Is mass a measure of inertia?

F. Herrmann and M. Pohlig, Karlsruhe Institute of Technology

Abstract: Mass is usually introduced as a measure of the inertia of a body. But what do we mean by inertia anyway? We introduce a measure of inertia. It turns out that for high, relativistic velocities neither the rest mass nor the relativistic mass fulfills the requirements for a meaningfully defined measure of inertia. But how are we going to talk about inertia in the physics lesson? How can we use students' everyday language and still arrive at a clear conceptualization? We will try to give an answer to these questions.

Any physical quantity is a measure of a property of a physical system. So, temperature tells us whether a body is hot or cold, pressure is a measure of the tendency to expand, momentum measures the drive or the impetus of a body, velocity tells us how fast an object is moving. If we know the property which is measured by a physical quantity, we usually have a good and intuitive understanding of the equations or formulas in which the quantity appears.

In the following we are concerned with the property inertia. It is known to be measured by the physical quantity mass. A body with a large mass is very inert, a body with a small mass is not. Since we are aware of this, mass appears to us to be a vivid and descriptive quantity.

However, the theory of relativity seems to question this vividness. When the velocity of a body becomes high, i.e. when we no longer have $|v| \ll c$, the mass becomes velocity-dependent and it is called relativistic mass. Does this mean that the inertia is velocity-dependent as well? The answer ist yes. So, is the (relativistic) mass a measure of inertia also in this case? Now, the answer is no. We will see that we have to revise some of our ideas about the relation between mass and inertia.

In order to do so, we must first explain what we mean by inertia. In fact, this is easier than giving a definition of mass. In the following, we propose such a definition.

For those readers who are interested in the issues of energy-mass-equivalence and in the different ways of using the terms mass, rest mass, relativistic mass, longitudinal and transverse mass, we recommend the article by Sandin[1], which gives a comprehensive and easy to understand overview. Roche[2] gives a survey of the historical development of the concept of mass. The question of what is meant by mass was also discussed in detail, for example by Hecht[3,4], Cuelho[5] and Schwarz[6].

Definition of a measure for inertia

Our definition should cover what we would call inertia in everyday life in the context of a moving object.

That is why we define:

$$T := dp/dv \qquad (1)$$

This quantity tells us whether one has to supply much or little momentum dp to a body in order to change its velocity by dv. If we need much momentum for a desired change of the velocity, the body is very inert; if we need little, it is less inert.* We restrict ourselves to those momentum changes whose direction is the same as that of the momentum which the body under consideration already had before the change.

Let us first consider a simple case: a body moving with non-relativistic velocity. Here, we have:

$$p = m \cdot v \qquad (2)$$

and equation (1) tell us that:

$$T = dp/dv = m \qquad (3)$$

The inertia is equal to the mass – a result that we are not surprised by.

We can also read the inertia from a diagram showing the relationship between momentum and velocity. There are two options for drawing such a diagram: either we plot p as a function of v, or v

as a function of p. We prefer v over p, so that p appears as the independent variable. In fact, we have a more direct influence on the momentum than on the velocity. This becomes especially clear when we later consider relativistic motions.

Figure 1 shows $v(p)$ for the non-relativistic case. The triangles tell us how much momentum dp is needed to change the velocity by dv. For the greater mass m_2 more momentum is needed than for m_1. We also see that for a given body, T is independent of the momentum or of the velocity.

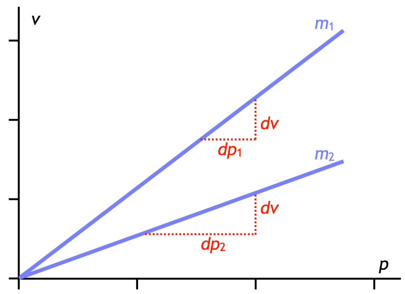

Fig. 1. The velocity is proportional to the momentum; the inertia of a body is the same for every velocity.

What is different at high velocities? – Inertia is defined by a characteristic curve

At high velocities, there is a significant change: the mass in equation (2) becomes velocity dependent:

$$p = m(v) \cdot v \tag{4}$$

with

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

$m(v)$ is called "relativistic mass". m_0 is the mass of the body when it is at rest, its "rest mass". Equation (4) can then be written as:

$$p(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} v \tag{6}$$

Solving for the velocity, we get

$$v = \frac{cp}{m_0\sqrt{c^2 + \frac{p^2}{m_0^2}}} \tag{7}$$

Figure 2 shows the velocity as a function of momentum for three different values of the rest mass.

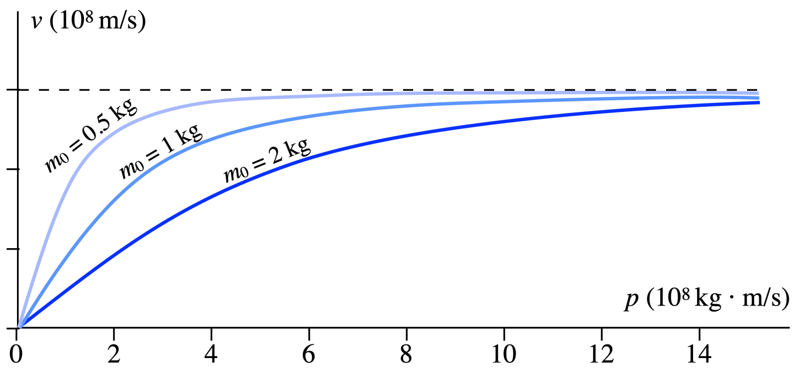

Fig. 2. Dependence of the velocity on the momentum for various rest masses. In all three cases, for increasing momentum, the velocity approaches the same limiting velocity.

We see that the inertia T of a body is no longer constant; it depends on the velocity. To get a desired increase in velocity, we need little momentum at the beginning, and then gradually more and more, Figure 3. The inertia increases with increasing velocity.

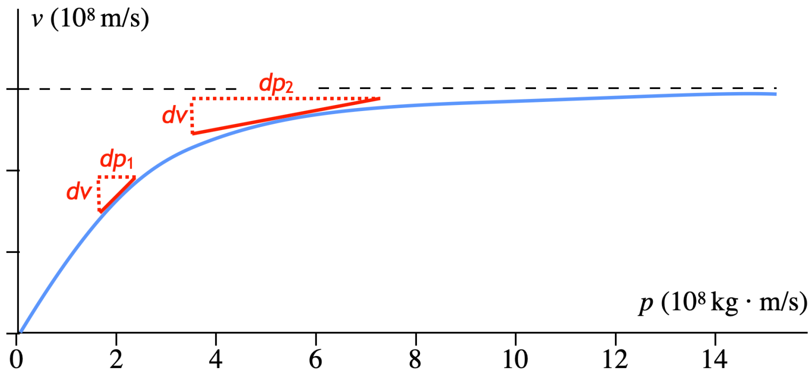

Fig. 3. The red triangles tell us how inert the body is. As the body is “charged” with momentum, its inertia increases.

The absolute value of the velocity cannot exceed a certain value c, which we call *terminal speed*.

Let us now calculate the inertia T, as defined by equation (1). All we have to do is derive the momentum, equation (6), according to the velocity. We obtain

$$T(v) = \frac{dp}{dv} = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \qquad (8)$$

It is obvious that inertia is no longer a property that can be defined by a single numerical value; it depends not only on the rest mass m_0 (which characterizes the body), but also on the velocity (which characterizes the body’s state of motion). Note that our inertia is neither equal to the rest mass nor to the relativistic mass, equation (5).

This result seems somewhat disappointing if one was expecting that inertia is an invariable property of a body. However, we should not be too surprised, because there are numerous other similar situations in physics. Consider electrical resistance. We are used to speaking of the resistance of a certain electrical component, which we call a resistor, e.g. 1000 kΩ. This value can be read from the I-V characteristic. However, we also know that the relationship between voltage and current cannot generally be described by a single number. One must specify a characteristic curve, Figure 4.

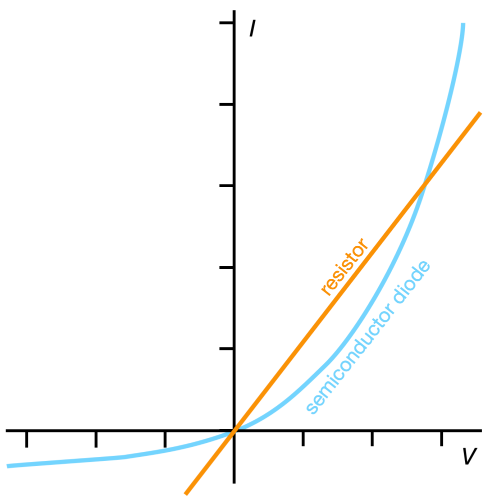

Fig. 4. I-V characteristic of a resistor and a semiconductor diode. To characterize the resistor, a single number is sufficient. For the diode a function graph is needed.

We can therefore say that Figures 2 and 3 represent the inertial characteristics of the bodies[7].

But even within the context of relativistic physics, the interpretation of mass as a measure of inertia still has its justification. How can that be? Didn't we just prove the opposite?

In our previous considerations, we had assumed structureless particles or bodies, without explicitly saying it, i.e. systems that could not be excited in any way, whose temperature or pressure could not be changed, etc.. We now want to abandon this restriction.

We consider a thought experiment. In a container there is a gas whose temperature is so high that its particles have relativistic velocities, see also Sandin[1]. With this container we now do classical experiments, i.e. experiments in which the velocity of the container remains in the classical range.

We first place the gas on a scale. We find that the weight, and thus the gravitational mass, is greater than the sum of the rest masses of the parts, and that it depends on temperature.

In addition, we carry out an acceleration experiment. However, we accelerate the gas in such a way that its (center-of-mass) velocity remains much smaller than c. In doing so, we find that the inertia is equal to the mass that we determined with the scale and that it is independent of the centre-of-mass velocity. The measured mass is therefore a measure of the inertia, although the particles of the gas move relativistically, and although they may have been photons, which have no rest mass at all. Thus, (relativistic) mass is a suitable measure of inertia as long as the centre-of-mass velocity is small compared to the terminal velocity – no matter how high the velocities of the components of the system are, and no matter whether they are particles with or without rest mass.

Conclusion

We were concerned with the question of whether mass is a measure of the property inertia.

For this purpose, we first settled what we want to understand by inertia, and we introduced a measure for it.

As long as one considers movements with center-of-mass velocities much smaller than the terminal speed c, relativistic mass measures inertia. In general, however, it does not.

If we allow for relativistic center-of-mass velocities, the mass loses this property. Neither the rest mass nor the relativistic mass is a measure of inertia. However, the inertial behavior can now be described by a characteristic curve.

* Alternatively, we could have defined inertia by the force F that is needed to obtain a desired acceleration a. Thus, we would write:

$$T := F/a \ .$$

Using $F = dp/dt$ and $a = dv/dt$ we come back to equation (1). Thus, the two definitions are equivalent. Note that $F = m \cdot a$ does not apply here because m is velocity dependent.

References

1. T. R. Sandin, "In defense of relativisitc mass", *AJP* **59**, 1032 - 1036 (November 1991), DOI: 10.1119/1.16642.
2. J. Roche, "What is mass?", *Eur. J. Phys*. **26**, 225-242 (2005), DOI:10.1088/0143-0807/26/2/002.
3. E. Hecht, "There is no Really Good Definition of Mass", *Phys. Teach.* **44**, 40-45 (January 2006), DOI: 10.1119/1.2150758.
4. E. Hecht, "Einstein Never Approved of Relativistic Mass", *Phys. Teach.* **47**, 336-341 (September 2009) DOI: 10.1119/1.3204111.
5. R. L. Coelho, "On the Definition of Mass in Mechanics: Why Is It So Difficult?", *Phys. Teach.* **50**, 304 (April 2012); DOI: 10.1119/1.3703550.
6. B. Schwarz, "On defining mass", *Phys. Teach.* **50**, L2 (July 2012), DOI: 10.1119/1.4744957.
7. In the literature, the expression on the right hand side of Eq. (8) is known to be the "longitudinal mass". This name was given in order to distinguish the expression from another one, the "transverse mass". The transverse mass is the inertia of a body that is accelerated transversely to its direction of motion. Since the velocity in the transverse direction is zero, it is simply equal to the (relativistic) mass. Thus, we can conclude that the inertia as defined in Ref. 1 is a tensor quantity.