is

$$d\boldsymbol{p}/dt = -m\omega^2 \boldsymbol{r}.$$

$\boldsymbol{r}$ is the radius of curvature of the circular path of the electron, ω is the corresponding angular frequency.

With the momentum balance equation

$$d\boldsymbol{p}/dt = \boldsymbol{F}$$

we get

$$-m\omega^2 \boldsymbol{r} = -e(\boldsymbol{v} \times \boldsymbol{B})$$

or

$$m\omega^2 r = evB$$

With $v = \omega r$ we finally obtain the so-called *cyclotron frequency*:

$$\omega = \frac{e}{m} B$$

Explanation of the induction process using the Lorentz force

A conductor loop is moved into a magnetic field as shown in Fig. 9.8. The magnetic field is constant in time. Nevertheless, an induced voltage occurs. This is easy to understand by looking at the process in the reference frame in which the conductor loop is at rest.

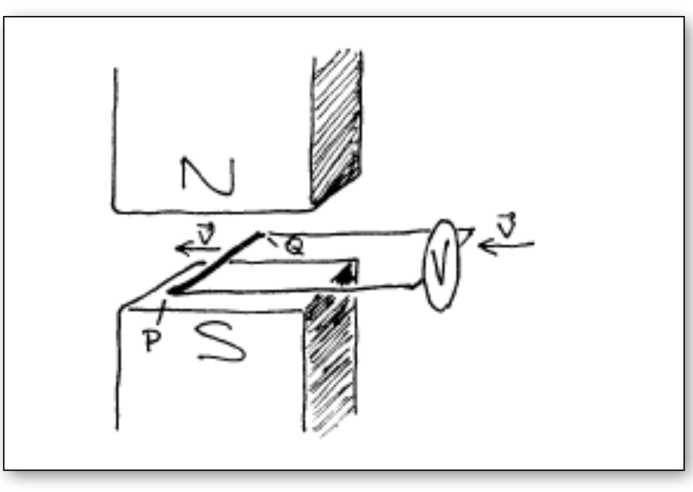

Fig. 9.8
In the reference frame of the permanent magnet the induction cannot be explained with Maxwell's 3rd equation.

Instead, the process can also be explained in the reference frame in which the magnet is at rest by using the Lorentz force: We consider the segment PQ of the conductor loop. Within the wire there are mobile charge carriers. They move in the magnetic field to the left with the velocity $\boldsymbol{v}$. Therefore, the Lorentz force is acting on them

$$\boldsymbol{F}_L = Q(\boldsymbol{v} \times \boldsymbol{B})$$

This shifts the charge carriers until $\boldsymbol{F}_L$ is compensated by $\boldsymbol{F}_{el} = Q\boldsymbol{E}$. Thus, an electric field of field strength is generated:

$$\boldsymbol{E} = \boldsymbol{v} \times \boldsymbol{B}$$

$\boldsymbol{E}$ has the direction of the conductor.

The Hall effect

In an electric conductor which is placed in a magnetic field, an electric current is flowing, Figure 9.9.

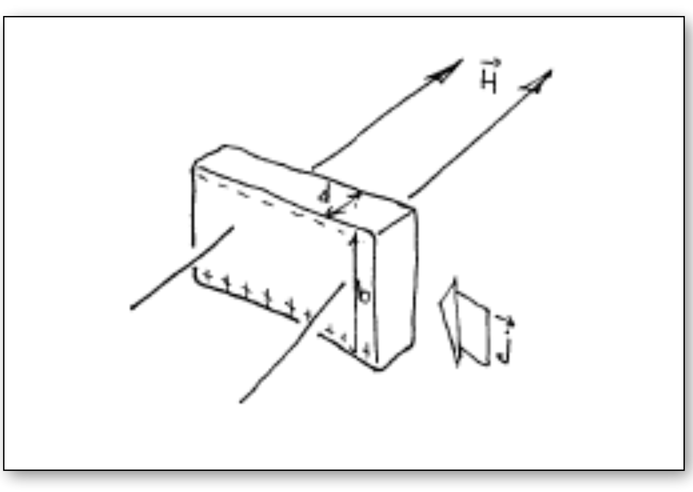

Fig. 9.9
The Hall effect

If the charge carriers move at velocity $\boldsymbol{v}$, they are subjected to the Lorentz force $\boldsymbol{F}_L = Q(\boldsymbol{v} \times \boldsymbol{B})$, perpendicular to the direction of the conductor. Since no electric current can flow in this direction, electric charge accumulates on the sides of the conductor, just in such a way that $\boldsymbol{F}_L$ is compensated by $\boldsymbol{F}_{el} = Q\boldsymbol{E}$. The voltage between the two sides of the conductor that is generated in this way is called the *Hall voltage* U_H. We have

$$Q(\boldsymbol{v} \times \boldsymbol{B}) = Q\boldsymbol{E}_H \quad \Rightarrow \quad \boldsymbol{E}_H = \boldsymbol{v} \times \boldsymbol{B} \quad \text{or} \quad |\boldsymbol{E}|_H = v \cdot B\,.$$

With $v = I/(\rho d b)$ and $U_H = |\boldsymbol{E}|_H b$ we get

$$U_H = vBb = \frac{1}{\rho}\frac{IB}{d}$$

By measuring I, B, U_H and d the density of the mobile charge carriers can be determined. In particular, the sign of the charge of the mobile charge carriers is obtained.

Forces on induced currents

If a closed conductor loop is moved into a magnetic field, Fig.9.10, a current is induced in the conductor loop. The Lorentz force in the magnetic field acts on the wire in which an electric current now flows. It is directed in such a way that it “tries to prevent” the movement of the conductor loop.

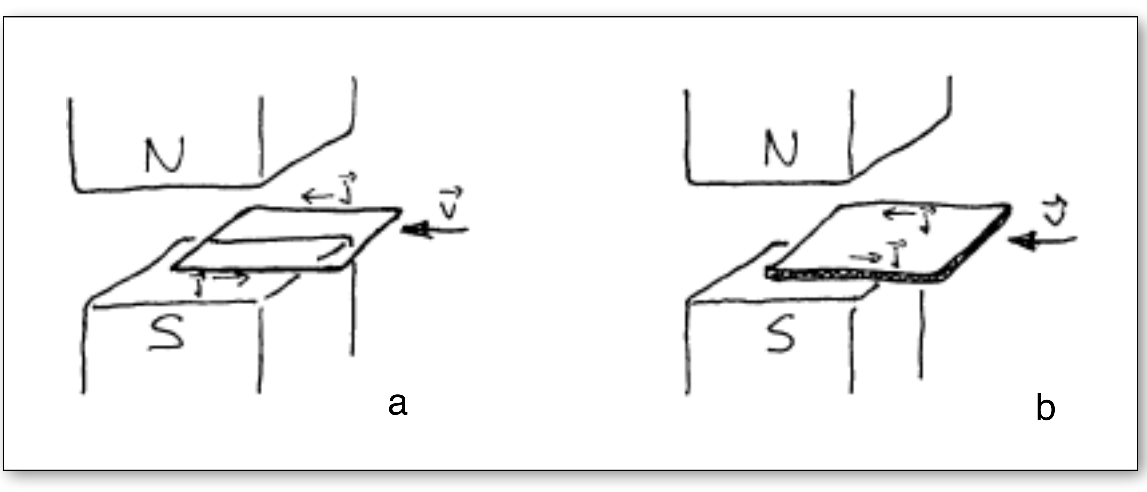

Fig. 9.10
Lorentz force on an induced current (a) in a conductor loop and (b) in a metal plate

Also, when a metal plate is moved into the magnetic field, an electric current is induced. Since the path of this current is not determined by the geometry of the conductor, it is called *eddy current*. Otherwise, however, the same applies as for the conductor loop: a Lorentz force acts on the metal plate, trying to prevent the movement.

10

Superconductivity

10.1 The superconducting phase

There are substances that differ significantly in their electromagnetic properties from those previously considered: superconductors. There are numerous variants of superconductivity. Here we limit ourselves to the consideration of a simple ideal case, which is realized by the so-called *type-I superconductors*.

Superconductivity is a state which is not irrevocably attached to the substance under consideration, but in which the substance is only under certain conditions, similar to the ferromagnetic state, or the solid, liquid or gaseous state. In particular, the temperature must not exceed a certain value; otherwise the substance loses its superconducting property. Just as a substance is only solid if the temperature is not higher than the melting temperature, or ferromagnetic as long as the temperature does not exceed the Curie temperature, a material is only superconductive if its temperature is not higher than the so-called *critical temperature* or *transition temperature*. Not all electrons that are responsible for electric conduction in the normal state participate in the phenomenon of superconductivity, but generally only some of them.

Substance	Critical temperature (K)
Hg	4.15
La	4.8
Nb	9.2
Ta	4.39
Tc	7.8
V	5.3

Table 10.1
Critical temperatures of some chemical elements

There are many substances that have a superconducting phase. Table 10.1 lists the transition temperatures for some chemical elements. There are also alloys and chemical compounds with much higher transition temperatures.

Here are the properties of the type-I superconductors:

(1) Their electric resistance is zero. Instead of the equation

$$\boldsymbol{j} = \sigma \cdot \boldsymbol{E}$$

the *1st London equation* applies

$$\Lambda \cdot \frac{d\boldsymbol{j}_S}{dt} = \boldsymbol{E}$$

$\boldsymbol{j}_S$ is the electric current density of the superconducting charge carriers, Λ measures a material property. The equation says that an electric field that is constant in time does not result in a constant electric current, but in a current whose intensity increases, a “uniformly accelerated current” so to speak.

(2) Superconductors expel magnetic fields from their interior. This property is called the *Meissner-Ochsenfeld effect*. The superconductor achieves the field displacement by electric currents flowing at its surface. The magnetic field of these currents is just such that it compensates the field inside the superconductor to zero. Just as an electric field penetrates into a body with free charge carriers to a certain depth if the charge carrier concentration is not very high, so the magnetic field penetrates into a thin surface layer of a superconductor. The thickness of this layer can be calculated using the *2nd London equation*

$$\Lambda \operatorname{rot} \boldsymbol{j}_S = -\boldsymbol{B}$$

Since Λ depends on the concentration of the superconducting charge carriers, the thickness of this layer also depends on it.

10.2 Ideal magnetic materials

Previously we had learnt about two ideal forms of magnetism: the magnetism of ideal soft magnetic materials and the magnetism of ideal hard magnetic materials. Superconductors are a third ideal magnetic material. Let us compare the three materials.

All three of them lose their magnetism at sufficiently high temperatures.

The soft magnetic material can be understood as the ideal form of the paramagnetic material, the superconductor as the ideal form of the diamagnetic material.

Soft magnetic and superconducting materials have in common that they do not allow a magnetic field inside them. However, they prevent the penetration of magnetic fields in different ways.

The soft magnet compensates a field that actually wants to be inside it by forming poles on its surface. The superconductor achieves the same result by allowing currents to flow on its surface. The consequence for the field strength distribution on the outside is very different in both cases.

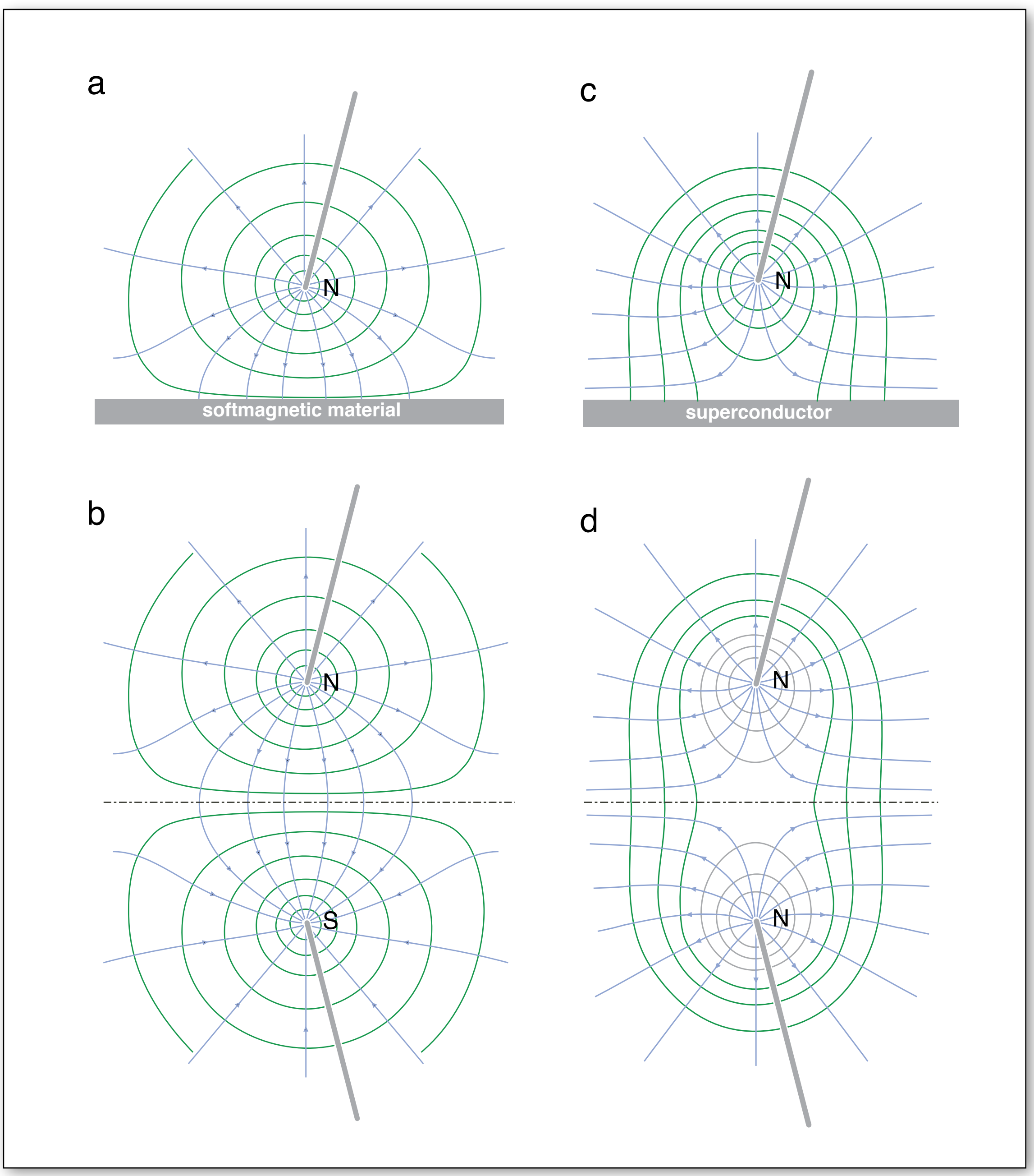

Fig. 10.1
A single magnetic pole is placed in front of the flat surface of a soft iron body (a). The field strength distribution remains the same if the soft iron is replaced by a mirror pole of opposite sign (b). A single magnetic pole is placed in front of the flat surface of a superconductor (c). The field strength distribution remains the same if the superconductor is replaced by a mirror pole of the same sign (d).

Fig. 10.1a shows the field strength distribution that results when a single magnetic pole approaches the flat surface of a soft iron body. The field strength distribution outside the iron is the same as when the iron is replaced by a magnetic “mirror charge”, a charge of the same magnitude as the first point charge, but of opposite sign, Fig. 10.1b.

Fig. 10.1c shows the magnetic pole in front of the flat surface of a superconductor. The currents at the surface cause the magnetic field lines to bend so that they are tangential to the surface. The same field strength distribution is obtained by replacing the superconductor by a mirror pole of the same magnitude and sign, Fig. 10.1d.

Fig. 10.2 shows two experiments analogous to each other. The soft iron piece in Fig. 10.2a has a temperature above its Curie temperature. It is therefore (almost) non-magnetic. The temperature is now lowered. As soon as the temperature decreases below the Curie temperature, the soft iron piece is attracted by the permanent magnet, Fig. 10.2b.

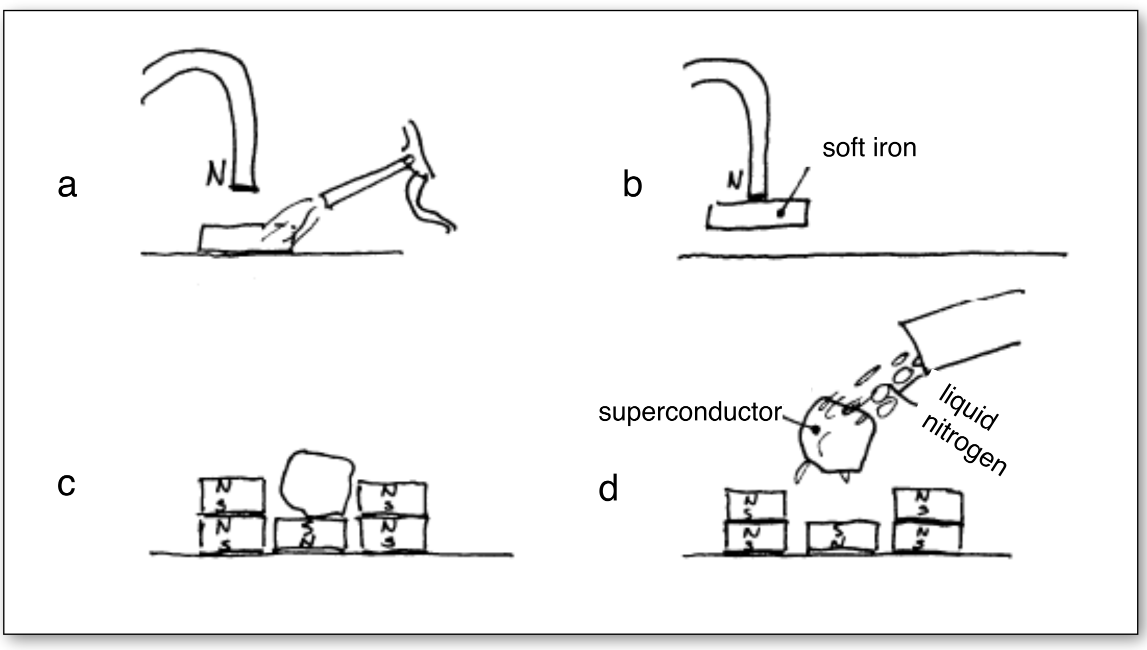

Fig. 10.2
The temperature of the soft iron piece is above the Curie temperature (a). The temperature of the soft iron piece has been lowered below the Curie temperature (b). The temperature of the superconductor is above the transition temperature (c). The temperature of the superconductor has been lowered below the transition temperature (d).

The temperature of the “superconductor” in Fig. 10.2c is higher than the transition temperature. Therefore, the material is not yet superconducting. Its temperature is now lowered. As soon as it has decreased below the transition temperature, the superconductor jumps up and remains suspended above the permanent magnet.

11

Energy current and momentum in the electromagnetic field

11.1 The energy current density

When a capacitor is charged, energy flows into the space between its plates. If an electromagnet is switched on, energy flows into the space between the poles. While an electric motor is running, energy flows into the rotor and then away through the motor shaft. With the generator it is the other way round. In all these cases, energy flows within the matter-free space: within the electromagnetic field. Since the field is unambiguously described by $\boldsymbol{E}$ and $\boldsymbol{H}$ (and $\boldsymbol{P}$ and $\boldsymbol{M}$), it must be possible to calculate the energy flow from these quantities. Since $\boldsymbol{E}$ and $\boldsymbol{H}$ are local quantities, it must be possible to calculate the local quantity energy flux density or energy current density $\boldsymbol{j}_E$.

We start with the well-known equation for the energy current

$$P = U \cdot I\,.$$

It is valid for the case that the energy flow is accompanied by a current of the electric charge. In principle, this equation cannot make any statement about the location where the energy flows, because it only contains "integral" quantities. To replace these by local quantities, we consider a geometrically particularly simple arrangement: the electric current flows through two flat, parallel conductors whose distance is small compared to the lateral extension, Fig. 11.1.

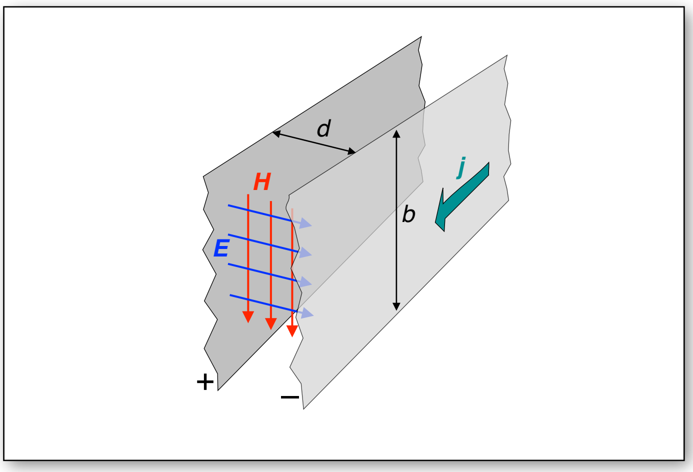

Fig. 11.1
To calculate the energy flux density

The relationship between voltage and electric field strength is obtained by applying equation (4.5):

$$U = |\boldsymbol{E}| \cdot d$$

and between electric current and magnetic field strength by applying equation (8.1):

$$I = |\boldsymbol{H}| \cdot b \tag{11.1}$$

We thus get

$$P = |\boldsymbol{E}| \cdot d \cdot |\boldsymbol{H}| \cdot b$$

and with $|\boldsymbol{j}_E| = P/(db)$

$$|\boldsymbol{j}_E| = |\boldsymbol{E}| \cdot |\boldsymbol{H}|\,.$$

The energy flows perpendicular to $\boldsymbol{E}$ and to $\boldsymbol{H}$, i.e. in the figure backwards. In our case it is

$$\boldsymbol{j}_E = \boldsymbol{E} \times \boldsymbol{H}. \tag{11.2}$$

This equation applies to every electromagnetic field, not only under the simple conditions of our calculation. The quantity $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ is also called the *Poynting vector*.

Equation (11.2) only contains local quantities. It not only makes a statement about whether and how much energy is flowing, it also says where it is flowing. $\boldsymbol{j}_E(\boldsymbol{r})$ is a vector field. The streamlines of this field illustrate the energy flow just as the $\boldsymbol{j}_Q$ streamlines illustrate the flow of the electric charge.

11.2 Examples of energy currents

Energy transmission with cables

A galvanic cell is connected to a resistor via two ideally conducting wires. The starting point of the energy is thus the cell, the terminating point is the resistor. We qualitatively describe the path of the energy between these two locations. Figure 11.2 shows the ***E*** and ***H*** fields. The j_E lines are perpendicular to ***E*** and perpendicular to ***H***. Three $\boldsymbol{j}_E$ lines are shown. It can be seen that their sources are located in the cell and their sinks in the resistor.

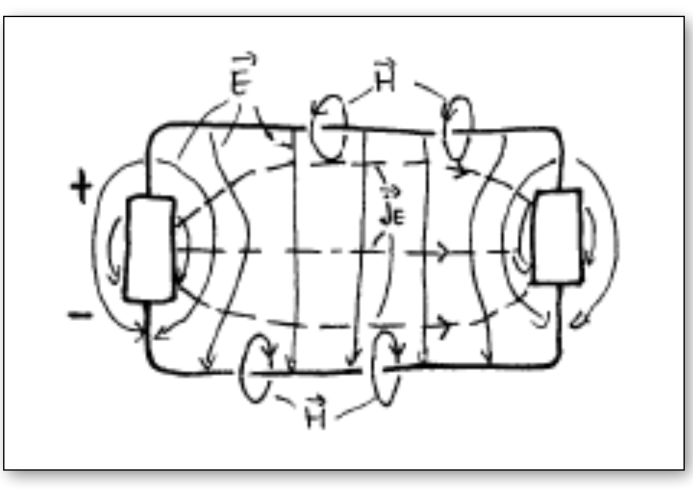

Fig. 11.2 Electric field strength, magnetic field strength and energy current density in a simple circuit: on the left the galvanic cell, on the right an ohmic resistor.

The incoming energy is dissipated in the resistor, i.e. entropy is generated. If the supply lines do not have zero resistance, the electric field strength has a component in the direction of the wire and some $\boldsymbol{j}_E$ lines terminate in the supply lines.

The energy flow when charging a capacitor

The electric current density decreases from the point where the supply lines of the capacitor plates are fixed, in such a way that it is zero at the edges of the plate, Fig. 11.3. The magnetic field therefore also decreases towards the edges, and thus also the energy current density $\boldsymbol{j}_E$. The energy current density therefore has sinks between the capacitor plates. This must be the case, because this is where the energy is deposited.

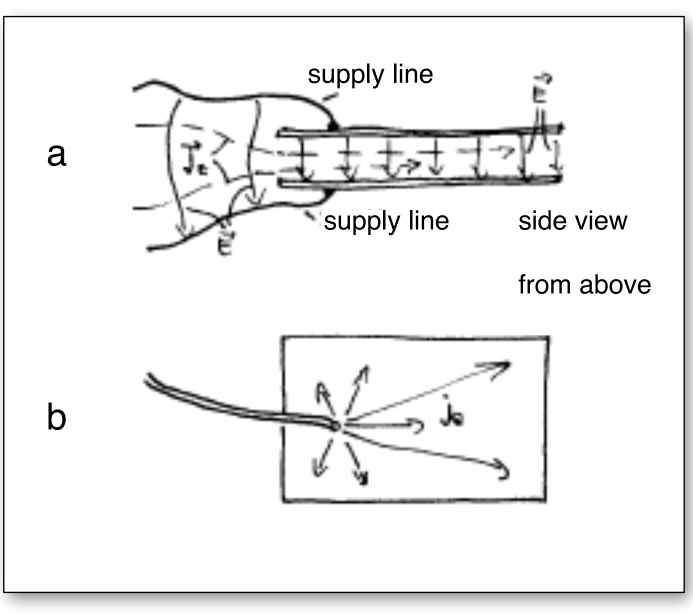

Fig. 11.3 (a) Electric field strength and energy current density; (b) charge current density

The moving capacitor

A capacitor with a charge per area $Q/A = \rho_A$ is moved parallel to its plate planes at velocity v, Fig. 11.4. We are interested in the $\boldsymbol{j}_E$ field. For this purpose, ***E*** and ***H*** must be calculated.

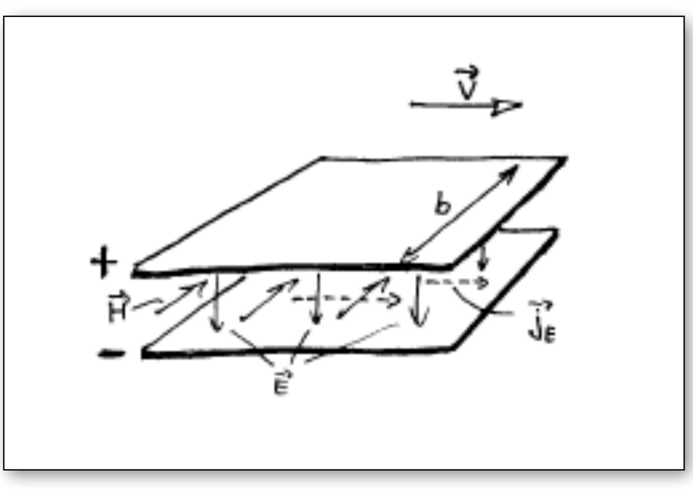

Fig. 11.4 A capacitor together with its energy is moved parallel to the plate direction.

Everywhere between the plates according to equation (4.9) we have

$$|\boldsymbol{E}| = \frac{\rho_A}{\varepsilon_0}$$

and according to equation (11.1)

$$|\boldsymbol{H}| = \frac{I}{b}$$

First we calculate I as a function of the known quantities. ρ_A is the charge per surface, $\rho_A b$ therefore the charge per length (in the direction of the motion). The current is charge per length times velocity:

$$I = \rho_A b\, v$$

We thus get

$$|\boldsymbol{H}| = \rho_A\, v$$

and finally with (11.2)

$$|\boldsymbol{j}_E| = \frac{\rho_A^2}{\varepsilon_0}|\boldsymbol{v}|$$

If we express the charge per unit area by the electric field strength, we obtain

$$|\boldsymbol{j}_E| = \varepsilon_0\, \boldsymbol{E}^2\, |\boldsymbol{v}|$$

This result is surprising, because if we assume that the field energy of the capacitor is simply shifted with the velocity ***v***, we get just half the value for the energy flux density, namely

$$\frac{\varepsilon_0}{2}\boldsymbol{E}^2|\boldsymbol{v}|$$

What happens to the other half? It flows mechanically back through the plates. The plates are under tensile stress and are moving. So an energy current

$$P = v\, F$$

is flowing through them.

The energy flow in the motor in the generator

The description depends on the choice of the reference frame. We choose the reference frame in which the magnet is at rest and the conductor is moving.

(a) *Generator*

The rod-shaped conductor c, Fig. 11.5, is moved to the right so that it slides on the two conductors a and b. Initially, the circuit is open. The Lorentz force acts on the free charge carriers in c, which we assume to be positively charged, and pushes them in the direction of conductor b, so that the electric potential of b increases in relation to that of conductor a. Between a and b there is now an electric field. The field lines run from b to a. We now introduce an energy receiver: a resistor whose resistance is large compared to the remaining resistance of the circuit. So, the voltage between a and b remains unchanged and an electric current is flowing through c against the electric field.

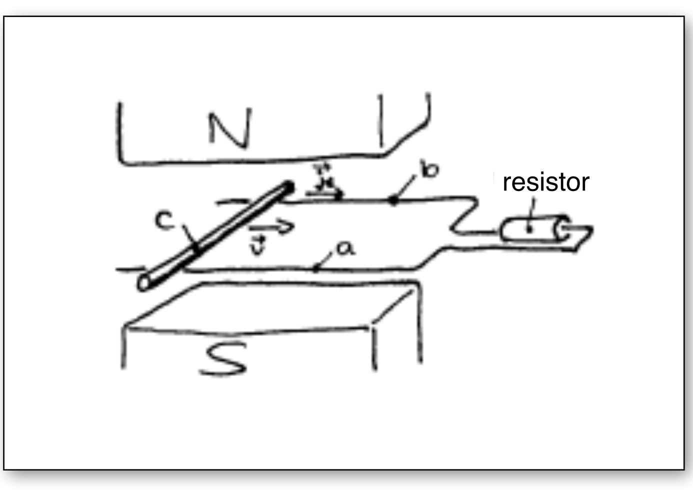

Fig. 11.5 Conductor c is moved to the right; it is sliding on conductors a and b.

Figure 11.6 shows qualitatively the ***E***, ***H*** and $\boldsymbol{j}_E$ fields. The energy flows out of the moving rod and through the electromagnetic field into the resistor.

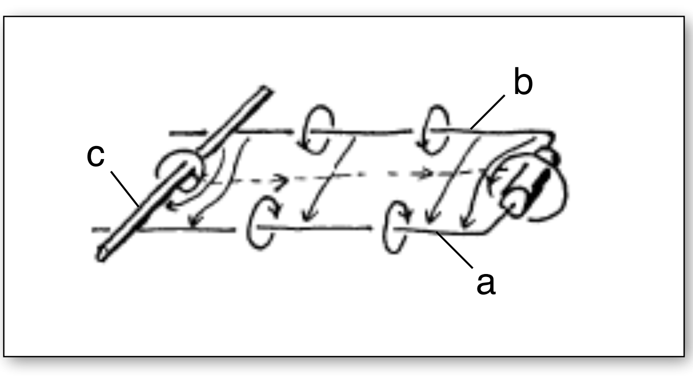

Fig. 11.6 ***E***, ***H*** and $\boldsymbol{j}_E$ field for the generator

(b) *Motor*

We replace the resistor with an electric energy source. The source is current stabilized. The current flows first through a, then through c and through b back to the source. A Lorentz force acts on the moving charge carriers in conductor c, the direction of which is parallel to the conductors a and b. Thus conductor c begins to move to the left. If c moves, then a Lorentz force acts on the charge carriers in c, which is directed parallel to c towards a. Thereby an electric field is generated, the field lines of which run from a to b.

The resulting $\boldsymbol{j}_E$ field lines run from the source to the conductor c, Fig. 11.7.

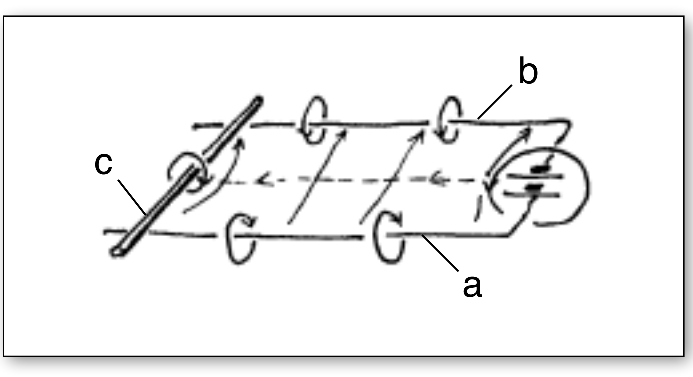

Fig. 11.7 ***E***, ***H*** and $\boldsymbol{j}_E$ field for the motor

11.3 Energy transmission with magnetic displacement currents

In the equation $P = U \cdot I$, instead of I, there should actually be the total electric current in Maxwell's sense, namely $I + \iint (\partial \boldsymbol{D}/\partial t) d\boldsymbol{A}$.

This can be seen from Fig. 11.8. The energy current flowing through the dotted surface is to be calculated. We consider the time shortly after the switch-on when the capacitors are not yet charged. The total potential difference of the source is then still at the resistor, and thus the total energy current coming from the source flows to the resistor. Its value is obtained with

$$P = U \cdot \iint \dot{\boldsymbol{D}}\, d\boldsymbol{A}$$

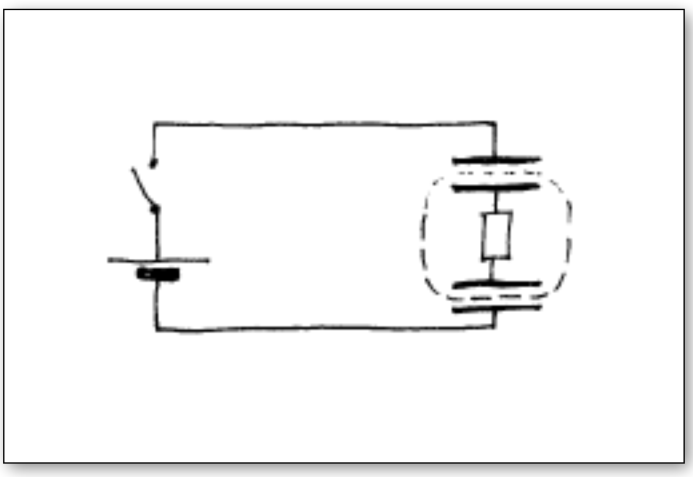

Fig. 11.8
Generalization of the equation $P = U \cdot I$

Energy can also be transmitted by means of *magnetic* displacement currents, and the analog equation applies:

$$P = U_m \cdot \iint \dot{\boldsymbol{B}}\, d\boldsymbol{A}$$

Figure 11.9a shows an example. A permanent magnet at A induces magnetic charges in a magnetic conductor (soft iron). If the permanent magnet is rotated, a magnetic alternating current flows in the soft iron. This causes the magnetic capacitor at B to be charged with an alternating sign. As a result, the permanent magnet at B is set in rotation.

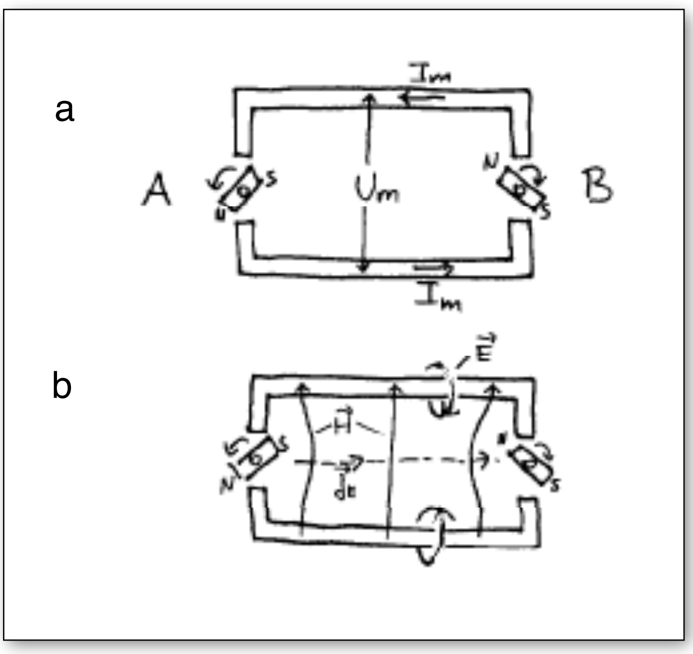

Fig. 11.9
Energy transmission with magnetic displacement currents.
(a) Magnetic potential difference and magnetic current;
(b) $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{j}_E$ field

The energy current transmitted by the soft iron conductors is given by the above equation. Each of the two magnetic conductors is at a spatially almost constant magnetic potential. U_m is the potential difference between them. Since

$$\mu_0 \partial \boldsymbol{H}/\partial t \ll \partial \boldsymbol{M}/\partial t$$

the equation simplifies to

$$P = U_m \cdot \iint \dot{\boldsymbol{M}}\, d\boldsymbol{A}$$

The integral extends over a cross-sectional surface of one of the two conductors.

Also this equation makes no statement about the path taken by the energy flow. This path is again given by $\boldsymbol{j}_E = \boldsymbol{E} \times \boldsymbol{H}$. There are closed electric field lines around each of the two conductors according to Maxwell's third equation. Magnetic field lines run from one conductor to the other. The $\boldsymbol{j}_E$ lines run from A to B, Fig. 11.9b.

If one of the two magnets is replaced by a coil through which an alternating current is flowing, Fig. 11.10a, an electric motor is obtained. If both magnets are replaced by coils, Fig. 11.10b, we get a transformer. In any case, the energy is transferred from left to right with the help of magnetic displacement currents, and the $\boldsymbol{j}_E$ current lines essentially run outside the magnetic conductors.

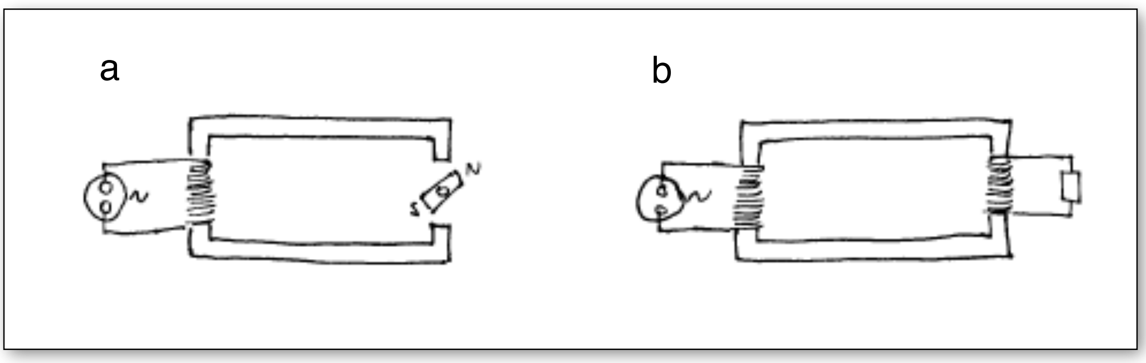

Fig. 11.10
Energy transmission using magnetic conductors; (a) electric motor; (b) transformer

11.4 Closed energy circuits within the electromagnetic field

A positive electric and a positive magnetic “point charge” are located next to each other, at rest, Fig. 11.11. Everywhere in the vicinity of the two charges is $\boldsymbol{E} \neq 0$ and $\boldsymbol{H} \neq 0$, and $\boldsymbol{E}$ and $\boldsymbol{H}$ are nowhere parallel to each other, apart from the line connecting the two charges. So there is everywhere $\boldsymbol{j}_E \neq 0$. The $\boldsymbol{j}_E$ field lines form concentric circles around the connecting line of the two charges. So the $\boldsymbol{j}_E$ field has no sources or sinks. This needs to be the case, because there is no other system whose energy decreases or increases.

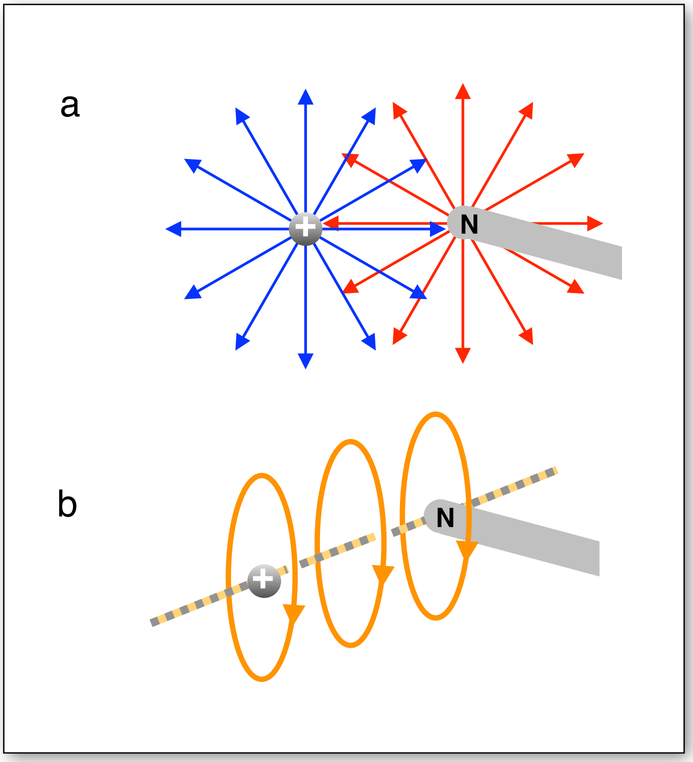

Fig. 11.11
Electric and magnetic point charge.
(a) $\boldsymbol{E}$ and $\boldsymbol{H}$ field in side view;
(b) $\boldsymbol{j}_E$ field in perspective

11.5 Momentum within the electromagnetic field

A charged capacitor is located in a homogeneous magnetic field, Fig. 11.12. The left plate is fixed, the right plate is moved to the right with constant velocity ***v***. As it moves, a force consisting of two components must be exerted on the right plate.

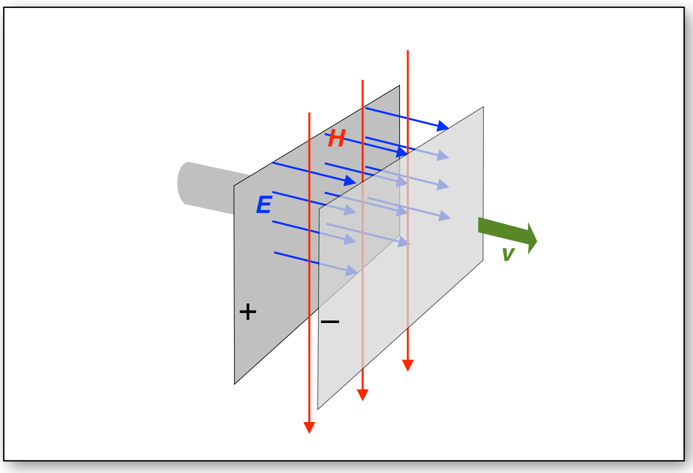

Fig. 11.12
As the right plate is moving to the right, the volume and the momentum of the electromagnetic field between the plates increases.

One component is parallel to the field strength vectors of the electric field. It is compensated by the electrostatic force exerted by the left plate on the right plate. The second component is parallel to the plate surface and points backwards in the figure. It balances the Lorentz force. In fact, a Lorentz force acts on the plate, because an electrically charged body is moved through a magnetic field. Here, however, there is no other body on which this force is exerted. Rather, it is the new field created during the movement that absorbs the corresponding momentum. So we conclude that the electromagnetic field between the plates has momentum. Since its volume increases linearly with time during the movement, its momentum also increases linearly with time. Let us calculate this momentum.

Since the relevant vectors are all parallel or perpendicular to each other, it is sufficient to calculate with the absolute values.

The momentum increase of the field in the time Δt is

$$\Delta p = F \Delta t$$

With the equation for the Lorentz force

$$F = Q \cdot v \cdot B$$

and

$$v = \frac{\Delta s}{\Delta t}$$

we get

$$p = Q \cdot B \cdot \frac{\Delta s}{\Delta t} \cdot \Delta t = Q \cdot B \cdot \Delta s$$

We replace

$$Q = \varepsilon_0 E A$$

and

$$B = \mu_0 H$$

and obtain

$$p = \varepsilon_0 E A \cdot \mu_0 H \cdot \Delta s = \varepsilon_0 E \cdot \mu_0 H \cdot \Delta V.$$

Here

$$\Delta V = A \cdot \Delta s$$

is the volume increase of the field. The momentum density thus becomes

$$\rho_p = \varepsilon_0 \mu_0 E H.$$

For vectors of an arbitrary orientation the cross product of the two field strengths is to be used

$$\rho_p = \varepsilon_0 \mu_0 (\boldsymbol{E} \times \boldsymbol{H}) \qquad (11.3)$$

The momentum density is therefore, except for a constant factor, equal to the energy current density.

11.6 Summary

We now have equations for the densities and current densities of both the energy and the momentum of the electromagnetic field. The energy density and the energy current density are given by equations (5.1), (7.1) and (11.2). The momentum current density is identical to the mechanical stress. It is given by equations (5.2), (5.4), (7.2) and (7.3). Finally, the momentum density is calculated by equation (11.3). Let us summarize these equations.

$$\rho_E = \frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 \qquad \rho_E = \frac{\mu\mu_0}{2}\boldsymbol{H}^2 \qquad \text{energy density}$$

$$\boldsymbol{j}_E = \boldsymbol{E} \times \boldsymbol{H} \qquad \text{energy current density}$$

$$\boldsymbol{\rho}_p = \varepsilon_0\mu_0(\boldsymbol{E} \times \boldsymbol{H}) \qquad \text{momentum density}$$

$$\sigma_{\parallel} = \frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 \qquad \sigma_{\parallel} = \frac{\mu\mu_0}{2}\boldsymbol{H}^2 \qquad \left\{\begin{array}{l}\text{mechanical stress parallel}\\ \text{to the field lines}\end{array}\right.$$

$$\sigma_{\perp} = -\frac{\varepsilon\varepsilon_0}{2}\boldsymbol{E}^2 \qquad \sigma_{\perp} = -\frac{\mu\mu_0}{2}\boldsymbol{H}^2 \qquad \left\{\begin{array}{l}\text{mechanical stress perpendicular}\\ \text{to the field lines}\end{array}\right.$$

Thus we have all components of the so-called *energy-momentum tensor* of the electromagnetic field, which plays an important role in the theory of relativity.

12

Structures within electromagnetism

12.1 The energy differential of the electromagnetic field

We consider a region of space of volume V in which there is a homogeneous electromagnetic field. From the energy density

$$\frac{E}{V} = \rho_E = \frac{\varepsilon_0}{2}\boldsymbol{E}^2 + \frac{\mu_0}{2}\boldsymbol{H}^2$$

we get

$$dE = V(\varepsilon_0 \boldsymbol{E} d\boldsymbol{E} + \mu_0 \boldsymbol{H} d\boldsymbol{H})$$

This relationship tells us by which amount dE the energy changes when the electric or magnetic field strength is changed.

We want to express this energy change by nonlocal quantities and first consider the magnetic term

$$dE = V\mu_0 \boldsymbol{H} d\boldsymbol{H}.$$

If dE is the change of the energy within a solenoid, we have

$$dE = V\mu_0 \boldsymbol{H} d\boldsymbol{H} = V\boldsymbol{H} d\boldsymbol{B} = V\frac{NI}{l}\frac{d\phi}{NA} = I d\phi$$

If, on the other hand, dE is the change of the energy in a "magnetic capacitor", then with $U_m/d = H$ and $Q_m/A = \mu_0 H$ we get

$$dE = V\frac{U_\mathrm{m}}{d}\frac{dQ_\mathrm{m}}{A} = U_\mathrm{m} dQ_\mathrm{m}$$

So in general we can write

$$V\mu_0 \boldsymbol{H} d\boldsymbol{H} = I\, d\Phi + U_\mathrm{m} dQ_\mathrm{m}.$$

Similarly, the local expression for the change of the energy of the electric field can be replaced by a nonlocal expression

$$V\varepsilon_0 \boldsymbol{E} d\boldsymbol{E} = I_\mathrm{m} d\Phi_\mathrm{el} + U dQ\,,$$

where Φ_el is the flux of D.

Thus, together we get

$$dE = I\, d\Phi + U_\mathrm{m} dQ_\mathrm{m} + I_\mathrm{m} d\Phi_\mathrm{el} + U dQ\,.$$

We call a relationship of this kind an *energy differential*.

Now two terms in this equation are not important:

$U_\mathrm{m} dQ_\mathrm{m}$ is rarely important because magnetic capacitors are not used as technical components.

$I_\mathrm{m} d\Phi_\mathrm{el}$ is practically not realizable because it is difficult to realize magnetic currents.

Therefore, in practical applications usually only the expression

$$dE = I\, d\Phi + U dQ\,.$$

is relevant.

It describes, for example, the energy change in a solenoid (1st term) or in an electrical capacitor (2nd term). This expression no longer has the symmetry between electric and magnetic quantities that we had considered previously, because the analog $I_\mathrm{m} d\Phi_\mathrm{el}$ to $I d\Phi$ and the analog $U_\mathrm{m} dQ_\mathrm{m}$ to $U dQ$ is missing.

However, between $I d\Phi$ and $U dQ$ there is a symmetry of another kind. This is exactly the symmetry that was exploited in the Physics I course and that we had called "dualism" (and that has its analogue in mechanics). In this mapping, not only quantities are to be replaced in an electric circuit, but also topological relationships, such as "junction ⟷ mesh", "parallel ⟷ series". Finally, also some quantities are to be replaced by their reciprocal value, namely "resistance ⟷ conductance", and from this follow the replacements "short circuit ⟷ open circuit", and "conductor ⟷ insulator".

While the consideration of the symmetry, in which $\boldsymbol{E}$ and $\boldsymbol{H}$ correspond to one another, is particularly helpful for understanding the physical fundamentals, the dualism, in which U and I correspond to each other, is useful for technical applications.

12.2 The analogy between charge density and current density

Another analogy is very useful in theoretical physics. It is based on a comparison between ρ_Q and $\boldsymbol{j}_Q$. Just as ρ_Q is the source of the $\boldsymbol{E}$ field, $\boldsymbol{j}_Q$ is the source of the $\boldsymbol{B}$ field. So here, $\boldsymbol{E}$ and $\boldsymbol{B}$ correspond to each other. In analogy to the electrical potential Φ the vector potential $\boldsymbol{A}$ is defined:

$$\boldsymbol{B} = \text{curl } \boldsymbol{A} \,.$$

In analogy to the Poisson equation

$$\Delta\ \Phi = -\ \rho/\varepsilon_0$$

it is (without proof)

$$\Delta \boldsymbol{A} = -\ \mu_0 \boldsymbol{j} \,.$$

13

Electromagnetic oscillations – alternating currents

13.1 The mesh rule in a circuit with inductance

In circuits in which a direct current is flowing, or in which induced voltages can be neglected against other voltages, $\int E\mathrm{d}r$ is zero on any closed path, particularly along any "mesh". A potential can be defined and the mesh rule (equation (2.7) applies:

$$\sum_i U_i = 0$$

However, the mesh rule no longer applies when inductances are present in the circuit. For the circuit shown in Fig. 13.1 we have to write (Equation (8.9)):

$$\oint_{\text{circuit}} \boldsymbol{E}\,d\boldsymbol{r} = -L\dot{I} \neq 0$$

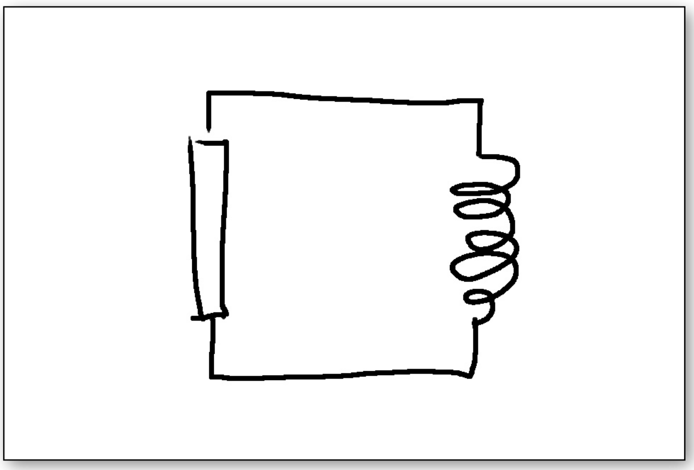

Fig. 13.1
Circuit with inductance

But since the mesh rule is very convenient, one can save it with the help of a trick. In the circuit of Fig. 13.1 the full contribution to the integral $\int \boldsymbol{E}d\boldsymbol{r}$ is due to the resistor. It is

$$-L\dot{I} = RI$$

or

$$RI + L\dot{I} = 0$$

One now acts as if there is a potential in the circuit; one acts as if the negative of ($-L\partial I/\partial t$), i.e. $L\partial I/\partial t$, is a voltage that is due to an electric field in the coil. So the equation $RI + L\partial I/\partial t = 0$ can be interpreted as follows: There is a potential drop $U_{\text{res}} = R\,I$ at the resistor, and a potential drop $U_{\text{coil}} = L\partial I/\partial t$ at the coil.

With this agreement we get

$$\sum_i U_i = 0 \qquad \begin{cases} \text{generalized mesh rule is valid} \\ \text{if we set } U_{\text{coil}} = +L\dot{I} \end{cases}$$

13.2 LC circuits

Figure 13.2 shows an electrical *resonant circuit* or *LC circuit*.

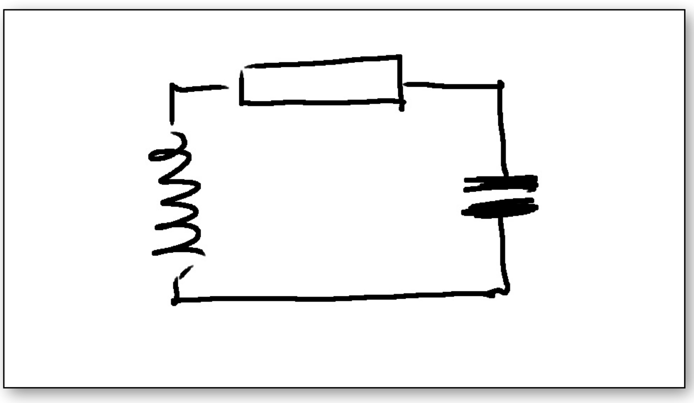

Fig. 13.2
LC circuit (series circuit)

For the mathematical treatment we use the generalized mesh rule:

$$L\dot{I}+RI+\frac{Q}{C}=0$$

We derive the equation with respect to time and use $dQ/dt = I$ (the relationship between a capacitor's charge and the charging current):

$$L\ddot{I}+R\dot{I}+\frac{I}{C}=0$$

Finally, we divide by L:

$$\ddot{I}+\frac{R}{L}\dot{I}+\frac{I}{LC}=0$$

This is a differential equation for damped oscillations. The solution follows the known recipe.

For $R = 0$, the oscillations are undamped, and we get

$$I=I_0\sin\omega t \quad \text{with} \quad \omega=\frac{1}{\sqrt{LC}}$$

ω is called the *angular frequency*. With $U = L\partial I/\partial t$ we obtain

$$U=U_0\cos\omega t \quad \text{with} \quad U_0=\sqrt{\frac{L}{C}}I_0$$

With the relation

$$E_{cap}=\frac{C}{2}U^2$$

which is obtained from (4.10) and (4.27) and with equation (8.11)

$$E_{sol}=\frac{L}{2}I^2$$

we get

$$E_{cap}=\frac{C}{2}U_0^2\cos^2\omega t=\frac{L}{2}I_0^2\cos^2\omega t=\frac{L}{2}I_0^2\frac{1}{2}(1-\cos 2\omega t)$$

and

$$E_{sol}=\frac{L}{2}I_0^2\sin^2\omega t=\frac{L}{2}I_0^2\frac{1}{2}(1+\cos 2\omega t)$$

We see that the sum

$$E_{cap}+E_{sol}=\frac{L}{2}I_0^2+\frac{C}{2}U_0^2$$

is constant in time.

The energy is flowing back and forth between the solenoid and the capacitor with the frequency 2ω. This can also be seen in the energy flow:

$$P=U\cdot I=I_0\cdot U_0\cdot\sin\omega t\cdot\cos\omega t=I_0U_0\frac{1}{2}\sin 2\omega t$$

For $R \neq 0$ we have

$$\omega=\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

The amplitudes of the electric current and the voltage decay according to

$$e^{-\frac{R}{2L}t}$$

and that of the energy flow according to

$$e^{-\frac{R}{L}t}$$

Figure 13.3 shows the resonant circuit obtained by "dual translation": $C \leftrightarrow L$, $R \leftrightarrow 1/R$, parallel $\leftrightarrow$ in series.

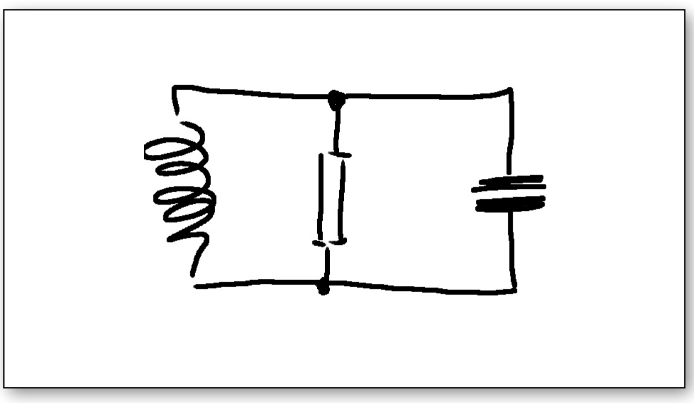

Fig. 13.3
LC circuit (parallel circuit)

Instead of calculating this circuit with the junction rule (which is dual to the mesh rule) we can simply translate the results of the previous calculation. The frequency, for example, becomes:

$$\omega=\sqrt{\frac{1}{LC}-\frac{1}{4R^2C^2}}$$

13.3 Alternating current and alternating voltage

If the current (or the voltage) changes in time according to a sine function, i.e. if $I(t) = I_0 \sin(\omega t + \phi)$ (or $U(t) = U_0 \sin(\omega t + \phi)$), we speak of an alternating current (or an alternating voltage). The alternating current (AC) has a great technical importance for various reasons.

High frequency alternating current is used as a data carrier and low frequency alternating current (usually 50 or 60 Hz) as an energy carrier.

The advantages of AC are largely due to the fact that the laws described by Maxwell's 3rd and 4th equations become relevant, because these equations contain time derivatives. Since the time derivative of a harmonic function is again a harmonic function, the relationships between the various electrical quantities are particularly simple.

If two points A and B of an electrical network are connected to each other by any arrangement of ohmic resistors, capacitors and solenoids, Figure 13.4, and if there is an alternating voltage

$$U = U_0 \sin \omega t$$

between A and B, an alternating current of the same frequency

$$I = I_0 \sin(\omega t + \phi).$$

will flow in the network.

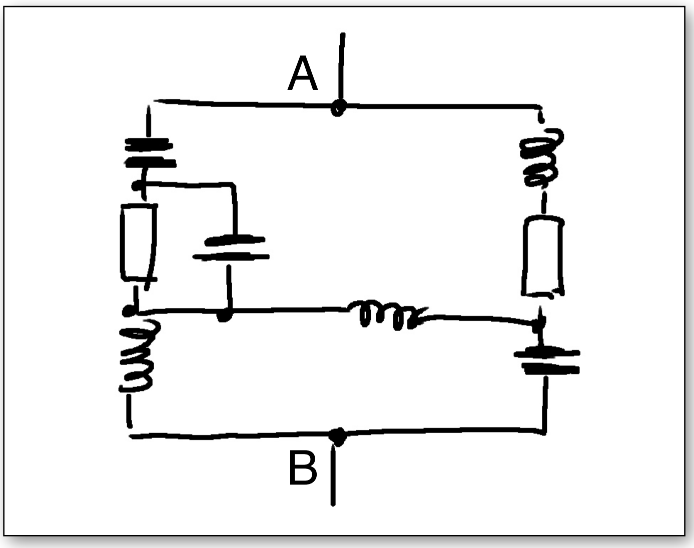

Fig. 13.4
Electrical network with solenoids, capacitors and resistors

In general, the current is not in phase with the voltage. We will see that I_0 is proportional to U_0.

In the following we investigate the relationship between $U(t)$, $I(t)$ and $P(t)$ for

- an ohmic resistor
- a capacitor
- a solenoid
- Resistor, capacitor and solenoid connected in series
- Resistor, capacitor and solenoid connected in parallel.

13.4 Resistance and reactance

(a) Ohmic Resistor

Ohm's law is

$$U = RI\,.$$

The alternating voltage $U(t) = U_0 \sin \omega t$ is applied between the ends of the resistor. The consequence is an alternating electric current

$$I(t) = \frac{U_0}{R}\sin\omega t = I_0 \sin\omega t$$

where $I_0 = U_0/R$.

The current of the energy that is dissipated in the resistor is

$$P(t) = U(t)\cdot I(t) = U_0 I_0 \sin^2\omega t = \frac{U_0^2}{R}\sin^2\omega t$$

The time average of $P(t)$ is

$$\overline{P(t)} = \frac{1}{2}\frac{U_0^2}{R} = \frac{1}{2}U_0 I_0$$

What value U_{rms} would have to have a DC voltage that causes the same dissipation in the resistor?

$$\frac{U_{rms}^2}{R} = \frac{1}{2}\frac{U_0^2}{R} \quad\Rightarrow\quad U_{rms} = \frac{U_0}{\sqrt{2}}$$

The corresponding electric current would be

$$I_{rms} = \frac{U_{rms}}{R} = \frac{U_0}{\sqrt{2}R} = \frac{I_0}{\sqrt{2}}$$

U_{rms} and I_{rms} are the *root mean square values* of the alternating voltage or alternating current. Alternating current and alternating voltage measuring instruments are calibrated in rms values. Therefore, the measured values can be used to calculate the average energy current as the product of (rms) voltage and (rms) current by using the formula $P = UI$.

The 220 V of the socket also represent the rms voltage.

(b) Capacitor

For a capacitor, equation (4.10) applies:

$$Q = CU\,.$$

Derivation with respect to time and insertion of I for dQ/dt results in

$$I = C\dot{U}$$

We apply the voltage $U(t) = U_0 \sin \omega t$ and get an electric current

$$I(t) = \omega C U_0 \cos\omega t = \omega C U_0 \sin\left(\omega t + \frac{\pi}{2}\right) = I_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

where $I_0 = \omega C U_0$.

If the voltage on the capacitor changes harmonically, the current through the capacitor also varies harmonically. However, $I(t)$ is phase-shifted by $\pi/2$ against $U(t)$: $U(t)$ lags behind $I(t)$ by $\pi/2$.

The quotient

$$X_C = \frac{U_0}{I_0} = \frac{1}{\omega C}$$

is the *reactance* of the capacitor.

We still calculate the energy current flowing into the capacitor:

$$P(t) = U(t)\cdot I(t) = \omega C U_0^2 \sin\omega t \cos\omega t = \omega C \frac{U_0^2}{2}\sin 2\omega t$$

Energy is alternately flowing into and out of the capacitor. The time average of the energy flow is zero.

(c) Solenoid

For the solenoid the equation

$$U = L\dot{I}$$

applies. With $I(t) = I_0 \sin \omega t$ we get

$$U(t) = \omega L I_0 \cos\omega t = \omega L I_0 \sin\left(\omega t + \frac{\pi}{2}\right) = U_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

If the current flowing through the solenoid changes harmonically, the voltage between its connections also varies harmonically. However, $U(t)$ is phase-shifted by $\pi/2$ against $I(t)$: $I(t)$ lags behind $U(t)$ by $\pi/2$.

The *reactance* of the solenoid is

$$X_L = \frac{U_0}{I_0} = \omega L$$

The energy current flowing into the solenoid is:

$$P(t) = U(t)\ I(t) = \omega L I_0^2 \sin\ \omega t\ \cos\ \omega t$$

Energy is alternately flowing into and out of the solenoid. The time average of the energy current is zero.

(d) Ohmic resistor, solenoid and capacitor in series, Fig. 13.5

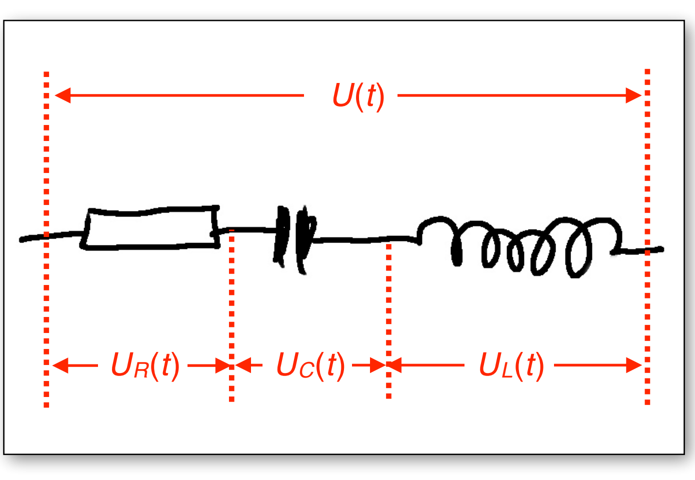

Fig. 13.5
Ohmic resistor, solenoid and capacitor in series

The electric current is the same in all three elements:

$$I(t) = I_0 \sin \omega t\,.$$

We are interested in the voltage

$$U(t) = U_R(t) + U_C(t) + U_L(t).$$

Since $U(t)$ is the sum of three harmonic voltages of the same frequency, it must have the following form

$$U(t) = U_0 \sin(\omega t - \Phi)\,.$$

We want to calculate U_0 and Φ.

$$U_0 \sin(\omega t - \phi) = U_R(t) + U_C(t) + U_L(t)$$

$$= RI_0 \sin\omega t - \frac{1}{\omega C} I_0 \cos\omega t + \omega L I_0 \cos\omega t$$

$$U_0(\sin\omega t \cos\phi - \cos\omega t \sin\phi) = I_0\left[R\sin\omega t + \left(\omega L - \frac{1}{\omega C}\right)\cos\omega t\right]$$

A comparison of the factors in front of $\sin\omega t$ and $\cos\omega t$ provides

$$U_0 \cos\Phi = I_0 R$$

and

$$U_0 \sin\phi = I_0\left(\frac{1}{\omega C} - \omega L\right)$$

Thus we get

$$\tan\phi = \frac{\frac{1}{\omega C} - \omega L}{R}$$

and

$$U_0^2 = I_0^2\left[R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right]$$

and finally

$$U_0 = I_0\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}$$

Sometimes the square root after I_0 is abbreviated

$$X = \sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}$$

The total energy current into the arrangement turns out to be

$$P(t) = U_0 \sin(\omega t - \Phi) I_0 \sin\omega t = U_0 I_0 \sin\omega t\,(\sin\omega t \cos\Phi - \cos\omega t \sin\Phi)$$

Its time average is

$$\overline{P(t)} = \frac{U_0 I_0}{2}\cos\phi = U_{eff} I_{eff}\cos\phi$$

The factor $\cos\Phi$ tells us which fraction of the product $U_{rms} \cdot I_{rms}$ is dissipated within the circuit.

(e) Ohmic resistor, solenoid and capacitor in parallel, Fig. 13.6

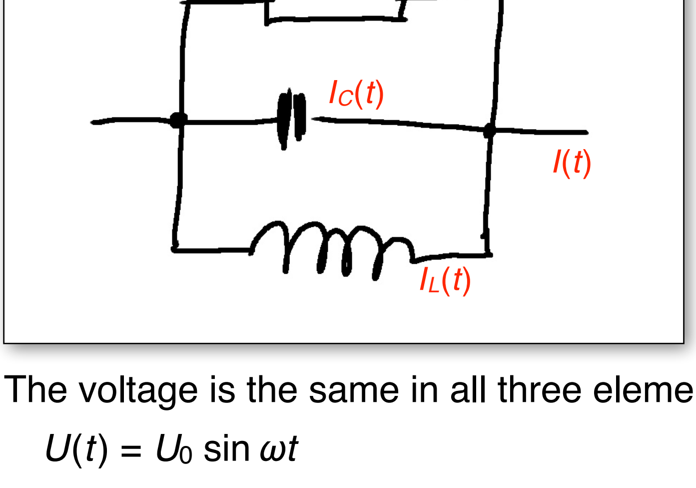

Fig. 13.6
Ohmic resistor, solenoid and capacitor in parallel

The voltage is the same in all three elements:

$$U(t) = U_0 \sin \omega t$$

We are interested in the electric current

$$I(t) = I_R(t) + I_C(t) + I_L(t).$$

Since $I(t)$ is the sum of three harmonic currents of the same frequency, it must have the following form

$$I(t) = I_0 \sin(\omega t - \Phi)\,.$$

We want to calculate I_0 und Φ.

$$I_0 \sin(\omega t - \phi) = I_R(t) + I_C(t) + I_L(t) = \frac{U_0}{R}\sin\omega t + \omega C U_0 \cos\omega t - \frac{U_0}{\omega L}\cos\omega t$$

$$I_0(\sin\omega t \cos\phi - \cos\omega t \sin\phi) = U_0\left[\frac{1}{R}\sin\omega t + \left(\omega C - \frac{1}{\omega L}\right)\cos\omega t\right]$$

A comparison of the factors in front of $\sin\omega t$ and $\cos\omega t$ provides

$$I_0 \cos\phi = \frac{U_0}{R}$$

and

$$I_0 \sin\phi = U_0\left(\frac{1}{\omega L} - \omega C\right)$$

Thus we get

$$\tan\phi = \left(\frac{1}{\omega L} - \omega C\right)R$$

and

$$I_0^2 = U_0^2\left[\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2\right]$$

and finally

$$I_0 = U_0\sqrt{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}$$

We again abbreviate

$$X = \frac{1}{\sqrt{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}}$$

We could have avoided this calculation if we dually translated the results of section (d).

The time average of the energy current is again

$$\overline{P(t)} = \frac{U_0 I_0}{2}\cos\phi = U_{eff} I_{eff}\cos\phi$$

13.5 The description of alternating current networks with complex quantities

If the frequency of an alternating current circuit is fixed once and for all, only two values are needed to characterize an alternating current: the amplitude and the phase. The same applies to AC voltages. The description by the time function

$I_0 \cos(\omega t + \Phi)$

is unnecessarily cumbersome. Now there is a calculus which operates only with these two numbers, namely amplitude and phase. This representation of alternating currents and alternating voltages is using complex numbers, and it is particularly simple.

$I(t) = I_0 \sin(\omega t + \Phi)$ is represented by $\underline{I} = I_0 e^{i\Phi}$

$U(t) = U_0 \sin(\omega t + \Phi)$ is represented by $\underline{U} = U_0 e^{i\Phi}$

We thus have

$I = \mathrm{Re}\left[\underline{I} e^{i\omega t}\right]$ and $U = \mathrm{Re}\left[\underline{U} e^{i\omega t}\right]$

Now we know that with the addition of two complex numbers, real parts and imaginary parts add up individually. Thus we get a simple method to graphically add currents (or voltages) which are phase-shifted against each other. We represent the currents (or voltages) in the complex plane by vector arrows and add them according to the rules of vector addition, Fig. 13.7.

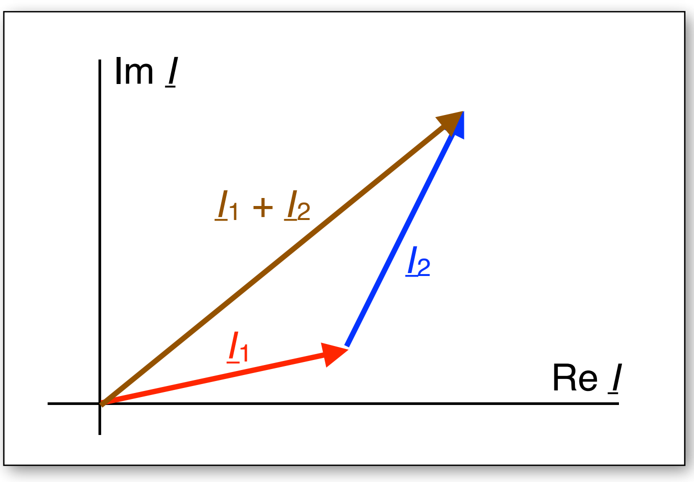

Fig. 13.7 Representation of the sum of two alternating currents in the complex plane

The representation of the time derivative of such a variable in complex notation is also very convenient. We consider

$I(t) = I_0 \cos(\omega t + \Phi)$

and thus

$\underline{I} = I_0 e^{i\Phi}$

The time derivative of I is

$$\dot{I}(t) = -\omega I_0 \sin(\omega t + \phi) = \omega I_0 \cos\left(\omega t + \phi + \frac{\pi}{2}\right)$$

In complex notation the time derivative becomes

$$\dot{\underline{I}} = \omega I_0 e^{i\left(\phi + \frac{\pi}{2}\right)} = \omega e^{i\frac{\pi}{2}} \underline{I} = i\omega \underline{I}$$

So we get the time derivative by multiplication with $i\omega$.

The relationship between current and voltage can also be described with complex numbers. We define the complex electrical resistance or *impedance* Z:

$$Z = \frac{\underline{U}}{\underline{I}}$$

For an ohmic resistor we get

$Z_R = R\,.$

For a capacitor at which the voltage $U = U_0 \cos\omega t$ is applied we have

$\underline{U} = U_0$

and

$$\underline{I} = C\dot{\underline{U}} = C\, i\omega \underline{U} = i\omega C U_0$$

thus

$$Z_C = \frac{1}{i\omega C}$$

For a solenoid through which a current of strength $I = I_0 \cos\omega t$ flows, the following applies we have

$\underline{I} = I_0$

and

$$\underline{U} = L\dot{\underline{I}} = L i\omega \underline{I} = i\omega L I_0$$

thus

$Z_L = i\omega L$

If several elements 1, 2, 3 … with the impedances Z_1, Z_2, Z_3, ... are connected in series, the complex total voltage is

$$\underline{U} = \underline{U}_1 + \underline{U}_2 + \underline{U}_3 + \ldots = Z_1\underline{I} + Z_2\underline{I} + Z_3\underline{I} + \ldots = (Z_1 + Z_2 + Z_3 + \ldots)\,\underline{I}$$

Therefore the total impedance is

$$Z = \frac{\underline{U}}{\underline{I}} = Z_1 + Z_2 + Z_3 + \ldots$$

Thus, the impedances add up when connecting in series. They can therefore also be added together in the complex number plane. For example, the impedance of the arrangement of Figure 13.8:

$$Z = \frac{1}{i\omega C} + R + i\omega L = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

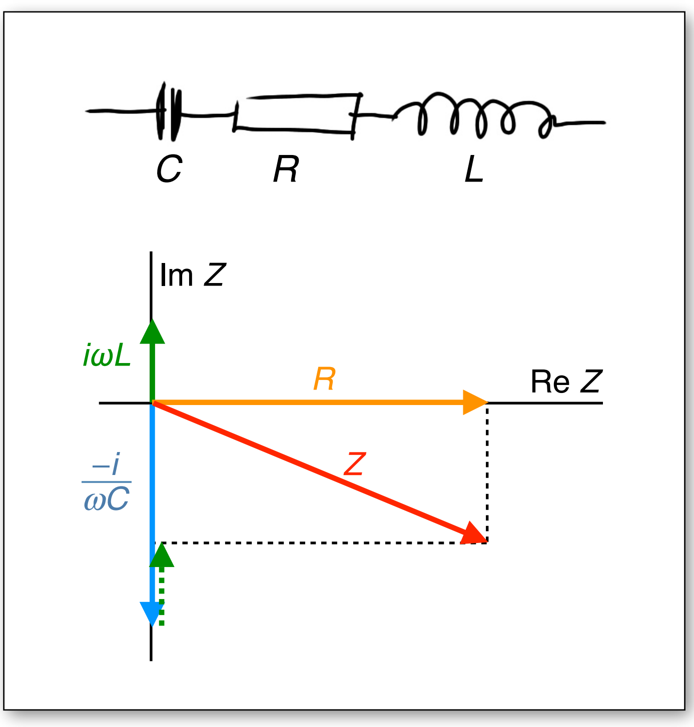

Fig. 13.8 Adding impedances in the complex number plane

The absolute value of the impedance of an arrangement of ohmic resistors, capacitors and solenoids is equal to the quantity X that was introduced in the preceding section. For the three devices connected in series we have

$$|Z| = \left|R + i\left(\omega L - \frac{1}{\omega C}\right)\right| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = X$$

13.6 The transformer

We use the complex representation of AC voltages, AC currents and AC resistances to calculate the transformer. A transformer consists of two coils wound on a common iron core, Fig. 13.9, so that the total magnetic flux Φ (flux of the $\boldsymbol{B}$ field) of one coil passes through the other coil, and vice versa.

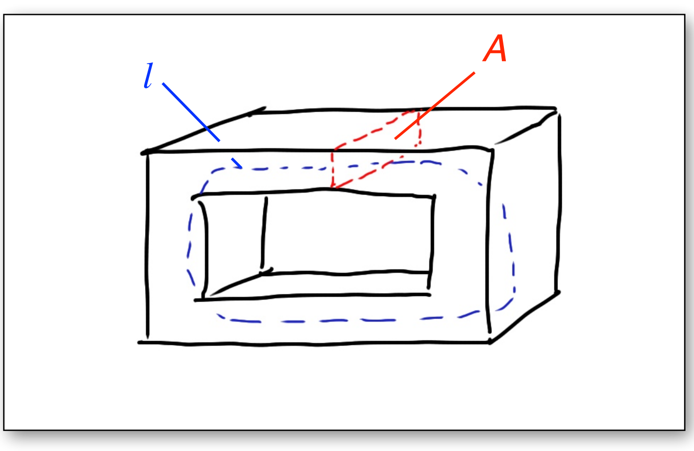

Fig. 13.9
Iron core of a transformer

The inductance of the two individual coils is (equation (8.10)):

$$L_1 = \mu\mu_0 \frac{N_1^2}{l} A \tag{13.1}$$

$$L_2 = \mu\mu_0 \frac{N_2^2}{l} A \tag{13.2}$$

If a current I_1 flows through coil 1, then a flux Φ_2 also results through coil 2:

$$\phi_2 = N_2 B_1 A = N_2 \mu\mu_0 \frac{N_1}{l} I_1 A = L_{12} I_1$$

The quantity

$$L_{12} = \mu\mu_0 \frac{N_1 N_2}{l} A \tag{13.3}$$

is called *mutual inductance*.

If a current I_2 flows through coil 2, a magnetic flux is also induced in coil 1:

$$\Phi_1 = L_{21} I_2$$

with

$$L_{21} = L_{12}$$

The induced voltages are

in coil 1 (primary coil) $\quad L_1\dot{I}_1 - L_{12}\dot{I}_2$

in coil 2 (secondary coil) $\quad L_2\dot{I}_2 - L_{12}\dot{I}_1$

In addition to an inductive resistance, the coils also have an ohmic resistance R_1 or R_2 respectively. This resistance behaves as if it were connected in series with the inductive resistance (why?), Fig. 13.10.

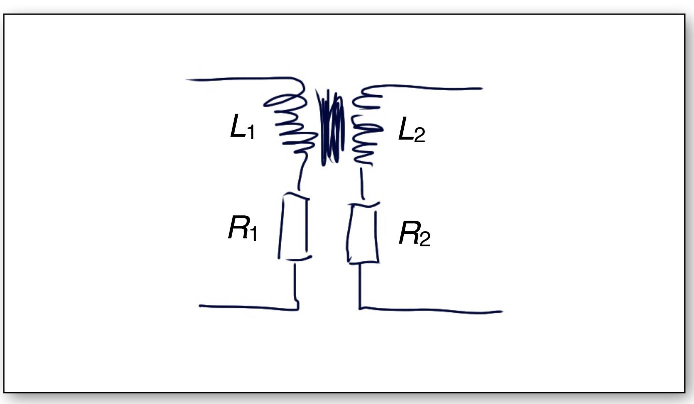

Fig. 13.10
Equivalent circuit of the transformer

Therefore for the complex voltage at the two coils we obtain

$$\underline{U}_1 = (R_1 + i\omega L_1)\underline{I}_1 - i\omega L_{12}\underline{I}_2 \tag{13.4a}$$

$$\underline{U}_2 = -i\omega L_{12}\underline{I}_1 + (R_2 + i\omega L_2)\underline{I}_2 \tag{13.4b}$$

The primary coil of the transformer is now connected to a source which supplies an alternating voltage $\underline{U}_1$ of constant amplitude (and of course also of constant frequency). An ohmic load of resistance R is connected to the secondary coil, Fig. 13. 11, so that

$$\underline{U}_2 = -R\,\underline{I}_2 \tag{13.5}$$

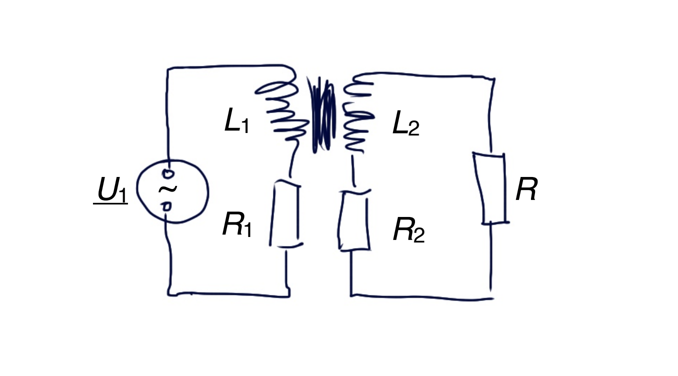

Fig. 13.11
The transformer is connected to a source of voltage $\underline{U}_1$ and a load of resistance R.

We are looking for the relationship between $\underline{U}_1$ and $\underline{U}_2$ and between $\underline{I}_1$ and $\underline{I}_2$.

We replace $\underline{U}_2$ in (13.4b) with (13.5):

$$0 = -i\omega L_{12}\underline{I}_1 + (R + R_2 + i\omega L_2)\underline{I}_2$$

and obtain

$$\frac{\underline{I}_1}{\underline{I}_2} = \frac{R + R_2 + i\omega L_2}{i\omega L_{12}} \tag{13.6}$$

We now ask for the quotient I_1/I_2 of the amplitudes of the currents. This is equal to $|\underline{I}_1/\underline{I}_2|$, i.e. equal to the magnitude of (13.6):

$$\frac{I_1}{I_2} = \left|\frac{R + R_2 + i\omega L_2}{i\omega L_{12}}\right| = \left|\frac{L_2}{L_{12}} - i\frac{R + R_2}{\omega L_{12}}\right| = \frac{1}{\omega L_{12}}\sqrt{\omega^2 L_2^{\,2} + (R + R_2)^2}$$

Usually transformers are constructed in such a way that at the frequency used the following applies:

$$R_1 << \omega L_1 \quad \text{and} \quad R_2 << \omega L_2 \tag{13.7}$$

(This is achieved by making the number of turns sufficiently large. The resistance R goes linear, the inductance L however quadratic with the number of turns.)

So we get

$$\frac{I_1}{I_2} \approx \frac{1}{\omega L_{12}}\sqrt{\omega^2 L_2^{\,2} + R^2}$$

If the load resistance is also small compared to the inductive resistance of the secondary coil, i.e.

$$R << \omega L_2,$$

we get

$$\frac{I_1}{I_2} = \frac{L_2}{L_{12}} = \frac{N_2}{N_1} \tag{13.8}$$

We now eliminate $\underline{I}_1$ and $\underline{I}_2$ in (13.4a) using (13.5) and (13.6):

$$\underline{U}_1 = (R_1 + i\omega L_1)\frac{R + R_2 + i\omega L_2}{i\omega L_{12}}\underline{I}_2 - i\omega L_{12}\underline{I}_2$$

$$= \left[(R_1 + i\omega L_1)\frac{R + R_2 + i\omega L_2}{i\omega L_{12}} - i\omega L_{12}\right]\left(-\frac{\underline{U}_2}{R}\right)$$

Using $L_1 L_2 = L_{12}^2$, which follows from equations (13.1), (13.2) and (13.3), we obtain

$$\frac{\underline{U}_1}{\underline{U}_2} = -\frac{(R_1 + i\omega L_1)(R + R_2) + i\omega L_2 R_1}{i\omega L_{12} R} = -\frac{L_1(R + R_2) + L_2 R_1}{L_{12} R} + i\frac{R_1(R + R_2)}{\omega L_{12} R}$$

Again we ask for the quotient of the amplitudes of the expression:

$$\frac{U_1}{U_2} = \frac{1}{\omega L_{12} R}\sqrt{\left[\omega L_1(R + R_2) + \omega L_2 R_1\right]^2 + \left[R_1(R + R_2)\right]^2}$$

With (13.7) this becomes approximately

$$\frac{U_1}{U_2} = \frac{L_1(R + R_2) + L_2 R_1}{L_{12} R}$$

If the coil resistances can be neglected, the relationship is further simplified. This simplification is possible if the following two inequalities are fulfilled

$$R >> R_2 \quad \text{and} \quad R >> \frac{L_2}{L_1} R_1 = \frac{N_2^{\,2}}{N_1^{\,2}} R_1$$

Under these conditions we get

$$\frac{U_1}{U_2} = \frac{L_1}{L_{12}} = \frac{N_1}{N_2} \tag{13.9}$$

Equations (13.8) and (13.9) apply simultaneously if

$$R_2, \frac{L_2}{L_1} R_1 << R << \omega L_2$$

14

Electromagnetic waves

14.1 Kinematics of harmonic waves

If the temporal variation of the value of a physical quantity f at any position x is the same as at $x = 0$, but shifted by x/v in time

$$f(x,t) = f(t - \frac{x}{v})$$

then this is called a wave. If f is a sine function, i.e. if

$$f(t,x) = f_0 \sin[\omega(t - x/v)] = f_0 \sin(\omega t - kx),$$

then the space-time distribution of f represents a *harmonic wave*.

If one looks at a fixed position x_1, then $f(t,x_1)$ describes a harmonic oscillation with the angular frequency ω, Fig. 14.1.

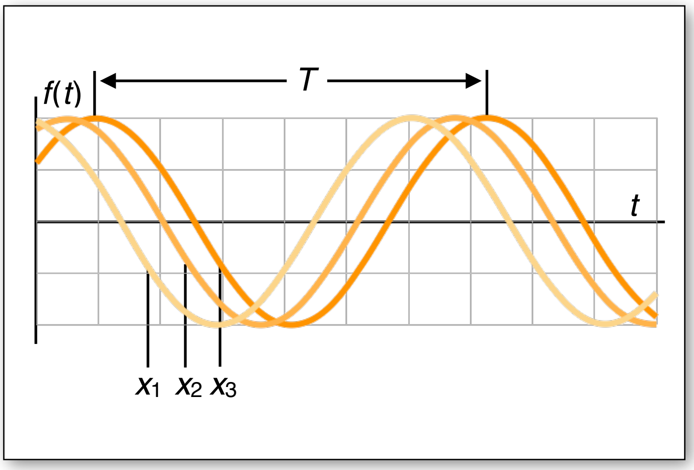

Fig. 14.1
The functions corresponding to the three positions x_1, x_2 and x_3 are phase-shifted against each other.

For another value of x_2 or x_3 it describes a sine oscillation of the same frequency, but phase-shifted against $f(t,x_1)$.

We now consider the variation of f at a certain instant of time t_1, Fig. 14.2. $f(t_1,x)$ represents a sinusoidal variation of f with the position x.

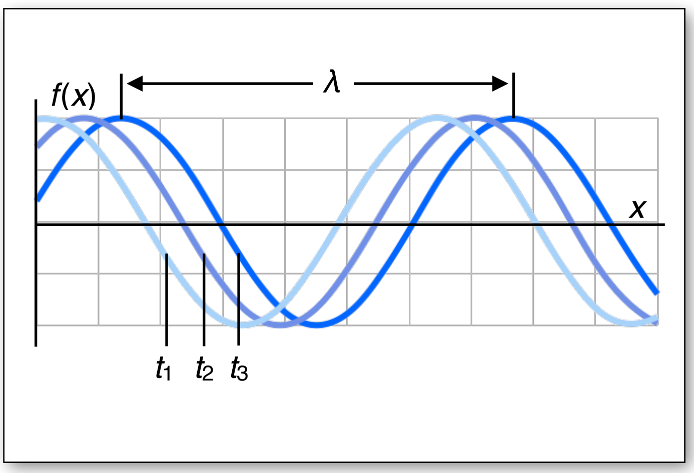

Fig. 14.2
The functions corresponding to the three instants t_1, t_2 and t_3 are phase-shifted against each other.

$k = 2\pi/\lambda$ is the *wavenumber*, λ is the *wavelength*. If we take two "snapshots" at a time interval Δt, we obtain twice the same variation as a function of x, only shifted in x direction by $\Delta x = v\Delta t$.

If many snapshots are taken one after the other, the sequence of these snapshots shows an apparent movement of the sine function in the x direction with the *phase velocity*

$$v = \frac{\omega}{k}$$

14.2 Harmonic waves as solutions of the Maxwell equations

The Maxwell equations have, among other things, harmonic waves as a solution. We try the solution

$\boldsymbol{E}(z,t) = (E_x(z,t),0,0) \quad E_x(z,t) = E_0 \cos(kz - \omega t)$

$\boldsymbol{H}(z,t) = (0,H_y(z,t),0) \quad H_y(z,t) = H_0 \cos(kz - \omega t)$

We check this solution by inserting it into the forth and third Maxwell equation. We integrate over the paths and surfaces shown in Figure 14.3.

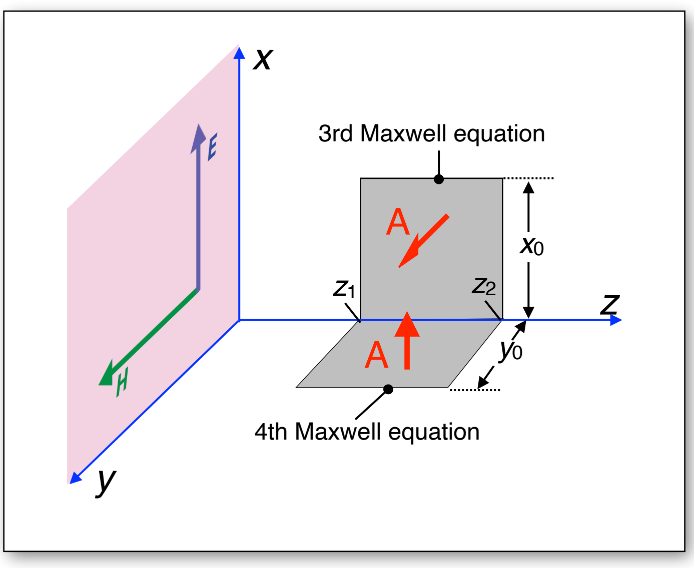

Fig. 14.3
Integration paths and surfaces for the verification of the wave solution

We assume that $j_Q = 0$ and $\chi_e = \chi_m = 0$, so that the Maxwell equations simplify to

$$\oint \boldsymbol{H}\,d\boldsymbol{r} = \varepsilon_0 \iint \dot{\boldsymbol{E}}\,d\boldsymbol{A}$$

and

$$-\oint \boldsymbol{E}\,d\boldsymbol{r} = \mu_0 \iint \dot{\boldsymbol{H}}\,d\boldsymbol{A}$$

4th Maxwell equation

$$y_0 H_0 \cos(kz_1 - \omega t) - y_0 H_0 \cos(kz_2 - \omega t) = \varepsilon_0 y_0 \int_{z_1}^{z_2} \omega E_0 \sin(kz - \omega t)\,dz$$

$$= \varepsilon_0 y_0 \omega E_0 \frac{1}{k}[\cos(kz - \omega t)]_{z_2}^{z_1}$$

$$H_0 = \varepsilon_0 \frac{\omega}{k} E_0 \qquad (14.1)$$

3rd Maxwell equation

$$-x_0 E_0 \cos(kz_2 - \omega t) + x_0 E_0 \cos(kz_1 - \omega t) = \mu_0 x_0 \int_{z_1}^{z_2} \omega H_0 \sin(kz - \omega t)\,dz$$

$$= \mu_0 x_0 \omega H_0 \frac{1}{k}[\cos(kz - \omega t)]_{z_2}^{z_1}$$

$$E_0 = \mu_0 \frac{\omega}{k} H_0 \qquad (14.2)$$

From (14.1) and (14.2) follows

$$\frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad (14.3)$$

$c = \omega/k$ is the *phase velocity* of the wave.

It is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Furthermore, from (14.1) and (14.2) follows

$$\varepsilon_0 E_0^2 = \mu_0 H_0^2 \qquad (14.4)$$

Our trial solution therefore satisfies the Maxwell equations, i.e. the Maxwell equations have harmonic waves as a solution, for which the relations (14.3) and (14.4) apply. These waves are called *electromagnetic waves*. They form a class of states of the system "electromagnetic field". We now discuss this solution of the Maxwell equations.

(a) Velocity

It turns out that the expression (14.3) is equal to the "propagation velocity" of the light. This is the strongest indication that light is an electromagnetic wave. The combination of optics with the theory of the electromagnetic field is Maxwell's merit. The first sections of his electromagnetic theory of the light are reproduced in section 14.4.

*(b) Phase relation between **E** and **H***

The electric and magnetic field strengths are in phase. ***E*** and ***H*** are perpendicular to each other. The wave is perpendicular to the direction of ***E*** and ***H***. It is said to be "transverse".

(c) Energy density, energy current density, momentum density and momentum current density

From equation (14.4) it follows that the electric field and the magnetic field contribute equally to the energy density of the wave. The energy density also constitutes a harmonic wave, its frequency is 2ω, the wave number is $2k$.

The energy current density vector $\boldsymbol{j}_E$ is perpendicular to ***E*** and ***H*** and points in the direction of propagation of the wave. The energy therefore flows in the same direction in which the phase of the wave is travelling. Also $\boldsymbol{j}_E$ forms a harmonic wave with the frequency 2ω and the wave number $2k$.

The momentum density is identical to the energy current density except for the factor $1/c^2$.

A momentum current flows in the direction of travel of the wave. (Adjacent areas of the wave "exert forces on each other".) The associated current density is

$$\sigma = \frac{1}{2}\left(\varepsilon_0 |\boldsymbol{E}|^2 + \mu_0 |\boldsymbol{H}|^2\right) = \varepsilon_0 |\boldsymbol{E}|^2 = \mu_0 |\boldsymbol{H}|^2$$

This momentum current corresponds to a compressive stress; it is also called *radiation pressure*.

14.3 Emission of Electromagnetic Waves – the Hertzian Oscillator

There are many ways to generate electromagnetic waves. Here we describe a possibility that is physically and technically particularly important. However, their computational treatment is more difficult than that of other methods, and we leave the calculation to the lecture of theoretical physics. The method works as follows: The point charges of an electric dipole are sinusoidally moved back and forth, so that the dipole moment varies according to

$$\boldsymbol{p}(t) = \boldsymbol{p}_0 \sin \omega t \tag{14.5}$$

Heinrich Hertz not only demonstrated the electromagnetic waves experimentally for the first time, he also calculated the field distribution for the vibrating dipole. We present here only a partial result of his calculations: We imagine the vibrating dipole to be infinitely small; this is equivalent to the fact that in the case of a dipole that is not infinitely small one only asks for the so-called *far field*. In polar coordinates with the dipole is oriented in z direction we get

$$\boldsymbol{E} = (E_r, E_\vartheta, E_\varphi)$$

$$E_r \approx 0$$

$$|E_\vartheta| = \frac{\omega^2 p_0}{4\pi\varepsilon_0 c^2 r} \sin\vartheta \sin\left[\omega\left(t - \frac{r}{c}\right)\right]$$

$$E_\varphi = 0$$

$$\boldsymbol{H} = (H_r, H_\vartheta, H_\varphi)$$

$$H_r = 0$$

$$H_\vartheta = 0$$

$$|H_\varphi| = \frac{\omega^2 p_0}{4\pi c r} \sin\vartheta \sin\left[\omega\left(t - \frac{r}{c}\right)\right]$$

We discuss this result.

(a) Local properties

Locally this field cannot be distinguished from that of the plane wave of section 14.2: $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to each other, and $\boldsymbol{E}$ and $\boldsymbol{H}$ are both perpendicular to the direction of propagation. In addition, $\varepsilon_0 \boldsymbol{E}^2 = \mu_0 \boldsymbol{H}^2$ applies everywhere.

*(b) Distribution of **E** and **H** in space*

The phase, i.e. the argument of the sine function, has a constant value on spherical shells (dipole in the centre of the sphere). The $\boldsymbol{H}$ field lines form closed "parallels of latitude", Fig. 14.4.

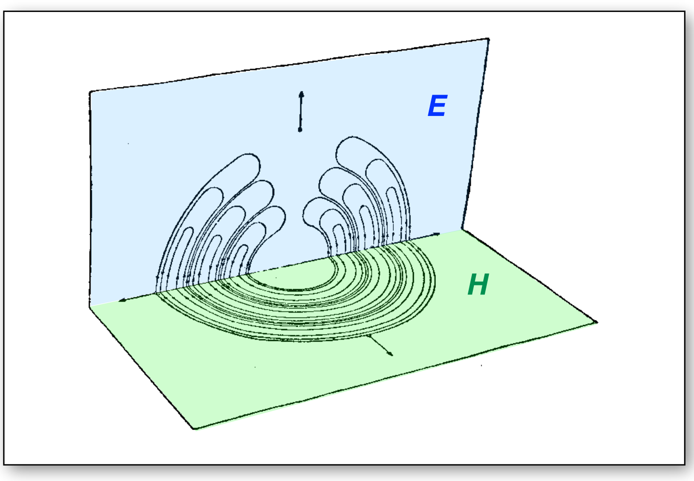

Fig. 14.4
E and ***H*** field of the radiating dipole

The $\boldsymbol{E}$ field lines follow "meridians" except near the poles. They turn back in places of weak field strength.

(c) Distribution of the energy current density

The $\boldsymbol{j}_E$ vectors point radially outwards. Their magnitude is

$$|\boldsymbol{j}_E| = \frac{\omega^4 p_0^2}{(4\pi)^2 \varepsilon_0 c^3 r^2} \sin^2\vartheta \sin^2\left[\omega\left(t - \frac{r}{c}\right)\right] \tag{14.6}$$

The amplitude decreases towards the outside with $1/r^2$, in accordance with the law of energy conservation.

The ϑ dependency for a fixed value of r, Figure 14.5, is such that $|\boldsymbol{j}_E|$ is maximum in the equatorial plane. In the direction of the dipole axis, the energy current density is zero.

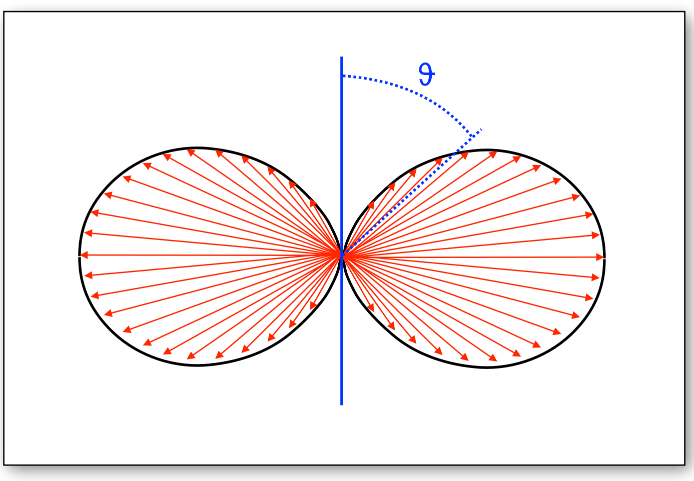

Fig. 14.5
Directional dependence of the energy current density of the dipole oscillator

(d) Frequency dependence of the energy current density

$|\boldsymbol{j}_E|$ is proportional to ω^4, equation (14.6), i.e. the radiated energy increases strongly with the oscillation frequency of the dipole, or in other words: If the dipole oscillates slowly, it does not radiate. A slowly oscillating dipole simply builds up and reabsorbs the dipole field known from electrostatics. The energy that is put into the field when it is built up is recovered when the field is removed.

A closer look shows that the second time derivative of the dipole moment is responsible for the radiation. It follows from this that for the generation of electromagnetic waves a harmonicsly oscillating dipole moment is not necessary, but that a uniformly accelerated charged particle also generates an electromagnetic field in which energy constantly flows away from the particle.

(e) Magnetic dipole radiation

One can also generate electromagnetic waves with a vibrating magnetic dipole. The field looks the same as that of the electric dipole, but the electric and magnetic field strengths are reversed. So one can tell from the field distribution whether a field originates from an electric or a magnetic dipole.

14.4 Maxwell's remarks on the electromagnetic theory of light

CHAPTER XX.

ELECTROMAGNETIC THEORY OF LIGHT.

781.] In several parts of this treatise an attempt has been made to explain electromagnetic phenomena by means of mechanical action transmitted from one body to another by means of a medium occupying the space between them. The undulatory theory of light also assumes the existence of a medium. We have now to shew that the properties of the electromagnetic medium are identical with those of the luminiferous medium.

To fill all space with a new medium whenever any new phenomenon is to be explained is by no means philosophical, but if the study of two different branches of science has independently suggested the idea of a medium, and if the properties which must be attributed to the medium in order to account for electromagnetic phenomena are of the same kind as those which we attribute to the luminiferous medium in order to account for the phenomena of light, the evidence for the physical existence of the medium will be considerably strengthened.

But the properties of bodies are capable of quantitative measurement. We therefore obtain the numerical value of some property of the medium, such as the velocity with which a disturbance is propagated through it, which can be calculated from electromagnetic experiments, and also observed directly in the case of light. If it should be found that the velocity of propagation of electromagnetic disturbances is the same as the velocity of light, and this not only in air, but in other transparent media, we shall have strong reasons for believing that light is an electromagnetic phenomenon, and the combination of the optical with the electrical evidence will produce a conviction of the reality of the medium similar to that which we obtain, in the case of other kinds of matter, from the combined evidence of the senses.

782.] When light is emitted, a certain amount of energy is expended by the luminous body, and if the light is absorbed by another body, this body becomes heated, shewing that it has received energy from without. During the interval of time after the light left the first body and before it reached the second, it must have existed as energy in the intervening space.

According to the theory of emission, the transmission of energy is effected by the actual transference of light-corpuscules from the luminous to the illuminated body, carrying with them their kinetic energy, together with any other kind of energy of which they may be the receptacles.

According to the theory of undulation, there is a material medium which fills the space between the two bodies, and it is by the action of contiguous parts of this medium that the energy is passed on, from one portion to the next, till it reaches the illuminated body.

The luminiferous medium is therefore, during the passage of light through it, a receptacle of energy. In the undulatory theory, as developed by Huygens, Fresnel, Young, Green, &c., this energy is supposed to be partly potential and partly kinetic. The potential energy is supposed to be due to the distortion of the elementary portions of the medium. We must therefore regard the medium as elastic. The kinetic energy is supposed to be due to the vibratory motion of the medium. We must therefore regard the medium as having a finite density.

In the theory of electricity and magnetism adopted in this treatise, two forms of energy are recognised, the electrostatic and the electrokinetic (see Arts. 630 and 636), and these are supposed to have their seat, not merely in the electrified or magnetized bodies, but in every part of the surrounding space, where electric or magnetic force is observed to act. Hence our theory agrees with the undulatory theory in assuming the existence of a medium which is capable of becoming a receptacle of two forms of energy*.