KPK

The Karlsruhe Physics Course

for the secondary school A-level

Mechanics

The Karlsruhe Physics Cours

A textbook for the secondary school A-level

- Electrodynamics
- Thermodynamics
- Oscillations, Waves, Data
- **Mechanics**
- Atomic Physics, Nuclear Physics, Particle Physics

Herrmann

The Karlsruhe Physics Course

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1 TOOLS

1.1 Physical quantities

A physical quantity is needed to describe a property quantitatively, i.e. with a numerical value. We look at the statement

$m = 5\,\mathrm{kg}$

Here is

m a physical quantity
5 a numerical value
kg a measuring unit

In mathematical terms, „5 kg" is to be regarded as the product of „5" and „kg". Please bear in mind when writing your own scientific text that the symbol of a physical quantity is written in italics while the measuring unit is written in a normal font. Hence, „*m*" stands for mass and „m" for meter.

1.2 What the value of a quantity refers to

We try to classify physical quantities, and this by looking at the geometrical entity they refer to: a point, a surface area or a region of space.

Value refers to a point

velocity, temperature, pressure, electric potential, density, …

Value refers to a surface area

all currents: force (momentum current), electric current, entropy current, energy current, …

Value refers to a region of space:

mass, momentum, electric charge, entropy, energy, …

If a quantity refers to a point, its value can change from point to point. This is evident for temperature and for pressure. Maybe it is not so obvious that also the velocity is part of this category. This is because all points of a moving body have the same velocity, or don't they? It is sufficient to look at a rotating body to convince yourself of the opposite. In such a body, each point has a different velocity. Also the velocity of water in a river chances from place to place.

Quantities that refer to a region of space are called *substance-like quantities.*

Not all quantities fulfill this pattern, e.g. time, but also the spring constant, electric resistance and capacitance.

> The value of a current strength refers to a surface area. The value of a substance-like quantity refers to a region of space.

1.3 Distributions

You are interested in the temperature at the location where you are currently at? You measure the temperature and find it to be 25 °C.

Hence:

$\vartheta = 25\ °\mathrm{C}.$

Sometimes you are interested in the temperature values along a line: how does the temperature above the point where you are standing decrease with the altitude z? Then, you ask about a unidimensional temperature distribution, i.e. about the function $\vartheta(z)$.

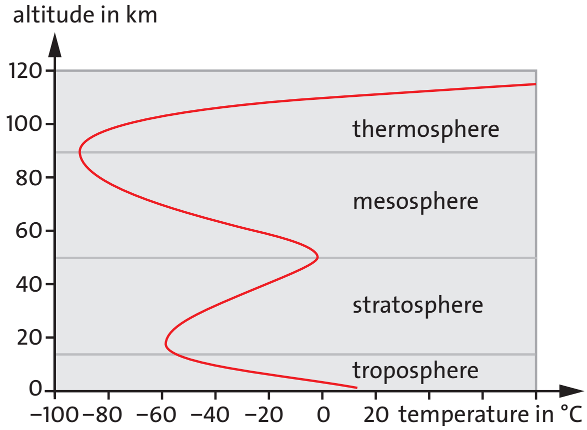


Fig. 1.1 Temperature as a function of the altitude

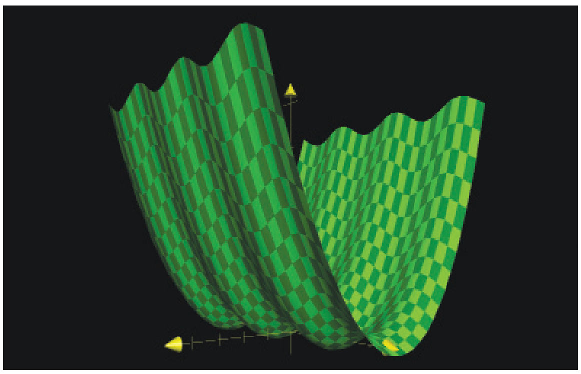


Fig. 1.2 3D plot of the function $z(x, y) = x^2 + \sin y$

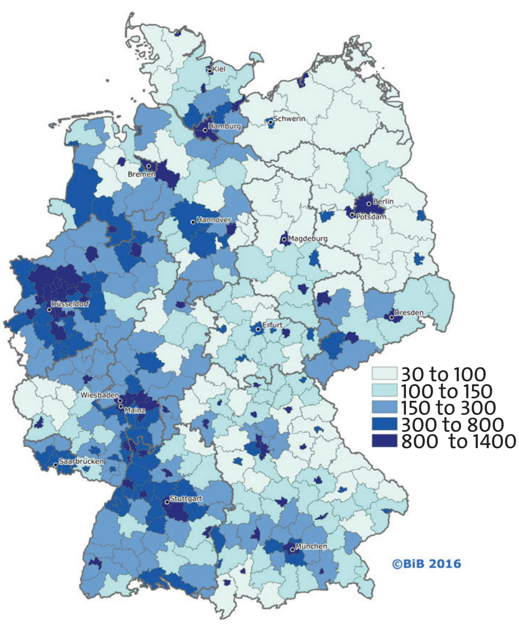


Fig. 1.3 Distribution of the population density in Germany

It happens that the values of a quantity that refers to a point are of interest in an entire plane, e.g. for the temperature or the air pressure on the surface of the Earth. The respective distribution will then be a function of two space coordinates: $\vartheta(x, y)$ and $p(x, y)$, i.e. a distribution in two dimensions.

To inform someone about the temperatures in a real three-dimensional region of space, he will have to be told a function $\vartheta(x, y, z)$ of the three space coordinates x, y and z, i.e. a three-dimensional temperature distribution.

The more dimensions are taken into account, the more difficult will be the graphical display of the distribution. Fig. 1.1 shows the temperature as a function of the altitude.

A two-dimensional distribution can be displayed graphically by means of a 3D plot. Fig. 1.2 shows the function $z(x, y) = x^2 + \sin y$. Try to understand why the diagram looks the way it looks.

Two-dimensional distributions can also be displayed by means of gray shadings or colors. Fig. 1.3 shows the population density as a function of the location in the Germany.

To display a three-dimensional distribution in a clear way, we will have to use some clever tricks. Fig. 1.4 shows the density distribution in a hydrogen atom.

We are often interested in how the value of a quantity will change over in time t. In this case also t appears as an independent variable.

Hence, we have to deal with functions such as

$\vartheta(x, t)$ or
$\vartheta(x, y, t)$ or
$\vartheta(x, y, z, t)$.

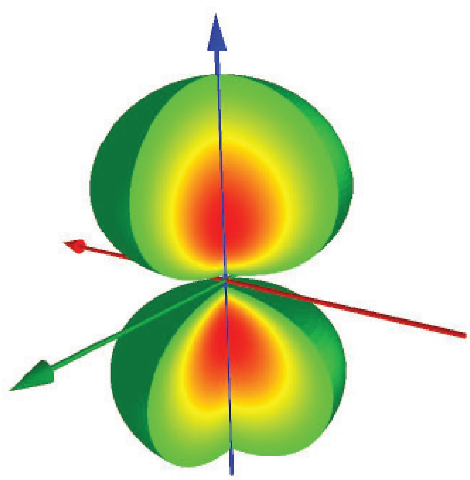


Fig. 1.4 Distribution of the mass density of the electron shell of an excited hydrogen atom

The function $\vartheta(x,t)$ can be graphically displayed as in Fig. 1.2. There, the time is put in the place of the space coordinate y. Such functions become very clear by making videos so that the time variable will also be perceived as time.

Sometimes, a quantity that refers to a point has the same value at every place. For example the temperature of a body can be in complete equilibrium: each point of the body has the same temperature. In this case, we say that the temperature distribution is *homogeneous*.

1.4 Substance-like quantities

Physical quantities whose values refer to a region of space are called substance-like quantities.

They include:

- energy
- momentum
- entropy
- electric charge
- amount of substance

If an object is doubled imaginarily, i.e. if a copy is made of it and placed next to the old one, the structure that consists of the two will have twice the amount of energy, twice the momentum, etc. (but not twice the temperature or twice the velocity).

Each substance-like quantity can be imagined as a measure for something that is contained in the respective object, like water in a recipient. Two recipients contain twice as much water as one, and three contain three times as much.

Momentum is a measure for the impetus or the verve of a body. The original name of the quantity describes the meaning of the physical quantity better than the name „momentum“: it was called *quantitas motus*, or quantity of motion. Two identical cars that drive with the same velocity have together twice as much momentum (impetus) as a single car.

Entropy is a measure for the heat contained in a body. Two identical bodies with the same temperature have together twice as much entropy (heat) as a single body.

The electric charge is a measure for something we do not have a colloquial name for. But we can still feel it. On two identical bodies that have the same electric potential, there is twice as much charge as on one.

Regarding the amount of substance, the corresponding statement is logical: for two identical bodies, the amount of substance – and hence the number of molecules of which it consists – is twice as large as for a single one.

Yet another particularity of the substance-like physical quantities: we can tell for each of them whether it is conserved or not, i.e. whether it can be produced, whether it can be destroyed or whether neither of the two is possible. Therefore, we can say:

> Energy can be neither created nor destroyed.
> Momentum can be neither created nor destroyed.
> Electric charge can be neither created nor destroyed.

But:

> Entropy can be created but not destroyed.

And finally:

> Amount of substance can be created and destroyed.

For a quantity that refers to a point it does not make sense to talk about its being conserved or not.

> We can tell for every substance-like quantity whether it is conserved or not.

1.5 Scalars and vectors

Hopefully you do not think that the classification of quantities is confusing – because things will become even more complicated now. Let us look once again at the different physical quantities but this time from another perspective.

We will compare two specifications at first: a temperature and a velocity. You can imagine it to be the air temperature and the wind velocity at a well-defined place and at a well-defined instant of time.

The specifications are

$\vartheta = 19\,°\mathrm{C}$
$v = 5\,\mathrm{m/s}$.

Do you notice that one of the two specifications is incomplete? Although we know how fast the air is moving, i.e. 5 m/s, we do not know yet in which direction it moves. Things are clear for the temperature as the temperature has no direction. Quantities such as the temperature that are defined by a single number are called *scalars*. Quantities for which a direction has to be specified in addition are called *vectors*. Here some examples:

scalars

energy, mass, electric charge, electric current strength, temperature, entropy

vectors

velocity, momentum, momentum current

Later you will get to know further vector quantities.

So what is the best way of telling someone a velocity value? There are several possibilities.

A graph, i.e. by means of a sketch, is the easiest way. The velocity is indicated by an arrow. The length represents the magnitude of the velocity, in our case 5 m/s, and the direction of the arrow corresponds to the direction of movement, Fig. 1.5.

This way, the wind velocity can indicated for any point on a map. Of course, it is crucial for this method to determine the length that corresponds to the velocity unit in the sketch. In Fig. 1.5, we have indicated the velocity unit 1 m/s as a straight segment.

To be able to tell that a physical quantity is a vector, a small arrow is drawn on top of the symbol of the quantity. Hence, we write

velocity: $\vec{v}$
momentum: $\vec{p}$
momentum current: $\vec{F}$.

In many cases, we would like to describe a vector quantity, e.g. the wind velocity, only with numerical values and not by means of a sketch. Fig. 1.6 shows how that can be done.

We choose a coordinate system whose axes are denominated v_x and v_y. Then, we draw the vector arrow of the velocity at any place of the coordinate system.

From the beginning and from the tip of the arrow, we draw straight lines perpendicular to the coordinate axis. Hence, we „project“ the vector arrow on the two coordinate axes. This way we obtain the x component v_{0x} of the velocity and the y component v_{0y}. In three-dimensional space, there would also be a z component.

These three components characterize the velocity vector unambiguously. In the left part of the image we have:

$v_{0x} = 3\,\text{m/s}$
$v_{0y} = 2\,\text{m/s}$,

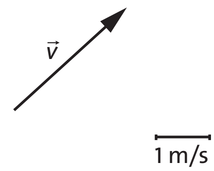


Fig. 1.5 A vector is graphically represented by an arrow. The length of the arrow corresponds to the magnitude of the vector.

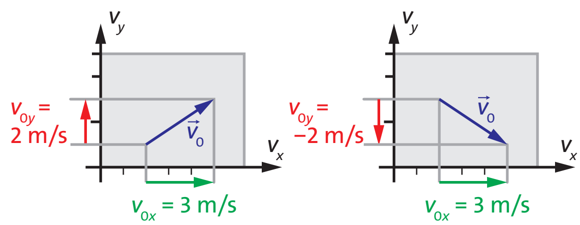


Fig. 1.6 A vector is broken down into its components

and in the right one:

$v_{0x} = 3\,\text{m/s}$
$v_{0y} = -2\,\text{m/s}$,

The components have a simple meaning: we could say that the air moves at the same time at 3 m/s in the x direction and at 2 m/s (left picture) or –2 m/s (right picture) in the y direction.

We have used two numbers to characterize the velocity vector. Strictly speaking, there have been even three, because there is also a z component, but this is zero in our case:

$v_{0z} = 0\,\text{m/s}$.

What has been said here for the velocity also holds true for other vector quantities. Also the momentum and the momentum current are defined each by an x, y and z component.

This was not relevant in our earlier considerations as we limited them to movements in one single direction. Hence, we were only dealing with one of the two components.

> The value of a scalar quantity is defined by one single number.
> The value of a vector quantity is defined by three numbers, i.e. the values of the x, the y and the z component.

1.6 Streamlines

The velocity is (1) a quantity that refers to a point and (2) a vector.

This means:

1. Its value can be different from point to point; it forms a distribution. Example: the velocity distribution of the air motion (of wind).

2. In each point it has a well-defined direction.

We would like to graphically illustrate the velocity distribution of water in a river (strictly speaking: on the surface of the river). One possibility is shown in Figure 1.7. Small vector arrows are drawn at as many points as possible. The vector refers to that place where its starting point (not the tip) is located.

Such an illustration is too confusing for certain purposes. Therefore, usually a *streamline picture*, Fig. 1.8, is drawn. A streamline is a line that has at each point the same direction as the velocity vector of the flow. Hence, the direction of the flow can be seen at each point. But we also learn something about the flow velocity: where the lines are very close to each other, it is high; where the lines are very distant, it is low.

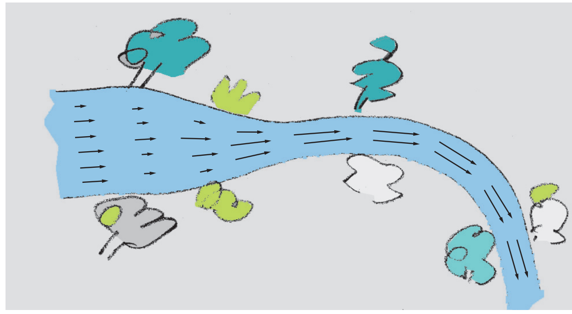


Fig. 1.7 Velocity distribution of the water in a river, illustrated with vector arrows

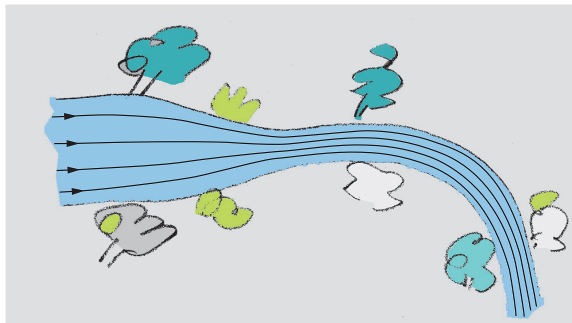


Fig. 1.8 Velocity distribution of the water in a river, illustrated with streamlines

Exercises

1. Check with Deutscher Wetterdienst [German Meteorological Service] how the distributions of the various quantities that are used to describe a weather are graphically illustrated. Explain.
2. Sometimes but not always, we can imagine a streamline to be like the trajectory of a small portion of water. Which precondition has to be met?
3. Describe methods for the illustration of an air current (velocity distributions) that have not been mentioned in the text.

1.7 Vector addition

Sometimes the values of physical quantities have to be added up.

A battery contains 10 kJ of energy, another one contains 12 kJ. Both batteries together have

$$10\ \text{kJ} + 12\ \text{kJ} = 22\ \text{kJ}.$$

The temperature in Stuttgart is 22 °C, in Karlsruhe it is 26 °C. The mean value of the temperatures is

$$\frac{22\,°\text{C} + 26\,°\text{C}}{2} = 24\,°\text{C}$$

A 4.5-volt battery and a 9-volt battery are connected in series. The newly created energy source has a voltage of

$$4.5\ \text{V} + 9\ \text{V} = 13.5\ \text{V}.$$

The examples show that there are different reasons for adding up values: calculation of a total amount, calculation of a mean value, connecting two devices in series.

All quantities of these examples were scalars. But sometimes we might also want to perform such operations with vector quantities, which means that vectors need to be added up. How does that work?

We look at an example in which velocities have to be added up. You walk in a forward direction on a train. The velocity of the train is 75 km/h, your velocity „relative to the train" is 4 km/h. Relative to the Earth („in the reference frame of the Earth"), you have a velocity of 75 km/h + 4 km/h = 79 km/h.

Here, the velocities to be added up had the same direction – the longitudinal direction of the train – and the addition was not hard to understand. But what will happen if the velocities to be added up have different directions? We assume that you walk on a ship (where there is more space than on a train) at 4 km/h and transversally to the direction of the ship. We suppose the ship has a velocity of 20 km/h. Fig. 1.9 (a) shows the respective vector arrows $\vec{v}_P$ and $\vec{v}_S$ (P for person and S for ship).

Seen from the Earth, the „resulting“ movement will no longer be in the longitudinal direction and not in the direction transversal to it either, but diagonal. The direction of the resulting velocity vector" is between the direction of $\vec{v}_P$ and of $\vec{v}_S$. Fig. 1.9 shows how the resulting velocity vector can be obtained. The two vector arrows are simply connected: the start of one is put on the end of the other. Then, an arrow has to be drawn from the start of the first to the tip of the second vector. The order in which the arrows are connected is not relevant, i.e. also the vector addition is commutative.

Vector addition: the arrows of the vectors to be added up are connected.

Fig. 1.10 shows the addition of two vectors.

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_3$$

Also the components of the three involved vectors are indicated on the axes of the coordinate system.

We can see that the following applies for the components:

$$\vec{v}_{x1} + \vec{v}_{x2} = \vec{v}_{x3}$$
$$\vec{v}_{y1} + \vec{v}_{y2} = \vec{v}_{y3}$$

Vector addition: the components are added up individually.

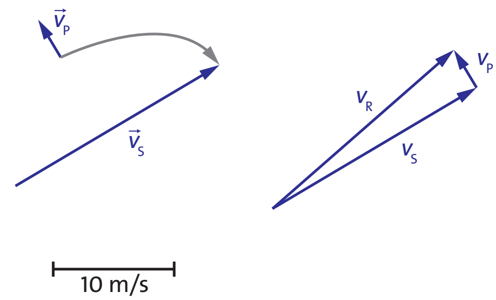


Fig. 1.9 (a) Velocity vectors of ship (S) and person (P). (b) Graphical addition of the vectors

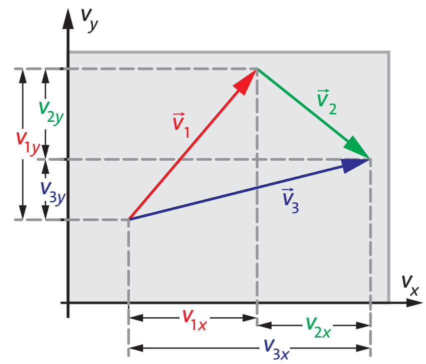


Fig. 1.10 Addition of vectors. The components are added up individually.

2 MOMENTUM AND MOMENTUM CURRENTS

2.1 Momentum

Momentum is something that is contained in a fast, heavy body. Colloquially, it can be described with the words „impetus“ or „verve“.

The relationship between momentum (impetus), velocity (how fast does the body move?) and mass *m* (how heavy is the body?) is as follows:

$$\vec{p} = m\,\vec{v}$$

This „vector equation“ is an abbreviation for the three equations of the components:

$$p_x = m \cdot v_x$$
$$p_y = m \cdot v_y$$
$$p_z = m \cdot v_z$$

As a measuring unit we use the Huygens (Hy) that fits in the SI system: if the mass is given in kg and the velocity in m/s, the equation will provide the momentum in Hy. Hence, we obtain

Hy = kg · m/s.

The momentum was introduced as a physical quantity by the philosopher, mathematician and natural scientist René Descartes (1596 – 1650), Fig. 2.1. Descartes called the quantity quantitas motus, in English: quantity of motion. (Back then, intellectuals communicated in Latin just as they do in English today.) However, the quantity introduced by Descartes could only take on positive values; and it was definitely not a vector. Hence, it was what we call today the magnitude of the momentum vector. Therefore, it was not yet very helpful back then.

Only Christiaan Huygens (1629 – 1695) introduced a plus/minus sign for the momentum. However, he was not aware either that the momentum is a quantity that is always conserved.

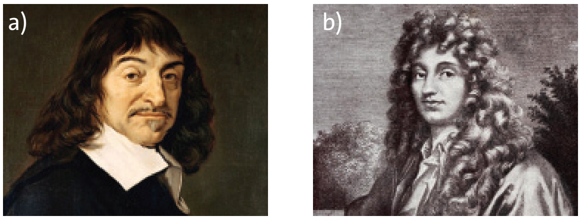


Fig. 2.1 René Descartes (a) and Christiaan Huygens (b)

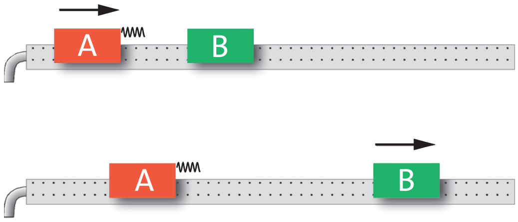


Fig. 2.2 In the collision, A transfers all its momentum to B.

2.2 Momentum currents

One-dimensional movements

Momentum can pass over or go or flow from one body to another. At first, we look at phenomena in which there is only momentum in one single direction. When doing so we can deal with the momentum as if it were a scalar. To prevent momentum from flowing into the Earth, we will make the experiments with vehicles that have low friction or with an air track.

A body A moves to the right and collides with B, Fig. 2.2. Through the spring bumper, momentum flows from A to B. If A and B have the same mass, all the momentum of A will go to B. If A has 2 Hy at the beginning (and B 0 Hy), B will have 2 Hy (and A 0 Hy) after the collision.

If A is heavier than B, Fig. 2.3, only a part of the momentum will be transferred.

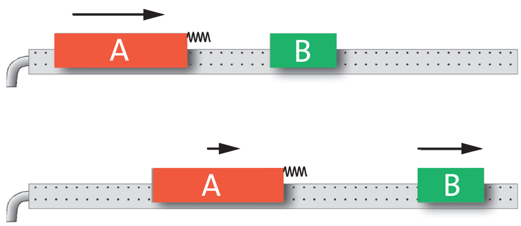


Fig. 2.3 If body A is heavier than body B, only part of its momentum will be transferred to B during the collision.

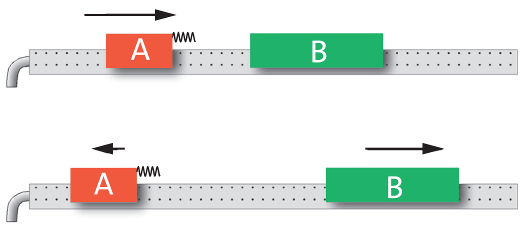


Fig. 2.4 Body A is lighter than B and releases more momentum than it actually has: its momentum becomes negative.

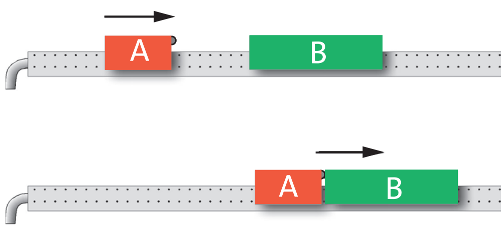


Fig. 2.5 If the bumper is inelastic, the vehicles will move with the same velocity after the collision.

And if A is lighter than B, Fig. 2.4, body A will transfer more momentum to B than it has. Therefore, A has to „enter in debt“: after the collision, its momentum is negative.

In all three cases, the law of momentum conservation is fulfilled. This can be verified by measuring the velocities before and after the collision and by calculating the momentum values.

But we can also influence the momentum transfer by changing the transfer process: instead of an elastic spring, we use a „bumper“ made of an inelastic material, e.g. of modeling clay. Then, the two gliders will hang on one another after the collision, i.e. they will have the same velocity – regardless of the masses of A and B, Fig 2.5.

Finally we replace collision partner B by the Earth, Fig. 2.6 and 2.7. If the momentum is transferred by means of a spring, body A will move to the left after the collision with the same velocity with which it moved to the right before. Hence, the Earth will receive twice as much momentum as A had at the beginning.

If an inelastic bumper is used once again, body A will simply transfer its momentum to the air track (from where it will continue flowing into the Earth) and stop.

A further variant of the experiment is shown in Fig. 2.8. The spring between A and B is stretched. A thread between the two bodies prevents it from being released. Then, the thread is cut. (To avoid that momentum can arrive on the overall arrangement from outside, it is burned through with the flame of a match.) As soon as the thread tears, the two bodies start moving in opposite directions: one of them has positive, the other negative momentum of the same absolute value. Together, both will therefore have 0 Hy before and after.

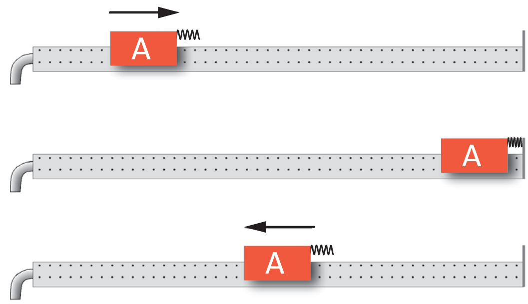


Fig. 2.6 Body A „collides with the Earth“. In this process, it releases twice as much momentum as it has. Therefore, it has negative momentum after the collision.

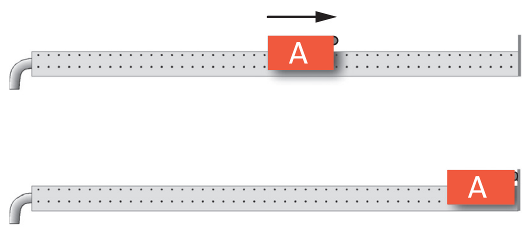


Fig. 2.7 Inelastic bumper. The Earth is the collision partner. After the collision both bodies (body A and the Earth) have the same velocity, i.e. 0 m/s.

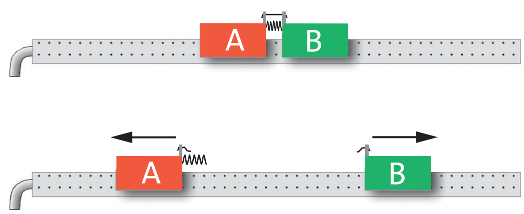


Fig. 2.8 Before and after cutting the thread the overall momentum is 0 Hy.

Two- and three-dimensional movements

In two- and three-dimensional movements, the vectorial nature of momentum becomes noticeable. In the following, we will examine movements that are limited to two dimensions. The results we will obtain, however, are also valid for three-dimensional processes.

An air hockey table is suitable for the experiments. In case you are not familiar with this device: it is a game machine similar to table soccer. An even, horizontal surface has many small holes of which air is flowing out. The air forms a cushion on which the round „pucks“ can float practically without friction. Alternatively, the experiments can also be made with coins that are shot against each other on a surface that is as smooth as possible. Although the coins will always stop quickly – they lose their momentum to the table – we can see quite well what the movement immediately after the collision is like.

Hence, we shoot a body (puck or coin) against another one. We do so in different ways, and observe. The colliding body A should always move in the y-direction at the beginning, Fig. 2.9–2.11. The straight line on which the center of A moves shall be called g.

If you play a bit with the pucks or coins, you will get a feeling of how the bodies behave during collision.

If the center of B is located on g, the whole movement will take place in one dimension, Fig. 2.9. Both A as well as B will only have y-momentum (or no x-momentum at all) after the collision. We have already analyzed such processes.

But if the center of B is located slightly apart from g, the momentum vectors will also have an x-component after the collision.

You see that the two bodies can now move in any direction after the collision; and it appears at first impossible to find a simple rule for their behavior.

However, if you consider how the bodies will certainly not move, you will get on the right track towards a principle on which the movement is based.

Just try to realize the behavior shown in Figure 2.10. More specifically, let's assume that body A has 0.1 Hy at the beginning. As it moves in the y-direction, this is a pure y-momentum. After the collision, it should stop, and B should have 0.1 Hy of x-momentum. But such a process does not exist.

Or the collision process from Fig. 2.11: again, at the start A has 0.1 Hy of pure y-momentum. After the collision, A and B each have 0.05 Hy y-momentum (hence, 0.1 Hy together). In addition, B also has a 0.05 Hy x-momentum. This process does not exist either.

There can be various reasons for the fact, that nature does not behave as we expect it should behave. In the present cases, there is a particularly simple reason: the law of momentum conservation would have been infringed.

Why? Isn't this reason accounted for in the example from Fig. 2.10? 0.1 Hy before and 0.1 Hy after the colli-

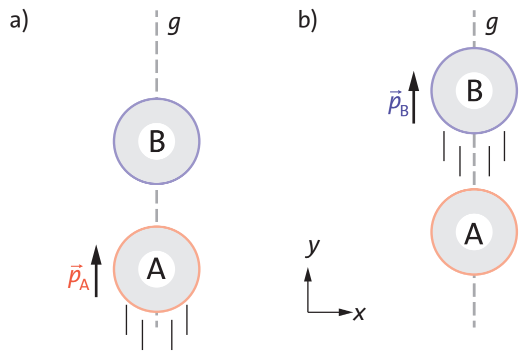


Fig. 2.9 Nothing new: the law of momentum conservation is respected. (a) prior to the collision, (b) after the collision

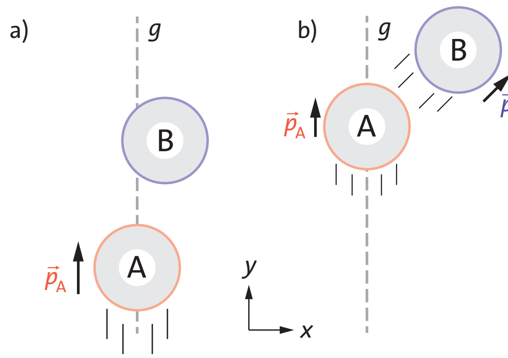


Fig. 2.10 Prior to the collision (a), A has only y-momentum, B does not have any momentum at all; after the collision (b), B would have pure x-momentum and A no momentum at all. Such a process does not exist!

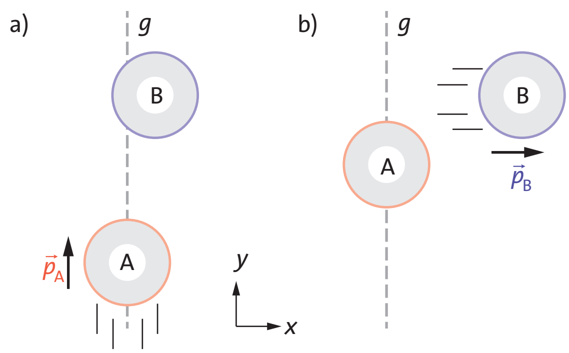


Fig. 2.11 Prior to the collision A has 0.1 Hy pure y-momentum; this momentum should distribute itself equally between A and B during the collision. In addition, B should have a remaining x-momentum of 0.05 Hy after the collision. Such a process does not exist!

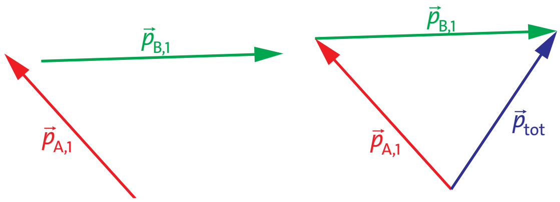


Fig. 2.12 The two momentum vectors prior to the collision (index 1), are added vectorially to the total momentum (index tot).

a) b)

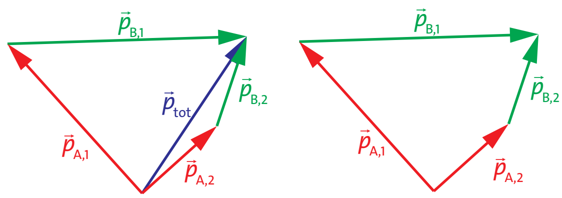


Fig. 2.13 (a) The vector sum of the momenta after the collision (index 2) is equal to the total momentum. (b) The momentum vector sum prior to the collision is equal to that after the collision.

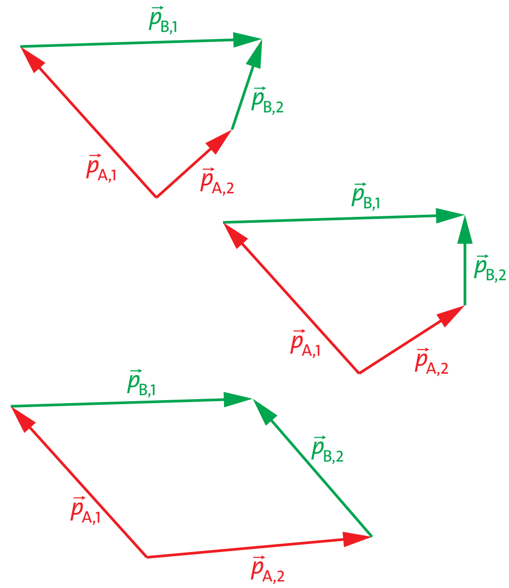


Fig. 2.14 Momentum balance for different collision processes

sion– isn't the law satisfied? But no! The law of momentum conservation for vector quantities was applied incorrectly. It is not sufficient that the absolute value of the momentum is the same before and after the collision; the law of momentum conservation has to be respected for each component individually. Or in other words: also the direction of the overall momentum must not change during a collision.

> The law of momentum conservation applies for each momentum component separately.

After the impossible processes, however, we would now like to examine the possible processes, too. Therefore, we make things a bit more complicated from the start. None of the two collision partners A and B is at rest at the beginning. Prior to the collision they have the momenta $\vec{p}_{A,1}$ and $\vec{p}_{B,1}$, respectively. („1" means before).

The sum of the momentum vectors prior to collision is the total momentum vector, Fig. 2.12:

$$\vec{p}_{A,1} + \vec{p}_{B,1} = \vec{p}_{tot}$$

Also the sum of the momentum vectors after $\vec{p}_{A,2}$ and $\vec{p}_{B,2}$ („2" means after) has to be equal, Fig. 2.13a. Or: the momentum sum prior to the collision is equal to that after the collision, Fig. 2.13b:

$$\vec{p}_{A,1} + \vec{p}_{B,1} = \vec{p}_{A,2} + \vec{p}_{B,2}$$

This also takes us to the following conclusion: the sum of the x-components remains as before during the collision, just as the one of the y-components.

Fig. 2.14 shows the momentum vectors for collisions that (as regards momentum conservation) are allowed.

Exercises

1. Willy (70 kg) and Lilly (52 kg) are rollerblading. Willy is standing still, Lilly arrives with 4.5 km/h from behind and holds on Willy. What is the velocity the two continue rolling at? (Give the result in km/h.)
2. Experiment with several coins of equal weight. Try to find a rule to describe the behavior of the coins.
3. Experiment with a light and a heavy coin. Try to find a rule to describe the behavior of the coins.
4. The momentum of the ice hockey puck is: $p_x = 3$ Hy, $p_y = 0$ Hy. Due to a hit, $\Delta p_x = -2$ Hy, $\Delta p_y = 2$ Hy is added to the puck. What will be its momentum afterwards? Calculate the components and find the result graphically.
5. A car with a weight of 1200 kg rolls with 30 km/h around a 90° curve. Friction can be neglected. Choose a coordinate system. What is the momentum of the car before the curve, what is it after the curve? By which momentum differ the start and the end momentum? Where does this momentum come from?

2.3 Momentum currents in friction processes

A block A slides over a board B, Fig. 2.15. In this process, A slows down and B starts moving. Momentum goes over from A to B. But the process quickly comes to an end: as soon as the velocities of A and B have become equal, no further momentum will flow from one body to the other. The following rule applies:

> During a process of friction, momentum flows from the body with the higher to the body with the lower velocity.

This rule probably sounds familiar to you. It is of the same type as the following statements:

- *Electric charge flows on its own from places of higher to places of lower potential.*
- *Entropy flows on its own from places of higher to places of lower temperature.*
- *A chemical reaction runs by itself from substances of higher to substances of lower chemical potential.*

We make a vehicle move to the right (in the positive x-direction) and then leave it up to itself. Due to the inevitable friction, it will stop very soon, i.e. it rolls until it comes to a halt. This behavior corresponds to our rule: momentum flows from the vehicle (velocity greater than zero) into the Earth (the velocity is zero).

Exercises

1. We could think that our rule will be infringed if the block in Fig. 2.15 slides leftwards over the board. Please show that the rule still applies here.
2. Can the rule be applied to the processes from Fig. 2.2 and 2.3?

2.4 Momentum pumps

We were thinking about the question of where the momentum of a body whose velocity decreases is going. We found out that the momentum flows into the Earth. Now we ask the reverse question: where does a vehicle get is momentum from when it is accelerated?

Willy pulls a trolley by means of a rope, Fig. 2.16. While pulling, the trolley accelerates: the momentum of the trolley increases. Where does this momentum come from? From Willy? Yes and no. Although it comes from Willy, his momentum is not reduced but it was and remains 0 Hy. Willy must take it himself from somewhere else.

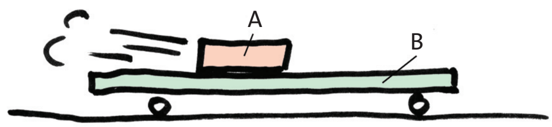


Fig. 2.15 Momentum flows from A (higher velocity) to B (lower velocity).

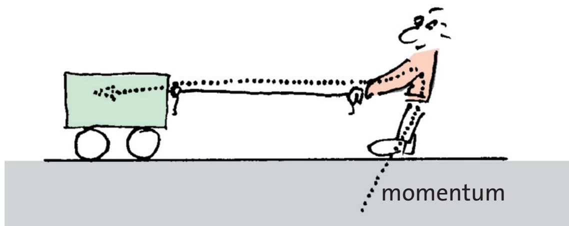


Fig. 2.16 Willy pumps momentum from the Earth into the trolley.

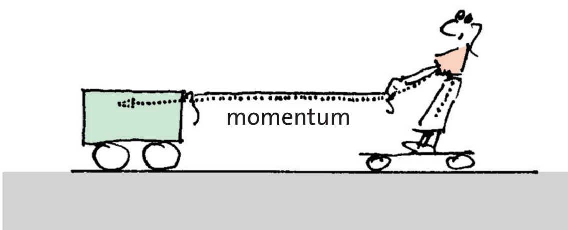


Fig. 2.17 Lilly pumps momentum out of herself into the trolley.

We change the experiment slightly, Fig. 2.17. Lilly pulls on the rope, the momentum of the trolley on the left increases. The trolley on the right, including Lilly, also starts moving – but to the left. Hence, the trolley on the right (including Lilly) receives negative momentum, or in other words: it releases positive momentum. In the process of pulling, momentum flows from the trolley on the right (including Lilly) through the rope into the one on the left. It was Lilly with her muscles that made the momentum flow from the right to the left. She acted as a „momentum pump".

Now we will also see what must have happened in the case of Figure 2.16. Willy has pumped momentum from the Earth through the rope into the trolley. We cannot see that the momentum of the Earth has become negative, just as we cannot see the increase of the momentum of the Earth while a vehicle is rolling to a halt (while releasing momentum to the Earth).

We will examine a few more situations where momentum is being pumped into another body.

In Fig. 2.18, Willy pulls the two trolleys A and B towards himself so that the trolleys accelerate. The momentum of A increases in the process while the momentum of B takes on increasingly higher negative values. Willy's momentum is and remains 0 Hy. Hence, he transports momentum from the trolley on the right to the one on the left. He is standing on a skateboard to make sure that no momentum comes from the Earth or escapes into the Earth.

A car drives with an increasing velocity, i.e. its momentum increases. Here, the engine works as a momentum pump. It transports momentum from the Earth into the car via the drive wheels (in passenger cars mostly via the front wheels), Fig. 2.19.

A toy car with remote control is standing on a piece of cardboard under which there are rollers such as drinking straws or pencils, Fig. 2.20. The car is started in a way that it moves to the right. Its momentum increases during the startup process. But the cardboard surface is rolling away to the left in the process, i.e. its momentum becomes negative. Hence, the motor of the car has pumped momentum from the cardboard into the car.

Go back once again and have a look at Fig. 2.8. After cutting the thread, the two trolleys start moving – the right one to the right and the left one to the left. Here, the trolley on the right has consequently received (positive) momentum, the one on the left has lost (positive) momentum. In this case the spring works as a momentum pump. While it loosens, it transports momentum from the left into the right trolley.

Just as any other pump, our momentum pump needs energy. The car engine that acts as a momentum pump gets energy from gas, the muscles from food. We will examine later where this energy will eventually go. At the moment, we would only like to remember:

> A „momentum pump“ (e.g. an engine) transports momentum from a body with a lower velocity to a body with a higher velocity. The momentum pump requires energy.

2.5 Momentum conductors and insulators

A necessary requirement for momentum being able to flow from A to B is a connection between A and B. But not any connection is sufficient. The connection has to be permeable for momentum. It has to be a connection that conducts momentum. How do such momentum-conducting connections look like? What type of objects conduct momentum? Which objects do not conduct momentum?

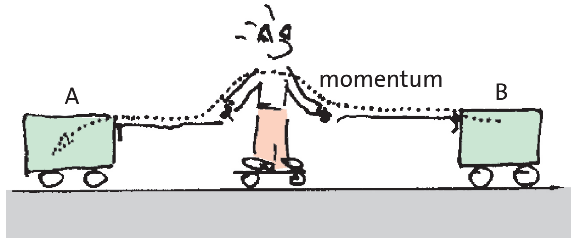


Fig. 2.18 Willy pumps momentum from the trolley on the right into the one on the left.

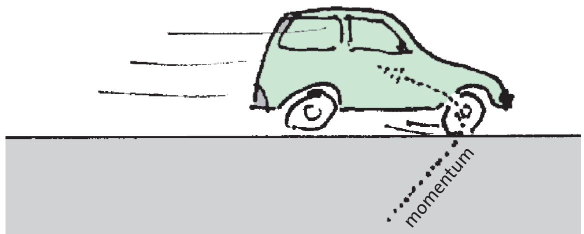


Fig. 2.19 The engine of the car pumps momentum from the Earth into the car.

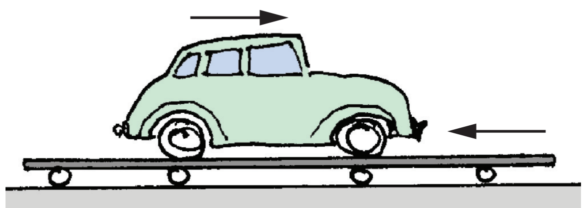


Fig. 2.20 The motor of the toy car „pumps“ momentum from the cardboard surface into the car.

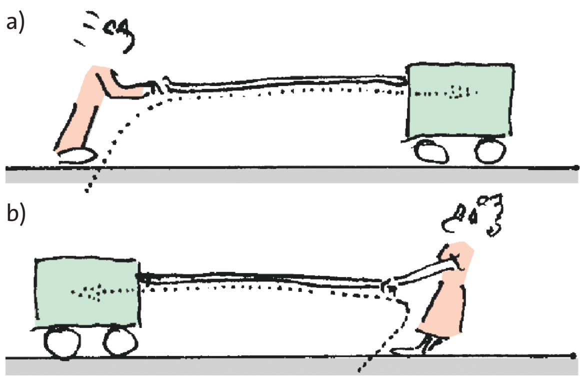


Fig. 2.21 Momentum is pumped from the Earth into the trolley. (a) The momentum flows in the bar to the right. (b) The momentum flows in the bar to the left.

In Fig. 2.21a, Willy pushes against a trolley by means of a bar. The trolley accelerates, its momentum increases. Hence, Willy pumps momentum from the Earth into the trolley. In the bar, momentum flows from the left to the right. In Fig. 2.21b, also a trolley is charged with momentum – this time by Lilly who pulls the trolley, i.e. once again by means of the bar. Here, momentum flows in the bar from the right to the left.

We can see in these two processes that the bar is a momentum conductor. It is clear that the exact shape of the bar is not relevant. Neither is the material the bar is made of, provided that it is a solid material. We conclude:

Solid materials are momentum conductors.

Fig. 2.22 shows Lilly trying to make the trolley move by pushing against the air to find out whether the air conducts the momentum up to the trolley; something that she does not seriously believe. She finds:

Air is not a momentum conductor.

This is taken advantage of for the air track: the air between the rail and the glider prevents the momentum of the glider from flowing away into the rail.

However, this principle applies only with restrictions. We will see later how we can outsmart it.

In Fig. 2.23, Willy charges the trolley with momentum by shoving a bar over the trolley. The bar thereby slides over the surface of the trolley; it is not fastened on the trolley. This way, Willy can actually transfer momentum into the trolley, albeit not very effectively.

We can see that the momentum transfer improves with an increasing friction between the bar and the trolley. If the bar slides easily over the trolley, the momentum current from the bar to the trolley will be low. If the friction is high, i.e. if for example the bar and the trolley have a raw surface, the momentum transfer will be good. We conclude:

If two objects rub against one another, momentum will flow from one to the other: the greater the friction, the more.

Basically, we have always taken the validity of this rule for granted: to prevent the momentum of an object from flowing into the Earth, we need to make sure that there is no momentum-conducting connection between the object and the Earth; we need to make sure that the friction is low.

The most important device that is used to reduce the friction between a body and the Earth is the wheel.

Wheels are used for momentum insulation.

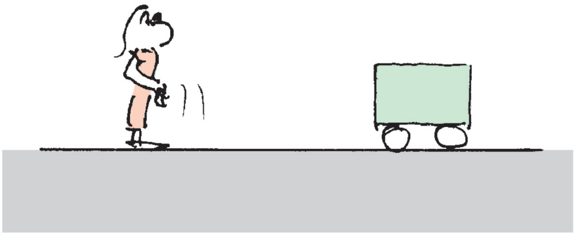


Fig. 2.22 Air is not a conductor for momentum.

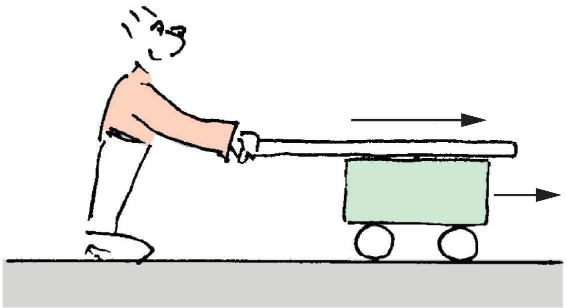


Fig. 2.23 Momentum transfer during a friction process

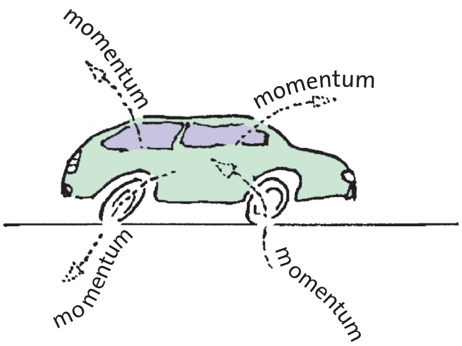


Fig. 2.24 A car that drives with constant velocity. All of the momentum that the engine pumps into the car flows back to the environment due to friction.

2.6 Flow equilibria

A car is accelerated: the engine pumps momentum out of the Earth and into the car. The faster the car drives, however, the stronger the friction of the air and the more momentum is lost. At a certain velocity, just as much momentum is pumped into the car as will flow back out due to friction. Hence, nothing will be left as a net momentum; the momentum of the car will no longer increase, Fig. 2.24.

This situation always exists when a car drives on an even ground and with a constant velocity. The inflow of momentum is equal to the outflow.

The situation can be compared with another one in which water takes on the role of momentum, Fig. 2.25: the bucket with the hole corresponds to the car. The bucket has a leak for the water just as the car has a momentum leak. New water is constantly flowing into the bucket, but just as much water is flowing back out through the hole so that the quantity of water in the bucket will not change.

Such a process, in which the outflowing current adjusts itself in a way as to be equal to the inflowing current, is called *flow equilibrium*.

> Flow equilibrium: the outflow adjusts in such a way that it is equal to the inflow.

If something moves with a constant velocity, there will often be a flow equilibrium.

For example, a cyclist pumps momentum into the bicycle (+ person) by pedaling. An equal current flows out via the air and the wheels due to friction. The same applies for planes and ships.

Exercises

1. Describe the following driving states of a car by indicating what is happening to the momentum. (a) The car starts driving. (b) The car rolls slowly at idling speed. (c) The car is slowed down by the brakes. (d) The car drives at a high, constant velocity.
2. Sometimes, a body moves at a constant velocity although there is no flow equilibrium. But why does the momentum remain constant in such cases?

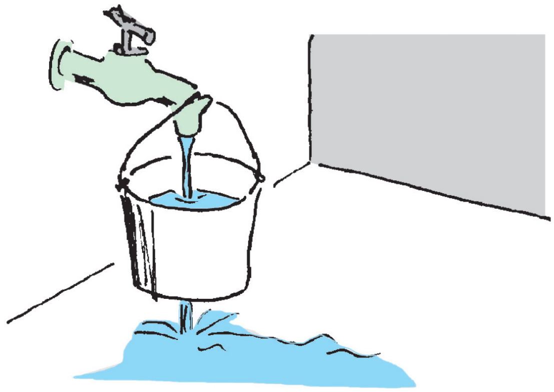


Fig. 2.25 An equal quantity of water flows in from the tap and out through the hole. The quantity of water in the bucket remains constant.

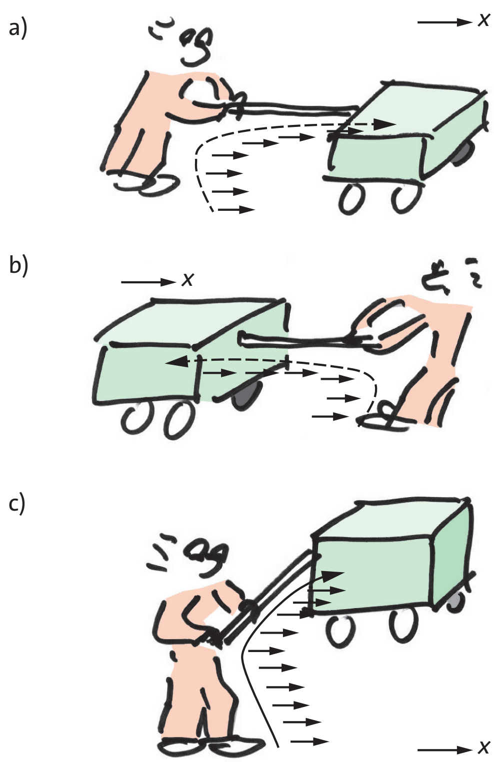


Fig. 2.26 (a) x-momentum is flowing in the bar to the right (in the positive x-direction). (b) x-momentum is flowing in the bar to the left (in the negative x-direction). (c) x-momentum is flowing in the bar to the rear (transversally to the x-direction).

2.7 Compressional, tensional and bending stress

In Fig. 2.26a, Willy makes a trolley move. Through the bar, x-momentum (the short arrows) flows from the left to the right, i.e. in the positive x-direction. In Fig. 2.26b, he pulls on the bar and xmomentum flows from the right to the left, i.e. in the negative x-direction. In Fig. 2.26c, he finally pushes the trolley ahead from the side. The x-momentum is now flowing transversally to the x-direction.

Now put yourself in the situation of the bar. Would you feel a difference in the three cases? Of course. In the first case, you would feel a compressional stress, in the second case a tensional stress and in the third case a bending stress.

We therefore have the following rule:

> x-momentum flows in the positive x-direction: *compressional stress*
> x-momentum flows in the negative x-direction: *tensional stress*
> x-momentum flows transversally to the x-direction: *bending stress*

Corresponding rules apply for the y- and the z-momentum.

Fig. 2.27a shows a truck that has just started moving. The engine pumps momentum from the Earth into the truck and through the trailer coupling leftwards into the trailer. We know that the coupling bar is under tensional stress – in accordance with our rule.

We now look at a truck that starts moving in a leftward direction, Fig. 2.27b. Here, the engine pumps negative momentum into the truck, i.e. positive momentum out of it. Therefore, (positive) momentum flows leftwards through the coupling bar. Of course, the coupling bar is again exposed to tensional stress. You see: our rule also applies in this case.

The type of tension a bar is exposed to cannot be seen, i.e. we cannot see either if and in which direction a momentum current is flowing in it. But there are objects for which we can tell what type of stress they are exposed to: all elastically deformable objects. They extend under tensional stress, shorten under compressional stress and bend under bending stress. Hence, we can also see whether and in which direction a momentum current is flowing through them:

> Shortening: *compressional stress*
> Extension: *tensional stress*
> Bending: *bending stress*

Exercises

1. A truck is driving to the right with a constant, high velocity. What type of tension (compression or tension) is the trailer coupling exposed to? Please sketch the path of the momentum.
2. Lilly accelerates the trolley in a leftward direction, i.e. by pushing it. In this process, there is a compressional stress in her arms. In which direction does the momentum current flow in the arms?
3. The high-speed train ICE 1 has respectively one traction unit (= locomotive) at the front and at the rear. On of them pulls, the other one pushes. Draw the momentum currents into a sketch of the train.

2.8 Momentum circuits

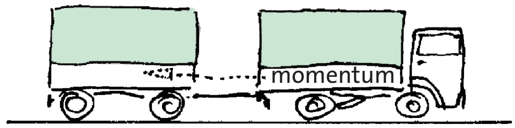
It is possible that a momentum current is flowing somewhere although nowhere the amount of momentum is changing. Fig. 2.28 shows an example: Lilly pulls a box over the floor at a constant velocity.

Again, we ask our old question: what is the path of the momentum? Hopefully, you can answer this question easily. Lilly pumps momentum out of the Earth via the rope into the box. Due to friction between the bottom of the box and the ground, it flows out of the box back into the Earth. Hence, we can say that the momentum flows „in a circuit", even if we do not know the exact way back through the Earth.

Fig. 2.29 shows a modified version of the experiment from Fig. 2.28: the box is not pulled over the ground but over a board on wheels.

The path of the momentum is even simpler in this case. As the board is installed on wheels, no momentum can flow into the Earth and Willy cannot pump any momentum out of the Earth. He therefore pumps momentum out of the board, the momentum continues flowing through the rope into the box, from the box it will flow back into the board. Hence, the momentum flows in a closed circuit also in this case. And this time, the path can be seen clearly at all points.

a)



b)

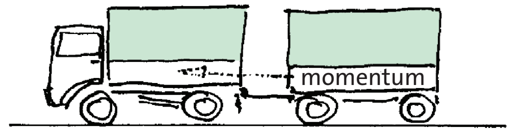


Fig. 2.27 A truck drives once to the right (a) and once to the left (b). Both times, the coupling bar is exposed to tensional stress and both times x-momentum is flowing in the negative x-direction.

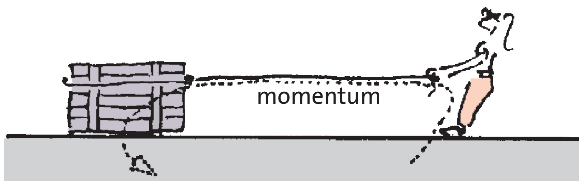


Fig. 2.28 Although a momentum current is flowing, there is no accumulation of momentum.

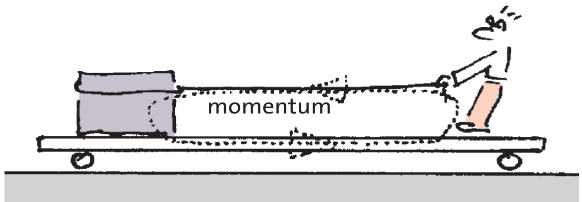


Fig. 2.29 Closed momentum circuit

We can also tell from the tensions that the momentum is really flowing to the left in the rope and to the right in the board: the rope is under tensional stress; therefore, the momentum is flowing to the left. The board is under compressional stress; thus the momentum goes to the right.

> Momentum can flow in a closed circuit. Then, the momentum does neither increase nor decrease at any point. A part of each momentum circuit is under compressional stress, another one under tensional stress.

The situation is even simpler in Fig. 2.30. Here, the momentum current is flowing in a closed circuit although nothing is moving anymore, and although we do not even have a „momentum pump“ anymore.

You might be surprised about the fact that now momentum is flowing without a drive. Hadn't we found out earlier that a drive is required to make a current flow? We now see that this rule does not always apply. There are currents without a drive. The fact that no drive is needed simply means that the current does not have to overcome a resistance.

There are also electric conductors that do not have a resistance, i.e. the superconductors. In an electric circuit that consists of a superconducting material, an electric current can flow without a drive.

Electric currents without resistance are rare; momentum circuits, in turn, are frequent. Figure 2.31 shows a further example.

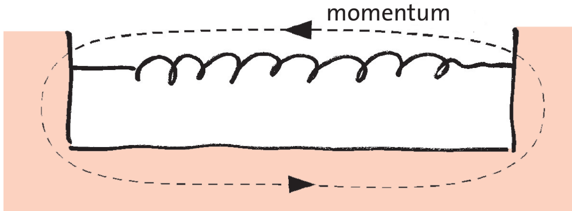


Fig. 2.30 Momentum circuit without drive

Fig. 2.31 Closed momentum circuit

Fig. 2.32 Dynamometer

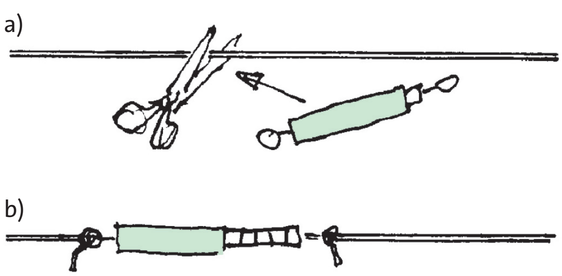


Fig. 2.33 (a) The strength of the momentum current in a rope should be measured. (b) The rope is cut and the dynamometer is connected with the newly formed ends.

2.9 The momentum current strength

A momentum current can be greater or smaller. A measure for such „great“ or „small“ of a current is the *momentum current strength*. It indicates how much momentum is flowing through an area per unit of time (how many Huygens are running through the area per second). The symbol for the momentum current strength is *F*. It is measured in Huygens per second (Hy/s).

If 12 Huygens are flowing per second through a rope, we have

$$F = 12 \text{ Hy/s}.$$

The unit Huygens per second (Hy/s) is usually abbreviated *Newton* (N):

$$\text{N} = \frac{\text{H}}{\text{s}}.$$

Hence, in our case:

$$F = 12 \text{ N}$$

The measuring unit was named after *Isaac Newton* 1643 – 1727).

Momentum current strengths can be measured easily with a so-called dynamometer. A particularly simple model is shown in Fig. 2.32.

However, we can only use it to measure „tensional momentum currents“. Fig. 2.33 shows how to use a dynamometer. The strength of the momentum current that flows through the rope in Fig. 2.33a should be measured. The rope is cut at any point and the new ends are connected to the two hooks of the dynamometer, Fig. 2.33b.

Exercises

1. A momentum current with a constant strength flows into a trolley. A momentum of 200 Huygens has accumulated within 10 seconds. What was the current strength? (Suppose there is no loss due to friction.)
2. When a truck starts driving, a momentum current of 6 000 N is flowing through the trailer coupling. What will be the momentum of the trailer after 5 s? (The friction losses of the trailer can be neglected.)
3. A constant momentum current of 40 N is flowing into a vehicle whose friction can be neglected. Represent the momentum graphically as a function of time.

2.10 Newton's law of motion

Let's look once again at Lilly who is charging a trolley with momentum, Fig. 2.34.

In a certain interval of time, a given amount of momentum is flowing through the cross-sectional surface S of the rope. As the momentum is flowing into the trolley, the trolley's momentum increases. The quotient of the momentum increase Δp of the trolley and the corresponding time period Δt is called rate of change of the momentum.

$$\frac{\Delta p}{\Delta t} = \text{rate of change of momentum}$$

If a momentum current of F = 5 Hy/s = 5 N is flowing through the rope, also the rate of change of the trolley's momentum will be 5 Hy/s:

rate of change of momentum = momentum current strength

Thus, we have:

$$\frac{\Delta p}{\Delta t} = F \qquad (2.1)$$

The Δ (delta) sign stands for a momentum portion and not the total momentum of a body and for a time interval and not the time of the day.

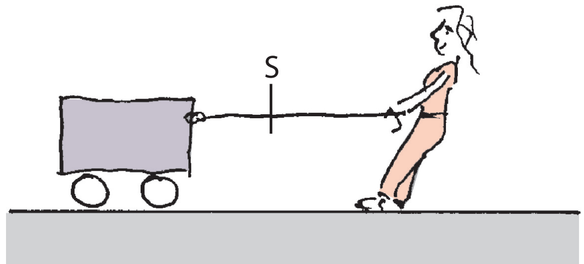


Fig. 2.34 A momentum current flows through the area S. Therefore, the momentum of the trolley increases

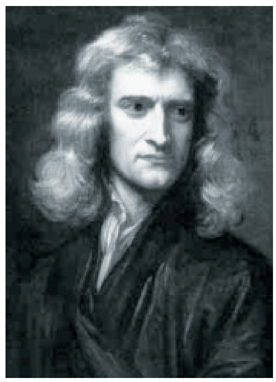


Fig. 2.35 Isaac Newton

Equation (2.1) is the famous Newton's law of motion. (There are actually three „Newton's laws of motion“ but the two others are only special cases of equation (2.1)).

Today it is hard to understand why it was so complicated to discover this law. Newton, Fig. 2.35, needed the law especially to describe the movement of celestial bodies: the planets and the Moon. This is because the momentum of the Moon is constantly changing at the expense of the momentum of the Earth. Today we know that momentum is flowing back and forth between the Earth and the Moon, i.e. through the gravitational field that surrounds all bodies. At Newton's time, however, nothing was known about fields yet. People imagined momentum to be transferred by a so-called action at a distance between the Earth and the Moon. Hence, there was not yet any idea about momentum currents. The quantity F therefore had a quite abstract meaning for Newton; he called it „force“ (in the Latin original „vis“).

Equation (2.1) is not yet complete. It does not take account the fact the momentum is a vector quantity. We can use only it if we are interested in a single momentum type – for example if we only deal with x-momentum.

The trolley in Fig. 2.36 is being charged with x-momentum, the one in Fig. 2.37 with y-momentum. Both

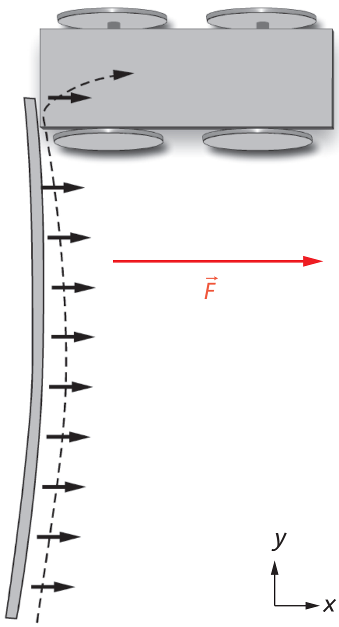


Fig. 2.36 *x*-momentum flows into the trolley through the bar.

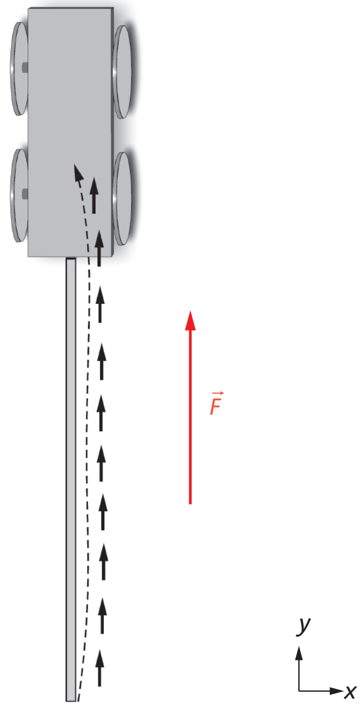


Fig. 2.37 *y*-momentum flows into the trolley through the bar.

times, 10 Hy/s (= 10 N) are flowing through a cross-section of the bar towards the trolley, but in the first case it is x- and the second it is y-momentum.

We can see that the momentum current is not defined unambiguously by saying that 10 Hy/s are flowing. In addition, we have to indicate the type of the flowing momentum; the direction of the flowing momentum has to be stated. Hence, the momentum current strength is, just as the momentum itself, a vector quantity. An arrow is therefore written on top of the symbol: $\vec{F}$. In the two figures 2.36 and 2.37, the vector arrow of the momentum current is shown. The length of the arrow indicates the magnitude of the momentum current – 10 N in this case – and the direction of the arrow indicates the direction of the transferred momentum. Caution: the direction of the momentum current vector arrow does not have anything to do with the flow direction, which is equal in both cases, i.e. through the bar from the bottom to the top. If we take into account the vectorial nature of the momentum, equation (1) will be transformed into:

$$\text{Newton's law of motion: } \frac{\Delta\vec{p}}{\Delta t} = \vec{F}$$

Exercises

1. Someone accelerates a trolley. (Friction can be neglected.) A dynamometer indicates the momentum current that flows into the trolley. It is pulled for 5 seconds. What will be the final velocity? (The mass of the trolley is 150 kg, the dynamometer indicates 15 N.)
2. A locomotive accelerates a train. Through the coupling between the locomotive and the wagons, a momentum current of 200 kN is flowing. What will be the momentum of the train (without locomotive) after 30 seconds? Now, the train has a velocity of 54 km/h. What is the mass of the train?
3. After standing still, a trolley with a mass of 42 kg is accelerated whereby a momentum current of 20 N is flowing through the pull bar. How much momentum will have flowed into the trolley after 3 seconds? At that time, its velocity will be 1.2 m/s. What will be its momentum? Where has the missing momentum gone?
4. Water with a velocity of 0.5 m/s is flowing in a straight tube with a length of 2 km and a diameter of 10 cm. The water is locked out by means of a valve at the end of the tube. Calculate the momentum that the water releases in this process. Where does that momentum go? The locking takes 2 s. What is the force of the water on the locking valve (the momentum current strength)? Note: calculate the water volume in liters first. 1 l of water has a mass of 1 kg.

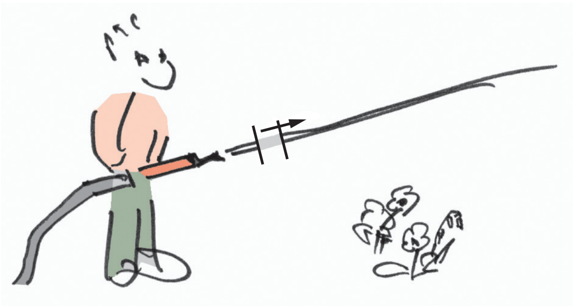


Fig. 2.38 The momentum of a water portion moves with the water portion: convective momentum current.

2.11 Convective momentum transports

In the momentum transports that we have analyzed so far, the momentum was always flowing through a material. The material only moved slightly or not at all in the process. But there is yet another way of transferring momentum: by a body or a substance that moves while simply taking along its own momentum.

We look at a small portion of the water in a water jet, Fig. 2.38. The water portion has momentum. At first, it is further on the left together with its momentum; later, it is further on the right. Hence, we have a momentum transport from the left to the right. Such a momentum transport is called *convective momentum current* (convectio = to carry along).

(Bear in mind: in the central heating system, the entropy is transported in a similar way. The water moves through the tubes together with the entropy that is contained in it. Here, we talk about a convective entropy current.)

When there is wind, the air transports momentum. Also in this case, there is a convective momentum current. The momentum that comes with the wind can be felt. It can be used to drive sailboats and we fear the damages that a storm can cause.

Also, rockets take advantage of a convective momentum current, Fig. 2.39. The rocket expels gases in a downward (or backward) direction with a high negative velocity and much negative momentum. The rocket itself receives the same value of positive momentum in the process.

Planes are driven in a similar way. Let's assume a plane flies in the positive x-direction. With the propeller or the jet engine, air that the plane „collects" at the front is charged with negative momentum and expelled on the rear. The plane receives the respective positive momentum, Fig. 2.40.

Here, we can see once again that the statement „air is not momentum-conducting" does not mean that no momentum can be transported by means of air.

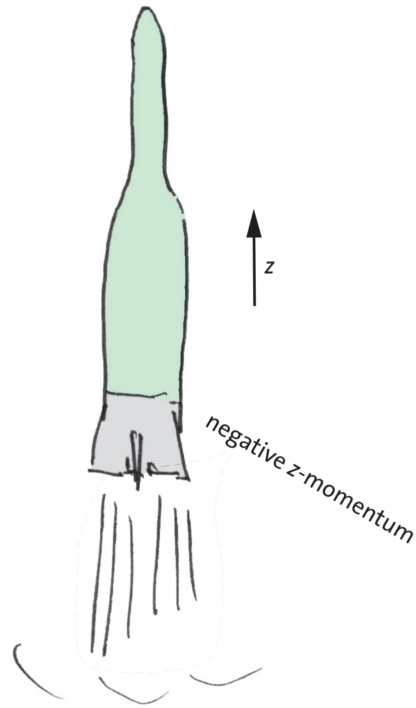


Fig. 2.39 The rocket expels combustion gases in a downward direction (in the negative *z*-direction). These combustion gases have negative momentum. The rocket receives the corresponding positive momentum.

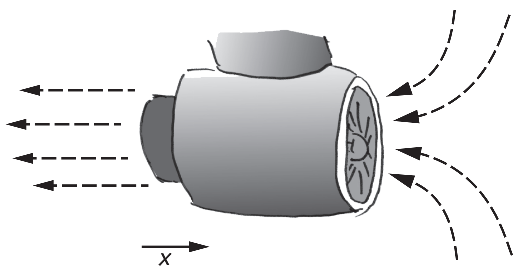


Fig. 2.40 The jet engine charges the air that it sucks in from ahead with negative ***x***-momentum. The plane receives the corresponding positive momentum.

> Convective momentum current: the momentum is transported together with a moving substance.

Exercises

1. 0.5 liters of water per second come out of a water sprayer with a velocity of 3 m/s. What is the momentum of a 1 m section of the water jet? What is the momentum current strength of the jet?
2. There is heavy wind with a wind velocity of 5 m/s. How much momentum does the air transport per second through an area of 10 m^2?

2.12 More about momentum conductors

So far, it has been rather easy for us to decide whether something conducts momentum or not. There have simply been good and bad conductors. As the momentum is a vector quantity, however, things are more complicated in most cases. We will look at a few examples:

Air

Yes, once again air. We have seen that it does not conduct momentum in a normal way but that it can be used to transport momentum convectively. In fact, there are other possibilities, too. If Lilly, Fig. 2.22, wanted to transfer momentum to the trolley, she would have to ensure that the air pressure increases on one side of the trolley. But she will not achieve a pressure increase in front of the trolley by simply pushing against the air ahead of her. The air escapes towards the side. But if it is prevented from escaping, a momentum transport will be possible. In Fig. 2.41, ahead of the block, i.e. inside cylinder A, the pressure can be increased by pushing against the piston of cylinder B. This effect is taken advantage of in pneumatic construction machines, e.g. in pneumatic hammers.

Another possibility to transfer momentum with the air is to perform a very fast pressure increase. This pressure increase will then continue moving as a sound wave. Lilly hits the tambourine, Fig. 2.42, and immediately after, the bead will bounce away from Willy's tambourine. The same principle applies for the transfer of momentum during an explosion that destroys the window glasses.

Wheels

The most important technical device that is used to prevent the momentum current from flowing from a body into the Earth is the wheel. Wheels are used for momentum insulation. However, this is only true for momentum with the same direction as the vehicle. In order to avoid having to stick to a coordinate axis, we call it longitudinal momentum. Momentum whose vector arrow is perpendicular to the vehicle is called transverse momentum.

In Fig. 2.43a, Willy tries to charge the board with momentum by means of a toy trolley. But it does not work, i.e. at least not with Willy's method. In Fig. 2.43b, he shows how it is done.

> Wheels do not conduct longitudinal momentum but they conduct transverse momentum.

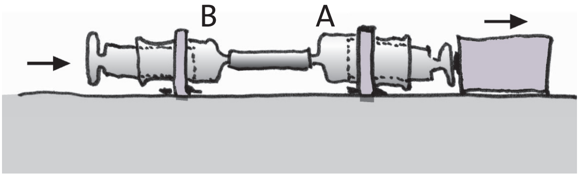


Fig. 2.41 Momentum flows through the air in cylinder A into the piston and further into the block.

Fig. 2.42 Lilly hits the tambourine. The bead on Willy's tambourine bounces away. Momentum has moved through the air.

a)

b)

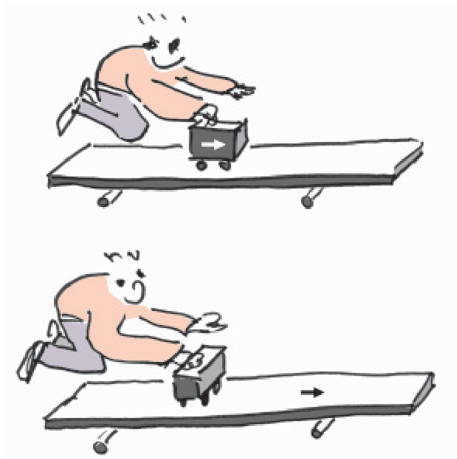


Fig. 2.43 (a) The wheels are nonconductors for longitudinal momentum. No momentum enters the board. (b) The wheels are conductors for transverse momentum. The momentum enters the board.

It is important for them to conduct the transverse momentum. Otherwise, cars could not go into a turn. Sometimes, this is even the case.

> In case of black ice, the wheels do not conduct any transverse momentum into the Earth.

Railway vehicles are safer in this respect. Here, transverse momentum is always transferred very well from the wheels to the railway track.

Ropes

In Fig. 2.44a, Lilly makes the trolley move to the right, i.e. in the positive x-direction, by means of the rope. She „pumps“ x-momentum into the trolley. In the rope, the momentum flows from the right to the left, i.e. in the negative x-direction. In the figure below, she tries it from the left – and of course it does not work. We see that x-momentum can only flow in the negative x-direction through the rope.

In Fig. 2.45, Willy tries in another way: he attempts to make momentum, whose direction is transversal to the rope, move through the rope – again without success.

We conclude:

> A rope is a conductor only for momentum whose vector arrow is parallel to the direction of the rope. The momentum flows in the direction opposite to that of the vector arrow.

We would like to apply the rule. Fig. 2.46 shows a top view of a trolley that is pulled by means of a rope. However, it is not pulled in a forward direction but a bit to the side. A momentum current of 40 N flows in the rope. How much momentum escapes into the Earth?

The momentum that flows in the rope must have the same direction as the rope. The corresponding current strength vector is called $\vec{F}$. We decompose this current in two parts, Fig. 2.47:

- one part $\vec{F}_{trans}$ that is transverse to the trolley direction and that escapes through the wheels;
- one part $\vec{F}_{long}$ that flows in the direction of the trolley and that causes the momentum increase of the trolley.

We have decompose this current in two parts, Fig. 2.47:

$$F_{trans} = F \cdot \sin 30° = 20 \text{ N},$$
$$F_{long} = F \cdot \cos 30° = 35 \text{ N}.$$

Hence: a transverse momentum of 20 Hy escapes per second into the Earth, and the momentum of the trolley increases by 35 Hy.

Fields

A magnet A is fastened on a small vehicle, Fig. 2.48. A second magnet B is approached to this magnet in a way that like poles are opposite to one another: north pole to north pole and south pole to south pole. If magnet B comes sufficiently close to magnet A, the vehicle will start moving, its momentum will increase.

The momentum conductor between A and B is the *magnetic field* that is attached at the poles of the magnets.

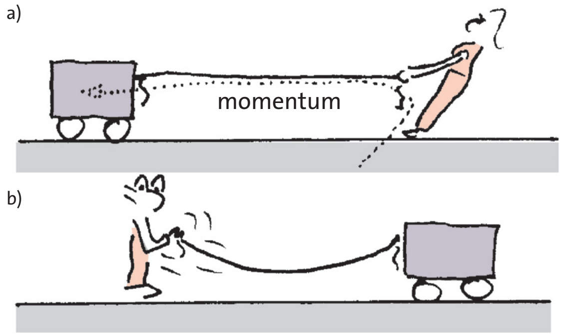


Fig. 2.44 (a) x-momentum flows leftwards through the rope – in the negative x-direction. (b) It cannot flow in the positive x-direction in a rope.

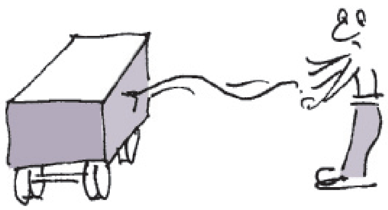


Fig. 2.45 No momentum whose vector arrow is transverse to the rope can flow through the rope.

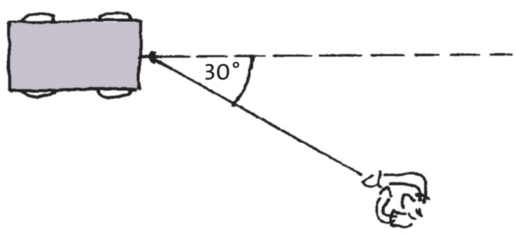


Fig. 2.46 Someone pulls on the rope. The x-momentum of the trolley increases.

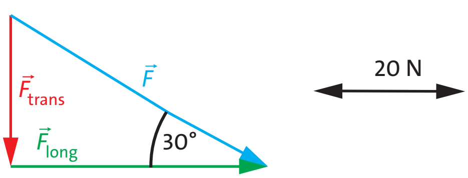


Fig. 2.47 Breaking down the momentum current of a rope into a longitudinal and a transverse component

Later, you will get to know some other fields, too: electric fields and gravitational fields. Just as the magnetic field, these fields are invisible, and just as the magnetic field, they conduct momentum. An electric field is attached to any body that bears an electric charge and a gravitational field is attached to any body that has a mass. As bodies always have a mass, there is consequently a gravitational field around every body.

Magnetic fields are momentum conductors.

Exercises

1. A cylindrical handle Z can slide back and forth on a bar S without any friction, Fig. 2.49a. Which momentum can permeate the connection between the handle and the bar and for which momentum is it nonconducting?
2. The cylinder Z_1 can slide back and forth on the bar S and the cylinders Z_2 and Z_3 can slide on the frame R, Fig. 2.49b. Which momentum can permeate the connection between Z_1 and the frame and for which momentum is it nonconducting?
3. A car is towing another one. The cars are driving in the same direction but are laterally offset against each other by 1 m, Fig. 2.50. The towing rope has a length of 3 m. A momentum current of 500 N is flowing through the rope. Which momentum current contributes to the movement of the towed car?
4. Ropes conduct x-momentum only in the negative x-direction. Invent a device that conducts x-momentum only in the positive x-direction.
5. Ropes are conductors only for momentum in one direction. There are devices that let air pass in only one direction; there are devices that can only be passed by people in one direction; there are devices that are conductors for electricity in only one direction. What are we talking about?

2.13 Hooke's law

We would like to build a momentum current meter ourselves. We suppose the device has not been invented yet and that the measuring unit of the momentum current strength has not yet been defined.

We need a large number of identical rubber rings. First, we define our own measuring unit. We hold a rubber ring in front of a ruler in a way as to unwind its total length but without stretching it beyond its normal length, Fig. 2.51, and measure its length. Let's assume we find 10 cm = 0.1 m. As the rubber ring is loose, no momentum current is flowing through it yet. Now we stretch it until it is 0.15 m long. Now, a momentum current is flowing. We define the strength of this momentum current to be our current strength unit. (As each ring consists of two rubber threads that are located next to each other, half a current strength unit is flowing in each of these threads.)

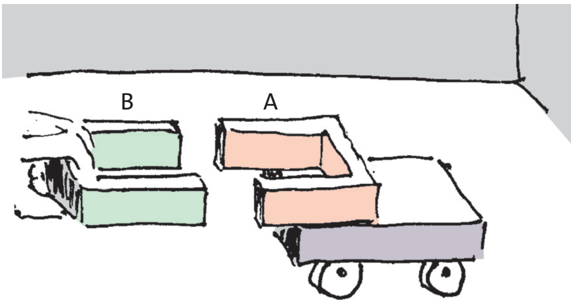


Fig. 2.48 The momentum goes through the magnetic field into the trolley Magnetic fields are momentum conductors.

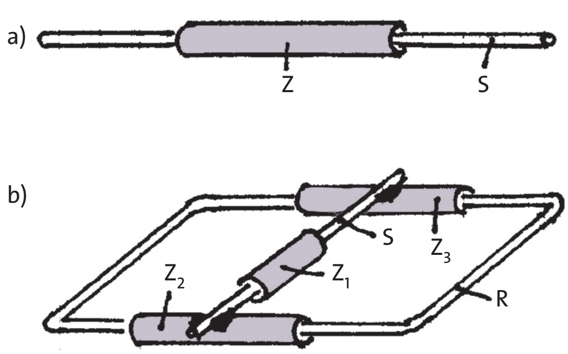


Fig. 2.49 For exercises 1 and 2

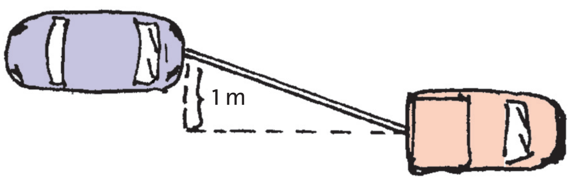


Fig. 2.50 For exercise 3

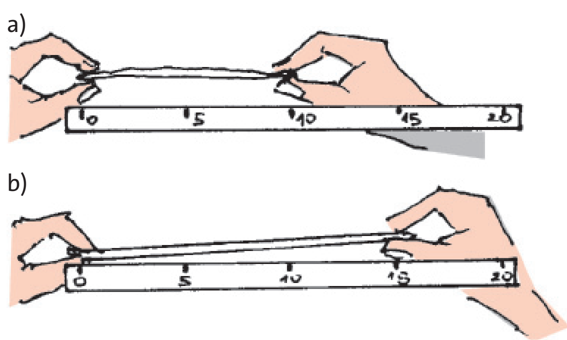


Fig. 2.51 Definition of a unit for momentum current strength. (a) The rubber band is unwound but not stretched. (b) The rubber band was extended by 5 cm.

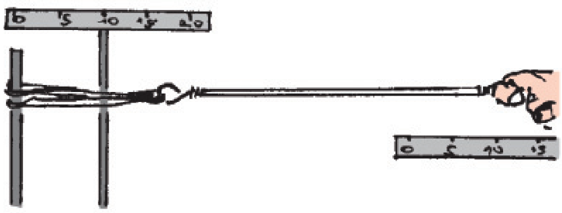


Fig. 2.52 An expander rope is calibrated with rubber band units.

Now we can create as many current strength units as we want with other rubber rings. Hence, we can create multiples of our unit. For example, if we hang 3 rings that are stretched to 15 cm next to each other, three current strength units will flow through all of them together.

By means of our rubber rings, we can also calibrate another elastic object, for example the rubber rope of an expander, Fig. 2.52. Therefore, we let one, two, three, and so forth… current strength units flow through the rope and measure the respective change of its length compared to the length in the unstressed state.

In Fig. 2.53, the momentum current strength F is indicated above the extension s. This curve is the calibration curve of the expander. If we wish to measure a momentum current strength now, we will no longer need to use our somehow complicated method with the uniform rubber rings. We can use the expander rope.

For example the strength of the current, which flows into a trolley that we are pulling, should be measured. Therefore, we simply pull the trolley by means of the expander and measure the extension of this rope. If the extension is for example 0.25 m, we can read from the calibration curve that the momentum current has a strength of 4 units.

We would now like to illustrate the relationship between the extension and the momentum current for yet another object: for a steel spring. The result is shown in Fig. 2.54.

The relationship is simpler than for the expander rope: it is linear. The extension s and the momentum current strength F are proportional to each other. We say that the spring complies with *Hooke's law*. As a formula, the law can be formulated as follows:

Hooke's law: $D \cdot \vec{s} = \vec{F}$

D is a constant for a given spring – the spring constant. Its measuring unit is N/m. In general, for

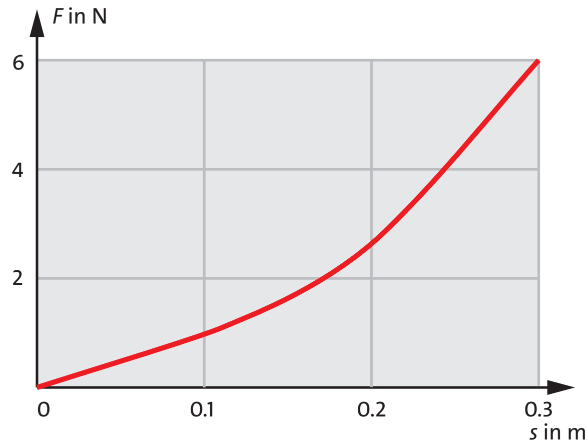


Fig. 2.53 Calibration curve of the expander rope: the momentum current strength F is plotted over the extension s of the rope.

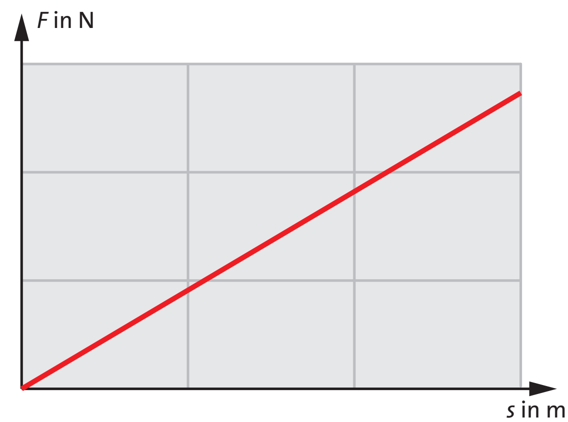


Fig. 2.54 For a steel spring, the relationship between the momentum current strength and the extension is linear.

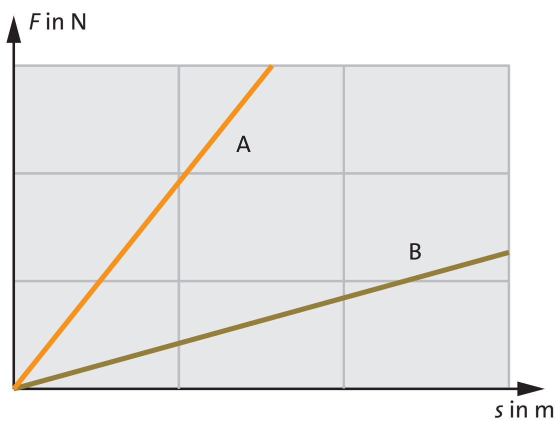


Fig. 2.55 The spring constant of spring A is greater than that of spring B. Spring A is harder.

different springs, the spring constant has different values. Fig. 3.55 shows the relationship between F and s for two different springs. For spring A, D has a higher value than for spring B. If spring A and spring B are stretched by the same value, the momentum current in spring A will be stronger than in spring B. But a stronger momentum current means a higher tensional stress. Therefore, the spring with the greater spring constant is the harder spring.

Many springs can be subjected not only to tension but also to compression. For such springs, Hooke's law applies both for extension (positive values of s) as well as for contraction (negative values of s).

Exercises

1. A spring has a spring constant of D = 150 N/m. What is its extension if a momentum current of (a) 12 N, (b) 24 N flows through it?
2. The F-s-relationship illustrated in Figure 2.56 was measured for a given rope.
(a) What is the extension of the rope if a momentum current of 15 N flows through it? What is the extension in case of a current strength of 30 N?
(b) What is the momentum current strength if the rope is extended by 20 cm?
(c) What can we feel when pulling the rope apart with our hands? Compare with a steel spring.
3. How could we build an arrangement whose F-s-relationship is that of Fig. 2.57?
4. Two springs are concatenated and built into a rope through in which a momentum current is flowing. One of the springs extends four times as much as the other one. How is the ratio of the two spring constants?
5. (a) Two identical springs are connected „in parallel“, Fig. 2.58a. Each one has the spring constant D. What is the spring constant of the overall system (gray box)? (b) The same for two springs „in a series“, Fig. 2.58b.

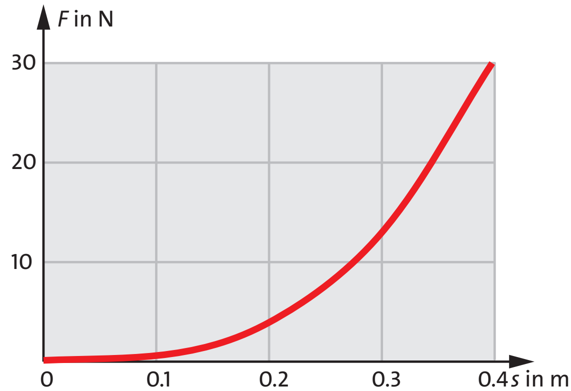


Fig. 2.56 For exercise 2

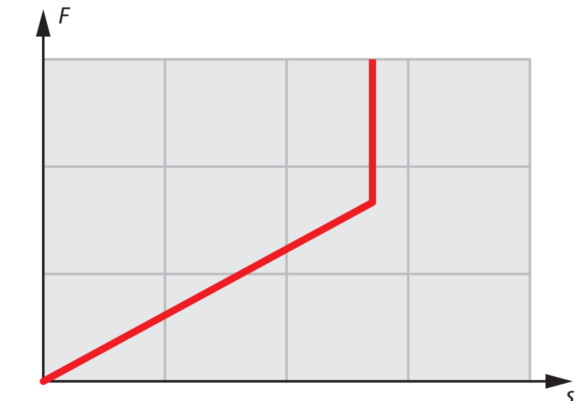


Fig. 2.57 For exercise 3

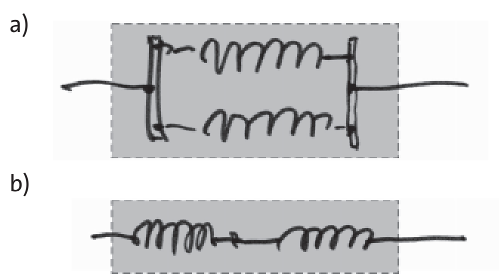


Fig. 2.58 For exercise 5

2.14 Velocity, acceleration, angular velocity

Velocity

Let's assume that a car (or another body) moves with a constant velocity v. v can be calculated from the traveled distance Δs and the time Δt required for it:

$$v = \frac{\Delta s}{\Delta t}$$

We use the deltas once again. While we denominate the position with s (in a given coordinate system), Δs refers to the distance between two points. Accordingly, Δt is not the time of the day but a duration: the time interval in which the body travels the distance Δs. If a car travels 200 m in 10 seconds, its velocity will be

$$v = \frac{200\ \text{m}}{10\ \text{s}} = 20\ \text{m/s}.$$

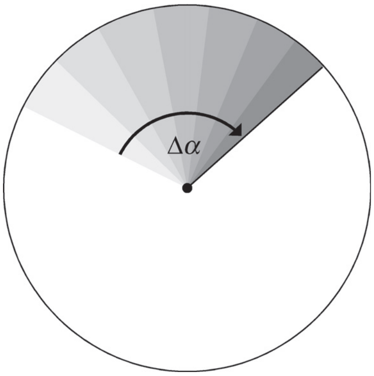


Fig. 2.59 In the time interval *Δt*, the radius moves over the angular interval *Δα*.

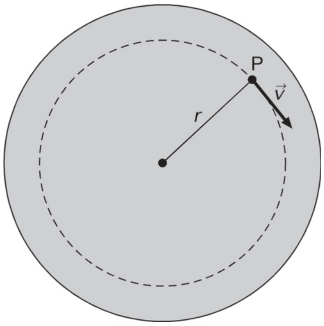


Fig. 2.60 The point P of the rotating body moves on a circular path.

2 MOMENTUM AND MOMENTUM CURRENTS

Acceleration

A train starts moving (in the positive *x*-direction). We assume that its velocity increases regularly: by 5 km/h every 10 seconds. Or by 30 km/h in a minute. Or by 150 km/h in 5 minutes. We can also say that the train is accelerated uniformly. The quotient of the velocity increase Δv and time interval Δt is called *acceleration a*:

$$a = \frac{\Delta v}{\Delta t}$$

In the case of our train, we obtain:

$$a = \frac{5\ \text{km/h}}{10\ \text{s}} = \frac{5000\ \text{m}}{3600\ \text{s} \cdot 10\ \text{s}} = 0.139\ \text{m/s}^2.$$

The SI measurement unit of the acceleration turns out to be m/s^2. In general, the acceleration of a vehicle is not constant as we have assumed here. If the velocity is constant, we have Δv = 0 and therefore also a = 0.

If a vehicle (that moves in the positive *x*-direction) slows down, Δv and also the acceleration will become negative.

Angular velocity

We assume, a wheel (or another body) rotates uniformly. One says that it rotates with a constant angular velocity ω. We mark a radius on the wheel and observe its movement, Fig. 2.59. The line rotates regularly around the center, it moves over the same angle every second. In the time interval Δt, it moves over the angular interval Δα. The angular velocity is calculated from the covered angle Δα and the time Δt required for it:

$$\omega = \frac{\Delta \alpha}{\Delta t}$$

The angular velocity is part of several other physical equations. In order to obtain from these equations results in the correct measuring units, i.e. the SI units, the angle α needs to be given in the radian measure. As no unit symbol is used for the radian, the measuring unit of the angular velocity turns out to be 1/s or s^{-1}.

The angular velocity is also referred to as rotational speed. If this name is used, we need to express it with another measuring unit: revolutions per minute (rpm).

We suppose that the shaft of an electric motor rotates with 2000 rpm. Then, the angular velocity will be:

$$\omega = \frac{2000 \cdot 2\pi}{60\ \text{s}} = \frac{212566}{60\ \text{s}} = 209\ \text{s}^{-1}.$$

We look at the point P of a rotating wheel, Fig. 2.60. P moves on a circular path. There is a relationship between the magnitude v of the point's velocity and the angular velocity ω of the wheel.

Calculating this relationship is most convenient when we use a full rotation as a reference. We denominate the orbital period with T. The full angle of 360° is equal to 2π when expressed in the radian measurement unit. Therefore, the angular velocity becomes:

$$\omega = \frac{2\pi}{T}$$

The distance traveled in the time T is $2\pi r$. Therefore, the normal, linear velocity becomes:

$$v = \frac{2\pi r}{T}$$

The last equation can be written in a slightly different way:

$$v = \frac{2\pi}{T} \cdot r$$

The quotient on the right side of the equation is equal to the angular velocity. We insert and obtain:

$$v = \omega r.$$

> Rotating body:
> $v = \omega \cdot r$
> v = velocity of a point P
> ω = angular velocity of the body
> r = distance between P and the center of rotation

Exercises

1. The flywheel of a car engine has a diameter of 30 cm. The engine runs at 3500 revolutions per minute. What is its angular velocity (in 1/s)? What is the value of the velocity of its external edge?
2. What is the angular velocity of the Earth's rotation around its axis? What is the absolute value of the velocity of a point on the equator?
3. What is the angular velocity for the movement of the Earth around the Sun? For this movement, the Earth has a velocity of 30 km/s. Calculate the distance Earth – Sun.

2.15 Momentum changes for circular movements

Willy lets an electric toy car drive around with a remote control, Fig. 2.61. Willy: „Now I'll let it drive in a circle with a constant velocity.“ Lilly: „Oops, you can't do that. Either in a circle or at a constant velocity!“. Willy: „Oh, you are right. What I meant is…“

What did Willy mean? That the absolute value or magnitude of the velocity vector is constant. As the car makes a circular movement, the direction of the velocity vector is constantly changing. Therefore, also the momentum of the car changes. The car is constantly receiving momentum from the Earth.

Fig. 2.62 illustrates the matter schematically. On the left, the momentum vectors are marked on the circular path at different instants of time. The momentum vector arrow rotates just like the direction of the car. On the right, the same momentum vector arrows are displayed once again, but in a way that their starting points are the same. Here we see that the momentum vector is rotating.

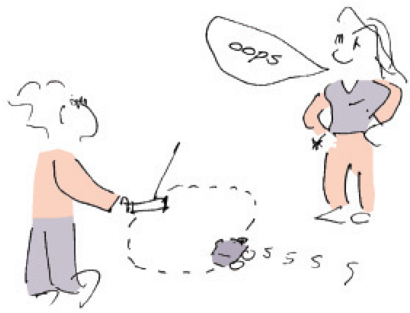


Fig. 2.61 The car is driving in a circle while the absolute value of its velocity is constant.

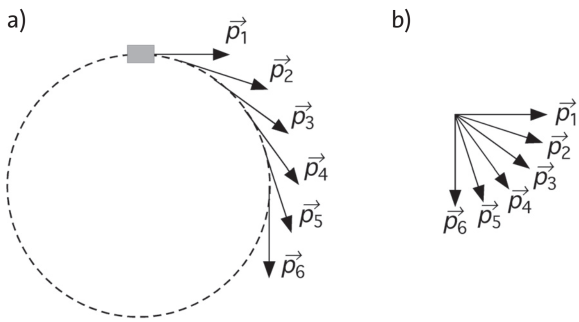


Fig. 2.62 (a) The momentum vector at different instants of time. (b) While the car is driving in a circle, the momentum vector is turning.

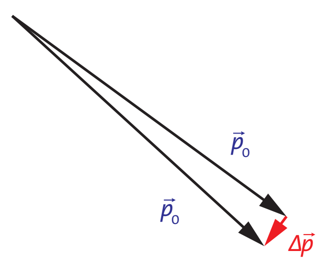


Fig. 2.63 The „momentum change“ vector is at a right angle to the momentum vector.

Fig. 2.63 shows two momentum vectors $\vec{p}_0$ and $\vec{p}_1$ at two instants of time that are very close to each other: t_0 and t_1. In the time interval

$$\Delta t = t_1 - t_0$$

the momentum changes by

$$\Delta\vec{p} = \vec{p}_1 - \vec{p}_0.$$

The additional momentum is at a right angle to the direction of movement ("transverse momentum").

The magnitude of the rate of change is calculated with the formula:

$$\frac{\Delta p}{\Delta t} = m\frac{v^2}{r}$$

Here, m is the mass of the body, v its velocity and r the radius of the circular trajectory. The proof of this formula is a bit confusing. We would like to skip it and only convince ourselves that the formula is plausible, i.e. that it supplies results that are in line with our expectations.

The dependence on *m*

Let's assume the car is travelling straight in the x-direction, i.e. that it has only x-momentum. After having traveled a quarter rotation, it must have released all of its x-momentum. The greater its mass, the more x-momentum it has at the beginning and the more it needs to release or to absorb per time interval.

The dependence on *r*

The car drives once a narrow and once a wide circle, while the absolute value of the velocity is the same both times. It is clear that the momentum change per second is smaller for the wide circle. Therefore, the r is in the denominator.

The dependence on *v*

The higher the velocity, the longer the momentum vector arrow and the greater the rate of change. In addition, the momentum vector rotates faster and the rate of change is higher for this reason. Hence, the velocity has a double effect on the rate of change. This is the reason for the proportionality to v^2.

> Rate of change of the momentum for a circular movement (with a constant absolute value of the velocity):
>
> $$\frac{\Delta p}{\Delta t} = m\frac{v^2}{r}$$
>
> The vector $\Delta\vec{p}$ of the momentum change is at a right angle to the momentum $\vec{p}$.

By means of $v = \omega \cdot r$ we can convert the equation into

$$\frac{\Delta p}{\Delta t} = m\omega^2 r.$$

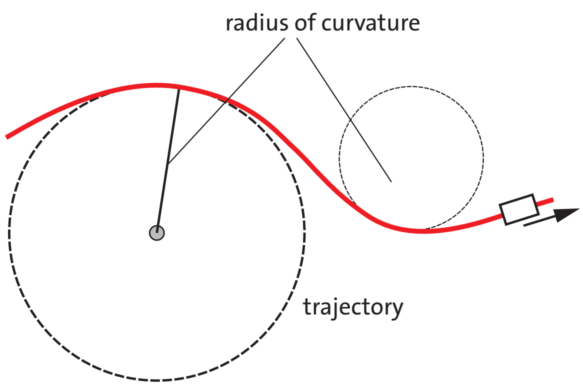


Fig. 2.64 Trajectory of a car. The radius of curvature changes each time the position of the steering wheel is modified.

At first it appears that the examined body that we consider – for example a car – would have to move on a circular path, i.e. to perform an entire circular movement. But this is not necessary, though. Rather, the formula applies in every instant of time. It also applies when the car only drives a very short bend (with a constant absolute value of the velocity). Then, r is no longer the radius of a traveled circular path but the radius of an imaginary circle that is associated to the bend at that instant of time. r is called the radius of curvature of the path at the point that we are considering, Fig. 2.64. Every time we change the orientation of the steering wheel of a car, we change the radius of curvature of the car's trajectory. As long as the steering wheel is held, the car drives on a path with a constant radius of curvature.

> Any given orientation of the steering wheel corresponds to a well-defined radius of curvature of the car's trajectory.

Exercises

1. The road from village A to village B has a 90° curve. Geometrically, the road consists of two straight line pieces that are connected by a quarter circle bend. Which steering wheel movement do the car drivers have to make while passing the curve? What is the rate of change of the momentum of the car as a function of time? The road is obviously poorly built. Curves of roads and highways are usually not circular bends. How does a well-designed curve look like? What is the rate of change of the momentum as a function of time in this case?
2. Willy has suspended a 500 g ball on a string and spins it around over his head in a circle. One rotation takes 0.8 s. The string has a length of 1 m. What is the magnitude of the momentum current that flows through the string?

2.16 Pulleys

A wheel with a groove that is used to deflect ropes is called pulley. Fig. 2.65 shows an application. Pulleys can also be found in cranes or in lifting tackles.

We would like to examine the behavior of pulleys. A momentum current meter is installed in each of the three rope sections A, B and C in Fig. 2.66. We pull on the loop on the right end of rope A so that the respective meter displays 12 N. What is indicated by the meters in sections B and C?

We can predict the result. On one hand, we know that the momentum current that passes through rope A continues to flow in the ropes B and C. Therefore, we must have:

$$F_A = F_B + F_C$$

As the whole arrangement is symmetric, the following has to apply in addition:

$$F_B = F_C.$$

With these two equations we obtain

$$F_B = F_A/2 \text{ and } F_C = F_A/2.$$

Or in our case: If $F_A = 12$ N we obtain $F_B = 6$ N and $F_C = 6$ N.

The dashed lines in Fig. 2.67 show the path of the momentum. It is a pure x-momentum (if we assume the positive x-direction to the right).

Let's take a look at a more complicated case, Fig. 2.68.

We pull on rope A and find: no matter how strongly we pull, the meter in A always indicates the same as that in B. But this does not mean that the momentum current strengths in A and B are equal. Although they have the same absolute value, they differ in their direction. We remember: the momentum that is flowing in a rope always has the same direction as the rope. In the rope section B, only x-momentum flows towards the pulley while x- and y-momentum flow towards the pulley in section A.

The sum of the two momentum types flows into the Earth through the support of the pulley. With sum we mean the vector sum, of course.

We summarize:

> When a rope runs over a freely rotatable wheel (a pulley), the momentum currents in both parts of the rope have the same absolute value..

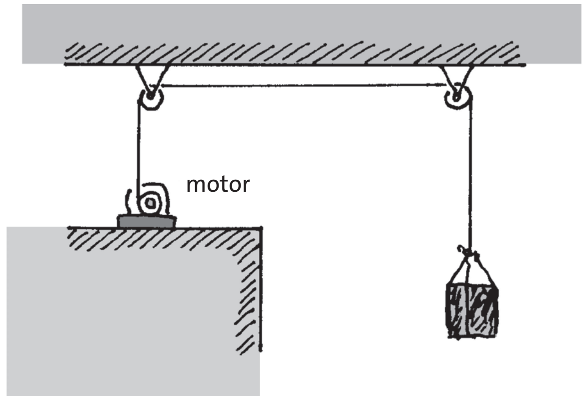


Fig. 2.65 The motor pulls up a load. The rope runs over two pulleys.

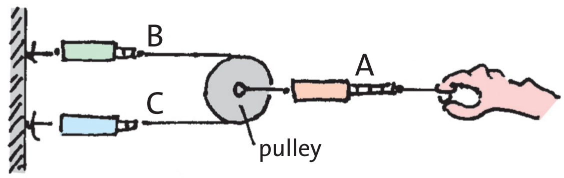


Fig. 2.66 The momentum current that flows through rope A is divided in the pulley into two equal currents.

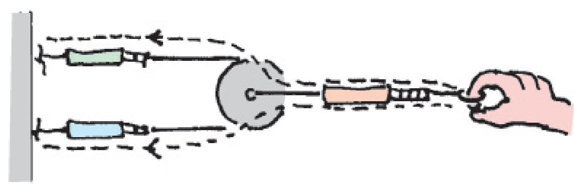


Fig. 2.67 Path of the x-momentum

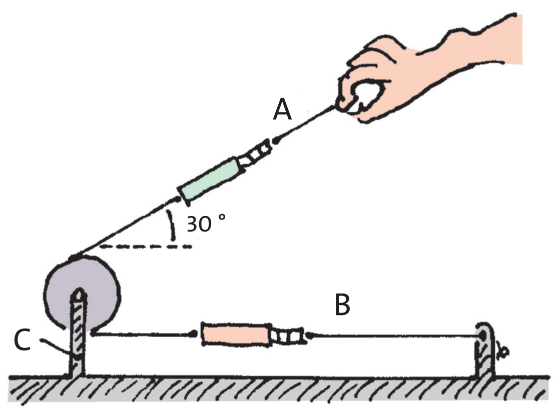


Fig. 2.68 The momentum currents in ropes A and B have the same magnitude but different directions.

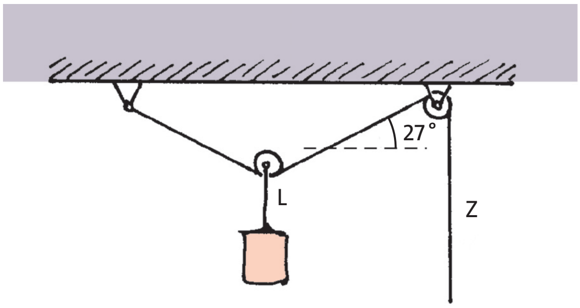


Fig. 2.69 For exercise 2

Exercises

1. The absolute value of the momentum current in rope A in Fig. 2.68 is 30 N. Draw vector arrows for the momentum currents in ropes A and B. Construct the current vector in C by means of vector addition.
2. A momentum current of 80 N is flowing in rope L in Fig. 2.69, on which a load is suspended. Construct the vector arrow for the current in section Z of the rope and the one in the suspension of the pulley on the right.

2.17 Relationship between pressure and momentum current

A block K is clamped between two walls by means of a spring, Fig. 2.70. A momentum current flows through the arrangement. The flowing momentum current is always connected to the mechanical stress that the conductor of the current is exposed to: compressional or tensional stress. You remember the rule: a momentum current to the right means pressure, a momentum current to the left means tension.

We look at the tension of the block. As the momentum current spreads out over the whole block, every part of the block is exposed to compressional stress; every part „feels“ the pressure, Fig. 2.71.

We compare the two blocks K_1 and K_2 in Fig. 2.72.

As the two springs are completely identical, momentum currents with the same strength are flowing in both cases – let's assume 200 Hy/s = 200 N. Block K_2 has a larger cross-sectional area than K_1. Therefore, the momentum current spreads out over a larger area. Thus, the momentum current per area is lower.

$$\frac{200}{25}\,\text{Hy/s} = 8\ \text{N}.$$

are flowing through each square centimeter of the cross-sectional area of block K_1.

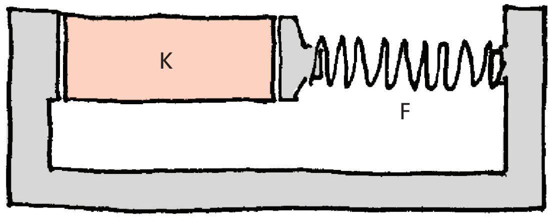


Fig. 2.70 The block K is exposed to compressional stress.

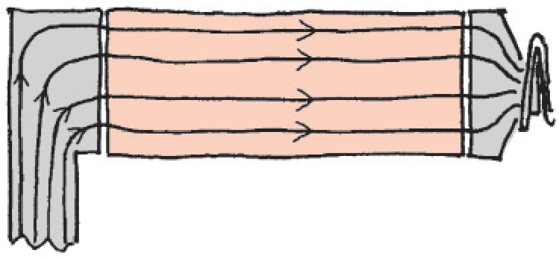


Fig. 2.71 The momentum current spreads out over the whole cross-sectional area of the block.

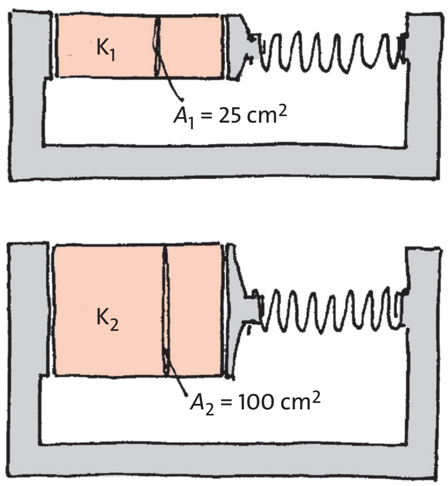


Fig. 2.72 The momentum currents in K_1 and K_2 have the same strength. The momentum current strength per area, i.e. the pressure, in K_1 is higher than in K_2.

$$\frac{200}{100}\,\text{Hy/s} = 2\ \text{N}.$$

are flowing through each square centimeter of the cross-section area of block K_2.

A piece of matter of K_1 therefore "feels" a higher pressure than a piece of matter of K_2 of the same size.

Hence, we can see: we can use the momentum current strength per area to characterize the mechanical stress at a defined point somewhere inside a body. This quantity, i.e. the quotient of the momentum current strength and the area through which the current is flowing, is called *pressure*. It is the same physical quantity that we have already seen earlier in a different context.

As the pressure is denominated with the letter p, we have

$$p = \frac{F}{A}.$$

If we insert the momentum current strength in Newton (N) and the area in m^2, we obtain N/m^2 as the measuring unit for the pressure. This unit is called *Pascal*, abbreviated as Pa. Hence,

$$\mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2}.$$

1 Pa is a very low pressure. Therefore, the larger units

1 kPa = 1000 Pa and 1 MPa = 1 000 000 Pa,

are often used; or also the *bar*:

1 bar = 100 000 Pa.

Once again back to our blocks: in block K_1, there is a pressure or, in other words, a *compressional stress* of

$$p_1 = \frac{F}{A_1} = \frac{200\ \mathrm{N}}{0.0025\ \mathrm{m}^2} = 80\,000\ \mathrm{Pa} = 80\ \mathrm{kPa}.$$

For block K2, we obtain

$$p_2 = \frac{F}{A_2} = \frac{200\ \mathrm{N}}{0.01\ \mathrm{m}^2} = 20\,000\ \mathrm{Pa} = 20\ \mathrm{kPa}.$$

(The areas A_1 and A_2 have to be expressed in m^2 so that we obtain Pa as a unit for the pressure.)

A momentum current of 200 N is flowing in the negative direction through the body K in Fig. 2.73.

This is taken into account in the calculation of the quantity p by adding a minus sign to the current strength value. Hence,

$$p = \frac{-200\ \mathrm{N}}{0.01\ \mathrm{m}^2} = -20\,000\ \mathrm{Pa} = -20\ \mathrm{kPa}.$$

A negative pressure value consequently stands for a tensional stress.

Summary:

> Pressure = momentum current strength divided by area.

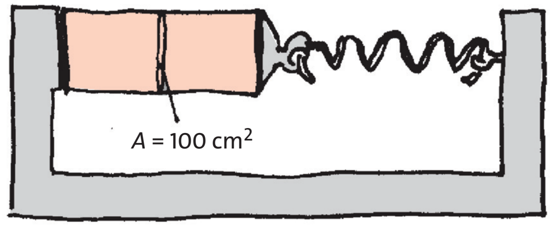


Fig. 2.73 The block is exposed to tensional stress, the pressure is negative.

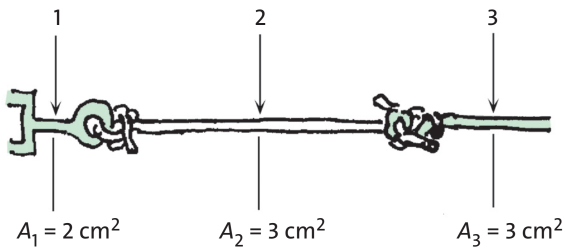


Fig. 2.74 For exercise 1

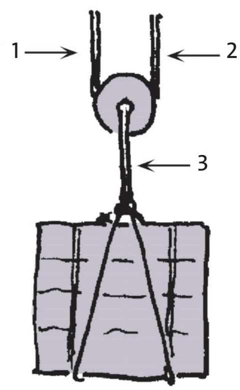


Fig. 2.75 For exercise 2

Exercises

1. A car is towed away. Fig. 2.74 shows a detail: the hook on the car that is pulled, a piece of wire rope and a plastic rope knotted to it. A momentum current of 420 N is flowing into the car in the process. Calculate the tension at the points 1, 2 and 3. Be aware of the sign: compressional or tensional stress?
2. The ropes in Fig. 2.75 have a cross-sectional area of 1.5 cm^2. The box has a mass of 12 kg. Calculate the tensional stress at the three points 1, 2 and 3.
3. You press a drawing pin into a wooden board. Estimate the pressure that exists in the middle, i.e. halfway up the pin. What is the pressure on the tip of the drawing pin?
4. Estimate the pressure that will arise on the tip of a hook if we hit the nail with a hammer.

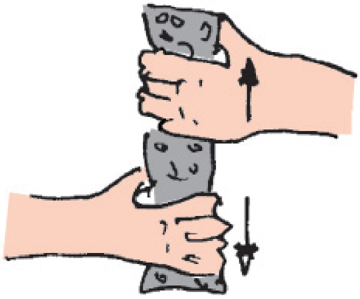


Fig. 2.76 The inside of a sponge is exposed to tensional stress in the vertical direction and to compressional stress in a horizontal direction.

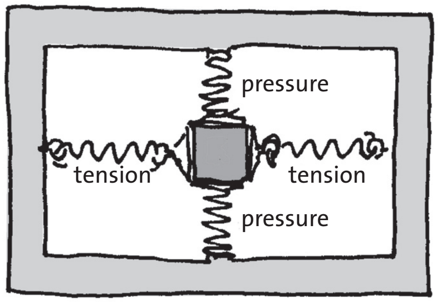


Fig. 2.77 The block is exposed to pressure in the vertical direction and to tension in the horizontal direction.

2.18 Stress in three directions

We want to expose a body to compressional and tensional stress at the same time. One could come up with the objection that this is impossible: „A body is either under compressional or under tensional stress; they exclude each other.“ We shall disregard this objection and simply try – and we are successful!

We take an object, for example a blackboard eraser, grab it with both hands and press the fingers together. Simultaneously, we pull the hands apart, Fig. 2.76.

The inside of the sponge is now actually exposed to both compression and tension. Pressure in the horizontal directions and tension in the vertical direction. Fig. 2.77 illustrates a similar situation: the block K is exposed to tensional stress in the horizontal direction and to compression in the vertical direction. Of course, it can also be put under tension or under compression in both directions. And the compressional or tensional stresses in the horizontal and vertical direction can have different magnitudes.

In the case of Fig. 2.78, the horizontal pressure has the value

$$p_1 = \frac{50\ \text{N}}{0.01\ \text{m}^2} = 5000\ \text{Pa} = 5\ \text{kPa}$$

and the vertical one

$$p_2 = \frac{300\ \text{N}}{0.015\ \text{m}^2} = 20000\ \text{Pa} = 20\ \text{kPa}.$$

Finally, the block can also be exposed to any compressional or tensional stress in the third direction in space, Fig. 2.79. For example, we can have the following:

$p_1 = 5000$ Pa
$p_2 = -2000$ Pa
$p_3 = -40\,000$ Pa

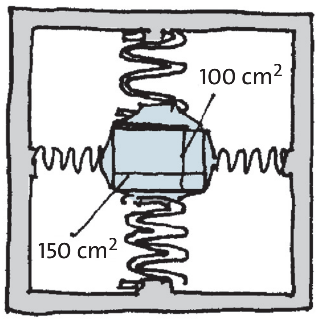


Fig. 2.78 The pressures in the horizontal and the vertical direction are different.

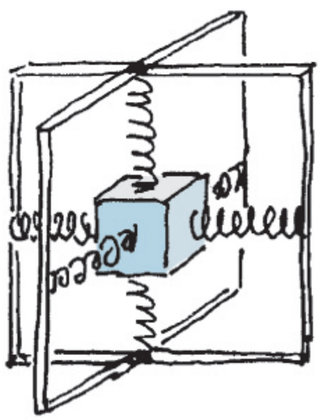


Fig. 2.79 The pressures can be imposed in three directions that are perpendicular to each other.

You might think that it would be possible to continue like this, i.e. that other, different pressure values could be created in further directions in space. Why not creating five different pressures (or tensional stresses) in five different directions, Fig. 2.80? Because it is impossible. Proving it is quite complicated. We would therefore simply accept the result:

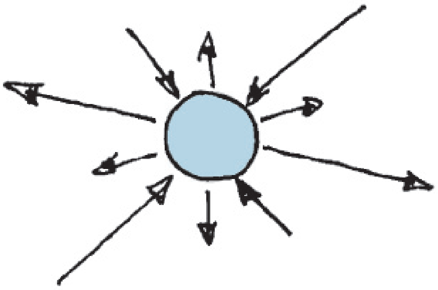


Fig. 2.80 More than three independent pressures cannot be applied in three dimensions (in two dimensions only two).

> Compressional or tensional stresses can be imposed in three directions that are perpendicular to each other.

As soon as we try to change the pressure in a fourth direction, the pressures in the first three directions will also change.

The result applies for every point within a body. But the mechanical stress can still change from point to point. In the compressed eraser from Fig. 2.76, the pressure or tension in the middle is certainly different from that on the upper or on the lower end.

If the pressure in three directions, which are perpendicular to one another, has the same value, i.e. 12 kPa, this pressure will also prevail in all other directions in space.

Every material can only withstand certain compressional or tensional stresses. In many cases, a material is much more enduring with regard to pressure than to tension.

For example concrete withstands compressional stresses of approximately 50 MPa, but tensional stresses of only 1/20 of this value. But sometimes a concrete support should be subjected to tension at certain points. Fig. 2.81 shows a concrete support that is supported on the outside and that bears a load in the middle – a typical situation. The concrete in the upper part of the support is exposed to pressure in a horizontal direction. In the lower area, it is exposed to tension in a horizontal direction. As the concrete does not withstand the tensional stress itself, it is equipped with steel tendons in the areas of tension because steel endures high tensional stresses.

For the same reason, i.e. to increase the resistance to tension of the material, certain plastics are reinforced with carbon fibers. Such materials are used for example to make skis, springboards for swimming pools and sailplanes.

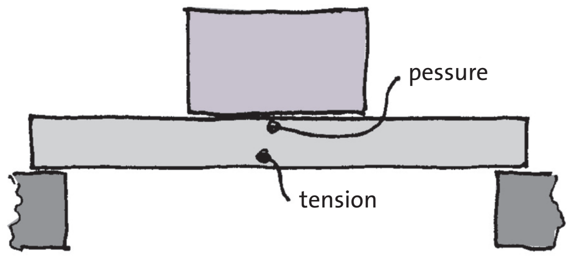


Fig. 2.81 There is pressure in the horizontal direction in the upper part of the support and tension in the lower part.

Many materials are not equally enduring in the different directions. A well-known example is wood. Coniferous wood endures a tensional stress of approximately 10 MPa in the direction of the grain, but only 1/20 of this value in the direction transversal to it.

Exercises

1. Name materials that endure high tensional stresses but only low compressional stresses.
2. Name materials that endure high compressional but only low tensional stresses.
3. Name materials that endure different compressional or tensional stresses in different directions.

2.19 The pressure in liquids and gases

So far, we have looked at the mechanical stress in solid objects. (Also a sponge is a „solid“ object as it is neither liquid nor gaseous.) We would now like to subject a liquid, for example water, to pressure. At first, we are a bit awkward on purpose and try to act similarly as for the block in Fig. 2.70: we press on the water in the middle from the top, Fig. 2.82. The obvious happens: the water escapes to the sides.

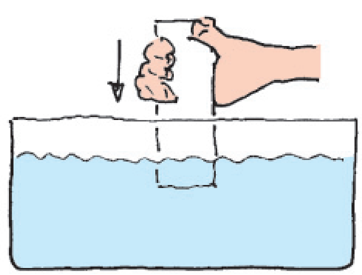


Fig. 2.82 The water cannot be put under pressure this way. It escapes to the sides.

Hence, we use a different technique: we lock the water so that it cannot escape, Fig. 2.83. If the cross-sectional area of the piston is $A = 5\ \text{cm}^2$ and the momentum current $F = 200$ N, there will be pressure of

$$p = \frac{F}{A} = \frac{200\ \text{N}}{0.0005\ \text{m}^2} = 400000\ \text{Pa} = 0.4\ \text{MPa}.$$

As the water tries to escape in the directions transversal to the piston, there will also be a compressional stress in these transverse directions that has the same value as that in the direction of the piston. In all other directions, there is a pressure of the same absolute value. The experiment illustrated in Fig. 2.84 shows this clearly.

> At a any point within a liquid, there is the same pressure in all directions.

This also applies for gases because gases will also escape laterally if they are not prevented from doing so.

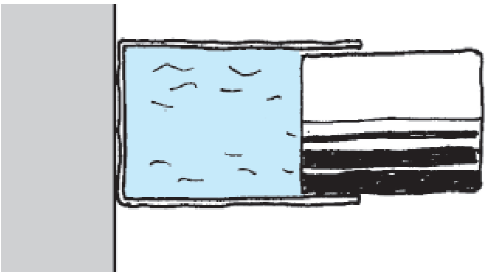


Fig. 2.83 The piston is subjected to pressure only in the horizontal direction, the water is under pressure in all directions.

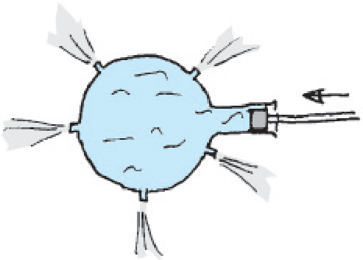


Fig. 2.84 As there is pressure in all directions, the water sprays in all directions.

2.20 Force

In this section, we simply talk about a different name for a term we already know.

As usual back then, Newton wrote his great piece „Principia Mathematica“ in Latin language. What we call momentum current today was called „vis“.

The name „momentum current strength“ for the quantity F has only existed since the beginning of the past century. But the name „force“ for the quantity F is still widely used today; in fact, it is used much more frequently than the name „momentum current“ or „momentum current strength“. We therefore have to become familiar with its use. However, there is a little problem: although „force“ denominates the same physical quantity as „momentum current strength“, the two terms are used in very different ways. We would like to call a description with momentum currents *momentum current model* and a description with forces *force model*.

We illustrate the application of the force model by means of Figures 2.85 and 2.86. In Fig. 2.85, Lilly pulls a trolley so that it starts moving to the right (no friction). Remember the description in the momentum current model:

- Lilly pumps momentum from the Earth into the trolley via the rope. Therefore, the momentum of the trolley increases.

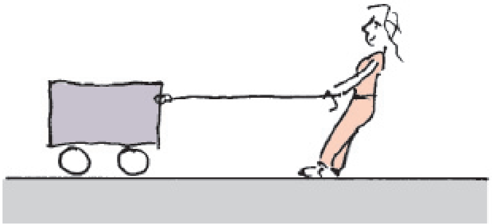


Fig. 2.85 Lilly exerts a force on the trolley. Therefore, the momentum of the trolley changes.

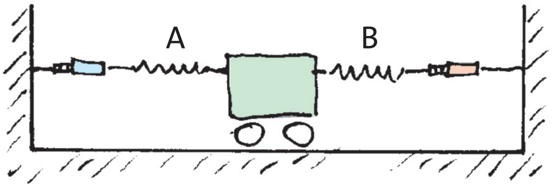


Fig. 2.86 Spring A exerts a leftward force on the trolley, spring B exerts a rightward force. As these forces have the same magnitude, the momentum of the trolley does not change.

The same process can be described with the force model as follows:

- A force is exerted on the trolley. Therefore, the momentum of the trolley increases.

Now, let's look at the situation from Fig. 2.86 with the momentum current model:

- We have a closed circuit. The momentum flows from the right through the spring into the trolley and back out on the left. As the whole momentum flows back out, the momentum of the trolley does not change.

The description is a bit more complicated in the force model:

- Spring A exerts a leftward force on the trolley, spring B exerts a rightward force of the same magnitude on the trolley. As the forces have the same magnitude but act in opposite directions, the momentum of the trolley does not change.

3 ANGULAR MOMENTUM AND ANGULAR MOMENTUM CURRENTS

In this chapter, we will look at a special type of movements: rotational movements. You are probably aware that rotational movements occur in many places and that they are particularly important.

We will make an interesting discovery: the description of rotational movements is very similar to the description of linear movements. We could also say that there is an analogy between the corresponding fields of mechanics. Thanks to this analogy, we can save ourselves much work.

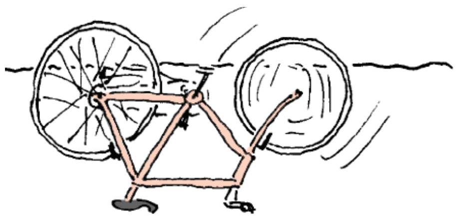


Fig. 3.1 The rotating wheel has a specific amount of angular momentum.

3.1 Angular momentum

We look at a wheel, that is rotating without (or with very little) friction, for example the wheel of a bicycle put upside down, Fig. 3.1. It rotates with a certain angular velocity, i.e. it performs a specific number of rotations per second. We can determine the value of the angular velocity by means of a stopwatch. With this method, we have described the rotary movement of the wheel.

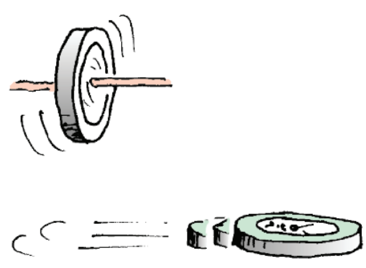
The angular velocity is for the rotary movement, while the usual velocity is for the linear movement. To describe the linear motion, however, we have also introduced a second quantity: the momentum. It is a measure for the „impetus“ of a body.

Likewise, we can say for our rotating wheel that it has impetus: something that is put in when making the wheel turn, and that comes back out when slowing the wheel down. This type of impetus is called *angular momentum*.

The symbol for the angular momentum is *L*. The measuring unit is Euler, abbreviated as E and named after the renowned mathematician and natural scientist Leonhard Euler (1707 – 1783), Fig. 3.2, who formulated the conservation principle of angular momentum for the first time.

Fig. 3.2 Leonhard Euler

a)



b)

Fig. 3.3 (a) The wheel has angular momentum. (b) The wheel has linear momentum.

Angular momentum and linear momentum are not the same. If the wheel in Fig. 3.3a had linear momentum, it would have to move in the way that is shown in Fig. 3.3b.

We make an experiment with two wheels, Fig. 3.4. The axis of one of them is fastened on the table, the other one can be carried around. The wheels can be connected with some sort of friction clutch. Then, one wheel will entrain the other.

The wheels are separated at first. One of them is set in motion, the other one is not. Next, the coupling discs are brought in contact. What happens?

The rotating wheel slows down and the other one, which did not rotate in the beginning, starts rotating. After the coupling discs have slid along one another for a while, they will eventually reach the same angular velocity.

So much for the observation. How can it be explained? What happened to the angular momentum during the process?

The angular momentum of the wheel that was rotating in the beginning has decreased. The angular momentum of the wheel that was not rotating has increased. Hence, angular momentum must have passed from one to the other.

> Angular momentum can pass from one body to another.

We consider again a sigle wheel that is connected firmly to its axis. The axis can rotate (almost) without friction. The wheel is set in motion; it is charged with angular momentum. Then, Willy grabs the rotating axis with his hand and slows down the wheel, Fig. 3.5. The wheel will come to a halt after a while. Where has the angular momentum gone?

The situation is very similar to one that you know: a vehicle that performs a linear movement slows down. Just as the momentum flows away into the Earth in case of the vehicle, the angular momentum of the rotating wheel flows into the Earth. The same would have happened if the wheel had not been slowed down on purpose. In that case, the angular momentum would have flowed away into the Earth through the bearings – just more slowly.

You see what wheel bearings are used for: they are meant to support a wheel or an axis and prevent rotary momentum from flowing away into the Earth.

> If a wheel bearing is not frictionless, i.e. if the wheel stops rotating by itself, its angular momentum flows away into the Earth.

Let's have another look at the experiment from Fig. 3.4. We set the wheel, which is fastened on the table, in motion. After, we also set the other wheel in motion – but in the opposite direction. We make sure that the absolute values of the angular velocity are identical for both wheels.

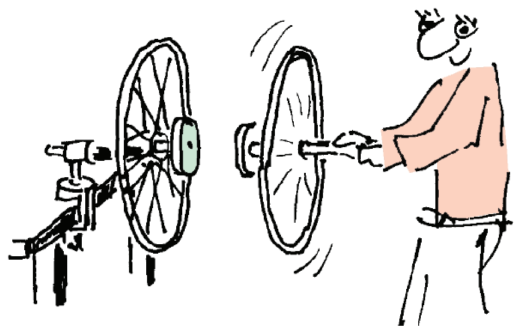


Fig. 3.4 As soon as the coupling discs are in contact with each other, angular momentum starts flowing from the right to the left wheel.

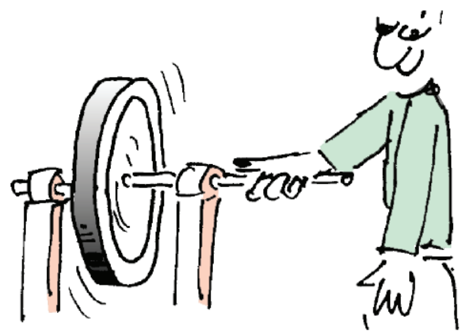


Fig. 3.5 The angular momentum flows away into the Earth.

Again, the wheels are connected by means of the friction clutch. How does the final state look like this time? Both wheels are standing still. How can this be explained?

There was angular momentum before. But where has it gone?

At the beginning, each wheel taken separately had an amount of angular momentum different from zero. If, however, opposite signs are attributed to the angular momenta of the two wheels, we see that the total angular momentum has already been zero at the beginning. The experiment leads us to the following conclusion:

> Angular momentum can assume positive and negative values.

We can define arbitrarily which of the two values is positive and which one is negative. But how can we formulate such a decision? A practical possibility is the right-hand rule, Fig. 3.6:

We grab the axis of rotation with the right hand in a way that the bent fingers point in the rotary direction. When our thumb points in the positive *x*-direction, the angular momentum is positive; when it points in the negative *x*-direction, the angular momentum is negative.

So far, we have carefully prevented the rotary axis from changing its orientation. Of course, this axis can have any orientation, and you can certainly imagine what this means: the angular momentum is a vector. We will get back to this matter later. For the time being, we always assume the direction of the axis to be fixed, i.e. parallel to the *x*-axis.

You see that dealing with angular momentum is very similar to dealing with linear momentum or also with the electric charge. The angular momentum is a substance-like quantity, too. And it has another important property in common with the momentum and the charge:

Angular momentum can neither be created nor destroyed.

Exercise

1. Formulate general statements about angular momentum and the corresponding statements for linear momentum and electric charge.

3.2 Angular momentum pumps

Just as linear momentum flows from the body with the higher to the body with the lower velocity in a friction process, angular momentum passes from the body with the higher angular velocity to the one with the lower angular velocity. Angular momentum flows out of a wheel, which contains positive angular momentum, by itself: via the bearings (that are never absolutely perfect) into the Earth.

In order to get angular momentum into the wheel, an effort has to be made. A wheel doesn't starts rotating by itself.

We can charge a wheel with angular momentum by hand, for example by means of a crank. Or we let a motor do the work, Fig. 3.7.

In both cases, we need something that forces the process of charging with angular momentum: an „angular momentum pump". In the first case, Lilly works as an angular momentum pump; in the second case the motor is the pump. But where does the angular momentum pump

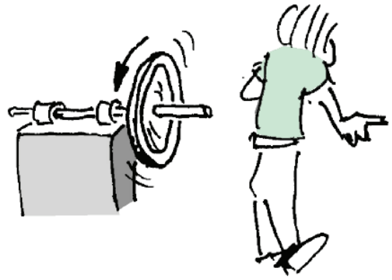


Fig. 3.6 The right-hand rule

take the angular momentum from? Similar to the linear momentum, angular momentum can be taken out of the Earth, too. An experiment shows this very clearly. We define the positive *x*-axis to be in the upward direction. we need a swivel chair and a wheel that can be held on its axis. Willy is standing next to the swivel chair, holds the wheel in a way that the axis points upwards and starts setting it in rotation. He then sits down on the swivel chair, Fig. 3.8, and brakes the wheel until it comes to a halt. Thereby he begins rotating himself. Why? While slowing down the wheel, angular momentum was flowing out of the wheel, into Willy and the swivel chair – but not any further. It could not flow into the Earth because the swivel chair is insulated from the Earth by the bearing.

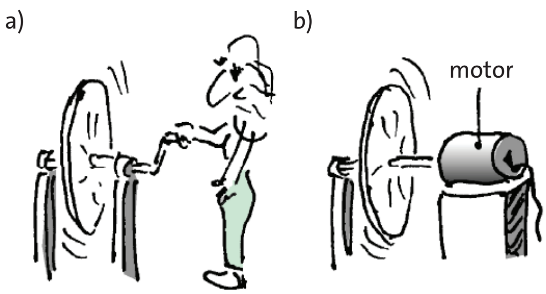


Fig. 3.7 (a) Lilly works as an angular momentum pump. (b) The motor works as an angular momentum pump.

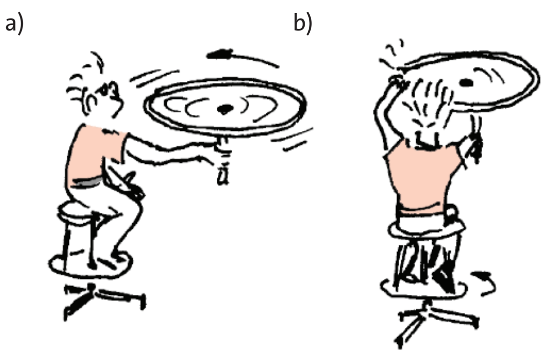


Fig. 3.8 (a) Only the wheel has angular momentum. (b) Angular momentum flows out of the wheel into Willy and the chair.

If Willy supports himself on the floor while slowing down the wheel, the angular momentum can flow directly into the Earth.

Here yet another variant of the experiment: Willy sits on the swivel chair and holds the wheel, Fig. 3.9. At first, the swivel chair and the wheel are at rest. Then, Willy sets the wheel in rotation. What happens? While the wheel starts turning, the whole swivel chair also begins to rotate, together with Willy – but in the direction that is opposite to the rotary direction of the wheel. Willy has obviously transferred angular momentum out of the chair and out of himself into the wheel. Now, Willy plus the chair have negative angular momentum.

If Willy supports himself again on the ground while charging the wheel, the chair will not rotate. The angular momentum will be pumped directly out of the Earth into the wheel.

a) b)

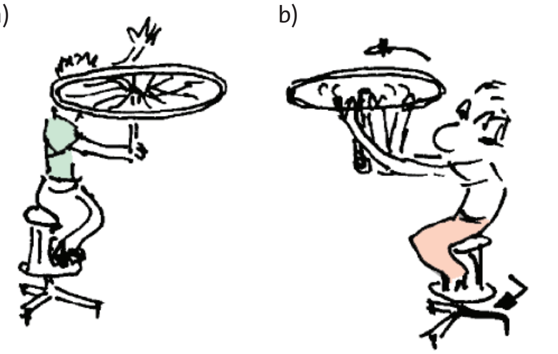


Fig. 3.9 (a) The wheel, Willy and the chair without angular momentum. (b) Angular momentum is pumped out of Willy and the chair into the wheel.

Exercise

1. In each of his hands, Will holds a rotating wheel with the axis pointing upwards, Fig. 3.10. The wheels are identical. Their angular velocities have the same absolute value but the rotary directions are opposite. While Willy is sitting on the swivel chair, he slows down both wheels at the same time. What happens? What will happen in the process of slowing down if the wheels have rotated in the same direction before?

Fig. 3.10 For the exercise

3.3 What angular momentum depends on – flywheels

A rotating wheel contains angular momentum. It is an angular momentum storage device. Some wheels are used exclusively to store angular momentum. They are called *flywheels*.

What are flywheels needed for? Steam engines and combustion engines (car engines) do not pump the angular momentum evenly but intermittently. A car engine produces approximately 50 angular momentum strokes per second. There are short time intervals between these strokes in which it does not „pump“. To bridge these delay times, the engine has a flywheel. While it is working, a part of the angular momentum goes into the flywheel; during the delay time, some of it comes back out. This is how the engine supplies a relatively even angular momentum current.

How can we store as much angular momentum as possible in a flywheel? We would like to examine what the angular momentum of a rotating body depends upon.

Let's use a very simple but somehow rough method to compare amounts of angular momentum. The body to be examined sits on a shaft with good bearings, Fig. 3.11.

Then, we clamp a clothes peg on the shaft, i.e. in a way that the peg will not rotate along. Hence, it acts as a brake. In other words: angular momentum flows out of the wheel through the clothes peg. Now we measure the time that the wheel takes to come to a hold, i.e. until the whole angular momentum has flowed out. The angular momentum that was in the wheel at the beginning is proportional to the time.

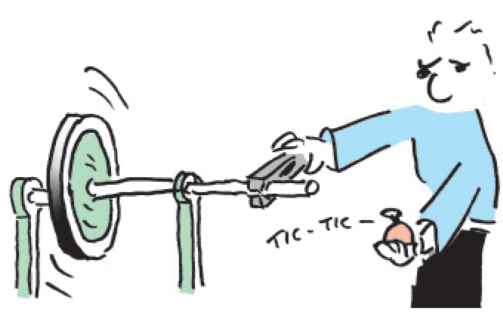


Fig. 3.11 Willy measures the time that the angular momentum needs to flow out of the flywheel.

(For this to be correct, the angular momentum current has to be constant during braking. This condition is met quite well for our clothes peg brake.) Now we compare respectively two rotating wheels or other bodies.

1. Two identical wheels. One of them is rotating fast, the other one slowly. Slowing down takes longer for the fast than for the slow wheel. Hence, the fast wheel contains more angular momentum than the slow one, Fig. 3.12. If we measure the angular velocity at the start of rotation, we can see that the angular momentum is proportional to the angular velocity:

$$L \sim \omega.$$

2. Two wheels have the same shape but are made of different materials. One is made for example of iron and the other one of aluminum. Hence, they have a different mass. Both are set to the same angular velocity. Slowing down the heavier wheel takes longer than slowing down the lighter one. This is because the heavy wheel has had more angular momentum than the light one, Fig. 3.13. We find:

$$L \sim m.$$

3. We finally compare two bodies that are neither different in the mass nor in the angular velocity. The only difference is that the mass in one of them is located further outside than in the other one, Fig. 3.14. We observe: the angular momentum changes very strongly with the distance of the masses from the axis of rotation. The exact relationship is:

$$L \sim r^2.$$

Of course, this relationship can only exist if the whole mass is located at a single distance r from the axis. This is approximately the case for the dumbbell-shaped structure from Fig. 3.14. Also in case of a typical flywheel, Fig. 3.15, the mass sits essentially at a specific distance from the axis of rotation.

If this is no longer the case, for example as in the massive wheels of Figure 3.13, the relationship will be more complicated. Then, masses that are located at diverse distances from the axis contribute to the total angular momentum. We would like to limit our analysis to the case of one single distance. We summarize the three proportionalities and obtain:

$$L \sim m \cdot r^2 \cdot \omega.$$

Fig. 3.12 The wheel that rotates fast has more angular momentum than the wheel that rotates slowly.

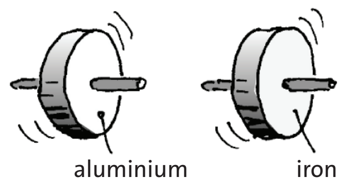


Fig. 3.13 The heavy wheel has more angular momentum than the light one.

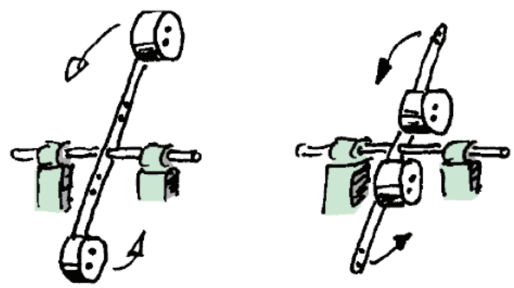


Fig. 3.14 Both dumbbells have the same mass, but the moment of inertia of the dumbbell on the left is greater than the one of the dumbbell on the right.

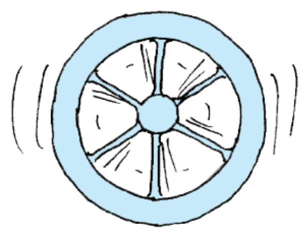


Fig. 3.15 Flywheel: the mass sits far outwards.

The measuring unit Euler is now defined in such a way that the proportionality symbol can be replaced by the equal sign:

$$L = m \cdot r^2 \cdot \omega. \qquad (3.1)$$

We compare this equation with the corresponding equation for the linear momentum:

$$p = m \cdot v.$$

Here, the mass m characterizes the body and v tells us how fast it is moving.

In equation (3.1), the term $m \cdot r^2$ characterizes the body and ω tells us how fast it is rotating. It is logical to assign the term $m \cdot r^2$ its own name: it is called moment of inertia of the body or of the wheel, abbreviated as J. The moment of inertia tells us how inert a body is with regard to rotational movements; how difficult it is to set it in motion or to slow it down. Therefore, instead of equation (3.1) we can write:

$$L = J \cdot \omega.$$

> The higher its angular velocity, the more angular momentum is contained in a body. The greater its moment of inertia (i.e. the larger its mass and the further outside the mass is located), the more angular momentum is contained in a body.

We now know how a flywheel has to look like: a large, heavy ring that is fastened on the wheel hub with thin spokes, Fig. 3.15.

Exercises

1. Wheels have different functions. Storing angular momentum is only one of them. What else are wheels used for? Name several different usages.
2. We cannot store any quantity of angular momentum in a flywheel simply by making it rotate increasingly faster. Why not?
3. The flywheel of a car has a mass of 8.5 kg. Although the mass is distributed over different distances, we can set r = 20 cm as a typical distance. Calculate the moment of inertia of the flywheel. How much angular momentum does it contain at 3 000 revolutions per minute?
4. Estimate the value to which the angular momentum of a figure skater will increase if she performs a spin. At first, she turns with 1 revolution per second while keeping one leg and both arms stretched out.
5. A star collapses and a neutron star is formed in a supernova explosion. The neutron star is much smaller than the original star, but its mass density is extremely high (approximately 10^{12} kg/cm^3) and it rotates extremely fast. We assume for the original star that its mass is located at a distance of 50 000 km from the center; for the neutron star, the distance should be 10 km. (In reality, its mass is of course distributed over a large distance range but we can calculate with a mean radius for a rough estimate.) The original star turns around its axis once per 120 days. How fast will the newly formed neutron star rotate?
6. Sit down on a swivel chair in a way that your legs neither touch the Earth nor the chair legs. Then, try to rotate with the chair. It will even be easier if you hold a heavy object in each hand. Cats do exactly the same to land on their four legs after falling. Explain.

3.4 Angular momentum conductors

In Fig. 3.16, a flywheel is charged with angular momentum. On the left, there is the angular momentum pump (an electric motor), on the right the flywheel and in between there is a long connection through which the angular momentum can move from the left to the right.

Such angular momentum conductors are called shafts. For example, cars have a motor shaft, a cardan shaft, drive shafts and other shafts.

Which quality of the shafts is responsible for the conductivity of angular momentum? Which material do they have to be made of? The only condition for the material is to be solid. Any solid bar can be used as an angular momentum conductor.

> Solid objects are conductors for angular momentum.

We would like to look at some other devices that have to do with the transport of angular momentum.

A bearing is used to hold a shaft while preventing angular momentum from flowing into the Earth. Fig. 3.17 shows a ball bearing.

> Bearings prevent angular momentum from flowing away.

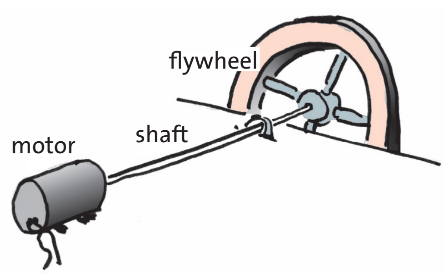


Fig. 3.16 Angular momentum flows through the shaft from the motor to the flywheel.

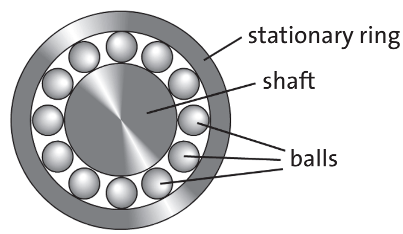


Fig. 3.17 Ball bearing (simplified illustration)

Fig. 3.18 shows a clutch. The connection between the motor and the flywheel can be interrupted and restored by means of a lever.

> With a clutch, two angular momentum conductors can be connected and separated again.

Every car has a clutch. It is located between the engine and the gearbox, Fig. 3.19. By stepping on the clutch pedal (the one on the left in the car) the connection between the engine and the gearbox is interrupted.

We have to release the clutch before we shift gears. If the clutch is not released during shifting the strong angular momentum current flows from the engine to the wheels, whereby the gearbox will be damaged.

Once again we let angular momentum flow through a shaft into a flywheel. Does it make a difference for the shaft whether an angular momentum current is flowing or not? And does it make a difference whether it flows from left to right or from right to left?

We cannot tell it by looking at the shaft, at least as long as the shaft is thick. Therefore, we use a flexible, elastic object, e.g. a plastic ruler, as a shaft, Fig. 3.20a. How will the ruler react if an angular momentum current flows through it? It will be twisted. We say that it is subjected to *torsional stress*. A solid object through which an angular momentum current is flowing is exposed to torsional stress – even if there is no visible twisting effect.

The direction of twisting depends on the flow direction of the angular momentum. In Fig. 3.20a, the wheel is charged with positive angular momentum, i.e. the angular momentum in the ruler is flowing from the left to the right.

Positive angular momentum is also flowing into the wheel in Fig. 3.20b. Here, it comes from the right; hence, it flows from the right to the left. What is the difference between the two rulers?

The edges of both rulers form a helical line. As you might know, there are two types of helices: right-hand and left-hand helices, Fig. 3.21. A right-hand helix is the one that looks like a corkscrew or like the thread of an ordinary screw. A left-hand helix forms the so-called left-hand threads or corkscrews viewed in a mirror.

Back to our angular momentum currents. In Fig. 3.20a, angular momentum flows from the left to the right. The ruler is twisted like a left-hand helix. In Fig. 3.20b, angular momentum flows from the right to the left. The ruler is twisted like a right-hand helix.

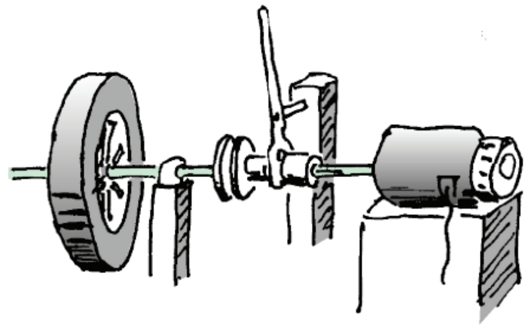


Fig. 3.18 The connection between the motor and the flywheel can be interrupted by means of the clutch.

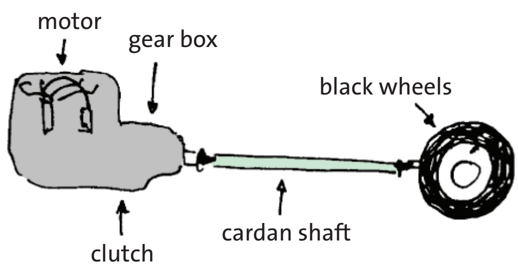


Fig. 3.19 The car clutch is used to interrupt the connection between the engine and the gearbox.

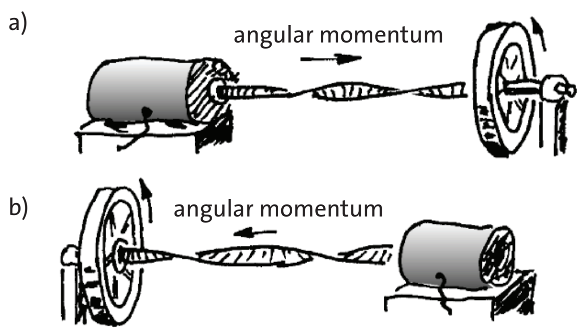


Fig. 3.20 (a) Angular momentum flows from the left to the right. (b) Angular momentum flows from the right to the left.

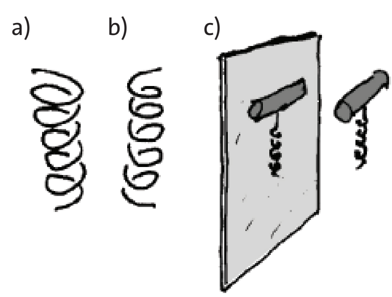


Fig. 3.21 (a) Right-hand helix (b) Left-hand helix (c) Corkscrew and mirror image

Angular momentum current to the right:
twisting forms a left-hand helix;
Angular momentum current to the left:
twisting forms a right-hand helix.

Exercises

1. Design an experiment that can be used to examine if water conducts angular momentum.
2. Design an experiment that can be used to prove that magnetic fields conduct angular momentum.
3. Air conducts almost no angular momentum. Just as there are convective transports of linear momentum in the air, there are also convective angular momentum transports. Give an example.
4. Shafts are angular momentum conductors. Cars contain a larger number of different shafts. They have different names according to their function. What are they used for?

3.5 Current strength and rate of change of the angular momentum

The angular momentum current through a shaft can be stronger or weaker. A measure of it is the angular momentum current strength. It indicates how much angular momentum is flowing through a cross-sectional area of the shaft per unit of time (how many Euler pass the area per second). The symbol for the angular momentum current strength is M, the measuring unit is Euler per second, abbreviated as E/s.

If 12 Euler are flowing through a shaft per second, we will have

$$M = 12 \text{ E/s}.$$

After some calculation we find that 1 E/s = 1 N · m. In engineering, angular momentum currents are mostly indicated in Nm, and the angular momentum current strength is called *torque*. The situation from Fig. 3.16 is then described as follows: „the engine exerts a torque to the flywheel."

Fig. 3.22 shows a section of the specification sheet of a car. In case of an angular velocity of 4000 U/min, the engine supplies its maximum angular momentum current, i.e. 145 E/s.

When a screw is tightened, an angular momentum current flows through the screwdriver. There are screwdrivers that allow us to set the maximum angular momentum current that they let pass (i.e. the maximum torque).

Die Motorvarianten	in der preisgünstigste		
Typ	**1.6 16V SX**	**1.9 JTD SX**	**Bipower SX**
Aufbau/Türen	GR/5	GR/5	GR/5
Zylinder/Hubraum [ccm]	4/1596	4/1910	4/1581
Leistung [kW(PS)]	76(103)	85(115)	68(92)
Max.Drehmoment [Nm]bei U/min	145/4000	203/1500	130/4000
0-100 km/h[s]	12,6	12,2	16,0
Höchstgeschwindigkeit [km/h]	170	176	157
Verbrauch pro 100 km [l/kg]	9,6S	7,5D	7,7G
Versicherungsklassen KH/VK/TK	15/17/26	17/20/32	15/17/26
Steuerbefreiung [Euro](Monate)	-	-	306(21)
Monatliche Gesamt-Kosten[Euro]	504	499	464
Grundpreis[Euro]	17550	19500	20300

Aufbau:

ST = Stufenheck	KB = Kombi	GO = Geländewagen offen
SR = Schrägheck	KT = Kleintransporter	GS = Geländew. geschlossen
CP = Coupe	TR = Transporter	PK = Pick-Up

Fig. 3.22 From the specification sheet of a car

Just as $\Delta p / \Delta t$ is the rate of change of the linear momentum, we have:

$$\frac{\Delta L}{\Delta t} = \text{rate of change of the angular momentum.}$$

If an angular momentum current of

$$M = 5 \text{ E/s}$$

flows through a shaft into a flywheel, the rate of change of the angular momentum of the flywheel will also be 5 E/s: Rate of change = angular momentum current strength.

Hence, we have:

$$\frac{\Delta L}{\Delta t} = M$$

Exercises

1. A flywheel's mass of 1200 kg sits in a ring at a distance of 1 m from the axis. The flywheel turns with 3 rotations per second. (a) How much angular momentum has the flywheel? The wheel is slowed down. The angular momentum is flowing into the Earth with a current strength of 120 E/s. (b) How long does it take until the wheel comes to a halt?
2. A single-cylinder four-cycle engine creates a reasonably even angular momentum current of 40 E/s on the average. It actually works only 1/4 of the time as only one of the four strokes is a working stroke (a stroke is half a spin, from one dead center of the piston to the next). The average angular velocity is 8 rotations per second. The engine has a flywheel with a moment of inertia of 2 kg · m^2. (a) How many working strokes does the engine perform per second? (b) How much angular momentum does the engine supply per working stroke? (c) What is the average angular momentum of the flywheel? (d) Estimate how much angular momentum the flywheel stores during the working stroke. Compare with the total angular momentum that it contains.

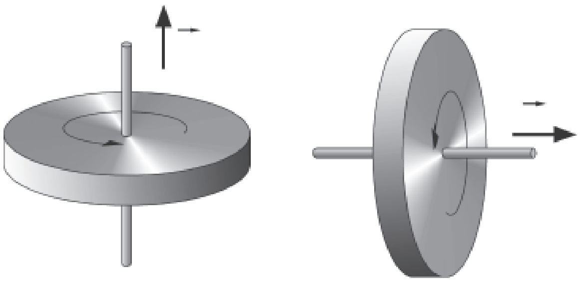


Fig. 3.23 Although the two identical flywheels rotate with the same numbers of revolutions per minute, they have different angular velocities and a different angular momentum.

3.6 Angular momentum and angular velocity as vectors

Two flywheels rotate equally fast, Fig. 3.23. However, the axes of rotation have different directions. To define an angular velocity unambiguously, the direction of the axis has to be indicated in addition to the absolute value the angular velocity. When a direction has to be indicated besides the absolute value to define a physical quantity, this quantity is a vector quantity.

> The angular velocity is a vector quantity.

The same statements also apply for the angular momentum.

> The angular momentum is a vector quantity.

We can therefore represent both angular velocity as well as angular momentum graphically by means of an arrow. The arrow direction, however, cannot be the direction of movement in this case as the individual parts of the rotating body are moving in a variety of directions. The arrow is therefore drawn in parallel to the axis of rotation. And what is the orientation of the arrow? On which side is the tip of the arrow?

Now, we can shorten the somehow complicated rule about the sign of the angular momentum (section 3.1):

> We grab the axis of rotation with our right hand in a way that the bent fingers point in the rotational direction. Then, the thumb points in the direction of the angular velocity vector and of the angular momentum vector.

If we wish to take into account the vector character of the angular velocity and of the angular momentum, we will have to modify some of our formulas. The relationship between the angular velocity and the angular momentum is now:

$$\vec{L} = m \cdot r^2 \cdot \omega$$

and that between the rate of change of the angular momentum and the angular momentum current:

$$\frac{\Delta \vec{L}}{\Delta t} = \vec{M}$$

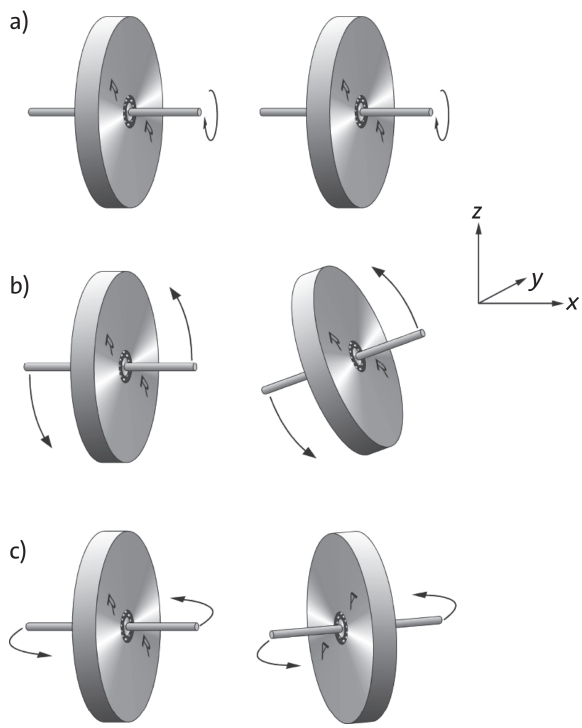


Fig. 3.24 The bearing prevents the transition of only one angular momentum component (in this case the *x*-component).

3.7 More about angular momentum conductors

We have found that bearings are used to prevent angular momentum from flowing away into the Earth.

This statement can now be formulated a bit more specifically. Fig. 3.24 shows a wheel. The bearing is located between the wheel and the axis. Hence, the axis can rotate freely in the wheel.

If the axis is turned as in image a, the wheel will not rotate along. We cannot supply any *x*-angular momentum to the wheel because the bearing will not let it pass. But the wheel can be turned or tilted in the *y*- or the *z*-direction by means of the axis; see images b and c. Hence, the bearing allows *y*- and *z*-angular momentum to pass. Or more generally speaking:

> A bearing is used to hold an axis in such a way that the angular momentum that has the direction of the axis will not flow away.

An object, a wheel for example, can also be positioned in a way that no angular momentum can flow away at all – no *x*-, no *y*- and no *z*-angular momentum, Fig. 3.25. Such a positioning is called *cardanic suspension*. The outer U-shapes support can be rotated or tilted as we like – the direction of the axis of the wheel in the middle will always remain the same. The angular momentum will always remain the same as angular momentum can neither be absorbed nor released through the suspension.

We have seen earlier that each component of the linear momentum vector is conserved. This applies accordingly for the angular momentum:

> The law of angular momentum conservation applies for each component separately.

Willy tries to convince us, Fig. 3.26.

He holds a fast-rotating flywheel. The angular momentum vector points in the *x*-direction at first. Hence, the flywheel contains *x*-angular momentum. Now he sits down on the swivel chair and tilts the rotary axis in an upward direction, i.e. in the *z*-direction. We can make two observations in the process:

1. Willy and the swivel chair start to rotate in the opposite direction of the wheel.

2. Willy feels a somehow unexpected reaction of the wheel.

What does this mean? At the beginning, the wheel had no *z*-angular momentum. After having tilted the wheel axis into the vertical direction, the wheel has had *z*-angular momentum. This angular momentum must have come from somewhere. It could not come from the Earth because Willy is, as far as the *z*-angular momentum is concerned, insulated from the Earth. Hence, the angular momentum is taken from Willy + the chair. In the end, Willy and the chair have as much negative *z*-angular momentum as there is positive *z*-angular momentum in the wheel.

And where has the *x*-angular momentum gone that the wheel had in the beginning? It could flow away into the Earth as the bearing of the chair is open for *x*- and for *y*-angular momentum. The peculiar reaction of the wheel axis is due to the angular momentum flow into the wheel and out of the wheel.

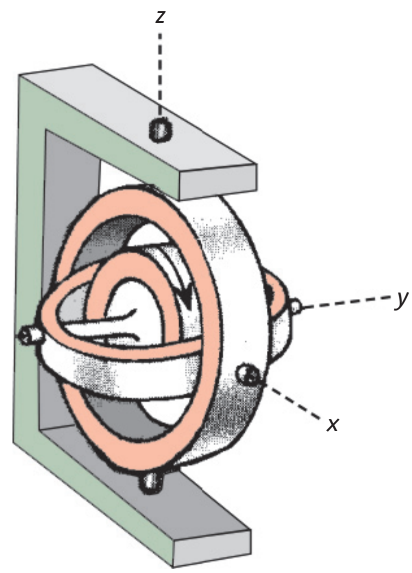


Fig. 3.25 Cardanic suspension. The wheel is completely insulated with regard to the support: neither *x*-, nor *y*-, nor *z*-angular momentum can flow away.

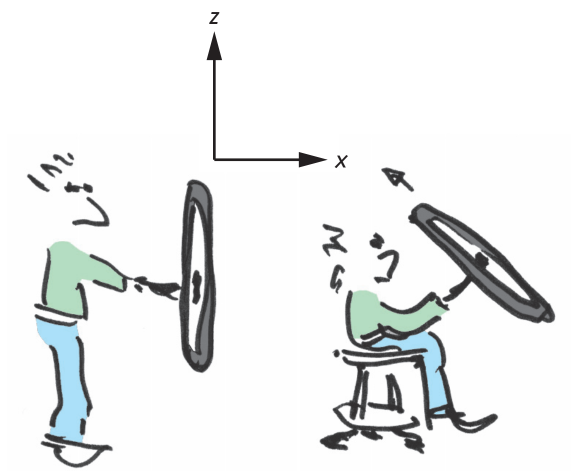


Fig. 3.26 While Willy turns the wheel axis in a vertical position, *x*-angular momentum is flowing from the wheel into the Earth and *z*-angular momentum is flowing from Willy + chair into the wheel.

Exercises

1. Sketch the chassis and the transmission system of a car with its essential parts: wheels, springs, shock absorbers and the drive shafts with the corresponding joints.
2. Lilly sits on the swivel chair and holds two rotating flywheels with the axes in the horizontal x-direction, Fig. 3.27. The swivel chair does not rotate at first. Lilly tilts the axes of the two wheels in the vertical z-direction. What happens? Discuss the case in which the x-angular momentum of the two wheels is equal at the beginning and the case that it is equal but opposite.

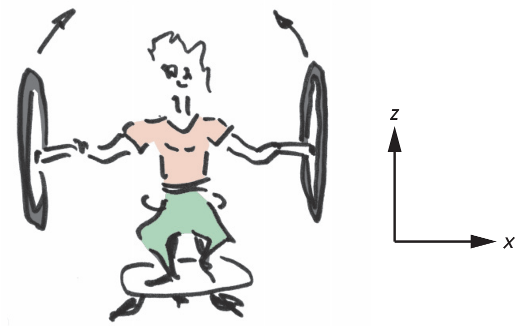


Fig. 3.27 For exercise 2

3.8 Angular momentum circuits

Fig. 3.28 shows a coffee grinder. Although a real coffee grinder is slightly more compact, it is essentially built as shown by the illustration. In the following, we will discuss the coffee grinder not because of its being particularly important but as an example for other machines in which something is driven by means of a rotating shaft: household machines such as washing machines, vacuum cleaners and electric mixers, different gardening tools such as lawn mowers, brush cutters and motor-driven hedge trimmers, all vehicles, numerous machines in factories and power plants.

In the coffee grinder, the grinder is driven by an electric motor. The motor pumps angular momentum to the grinder via a shaft. Does the angular momentum of the grinder increase in this process? No, because the shaft would have to rotate increasingly faster, but it does not do so.

So where has the angular momentum gone? It has to flow away from the grinder. This is not surprising as there is very strong friction between the rotating inner part of the grinder and the fixed outer part. Friction is like a bad bearing, i.e. a bearing through which the angular momentum flows away.

Hence, we have a closed angular momentum circuit: the motor pumps angular momentum out of its support to the grinder via the shaft. From there, it flows back to the motor through the housing or the support.

In all devices where something is driven by a rotating shaft, angular momentum flows in a closed circuit. Fig. 3.29 shows the turbine and the generator in a power plant.

Transmissions based on rotating shafts are often more complicated than in Figures 3.28 and 3.29.

Fig. 3.30 shows a transmission in which the shafts of the motor and the grinder form an angle of 90°.

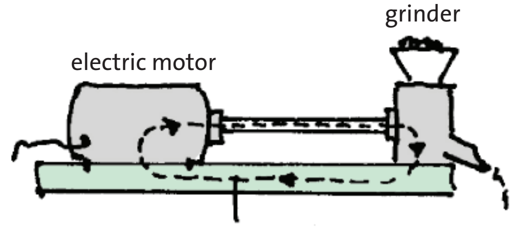


Fig. 3.28 Coffee grinder. The angular momentum grinder flows in a closed circuit.

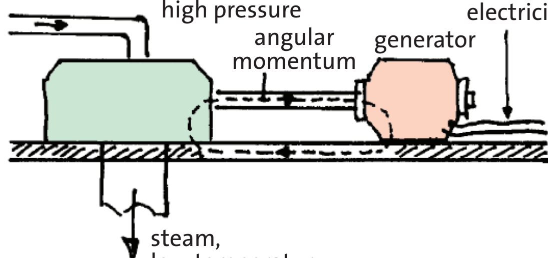


Fig. 3.29 Turbine and generator in a power plant. The angular momentum flows in a closed circuit.

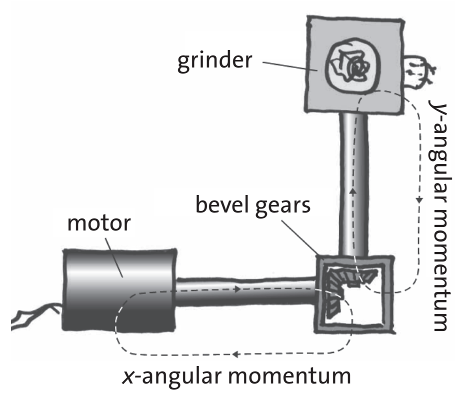


Fig. 3.30 Drive by means of bevel gears

The two shafts are connected to each other with bevel gears. Two angular momentum types are involved in this drive. x-angular momentum flows in a circuit between the motor and the bevel gear, and y-angular momentum between the bevel gear and the grinder.

Fig. 3.31 Bevel gearbox

4 THE GRAVITATIONAL FIELD

4.1 Gravitational attraction

The Earth attracts all objects. This can be noticed in two phenomena:

- We take an object in our hand and release it. It will fall down.
- Every object has weight.

Both phenomena show that the object receives momentum from the Earth. A falling body becomes increasingly fast while falling: its momentum increases. The fact that a body that does not fall also receives momentum can be seen if we suspend it on a dynamometer, Fig. 4.1. The dynamometer indicates that a momentum current is constantly flowing away from the body into the Earth via the suspension. This momentum has to be replaced continuously. Hence, momentum is flowing continuously into the body, albeit via a connection between the body and the Earth that is absolutely invisible.

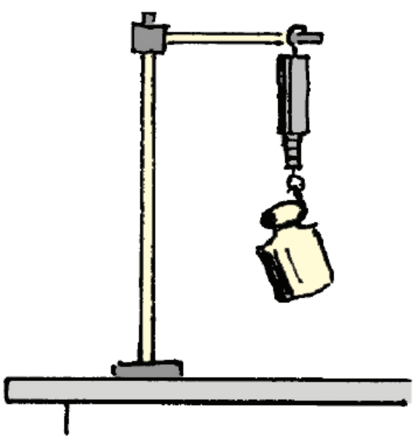


Fig. 4.1 The dynamometer indicates that a momentum current is flowing upwards and away from the hanging body. This momentum has flowed into the body through the gravitational field.

We have already learned about a similar momentum-conducting connection earlier, i.e. about an invisible connection: the magnetic field. In the case that interests us at the moment, however, there cannot be a magnetic field because this would mean that only magnets or iron bodies would be attracted by the Earth. The connection therefore consists of an entity or object that, although it is not a magnetic field, is similar to the magnetic field. It is called gravitational field. Just as a magnetic pole is surrounded by a magnetic field, every object that has a mass, i.e. every body, is surrounded by a gravitational field. The larger the mass of the body, the higher the density of this field.

> Every body is surrounded by a gravitational field. The larger the mass of the body, the higher the density of the field. Momentum flows through the field from one body to another. Gravity means that there is a momentum current from the Earth to the respective body.

4.2 What gravity depends upon

We try. At first, we suspend a body A made of iron with a mass of 1 kg on a dynamometer and then a wooden body B that also has a mass of 1 kg. Hence, the following applies:

$$m_A = m_B \, .$$

The dynamometer indicates the same both times:

$$F_A = F_B \, .$$

The bodies have the same weight. This does probably not sound surprising to you; but still, it cannot be taken for granted.

We would like to understand it. What does it mean that the piece of wood and the piece of iron have the same mass? To answer this question, we have to remember the equation

$$p = m \cdot v$$

The mass is the proportionality factor in the relationship between momentum and velocity. It tells us how much momentum is needed to accelerate a body to a given velocity. It tells us how inert the body is. But from the fact that two bodies have the same inertia we can still not conclude at this point that they have the same weight. From

$$m_A = m_B$$

we cannot simply conclude that

$$F_A = F_B .$$

We can only try it out. The experiment shows that it is actually the case:

> Bodies of equal inertia also have the same weight.

We have gotten used to this fact and can hardly imagine that it could be otherwise. Still, this observation has been discussed by physicists for a long time. At first, it seemed to be a mere coincidence. It was considered to be possible that, simply by performing an exact measurement, a difference between inertia and weight might be detected. Only the theory of relativity has demonstrated that inertia and weight actually have to match.

Now we take two bodies, each with a mass of 1 kg. We can consider both of them together to be a single body with a mass of 2 kg. A momentum current twice as high as for a single body flows into the arrangement of the two. You might also take this for granted. But we could definitely imagine that adding a second body would influence the momentum current that flows into the first one. This is not the case, though. Hence, the following applies for the momentum current that flows from the Earth into another body:

$$F \sim m,$$

or written as an equation:

$$F = m \cdot g . \qquad (4.1)$$

For the factor of proportionality we find:

$$g = 9.8 \text{ N/kg},$$

or approximately

$$g = 10 \text{ N/kg}.$$

Our result is not yet complete.

At first, we notice that g must be the absolute value of a vector. This can already be concluded for mathematical reasons. As the quantity on the left of equation (4.1) is a vector quantity, there must also be a vector on the right. Expressed in vectorial terms, equation (4.1) becomes

$$\vec{F} = m \cdot \vec{g} . \qquad (4.2)$$

We now make the following thought experiment: we think of an object in a variety of places: here in Europe, in Japan, at an altitude of 1000 km above the surface of the Earth, on the Moon, on Mars or far away from all celestial bodies. The momentum current is different each time. Although the proportionality

$$F \sim m,$$

still applies for each of these places, the factor of proportionality is different in each case.

The momentum that flows in a body in Japan is tilted by approximately 90° with regard to the one that flows into a body with the same mass in Europe. Consequently, the direction of $\vec{F}$ depends on the place on Earth and therefore also that of $\vec{g}$. But also the absolute value of $\vec{g}$ is location-specific. If we move away from the Earth, it will become increasingly smaller. At a great distance from any celestial body, it is practically zero. The absolute values for some prominent places are listed in Table 4.1.

The vector quantity $\vec{g}$ tells us about the gravitational field at the respective point. Its absolute value tells us about the density of the field. From the fact that $\vec{g}$ is a

Location	*g* in N/kg
surface of the Earth	9.8
1000 km above surface of the Earth	7.3
surface of the Moon	1.62
surface of Mars	3.8
surface of the Sun	274
surface of a neutron star	1000 000 000 000

Table 4.1 Absolute value of the gravitational field strength at different locations

vector, we conclude that the gravitational field has a particular direction in each point of space. In this respect, it is similar to wood. Also in wood, a singular direction can be indicated at each point: that of the texture. $\vec{g}$ is called the *gravitational field strength*.

We can measure the gravitational field by means of equation (4.2). Before we look at very extended gravitational fields, we would like to examine the consequences of the field at a place that is close to the surface of the Earth.

By the way, what do we mean by saying that an object is heavy? Probably that it is hard to lift it off the ground. Does this mean that it has a large mass? Strictly speaking no; on the Moon, it would obviously not be hard at all to lift this „heavy" object off the Moon surface. Therefore, „heavy" rather means that a strong momentum current flows into the body. The same object can be heavy or light, depending on where it is located.

Let's also describe gravity in the force model: if a force is calculated according to equation (4.2), it will be called gravitational force, and we say that the gravitational force acts on the body.

> Relationship between the gravitational field strength $\vec{g}$ and the momentum current $\vec{F}$ flowing into a body with mass m:
>
> $$\vec{F} = m \cdot \vec{g}$$

Exercises

1. Which momentum current flows out of the Earth into your own body? (Which force of weight acts on your body?) What would be the strength of this momentum current on the Moon, what would it be on a neutron star?
2. During an expedition on the Moon, astronauts determine the force of weight force on a body by means of a dynamometer. They find $F = 300$ N. What is the mass of the body?

4.3 Free fall

If we deal only with movements in the vertical direction, we will only need to look at the vertical component of momentum and velocity. We denominate them with the letters p and v and choose the downward direction to be the positive direction. Thus, a body that moves downwards has positive momentum and a positive velocity.

The phenomena that we will now examine all take place near to the surface of the Earth. We will neither go to an altitude of 1000 km nor 1000 km east-, west-, south- or northwards. Under these conditions, we can consider the gravitational field strength as constant, i.e. independent of the position. We say the gravitational field is *homogeneous*.

We take an object in our hand and release it. It will fall to the ground. Now we are able to explain this phenomenon: a momentum current of the strength $m \cdot g$ flows into the object, i.e. its momentum increases continuously. The longer it falls, the faster it moves.

However, there is something peculiar about this process. If we release two objects – a heavier and a lighter one – at the same time from the same altitude, we will find that they arrive at the ground simultaneously. Isn't the heavier one supposed to fall faster as it receives more momentum from the Earth?

We calculate the law according to which the momentum of the two bodies increases. We assume that the mass of the heavier body to be 4 kg, that of the lighter body 1 kg.

We insert $F = m \cdot g$ into $p = F \cdot t$ and obtain

$$p = m \cdot g \cdot t\,. \qquad (4.3)$$

Here we insert the mass and the gravitational field strength and obtain for the heavy body

$$p = 4\ \text{kg} \cdot 9.8\ \text{N/kg} \cdot t = 39.2\ \text{N} \cdot t$$

and for the light one

$$p = 1\ \text{kg} \cdot 9.8\ \text{N/kg} \cdot t = 9.8\ \text{N} \cdot t\,.$$

These two p-t relationships are illustrated in Fig. 4.2. We can see that the momentum increases regularly for

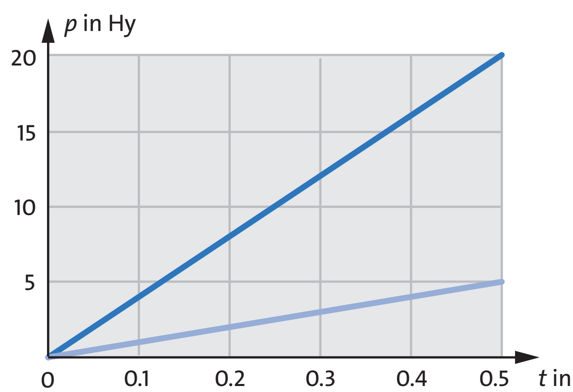


Fig. 4.2 Momentum as a function of time for two falling bodies with different masses

the two objects. But the momentum of the heavy body increases faster than that of the light one. The heavy body has four times as much momentum as the light one at all times.

So why do the two bodies fall equally fast? To find the answer to this question, we need the formula

$$p = m \cdot v \,. \tag{4.4}$$

It allows us the following conclusion: to bring the heavy body to a given velocity, we need four times as much momentum as required to bring the light one to the same velocity. The body with the larger mass has a greater inertia than that with the small mass.

We can also obtain this result by means of a simple calculation. We equate the right sides of equations (4.3) and (4.4) and obtain

$$m \cdot g \cdot t = m \cdot v \,.$$

Division of both sides of the equation by m leads us to

$$v = g \cdot t \,. \tag{4.5}$$

As the mass has disappeared from the equation, it tells us that the velocity of a falling body does not depend on its mass. In Fig. 4.3, the velocity of an arbitrary freel-falling body is graphed as a function of time.

Equation (4.5) also tells us that the velocity of a falling body increases at a constant rate. This means that its acceleration is constant.

The acceleration can be calculated easily. We look at the time interval from $t = 0$ to $t = t_0$. In this time period, the velocity increases from $v = 0$ to $v = v_0 = g \cdot t_0$. We therefore obtain:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_0}{t_0} = g.$$

Consequently, the acceleration of a falling body is equal to the gravitational field strength.

The fact that the gravitational field strength appears in equation (4.5) means that the falling velocity depends on the location of the falling body. For example, on the Moon all bodies fall approximately six times as slowly as on Earth.

Our discussion was based on the assumption that the body is not losing any momentum while falling. We have therefore simplified the actual situation: in reality, it loses momentum through friction with the air. If a body is not too light and if it only falls over a short distance, our simplification is justified, though.

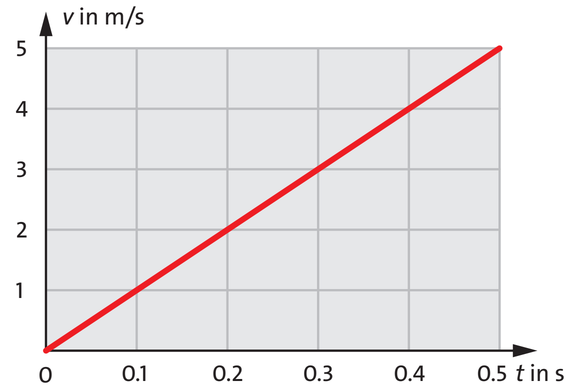


Fig. 4.3 The velocity of a free-falling body increases linearly over time.

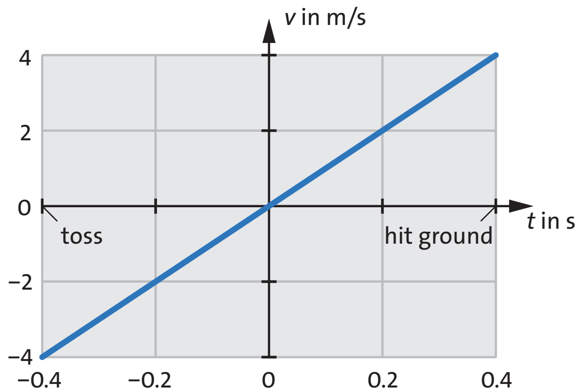


Fig. 4.4 The velocity of a body, which has been thrown upwards, as a function of time. While flying upwards, the velocity is negative; while falling down, it is positive.

Such a movement is referred to as a *free fall*.

Free-falling bodies:
- The velocity increases at a constant rate.
- All bodies fall with the same velocity.
- The acceleration is equal to the gravitational field strength.

We look at another variant of the free fall: the object is not simply let fall from a state of rest but it is thrown vertically upwards. It has negative momentum at the start in this case. Still, it is provided continuously with fresh momentum by the Earth which leads to a gradual reduction of its negative momentum: the object flies increasingly slowly, comes to a halt and finally starts moving in the positive (downward) direction.

The upward movement is the mirror image of the downward movement in this case. While falling down, the momentum of the body increases regularly; while flying upwards, its negative momentum decreases regularly. This applies accordingly for the velocity: while flying upwards, the negative velocity decreases linearly with time; while falling down, the (positive) velocity increases linearly with time.

Fig. 4.4 shows the velocity as a function of time. Here, we have chosen the time of reversal as a zero position of the time axis. For this method of counting, the object is thrown at the time „minus 0.4 seconds". We can see in this diagram that the object needs the same time for the upward movement as for the downward movement.

Exercises

1. You jump in the water from a 3-meter diving board. The free fall takes 0.77s. What is your momentum when you hit the water surface? What is your velocity?
2. What is the velocity of a free-falling body after a falling time of 1/2 s on Earth and on the Moon? What would it be on the Sun if a body existed there?
3. A stone is thrown upwards. Its initial velocity is 15 m/s. After what time will it hit the surface of the Earth?
4. A stone is shot upwards by means of a slingshot. After 5 seconds, it hits the ground. What was its initial velocity?

4.4 Falling with friction

The air friction can often be neglected. How strong it is depends on

- the shape of the body
- its velocity.

You certainly know how this works when you think of a car:

- The shape of the car body is designed in a way as to minimize air friction.
- If we drive fast, the friction and hence the fuel consumption (per kilometer) will be higher than if we drive slowly.

Figures 4.5 and 4.6 show that the friction, i.e. the momentum current that flows away into the air, increases strongly with a growing velocity.

In both pictures, the momentum loss due to friction is illustrated as a function of the velocity, in Fig. 4.5 for a typical passenger car and in Fig. 4.6 for a much smaller object: a ball with a diameter of 30 cm.

We have seen: if there are no friction losses or as long as they can be neglected, all bodies will fall equally fast. But what will the falling velocity be like if friction can no longer be neglected?

We drop a large, light ball, Fig. 4.7, left side. Its mass shall be

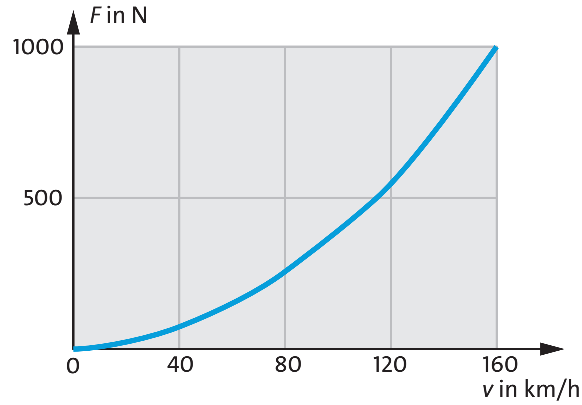


Fig. 4.5 Momentum current that flows away into the air as a function of the velocity for a typical passenger car

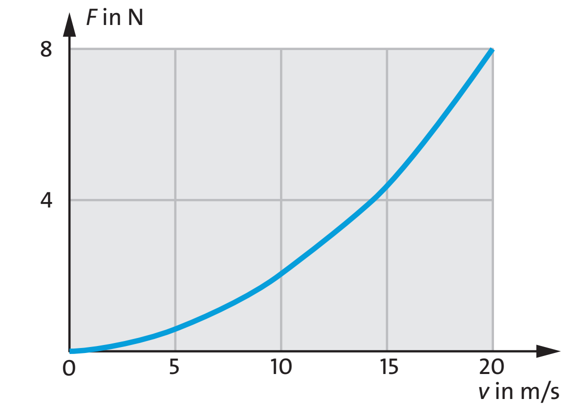


Fig. 4.6 Strength of the momentum current that flows away into the air as a function of the velocity for a globe with a diameter of 30 cm

$$m = 100 \text{ g} = 0.1 \text{ kg},$$

and its diameter

$$30 \text{ cm} = 0.3 \text{ m}.$$

A momentum current of

$$F = m \cdot g = 0.1 \text{ kg} \cdot 10 \text{ N/kg} = 1 \text{ N}$$

flows continuously from the Earth into the ball. At the beginning of the fall, its velocity is still slow, and consequently the loss of momentum to the air is small. At a velocity of 2 m/s, the momentum current that flows into the air still has a strength of less than 0.1 N, see Fig. 4.6. The loss is still small compared to the momentum current of 1 N that comes from the Earth. However, the loss will increase and the ball will eventually lose as much momentum to the air as it receives from the Earth per second. From that moment, its momentum will no longer increase. From Fig. 4.6 we can read that the ball will now have a velocity of approximately 7 m/s.

Fig. 4.8 shows the velocity of our ball over time: at the very beginning, its velocity increases linearly over time; it behaves like a freely falling ball. But the loss will gradually increase. Finally, i.e. when the amounts of momentum that flow towards and away from the ball are equal, its momentum and consequently also its velocity will no longer increase. It has reached its *terminal velocity*. The ball is in the state of *flow equilibrium*.

We now drop another ball. It has the same diameter (30 cm) but four times the weight of the first one, Fig. 4.7, right side:

$m = 0.4$ kg.

A momentum current of

$F = m \cdot g = 0.4 \text{ kg} \cdot 10 \text{ N/kg} = 4 \text{ N}.$

flows into the ball from the Earth via the gravitational field. At which velocity will this ball stop becoming faster? We look once again at the diagram from Fig. 4.6. At a velocity of 14 m/s, the lost momentum is equal to the momentum that comes from the Earth. Therefore, the heavy ball reaches the flow equilibrium at a higher velocity than the light ball.

> At high velocities, the friction of the air can no longer be neglected.
> The velocity of a falling body increases up to a terminal velocity.
> The terminal velocity depends on the shape of the body. It is higher for heavier bodies than for light ones.

Parachuting is an interesting way of applying our reflections. Lilly jumps out of the plane and reaches her terminal velocity of approximately 50 m/s after a few seconds. Then, she will „fall" at this velocity for a longer time. The momentum current that flows into Lilly via the gravitational field has the same strength as the one that flows away due to the air friction.

The parachute opens around 400 m above the ground. But opening the parachute means that the air friction suddenly increases strongly. The momentum current that is flowing away becomes suddenly much

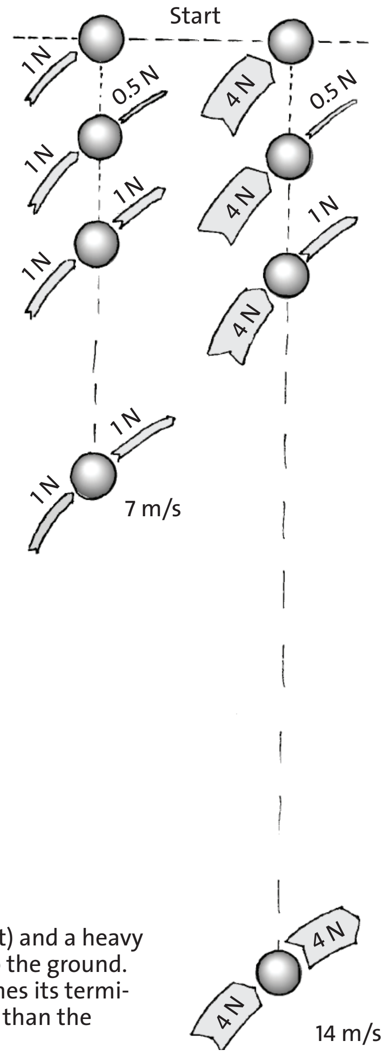


Fig. 4.7 A light (left) and a heavy (right) globe fall to the ground. The light one reaches its terminal velocity earlier than the heavy one.

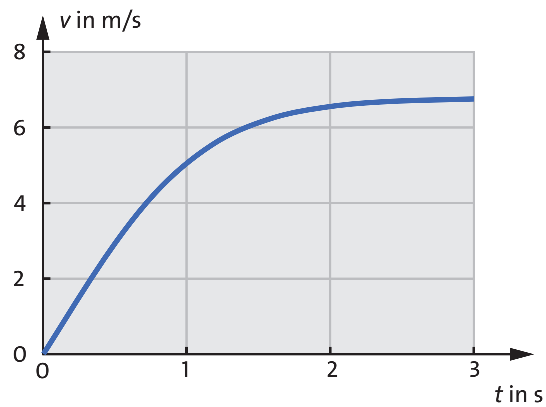


Fig. 4.8 If there is air friction, the velocity of a falling body will increase up to a terminal velocity.

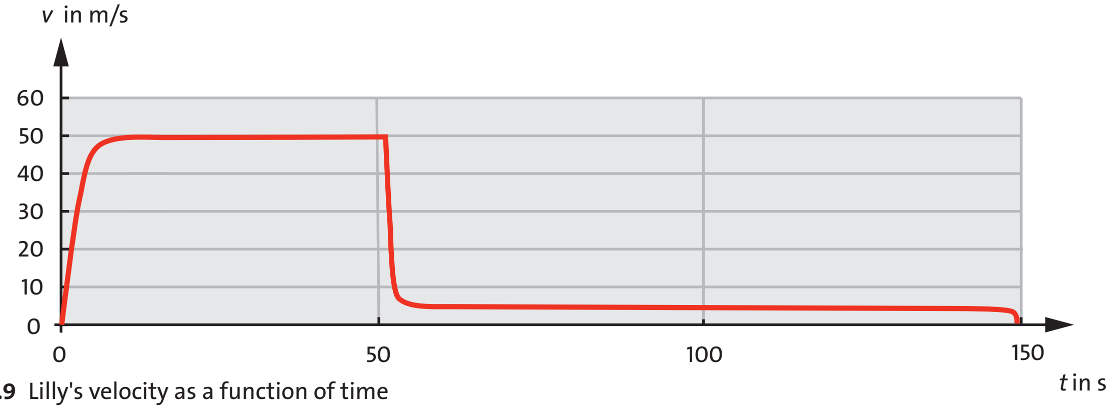


Fig. 4.9 Lilly's velocity as a function of time

stronger than the current flowing into Lilly. Hence, Lilly's momentum decreases, and so does her velocity and consequently also the friction loss. Eventually, the friction momentum current again reaches the same value as the gravitational momentum current, but at a relatively low velocity of approximately 4 m/s. The parachute is now floating with Lilly towards the Earth at a constant, low velocity. In Fig. 4.9, Lilly's velocity is graphed as a function of time.

If there is no air or another medium that causes friction, there will not be any terminal velocity either. The Moon has no atmosphere. Therefore, absolutely all bodies fall with the same velocity there: a sheet of paper falls to the ground as fast as a large stone. But we can also make the same observation on Earth. Therefore, the experiments have to be made in a recipient of which the air has been pumped out. We let some small objects with different masses fall in an evacuated glass tube. As expected, all of them fall equally fast.

Exercise

1. What is the terminal velocity of a falling globe with a diameter of 30 cm and a mass of 0.8 kg?

4.5 Weightlessness

Willy, Fig. 4.10a, feels his heaviness; his body has to bear the weight of his heavy head and his feet have the hardest part: they have to carry the whole body. Willy has an idea: see Fig. 4.10b. The legs are disburdened. But now the arms have to bear the whole weight. In Fig. 4.10c, we see his third attempt to get rid of his weight – once again without any success.

Willy is bothered by the feeling of weight or heaviness. We will try to define this feeling in physical terms. In each of the three cases, Willy feels momentum currents that flow within his body. Momentum flows into every part of his body via the gravitational field, and this momentum has to be conducted back into the Earth. These currents are sketched for a standing person in Fig. 4.11: momentum flows into the head, the arms, the upper part of the body, etc. All this momentum has to flow downwards through the legs and the feet into the Earth. Hence, the momentum current is strongest in the feet.

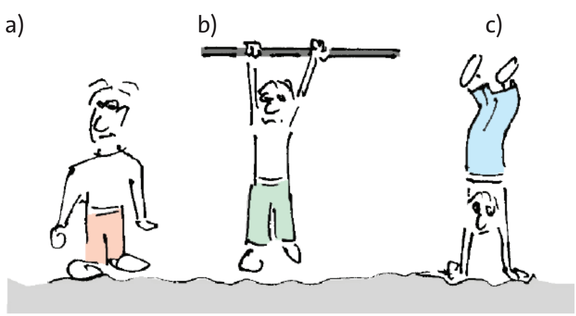


Fig. 4.10 No matter how hard he tries – Willy won't get rid of his feeling of heaviness.

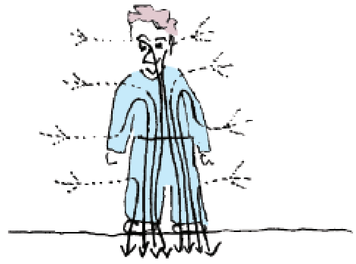


Fig. 4.11 The momentum currents that flow into the person via the gravitational field have to flow back out.

In the following, we examine a model of a person: it consists of two blocks lying on top of each other (to symbolize the upper and the lower part of the body), Fig. 4.12. We can see that the momentum current at the bottom of the lower block is twice as strong as at the bottom of the upper block.

We would now like to put this „person" into a state of weightlessness: a state in which no momentum currents are flowing through it. Or in other words: a state in which none of its parts is subjected to compressional or tensional stress.

You will probably think that this person would have to be sent far away from the Earth, to a place where the gravitational field of the Earth cannot be felt anymore. There, no momentum would flow into our person and no momentum could therefore flow through it either. This would actually be an option. But there is another much simpler method: we let the momentum flow into the person but not back out of it. Also in this case, no momentum will flow through the person and it will feel weightless.

But how can that be achieved? Very simply. To prevent the momentum from flowing back out of the person, i.e. from flowing away into the Earth, it is sufficient to interrupt the connection to the Earth. We can do that by letting our person fall freely, Fig. 4.13. Although momentum is now flowing via the gravitational field into every block (in every part of the person) and into each point of the blocks, it does not flow around in the blocks anymore. Also, no momentum is flowing from one block into the other anymore. The consequence: there are no longer compressional or tensional stresses. The lower block does not feel the weight of the upper one anymore.

Of course, the same applies for yourself, i.e. for a real person: if you jump down from some place, you will be weightless during your fall. Even if you jump upwards, you will be weightless as soon as you lose contact to the Earth, and you will remain weightless until you touch the Earth again.

However, the time that we spend in the air while falling is so short that we hardly notice the feeling of weightlessness. Therefore, we make an experiment with our model person, Fig. 4.14. The two blocks are standing on a plate that is suspended with threads, similar to a weighing pan. There is a thin board, which is connected to the wall by means of a thin stretched rubber strap, between the lower and the upper block. The rubber strap would pull out the board if it were not trapped by the weight of the upper block.

Here is the experiment: we cut the thread on which the whole arrangement is suspended. In the same moment, the board pops out while being pulled by the rubber strap. Why? The tower of blocks was falling freely for a short time. It was weightless during this short time. The upper block did no longer press on the lower one; it released the board. We summarize:

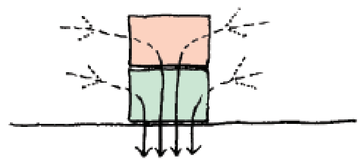


Fig. 4.12 The „body" of our model person consists of an upper and a lower part.

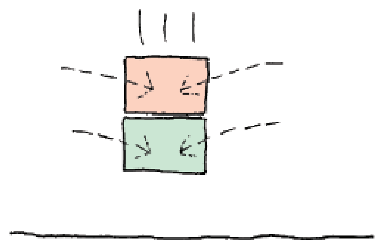


Fig. 4.13 A freely falling body is weightless. No momentum currents are flowing in it.

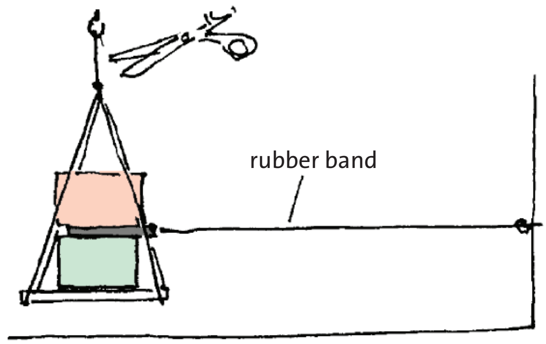


Fig. 4.14 The blocks are weightless during the free fall. The trapped board is released.

> Freely falling bodies are weightless.

4.6 Circular orbits in the gravitational field

Satellites and space platforms move drivelessly on a circular (or nearly circular) orbit around the Earth.

But why don't they fall down towards the Earth? This is exactly what they do. Without the continuous momentum current from the Earth, a satellite would fly straight ahead. However, as it receives momentum from the Earth, its path is bent towards the Earth. If the Earth was flat, it would fall down onto the Earth. But the Earth is round and the satellite always falls in a way as to follow the curvature of the surface of the Earth. In order to fly on a circular orbit around the Earth, the satellite has to have a specific velocity. The direction of the velocity vector has to be parallel to the surface of the Earth, i.e. perpendicular to the connection line between the satellite and the center of the Earth and its absolute value has to have a very specific value.

If these conditions are not met, the satellite will move on a different orbit: an ellipse, a parabola, a hyperbola or a straight line.

We would like to calculate the velocity that a satellite must have so as to have a circular orbit.

We remember: the momentum change per time interval for a body that moves on a circular orbit with a radius r and the velocity v is:

$$\frac{\Delta p}{\Delta t} = m\frac{v^2}{r}.$$

The momentum change of a satellite is caused by the momentum current from the Earth. Therefore, the following has to apply:

$$m\frac{v^2}{r} = m \cdot g.$$

Dividing both sides of the equation by m, we obtain:

$$\frac{v^2}{r} = g.$$

This results in:

$$v = \sqrt{r \cdot g}. \qquad (4.6)$$

The equation also applies approximately to the Moon that orbits around the Earth and to the planets that move around the Sun because the respective orbits are almost circular.

> If a satellite or a celestial body moves on a circular orbit around another one whose mass is much larger, its velocity is (r = radius of the orbit, g = gravitational field strength)
>
> $$v = \sqrt{r \cdot g}.$$

We can also „reverse“ the statement: if the velocity of a satellite is given by equation (4.6), its orbit is circular. At the start, however, a satellite can be given any velocity: any arbitrary absolute value and any direction. So what will the satellite do if the velocity at the start is not that given by equation (4.6) or if it does not have the right direction? The satellite will not move on a circular orbit. So can we let it fly around on any arbitrary orbit? Definitely not. The possible orbits belong to a very specific class of curves, the so-called conic sections. They include:

- circle
- ellipse
- parabola
- hyperbola
- straight line

You certainly understand that both the circle as well as the straight line are nothing else than special cases of the ellipse.

We would like to ask two questions but answer only one of them:

Question 1: why does a satellite fly on a circular orbit?

Answer: it was a stupid question. It flies on a circular orbit because it has been set on a circular orbit.

Question 2: why do the Moon and the planets fly on circular orbits?

Answer: good question. It is hard to answer in this context. Apart from this, these orbits are not exactly circular in a strict sense. The deviation from the circle is relatively large in case of the planet Mercury.

We also calculate the velocity of the *International Space Station* (ISS). It is located at an altitude of 400 km. There, the absolute value of the gravitational field strength is g = 8.7 N/kg. r is equal to the radius of the Earth plus 400 km, hence r = 6770 km. This leads us to

$$v = \sqrt{6.770 \cdot 10^6 \text{ m} \cdot 8.7 \text{ N/kg}}$$
$$= 7.675 \cdot 10^3 \text{ m/s} = 27630 \text{ km/h}.$$

You know that astronauts have a feeling of weightlessness in their spaceship. How can this be explained? Is it because they are so far away from the Earth? Definitely not. We have seen that the ISS is located at an altitude of approximately 400 km. Compared to the radius of the Earth, this is a very short distance. The ISS practically flies closely above the surface of the Earth, Fig. 4.15.

There, the absolute value of the gravitational field strength is only around 10 % lower than on the surface of the Earth. The explanation of weightlessness is exactly the one that we have found for falling bodies: a spaceship with its crew and all its accessories is a freely falling body. And freely falling bodies are weightless. Bear in mind that weightlessness does not mean that the gravitational field strength is zero.

Exercises

1. An astronaut has two objects in his spaceship that look equal but that have different masses. Can he find out which of the bodies has the larger mass and if so, how?
2. A spaceship is located so far away from the Earth that the gravitational field strength is practically equal to zero. Now the astronauts would like to feel their weight again. What can they do without flying to the Earth or to another celestial body?
3. Derive a formula that can be used to calculate the angular velocity of the circular movement of a satellite from the orbital radius and the gravitational field strength.
4. Also the Moon is a satellite of the Earth. It moves around the Earth on a circular orbit with a radius $r = 384{,}000$ km. Calculate the field strength of the gravitational field of the Earth at that distance. Hints: (a) Calculate the circumference of the circular trajectory of the Moon. (b) Calculate the time needed by the Moon for one revolution in seconds. (c) Calculate the velocity of the Moon. (d) Calculate the gravitational field strength.
5. A satellite moves at first on a circular orbit. How will the orbit change if the absolute value of the velocity is suddenly reduced? How will the orbit change if it is suddenly increased? What needs to be done to get a hyperbolic orbit?

4.7 The field of spherically symmetric bodies

The gravitational field strength decreases in the outward directions from the Earth. This decrease is described by

$$g(r) = G\frac{m_A}{r^2}. \qquad (4.7)$$

The equation does not only apply for the Earth but for any spherically symmetric body, i.e. in particular also for other celestial bodies: stars, planets and Moons because they are nearly spherically symmetric.

Let us have a closer look at the equation:

- In the way it is written here, it applies for the absolute value of the field strength. The direction of the field strength vector is the direction towards the center of the spherically symmetric body, i.e. towards the center of the Earth in case of the Earth.
- G is the *gravitational constant.* It is:

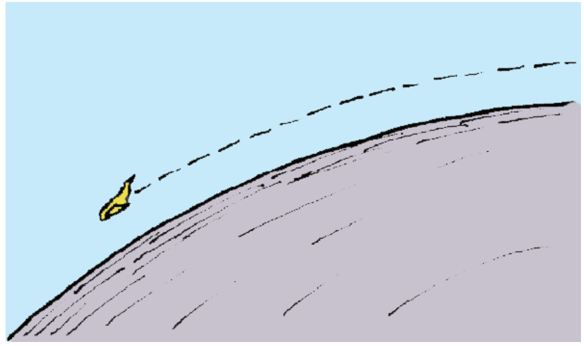


Fig. 4.15 The ISS flies at an altitude of only 400 km, i.e. closely above the surface of the Earth. The absolute value of the gravitational field strength is only 10 % lower than at the surface of the Earth.

$$G = 6.67 \cdot 10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}.$$

G has the same value for the Earth, the Moon, the Sun, any other celestial body and also any small terrestrial body.

- It is not surprising that g is proportional to the mass of the body: a body with a small mass has a weak gravitational field; a body with a large mass has a strong gravitational field.
- r is the distance to the center of the body. The field strength decreases with the square of the distance, i.e. quite fast.

Figure 4.16 shows the field strength – illustrated by vector arrows – around a spherical body. The point an arrow relates to is the respective starting point.

If the distance r is great compared to the size of the body, we will no longer have to require the shape to be spherical. The field at a great distance is the same, regardless of whether the body is spherical or not.

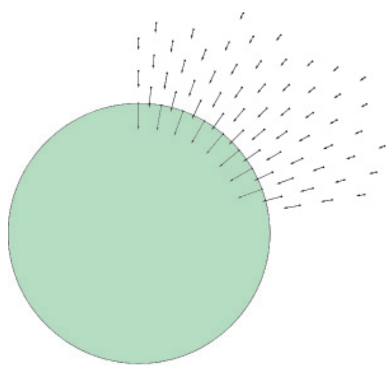


Fig. 4.16 Gravitational field strength vectors in the vicinity of a spherical body

We will look once again at a result of the previous section:

$$v = \sqrt{r \cdot g}. \qquad (4.8)$$

To calculate the velocity that a satellite or another celestial body must have in order to move on a circular orbit with the radius r, we need to know the field strength g of the gravitational field. With equation (4.7) we are now able to calculate it. If we also insert the g from equation (4.7) in equation (4.8) we obtain:

$$v = \sqrt{r \cdot g} = \sqrt{r \cdot G \cdot \frac{m}{r^2}} = \sqrt{\frac{G \cdot m}{r}},$$

and thus:

$$v = \sqrt{\frac{G \cdot m}{r}}. \qquad (4.9)$$

Please bear in mind that m is the mass of the central body and not the mass of the body that circulates around it. Therefore, we can have:

m = mass of	r = orbital radius of	v = velocity of
Earth	satellite	satellite
Earth	Moon	Moon
Sun	Earth	Earth

We assume our central body to be the Earth. To calculate the velocity of a satellite or of the Moon, we only need the respective orbit radius r in addition to the mass of the Earth. The mass of the Sun, the planets and the Moon of the Earth are listed in Table 4.2.

The orbits on which the planets move around the Sun are all situated approximately in one plane, i.e. the *ecliptic plane*. Also the orbit of the Moon is situated in this plane.

We keep in mind:

> When a satellite or a Moon or a planet circulates around a central body, the following applies:
> - the larger the mass of the central body and
> - the smaller the orbit radius, the higher the velocity.

Example

A satellite should describe a circular orbit around the Earth at an altitude of 10,000 km above the surface of the Earth. What has to be its velocity?

$r = 10\,000 \text{ km} + 6\,370 \text{ km} = 16.37 \cdot 10^6 \text{ m}$

$m = 5.97 \cdot 10^{24} \text{ kg}$

$G = 6.67 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$

	m in masses of the Earth	m in kg
Sun	$3.33 \cdot 10^5$	$2.0 \cdot 10^{30}$
Mercury	0.053	$0.317 \cdot 10^{24}$
Venus	0.82	$4.9 \cdot 10^{24}$
Earth	1	$5.97 \cdot 10^{24}$
Mars	0.107	$0.64 \cdot 10^{24}$
Jupiter	318	$1900 \cdot 10^{24}$
Saturn	95.2	$569 \cdot 10^{24}$
Uranus	14.6	$87 \cdot 10^{24}$
Neptune	17.2	$103 \cdot 10^{24}$
Moon of Earth	0.0123	$7.35 \cdot 10^{22}$

Table 4.2 Mass of Sun, planets and Moon

With equation (9) we obtain:

$$v = \sqrt{\frac{6.67 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2) \cdot 5.97 \cdot 10^{24} \text{ kg}}{16.37 \cdot 10^6 \text{ m}}}$$
$$= 4930 \text{ m/s}.$$

Exercises

1. Continuation of exercise 4 from the previous section: calculate the mass of the Earth.
2. The distance between the Sun and the Earth is almost exactly 150 million kilometers. (a) Calculate the velocity of the Earth on its orbit around the Sun. (b) Calculate the field strength of the gravitational field of the Sun at the place of the Earth's orbit. (c) Calculate the mass of the Sun.
3. Every Moon circulates around a planet and the planets circulate around the Sun. We can observe these movements very well by means of telescopes, i.e. we can measure the orbital radii and the times of orbital revolution. The masses of celestial bodies can be determined on this basis. Which data is needed to determine the mass of a planet? (You only have to look at the two previous exercises.)
4. Television satellites circulate around the Earth in a way that they have the same angular velocity as the Earth itself. Why? At which altitude does a television satellite fly? Which orbital velocity does it need to have?

4.8 Galilei, Kepler and Newton

The physical explanation and the mathematical description of the gravitational phenomena – the free fall and the movement of celestial bodies – has been one of the great achievements of physics. This development took place in the 16th and 17th century. Many scien-

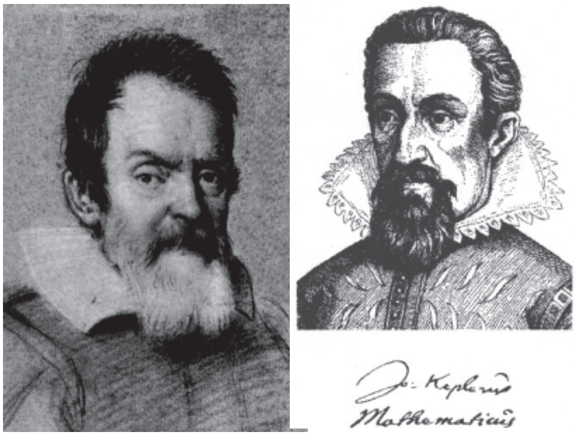


Fig. 4.17 Galileo Galilei (left) and Johannes Kepler (right)

tists were involved, but the most important contributions were made by only three of them: Galilei, Kepler and Newton.

Galileo Galilei (1564-1642), Fig. 4.17, made numerous discoveries and inventions. He found, inter alia, that the velocity of falling bodies increases at a constant rate when the friction of the air can be neglected and that all bodies fall with the same velocity.

Johannes Kepler (1571-1630) succeeded to describe the planetary orbits in a mathematically exact way. Among other things, he found:

The quotient

$$\frac{T^2}{r^3}$$

has the same value for all planets of the solar system (T = time of revolution, r = orbit radius).

Isaac Newton (1643-1727), Fig. 2.35, discovered that the fall of an object onto the Earth is basically the same phenomenon as the movement of the Moon around the Earth and of the planets around the Sun.

In addition, he found the relationship that we described with equation (4.7). However, he had to formulate this relationship slightly differently as no fields – and consequently no field strength – were known back then. But we can easily derive Newton's equation from our equations.

We apply equation (4.7) to a body A with the mass m_A (e.g. the Earth):

$$g(r) = G\frac{m_A}{r^2}. \qquad (4.10)$$

The momentum current from body A into another body B (with the mass m_B) is

$$F = m_B \cdot g(r).$$

We replace $g(r)$ by means of equation (4.10):

$$F = m_B \cdot G\frac{m_A}{r^2} = G\frac{m_A \cdot m_B}{r^2}.$$

Thus, we get:

$$F = G\frac{m_A \cdot m_B}{r^2}.$$

This is Newton's law of gravitation.

The centers of two bodies (masses m_A and m_B) have the distance r from each other. The momentum current that flows through the gravitational field from one body to the other is
- proportional to m_A and to m_B;
- inversely proportional to r^2.

Once again the conditions under which the law of gravitation applies: each of the bodies either has to have a spherical mass distribution or be small in relation to the distance r.

Exercises

1. Derive Kepler's law that was quoted in the text, starting from equation (4.9). Convert the velocity into the time of revolution.
2. The gravitational constant should be determined experimentally by measuring the momentum current between two bodies that have a mass of 1 kg and whose centers have a distance of 10 cm. What is the problem in this measurement?

4.9 The tides

The velocity of a falling apple in the gravitational field of the Earth increases at a constant rate:

$$v = g \cdot t,$$

the acceleration a is:

$$a = g.$$

g is the field strength of the gravitational field of the Earth. The acceleration of the apple does not depend on its own mass, but it depends on the field strength g and g depends on the mass of the Earth and on the location:

$$g(r) = G \cdot \frac{m}{r^2}.$$

This leads us to an interesting problem: what will happen if the falling body is so large that g has different values at different points of the body, Fig. 4.18?

A slightly clearer situation is shown in Fig. 4.19. Instead of an extended body, we look at a sort of dumbbell. Two bodies K_1 and K_2 with the same mass

$$m = 4 \text{ kg}$$

are connected by a bar.

The mass of the bar is so small that we can neglect it in relation to the one of the two bodies. The field strength has the different values g_1 and g_2 at the two bodies. We assume the following:

$$g_1 = 11 \text{ N/kg and } g_2 = 12 \text{ N/kg}.$$

Different momentum currents are consequently flowing into the two bodies:

$$K_1: \quad F_1 = m \cdot g_1 = 4 \text{ kg} \cdot 11 \text{ N/kg} = 44 \text{ N}$$
$$K_2: \quad F_2 = m \cdot g_2 = 4 \text{ kg} \cdot 12 \text{ N/kg} = 48 \text{ N}$$

Hence, K_2 receives more momentum per second from the Earth than K_1. If there were no bar, the momentum increase (i.e. the rate of change) of K_2 would be 48 N and that of K_1 would be 44 N. K_2 would fall faster than K_1 at all times.

But the bodies are connected by the bar and cannot fall with different velocities. Therefore, a momentum current F_G, which ensures that the momentum increase of the two bodies will become equal, has to flow from K_2 to K_1. In our example,

$$F_G = 2 \text{ N}.$$

Thus, the momentum increase is:

$$\text{body } K_1: \quad \frac{\Delta p_1}{\Delta t} = 44 \text{ N} + 2 \text{ N} = 46 \text{ Hy/s}$$
$$\text{body } K_2: \quad \frac{\Delta p_2}{\Delta t} = 48 \text{ N} - 2 \text{ N} = 46 \text{ Hy/s}$$

The bar is subjected to tensional stress. This can also be expressed as follows: K_2 pulls on K_1 so that K_1 becomes faster, or K_1 pulls on K_2 so that K_2 becomes slower.

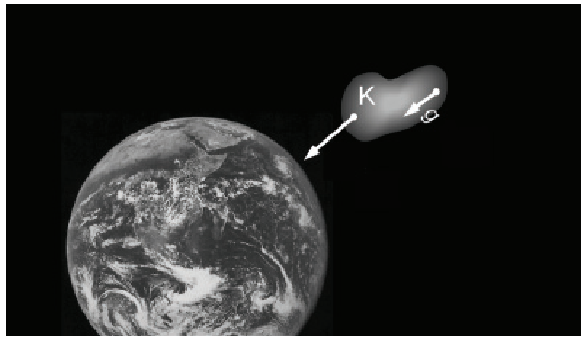


Fig. 4.18 The strength of the field of the Earth has different values in the area of the body K.

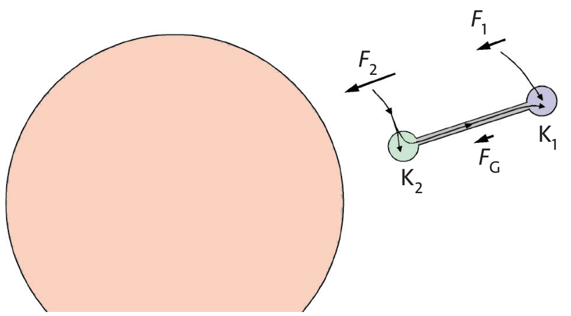


Fig. 4.19 In K_1, less momentum flows in via the gravitational field than in K_2. Momentum therefore has to flow from K_2 to K_1.

Here once more the reason behind the flowing of the momentum current F_G: the field strength in the area of the body has different values. The field is not homogeneous.

The result that we have obtained by means of the dumbbell body is also valid for any other body:

> If the gravitational field in the area of a falling body is not homogeneous, momentum currents will flow within the body. The body is subjected to tensional stress in the falling direction.

It is as if someone tried to stretch the body.

We compare this statement to the result that we found earlier: free-falling bodies are weightless. This means that no momentum currents are flowing within such a body. Now we can see that this rule only applies for a field that is homogeneous in the area of the examined falling body. But where do such tensional stresses actually play a role, and what do these considerations have to do with our headline?

Every body that is exposed to the gravitational field of another one will be subjected to tensional stress if the field of that other body is not homogeneous. The Earth is located in the gravitational field of the Sun and the Moon. The fields of these two celestial bodies are almost homogeneous in the area of the Earth, but only almost. And as they are not perfectly homogeneous, the Earth as a whole is exposed to tensional stress. The influence of the Moon is thereby greater than that of the Sun. The inhomogeneity of the Moon's field as greater at the location of the Earth than that of the Sun's field.

In the following, we will only focus on the influence of the Moon. The Moon tries to stretch the Earth in the direction of the connection line Earth – Moon. This does not have much effect on the solid Earth, but the water of the oceans can react to this tensional stress. Water accumulates respectively on the two opposite sides of the Earth, Fig. 4.20, which leads to a high tide. On the sides of the Earth, the tide is low. While the Earth rotates around its own axis, those accumulations of water move around the Earth, or rather: the Earth rotates away under the accumulated water. If we remain at a fixed place of the Earth, the tide will rise and fall over a period of 12 hours. The falling and rising tides are also referred to as ebb and flow.

These tensional stresses of the tides are a minor effect on Earth. But there are also areas in the world where they become very strong, i.e. on the surface of a neutron star. A neutron star is not a convenient place for several reasons. The gravitational field strength is approximately 10^{12} N/kg so that humans would be crushed immediately by their own weight. But also during a free fall, i.e. while being in a state of weightlessness under terrestrial conditions, we would be torn apart by the tensional stress of the tides on a neutron star.

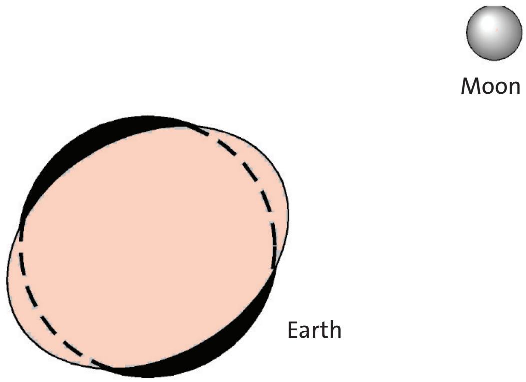


Fig. 4.20 The gravitational field of the Moon tries to stretch the Earth. Two „water hills“ arise on opposite sides of the Earth. (The image is not drawn to scale.)

Exercises

1. Calculate the gravitational field strength of the field of the Moon at the location of the Earth. How big is the difference between the side that faces the Moon and the one that faces away from it?
2. A freely falling body in a non-homogeneous field is exposed to tensional stress in the direction of fall. But this is not the full truth. It is also subjected to mechanical stress in the direction transversal to it. Why? Explain once again by using a dumbbell-shaped body but align it transversally to the fall direction. What type of stress exists: tensional or compressional?

5 MOMENTUM, ANGULAR MOMENTUM AND ENERGY

5.1 What is energy?

Let's repeat: what is momentum? The answer is easy: it is essentially a measure for what we colloquially call impetus or verve. Impetus is contained in a body, i.e. a body is „filled with impetus“. Thus, momentum is a substance-like quantity.

And energy? We can also think of it as „stuff“ that is contained in things – in solid bodies, in liquids, in gases and in fields. It is a substance-like quantity as well. However, we cannot say that it corresponds to anything we have a name for – like saying „impetus“ if we mean momentum or „heat“ to refer to entropy. The reason: energy has no property by which it can always be recognized easily. A body has much energy when it is hot, when it moves fast, when it rotates fast or when it is under high pressure. And its energy depends on the chemical composition. Unfortunately, the energy content is not simply proportional to the velocity, to the temperature, to the pressure, etc..The relationship is more complicated.

Therefore, some skill is needed to tell whether a system contains much or little energy and it is often complicated to calculate the energy. We would like to analyze this problem in the following.

Energy also has simple characteristics after all:

> Energy is a substance-like quantity. Energy can neither be created nor destroyed.

For general orientation we would also keep in mind:

> A body has much energy when it is moving or rotating fast, when it is hot or when it has a high pressure.

There are restrictions for each of these criteria, but you will only get to know them little by little.

We will only get a final answer to the question „what is energy?“ in chapter 7. At the moment, we could do much with this answer yet.

Please remember: the symbol of the energy is E, the measuring unit is Joule, abbreviated J.

5.2 Momentum as an energy carrier

Willy, Fig. 5.1, once again pulls a box over the floor. He makes an effort, hence releases energy. The energy comes from his muscles. Where does this energy go? It goes to the bottom of the box, creates entropy there and spreads out in the environment together with the entropy.

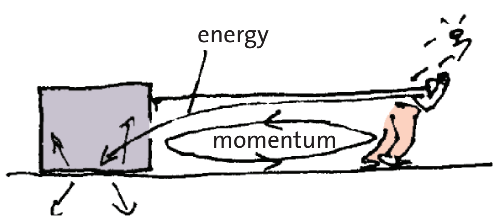


Fig. 5.1 With momentum from Willy's muscles as an energy carrier, the energy flows to the bottom of the box. From there, it will move into diverse directions with entropy as an energy carrier.

We would like to examine the energy transport between Willy and the box. The first point to clarify: what is the energy carrier? A momentum current is flowing in the rope between Willy and the box at the same time as the energy current. We conclude that the momentum is the energy carrier we are looking for.

> Momentum is an energy carrier.

Energy and momentum current are illustrated schematically in the flow chart of Fig. 5.2.

Not every momentum current comes with an energy current: the momentum current in Fig. 5.1 flows, as we know, from the box through the Earth and back to Willy. But the energy goes its own way from the bottom side of the box. The momentum current that flows back does not carry any energy.

So, what does the strength of the energy current depend upon? Or in more general terms: what do we have to do to transfer as much energy as possible with a rope or a bar?

If we attach a tight rope with hooks on a wall, Fig. 5.3, a momentum current but no energy current will flow. What is the difference between the ropes in Fig. 5.1 and Fig. 5.3? The first rope moves, the second one doesn't. Hence, we can see that the movement is important for the energy transport; or more precisely: the velocity at which the momentum conductor moves.

Of course, the strength of the energy current also depends on the strength of the momentum current because if the rope is not subjected to mechanical stress, it cannot be used to transfer energy.

We therefore have an important result:

> The energy current P through the rope depends on
> - the momentum current F in the rope,
> - the velocity v of the rope.

We want to find out the quantitative relationship. By which equation are the three quantities P, F and v linked to each other?

At first we look at the relationship between the energy current P and the momentum current F. Fig. 5.4 shows a top view of two boxes being pulled over the floor.

We compare the two rope sections A and B. Both are moving with the same velocity. The momentum current as well as the energy current split up evenly at the intersection point P: the momentum current in rope B is half as strong as in A, and so is the energy current. Hence, the energy current strength is proportional to the momentum current strength in case of equal velocity:

$$P \sim F.$$

To find the relationship between P and v we make an experiment. A box is pulled by means of a pulley, Fig. 5.5.

We compare the rope sections A and B. Let's first consider the energy current: all the energy that flows into the rope from the right continues from the pulley through rope A. No energy can flow in rope C because C is not moving. We consequently have:

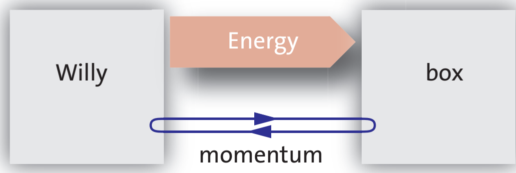


Fig. 5.2 Flow chart for the energy and momentum currents in Fig. 5.1

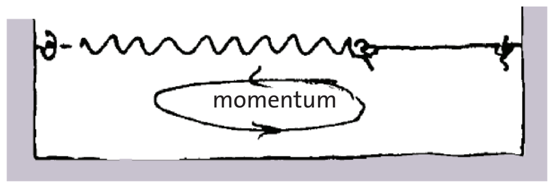


Fig. 5.3 A momentum current but no energy current is flowing.

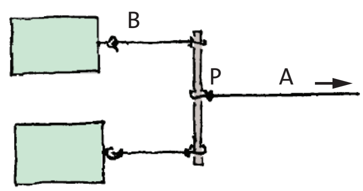


Fig. 5.4 Two boxes are pulled over the floor. Top view

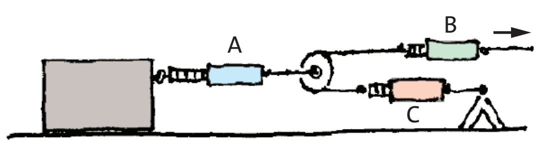


Fig. 5.5 The momentum current strength in rope A is twice that in rope B. The velocity of rope A is half the velocity of rope B.

$$P_A = P_B.$$

Next, we compare the velocities of A and B. If the box, and therefore rope A, moves over a certain distance to the right, the right end of B will move to the right by twice this distance. This means that the velocity of B is twice that of A. Therefore we have:

$$v_B = 2\,v_A.$$

Finally, we compare the momentum currents in A and B. This can only be done by means of a measurement. It turns out that the momentum current in B is just half that in A. (By the way: in C it is the same as in B so that the junction rule is fulfilled.) We can therefore write:

$$F_A = 2\,F_B.$$

All these results together will be described correctly if we set:

$$P \sim v \cdot F.$$

This proportionality tells us on one hand that P is proportional to F if the velocity is kept constant. On the other hand it states: if v is doubled and F is halved, P will remain constant. This is exactly what we found in our experiment with the pulley.

> If energy is transferred with momentum as an energy carrier, the energy current is proportional to the momentum current and to the velocity at which the conductor is moving.

To get an equation from this proportionality, a constant of proportionality would normally have to be introduced. But the SI measurement units of the three quantities have been chosen in a way that simply the following applies:

$$P = v \cdot F.$$

This is the result we were looking for. We can use it to calculate the energy current in our rope if we know the momentum current in the rope and the velocity of the rope.

Example

We pull on a rope in which a dynamometer is installed. The dynamometer indicates 120 N, the rope moves at 0.5 m/s. For the energy current we get:

$$P = v \cdot F = 0.5\ \text{m/s} \cdot 120\ \text{N} = 60\ \text{W}.$$

Notice that the velocity has to be inserted in m/s and the momentum current in N in order to obtain the energy current in the SI unit watt.

Similar formulas apply when the energy flows with other carriers. If the electric charge is the energy carrier, we have the following relation:

$$P = U \cdot I\,,$$

i.e. the energy current is proportional to the electric current I. If the entropy is the energy carrier, we will have:

$$P = T \cdot I_S,$$

i.e. the energy current is proportional to the entropy current I_S. From the formula

$$P = v \cdot F$$

we can derive an equation that is more practical in certain cases.

We replace

$$P = \frac{\Delta E}{\Delta t}$$

on the left and

$$v = \frac{\Delta s}{\Delta t}$$

on the right, hence:

$$\frac{\Delta E}{\Delta t} = \frac{\Delta s}{\Delta t} \cdot F.$$

Therefore, we obtain:

$$\Delta E = F \cdot \Delta s.$$

The equation tells us for example: if we push against a bar and move the bar by the distance Δs, the energy $F \cdot \Delta s$ will flow through the bar. F is the strength of the momentum current that flows through the bar while pushing. Of course, F must be constant during the process.

Example

We pull on a rope in a way that a momentum current of 120 N is flowing and that the rope moves by 2 m. How much energy is transferred through the rope in the process? With

$F = 120$ N and $\Delta s = 2$ m

we obtain

$$\Delta E = F \cdot \Delta s = 120 \text{ N} \cdot 2 \text{ m} = 240 \text{ Nm} = 240 \text{ J}.$$

Exercises

1. A tractor pulls a trailer on an even road at a velocity of 20 km/h. A momentum current of 900 N is flowing through the trailer hitch. What is the energy consumption of the trailer? (What is the strength of the energy current from the tractor to the trailer?) Where will the momentum that flows to the trailer go? Where will the energy go?
2. The drive belt of a machine runs at a velocity of 10 m/s. The current strength of the energy transferred with the belt is 800 W. Which is the force applied by the belt to pull on the belt disc? (What is the strength of the momentum current in the belt?)
3. A crane lifts a load of 50 kg with a velocity of 0.8 m/s. What is the strength of the energy current through the crane rope? The load is lifted by 5 m. How long does this process take? How much energy is flowing through the rope during this time?
4. A truck pulls a trailer on an even road from one town to another. The distance between the towns is 35 km. A momentum current of 900 N flows through the trailer hitch. How much energy will have flowed from the truck to the trailer in total?

5.3 Angular momentum as an energy carrier

Figure 5.6 shows once again the coffee grinder that we have already analyzed in chapter 3. Also here, energy is transferred: from the motor to the grinder by means of the shaft. Only the angular momentum can be an energy carrier in this case.

Angular momentum is an energy carrier.

The corresponding flow chart is shown in Fig. 5.7.

Fig. 5.8 shows the flow chart of a hydroelectric power plant. The generator is driven by a water turbine.

In both cases, the angular momentum flows in a closed circuit.

The relationship between the energy current P and the angular momentum current M is of the same type as that between the energy current and the momentum current (or energy current and electric current):

$$P = \omega \cdot M.$$

If energy is transferred by means of a rotating shaft (with angular momentum as an energy carrier), the energy current strength will be proportional to the angular momentum current strength and to the angular velocity at which the shaft is moving.

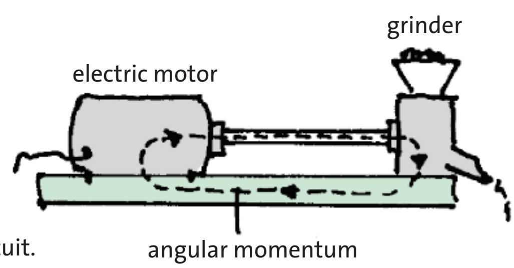


Fig. 5.6 Coffee grinder. The angular momentum flows in a circuit.

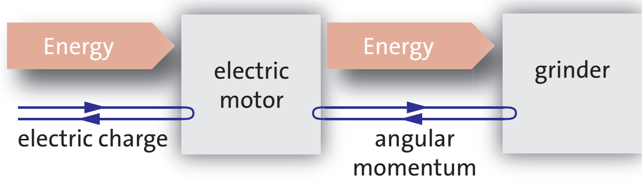


Fig. 5.7 Flow chart of the coffee grinder from Fig. 5.6

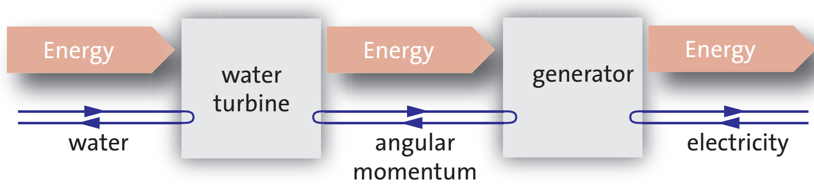
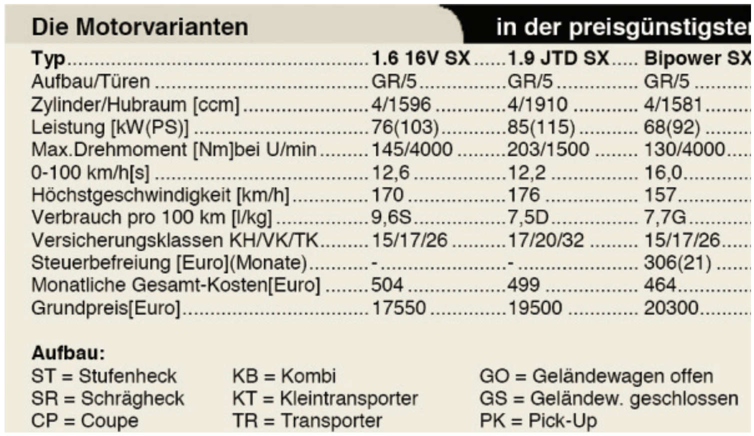


Fig. 5.8 Flow chart of a hydroelectric power plant



Die Motorvarianten — **in der preisgünstigste**

Typ	**1.6 16V SX**	**1.9 JTD SX**	**Bipower SX**
Aufbau/Türen	GR/5	GR/5	GR/5
Zylinder/Hubraum [ccm]	4/1596	4/1910	4/1581
Leistung [kW(PS)]	76(103)	85(115)	68(92)
Max.Drehmoment [Nm]bei U/min	145/4000	203/1500	130/4000
0-100 km/h[s]	12,6	12,2	16,0
Höchstgeschwindigkeit [km/h]	170	176	157
Verbrauch pro 100 km [l/kg]	9,6S	7,5D	7,7G
Versicherungsklassen KH/VK/TK	15/17/26	17/20/32	15/17/26
Steuerbefreiung [Euro](Monate)	-	-	306(21)
Monatliche Gesamt-Kosten[Euro]	504	499	464
Grundpreis[Euro]	17550	19500	20300

Aufbau:

ST = Stufenheck	KB = Kombi	GO = Geländewagen offen
SR = Schrägheck	KT = Kleintransporter	GS = Geländew. geschlossen
CP = Coupe	TR = Transporter	PK = Pick-Up

Fig. 5.9 For exercise 2

Exercises

1. The Iffezheim power plant on the Rhine in the Baden-Baden area has four water turbines with one generator each. We look at such a power unit. A turbine provides its generator with an energy current of 27 MW. The shaft that connects the turbine to the generator turns with 100 rotations per minute. What is the angular momentum current of the shaft?
2. Fig. 5.9 shows the specification sheet of a car. Which energy current (= power) is supplied by one of the engines with a maximum angular momentum current (maximum torque)? Compare with the maximum energy current indicated on the specification sheet. Where could the difference come from?

5.4 Mechanical energy storage

a) Elastically deformed bodies as energy storage devices

We stretch a long, strong spring, Fig. 5.10. This is exhausting because much energy is needed.

We look at the right end of the spring (point A in Fig. 5.10). This end of the spring is exposed to mechanical stress, i.e. a momentum current F is flowing in it, and it moves at a velocity v. According to the formula

$$P = v \cdot F$$

also an energy current is flowing in it. Now we look at the left end of the spring (point C). Here, the momentum current is the same as at A. But as C does not move, no energy current is flowing here. The energy that flows into the spring at A does not come out at C. It is stored in the spring.

We can also check the currents at other points of the spring, e.g. in the middle of the spring. There, the momentum current is the same as at A and at C again. The velocity, however, is only half that of A. Therefore, also the energy current is only half as strong as the one that enters into the spring at A. This is logical: half of the energy is stored in the right half of the spring and the rest continues flowing into the left spring half. We can also take this reflection further: only a third of the energy is stored in each third of the spring, a fourth of the energy is stored in each fourth of the spring, etc.. Or in short: the energy spreads evenly over the total length of the spring.

If a spring can be compressed without buckling to the side, it can also be used as an energy storage device in this way.

These observations are not only valid for springs, but also for any other elastically deformable objects: a stretched expander contains energy, just as a taut slingshot, a bent springboard or a compressed soccer ball.

Of course, we would like to know how much energy is contained in a specific energy storage device. We would like to calculate how the stored amount of en-

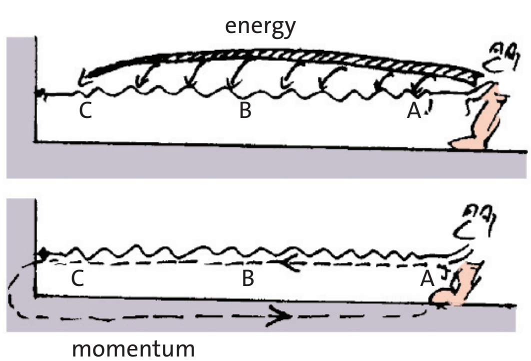


Fig. 5.10 When the spring is stretched, energy flows into the spring via the right end.

ergy depends on the prolongation (or shortening) of the spring. The problem is more complicated than it might seem at first.

We stretch a spring by moving one end with a constant velocity v.

We know the relationship:

$$P = v \cdot F. \tag{5.1}$$

We could think that it can be used to calculate the stored energy E of the spring in the following way: multiplication of the energy current (the joules per second) by the time t_0 during which it was flowing, i.e.

$$E_{\text{spring}} = P \cdot t_0. \tag{5.2}$$

But this would give us the correct result only if the energy flows evenly, i.e. if the energy current does not change over time. Unfortunately, this is not the case here; the momentum current F in equation (5.1) is not constant. However, we can still apply equation (5.2) if we insert the time average value of P:

$$E_{\text{spring}} = \overline{P} \cdot t_0. \tag{5.3}$$

Hence, we need to find the average value of the energy current.

We start from equation (5.1). We replace F by means of

$$F = D \cdot s$$

(see section 2.13) and obtain:

$$P = v \cdot D \cdot s.$$

This equation tells us that the energy current is stronger the more the spring has been stretched. As we pull on the spring with a constant velocity, we can replace s by $v \cdot t$:

$$P = v \cdot D \cdot v \cdot t = D \cdot v^2 \cdot t.$$

Hence, the energy current increases linearly over time, Fig. 5.11.

We can read the mean energy current from this graph: it is equal to the energy current at the time $t_0/2$, i.e.

$$\overline{P} = \frac{D}{2} v^2 t_0.$$

By inserting in equation (5.3) we obtain:

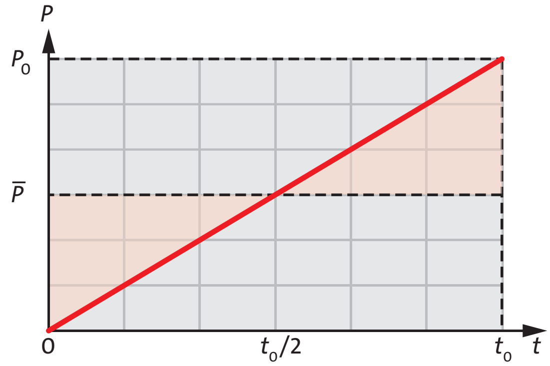


Fig. 5.11 The energy current increases from zero to P_0 in the period from $t = 0$ to t_0. The average energy current in this time interval is equal to the energy current at the time $t_0/2$. The downward deviation before will be compensated by the upward deviation after.

$$E_{\text{spring}} = \frac{D}{2} v^2 t_0^2. \tag{5.4}$$

Now, $v \cdot t_0 = s_0$, i.e. equal to the extension of the spring. If we insert this in equation (5.4), we obtain our final result:

$$E_{\text{spring}} = \frac{D}{2} s_0^2.$$

The index „0“ is no longer needed as there is no risk of confusion:

> If we stretch a spring, its energy will increase by
>
> $$E_{\text{spring}} = \frac{D}{2} s^2.$$
>
> If the spring is released, the energy will flow back out.

We assumed that the spring had been extended in order to store energy. If a spring can be compressed, however, the same equation will apply for it. Then, s will be the distance by which the spring is shortened.

Although the derivation was cumbersome, the result is simple and also logical. If you have forgotten the equation, you can recover its essential aspects through skilled guessing.

To begin with: what does the energy stored in a spring depend on after all? First, on the spring itself, i.e. on the spring constant D. And second, on how strongly it has been extended or shortened, i.e. on s. For a given extension s, a hard spring contains more energy than a soft one. This is ensured by the D in the formula.

It is also clear that the more the spring has been extended, the more energy it contains – therefore the s.

But why is the s squared? There is a good reason: the spring stores positive energy (negative energy does not exist), regardless of whether it is extended or shortened, i.e. regardless of whether s is positive or negative. As s is squared, the result is always a positive energy, both for a compressed as well as for a stretched spring.

The only part of the formula that cannot be guessed as easily is the factor 1/2. You therefore have to memorize it.

Later you will learn about several other equations that have this structure.

To conclude, please bear in mind: the energy that is calculated according to our equation is not the entire energy of the spring. It is only a tiny part of it; i.e. that part that is put in during stretching and that comes back out during release.

b) Moving bodies as energy storage devices

We charge a trolley or a car with momentum, just as we have already done many times before, Fig. 5.12. However, we have learned in the meantime that not only momentum but also energy is flowing in the rope. Neither the energy nor the momentum can leave the car. If we pull, both momentum as well as energy will consequently be accumulated in the car.

The amount of energy that a body contains due to its movement is called *kinetic energy.*

If we let a moving trolley roll until it stops, its momentum will flow away into the Earth. The energy will take a different path. It will be used (or rather: wasted) for entropy production. Entropy is produced wherever friction takes place. The energy thereby spreads out in the environment: partially in the ground and partially also in the trolley and in the air.

Again, there is a simple equation that can be used to calculate the stored (kinetic) energy. The derivation is similar to what we have seen for the stretched spring. We skip it because the result can almost be guessed.

> If a body is charged with momentum, its energy will increase by
>
> $$E_{kin} = \frac{m}{2}v^2.$$
>
> If the body releases the momentum, also the energy will flow back out.

The fact that the velocity is squared ensures that the energy will always be positive. By means of $p = m \cdot v$ we can express the velocity by the momentum and obtain:

$$E_{kin} = \frac{p^2}{2m}.$$

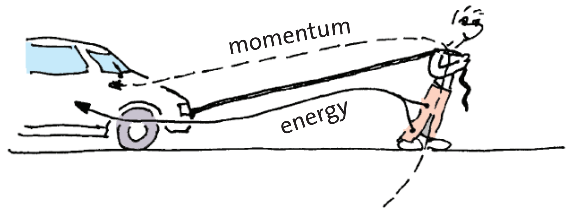


Fig. 5.12 The car is charged with momentum and energy.

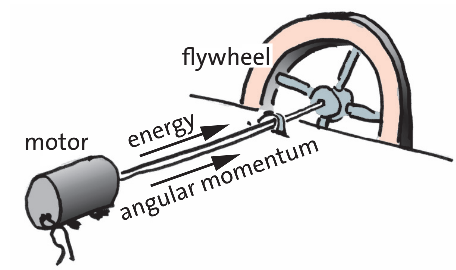


Fig. 5.13 In the flywheel angular momentum and energy is stored.

c) Rotating bodies as energy storage devices

A flywheel is charged with angular momentum, Fig. 5.13. However, not only angular momentum but also energy flows through the shaft. Both the angular momentum as well as the energy are stored in the flywheel.

The energy E_{rot} stored in the flywheel can be calculated from the moment of inertia and the angular velocity. The formula can be obtained very easily by replacing in

$$E_{kin} = \frac{m}{2}v^2$$

the mass by the moment of inertia and the velocity by the angular velocity.

> If a body is charged with angular momentum, its energy will increase by
>
> $$E_{rot} = \frac{J}{2}\omega^2$$
>
> If the body releases the angular momentum, also the energy will flow back out.

By means of $L = J \cdot \omega$ we can express the angular velocity by the angular momentum and obtain:

$$E_{rot} = \frac{L^2}{2J}.$$

Exercises

1. A trolley with a weight of 30 kg is charged with momentum. A momentum current of 20 N is flowing for 6 s. There are no losses due to friction. What will be the kinetic energy of the trolley in the end?
2. A trolley with a weight of 200 g is accelerated through a spring that is being released. It reaches a velocity of 0.8 m/s. The spring has extended from 10 cm to its normal length of 15 cm. What is the value of the spring constant?
3. A glider on the air track collides with a glider that is twice as heavy and that does not move at the beginning. The collision is completely inelastic, i.e. the gliders are attached to each other after the collision. Compare the kinetic energies before and after the collision. Explain.
4. A trolley (m = 20 kg) moves with v = 0.5 m/s in the x-direction, hits a spring bumper and bounces back. (a) What is its momentum before and after the collision? (b) What is its kinetic energy before and after the collision? (c) By how much is the spring compressed (spring constant D = 60 N/m)? (d) What will be the velocity of the trolley if the spring is compressed halfway?
5. A glider on the air track (mass 300 g) is attached to a spring (D = 7.5 N/m), Fig. 5.14. If the glider is moved out of the equilibrium position and subsequently released, it will perform an oscillation. Describe the path of the energy and the momentum during the oscillation process. While passing through the „equilibrium position“ (the spring is released), the glider has a velocity of 0.5 m/s. How far does the glider oscillate to the right and the left?
6. A vehicle with a mass m and a velocity v moves (without friction) around a 90° curve. Establish an energy and momentum balance. (How much goes in, how much comes out?)
7. A steam engine has a flywheel with a diameter of 2.2 m and a mass of 1.8 tons. Assume that the entire mass is located in the outer ring. The engine runs at 2 rotations per second. How much angular momentum and how much energy are stored in the flywheel?
8. Two flywheels A and B each have a moment of inertia of 2 kg·m^2. Wheel A turns at 2 rotations per second, wheel B does not turn at first. (a) How much angular momentum does wheel A have? (b) How much energy does A have due to the rotary movement? The wheels are connected to each other by means of a clutch. (c) What is the angular momentum of each of the two wheels? (d) How much energy is contained in each of the two wheels? What is the energy of the two wheels together? (e) Comment on the result.
9. For fusion experiments at the Max Planck Institute for Plasma Physics in Garching (Munich area), a very strong current of electric energy is needed for short time intervals. The required energy current is so strong that the normal electricity grid is not sufficient. Therefore, large flywheels that are charged slowly (with a weak energy current) by means of an electric motor are used. The whole accumulated energy can then be released within a few seconds through a generator. Such a flywheel supplies an energy current of 150 MW during 10 seconds. If it is charged, it will turn at 1600 rotations per minute. What is its moment of inertia?

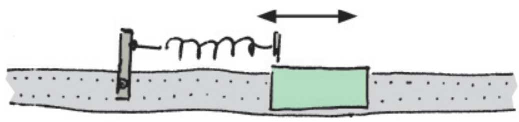


Fig. 5.14 For exercise 5

5.5 Energy storage in the gravitational field – the gravitational potential

A heavy object is pulled upwards, Fig. 5.15. Besides momentum, energy is flowing in the rope. As we know, the momentum comes into the body from the Earth via the gravitational field. We can also think of the field as an invisible spring that pulls on the body. Just as energy is stored in the spring while stretching it, energy is stored in the gravitational field when an object is lifted. If the object is lowered again, the energy can be recovered from the field.

We need more energy to lift a heavy object than to lift a light one. Hence, the larger the mass of the object, the more energy is stored in the field. And the greater the difference of altitude $h = h_2 - h_1$ by which the object is lifted, the more energy is needed.

We calculate the energy E_{grav} that is stored in the gravitational field as a function of m and h. Again, we start from the equation

$$P = v \cdot F$$

for the energy current. As we move the body upwards with a constant velocity, we can write:

$$v = \frac{h}{t_0}.$$

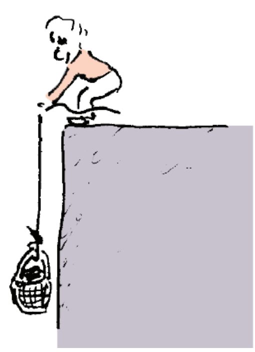


Fig. 5.15 Energy is stored in the gravitational field while the bucket is pulled upwards.

t_0 is the time that the process takes. For the momentum current we have:

$$F = m \cdot g.$$

The momentum current is constant in time. Therefore, we do not need to calculate the average value as in the case of the spring. The energy current during the pulling process will therefore be:

$$P = v \cdot F = \frac{h}{t_0} \cdot m \cdot g.$$

We obtain the stored energy as energy current strength multiplied by time:

$$E_{grav} = P \cdot t_0,$$

i.e.

$$E_{grav} = m \cdot g \cdot h = m \cdot g \cdot (h_2 - h_1).$$

When the altitude of a body on Earth increases, energy is stored in the gravitational field:

$$E_{grav} = m \cdot g \cdot (h_2 - h_1).$$

When the altitude of the body decreases again, the energy is recovered.

We introduce a new physical quantity that simplifies the description: the gravitational potential ψ (psi):

$$\psi = g \cdot h.$$

Further up, the gravitational potential on Earth has a higher value than further down. The potential difference between two places that differ by 2 m of altitude is greater on Earth than on the Moon.

We can think of a difference of the gravitational potential as a drive for a mass flow. All bodies fall from the top to the bottom, from the high to the low gravitational potential. On a bicycle, we roll down the hill without pedaling. Water flows (because it has a mass) from places of high to places of low gravitational potential. All these movements or currents exist because the respective body or the liquid has a mass.

A difference of the gravitational potential is a drive for a mass flow.

Now we can write the energy that is stored in the gravitational field in a shorter form:

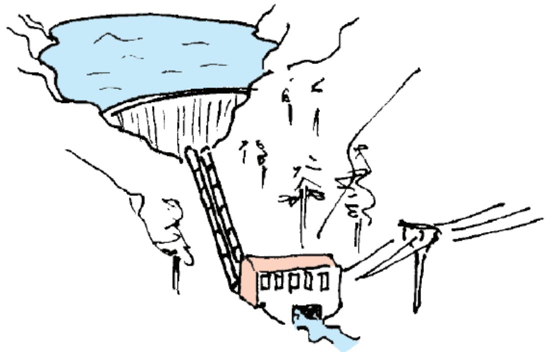


Fig. 5.16 Hydroelectric power plant. The water flows down in the pipes and thereby gets energy from the gravitational field.

If a body is brought from the gravitational potential ψ_1 to the higher potential ψ_2, the energy

$$E_{grav} = (\psi_2 - \psi_1) \cdot m$$

will be stored in the gravitational field.

The energy of the gravitational field is made available in hydroelectric power plants, Fig. 5.16.

At high altitudes in the mountains, water is collected from streams and rivers and led downwards through pipes. While flowing downwards, the water receives energy from the gravitational field. Then, it flows through the turbine of the power plant and releases its energy there. The turbine drives a generator by means of a shaft, i.e. the energy goes with the angular momentum from the turbine to the generator.

Pump storage plants are a special type of hydroelectric power plants. A pump storage plant is used as an energy storage device for the electric grid. It can absorb energy from the grid and release it again to the grid. Such plants are needed because most other power plants are unable to react quickly to the varying demand. Coal-fired power plants, nuclear power plants and run-of-river power plants (hydroelectric power plants that take advantage of the slope of a river) can only change their energy output slowly or not at all. Wind turbines supply energy when there is wind, and their energy output does not correspond to the demand. Therefore, a storage medium, which can absorb and release energy quickly, is needed – similar to a car battery but much larger.

Two large water reservoirs at different altitudes are part of a pump storage plant. When electric energy (energy carried by electric charge) is needed (when energy is expensive), water is let flow from the high reservoir into the low one through a turbine. The turbine drives a generator. When there is energy left in the grid (when power is cheap), the generator is made work as an electric motor and the turbine as a pump, and the water is pumped back up.

Exercises

1. Draw the flow chart of a pump storage plant for its two types of function. The pump storage plant Goldisthal (Thuringian Slate Mountains) has an upper basin at an altitude of 880 m above sea level and a lower basin at 550 m. What is the gravitational potential at the top and at the bottom (in relation to sea level)? The exploitable water quantity is 12 million m^3. How much energy can be stored? The generators can supply a maximum energy current of 1060 MW. How long will the energy stock last?
2. The Itaipú hydroelectric power plant on the Paraná river is located on the border between Brazil and Paraguay. It is the largest hydroelectric power plant of the world. It has 20 turbines and 20 generators. The difference in altitude between the input and the output of the water is 120 m, the water flow is 12 000 m^3/s on average. How much energy current is supplied by the power plant?
3. The water leaves the nozzle of a fountain with a velocity of 5 m/s. How high does it spray?
4. A stone is thrown vertically upwards. Sketch the path of the energy and the momentum (a) while the stone is being thrown off; (b) while it is flying upwards; (c) while it is coming back down.
5. A hollow cylinder (r = 10 cm, m = 2 kg) is nudged so that it rolls at a velocity of 0.8 m/s on a surface that is even at the start, Fig. 5.17. Then, the surface becomes an ascending slope. Up to which altitude will the cylinder roll?

5.6 Tackle, gear drive, chain and belt drive

The tackle

A tackle is often used to lift a load: an arrangement of ropes and pulleys. Fig. 5.18 shows a particularly simple tackle. What are the benefits of it?

We remember: the pulley divides the momentum current, which flows through the rope A from the bottom, into two partial current of equal magnitude in the rope sections B and C. The momentum current in B is therefore only half as strong as that in A.

We assume the mass of the load to be 200 kg. Then, the momentum current in A is

$$F_A = m \cdot g = 200 \text{ kg} \cdot 10 \text{ N/kg} = 2000 \text{ N}.$$

For the momentum current in B we obtain

Fig. 5.17 For exercise 5

$$F_B = F_A/2 = 1000 \text{ N}.$$

Consequently, the motor (the „momentum pump") only needs to create a momentum current of 1000 N and not of 2000 N that would be the case without the pulley. We almost have the impression of getting something for free here. We will see that this is not the case if we compare the energy currents

$$P = v \cdot F$$

in the rope sections A and B. While the momentum current in B is half of that in A, the velocity of B is twice as high as the one of A:

$$F_B = F_A/2\,,$$
$$v_B = 2\,v_A\,.$$

Therefore:

$$v_A \cdot F_A = v_B \cdot F_B\,.$$

Hence, the energy currents in A and B are equal. We could also have explained it this way: no energy is flowing through rope C because $v_C = 0$. The whole energy that comes from the motor via rope B therefore has to continue flowing to the load through rope A. We can conclude:

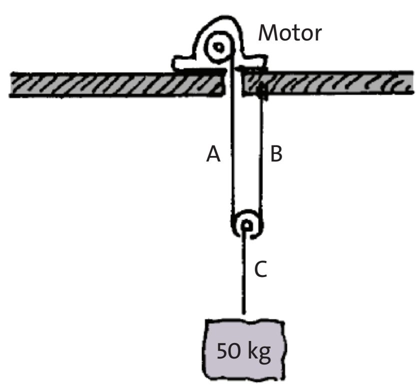


Fig. 5.18 Simple tackle

> *Tackle*: One factor in the equation $P = v \cdot F$ changes at the expense of the other one.

Maybe this result looks familiar to you. We have already come across a similar situation in connection with the electric transformer. The equation

$$P = U \cdot I$$

applies for an electric energy transport (energy current equal to voltage multiplied by electric current.) The energy current that flows into the transformer with the electric charge as an energy carrier (index A) is equal to the outflowing one (index B):

$$U_A \cdot I_A = U_B \cdot I_B \,.$$

Hence, a tackle can also be considered a „momentum current transformer".

Gear drive

Fig. 5.19 shows a photo of a simple gear drive.

Such a drive is illustrated schematically in Fig. 5.20. The energy comes through the shaft on the left (A) and leaves the drive through the one on the right (B). The energy carrier of the arriving and the outflowing energy is angular momentum. The relationship between the energy current and the angular momentum current is $P = \omega \cdot M$.

As all the energy that arrives via A flows away via B, the following has to apply:

$$\omega_A \cdot M_A = \omega_B \cdot M_B \,.$$

We divide both sides of the equation by ω_A and M_B:

$$\frac{M_A}{M_B} = \frac{\omega_B}{\omega_A}. \qquad (5.5)$$

The ratio ω_A/ω_B of the angular velocities, i.e. the gear ratio, can be determined easily: if gear wheel A has twice as many teeth as gear wheel B, gear wheel B will turn twice as fast as A. Therefore we have:

$$\frac{M_A}{M_B} = \frac{z_A}{z_B}.$$

Here, z_A and z_B are the number of teeth of the two gear wheels.

We summarize:

> *Gear box*
>
> One factor in the equation $P = \omega \cdot M$ changes at the expense of the other one.

Fig. 5.19 Gear drive

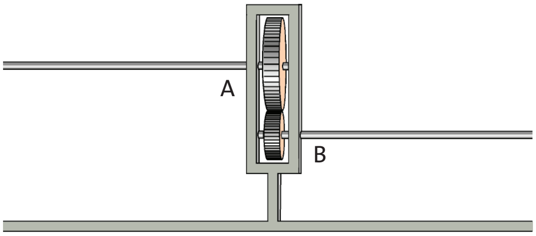


Fig. 5.20 Simple gear drive

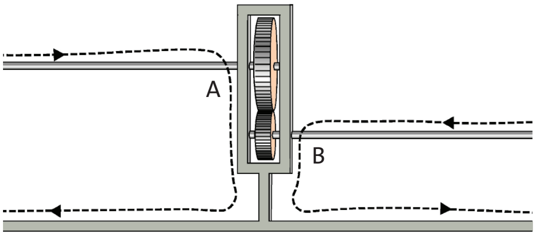


Fig. 5.21 The angular momentum in the shafts A and B flows in opposite directions.

A gear drive can be considered an „angular momentum current transformer".

The gear ratio can be changed for the shifting gear box of the car.

We would like to take a closer look at the path of the angular momentum currents in case of the gear box, Fig. 5.21.

At first, we have to notice that in our simple gear box, the direction of the currents in the two shafts A and B is opposite. Again, the support, the foundation or the chassis ensures that the angular momentum will flow back. The path of the angular momentum current within the gear box is only displayed schematically here. The current flows both over the gear wheels as well as over the vertical supports.

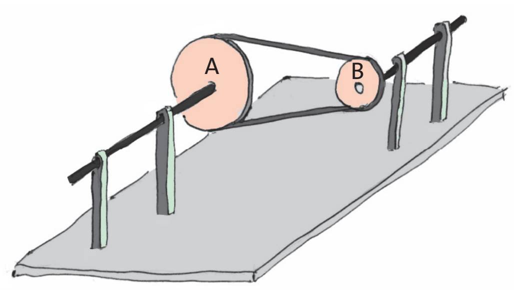


Fig. 5.22 Belt drive. Wheel B turns faster than wheel A.

Chain drive and belt drive

You know the chain drive from the bicycle. A belt drive, Fig. 5.22, is essentially the same, only that a flexible belt, often equipped with teeth, is used instead of the chain made of steel links.

Such drives are installed in many machines. You can find them, inter alia, under the engine hood of every car. Such drives have two functions:

- they transport energy (with angular momentum as an energy carrier) from one place A to another place B;
- if the two chain wheels (or belt discs) have different diameters, they will act as angular momentum current transformers.

The energy flows to wheel A with angular momentum as an energy carrier, and it flows away from wheel B with angular momentum.

The equation

$$\omega_A \cdot M_A = \omega_B \cdot M_B \,.$$

applies once again for the respective current strengths. In case of a bicycle with derailleur gears, the gear ratio can be changed, Fig. 5.23.

Exercises

1. Draw the path of the momentum for the tackle into Fig. 5.24. By which factor is the momentum current at point A smaller than at point B?
2. Draw the path of the momentum current for the tackle in Fig. 5.25. By which factor is the momentum current at point A smaller than at point B?
3. What is the gear ratio for the chain drive of your bicycle? In case the bicycle has a derailleur gear: what is the gear ratio for the different gears? In case it has a hub gear: please try to determine the gear ratios of the hub gear as accurately as possible.

Fig. 5.23 Derailleur gears of a bicycle

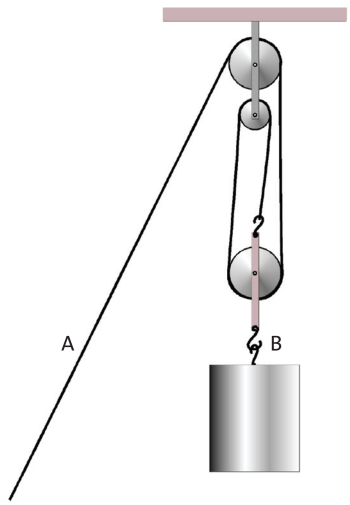


Fig. 5.24 For exercise 1

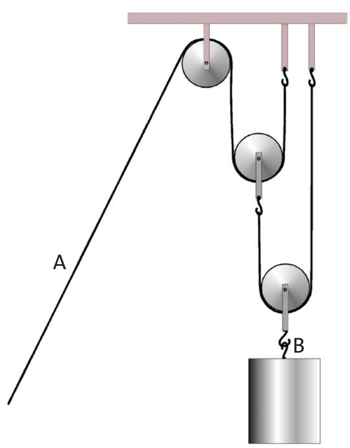


Fig. 5.25 For exercise 2

5.7 Friction

In a friction process, momentum flows from a place of higher velocity to a place of lower velocity. The two „places“ can also be two points within a liquid or a gas. We are interested in the case where they are located in two bodies. Let's call them A and B, Fig. 5.26.

The contact between the two bodies can differ strongly:

- they can be in direct contact. Example: you push a book over the table or a chair over the floor.
- There is a lubricant between A and B. Example: a rotating axis in a bearing. Without lubrication, we would have case 1. The lubricant is used to reduce friction.
- There is a gas, air in most cases, between A and B. Example: the friction of the air for the driving car or bicycle.

The energy balance seems to be incorrect in a friction process. Energy is lost. We call the velocity of one body v_A, the velocity of the other one v_B and the momentum current strength, as always, F.

Momentum is an energy carrier. In body A, on its path towards the area of friction, it carries the energy current

$$P_A = v_A \cdot F.$$

In body B, where it flows away from the area of friction, it carries the energy current

$$P_B = v_B \cdot F.$$

As $v_A > v_B$, more energy flows to the place of friction than away from it. What happens to the difference

$$P = P_A - P_B = (v_A - v_B) \cdot F\ ?$$

Entropy (that we perceive as heat) is created during each friction process. Also the entropy is an energy carrier. The entropy created during friction necessarily carries energy away. The respective energy current can be written as:

$$P = T \cdot I_S.$$

Here, T is the absolute temperature (measured in Kelvin) and I_S the entropy current strength.

We can therefore write for the friction process:

$$T \cdot I_S = (v_A - v_B) \cdot F = \Delta v \cdot F\ .$$

We have abbreviated $(v_A - v_B)$ as Δv.

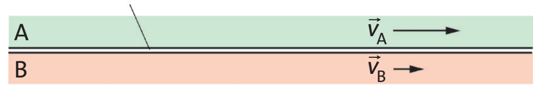


Fig. 5.26 During the friction process, *x*-momentum flows from body A to body B. (Positive *x*-direction to the right)

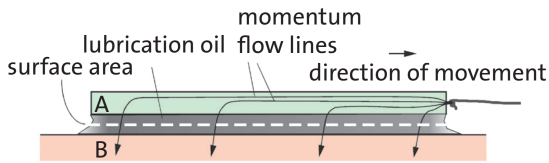


Fig. 5.27 There is an oil film between plate A and the support surface B. A is moved to the right. A current of *x*-momentum flows through the oil film from the top to the bottom.

> Entropy is created in a friction process. Together with this entropy energy disappears to the environment.

Friction is often an undesired phenomenon – because of the energy loss associated with it. It would be great to get rid of the atmospheric friction of cars or the friction in the bearings of a rotating shaft.

Certain technological devices, however, are based precisely on friction. There, it is indispensable. Examples are brakes, clutches and shock absorbers of cars.

Let's repeat: in a friction process, momentum flows from a body A to a body B. A and B have different velocities. Not the absolute values of the two velocities are important for the friction process, but only the velocity difference Δv. The strength F of the momentum current between A and B therefore depends on Δv.

Depending on how the bodies rub on each other, this relationship is varies. We have to look at these different Δv-F relationships to understand the different friction processes. We will plot F over Δv in a diagram. The respective graph is called characteristic curve of the friction process.

Characteristic curves can have a variety of forms. But we can identify three fundamental patterns.

Viscous friction

When there is a viscous medium, i.e. lubricating oil, between the bodies A and B, Fig. 5.27, the characteristic curve is particularly simple.

The momentum current is proportional to the velocity difference:

$F \sim \Delta v$, or
$F = k \cdot \Delta v$.

Fig. 5.28 shows the corresponding graph.
k depends on:

- the viscosity of the liquid. The more viscous it is, the greater is k and the greater the momentum current strength.
- The geometry of the arrangement. When A and B slide past each other as illustrated in Fig. 5.27, k is proportional to the area of the oil film and inversely proportional to the distance between A and B.

> Viscous friction:
> $F = k \cdot \Delta v$.

Maybe this appears familiar to you. There is a strong similarity to the electric current that flows through a resistor. Here, the electric current strength is proportional to the electric potential difference:

$I \sim \Delta\varphi$, or
$I = G \cdot \Delta\varphi$.

The proportionality factor is the electric conductance (the inverse of resistance) and depends 1. on the material (on the electric conductivity) and 2. on the geometric dimensions of the electric resistance, i.e. on the cross-sectional area and on the length.
Back to viscous friction: where does it occur?

Lubrication of machine parts

Machines have parts that touch and that rub against one another. To reduce the effect of momentum or angular momentum flowing away and to avoid energy losses, such parts are lubricated: a thin layer of lubrication oil is applied in between as illustrated in Fig. 5.27. Lubrication is required also for other reasons: material wear is reduced and there will be less noise. You have certainly heard the squeaking noise of door hinges at some point.

Shock absorbers

A car has a spring on each wheel so that the passengers are not jolted by every pothole, Fig. 5.29. There is a shock absorber next to (or in) the spring. Without a shock absorber, the car would perform long-lasting vibrations. In addition, the wheels would bounce on the road, making the car lose contact with the ground.

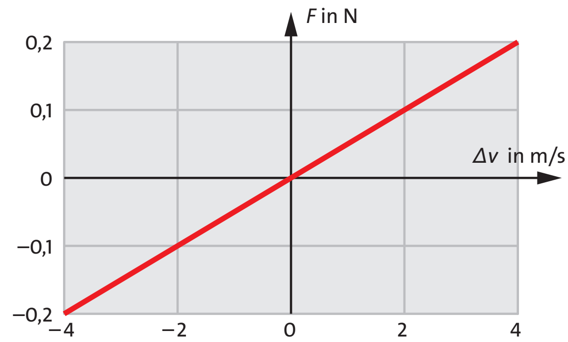


Fig. 5.28 Characteristic curve for viscous friction: the momentum current is proportional to the difference of the velocities of the involved bodies.

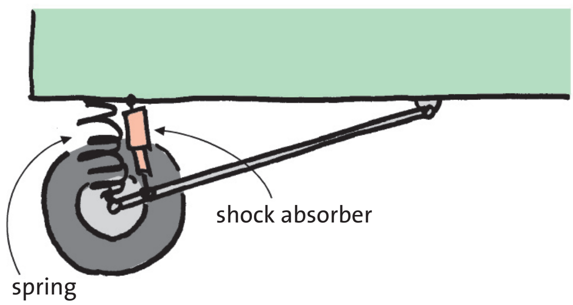


Fig. 5.29 Simplified wheel suspension with a spring and a shock absorber.

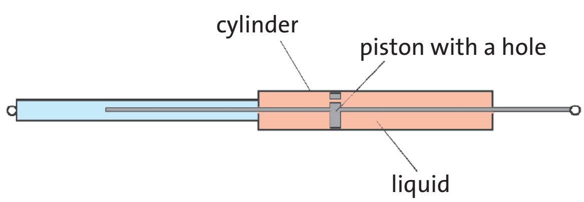


Fig. 5.30 Shock absorber. If the two connections are moved opposite to each other, the liquid will be pressed through the hole in the piston.

It could no longer be steered and slowed down properly. Hence, the „loss“ of energy is desired in case of the shock absorber.

The structure of a shock absorber is shown in Fig. 5.30. When the two „connections“ on the right and the left are moved against each other, the liquid has to flow through a small hole in the piston. Friction takes place at that point. The effect of a shock absorber can be understood best if we take it in our hand and pull or push on the connections. The faster they are moved, the harder it will be to move one end against the other.

The following also applies for shock absorbers:

$$F = k \cdot \Delta v \,.$$

Δv is the difference between the velocities of the two connections of the shock absorber.

Friction between two solid bodies

A wooden block slides over an even horizontal wood plate. This can be achieved particularly conveniently in the way done by Willy in Fig. 5.31: the block is held and it is ensured that the support surface moves away underneath. A momentum current flows from the rotating tabletop into the block in the process.

Willy modifies the velocity of the rotation. But the momentum current meter indicates always the same momentum current. This observation also holds true for other solid bodies that rub on each other:

> When two solid bodies rub against each other, the momentum current, which flows from one body to the other, is independent of the velocity difference.

The Δv-F characteristic curve is shown in Fig. 5.32. Here, the velocity difference was assumed to be positive.

But what happens in case of negative Δv values? Willy has to turn the table in the other direction. Before, however, the block needs to be suspended on the other side, i.e. on the right wall (because our momentum current meter is installed in a cord, which only lets the momentum current pass in one direction). The result of the experiment is not surprising: what happens is the same as before: the momentum current is independent of the velocity difference, but now it flows in the direction that is opposite to the previous direction. The Δv-F characteristic curve for positive and negative Δv values is shown in Fig. 5.33.

There is still something missing on the characteristic curve. What will be the value of F if the velocity difference is equal to zero? We pull on a block, which lies on a (solid) table, by means of a momentum current meter and observe that the momentum current can be increased from zero to a well-defined value without the block starting to move. Hence, it behaves as if it were glued to the tabletop. Only if the momentum current exceeds this limit value, the block will be detached and start moving. This limit momentum current is greater than the one that flows when the body is moving.

We can take this fact into account in our characteristic curve, Fig. 5.34. You certainly know the following

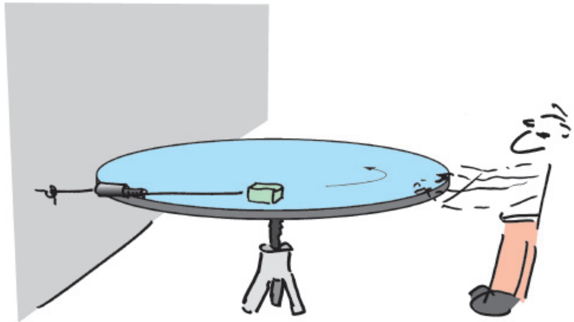


Fig. 5.31 Willy makes the table rotate. The momentum current between the block and the table is independent of the velocity difference.

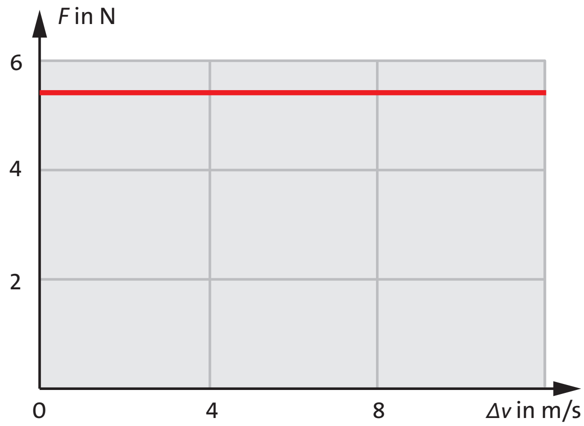


Fig. 5.32 Positive part of the Δv-F characteristic curve for the friction between two solid bodies

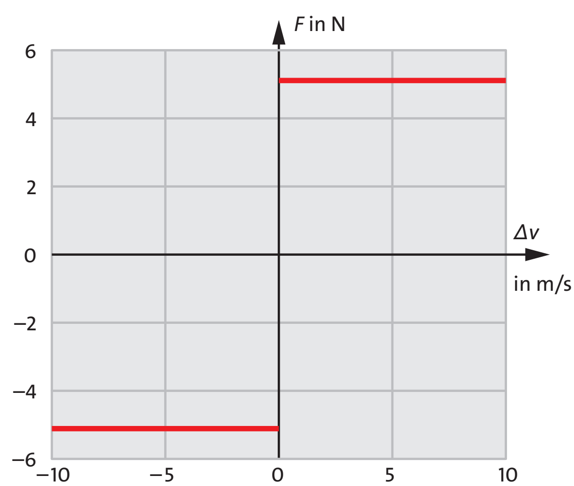


Fig. 5.33 Positive and negative part of the Δv-F characteristic curve for the friction between two solid bodies

phenomenon: if you wish to move a heavy piece of furniture, i.e. a closet, you will have to pull or push very strongly at first, i.e. until you reach the limit value of the momentum current. As soon as the closet moves, it will be easier.

After having obtained an overview of the entire characteristic curve, we would like to look once again at Willy and his rotating table and the positive part of the characteristic curve from Fig. 5.32. We call the momentum current, which flows from the table into the block and from the block through the cord to the left, F_F (F for friction), Fig. 5.35. It is a pure x-momentum. We saw earlier that this x-momentum current is independent of the velocity difference.

But it depends on the weight of the block, and this means: on another momentum current F_T (T for transverse), which comes from the Earth, flows through the gravitational field into the block, subsequently into the table and from there back into the Earth.

The transverse momentum current F_T is a pure z-momentum current.

The relationship between F_F and F_T is simple:

$$F_F = \mu \cdot F_T \,.$$

> The stronger the transverse momentum current, the stronger the friction.

The factor of proportionality μ depends on the nature of the two surfaces. Fig. 5.36 shows the characteristic curve for a friction process for 3 different values of the transverse momentum current.

The last equation tells us that a momentum current can be controlled by means of another one: if we change F_T, F_F will also change. This property is used in technical applications.

Brakes

Fig. 5.37 shows a schematic top view of the disc brake of a car. (It could also be the rim brake of a bicycle.) The disc rotates between two blocks. If someone steps on the brake pedal, the blocks will be pressed more or less strongly against the brake disc, i.e. a z-momentum current is created through the brake disk. This momentum current leads to a more or less strong x-momentum current out of the brake disc.

Now we can understand something for which car drivers have a feel: the momentum decrease of the car during braking does not depend on how fast the car is driving but only on how strongly we step on the brake pedal. In other words: the brake works just as well (or poorly) at a high velocity as at a low velocity. This

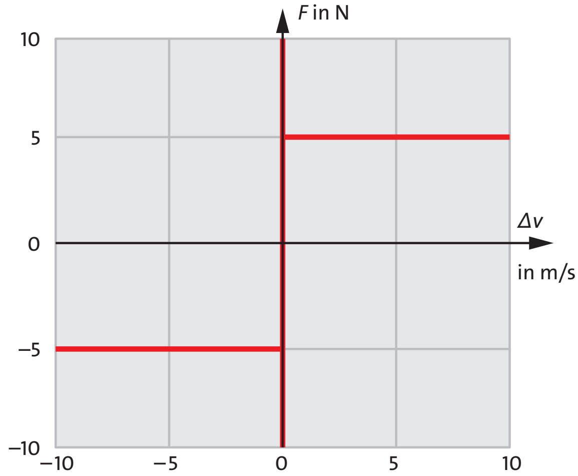


Fig. 5.34 For $\Delta v = 0$, the characteristic curve has a discontinuity.

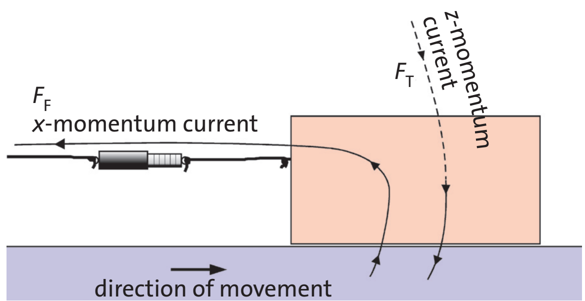


Fig. 5.35 The x-momentum current through the cord is proportional to the z-momentum current that arrives via the gravitational field.

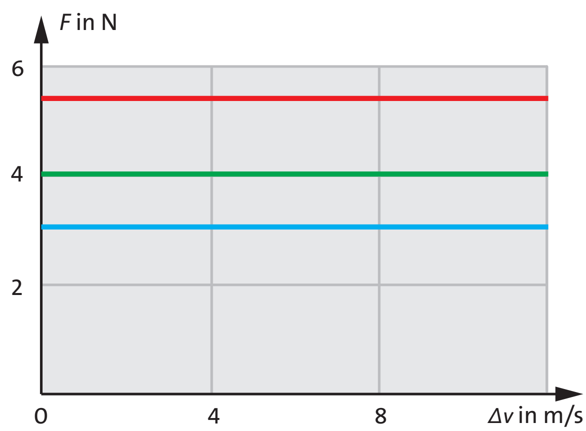


Fig. 5.36 Δv-F characteristic curves for three different values of the transverse momentum current

should not be taken for granted. There are brakes whose effect depends on the velocity. For example the effect of eddy current brakes increases with a higher value of Δv. Such brakes can be found in cablecars, in the ICE3 (high speed train), in rollercoasters or in the free fall tower.

The clutch

We have already seen it before. It is used to interrupt and to restore a connection for an angular momentum current; see Fig. 3.18 and Fig. 3.19. When we hold down the clutch pedal (the left pedal) in the car with our foot, the connection between the engine and the gear box is interrupted. To engage the clutch, we slowly release the pedal. In this process, the two clutch discs are increasingly pressed together and an angular momentum current is flowing. This current does not depend on the difference of the discs' angular velocities but only on how far the clutch pedal has been released. In case of a half-engaged clutch, the angular momentum current is therefore independent of how fast the engine runs. This can be detected easily: to start driving, it is not important how strongly we step on the accelerator pedal. Only if the clutch is fully engaged, i.e. if the clutch discs do no longer rub against each other, the position of the accelerator pedal will have an effect on the car's momentum change.

Turbulent friction

Turbulent friction is a third type of friction. When a body moves through a medium with a very low viscosity, e.g. through the air, this medium is set in turbulent motion. It receives momentum, which it carries away convectively.

Viscosity is no longer important for this momentum current. But the momentum current depends on the inertia, i.e. the mass density, of the medium.

In addition, it does not grow proportionally to the velocity difference but it is proportional to the square of Δv, Fig. 5.38.

Hence:

$$F \sim \Delta v^2 .$$

The momentum loss of a car at higher velocities is of this type. We can learn from this example how to drive if we would like to save fuel and therefore energy. We could think at first: I rather drive at 120 km/h instead of 60 km/h. Although the engine needs more gas per second, we can make the trip in half the time. If the momentum loss were proportional to Δv, the two effects would just offset each other. But this is not the case. As the momentum loss increases with the square of the velocity, the gas consumption per second at 120 km/h is more than twice as high as at 60 km/h.

This argumentation is not valid for low velocities as, in that case, the turbulent friction is not as significant in relation to the other friction types anymore. The car requires an unnecessarily high amount of energy at velocities of around 80 km/h and more.

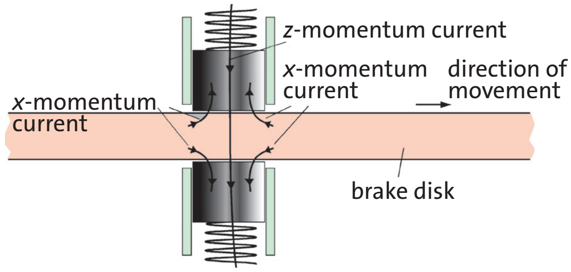


Fig. 5.37 Disc brake or rim brake. For braking, a z-momentum current is sent through the brake disc from the top to the bottom. This causes an x-momentum current flowing out of the disc.

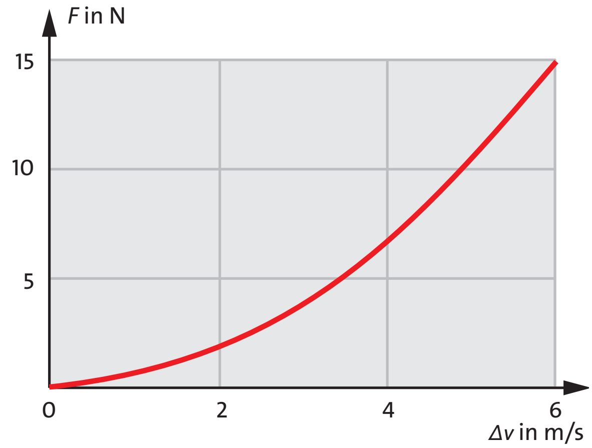


Fig. 5.38 Δv-F characteristic curve for turbulent friction

Exercises

1. With a brake pedal held constantly, a car is slowed down from 80 km/h until it stops. Indicate in one single graph as a function of time: the momentum of the car, the momentum current out of the car, the kinetic energy of the car, the energy current out of the car. The mass of the car is 1.2 t, the braking momentum current is 3600 N.
2. There are brakes for which the momentum current is not constant but proportional to the velocity of the vehicle to be slowed down. Which problem will arise? How can it be solved?

6 REFERENCE FRAMES

6.1 Reference frame and zero of a physical quantity

The man who floats down the Rhine in a boat says that the landscape is passing by, Fig. 6.1, while we on the shore say that the boat with the man is passing by.

This is an example of the world being described or contemplated in different reference frames.

We would like to describe the movement of a body K: how fast does it move? In which direction does it move? We cannot answer these questions without having clarified what the movement refers to.

You walk forward through the corridor on a moving train. In relation to the carriage, you move at 3 km/h; in relation to the Earth, you move perhaps at 203 km/h.

Hence, when talking about a movement, it always needs to be clear which other body the movement refers to. This other body is called reference body. But most characteristics and particularities of the reference body are not relevant. We can therefore think of it as replaced by a coordinate system that we attach to it. This coordinate system is called *reference frame.*

The reference body is at rest in its own reference frame. When using the Earth as a reference body, we call the corresponding reference frame „reference frame of the Earth"; when using a train as a reference body, we talk about the „reference frame of the train".

The man in the boat from Fig. 6.1 has chosen the boat as a reference body or a coordinate system, which is attached to the boat, as a reference frame. The boat itself is at rest in this reference frame, its velocity is 0 km/h while the shore (and consequently the Earth) moves at –20 km/h. We can also say that the shore moves *with respect to the boat* at –20 km/h. Someone standing on the shore would choose the Earth as a reference body. For him, the Earth has the velocity 0 km/h while the boat moves with 20 km/h.

Fig. 6.1 The man in the boat chooses the boat as a reference body. In his reference frame, the shore is moving. In the reference frame of the man on the shore, the boat is moving.

Later, we will be interested in reference frames in which there is weightlessness. We have seen that this holds true for freely falling bodies. We look at an apple that is falling from the tree, Fig. 6.2.

At first, we use the reference frame of the Earth. The Earth itself is not accelerated in this case. Hence,

$$a_{\text{Earth}} = 0 \text{ m/s}^2 \,, \; a_{\text{apple}} = 9.81 \text{ m/s}^2.$$

If the apple is the reference frame, we will obtain

$$a_{\text{Earth}} = -9.81 \text{ m/s}^2 \,, \; a_{\text{apple}} = 0 \text{ m/s}^2.$$

The specification of the reference frame can also be more complicated, e.g.: the reference frame is „attached“ to the center of mass of the solar system, the *x*-axis points towards the polar star, the *y*-axis towards ... etc.

> In order to describe a movement, a reference frame has to be chosen, i.e. a coordinate system that we imagine to be attached to a reference body.

The fact that a reference frame has to be chosen can also be seen as follows: we need to define or to choose the zeros of position, velocity and acceleration.

Maybe you recall that the zero also has to be specified for other physical quantities. Hence, indicating an electric potential value or also a temperature does not make sense unless we know where the zero is on the respective scale.

Changing the reference frame means that the zero of position, velocity and acceleration is chosen anew.

> *Change of the reference frame:*
> shift of the zero of position, velocity or acceleration

The following part deals with the question of how the description of the world will be modified if the reference frame is changed, i.e. if it is described in another reference frame S' instead of the original reference frame S.

A change of the zeros of velocity and acceleration is particularly interesting. We will not be interested to the zero of position.

6.2 Phenomena in different reference frames

When we change the reference frame, the values of velocity or acceleration will change. But not exclusively. Many other quantities also change their values, and there are phenomena that have to be interpreted in a completely different way after the reference frame has been changed. After a change of the reference frame, the world looks quite different.

In the following, we will at first assume one of the reference frames, i.e. S, to be the Earth while another reference frame S' moves against S at a constant velocity v_0.

In section 6.3, S' will move against to S with a constant acceleration.

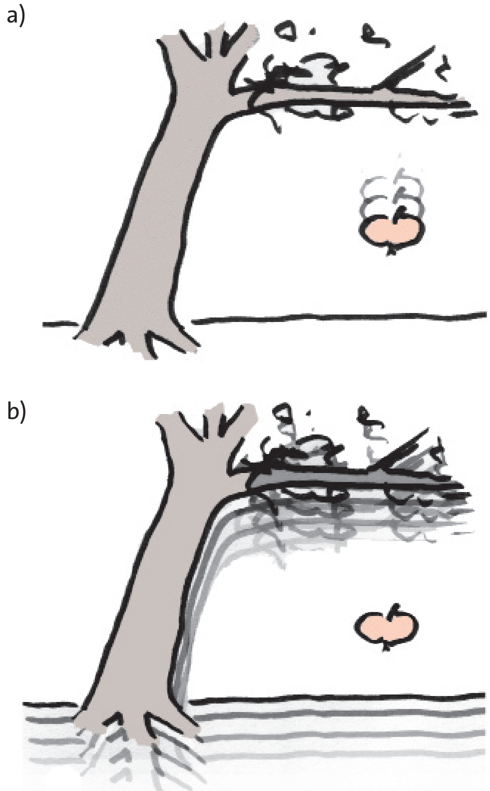


Fig. 6.2 (a) From the perspective of the Earth, the accelerated apple moves downwards. (b) In the reference frame of the apple, the accelerated Earth and the tree move upwards.

Velocity, momentum and kinetic energy

We look at linear, horizontal movements. The *x*-direction is the direction of movement. Hence, we only need to consider the *x*-component of the velocity vector, which we denominate with v.

Let's assume a car moves at a velocity v with respect to the Earth (reference frame S). In a reference frame S', which moves against the Earth at the velocity v_0, the velocity of the car is

$$v' = v - v_0 \,.$$

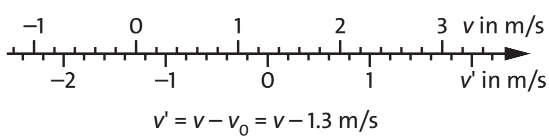


Fig. 6.3 In the reference frame S, the car has the velocity v, in S', it has the velocity $v - v_0$.

We can also say that the zero of the velocity has been shifted by v_0, Fig. 6.3.

We now ask for the values of other quantities; at first of momentum. In S, we have:

$$p = m \cdot v,$$

and in S':

$$p' = m \cdot v' = m \cdot (v - v_0).$$

And the kinetic energy. In S, it is:

$$E_{kin} = \frac{m}{2} v^2,$$

and in S':

$$E_{kin} = \frac{m}{2} v'^2 = \frac{m}{2} (v - v_0)^2.$$

We see:

> The values of momentum and kinetic energy depend on the reference frame.

On might worry that the laws of nature known to us would no longer be valid in S'. We examine the consequences of a change of the reference frame for the law of momentum conservation principle and the law of energy conservation.

Two bodies A and B (each with a mass of 2 kg) move towards each other without friction and collide, Fig. 6.4. A is equipped with a spring bumper. The velocities at the beginning are 3 m/s and –3 m/s, respectively.

Table 6.1 lists velocity, momentum and kinetic energy for both bodies before and after the collision. Please check!

The total momentum prior to the collision is 0 Hy – just as after the collision. For the total kinetic energy before and after the collision we obtain 18 J.

We now describe the same process in a reference frame that moves to the right at v_0 = 3 m/s, Fig. 6.5.

In this reference frame, body A has the velocity zero before the collision and body B after the collision, Table 6.2.

The values of all quantities are now different from those in our original reference frame; but also in the new one, the total energy and the total momentum do not change during the collision process. The result confirms a generally rule:

> S' moves against S with v_0:
> The change of the reference frame does not affect the laws of energy and momentum conservation.

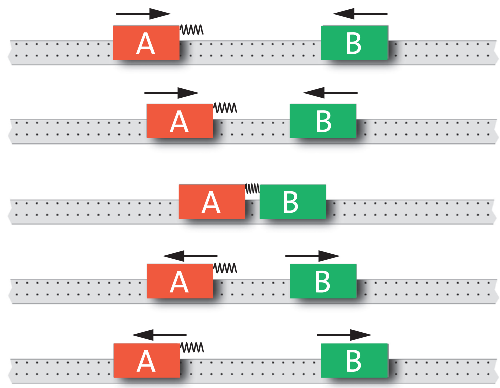


Fig. 6.4 Momentum and kinetic energy are not changed during collision.

	A	B	total
before			
v	3 m/s	−3 m/s	
p	6 Hy	−6 Hy	0 Hy
E_{kin}	9 J	9 J	18 J
after			
v	−3 m/s	3 m/s	
p	−6 Hy	6 Hy	0 Hy
E_{kin}	9 J	9 J	18 J

Table 6.1 The values of velocity, momentum and kinetic energy in the reference frame S (Earth)

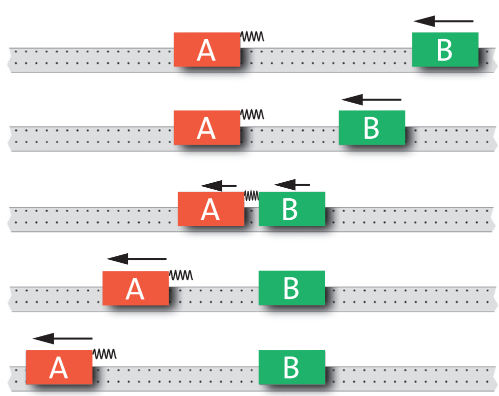


Fig. 6.5 The same process as in Fig. 6.3 but in another reference frame. Momentum and energy conservation laws still apply

Water current strength

A water current of 5 liters per minute is flowing in a pipe. We assume the water to have the same velocity everywhere in the pipe.

Now we use the water as a reference body. While the water does not move in the corresponding reference frame, the pipe does. The water current strength is now zero liters per minute.

Hence, the value of the water current strength depends on the reference frame.

Electric current strength

We begin with a somewhat peculiar electric current, i.e. we look at the electron beam in an old television tube. The electrons of the electron beam move at a velocity of approximately 10^8 m/s – relative to the Earth. As the electrons are charged, an electric current belongs to the electron beam, and this electric current creates a magnetic field around itself, Fig. 6.6.

Now we change the reference frame. We describe the situation in a reference frame that moves along with the electrons. In other words: in our new reference frame S', the electrons do not move. Hence, there is no electric current and no magnetic field either. We conclude that the electric current strength in a current of charged particles depends on the reference frame. In addition, we learn something much more interesting: magnetic fields depend on reference frames.

Later you will learn that magnetic fields are described quantitatively by the *magnetic field strength.*

> S' moves against S at v_0:
> The magnetic field strength depends on the reference frame.

	A	B	total
before			
v'	0 m/s	−6 m/s	
p'	0 Hy	−12 Hy	−12 Hy
E'_{kin}	0 J	36 J	36 J
after			
v'	−6 m/s	0 m/s	
p'	−12 Hy	0 Hy	−12 Hy
E'_{kin}	36 J	0 J	36 J

Table 6.2 The values of velocity, momentum and kinetic energy in the reference frame S'

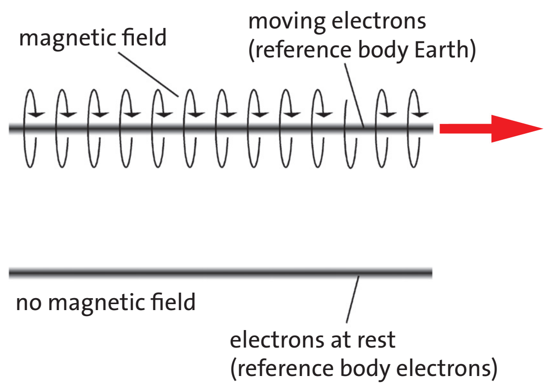


Fig. 6.6 The magnetic field strength is zero in the reference frame in which the electrons are at rest.

Now we take another look at a normal electric current in a copper wire. Here, the charge carriers move much, much more slowly. We could think that it is much easier in here to eliminate the magnetic field by changing the reference frame. But this is not the case.

Each atom of the copper wire has an electron that is not firmly bound to the atomic nucleus. If an electric current flows through the wire, these mobile electrons will slide past the positive remainder.

In a reference frame in which the wire is at rest, the mobile electrons cause an electric current. If we now switch to the reference frame in which the mobile electrons are at rest, the corresponding electric current will become zero. But in this reference frame, the positive remainder moves. And it causes an electric current. For the current strength it is not relevant whether positive charge carriers move in one direction or negative ones in the opposite direction. Hence, no matter how the reference frame is chosen: the electric current strength and the magnetic field will remain unchanged.

> S' moves against S at v_0:
> Upon the change of the reference frame, the electric current strength in a beam of charged particles changes. It does not change in a neutral conductor.

The bicycle chain

We describe the energy transport through the bicycle chain from the front chain wheel (next to the pedals) to the one at the rear (on the rear wheel). As a momentum current only flows in the upper part of the chain, this current is the only one we need to look at. More precisely, we assume:

Velocity of the bicycle against the Earth:

$$v_{\text{bicycle}} = 18 \text{ km/h} = 5 \text{ m/s}$$

Velocity of the upper part of the chain against the bicycle:

$v_{chain} = 0.8$ m/s

Momentum current in the upper part of the chain:

$F = 80$ N.

We start with the description in the reference system S of the bicycle, Fig. 6.7. This means that the bicycle is at rest, the Earth moves at 5 m/s to the left and the chain at 0.8 m/to the right.

Therefore, we obtain for the energy current through the chain:

$$P = v_{chain} \cdot F = 0.8 \text{ m/s} \cdot 80 \text{ N} = 64 \text{ W}$$

Now we switch to the reference frame S' of the Earth, Fig. 6.8. With respect to the bicycle, the Earth moves at

$$v_0 = -v_{bicycle} = -5 \text{ m/s} .$$

We therefore obtain for the energy current in the chain:

$$P = (v_{chain} - v_0) \cdot F = (v_{chain} + v_{bicycle}) \cdot F$$
$$= (0.8 \text{ m/s} + 5 \text{ m/s}) \cdot 80 \text{ N} = 464 \text{ W}$$

But now we have a problem: the energy current is too great. A pedaling person cannot create such an energy current with his or her muscles. Something must be incorrect. In fact, we have forgotten something. The momentum current that flows through the upper part of the bicycle chain (tensional stress, i.e. from the right to the left) obviously has to flow back. And this is what it does – i.e. through the bicycle frame (essentially through the horizontal link between the rear and the front chain wheel). As the bicycle moves in the reference frame S', an energy current is connected to this momentum current, i.e. from the rear to the front. It is not difficult to calculate this energy current:

$$P = v_0 \cdot (-F) = -5 \text{ m/s} \cdot 80 \text{ N} = -400 \text{ W}.$$

(We have counted the momentum current from the left to the right as negative.) The energy current flows in the bicycle frame from the rear to the front. The total energy current of the chain and the frame is therefore

$$P = 464 \text{ W} - 400 \text{ W} = 64 \text{ W}.$$

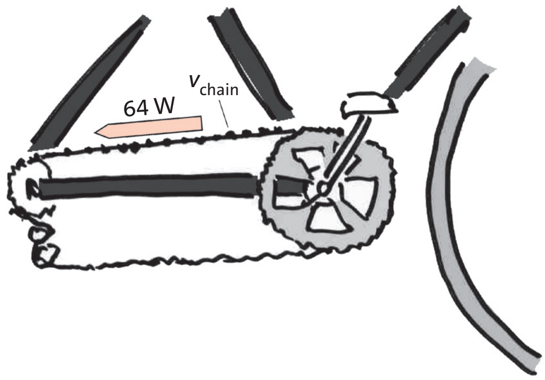


Fig. 6.7 The bicycle as a reference body: an energy current of 64 W (red arrow) flows through the chain from the right to the left.

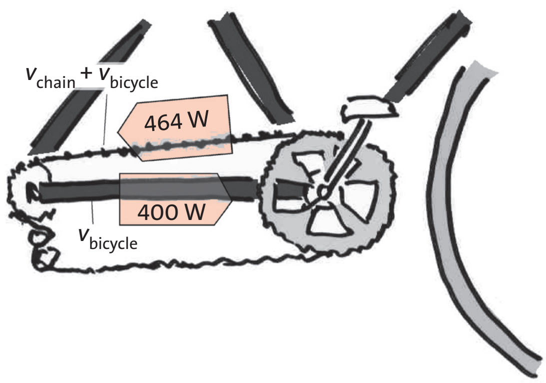


Fig. 6.8 The Earth as a reference body: an energy current of 464 W flows through the chain to the left. 400 W flow through the bicycle frame back to the right.

In any case, the pedaling person releases 64 W from his or her muscles, and this amount arrives at the rear. The path the energy current takes, however, depends on the chosen reference frame.

> S' moves against S at v_0:
> Mechanical energy currents depend on the reference frame.

We could also examine other physical quantities: will their values change if we change the reference frames? In doing so, we would find at first: while some quantities depend on the reference frame, others seem to be independent of it. Therefore, we would at first find that the values of the following quantities will not change if we change the reference frames:

length, duration, mass, pressure, electric charge, temperature, entropy…

However, this list actually contains a few quantities that should not be included. If we choose a reference frames with a very high v_0, we will find that most quantities are no longer included in the list.

Hence, the values of lengths, time intervals and masses change in case of a change of the reference frame. Only very few will be left, for example electric charge and entropy.

In the next chapter, we will examine phenomena that occur in case of high velocities. They are the objects of the *theory of relativity.*

We summarize:

It looks as if a change of the reference frame would change the world. But this is not true. The world remains the way it is. Only our description, our point of view, changes and makes the world look different to us.

> The world is not changed by a change of the reference frame. Only our description of the world changes.

Exercises

1. Two bodies A and B (each with a mass of 2 kg) collide in a similar way as illustrated in Fig. 6.4 – however, they do not have a spring bumper but a plastic bumper so that they are attached to one another after the collision. The initial velocities are as in Fig. 6.4, i.e. v_A = 3 m/s and v_B = –3 m/s. What is the momentum of body A, the momentum of body B and the overall momentum before the collision and after the collision? What is the kinetic energy of body A, the kinetic energy of body B and the overall kinetic energy before the collision and after the collision? Describe the process in a reference frame that moves against the one previously used at v_0 = 3 m/s.
2. A locomotive pulls 4 wagons at a constant velocity on a level path. Sketch the path of momentum and energy (a) in the reference frame of the Earth and (b) in the reference frame of the train.

6.3 Free float frames

Willy is in an elevator that is not suspended on a rope but that falls freely downwards, Fig. 6.9. (The elevator will land smoothly at the end.) It moves in an accelerated way: its velocity increases linearly over time.

Willy finds that he is weightless. He also experiments with different objects. Everything he releases simply keeps floating in front of him and does not fall to the ground (of the elevator). He concludes: the gravitational field strength is zero.

Lilly observes from outside and comes to a different conclusion. The gravitational field strength is not zero at all. The objects that Willy releases fall to the ground in an accelerated motion. Willy does not perceive anything of it because he is also falling himself. Afterwards, Willy and Lilly discuss their observations and finally agree on the following:

> S' moves against S with a constant acceleration:
> The gravitational field strength depends on the reference frame.

Willy takes another ride or flight in his falling elevator. In the meantime, Lilly holds a spring dynamome-

Fig. 6.9 Willy: „The ball floats in front of me. The gravitational field strength must be zero." Lilly: „The gravitational field strength is not zero; Willy and the ball fall equally fast."

Fig. 6.10 Willy: „The field strength is zero. The spring is stretched because momentum is flowing into the block so that it becomes faster." Lilly: „The block does not become faster. A momentum current is flowing into the block via the gravitational field, and this momentum flows back out through the spring."

6 REFERENCE FRAMES

ter on which a heavy block is suspended, Fig. 6.10. The spring is stretched, it is subjected to tensional stress. For Lilly it is clear that a momentum current flows into the block via the gravitational field, and this momentum current has to flow away through the spring. This is usually expressed as follows: the block is heavy. Its mass is a measure for „heaviness".

Things are different from Willy's perspective: the field strength of the gravitational field is zero. He explains the fact that Lilly's spring is stretched as follows: the block becomes increasingly faster, its momentum increases continuously. It receives momentum through Lilly's spring (and Lilly gets it from the ground). This can also be expressed as follows: the block opposes to the acceleration because it is inert. Its mass is a measure for this inertia.

We found earlier (section 4.2) that a body has two characteristics due to its mass: it is heavy and it is inert. Also, we have already seen that the two characteristics are linked to each other. If body A is twice as heavy as body B, A will also be twice as inert as body B. Now we see that this is no coincidence.

> Depending on the reference frame, mass manifests itself differently: once as heaviness and once as inertia.

The physical laws apply in both reference frames. But the numerical values of the physical quantities are different, for example the field strength is zero in one frame and different from zero in the other one. Also, the physical interpretation of the experiments is different: once, the spring is stretched due to the weight of the body that is suspended on it, and another time due to its inertia.

When describing a phenomenon in physical terms, we are always free to choose a reference frame. According to our reflections up to now, it looks as if our choice was completely irrelevant as physical principles apply in any coordinate system. But there is another argument: possibly, the description in one reference frame is simpler than in another one – this is precisely the case.

In our example, we might believe that Lilly's reference frame is the simplest one. She stands on the ground just as we do, and we know how the world looks like when described from this perspective. But if we look more closely on the matter we may come to a different conclusion: Willy's reference frame is the more convenient one, because the world could not be simpler than what is described by Willy: if a body is put somewhere, it will remain there; it will not start moving and become increasingly faster in the way in which we terrestrial human beings constantly perceive it. For Willy, things either remain where they are or, when set in motion – i.e. when given momentum – move straight ahead at a constant velocity. It could not be simpler.

Reference frames in which bodies, which are left up to themselves, move at a constant velocity or do not move at all, are called *free float frames*.

> Free float frame:
>
> A body that is left up to itself does not move or moves at a constant velocity.
>
> Physics is particularly simple in a free float frame.

Earlier, we referred to a movement as performed by Willy with his elevator cabin as a „free fall". Such a movement normally ends after a short time.

But in section 4.6 we learned that the state of the free fall can also be maintained permanently. Any satellite that circulates around the Earth or also the space station ISS is constantly in a state of freely falling.

Now we will look at a spaceship that is floating in space, far away from all planets and stars. Willy and Lilly feel weightless. Hence, also this spaceship corresponds to a free float frame, Fig. 6.11.

Willy and Lilly are now longing for the Earth – especially because this is where they have the wonderful feeling of heaviness. As the Earth is too far away to just stop by for a short while, they create this feeling in a different way: they turn on the jet engines, Fig. 6.12.

From an outside perspective (i.e. from a free float frame) we would say: the momentum of the spaceship increases, and so does the momentum of Willy and Lilly. Hence, a momentum current is flowing into both

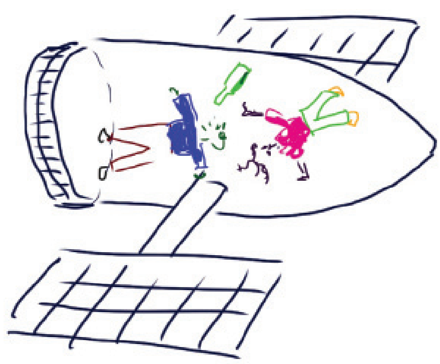


Fig. 6.11 The spaceship floats without drive in space. Willy, Lilly and everything else in the spaceship are floating. The reference frame of the spaceship is a free float frame.

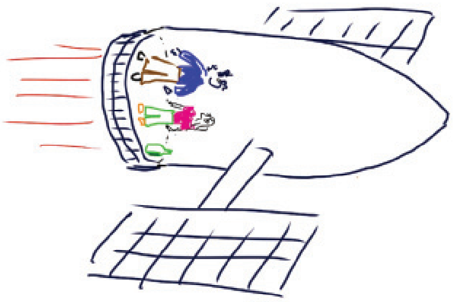


Fig. 6.12 The drive is switched on. Willy, Lilly and the bottle „are standing on the floor". The reference frame of the spaceship is not a free float frame.

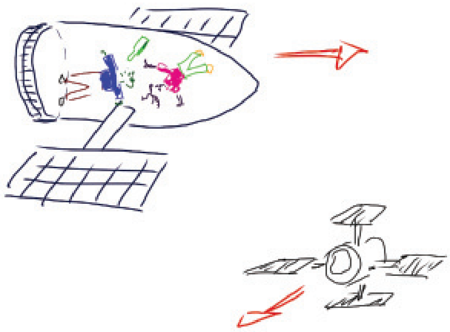


Fig. 6.13 The spaceship of Willy and Lilly is driveless, its reference frame is a free float frame. But also the reference frame of the other spaceship, which is also driveless and which moves at another velocity, is floating.

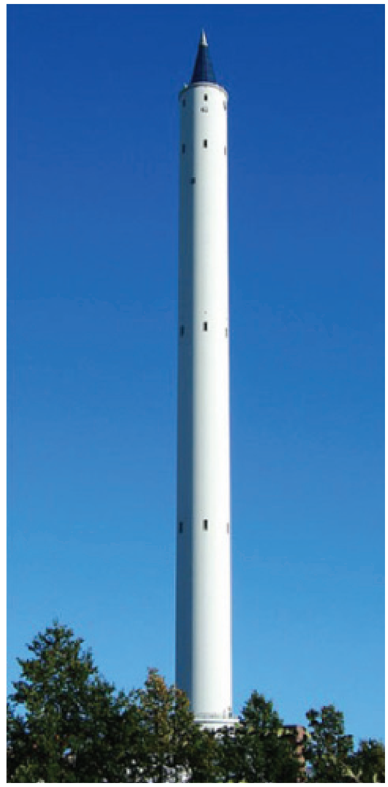


Fig. 6.14 Drop tower of the University of Bremen

of them. Willy and Lilly, however, see this differently, because they describe the process in the reference frame of the spaceship that is not a free float frame any longer. They notice that they have become heavy. They „stand" on the „ground", and every object they drop will „fall" „downwards".

Once again back to the driveless spaceship. Willy and Lilly look through the window and see that another spaceship is moving nearby, notably also without any drive. However, it flies in another direction, Fig. 6.13.

The passengers of the other spaceship are weightless as well, meaning that the other spaceship also defines a free float frame. Hence, there is not only one free float frame at a specific location. How many are there? Two? No. Every spaceship that moves against that of Willy and Lilly at a constant velocity defines a free float frame. Thus, there is an infinite number of free float frames.

Every reference frame that moves against a free float frame at a constant velocity is a free float frame, too.

Exercises

1. A „drop tower" was built at the University of Bremen for physical experiments, Fig. 6.14. Inside of it, there is a tube with a length of more than 100 m. This tube can be evacuated for the experiments. Willy climbs in an experimenting capsule and is dropped at the top of the drop tower*. A thick layer of Styrofoam balls at the bottom of the tower ensures a smooth landing. The falling process takes almost 5 seconds. Lilly observes the experiment. Which maximum velocity did she measure for Willy? How does Willy describe his movement? How can the duration of the „falling process" be doubled?
   * The drop tower actually exists. The story of Willy and Lilly, however, is entirely fictitious.
2. Lilly now enters a capsule herself and lets herself be catapulted upwards in the drop tower with a velocity of v = 50 m/s. At the same time, Willy is released with his capsule in the tip of the tower. Willy and Lilly observe each other during the experiment that takes 5 s. How does each of them describe the movement of the other one? You can make all statements without calculating.
3. A skydiver floats towards his landing site. His velocity is constant. Why does this floating have nothing to do with the floating of Willy and Lilly in the drop tower?

7 LIMITING VELOCITY

7.1 Mass is identical with energy

In physics, sometimes it was discovered that two apparently different things actually are not different. The most spectacular case of this type was the discovery by Albert Einstein (1879-1955), Fig. 7.1, that the two well-known quantities mass and energy are actually the same physical quantity. Mass or weight had already been known in ancient times, just as precursors of the energy can be traced back to the time of Aristotle. The fact that we are talking about the same quantity is stated in Einstein's famous publication from 1905: „The mass of a body is a measure of the body's energy content."

As the mass is measured in kilogram and the energy in Joule, we cannot simply write: mass = energy. We also need a conversion factor k:

$$E = k \cdot m\ ,$$

where $k = 8.987 \cdot 10^{16}$ J/kg.

Hence, this equation can be used to convert a specification in kg to a specification in Joule, just as a specification of length can be converted from kilometers to miles. Just as an indication in meters and an indication in miles relate to the same physical quantity, also an indication in Joule and another one in kilogram refer to the same physical quantity.

Actually, we would only need a single symbol and a single name for the quantity mass/energy. According to the old habit, however, it is called mass when stated in kilogram and energy when the measurement unit Joule is used.

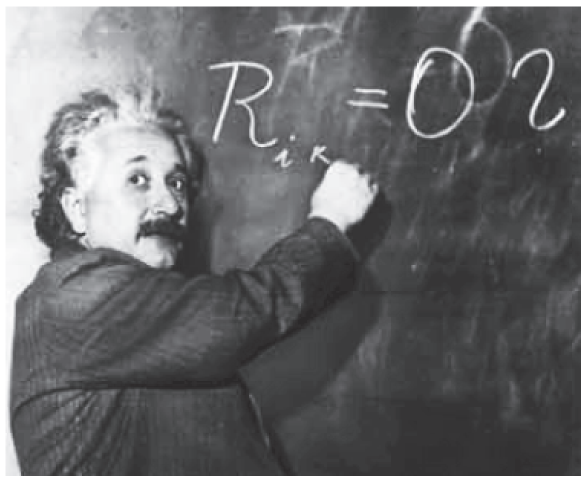


Fig. 7.1 Albert Einstein

If we use the conversion factor k in the following, it will mostly be sufficient to use the approximate value

$$k = 9 \cdot 10^{16} \text{ J/kg}.$$

> Mass and energy are the same physical quantity. It is called mass (m) when measured in kg and energy (E) when measured in J.
> $E = k \cdot m \qquad k = 8.987 \cdot 10^{16}$ J/kg

If this statement is true – and it is indeed true –, we can draw two conclusions:

1. Energy has to describe the properties that we have only known in relation to mass up to present: inertia and gravity. We look at a battery: it would have to

be heavier in the charged state (i.e. when containing more energy) than in the uncharged state. Or any other body that we heat up would have to be heavier in the warm than in the cold state.

2. Mass has to describe the properties that we have only known in relation to energy up to present: It must be possible to drive something with it, for example an electric generator. Hence, we could take any sort of material without any value such as sand. Just because it has a mass, the sand would have to be suitable to drive something.

Both statements seem not to make sense at first. We normally do not notice that they are true. Now we will see why.

7.2 Energy has the properties of mass

According to Einstein's discovery, energy has weight. The equation $E = k \cdot m$ tells us how many joules are equivalent to one kilogram.

According to this statement, for example the following would have to be true (Fig. 7.2):

- A full battery is heavier than an empty one.
- Warm water is heavier than cold water.
- Two separated magnets are heavier than two connected ones.
- A car becomes heavier with an increasing velocity.

You will understand the reason why we normally do not notice these phenomena if you check by how many kilograms the mass of the mentioned objects will change.

We look at a mono-cell as an example. In the process of discharging, it releases an amount of energy of approximately 10 kJ. What is the weight it loses in this process?

We calculate

$$m = \frac{E}{k} = \frac{10\ \text{kJ}}{9 \cdot 10^{16}\ \text{J/kg}} = 1.1 \cdot 10^{-13}\,\text{kg}.$$

The mass of the battery decreases by an amount that is smaller than the mass of a small dust particle. (A typical dust particle has a mass of approximately 10^{-12} kg.) This mass cannot be determined with a common scale.

Similarly small values are found for the mass difference between a slow and a fast car or between cold and warm water.

The discovery of energy having the properties of mass therefore seems to have no practical implica-

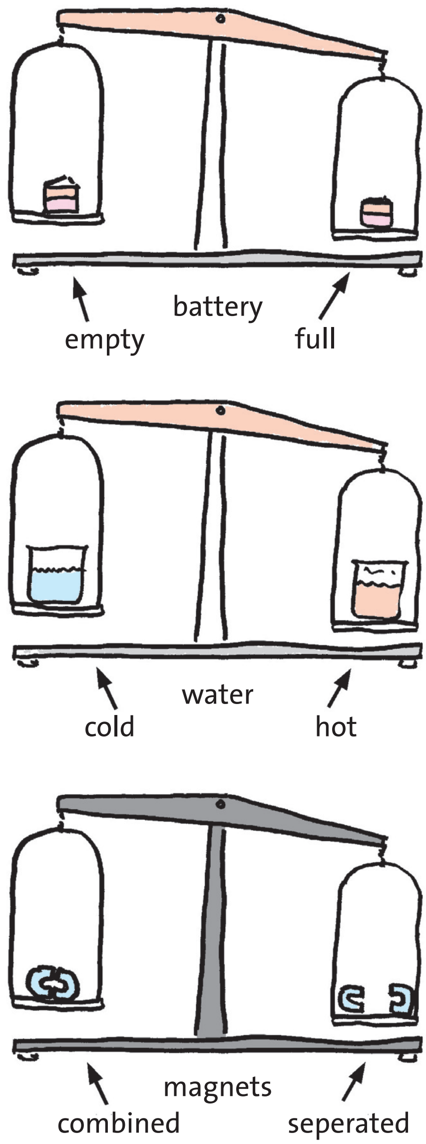


Fig. 7.2 A full battery is heavier than an empty one; warm water is heavier than cold water; separate magnets are heavier than connected ones.

tions. But is there any situation in which the mass change can be noticed? Otherwise, the statement could not be proven. In fact, such situations exist, for example:

- when charging particles such as electrons or protons in a „particle accelerator“ with a very great amount of momentum;
- when separating the protons of an atomic core, which are held together by very strong fields.

Exercises

1. The annual consumption of electric power by the city of Hamburg is approximately $5 \cdot 10^{16}$ J. What is the amount of this energy in mass units?
2. The Sun releases energy with the light. By how many kg will the mass of the Sun decrease per second? (Here is what you need for the calculation: approximately 1400 W arrive at the Earth per square meter with the light. The distance Earth-Sun is 150 million kilometers.
3. The sunlight that hits a square meter (perpendicular to the Sun rays) per second carries an energy of approximately 1400 Joule. What is the mass of the corresponding amount of light? How long would we have to wait until 1 gram of light has fallen onto the square meter?
4. Approximately 500 kJ of energy is needed to accelerate a car to 100 km/h. What is the mass increase of the car in the process? During acceleration, the car also loses mass because of its fuel consumption. Estimate whether the car becomes heavier or lighter.

7.3 Mass has the properties of energy

Provided that mass and energy are identical, it would be possible to use any substance – just because it has mass – for any useful things for which energy is needed, for example to drive vehicles and machines or to heat buildings. The substance would not have to be a specific fuel or propellant. The fact that it has a mass should be sufficient – and all substances have a mass.

Hence, we should be able to use for instance sand as a fuel. We would like to check how much sand is needed to drive a car.

The equation

$$E = k \cdot m$$

tells us that 1 kg of sand (or 1 kg of any other substance) contains an amount energy of

$$E = 9 \cdot 10^{16} \text{ J/kg} \cdot 1 \text{ kg} = 9 \cdot 10^{16} \text{ J} .$$

During the combustion in a common combustion engine, $4.3 \cdot 10^7$ J can be obtained out of 1 kg of gas. Since we have

$$2\,000\,000\,000 \cdot 4.3 \cdot 10^7 \text{ J} \approx 9 \cdot 10^{16} \text{ J} ,$$

an amount of energy that is two billion times higher corresponds to the kilogram of sand. But is that possible? Isn't it a fallacy?

In fact, the calculation is correct. Only the conclusion that sand can be used to drive a car is wrong. The fact that energy is not always useful to drive something is perfectly known.

Here is an example that will certainly seem logical to you: To heat a building, it is not sufficient to have enough fuel oil. In addition, oxygen is required for combustion of the fuel oil. If there was no oxygen, the fuel oil would be worthless. We would not be able to transfer the energy to another carrier – and this is what matters. Hence, we need a suitable reactant in addition to the fuel oil.

Something very similar holds true for the enormous amounts of energy that every substance contains due to its mass. To make use of this energy, i.e. to transfer it to another energy carrier, a suitable reactant is required.

The reactant that is needed here is the so-called *antimatter*. Antimatter is a form of matter that does almost not exist in nature.

Antimatter can be produced artificially but this process requires a high amount of energy: just as much energy as corresponds to the mass of the produced antimatter. Hence, nothing is gained.

In addition, it is practically impossible to store antimatter for longer than fractions of a second. It reacts very fast with common matter.

There have been speculations about whether parts of space that are very distant from us consist of antimatter. However, this has not been confirmed.

7.4 Rest mass and rest energy

The identity of mass and energy has implications for physics as a whole. It turns out that many well-known equations have to be replaced by new ones.

But can this be possible? Have the old equations not been tried and tested? Haven't they described the world correctly? If they were wrong, wouldn't we have had to become aware from the start? The same is true as for the phenomena that we discussed in the previous section. We can only see under extreme conditions that the equations are incorrect: if we measure very accurately or if the velocity of the bodies is extremely high. Hence, the old „classical“ equations are good approximations of the more correct „relativistic“ equations under normal circumstances.

Especially peculiar consequences arise for mechanics. They will be discussed in the following.

From „non-relativistic“ or „classical“ mechanics, we know the relationship between kinetic energy and velocity:

$$E_{\text{kin}}(v) = \frac{m}{2}v^2.$$

But you know that the kinetic energy is only a part of a body's total energy. The total energy would have to be written as follows:

$$E(v) = \frac{m}{2}v^2 + E_0. \qquad (7.1)$$

Here, E_0 is the energy that the body has when its velocity and its momentum are zero, i.e. its *rest energy*. But classical physics does not tell us the amount of E_0. Therefore, it will not tell us the total energy E of a system either. So far, you have certainly not been aware of this defect, though.

If we now take into account

$$E = k \cdot m$$

this defect will disappear. The energy for $v = 0$ is equal to the mass for $v = 0$, (multiplied by k) and we know this mass of course. The mass for $v = 0$ is called *rest mass* m_0 of the body.

Rest energy E_0:	energy at $v = 0$
Rest mass m_0:	mass at $v = 0$

$$E_0 = k \cdot m_0$$

Do not get confused by the word "rest energy". $v = 0$ means that the velocity of the center of mass is zero. However, a system whose center of mass is at rest can consist of moving particles. This is for example the case in the solar system. The Sun, planets and Moons rotate around their own axis and around each other. This means that the rest energy of the entire solar system is higher than the sum of the rest energies of its parts. For the solar system, however, this difference is so tiny that it is practically irrelevant. But still, the rest energy is sometimes also called *internal energy* for the sake of clarity.

7.5 How the velocity depends on the momentum

We would now like to examine the effects of the identity of mass and energy on the relationship between momentum and velocity:

$$p = m \cdot v.$$

What will happen can be seen better if we write the equation as:

$$v = \frac{p}{m}. \qquad (7.2)$$

According to our old classical concepts, the equation says: if a body is provided with momentum, its velocity will increase. If m is small, the velocity will increase strongly; if m is large, it will only increase slightly.

Now we charge a body with momentum, at best in portions: a momentum portion at a time. Also the energy of the body will increase with each momentum portion. But this leads to an increase of the mass in the denominator of equation (7.2). Hence, the body becomes increasingly inert. But the larger m becomes, the smaller will be the velocity increase with each momentum portion. After having supplied a very large quantity of momentum, the velocity will no longer increase. In other words: if momentum is supplied, the velocity approaches a *limiting velocity* „asymptotically“. This limit value is the same for all bodies, particles or other objects, i.e.

$$v_{\text{limiting}} = \sqrt{k}$$

Like k, the limit value is an universal natural constant. It is denominated with the symbol c. It is:

$$v_{\text{term}} = c = 3 \cdot 10^8 \text{ m/s}.$$

The mathematical relationship between v and p is:

$$v(p) = \frac{p}{\sqrt{m_0^2 + \left(\frac{p}{c}\right)^2}}. \qquad (7.3)$$

In Fig. 7.3, it is displayed for 3 bodies that differ from each other in their rest mass.

You can see that the velocity for very high values of the momentum approaches c in each of the three cases. This is also seen from equation (7.3). For very great values of p, the first addend under the root can be neglected with respect to the second one. We therefore obtain:

$$v(p) \approx \frac{p}{\sqrt{\left(\frac{p}{c}\right)^2}} = c.$$

For movements that we know from our everyday experience, the velocities are much lower than c, i.e. $v \ll c$. Even velocities that appear high to us, for example the velocity of a high-speed train or the velocity of the Earth on is orbit around the Sun, are still tiny compared to c. The Earth orbits the Sun at 110 000 km/h ≈ 30 000 m/s. This is only a ten thousandth part of the terminal value. In Fig. 7.3, the respective point cannot even be distinguished from the zero of the graph. If we apply equation (7.3) to such movements, we can neglect the second addend under the root with respect to the first one, and the v-p relationship becomes:

$$v(p) \approx \frac{p}{m_0}.$$

Hence, for velocities $v \ll c$ the relativistic relationship turns into the classical relationship.

We can also see this in the graph of Figure 7.3: in the vicinity of the zero, the curves look like straight lines with different slopes. Fig. 7.4 shows a section that is enlarged 100 000 times. (Bear in mind that the axes are calibrated differently.)

$$v(p) = \frac{p}{\sqrt{m_0^2 + \left(\frac{p}{c}\right)^2}}.$$

great values of momentum:
velocity is independent of momentum
$v \approx c$;
small values of momentum:
velocity is proportional to momentum

$$v(p) \approx \frac{p}{m_0}.$$

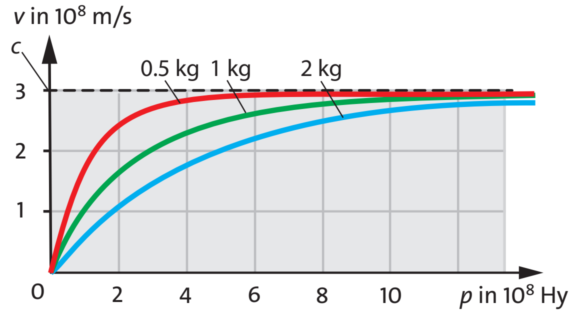


Fig. 7.3 Dependence of the velocity on the momentum for bodies with a rest mass of 0.5 kg, 1 kg and 2 kg. For high momentum values, the velocity of all bodies approaches the limiting velocity c.

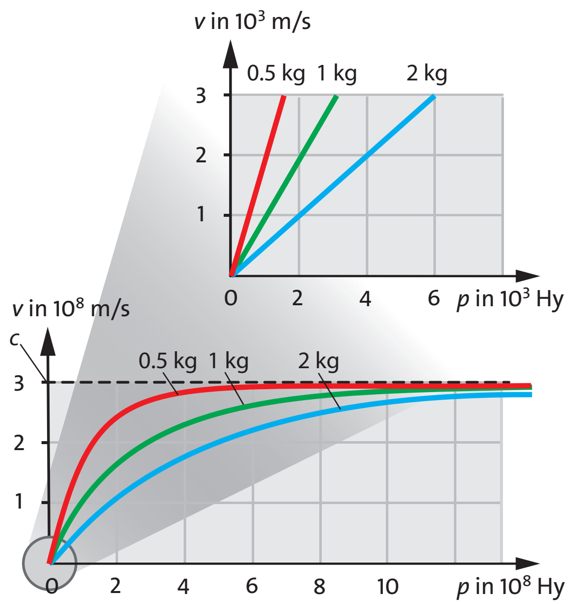


Fig. 7.4 Dependence of the velocity on the momentum. The section shows an enlarged view of the beginning of the curves. Here, the velocity is proportional to the momentum.

7.6 What happens to the velocity when the reference frame is changed

Let's have another look at the equation

$$E = k \cdot m\,.$$

We had concluded from it that there is an upper limit for the velocity

$$v_{\text{limiting}} = \sqrt{k} = c\,.$$

Now we come across a problem with this limiting velocity.

Let's imagine the following: a car drives on a long and wide conveyor belt, Fig. 7.5. Relative to the conveyor belt, it drives at 15 m/s, and the conveyor belt moves in the same direction at 8 m/s.

We are standing next to it, watch and see that the car moves relative to us at 23 m/s. This could also be expressed as follows: in the reference frame of the conveyor belt, the car moves at the velocity

$$v' = 15\text{ m/s}.$$

In the reference frame of the Earth, the conveyor belt moves at

$v_0 = 8$ m/s

and the car at

$v = v' + v_0 = 15$ m/s + 8 m/s = 23 m/s.

Fair enough up to this point. But now we let our (imaginary) car drive at a velocity of 0.6 c relative to the conveyor belt. There are no objections to this as the velocity is lower than the limiting velocity. But we now also let the conveyor belt move faster, e.g. at 0.8 c. This cannot be forbidden either as it is less than c. But there will be a problem if we want to know the velocity of the car relative to the Earth. According to our old equation

$$v = v' + v_0 \tag{7.4}$$

we obtain 1.4 c, which cannot be not a valid result. If we do not want to be in doubt about c being the limiting velocity, we can only conclude that equation (7.4) must be incorrect. And it is actually incorrect. The correct equation can be derived from the requirement of c being the limiting velocity. As the derivation is somewhat tedious, we look immediately at the result and check whether it supplies the expected velocity values. Instead of (4) the correct relation is

$$v = \frac{v' + v_0}{1 + \frac{v' v_0}{c^2}}.$$

We look at different special cases for which we already know the result in advance.

$v' \ll c$ and $v_0 \ll c$

If both v' as well as v_0 are very small compared to c, the term

$$\frac{v' v_0}{c^2}$$

in the denominator will become much smaller than 1 and can be neglected against 1. Therefore, we obtain

$$v = v' + v_0 \,,$$

i.e. our old formula for „non-relativistic“ velocities.

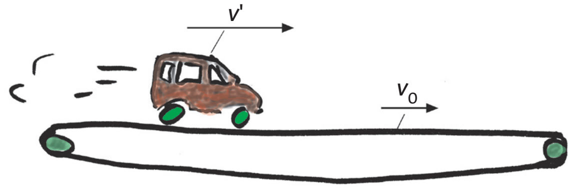


Fig. 7.5 The velocity of the car relative to the Earth is equal to the velocity of the car relative to the conveyor belt v' plus the velocity of the conveyor belt v_0, – but only approximately.

$v' \approx c$ and $v_0 < c$

Now we let the car drive relative to the conveyor belt at almost the terminal velocity: $v' \approx c$, assuming that the conveyor belt moves at a velocity $v_0 < c$. We obtain

$$v = \frac{v' + v_0}{1 + \frac{v' v_0}{c^2}} = \frac{c + v_0}{1 + \frac{c v_0}{c^2}} = \frac{c + v_0}{1 + \frac{v_0}{c}} = \frac{c \cdot (c + v_0)}{c + v_0} = c.$$

i.e. seen from the Earth, the car also moves at the limiting velocity. The limiting velocity is not exceeded.

$v' \approx c$ and $v_0 \approx c$

Finally, we also let the conveyor belt move at almost the terminal velocity.

We obtain

$$v = \frac{v' + v_0}{1 + \frac{v' v_0}{c^2}} = \frac{c + c}{1 + \frac{c c}{c^2}} = \frac{c + c}{2} = c.$$

Again, the car only moves at the velocity c in the reference frame of the Earth.

We can also describe our thought experiment as follows: For the movement of the car relative to the Earth we combine the movement of the car relative to the conveyor belt and the movement of the conveyor belt relative to the Earth.

> When combining movements, the velocities v' and v_0 must not be added, but the following applies:
>
> $$v = \frac{v' + v_0}{1 + \frac{v' v_0}{c^2}}.$$

Exercises

1. The spaceship Uranus flies at 0.9 c relative to the Earth. Wostok moves towards it in the opposite direction. Wostok has a velocity of 0.5 c relative to the Earth. How fast does Wostok appear to the Uranus crew? Uranus overtakes Shenzhou that flies at 0.5 c in the same direction as Uranus. How fast is Shenzhou relative to Uranus?
2. A car drives at v' = 140 km/h, coincidentally in the same direction in which the Earth orbits the Sun (at v_0 = 30 km/s). By how much is the car faster than the Earth from the perspective outside of the Earth? The solution will become easier if you calculate at first $v - v_0$ without inserting numbers. Neglect summands whose values are nearly zero.

7.7 How energy depends on momentum

For the relationship between energy and velocity, we have found (equation (7.1)):

$$E(v) = \frac{m}{2}v^2 + E_0.$$

We transform the equation by means of $p = m \cdot v$ and obtain:

$$E(p) = \frac{p^2}{2m} + E_0. \tag{7.5}$$

Also this equation is valid only as a classical approximation, i.e. for momentum values that are not too large. The relativistic relation is:

$$E(p) = \sqrt{c^2 \cdot p^2 + E_0^2}. \tag{7.6}$$

Again, we would like to examine the compatibility of the relativistic equation with the classical one.

Fig. 7.6 shows the function graph of equation (7.6) for three different rest masses: 1 kg, 2 kg and 3 kg.

The vertical axis is the mass/energy, on the left in the unit kilogram and on the right in Joule. The horizontal axis reaches up to very large momentum values.

We look once again at the two limit cases: very large momentum and small momentum. For large momentum values, we can neglect E_0 against $c \cdot p$ in equation (7.5) and obtain

$$E(p) \approx c \cdot p.$$

The equation of the asymptote is

$$E(p) = c \cdot p.$$

Hence, the curve approaches the asymptote for large p values. In other words: for large p values, the energy is proportional to the momentum.

Let's now have a look at small momentum values. Fig. 7.7 shows the classical and the relativistic energy-

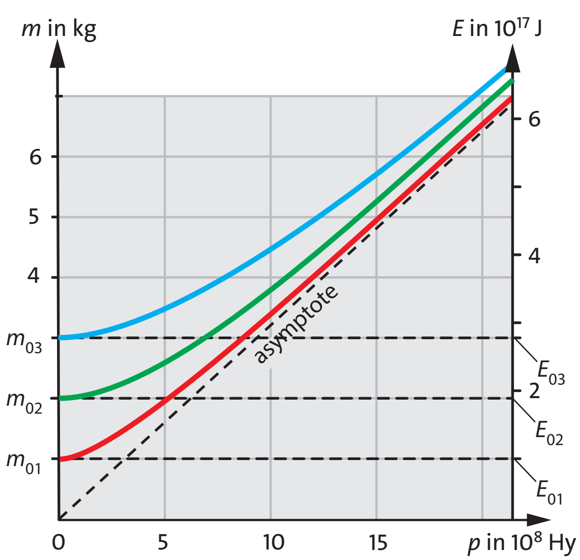


Fig. 7.6 Relationship between energy and momentum for bodies with the rest masses 1 kg, 2 kg and 3 kg. All three curves approach the asymptote $E = c \cdot p$.

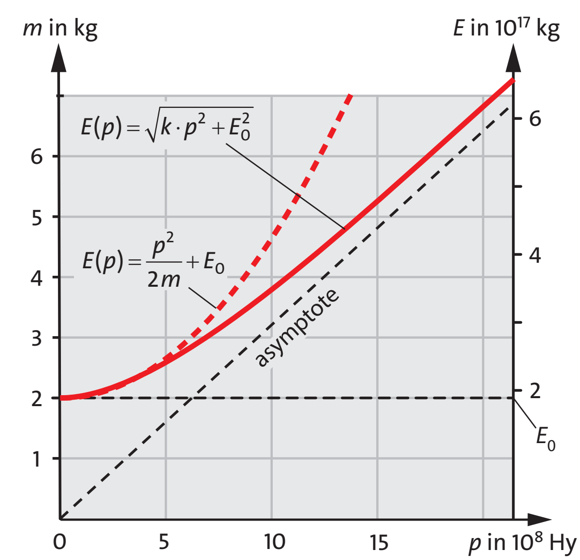


Fig. 7.7 For small momentum values, the relativistic curve is well approximated by the classical one.

momentum relationship, i.e. the graphs of the functions (7.5) and (7.6).

We can see that the functions for small momentum values are nearly congruent. The classical formula (7.4) is a good approximation for small velocities, i.e. for $v \ll c$.

$$E(p) = \sqrt{c^2 \cdot p^2 + E_0^2}$$

great values of momentum:
energy is proportional to momentum
$E(p) \approx c \cdot p$
small values of momentum:
energy changes with the square of momentum

$$E(p) = \frac{p^2}{2m} + E_0$$

Exercises

1. Calculate $E(v)$ starting with the expressions $v(p)$ and $E(p)$. Illustrate the relationship graphically. Interpret graph. At which velocity will a body be twice as heavy as in the state of rest?
2. A particle (e.g. an electron) is charged at a constant rate with momentum by means of an electric field. (A constant momentum current flows into the particle.) Illustrate graphically: $p(t)$, $E(t)$ and $v(t)$.

7.8 Particle accelerators

What are particle accelerators for?

Matter consists of molecules, molecules consist of atoms, atoms consist of protons, neutrons and electrons, protons and neutrons are made up of quarks. Hence, there is a hierarchy of components or particles. Apart from the particles listed here, there are many others that are normally hardly noticed though.

Some particles are not noticed because they are very rare in nature: for example muons, antielectrons, antiprotons or antineutrons. These particles, however, can be produced artificially in larger quantities. Other particles are not perceived as they do almost not „interact" with the normal matter. They fly through the matter in an almost unencumbered way. This is true, inter alia, to the neutrinos that come from the Sun and to the particles of the dark matter of which not much is known yet.

Particle accelerators are important devices to explore the components of matter. Particles – mostly electrons or protons – are charged with very much momentum and energy and shot onto a target, i.e. any matter at rest or, even better, particles with opposite momenta are shot onto one another. New particles, i.e. a very large quantity of them, are formed in the process. Some of these new particles only have an extremely short lifespan and decay into other particles. Hence, a sequence of particle transformations takes place. The newly formed particles are examined: its energy, its momentum and its electric charge are measured and the frequency at which they arise in a reaction is checked.

About the structure of an accelerator system

The particles (protons or electrons) move in an evacuated tube. As they are electrically charged, they can absorb momentum – and consequently also energy – in an electric field (just as a body absorbs momentum and energy in the gravitational field due to its mass).

In large accelerators, the particles do not form a continuous beam but they move in bunches through the machine. During each circulation, a bunch moves through several „accelerating cavities", i.e. areas with an electric field. In each accelerating cavities, the bunch receives a „kick", i.e. a portion of momentum and a portion of energy.

(The electric field has to be switched on and off repeatedly in this process. The particles cannot be accelerated on an orbit with a constant electric field. This is because in that case they would be accelerated by the field during one part of the round trip and slowed down again during another part.)

Magnetic fields are needed for the particles to move on a bent orbit and not straight ahead. Therefore, electromagnets are located all over the ring.

The LHC

A large proton accelerator system is located at the CERN in Geneva. The main ring is the LHC ring (Large Hadron Collider). Four smaller accelerators are placed ahead of it. The first of them is a linear accelerator („Linac").

Fig. 7.8 shows a view from above of the system. It is located at a depth of approximately 100 m under the surface of the Earth. The way that the protons take through the 5 accelerators is shown. The main ring has a circumference of 27 km. From the penultimate accelerator ring, the protons are fed into the LHC in opposite directions. The figure also indicates the energy that the particles have after running through each of the four rings. The „proton-synchrotron booster" brings them to the 1.5-fold value of their rest mass m_0, when leaving the „proton-synchrotron" they have an energy of 20 m_0, the „super-proton-synchrotron" brings the mass/energy to 400 m_0 and the LHC eventually to 7000 m_0.

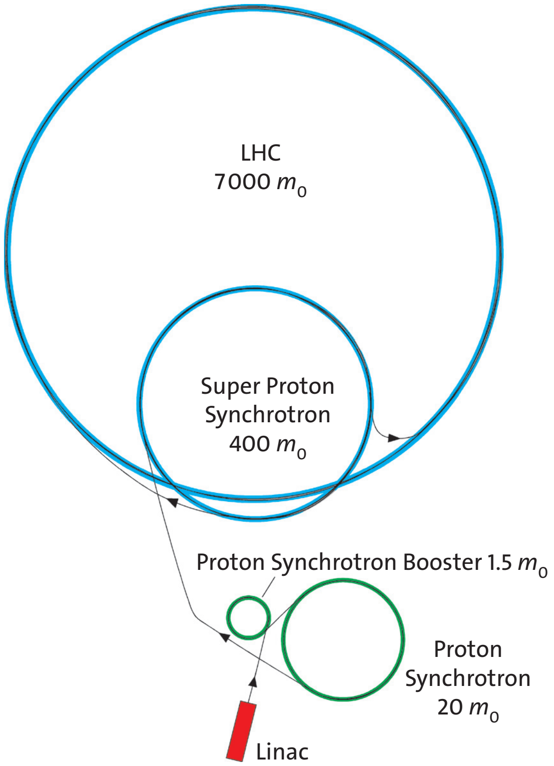


Fig. 7.8 LHC accelerator system at CERN. The large ring has a circumference of 27 km. The system is located in a tunnel under the surface of the Earth.
Not shown are:
- the magnets that force the beam onto the nearly circular orbit
- the „accelerating cavities" that are distributed over the rings
- the detectors

In the LHC, the proton bunches are brought to their final energy of $7000 \cdot E_0$ in approximately 20 minutes. Then, they circulate in the ring for several hours. There are always 2808 bunches in the LHC ring at the same time. At the beginning, each bunch contains approximately 10^{11} protons.

At some points, bunches moving opposite to one another cross their way. Here, new particles can be formed out of the protons, that move in opposite directions, and huge detectors are therefore located here in order to detect these new particles and to measure their properties.

> *Particle accelerator system*
> Particles are charged with energy and momentum and brought to collision. New particles are formed, inter alia particles with a rest energy that is much higher than that of the initial particles.

Possible reactions

Without knowing the details of the particle reactions, we can make some statements about such processes: the general conservation laws have to be fulfilled. Energy, momentum and electric charge of initial particles and product particles have to be equal. But there are yet other conservation laws that have to be obeyed.

As each of two colliding protons brings an energy of $7000 \cdot E_0$, $14\,000 \cdot E_0$ is available for the reaction. Hence, particles with a much higher rest mass than that of the proton can be created.

A note regarding the name „accelerator": it is not quite suitable but we have used it because it is a historic custom. When coming out of the third accelerator stage, the protons have a mass of 20 m_0. Their velocity is therefore almost equal to the limiting velocity. Consequently, they hardly become any faster in the two subsequent „accelerator stages". They only become heavier.

In addition, the protons are charged with energy for 20 minutes and will then only circulate at a constant energy in case of the LHC. Therefore, a ring as the LHC is also called *storage ring*. Highly energetic particles are stored in it.

Exercise

1. What is the velocity of the protons after the 2nd, 3rd, 4th and 5th acceleration stage of the LHC system? The result from Exercise 1, Section 7.7 is needed. Give the result in units c, i.e. how many times c is the respective velocity?

7.9 Light

There are particles whose rest mass is zero: the photons, i.e. the particles light consists of. If we set $m_0 = 0$ in equation (7.3), we will obtain:

$$v = c.$$

Hence, photons always move at the limiting velocity.

But rest mass or rest energy zero does not mean that the photons have no energy. Rather, the appearance of light clearly tells us the energy of its photons: the higher the frequency f of the light, the higher the energy of the respective photons:

$$E = h \cdot f.$$

h is a fundamental physical constant, the *Planck constant:*

$$h = 6.626 \cdot 10^{-34}\ \text{Js}.$$

The photons of violet light (high frequency) have more energy than those of red light.

The energy-momentum relationship of the photons is very simple. With $E_0 = 0$, equation (7.6) becomes

$$E(p) = c \cdot p\ .$$

Written with m instead of E, we have:

$$m \cdot c^2 = c \cdot p\ ,$$

or

$$m = \frac{p}{c}.$$

The phenomenon that we will discuss now looks quite innocuous at first. But we will see that it has peculiar consequences.

When a body is moved upwards in the gravitational field of the Earth, i.e. if it is brought to a higher gravitational potential, the body has to be provided with energy. The body will not retain this energy but release it immediately to the gravitational field.

You got to know the formula

$$\Delta E = m \cdot (\psi_2 - \psi_1)\ .$$

m is the mass of the body and

$$\psi = g \cdot h$$

the gravitational potential (g = field strength, h = height).

However, by means of our new equation $E = k \cdot m$ we now can replace the mass and obtain:

$$\Delta E = \frac{E}{k} \cdot \Delta\psi.$$

Transforming the equation, we obtain the „relative energy change“, i.e. the energy change divided by the total energy:

$$\frac{\Delta E}{E} = \frac{\Delta\psi}{k}. \qquad (7.7)$$

We apply this equation to light. We know that the energy of the photons is related to the frequency of the light:

$$E = h \cdot f\ .$$

Therefore, we can write

$$\Delta E = h \cdot \Delta f\ ,$$

and we can transform equation (7.7):

$$\frac{\Delta f}{f} = \frac{\Delta\psi}{k}. \qquad (7.8)$$

At first, the result is not very thrilling. We look at an example: the light of a street lantern. The lantern is located at an altitude of 4 m. On its way downwards, the frequency increases according to our formula. We calculate by how much. With Δh = 4 m and g = 10 N/kg, the difference of the gravitational potential becomes:

$$\Delta\psi = 40\ \text{Nm/kg} = 40\ \text{J/kg}.$$

With $k = 9 \cdot 10^{16}$ J/kg we obtain:

$$\frac{\Delta f}{f} = 4.4 \cdot 10^{-16}.$$

$$\Delta f = 4.4 \cdot 10^{-16} f.$$

The frequency increases by a tiny fraction of the initial frequency f. After all, this effect could be determined experimentally, albeit not with the light of a street lantern.

7.10 Clocks in the gravitational field

To better understand the peculiar consequences of this effect, we imagine the frequency change to be greater than it actually is. Willy lives on the top floor of a skyscraper, Lilly on the ground floor, Fig. 7.9.

Willy and Lilly want to find out in an experiment whether the formula for the frequency change of the light is correct.

Willy points downwards with a green laser. On its way, the energy of the light and consequently the frequency increases, and blue light (higher frequency) arrives down at Lilly's place. Lilly also has a green laser that she points upwards. This light loses energy while rising; its frequency decreases. It is red when it arrives at Willy's place.

Willy now has a clock whose rate is controlled by the frequency of the light. Lilly has such a clock, too. Willy (top) therefore concludes that Lilly's clock runs more slowly than his own. And Lilly (bottom) comes to the same conclusion: Willy's clock runs faster. They want to check whether this conclusion is correct. Willy and Lilly meet halfway and compare their clocks, i.e. they set them in a way that they indicate the same time. Then, Willy goes back upwards and Lilly back downwards. After a while, they meet halfway once again and compare their clocks. As expected from our formula, the clocks do no longer indicate the same time. Willy's clock is fast compared to Lilly's clock. The reason is not a defect of the clocks. At Willy's place, more time has passed than at Lilly's place. Let's assume that Willy and Lilly are twins, i.e. born at the same time. Willy who lives at the top of the skyscraper would, seen from Lilly's perspective, age faster than Lilly, or Lilly would age more slowly compared to Willy.

This phenomenon is known by the name *twin paradox* (in most cases, a slightly different story is told to illustrate the context). A paradox is an assertion that seems to be contradictory but that is actually not.

Back to reality: the only incorrect piece of our story is the effect being so strong.

Hence, another one of those relativistic effects that are funny but completely irrelevant? Not exactly. On Earth, it is relevant in cases where accurate frequency measurements are important. This is the case for GPS.

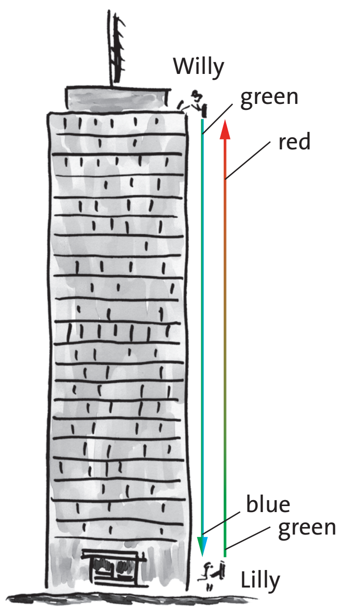


Fig. 7.9 The energy (and the frequency) of upward-moving light decreases. The energy of the downward-moving light increases. Lilly has the impression that Willy's time passes faster, and Willy feels that Lilly's time runs more slowly.

In GPS-based position calculations, this effect has to be taken into account.

But there are also places in the world where the rates of clocks in the gravitational field are very different: in the vicinity of black holes. Black holes are celestial bodies with very uncommon properties. Prior to looking at time effects in the vicinity of black holes, we will get an overview of the most important celestial bodies.

> Two people separate, move to places of different gravitational potential and meet again. More time has passed for the person who was on the high gravitational potential.

Exercises

1. The skyscraper has a height of 400 m. Willy and Lilly live there for two years. How much more than Lilly has Willy aged during this time?
2. Assume that the effect of faster aging is much stronger: Willy ages twice as fast as Lilly. What are the consequences of this aging process for Willy's and Lilly's everyday life?

7.11 Celestial bodies

Single objects

There are celestial bodies of very different sizes, masses, composition and temperature. Starting from a certain mass, every celestial body is nearly spherical. Any large deviation from the spherical shape would smooth out just as a "mountain" of water would do: the water flows to places where the gravitational potential is lower, Fig. 7.10. Also the mountains that continuously form on Earth only reach a height of less than 10 km. They flow (very slowly) towards the lower gravitational potential, too.

Most celestial bodies that can be seen at night with the naked eye are stars. Only the Moon and the planets are no stars.

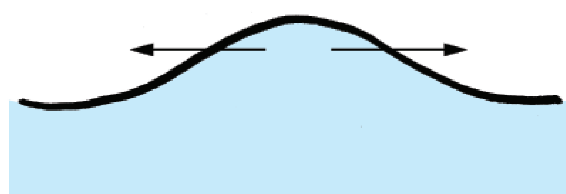


Fig. 7.10 The water flows to places of lower gravitational potential.

Planets

The eight planets Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and Neptune (enumeration from the inside to the outside) move around the Sun on nearly circular orbits. Almost no heat is produced inside a planet. It is relatively cold on its surface. The surface of Venus has a mean temperature of approximately 500 °C, Neptune's surface temperature is –200 °C. 500 °C is a low value compared to the temperature at the surface of the Sun.

The masses of the planets are very small compared to the mass of a typical star (see Table in Section 4.7). Other stars also have planets, but observing them is difficult because they are so small and because they do not glow themselves.

Moons are celestial bodies similar to the planets. A Moon always orbits around a planet.

Planet
- orbits around the Sun
- is small compared to the Sun
- is relatively cold

Sun-like stars

A typical example for a star is the Sun. The fact that we perceive it as much larger and brighter than other stars is simply due to it being much closer to us. The masses of the stars are in the range between 1/100 of and 100 times the mass of the Sun.

By far the biggest part of a star's mass is located in a small internal area. In case of the Sun, 90 % of the mass are within half of the Sun's radius. Hence, we could almost say that „the Sun looks bigger than it actually is".

In a much smaller inner core of the Sun, a nuclear reaction occurs at a temperature of 10 million Kelvin: hydrogen is transformed very slowly into helium. Later, also higher elements are formed. The entropy that is created during the reaction flows outwards together with the associated energy: from the high temperature of the reaction zone to the „low" temperature at the surface (approximately 5800 K). Further entropy is produced on the way to the outside. From the surface, entropy and energy leave the Sun with the light. Due to its high temperature, the material of the Sun is gaseous.

Sun-like star
- is gaseous
- a nuclear reaction takes place inside

White dwarves

A white dwarf is formed from a Sun-like star, which is not too heavy, when its „core fuel" is no longer sufficient to maintain the nuclear reaction, i.e. when it is „burnt out". The white dwarf is not gaseous but in a state that is rather similar to that of the Earth; only is the matter compressed very strongly due to its own weight, i.e. the gravitational field. Its density is approximately 10^3 kg/cm^3.

The masses of the white dwarves are in the range between 0.5 and 0.7 sun masses, their diameters amount to several thousand kilometers. They still glow, but only due the fact that they are cooling down. Because of their small size, white dwarves cannot be seen on the night sky with the naked eye. A white dwarf consists of carbon and oxygen, the reaction products of the reaction in the star out of which it was formed. An unusual relationship applies for white dwarves: the larger the mass of a white dwarf, the smaller its diameter.

White dwarf
- is a burnt out star
- matter is strongly compressed by the gravitational field
- glows in the process of cooling down
- the larger the mass, the smaller the diameter

Red giants

They represent an intermediate stage in the development from a Sun-like star to a white dwarf. The diameter of a red giant is approximately a hundred times as large as that of the star of which it was formed. However, its size only arises due to the fact that the already light shell of the initial star is strongly enlarged. This expansion is caused by the strong light flow that comes from the small core. This radiation literally inflates the shell and eventually blows it completely away so that only the core, which has been transformed into a white dwarf, is left. Hence, it is even more true for a red giant that it „looks much bigger than it actually is".

Red giant
- shell is strongly inflated due to radiation
- intermediate stage on the way to the white dwarf

Neutron stars

A Sun-like star that runs out of nuclear fuel will shrink. We have seen that a white dwarf can be formed in this process. However, if the initial star is very heavy, the pressure will become so high that no stable white dwarf will result. In case of sufficiently high pressures, the electrons react with the protons of the atomic nucleus to form neutrons and neutrinos:

proton + electron → neutron + neutrino .

Normally, i.e. in case of lower pressures, no neutrons are formed out of protons and electrons. On the contrary: neutrons disintegrate spontaneously into protons and electrons and antineutrinos (neutrinos and antineutrinos are very light and fugitive particles.)

The neutrons formed out of the protons and electrons take up much less space than the electrons and protons. Hence, much space is created and the star implodes until the neutrons are packed together very closely. It becomes a *neutron star*. The neutron star has a diameter of only approximately 20 km at a mass between 1.3 and 2 sun masses. For this reason, it has a very high density: approximately 10^{12} kg/cm^3.

During the implosion of the star, the gravitational field releases a huge amount of energy – more than the rest energy of the Sun – and a gigantic explosion takes place: a *supernova*. A part of the matter of the initial star is thereby catapulted away to the outside. A supernova is a process that does not take long and that can therefore be observed on rare occasions only. In our galaxy, the Milky Way, approximately 20 supernovas take place per 1000 years.

A supernova can also occur in another way. A white dwarf forms a binary star with a red giant. Then, matter flows continuously from the red giant to the white dwarf. The mass of the white dwarf increases in the process and so does the pressure inside. Finally, the pressure becomes so high that the reaction

proton + electron → neutron + neutrino

starts and the white dwarf „implodes", similar to a house that is built higher and higher until it can no longer bear its own weight and collapses.

> *Neutron star*
> - is a burnt out star with a large mass
> - neutrons were formed out of protons and electrons
> - a supernova explosion takes place during its formation

Black holes

If the mass of the star that is running out of nuclear fuel is even larger, also the neutrons will no longer withstand the high pressure. Then, nothing will be left to keep the matter from collapsing further. The final stage of the star is a *black hole*. Its mass is approximately 5 to 15 sun masses.

Seen from the outside from a long distance, a black hole appears as a sphere with a diameter of approximately 10 to 20 km that does not emit any radiation. How is that possible? If the black hole is approached from the outside, the gravitational potential will decrease. Seen from the outside, a clock runs more slowly the closer it comes to the center of the black hole. At a defined distance from the center, it finally stops completely. The corresponding spherical surface is called *event horizon*. It appears to us as outsiders as if the time at the event horizon is standing still. If we were to let an object fall into the black hole, we would find the object to move increasingly slowly and to never reach the event horizon.

As a consequence, nothing – no light and not matter – can reach us from the event horizon, and especially not from further inside either.

> *Black hole*
> - mass is so large that even the neutrons cannot bear the pressure
> - an object that falls into a black hole never reaches the event horizon from an outside perspective.

What we have described here is called *stellar black hole*. There is yet another class of black holes. At the center of every galaxy, there is a black hole that is much bigger and heavier than a stellar black hole. The mass is 10^6 to 10^{10} sun masses.

Systems of stars

Binary star

Two stars move around one another on closed orbits. Approximately half of the stars are partners in a binary star. The Sun is not. The two partners of a binary star can be of a different nature, i.e. for example: one of them is a normal star and the other one a white dwarf; or one is a red giant and the other one a neutron star.

> *Binary star*
> - two stars orbit around one another

Galaxies and galaxy clusters

The stars are not distributed evenly in space but pooled in *galaxies*.

Galaxies have very different sizes. The galaxy our solar system belongs to is the *Milky Way* system. At night, it appears to us as a bright band that extends over the sky: the Milky Way. From the outside, we would see that it is approximately disc-shaped. It consists of around $3 \cdot 10^{11}$ stars and has a diameter of approximately 100 000 light years.

Our next neighbor at a distance of approximately 2.5 million light years is the Andromeda galaxy. It is hardly visible with the naked eye.

There are galaxies that emit much more electromagnetic radiation than other, normal galaxies: the *quasars*. The radiation mostly comes from a heavy black hole that is located at the center of the quasar and where matter is constantly falling in. We will discuss later what a black hole is.

Also the galaxies are not spread evenly over the universe. They also exist in bundles, i.e. in so-called *galaxy clusters*.

Galaxy
- cluster of many stars

Quasar
- galaxy that contains a black hole where matter falls in
- emits very much radiation

Further components of the universe

Two important components of the universe that do not fit in our current classification still have to be mentioned.

Cosmic background radiation

When looking at the particle number of the different components of the universe, we find that by far the greatest contribution is made by the photons, i.e. the particles of light. The entire space is filled with electromagnetic radiation in the microwave range, i.e. with wavelengths of several millimeters to centimeters (i.e. the same radiation that is used for mobile phones). The number of photons is approximately 10^{10} times larger than the one of the matter particles, i.e. of protons and neutrons in the universe. This radiation is called *cosmic background radiation.*

However, their contribution to the mass of the cosmos is very small compared to that of matter.

Cosmic background radiation
- microwave radiation
- 10^{10} times as many photons as matter particles

Dark matter

But also the matter the visible and invisible stars consist of does not make to biggest contribution to the overall mass of the universe. A contribution that is approximately six times bigger comes from *dark matter*. It consists of particles that do almost not interact with the „normal" particles, i.e. protons, neutrons, electrons and photons. It can only be felt significantly through its gravitational field.

Dark matter
- total mass is around six times that of normal matter
- manifests itself through its gravitational field

8 SPACETIME

Space can be defined as „room for something". It can be filled or empty. We can indicate its quantity that we call volume and that we measure in m^3. We can describe a point in space by means of three coordinates; then, we talk about the position.

Also for the time we can indicate a „point" on a time scale and a sort of amount of time by an interval on the time scale. This is usually called duration. Points in time and intervals are measured in seconds.

This is how the terms space and time are used in everyday life. In the following, we will see that modern, i.e. relativistic physics, teaches us that space and time are more than just the stage where the physical processes take place.

We will see that

- space and time are coupled and form a unit: the spacetime. The physics of spacetime is the subject of the so-called *special relativity theory.*
- Spacetime has properties that differ from point to point. This is the subject of the *general relativity theory.*

Both theories or descriptions of nature were formulated by Einstein.

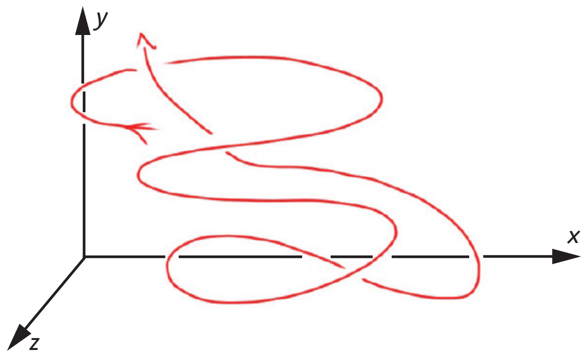


Fig. 8.1 Trajectory of a body

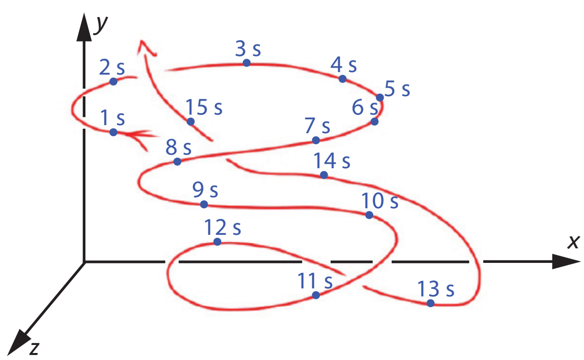


Fig. 8.2 Trajectory of a body with time specifications

8.1 Problems of presentation and designation

We would like to graphically illustrate the movement of a body in space, for example a helicopter, a bird or a portion of water in the swirling flow of a river. This can be done in several ways.

Fig. 8.1 shows the trajectory of the body. Here, we learn something about the movement: the position where the body was located and the order in which it has passed these points.

But this does not yet tell us everything about the movement. We do not know at what times the body was at the different locations. There is a way to close this information gap, Fig 8.2.

The picture tells us at what instant of time the body is located at which position. Now, 4 numerical values are associated with each point on the red curve: three space coordinates and one time indication. We have used a trick to display the three space coordinates: perspective display. You can imagine the curve to be displayed in a real three-dimensional coordinate system, too.

It would actually be nice if we were able to also indicate the time on a coordinate axis. For this purpose, we would need a fourth dimension – which we do not have. But we often have to work with movements in a plane. This means that we only need two dimensions for the trajectory. In this case, we can use the third axis for the time.

Things will become even simpler if the movement takes place in only one direction, for example like in case of a car that drives on a long straight road – while its velocity may change arbitrarily. In such cases, we can graph the movement in a two-dimensional coordinate system.

Fig. 8.3 shows the movement (and also the standstill) of a car. Can you describe the movement in words?

You might wonder why the time axis was set as a vertical axis and the position axis as a horizontal axis. Up to present, you have certainly seen it inversely. It does not have any deeper reason; it is simply a usual practice in spacetime physics, which will be addressed in the following.

The gray line that describes the movement of a car is not the trajectory of the car; compare with the above statement. We need a new name for it. It is called the *world line* of the car.

A point on the world line, for example the point P, is called *spacetime point*. It is characterized by two numerical values: the specifications for the position and for the time.

Four numerical values are generally needed to describe three-dimensional movements: three space coordinates and one time coordinate. A world line would be a line in a four-dimensional coordinate system.

> A world line describes the movement of a body. It tells us at which position a body is located at different instants of time.

By indicating the coordinates of space and time, i.e. by indicating the spacetime point, we can describe when and where an „event“ takes place.

Willy and Lilly make an appointment: they would like to meet at 11.00 h in front of the canteen. The event is: „Willy and Lilly meet up“. The spacetime point is:

t = 11 h

x = in front of the canteen.

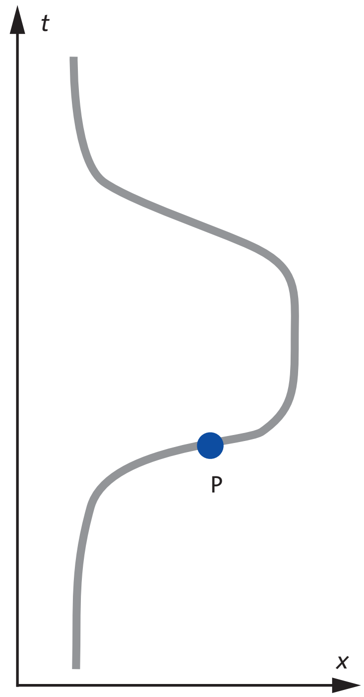


Fig. 8.3 World line of a moving body

Exercises

1. A model railway moves on a circular track. The radius is 1 m. The locomotive needs 10 seconds for a round. A further locomotive drives twice as fast. Draw the two corresponding world lines in a common coordinate system.
2. A construction vehicle drives on a straight path. The trajectory with the time specifications, Fig. 8.4, tells you how it drives. Draw the world line of the vehicle into an appropriate spacetime diagram. Include some spacetime points.

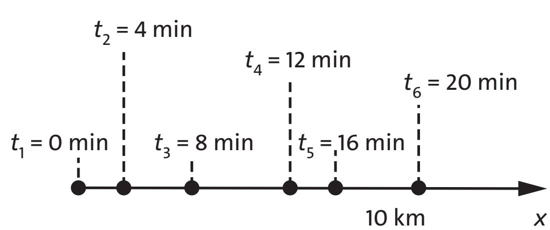


Fig. 8.4 For exercise 2

Exercises

3. Four world lines are drawn into the t-x diagram from Fig. 8.5. They show the history of 4 bodies. Two of the bodies move at a constant velocity. Which one is faster? One of the two becomes faster, the other one slower. Which ones?
4. Lilly experiences weightlessness during a parabolic flight. She floats. As she has not held her mobile phone properly, it receives a little shove and moves upwards (seen from Lilly's perspective) at a constant velocity. Draw the world lines of Lilly and her mobile phone into a common spacetime diagram. Lilly should stay at rest in this diagram.
5. Lilly wants to do early morning exercise and starts riding her bike at 8.00 h. She drives on a straight route at a velocity of 15 km/h. She starts her return trip in a way as to meet Willy, who has stayed at home, at 9.00 h. (a) Draw the world lines of Lilly and Willy in an appropriately chosen coordinate system. (b) Indicate the coordinates of the spacetime points of the following events: Lilly starts her return trip; Willy and Lilly meet again.
6. Figure 8.6 shows two world lines. Make up a corresponding story.

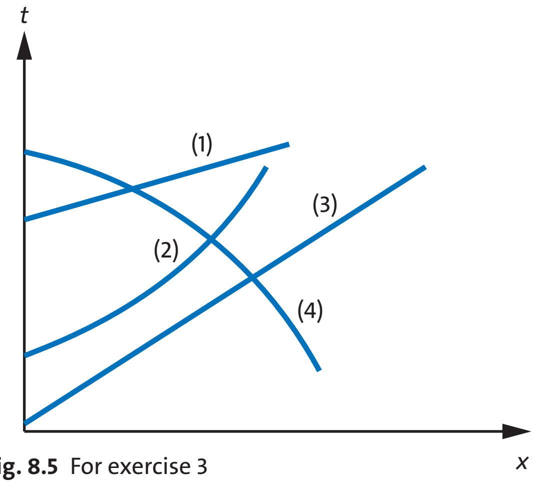


Fig. 8.5 For exercise 3

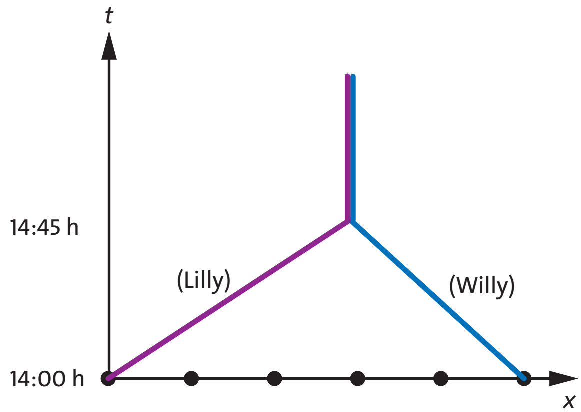


Fig. 8.6 For exercise 6

8.2 The time interval between two spacetime points

For ordinary people, space and time are independent of each other. This can be seen as follows:

Willy and Lilly each have a stopwatch. They start their watches at the same time, separate for a longer time and meet up again in order to compare the displays of their watches. They indicate the same – of course, you will say. This is how time works; it runs equally fast for all of us.

Strangely, this trivial statement is not correct. Here is what Willy and Lilly would find if they were to do a much more accurate measurement: They start their watches at the same time, separate for a longer time and meet up again in order to compare the displays of their watches. The watches indicate something different. Depending on how the two have moved in the meantime, Willy's watch indicates a bit more or a bit less than Lilly's.

We would like to analyze this phenomenon.

We describe the process by means of world lines.

To keep things simple, we imagine Willy and Lilly to move in an area of the universe in which no celestial bodies exist, i.e. far away from stars and planets. The coordinate system we use is a free-floating one.

Once again: Willy and Lilly rest in their free-floating coordinate system and start their stopwatches.

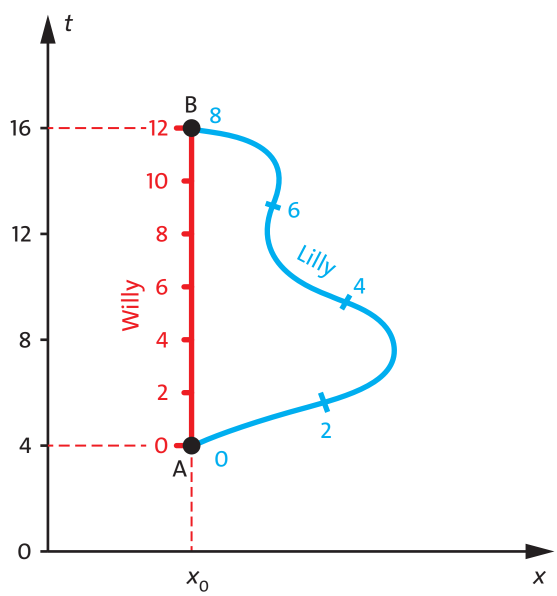


Fig. 8.7 Lilly's stopwatch indicates less than Willy's.

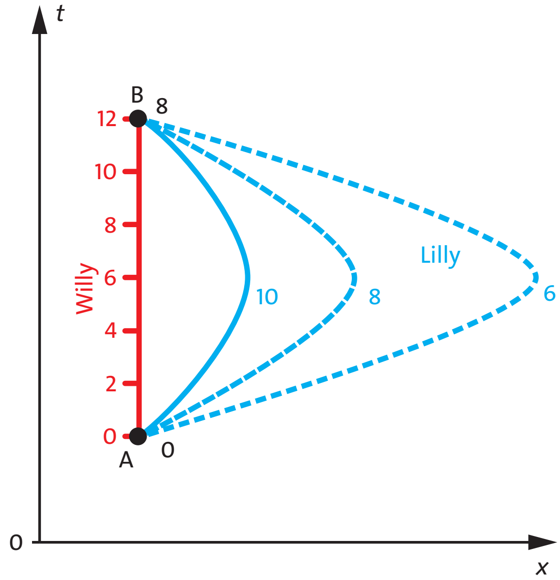


Fig. 8.8 The stopwatch indicates the greatest value on the straight world line.

Now we assume Lilly to move away in a spaceship; she flies around while Willy is a bit lazy; he remains where he is.

The world lines of Willy and Lilly are shown in Figure 8.7. As Willy does not move, his space coordinate remains constant all the time: $x = x_0$. It turns out that Lilly's stopwatch indicates less than Willy's.

They repeat the experiment once, and many times, in order to find out what to do to make the stopwatch indicate as much as possible and as little as possible. The result is surprising for both of them. The longest time is displayed when no movement is made at all as in case of Willy, Fig. 8.8.

Lilly's watch always displays a lower value than Willy's, and the faster she moves while increasing her distance to Willy and returning, the lower the indication of the watch. Eventually, she manages to move away and back at almost the limit value and it becomes evident that her watch displays almost zero in this case, Fig. 8.9. As she was so fast, she also moved far away from Willy.

Do not let yourself be fooled by the Figure: the vertical time axis corresponds to Willy's time because he is located in a floating reference frame the whole time.

Here, we have always talked about the values that a stopwatch displays. Lilly's watch indicates less than Willy's. However, there is more than just the pointer position of two watches. Less time passes for Lilly's watch because the watch is moving. Though, not only the watch is moving but also Lilly herself and her spaceship. Less time consequently passes for Lilly and her spaceship, too. And this also means that Lilly ages less than Willy between the spacetime points A and B.

We can summarize the observations:

> If two people separate with their watches (event A) and meet again later (event B), their watches will indicate different times.
>
> For the person (watch) who does not move at all, the longest time passes. For a person (watch) who gets from A to B at (almost) the limit velocity, (almost) no time passes.

Yet another generalization: Willy's watch displays the longest time not only when he does not move at all, but also when he moves free-floating. Then, spacetime B in the world line diagram is no longer situated straight above A, Fig. 8.10.

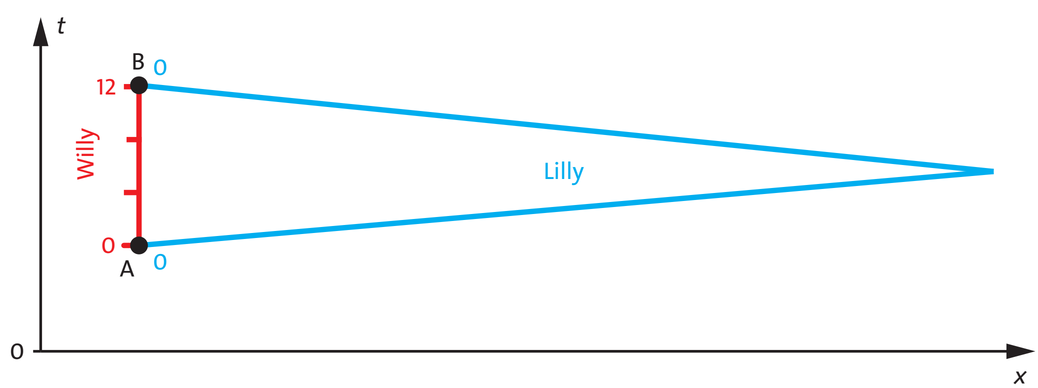


Fig. 8.9 If Lilly moves (almost) at the limit velocity, she will (almost) not age at all.

Willy's world line is still a straight line, but this straight line is now inclined against the t-axis. Hence, we have the following general rule:

> For the person (watch) who moves free-floating, the longest time passes. For a person (watch) who gets from A to B at (almost) the limit velocity, (almost) no time passes.

Or formulated a bit more succinctly:

> The longest time passes on the straight world line.

In this simple form, however, the rules only apply as long as there are no gravitational fields caused by heavy bodies, i.e. stars or planets.

The fact that the watches indicate different values after having moved for a while on different paths at different velocities is not in line with our normal experience. And you might find this difficult to understand. However, you are very familiar with a analogical phenomenon. This might make the watch story appear a bit more plausible.

So far, we have talked about two people with respectively one watch who separate from each other at a *spacetime point* A and who meet again at another *spacetime point* B.

Our other story goes as follows: Willy and Lilly separate at a *place* A and meet again at a *place* B, Fig. 8.11. Attention: now, A and B do not stand for spacetime points but for positions in the usual space – for example two cities. Willy and Lilly are both driving each with one car from A to B, but on different routes.

This time, they do not look at the watch but at the mileage counter. Of course, the mileage counters indicate different distances when Willy and Lilly have arrived at B. They can also try a variety of paths and find: the mileage counter shows a different value each time. But there is also a distinguished connection in this case: the road that is completely straight between A and B. The distance is smallest for it. All other paths are longer. The longest way that we could think of would be infinitely long. Hence, we could formulate the following rule:

> If two people separate with their mileage counters (place A) and meet again later (place B), their mileage counters will indicate different path lengths.
>
> For the person (mileage counter) who moves straight ahead, the distance is shortest.

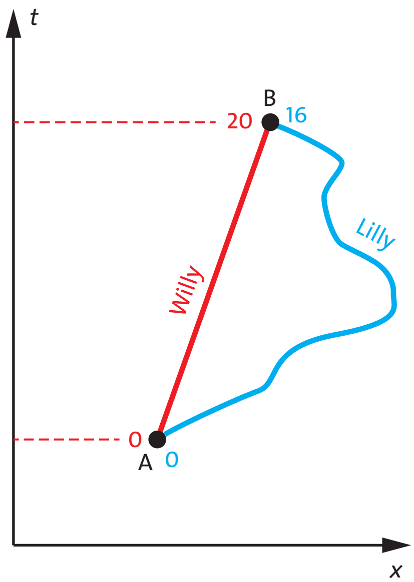


Fig. 8.10 The stopwatch indicates the greatest value on the straight world line.

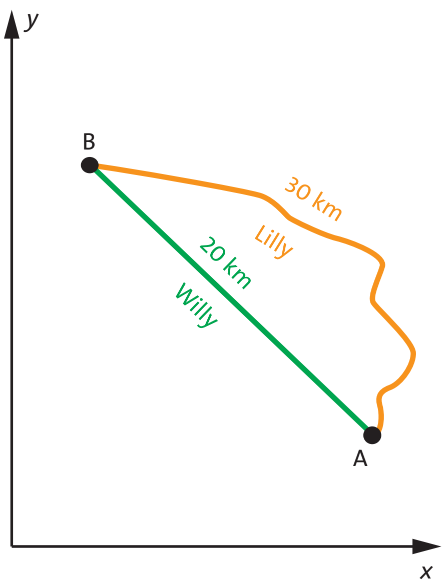


Fig. 8.11 The mileage counter displays the smallest value on the straight line.

normal space	spacetime
position	spacetime point
milage counter	watch
movement on a straight line: smallest distance	free-floating movement: greatest time interval

Table 8.1 Analogy between the trajectory in space and the world line in spacetime

We are consequently dealing with an „analogy“, Table 8.1.

Exercises

1. Five different world lines describe 5 „trips“, Fig. 8.12. All of them start at time t_1 from a common position and meet again at the same position at the time t_2. The dashed line is the world line of a body that moves at nearly the limit velocity. (a) Which world lines are physically permitted and which are not? (b) Sort the permitted world lines according to the time that passes for the traveler.
2. To measure the trajectory of the Moon exactly, laser pulses are sent from the Earth to the Moon, reflected there by a mirror that was set up during a Moon landing in 1969 and received back on the Earth. This way, Willy measures the runtime of the laser pulse. It is 2.55 seconds. Imagine Lilly could travel together with such a light pulse. (a) What are the important events A and B for the measurement? (b) Draw the world lines for Willy and Lilly between the two events. Choose the reference frame in which Willy is at rest. (c) Which travel duration (between the events A and B) is indicated by Lilly's watch? (d) Both Willy as well as Lilly know the limit velocity. What did Lilly find out regarding the distance between the Moon and the Earth? Formulate hypotheses.
3. Willy and Lilly have the same birthday. Willy is 15 years old today, Lilly 16 years. On this birthday, Lilly goes on a trip together with a light pulse. One year later – Willy is celebrating his 16th birthday – Lilly comes back. (a) Willy has calculated the route traveled by Lilly. What is the result of his calculation? (b) Lilly also arrives on her birthday. But how old is she?

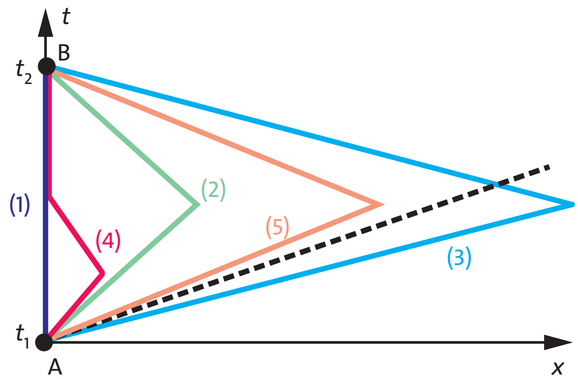


Fig. 8.12 For exercise 1

8.3 Time travels – the twin paradox

The stories of Willy and Lilly in the previous section were purely fictitious and we have not paid attention about the numerical values on the curves in Fig. 8.7 and 8.8 being realistic. They were only meant to explain the principle.

Now we would like to examine the magnitude of these effects in greater detail, i.e. to see which circumstances are required to observe them at all. In general, this is a complicated task, but if we assume Lilly to move at a constant velocity v during her trip, things will become simple again. Let's call the time that passes for Willy T_s (s for straight world line) and the time that passes for Lilly T_b (b for bent).

T_b is calculated from T_s according to:

$$T_b = T_s\sqrt{1-\frac{v^2}{c^2}}. \tag{8.1}$$

The factor

$$\sqrt{1-\frac{v^2}{c^2}},$$

which appears in many formulas in relativistic physics, tells us that the effect is small under normal conditions. If the velocity v is low compared to the limit velocity c, this factor will be nearly equal to 1, and this means in our case that we have

$$T_b \approx T_s.$$

We would like to test this statement for a particular case. Once again back to Willy and Lilly: They start their stopwatches. Willy does not move, whereas Lilly drives away by car for an hour at 90 km/h, turns around and drives back for an hour at 90 km/h. (Attention: the hour is read from Willy's stopwatch.) What will the stopwatches display when the two meet again? Of course, Willy's watch will indicate 2 hours (as this was the duration of the trip). We obtain the indication of Lilly's watch by means of equation (8.1).

Now directly calculate the difference between the two time indications, i.e. Willy's minus Lilly's time:

$$\begin{aligned}\text{difference} &= T_s - T_b = T_s - T_s\sqrt{1-\frac{v^2}{c^2}}\\ &= T_s\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)\end{aligned} \tag{8.2}$$

With

$$T_s = 2\cdot 1\ \text{h} = 7200\ \text{s}$$

$v = 90$ km/h $= 25$ m/s
$c = 3 \cdot 10^8$ m/s

we obtain

$$\text{difference} = T_s - T_b$$
$$= 7200 \text{ s}\left(1 - \sqrt{1 - \frac{25^2}{(3\cdot 10^8)^2}}\right)$$
$$= 25 \cdot 10^{-12} \text{ s.}$$

The difference is 25 picoseconds. This is much less than the measuring accuracy of the stopwatches. Hence, it is not surprising that such effects cannot be felt in normal life.

We have assumed Willy and Lilly to be on Earth, i.e. not in a freefloat frame. This is allowed in our case as the movement only takes place in the horizontal plane while the gravitational field strength vector is perpendicular to the surface of the Earth.

In order to make the effect stronger, we now let Lilly travel faster: She is not driving leisurely at 90 km/h but accelerates by means of a rocket in space to 90 % of the limit velocity, i.e. 0.9 *c*, and she is not flying for two hours but for 20 days (seen from the perspective of Willy's free-float-frame): 10 days in one direction, and 10 days back. Thereby, she moves away over $2.3 \cdot 10^{11}$ km (please check). How many days are passing for Lilly during this trip? We insert in equation (8.1):

$$T_b = T_s \cdot \sqrt{1 - \frac{v^2}{c^2}}$$
$$= 20 \text{ days} \cdot \sqrt{1 - \left(\frac{0.9c}{c}\right)^2} \approx 9 \text{ days}$$

While Willy has aged by 20 days, Lilly has only become 9 days older.

The trip that Lilly makes in our fictitious and slightly unrealistic story is also called *time travel*.

Let's assume that Lilly starts her trip on July 1. She will then be traveling for 9 days and arrive in a world in which the calendar already indicates July 21. Some people would say that she has made a trip to the future. But this is actually not a smart expression as Willy has arrived on July 21, too. Hence, everybody „travels“ to the future in any case.

The fact that one person ages less then someone else is also known as the *twin paradox*. At first, we cannot see anything paradoxical about this statement. The argumentation goes as follows: seen from Willy's perspective, Lilly moves. We can also say that Lilly moves in Willy's reference frame. But shouldn't we obtain the opposite result, i.e. that Willy ages less, if we go to Lilly's reference frame?

We must not draw this conclusion because Lilly's reference frame is not a free-float frame due to the accelerations during her trip. And in this case, equation (8.1) is no longer valid.

> Two people W and L separate (event A) and meet again (event B). W moves freely floating, L moves at a constant velocity *v* (same amount of *v* on the outward and return path). If the time T_s passes for W, the time
>
> $$T_b = T_s\sqrt{1 - \frac{v^2}{c^2}}.$$
>
> will pass for L.

Our new formula also provides another result that might look surprising at first but that was actually to be expected. Lilly wants to age as little as possible during her trip. What does she have to do? Fly away and return as fast as possible. But now we know: she cannot move faster than at the limit velocity *c*. Hence, she travels at (almost) the limit velocity. Inserting *c* in our equation shows that the best we could wish for is achieved: the term under the root and consequently Lilly's aging process is zero.

Exercises

1. This time, Lilly stays at home and Willy moves away and returns back. When the two of them meet again, they compare the indications of their watches. The travel time measured by Willy is half of that measured by Lilly. What was Willy's velocity of travel?
2. Lilly travels to a star that is 99 light years away from the Earth according to the star atlas. She reaches a velocity of 0.98 *c* with her spaceship. Willy, who does not want to join her trip, calculates how long it will take to see Lilly again (Lilly does not plan to stay at destination) and is very concerned. (a) Why is Willy concerned? (b) Which travel time is indicated by Lilly's board clock when she reaches her destination? Based on the distance of the star from the starting point, i.e. 99 light years, and the travel time indicated by her watch, Lilly calculates the travel speed and is surprised for a moment. (c) What is the travel velocity resulting from her calculation? She is surprised at first, thinks for a while and finds a peculiar explanation for her result. (d) Which explanation?

8.4 Movement on a circular orbit, GPS

We would like to apply equation (8.1) to a special situation: a circular movement. We imagine Lilly to sit on a carousel and move in a circle while Willy stands next to it and sees Lilly pass time and again, Fig. 8.13.

Lilly's movement is not a simple back-and-forth movement anymore. Her movement does not take place in one dimension of space but in two dimensions. We therefore need two space coordinates in addition to the time. Therefore, the spacetime diagram becomes three-dimensional and we display it in perspective, Fig. 8.14.

Lilly's trip starts in the spacetime point A. Willy and Lilly start their stopwatches. After four turns of the carousel, both will be located in the spacetime point B and compare their watches.

We would like to calculate the „aging difference“, i.e. for one rotation at first.

Given is the „turn around time“ T of the carousel (measured with Willy's watch!) and the radius r of Lilly's orbit.

Hence, we have:

$$T_s = T$$

and Lilly's velocity is

$$v = \frac{2\pi r}{T}.$$

Inserted in equation (8.1), we obtain what is indicated by Lilly's stopwatch:

$$T_b = T \cdot \sqrt{1 - \frac{4\pi^2 r^2}{c^2 T^2}}$$

The difference between the watch displays, i.e. the aging difference Willy-Lilly, will then become (see equation (8.2)):

$$\text{difference} = T_s - T_b = T \cdot \left(1 - \sqrt{1 - \frac{4\pi^2 r^2}{c^2 T^2}}\right) \qquad (8.3)$$

We assume:

$$T = 5\ \text{s}$$

and

$$r = 3\ \text{m}$$

Then, we will obtain

Fig. 8.13 Lilly moves, Willy does not (in the reference frame of the Earth).

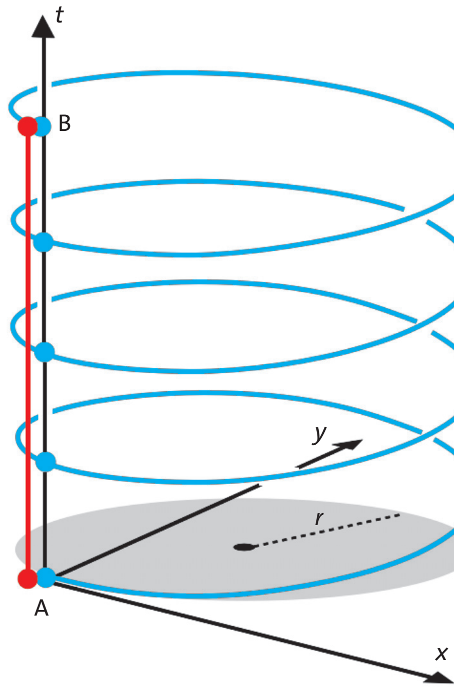


Fig. 8.14 World lines of Willy (red) and Lilly (blue) that connect the spacetime points A and B. Willy ages more than Lilly.

$$\text{difference} = 5\ \text{s} \cdot \left(1 - \sqrt{1 - \frac{4\pi^2 \cdot 3^2}{9 \cdot 10^{16} \cdot 5^2}}\right)$$
$$\approx 4 \cdot 10^{-16}\ \text{s}.$$

Hence, the difference is tiny again as expected.

We could conclude the aging difference to be immeasurably small and without any significance.

But the effect can be perceived

- if the measurement accuracy is very high;
- if the velocity is very high.

We have already come across these conditions: they also need to be fulfilled in order to find that energy and mass are the same physical quantity.

So is all this lacking practical significance? Not at all!

In some technical applications, extreme time measuring accuracy is important, for example in determining a position with the GPS system. And in some physical experiments with particle accelerators, the particles move at a velocity that is no longer low in relation to the limiting velocity. We will examine both cases in greater detail.

Example: muon accelerator

Muons are „elementary particles“ that are very similar to electrons. They have the same charge and the same angular momentum as electrons; their mass, however, is approximately 200 times larger than that of the electrons. In contrast to the electrons, muons are not stable. They decay into electrons and neutrinos. Such a decay is a statistical process. We cannot tell in advance from an individual muon when it will disintegrate; but the mean lifetime τ of a large number of muons has a very specific well-known value. We know:

$\tau = 2.2\ \mu s$ (microseconds)

Muons can be brought to a high velocity by means of a particle accelerator. In an experiment at the CERN laboratories, muons that formed a beam were brought to a velocity of

$v = 0.9995\ c$

i.e. very close to the limiting velocity. Then, they circulated – guided by a magnetic field – in a so-called storage ring, similar to Lilly on the carousel but much faster. The mean lifetime of these muons can be measured easily by detecting the electrons that are formed in the decay process. This lifetime can be imagined as a kind of clock that moves along with the muons. This moving clock has to indicate less than a clock that is at rest in the laboratory, or in other words: the laboratory clock has to indicate more than the muon clock. While 2.2 µs pass between the formation of the muons and their decay, the time between these events must be greater in the laboratory. We calculate it by means of equation (8.1). Given are

$$T_b = 2{,}2\ \mu s$$

and

$$v = 0.9995\ c\,.$$

As we would like to calculate T_s, we transform equation (8.1):

$$T_s = \frac{T_b}{\sqrt{1-\frac{v^2}{c^2}}},$$

insert and obtain:

$$T_s = \frac{2.2\cdot 10^{-6}}{\sqrt{1-\frac{0.9995c^2}{c^2}}} = 69.6\ \mu s.$$

32 times as much time as for the „clock of the muon“ has passed for the laboratory clock.

Example: GPS

To determine a position, the GPS device (GPS = *Global Positioning System*) calculates to distance to several satellites from on the runtime of electromagnetic signals emitted by the satellites. To enable this, very accurate clocks are located both on Earth as well as in each satellite. However, as another world line is associated with the satellites than with any clock on Earth, the „satellite time“ would deviate more and more from the „Earth time“. Therefore, it is corrected continuously.

The runtime difference has two causes. We have already addressed one of these effects earlier (Section *7.10 Clocks in the gravitational field*). We will get back to it once again at a later time. At the moment we are interested in the second effect. It is connected to the fact that the satellite clock moves against the Earth.

We calculate the runtime difference that would accumulate on one day. In this case, the clock in the GPS device on the Earth corresponds to Willys watch in the previous example, the clock in the satellite corresponds to Lilly's watch. We need the following data:

radius of the satellite orbit: $r = 26{,}600$ km
turn around time 12 h = 43,200 s

Inserted in (8.3), we obtain:

$$T_s - T_b = 12\ \text{h} \cdot 0.83 \cdot 10^{-10} = 3.6 \cdot 10^{-6}\ \text{s}.$$

As the satellite performs two circular trips per day, the satellite clock would measure a value that falls short of 7.2 microseconds per day.

You might have questions about our method of calculating this time difference: Don't we have to consider that the clocks are also moving on Earth? Basically yes. The velocity of the Earth clocks, however, is much lower than that of the satellite. Its contribution to the overall effect is therefore very small and can be neglected.

In addition, we have pretended the clock to be free-floating on Earth in contrast to the satellite. But isn't it actually just the opposite? Shouldn't the satellite clock measure the higher value in that case? No. Here, we cannot apply our rule that the floating clock indicates the highest value; it only applies if there are no celestial bodies with their gravitational fields nearby.

Once again back to the other effect that distorts the rhythm of clocks. This „effect of altitude“ causes the satellite clock to measure 45.6 µs in excess per day. Its effect is therefore opposite to the „velocity effect“.

Hence, both effects in combination cause the satellite clock to indicate

$$45.6\ \mu\text{s} - 7.2\ \mu\text{s} = 38.4\ \mu\text{s}$$

in excess per day.

As the „error“ is known, it can be corrected easily by simply making the satellite clock run at an appropriate slower rate: if it were standing next to an Earth clock, it would indicate 38.4 µs less per day. Conversely, if it is located on board of the satellite, it will run synchronously to the Earth clocks.

Exercises

1. Neptune and Mercury have orbited the Sun for several billion years. Find the radii and turn around times of the two planets on the Internet. We imagine that clocks, which have the same structure and which run eternally, were installed 100 million years ago on both planets. Which time difference has accumulated during these last 100 million years? (Gravitational effects shall be disregarded.)
2. Two equal amounts of the same radioactive material are deposited at the same time on the Moon and on Earth. After a certain time, only half of the radioactive material still exists on Earth. What can you say about the amount of material on the Moon? Would a scientist on the Moon say that his material has a different half-life in spite of being chemically identical to the one on Earth?

8.5 Clocks at different altitudes – from another perspective

We have seen that the „incorrect pace“ of the clocks in the GPS satellite has two causes. It looks as if these causes were two fundamentally different phenomena.

We will now like examine one of these effects from a new perspective using our new knowledge about spacetime.

Once again back to the old story: a skyscraper; Willy and Lilly are located at half its altitude and compare their watches. Then, Willy goes upwards and Lilly goes downwards. After a certain time, they meet again halfway and compare what the watches indicate. It turns out that Willy's watch indicates more than Lilly's.

The deviation apparently had something to do with the gravitational field. But now we know a trick to get rid of the gravitational field; or better, to make its field strength become zero: describing the process in a free-float frame. How could that work in case of our skyscraper?

We need a third person, Lilly's sister Milly, and the whole process is described in Milly's reference frame. What does Milly have to do? As soon as Willy and Lilly have compared the indication of their watches, she vigorously jumps upwards; she flies very high and

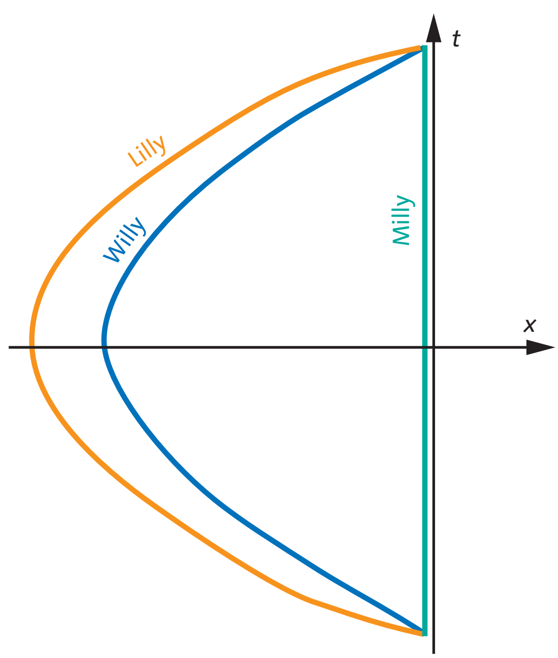


Fig. 8.15 World lines of Milly, Lilly and Willy. The longest time passes for Milly, the shortest one for Lilly.

will of course come back down at some time. She jumps as high as to land precisely at the moment in which Willy and Lilly compare their watches for the second time. Like any thrown stone or „jumping person", Milly is weightless; she flies or floats freely. Her reference frame is therefore ideal to describe the world because everything looks much simpler in this reference frame. In particular, the gravitational field strength is zero.

How does the world look like from Milly's perspective? Milly does not say that she flies upwards first and then comes back down, but the skyscraper with Willy and Lilly moves downwards and then comes back up, Fig. 8.15. (In the Figure „downwards" is leftwards.) In Milly's reference frame, Willy and Lilly each go on a trip, but on different world lines. On the largest part of her trip, Lilly is further away from Milly than Willy. We know what that means: When meeting the second time, Willy's watch displays more than Lilly's watch. And the highest value is indicated by Milly's watch because Milly does not move at all (in her free-float frame).

8.6 Simultaneity is no longer what it used to be

Everyone knows the meaning of „simultaneous". A bicycle fell over in Berlin and simultaneously a dog escaped in Stuttgart. The phrase seems to be clear. But we are by now used to surprises; and indeed, we will have another problem if we take spacetime seriously.

We make a short detour and try to turn things around, i.e. to interchange time and space. „Simultaneous" means „at the same time". Let us first ask for events occurring „at the same place". Hence, we replace

two events that occur at the same instant of time (simultaneously) at different places

by

two events that occur at the same place at different times

Example

Lilly travels on a high-speed train (ICE), wagon 8, seat 28. She plays a computer game. Event A shall be the start of the game, event B the end. Do the events occur at the same place?

- Yes, because they take place in wagon 8, seat 28.
- No, because Lilly was in Karlsruhe at the start of the game and in Mannheim at the end.

There is no contradiction. The question has a different answer depending on the reference frame. In the reference frame of the highspeed train, the events occur at the same place while they occur at different places in the reference frame of the stationary Earth. This will certainly not be questioned by anyone.

Now, we simply have to get used to the idea that simultaneity also depends on the reference frame. How can that look like? Here is another example:

The minute hands of the train station clocks in Karlsruhe and Mannheim advance simultaneously by one bar – in the reference frame of the Earth. In the reference frame of the train that drives northwards, the hand in Mannheim advances a bit earlier than that in Karlsruhe and vice versa for trains driving southwards.

Of course, the effect in our example is tiny again.

> Two events that occur simultaneously in one reference frame are not simultaneous in another reference frame.

Fig. 8.16 shows an analogous situation that will certainly not appear problematic to you either.

What can we learn from these examples? It is very simple: choose the most suitable reference frame for any situation that you wish to describe or any problem you wish to solve, i.e.:

The events „Lilly starts the computer game" and „Lilly ends the game" in the reference frame of the train (and not in the reference frame of the Earth).

The events „a bicycle falls over in Berlin" and „a dog escapes in Stuttgart" in the reference frame of the Earth (and not in a rocket that flies past at almost the limit velocity).

The arrangement of the floors in a skyscraper in Frankfurt with an altitude axis that points upwards in Frankfurt (and not with an axis that points upwards in Shanghai or Sydney).

Of course, each of these situations can also be described in an inappropriately chosen reference frame or coordinate system, but then things will become complicated.

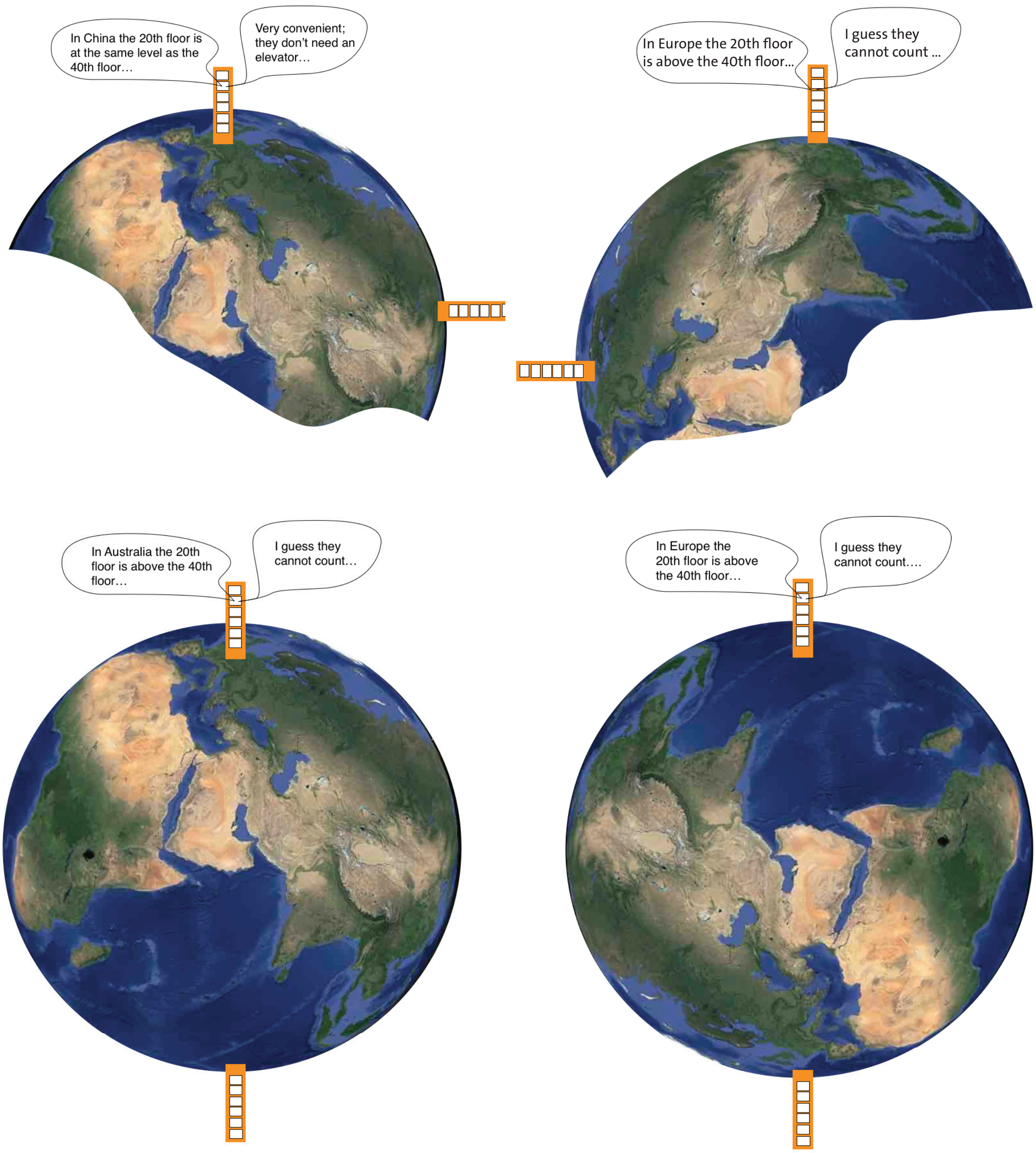


Fig. 8.16 The meaning of „top“ and „bottom“ depends on the perspective.

9 CURVED SPACE

9.1 Space – more than an empty recipient

We have dealt with the term "space" and discovered peculiar things: space is connected with time, and this is a way that we have not come across in our daily experience.

But we still had the impression of space and time only being needed to determine the coordinates of events. Space would be a sort of empty recipient in which the world's events take place.

In the following, we would like to understand that there is more to say about space and time. We will see in particular that the space has characteristics that change from one point to another, also in places where no matter exists. This means that it is not only similar to an empty recipient but also to an object.

But what exactly is space? The object or the recipient in which the object is located? Both terms are not suitable because space is both in one: an entity with properties (like an object) and the place where the object is located (like the inside of an empty recipient).

The theory we use to describe this entity is called Einstein's theory of gravitation, usually referred to as *general theory of relativity.*

9.2 Mass curves space – geodesics

The space around us has a characteristic we take for granted up to the point that we do not even imagine that it could also be different: it is „flat“. Actually, it is not flat everywhere. In the environment of heavy celestial bodies it is „curved“.

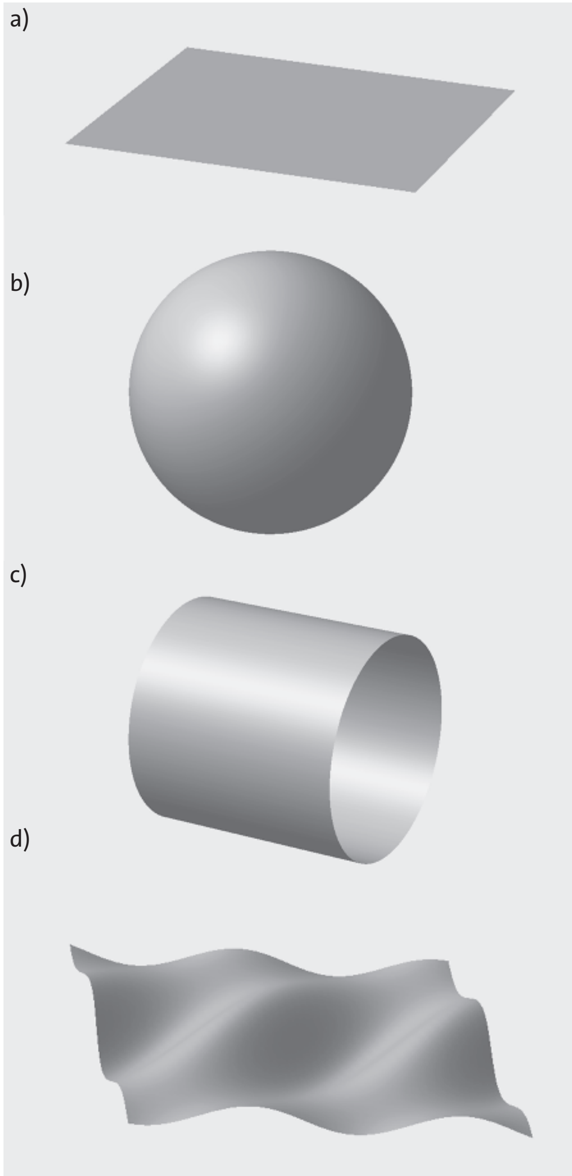


Fig. 9.1 Different, „two-dimensional“ worlds: (a) plane, (b) surface of a sphere, (c) cylinder surface, (d) hills and valleys

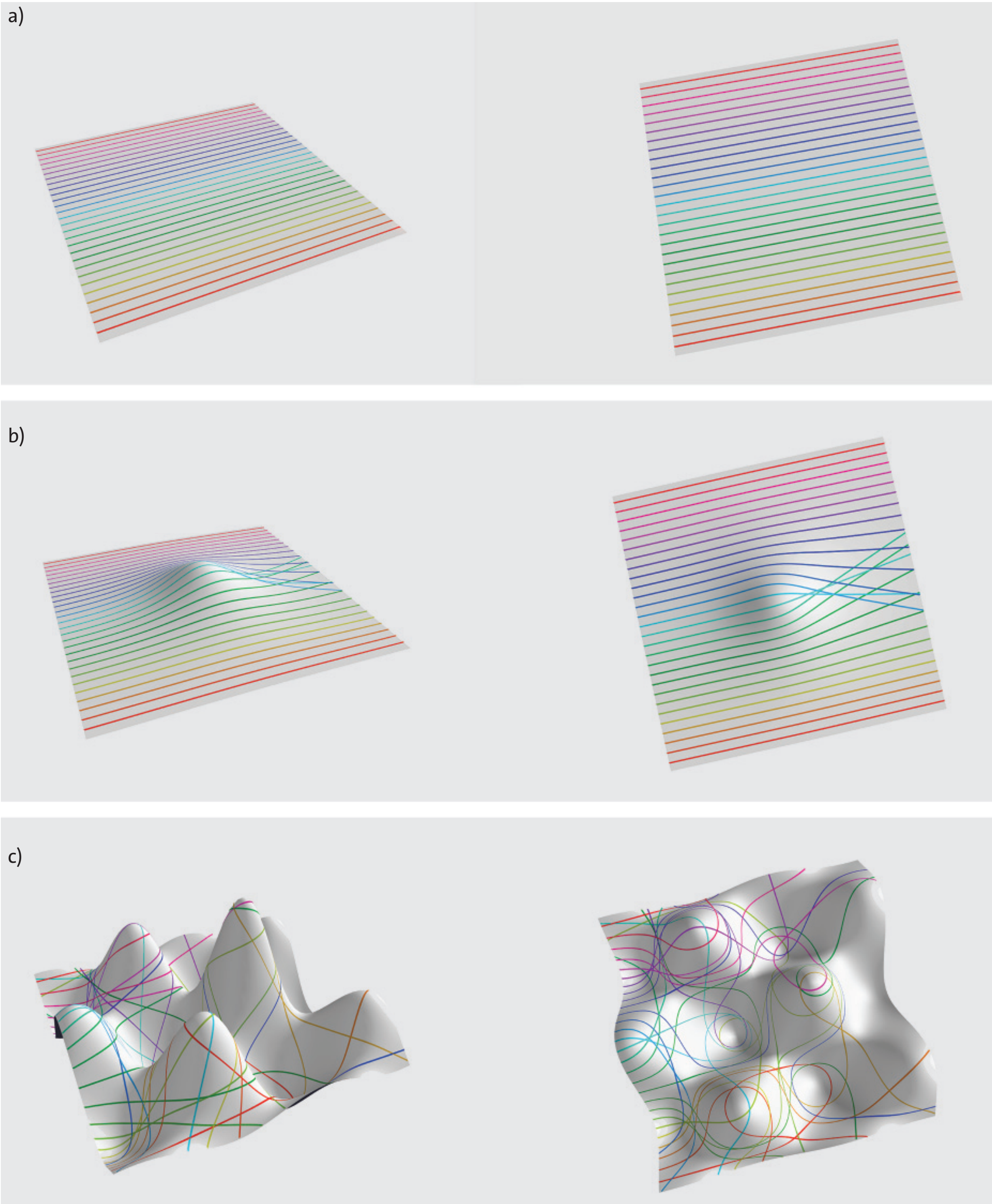


Fig. 9.2 Geodesics in a „two-dimensional world“. (a) flat world; (b) and (c) curved worlds

What do we mean by „flat space“ or „curved space“? It can be understood better if we imagine a two-dimensional world instead of the three-dimensional world we are living in: everything that exists is two-dimensional, including the beings that live in this world. We call them 2D people.

We could now imagine various different 2D worlds: even ones but also diverse types of arched or bent ones, Fig. 9.1.

As three-dimensional beings, we can tell from each of these surfaces if and at where and how strongly the surface is curved. We can see it because we have *embedded* the two-dimensional world in our three-dimensional world.

To familiarize ourselves with curved worlds, we need an important concept: the straight line.

But what is a straight line in a curved space? We ask our 2D people and they explain: you will get a straight line if you always drive straight ahead in a car, i.e. if the steering wheel is set to a straight direction and held. By the way, we can safely make real, three-dimensional people drive around in a car on the two-dimensional surface of the Earth for this purpose. Hence, we imagine a hilly landscape and always drive straight ahead with an off-road vehicle.

The line or route on which the car drives is the straightest line we can achieve on a two-dimensional surface. This is because we neither deviate to the right nor to the left. Hence, we drive on a „straightforward line“. The technical term for such a line is *geodesic*.

Fig. 9.2 shows geodesics in three different "landscapes", on the left side a perspective view and on the right side a view from above. These geodesics could possibly be trajectories of cars driving straight ahead and starting parallel to each other on the left edge.

The 2D people can now find out how the surface is curved without leaving their flat world, and would come to the same result if the third dimension were not to exist. How can they tell that their "space" is curved? By means of the geodesics: if two adjacent geodesics that start in parallel to one another do not remain parallel.

The world of Fig 9.2a is flat. We can see that the geodesics remain parallel to each other.

The pictures in the second row (Fig. 9.2b) show a landscape with a hill. Although the cars start in parallel to each other and always drive straight ahead, their routes do not remain parallel.

The landscape of Fig. 9.2c has high hills and valleys leading to a quite chaotic course of the straightforward lines (geodesics).*

However, the 2D people do not have the same notion of curvature as ourselves (i.e. three-dimensional people).

Fig. 9.3 shows a flat world that we (as 3D people) would describe as curved. The 2D people, in turn, do not classify it as curved because the geodesics that start in paralle on the left will remain parallel.

Finally, we look at a particularly simple two-dimensional world: the surface of a sphere (Fig. 9.1b). It essentially corresponds to the world we are living in, i.e. the surface of the Earth. Imagine that we start with two cars that are parallel to one another and drive always straight ahead. (There are no mountains, valleys and oceans and we can drive everywhere, i.e. we do not need any roads). More specifically: we start at the equator, each car drives northwards on a meridian. Of course, the car routes will not remain parallel; they will finally intersect at the North Pole. They intersect because the two-dimensional space in which they are situated, i.e. the surface of the sphere, is curved, Fig. 9.4a.

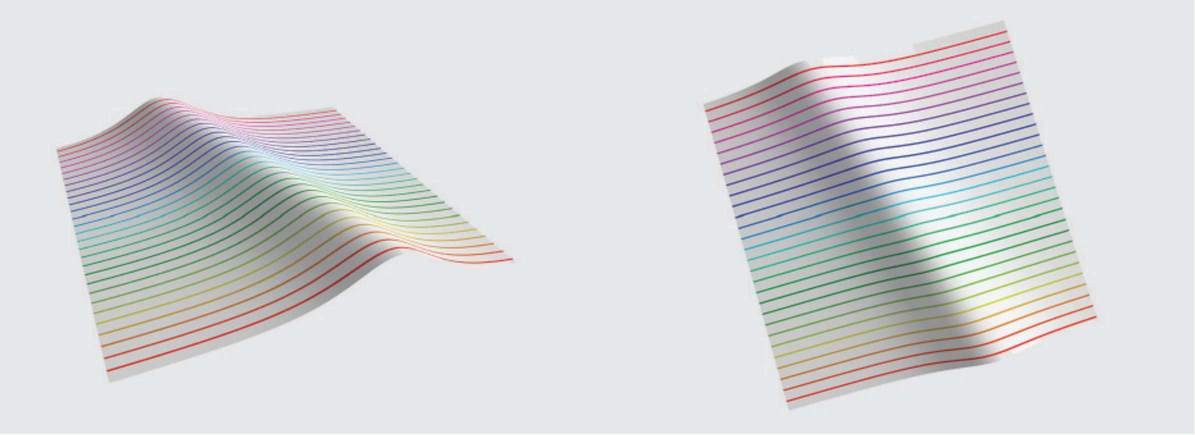


Fig. 9.3 The two-dimensional world is flat although its embedding is curved.

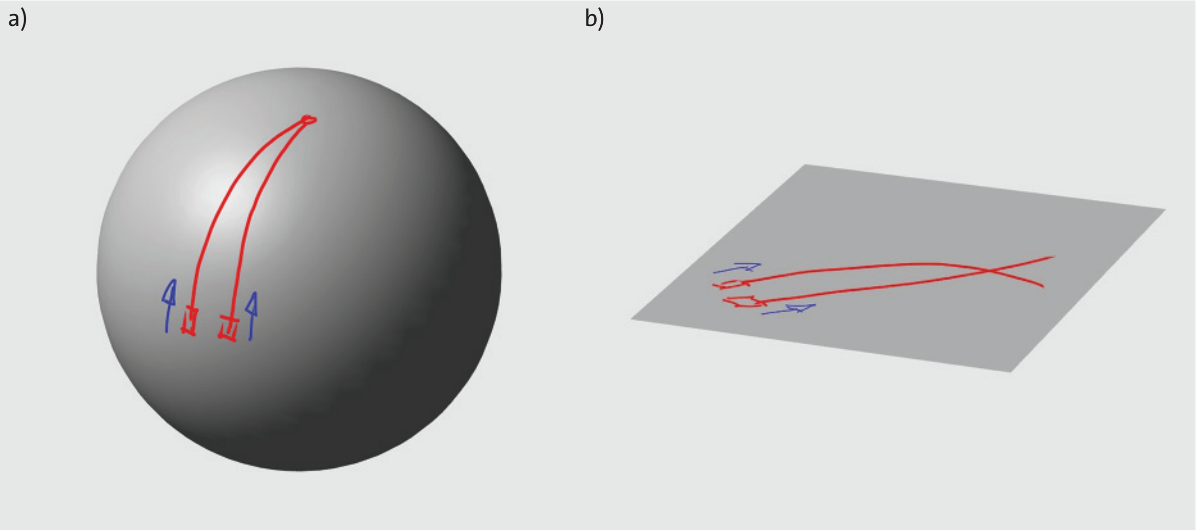


Fig. 9.4 (a) The initially parallel lines intersect because the „two-dimensional space“ is curved. (b) The initially parallel lines intersect because these lines are curved.

Also in Fig. 9.4b two lines that are initially parallel intersect at some point. They are situated in a flat space. They intersect because they are curved themselves. (Hence, they are no geodesics.) We find that the fact that two lines which are parallel at first, but do not remain parallel can have two causes: first, the space is curved and second, the lines are curved.

It is of course also possible that both the space as well as the lines are curved.

Everything we have found in this context applies for the three-dimensional space as well. Also here, there are „straightforward lines“ or geodesics. And if the space is „curved“, parallel geodesics will not remain parallel. If we were able to embed the three-dimensional space in a four-dimensional one, it would possibly be easier for us to get a clear idea of its curvature – but there is no fourth dimension of space, i.e. no embedding either. In any case, the following also applies for our three-dimensional space:

> Two initially parallel lines can intersect because:
> - the lines are curved;
> - space is curved.

* Here you can have geodesics drawn in a landscape defined by yourself: http://www.physikdidaktik.uni-karlsruhe.de/software/geodesiclab/a3.html

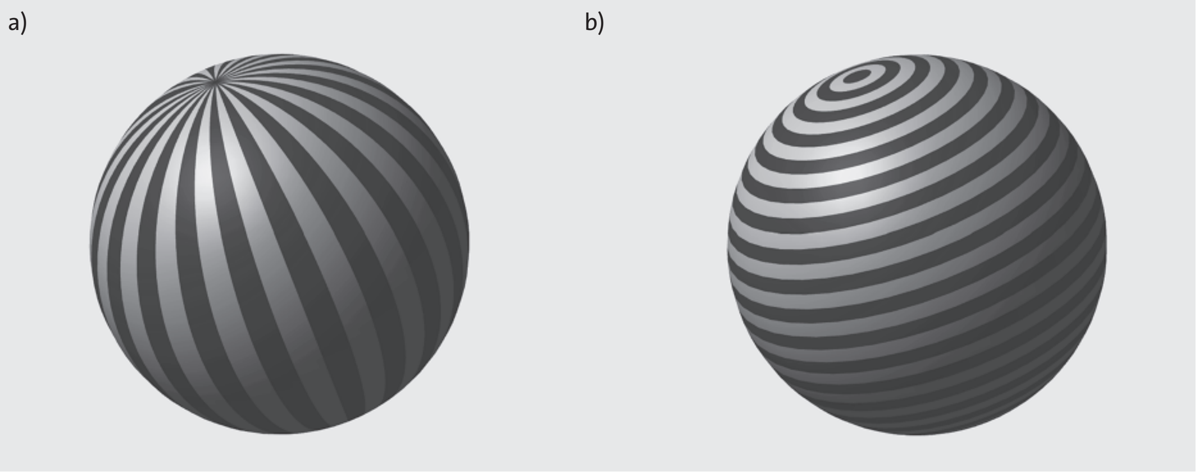


Fig. 9.5 For exercise 1

Exercises

1. Each of the black segments in Fig. 9.5 is limited by two lines. Do those lines intersect? If yes, why? If no, why not? Please answer the questions for both image sections.
2. Try to imagine a one-dimensional world. Can the inhabitants find a curvature of this world? How would the 2D people from the two-dimensional world comment the opinion of the 1D people?
3. One- and two-dimensional worlds are not very suitable living spaces. Try to imagine what the structure of living beings, their roads, houses, vehicles etc. would have to be like. What are the problems?
4. 2D people know the formula for the circumference of a circle: $U = 2\pi r$. They would like to know whether the formula also applies for very large circles, Fig. 9.6. They move away from M on a line that is straight for them. Upon arrival in point A, they turn at a right angle and continue to move on a curve whose points are located at a constant distance from M, i.e. they move on a curve with a constant curvature. After having traveled long enough, they return to A and find that measurement and calculation for the circumference of the circle have different values. (a) Explain how the discrepancy arises. (b) How will the discrepancy between the measured and the calculated value change if the 2D people choose increasingly large radii?

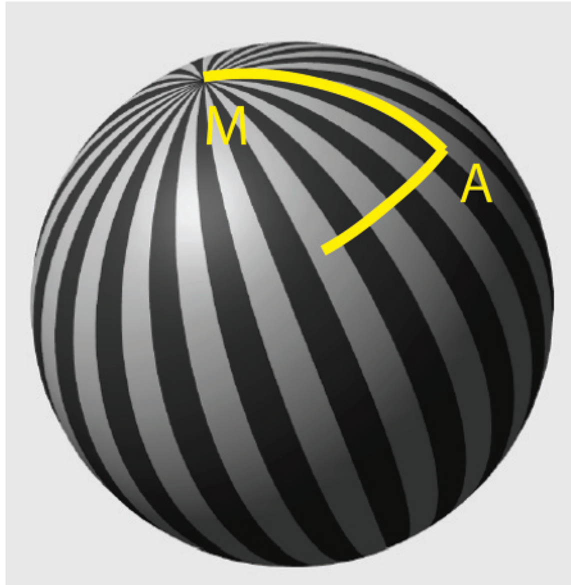


Fig. 9.6 For exercise 4

9.3 Space curvature in the environment of celestial bodies

In the previous section, we did not talk about physics but about geometry. So let's get back to physics.

Now, the question is whether the three-dimensional space we are living in is curved. The answer: it is curved; however, the curvature is very slight almost everywhere. It is only strong in conglomerations of energy/mass and in its environment. The curvature close to the Earth is still so weak that it cannot be detected. Things are different in close proximity to the Sun. There, the curvature – although still being very slight – can be measured. However, it is much stronger in the environment of neutron stars; and the space is downright crumpled in the close neighborhood of a black hole.

Space is curved by energy/mass.

But how can the curvature of our three-dimensional space be detected after all? In principle, it is simple: just the way we did in the previous section in the two-dimensional space: we choose two neighboring parallel geodesics and move forward on them. When their distance changes, the space is curved. But this is easier said than done.

And how can the geodesics be found in three-dimensional space? How does the „car", which drives always straight ahead, has to be in this case?

Every body K flies straight ahead as long as no momentum flows into it or comes out of it. And this is where the problem arises. We would like to determine the geodesics for the example close to the Sun, but precisely there, momentum flows out of the Sun into K. This is commonly called gravitational attraction. But we can get rid of it using a trick: by charging the Sun and the body K electrically, i.e. in a way that the result is repulsion. If attraction and repulsion just neutralize one another, K will receive just as much momentum through the gravitational field as it releases via the electric field. Of course, this can only be done in our mind, it is a *thought experiment.*

Consequently, K would now fly on a geodesic in three-dimensional space. Due to the space curvature, its orbit would be deviated in compared with a straight line in the case that space would be flat.

Once again: this method does not work in practice, but it is conceivable in principle.

Space curvature also manifests itself in another way. At first, we look at a large, cube-shaped region of space in which there is no celestial body; hence, the space in the cube is not curved, top of Fig. 9.7.

If the edge length of the cube is a, the cube will have a volume of:

$$V = a^3 .$$

Now we assume that there is a celestial body at the center of the cube, bottom of Fig. 9.7. The cube should be so large, i.e. the cube surface should be so far away from the celestial body, that the space is flat on the outside there; four of the cube edges are parallel sections of straight lines. Now the peculiarity: the volume of the space region within the cube is no longer equal to a^3 but larger. More can be packed into the cube than into the cube without the celestial body. We can therefore say:

> The volume of a region of space is enlarged by energy/mass.

Once again something that cannot be imagined? Not necessarily. We would find the same matter completely normal in a two-dimensional world. Here is a story that might be somehow unrealistic but that will help you understand our new theorem.

A farmer bought one hectare of land to create a meadow for his sheep. He would like to have more land but he cannot find anyone who would sell him some. But he has an idea: he enlarges the area of the meadow by piling up a hill on the purchased hectare, Fig. 9.8.

This is easy to imagine for us because we see the two-dimensional cultivation area „embedded“ in the three-dimensional space. Such an embedding, however, is not necessary in principle.

Exercises

1. The text of this section mentions a cube with a well-defined edge length. Depending on whether there is a heavy star within the cube or not, it has a different volume. Also a sphere could be examined instead of a cube. To calculate its volume we need to start from its surface. More can be packed into a sphere with a specific surface if there is a heavy body, i.e. for example a stone, inside it. Establish a formula that can be used to calculate the volume of an empty sphere if its surface area is given.
2. The square-shaped meadow in Figure 9.8 illustrates in a two-dimensional world the difference of the surface areas of a two squares with the same edge length. We can also choose a circle instead of the square-shaped edging. We then find that, in case of an identical circumference, the surface area depends on whether or not there is a hill in the circle. Illustrate this phenomenon in a sketch.

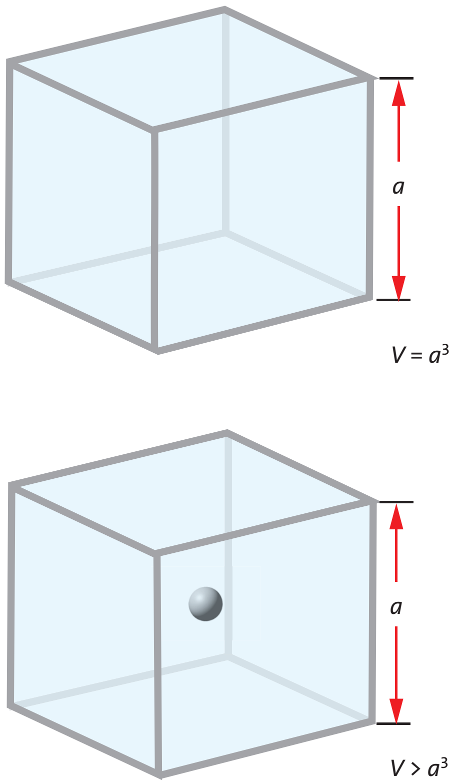


Fig. 9.7 The volume of the cube that contains a heavy celestial body is larger than a^3.

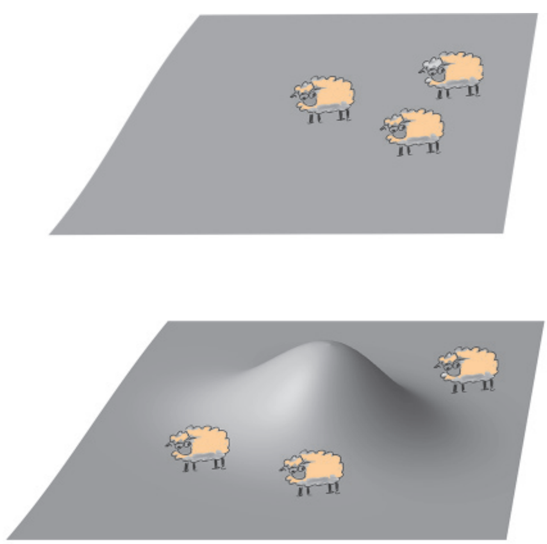


Fig. 9.8 The square-shaped meadow with the hill has a larger surface area than the one without a hill.

9.4 Trajectories in the gravitational field

We are interested in the trajectory of a light „object" in the vicinity of a heavy one, i.e.:

- of a planet that orbits the Sun;
- of a satellite in an orbit around the Earth;
- of light that comes from a remote star and that passes the Sun at a short distance.

We call the light body L and the heavy one H.

If the heavy body did not exist, L would move on a straight line. If we sent out two light bodies L_1 and L_2 instead of L, i.e. in a way that both of them start in parallel to each other with the same velocity, their trajectories would remain parallel to one another.

Now we include our heavy body H. Two things will happen:

- due to gravitation, the light body is deviated because momentum flows through the gravitational field from H to L. Its trajectory will be curved.
- The space will be curved.

Both effects contribute to the deviation of the trajectories. They will not remain parallel.

The first effect, i.e. deviation of the trajectory due to momentum transfer, depends on the velocity of L, the second effect does not.

Objects that fly by fast

We start with an „object" that flies closely past the Sun at a maximum velocity, i.e. at (almost) the limiting velocity. This object can also be light that comes from another star and that moves precisely at limiting velocity.

At first, we disregard space curvature. It is clear that the trajectory is curved because the Sun attracts the body or the light, Fig. 9.9.

In other words: the body or the light receives momentum from the Sun via the gravitational field. The deviation angle (in the radian measure) is

$$\alpha = 2\frac{G}{c^2}\cdot\frac{m}{r}. \qquad (9.1)$$

Here, G and c are universal constants (the gravitational constant and the limiting velocity).

We do not derive the formula because the calculation is a bit tricky. But the equation is plausible: the deviation is higher

- the greater the mass m of the central body;
- the shorter the distance r from the center of the central body.

This result is not yet complete because we pretended that the space was flat (not curved), i.e. that the geodesics were straight lines.

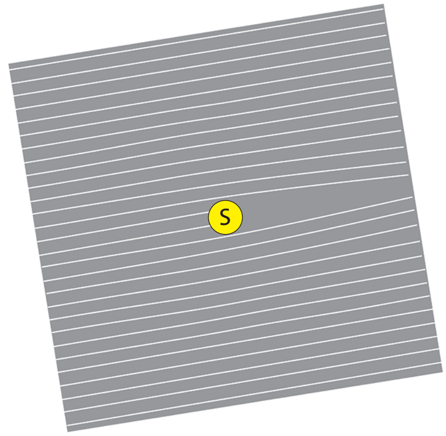


Fig. 9.9 This is how the light would be deviated if the space were flat.

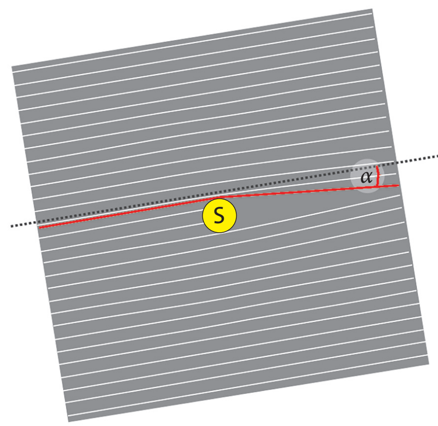


Fig. 9.10 Two effects contribute to the deviation of light from the straight trajectory:
• deviation due to momentum transfer from the Sun (bright lines);
• space curvature.

But the curvature of the space in the near surroundings of S alone already ensures that the geodesics are no straight lines. This entails a further contribution to the deviation, Fig. 9.10. The calculation (which is complicated again) shows that this second contribution is equal to the first one. Thus, the total deviation is twice that of equation (9.1):

$$\alpha = 4\frac{G}{c^2}\cdot\frac{m}{r}. \qquad (9.2)$$

We calculate this deviation angle for light that comes from a star and that moves closely past the surface of the Sun, Fig. 9.11.

We need the following data:
gravitational constant $G = 6.67 \cdot 10^{-11}$ m^3/(kg · s^2)
limiting velocity $c = 3.00 \cdot 10^8$ m/s
mass of the Sun $m = 1.99 \cdot 10^{30}$ kg
radius of the Sun $r = 696\,000$ km

Inserted in equation (2), we will obtain in the radian measure

$$\alpha = 4\frac{G}{c^2}\cdot\frac{m}{r} = 0.85 \cdot 10^{-5}.$$

Multiplied by (180°/π) we will obtain the angle in degrees:

$$\alpha = 4.87 \cdot 10^{-4} \text{ degrees} = 1.75'' \text{ (angular seconds)}$$

Hence, the effect is very weak.

You will wonder how the deviation of the light from the star that is located directly next to the edge of the Sun in the sky can be measured after all. This might sound confusing because the light of the stars is completely overshone by the Sun during the day. One therefore had to use a trick: measuring during a solar eclipse, i.e. when the bright light of the Sun is shielded by the Moon.

Such a measurement was performed for the first time in 1919, i.e. shortly after Einstein's publication of his theory of gravitation. That the light should be deviated had already been predicted before Einstein's publication, but at that time a deviation according to equation (1) had been expected. However, the measurement has resulted in twice that value, i.e. the one corresponding to equation (2). It thus confirmed that space is curved. At the same time the general theory of relativity was confirmed.

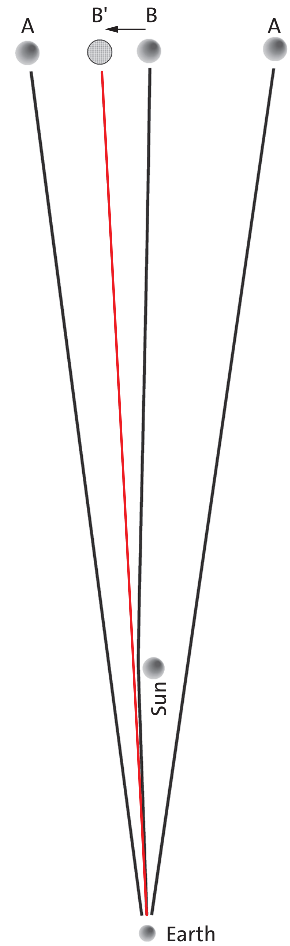


Fig. 9.11 The light of star B that passes closely by the Sun is deviated and hits the Earth. Therefore, star B seems to be located further on the left than what corresponds to its actual position. (The sketch is not drawn to scale.)

Trajectories of satellites, planets and Moons

We examined the trajectories of satellites, planets and Moons earlier. Such trajectories are ellipses or – as a special case – circles.

However, the trajectories are ellipses only if the space is not curved. Space curvature in the surroundings of the Sun and the stars is so weak that the trajectories of planets and Moons are actually a very close approximation of elliptical trajectories.

But the space is not exactly flat, and therefore there is a slight deviation from the elliptical form of the trajectories. Just as the trajectory of the light, which passes by the Sun, is additionally bent towards the Sun due to space curvature, the trajectories of the planets are also bent further towards the Sun. Where the planet is closer to the Sun this bending effect is stronger than where the planet is farther away. The result can be described as follows: an ellipse that rotates very slowly. It rotates in the same direction as the planet's direction of movement, Figure 9.12. Such a rotation of the elliptical trajectory is called *periphelion precession*. (The *periphelion* is the point of the elliptical trajectory that is closest to the Sun; the point at the largest distance to the Sun is called *aphelion*.)

In case of the planets of the Sun, the periphelion precession caused by space curvature is very weak and can only be observed for the planets that are closest to the Sun: Mercury, Venus, Earth and Mars. It is strongest (but still very weak) for Mercury. The large axis of the Mercury ellipse rotates by 43 angular seconds per century. (In fact, ellipses also rotate for another reason: because the trajectory of every planet is disturbed by the other planets. Hence, the 43" are only the contribution caused by space curvature.)

The effect of periphelion precession is much larger for celestial bodies that curve the space more strongly: in the environment of neutron stars and black holes.

Exercises

1. Calculate the deviation of light, (a) that passes by the edge of a neutron star with the mass $m = 3 \cdot 10^{30}$ kg and a radius of $r = 10$ km. (b) That passes by the edge of the Earth.
2. We look at a close celestial body. It appears to us as a disc. (a) When the celestial body is the Sun or the Moon, we see less than half of the surface. Explain. (Help: the effect becomes stronger the closer we are to the celestial body.) (b) When the star is a neutron star, we see more than half of its surface. Explain.

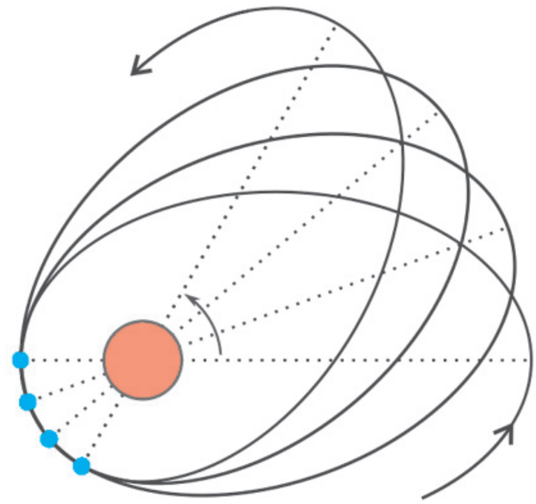


Fig. 9.12 Trajectory of a light celestial body in the surrounding of a heavy one. If space were flat, the trajectory would have an elliptical shape. Due to space curvature, it becomes similar to a rotating ellipse. The effect is very weak for the planets of the Sun.

9.5 The Schwarzschild radius

There is not much to understand in this section as it is only about simplifying an expression.

Once again back to equation (9.2) that tells us how a light beam is deviated by a celestial body with the mass m:

$$\alpha = 4\frac{G}{c^2} \cdot \frac{m}{r}.$$

Now we introduce an abbreviation:

$$r_S = 2\frac{G}{c^2} \cdot m. \qquad (9.3)$$

Hereby, the equation for the deviation angle is simplified:

$$\alpha = 2\frac{r_S}{r}.$$

We can see that the new quantity r_S is measured in meters.

This quantity is called *Schwarzschild radius*. As G and c are natural constants, r_S is nothing else than a measure for the mass – just expressed in meters.

We insert the values of G and c in equation (9.3) and obtain:

$$r_S = m \cdot 1.48 \cdot 10^{-27} \text{ m/kg}.$$

But why is r_S introduced in the first place? Because the term on the right side of equation (9.3) is part of many other formulas and those formulas become clearer when the term is abbreviated as r_S. In addition, r_S has another special meaning in connection with black holes – but this will be addressed later.

body	mass	Schwarzschild radius
book	0.5 kg	$0.74 \cdot 10^{-27}$ m
Moon	$7.35 \cdot 10^{22}$ kg	0.11 mm
Earth	$5.97 \cdot 10^{24}$ kg	9 mm
Sun	$1.99 \cdot 10^{30}$ kg	3 km

Table 9.1 Examples for Schwarzschild radii

Schwarzschild radius = mass $\cdot$ $1.48 \cdot 10^{-27}$ m/kg

To get an idea of typical values of the Schwarzschild radius, Table 9.1 shows a few examples.

9.6 Temporal and spatial intervals

We know that clocks in the gravitational field have a different pace depending on where they are located. We have seen that, in close proximity to the surface of the Earth, the higher clock runs faster than the lower one.

The simple formula

$$\Delta t = \sqrt{1 - \frac{r_S}{r}}\,\Delta t_0 \qquad (9.4)$$

applies for a spherical celestial body.

It is valid everywhere outside of the celestial body. Δt_0 is the time interval between two events measured at a long distance from the celestial body. Δt is the time interval of the same events that we measure when we are located at the distance r from the center of the celestial body. In other words: time passes more slowly at a close distance to the celestial body. r_S is the Schwarzschild radius. Normally, i.e. at the surface of the Earth or the Moon or also of the Sun, r_S is much smaller than r, see Table 9.1. This means that the root in equation (9.4) is almost equal to 1, and this means in turn that the two time intervals are practically equal – just according to what we know from everyday life.

We have already addressed the slight deviation that will arise earlier in connection with the GPS.

But the difference can also become large. Radius and mass of a typical neutron star are

$$r = 10 \text{ km}$$
$$m = 3 \cdot 10^{30} \text{ kg}\,.$$

Therefore, we obtain the Schwarzschild radius

$$r_S = 4.5 \text{ km}\,.$$

Inserted in equation (9.4) we obtain:

$$\Delta t \approx 0.75\,\Delta t_0\,.$$

Willy, who lives on the surface of the neutron star, does of course not notice anything about the time running more slowly at his location; rather, he finds that the clocks far away, i.e. where Lilly lives, run much faster. When Lilly sends him two short light signals with her laser at an interval of 1 minute, only 45 seconds pass at Willy's place between the arrival times. And Lilly sees that Willy's watch runs more slowly than her own one. By the way, Willy uses a second watch besides his normal one, which runs faster so that it displays Lilly's time, i.e. Lilly's local time. And Lilly also has a second watch besides her own, normal watch, which runs more slowly so as to display Willy's local time.

You have certainly noticed that, once again, this story is completely unrealistic. Willy can under no circumstances stay on the surface of a neutron star, no matter how well he protects himself against radiations and high temperatures. He would be crushed by the strong gravitational field because of the field strength, see equation (4.7) in chapter 4:

$$g = G \cdot \frac{m}{r^2} \approx 2 \cdot 10^{23} \text{ N/kg}.$$

> Seen from outside, the time close to a heavy body passes more slowly than at a long distance from the center:
>
> $$\Delta t = \sqrt{1 - \frac{r_S}{r}}\,\Delta t_0$$

Equations (9.4) is tricky for another reason.

We look at a circle whose center matches the center of the central body.

In our non curved world, the relationship between the circumference U and the distance r to the center (which is called radius) is

$$r = \frac{U}{2\pi}.$$

In the vicinity of a heavy celestial body, this does not apply anymore. Here, the distance to the center is larger. We will call it ρ. Hence, we have:

$$\rho > r\,.$$

Attention: ρ is not the radius of the circle.

To read the equations (9.4) and (9.5) correctly, you have to know that they contain r and not ρ. The equation does not contain the distance to the center but the circumference of a circle divided by 2π.

Exercises

1. Willy is on a neutron star, films a video and sends it to Lilly who is far out. What does Lilly see in the video? (We assume that Willy is not harmed by the strong gravitational field.)

9.7 Black holes

As the influence of the mass/energy on space and time is particularly dramatic in the vicinity of black holes, we will to address these objects in greater detail.

The event horizon

So far, we have considered r_S to be not more than a measure for the mass of a body or a quantity that helps us simplify some equations. Now we will see that the Schwarzschild radius gets a very specific meaning. A black hole is formed when a star, whose core fuel has been consumed, shrinks and its radius approaches the Schwarzschild radius r_S.

Let's start with a side note on this matter: earlier (chapter 4, equation (4.7)) we saw that the gravitational field strength for a spherically symmetric celestial body can be calculated with the equation

$$g(r) = G \cdot \frac{m}{r^2}. \tag{9.6}$$

Here, G is the gravitational constant, m the mass of the body and r the distance from the center.

The formula only applies for the area outside of the body. If we imagine that the entire mass is concentrated in one point, the equation – with the exception of the point – would apply everywhere. But if we approach this point, i.e. if r approaches zero, the field strength will approach infinity.

Back to the black hole. Also here, the field strength increases when we approach the center; however, it already approaches infinity further outside, namely at the *event horizon*. The event horizon is the spherical surface for which

$r = r_S$.

We can already tell from equations (9.4) and (9.5) in the previous section that something must happen at this distance from the center. When r becomes equal to the Schwarzschild radius, we get $\Delta t = 0$ and $\Delta s = 0$. For outsiders, time stands still at the event horizon and the vertical length of all objects becomes zero.

In addition, also the gravitational field strength approaches infinity. This means that every object that has come close to the event horizon is attracted by the black hole and cannot come back outwards anymore: no body, no particle and no light either.

> No body, no particle, no light can leave the event horizon or cross the event horizon from inside.

The black hole seen from outside

As time at the event horizon comes to a halt compared to the time far outside, a body that is captured by the black hole or that falls towards the black hole becomes increasingly slow until it stops completely at the event horizon. For this reason, a black hole is also referred to as a *frozen star*. We as outsiders never experience that something that falls towards the center of the black hole actually falls into the black hole. Already during formation of the black hole, when the matter of a burnt-out star moves in an inward direction, we can only see the state of this matter shortly before reaching the event horizon. Hence, we can see the past – up to the time when the black hole was formed.

However, „seeing" bears yet a problem because in order to see something, light has to come to the outside. The wavelength of light that moves „upwards" in the gravitational field becomes longer while the frequency becomes lower. Light that comes from the event horizon has an infinitely long wavelength or the frequency zero, which means that no light comes to the outside from the event horizon anymore. For this reason, a black hole appears as black. For us, the event horizon is to a certain extent a border of the world. The area „behind" the event horizon is cut off from the rest of the world.

The outside world seen from the black hole

We imagine to be in close proximity but outside of the event horizon. Here, exactly the opposite of what we have just found for the observation from outside applies. From this perspective, the time outside will run faster, i.e. infinitely fast, if we approach the event sufficiently. Therefore, we would see infinitely far into the future of the outside world. But also here, we have a problem with seeing because the „light" that comes from the outside has an infinitely short wavelength. Instead of visible light, there is UV light, X-rays or even gamma radiation depending on our distance from the event horizon – but we have already been aware that the close environment of the event horizon is not a very comfortable place.

9.8 Gravitational waves

That space is more than only room for something can be seen particularly clearly by the fact that *gravitational waves* exist: distortions of space that move through the space similar to a change of the density that runs through the air in case of a sound wave.

Gravitational waves move at the same velocity as electromagnetic waves, i.e. at the limiting velocity c.

Just as electromagnetic waves are created by oscillating electric charge or sound waves by a vibrating speaker membrane, gravitational waves are formed by the oscillation of mass, i.e. of bodies.

In case of movements of bodies under terrestrial conditions, the created waves are extremely weak. To create waves that can be detected, huge masses have to oscillate very fast. Actually, such processes occur in the universe.

Two stars that orbit around each other form a binary star system. Due to their movement, they radiate gravitational waves. However, in „normal“ binary star systems, this radiation is still so weak that we are unable to detect it. Things are different for stars that are heavy and small so that they can move around each other at a very short distance and therefore at a high velocity. This is the case for binary star systems whose partners are white dwarfs, neutron stars or black holes. We can observe that the rotational period gradually becomes slightly shorter and that the distance of the two stars becomes smaller. This shows that the system loses energy. This energy moves away with gravitational waves.

Here is an example that shows how small such effects are:

The two white dwarfs J065133.338 and 284423.37 circulate around each other. Their masses are 0.26 and 0.5 Sun masses, respectively. The rotation period is 12.75 minutes. It can measured from the Earth that the rotation period decreases by 310 microseconds per year. The waves themselves, however, are still so weak that they cannot be detected.

But there are events in the universe in which waves are created that are strong enough as to be directly detectable on Earth. Two black holes that move around each other gradually lose energy, their distance becomes smaller and smaller and their rotational period becomes shorter and shorter. Thereby, the radiation becomes stronger. But there is an end at some point: the black holes merge. During the last rotations that take place at a very high velocity, a strong gravitational wave is emitted. It is strong enough as to be measurable on Earth, even though the event took place at a distance of 10^9 light years, i.e. in a remote galaxy. (Have you noticed that we said „took place“ and not „takes place“? A distance of 10^9 light years means that the event happened far in the past.)

How does a gravitational wave look like when it reaches us? It is a distortion of space: distances between two bodies are being changed: increased or reduced. How this happens is shown in detail by the animation of Fig. 9.13. (You need an Internet connection to view the animation.)

Reading the figure correctly is quite a challenging task. The wave moves perpendicularly to the drawing plane. We are at the center and examine the distances to bodies that are arranged in a circle around us. The wave causes the distances to increase and decrease periodically. If they increase in one direction, they will decrease in the respective orthogonal direction. We can also say that the volume of a space remains constant while the wave passes. Fig. 9.14 shows the process in three dimensions. Here, also the progression of the wave can be seen.

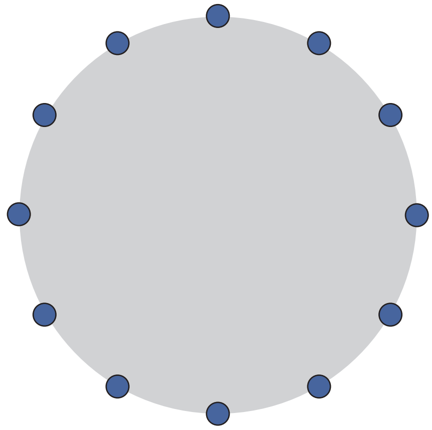


Fig. 9.13 Seen from the center, the distances of the blue bodies increase and decrease. (The wave moves perpendicularly to the drawing plane.) See the video on the website: http://www.physikdidaktik.uni-karlsruhe.de/kpk/Videos_linked_from_KPK_books.html

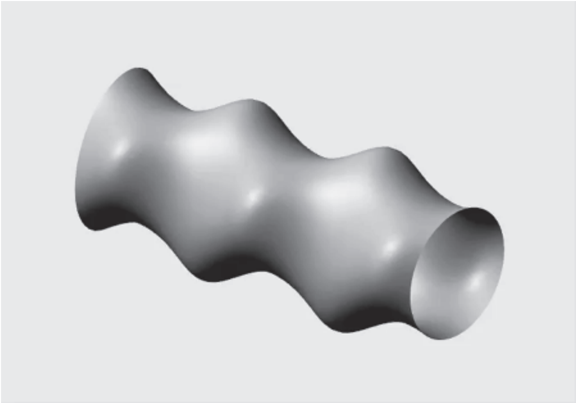


Fig. 9.14 Deformation of the space seen from the central axis. See the video on the website: http://www.physikdidaktik.uni-karlsruhe.de/kpk/Videos_linked_from_KPK_books.html

What is shown in Fig. 9.13 does of course not only apply for the points on the central line. We can move the axis in parallel to any other place and draw a „party balloon“ precisely this way.

It is clear that the image is a drastic exaggeration of the proportions. The elongation or compression of the space is much smaller in reality: the distance of two bodies at a spacing of 1 m typically changes by 10^{-22} meters. For the waves that can nowadays be detected directly, the wavelength amounts to more than 1000 km.

Gravitational wave: space is stretched and compressed periodically in the direction perpendicular to the direction of the propagation of the wave. The distance between two bodies changes accordingly.

Exercises

1. Compare the wavelength of red light with that of a gravitational wave mentioned in the text above. (Calculate the ratio of them.)
2. A gravitational wave (perpendicular to the drawing plane of Fig. 9.15) is passing by Lilly and Willy. The figure shows a snapshot of an instant of time when horizontal distances are just increasing. Willy says, that points A and B are receding from him. Lilly asserts that B and C recede from her. How does that go together?

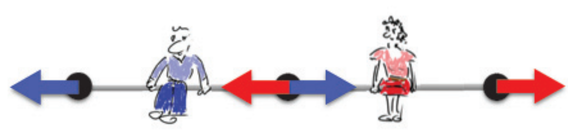


Fig. 9.15 For exercise 2

10 COSMOLOGY

10.1 The stars in motion

Cosmology addresses the structure and the development of the *universe*.

The night sky creates an ambience of peace and stability. The stars always seem to be at the same place and to shine with constant intensity.

In fact, however, the universe undergoes a constant evolution: the stars move against each other. New stars are formed and others disappear, explode or fall into black holes. Also the galaxies move, rotate and collide with other galaxies.

For example, the Sun moves at 220 km/s on an orbit around the center of the Milky Way. (For reference: the Earth orbits the Sun at 30 km/s.)

If we observe the sky on several days and compare the position of the celestial bodies, we will find that the Moon and the planets move against the background of the stars. The stars apparently do not move on the other hand. Why can't we see anything of the movement of the stars? Because they are too far away. An airplane in the sky seems to move very slowly although its actual velocity is approximately 800 km/h. We perceive a movement as more slowly the longer the distance between us and the moving object. What we perceive is the change of the direction in which we see the object. We can also say that we perceive the angular velocity

$$\omega = \frac{v}{r}$$

and, at a given velocity v, this angular velocity decreases with a growing distance r to us.

We cannot see any other drastic cosmic events on the night sky because they do not happen often enough in our closer neighborhood. The next quasar is (or was?) several billion light years away from us and could therefore not be seen with the naked eye.

Even more interesting but certainly not visible with the naked eye is another process: the universe expands. It seems as though the galaxies moved away from us. We will examine especially this phenomenon in the following.

But for now, something else about the method.

We will be dealing with very long distances in the following. Thereby it is useful to not indicate distances in meters but in a much larger measurement unit, the light year (ly). A light year is the distance that light travels in one year. In meters we have:

$$1\text{ ly} = 9.461 \cdot 10^{15}\text{ m.}$$

Besides the light year, also the units light hour, light minute or light second are sometimes used. Here a few examples for cosmic distances:

• Earth — Moon	1.3 light seconds
• Sun — Earth	8.3 light minutes
• diameter of the solar system	150 light hours
• distance of the star that is closest to the Sun (Proxima Centauri)	4.2 ly
• diameter of the Milky Way	100 000 ly

We would even like to continue: distance of the next quasar, etc. But in doing so, we would already come across the first difficulty: what exactly do we want to understand by distance: the distance at the instant of time at which it emitted the light that is now reaching us; or the distance from us that it has today in case it still exists?

10.2 The cosmological principle

A fundamental rule that has proved correct is the cosmological principle. It states that we are not located at any special place of the universe. This includes:

- The Earth is not the center of the world as was assumed at earlier times. Rather, the world neither has an edge nor a center.
- The Earth is not a unique planet but rather one of many similar planets that orbit other stars.
- The Sun is not a unique star, but one of countless other similar stars.
- Our Milky Way in not a special galaxy but one of countless other similar galaxies.
- Our galaxy cluster is not a special cluster but one of countless other similar galaxy clusters.

The galaxy clusters are the largest structures that exist in the universe. Now we would like to imagine a (huge) area, which contains a very large number of galaxy clusters, to be cut out of the universe – and then also a second one at a different place of the universe. We can now say that these two areas look alike on average.

In other words, we can say that the universe is homogeneous „on a large length scales".

This characteristic of the universe is comparable to that of a gas – for example of air. A portion of air in a cube with an edge length of 1 cm can at first not be distinguished from the air in an adjacent cube: in both cubes, the air has the same density, the same temperature and the same pressure. Only if we examine the air in a strongly enlarged view, we can see that the nitrogen and oxygen molecules fly around completely irregularly and that „the world" looks different in each place. Thus, the air is already homogeneous in a cube with a volume of

$$1\ \text{cm}^3;$$

in case of the universe, we need to examine a cube with an edge length of at least

$$10^8 \text{ light years.}$$

> The universe is homogeneous on large length scales.

We could assume that we are not located at a particular place on the time scale either, i.e. that, on average, the universe has been at all times what it is today. But this is not true as we will see in the following.

10.3 Curved or not curved?

We have seen that the space in the neighborhood of heavy celestial bodies is „curved". Beyond this neighborhood, it is „flat". Our daily experience confirms that it is also flat at a larger distance of heavy celestial bodies. But we have to expect space to be only approximately flat. We know: curvature is caused by energy (= mass). The universe contains energy, i.e. we could expect it to be curved, albeit only to a very limited extent. So is it curved or not?

Before answering this question, we would like to upgrade our previous observations. In section 9.3 we found that the volume of an area of space with a given surface will increase if a celestial body with a large mass is put inside. We had chosen a cube-shaped area of space. Instead of the cube, we will now choose a spherical region in order to simplify the arguments. In addition, we imagine the sphere to be very large so that many galaxies can find space in it.

Of course, we also expect the volume of the sphere to be slightly „too large" in this case. In the normal, flat space, the volume of a sphere is calculated according to

$$V = \frac{4}{3}\pi r^3.$$

In our case, where the spherical region contains matter, the volume should be slightly larger:

$$V > \frac{4}{3}\pi r^3.$$

Attention! We have already seen earlier that r does not stand for the distance from the center, but for the circumference of a large circle divided by 2π, see section 9.6.

We would now also like to imagine the possibility that the mass of what is located in the spherical region of space is negative. The assumption might seem absurd at first, but who knows? There is nothing wrong about a thought experiment. In this case, we would expect the volume of our sphere to be smaller than the volume of a sphere in flat space:

$$V < \frac{4}{3}\pi r^3.$$

In any case, we would like to address the question without prejudice: how is the curvature of space? Does it cause the volume to be too big? Or rather too small?

As a matter of principle, there are two ways of answering this question:

1. If we know exactly what exists in the sphere, we can calculate (using Einstein's theory) how space is curved;
2. by simply looking it up.

For the time being, the calculation does not quite work because there is still some confusion about the content of the sphere. The problem arises due to the fact that there are not just the visible celestial bodies besides the so-called dark matter that can also be seen indirectly. What is sometimes referred to as an empty space is not as empty as it seems. This means that the idea of the negative mass cannot simply be discarded.

Hence, we use the other method and look things up. But there is a problem, too: we need to calculate distances of galaxies very accurately to be able to detect even small deviations from the flatness. The present result of such measurements is astonishing: the universe is flat.

> On large length scales, the universe is flat.

You might say that you have always imagined it to be this way. Correct. But after having learned that space can be curved not only in principle but that it is actually curved, namely in the vicinity of every heavy celestial body, the observation of the entire universe being flat is rather unexpected and requires an explanation. In fact, it is currently one of the major open questions of physics.

10.4 The expansion of the universe

We have seen that the spatial structure of the universe is very simple:

- the universe is the same everywhere (on large scales)
- the universe is flat (as far as we are now able to say with our current measurement accuracy).

In another respect, however, it is not simple at all, namely with regard to its evolution in time: the universe expands.

Does this mean that it grows bigger and bigger? Saying this would be thoughtless. We do not know how large it is, and in case it were infinitely big, we could not just say that it would grow even more. So what does it mean to say that the universe expands?

We look at an elastic rope that expands or that is stretched, Fig. 10.1. We are not interested in knowing who is pulling on it, whether anyone is pulling on it at all or wherever and why it expands.

The rope has knots A, B, C... at equal spacings so that we can see how it stretches.

Willy, Fig. 10.1a, is at knot D. What does he see? He sees that the neighboring knots C and E move away from him, and he sees that the knots B and F move away twice as fast as C and E, the knots A and G three times as fast, and so forth. Figure 10.1b shows the same rope once again: from the perspective of Lilly who is at knot E. How does she perceive the environment? For her, knot E is at rest, the knots D and F move outwards, C and G move twice as fast and so forth.

We can see: each of the two thinks of his/her place as the center of the world and believes that the rope expands from this place to the left and the right. Anyone else who is standing at any other place of the rope has the same perception, too.

Now you can understand what we mean by saying that the universe expands. Seen from any position, the stars and galaxies appear to move outwards, and the further they are away, the faster they seem to move.

However, „movement“ is not quite the right expression in this context. The reason for the increasing dis-

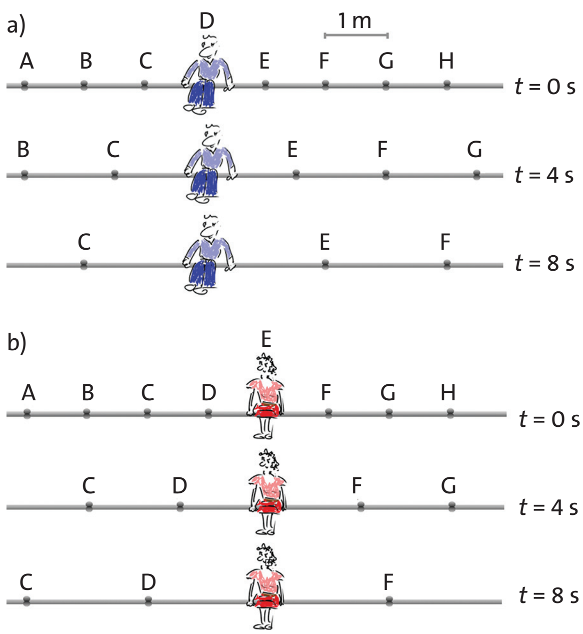


Fig. 10.1 The rope stretches. For Willy, the knots A, B and C move to one side while the knots E, F, G and H move to the other side (a). For Lilly, the knots A, B, C and D move to one side while the knots F, G and H move to the other side (b).

tance is simply the fact that new space is being formed between the galaxies.

> The universe expands: new space is formed everywhere.

This sounds interesting. Does this mean that someone's piece of land would have grown after a year? Could we possibly make a new business model out of that? No. Although space expands, everything that is located inside that space and that is connected in some way – i.e. that is tied together by some fields – keeps its size: all objects on Earth, the Earth itself, the solar system, our galaxy, our galaxy cluster. Only the distances between the galaxy clusters grow.

We can imagine it to be as follows: Willy and Lilly sit at a short distance from one another on the expanding rope, Fig. 10.2a. Seen from Willy's position, Lilly is moving; from Lilly's perspective, Willy is moving. The velocity at which Willy moves away from Lilly, or also Lilly from Willy, shall be called expansion velocity. In Fig. 10.2b, the two are holding hands. Now they are no longer moving away from each other. The rope slides away under them. We could also put it this way: in order not to move away, they have to move relative to the rope, i.e. in a way that the expansion velocity is just being compensated.

In case of the universe, the distances within all structures up to the galaxy cluster do not increase due to the expansion of the space. But how can the expansion still be seen then? By observing the distances between the largest structures of the universe, i.e. the galaxy clusters.

We look at a point at the distance d (from us). The velocity at which the point seems to move was called expansion velocity before. If the distance changes by Δd in the time interval Δt, the expansion velocity will be

$$v_e = \frac{\Delta d}{\Delta t}.$$

For Willy in Fig. 10.1, the expansion velocity of the knots E and F is:

$$\text{point E: } v_e = \frac{\Delta d}{\Delta t} = \frac{0.5\text{ m}}{4\text{ s}} = 0.125\text{ m/s},$$

$$\text{point F: } v_e = \frac{\Delta d}{\Delta t} = \frac{1\text{ m}}{4\text{ s}} = 0.25\text{ m/s}.$$

We can see that v_e is proportional to the distance of the knots:

$$v_e \sim d\,.$$

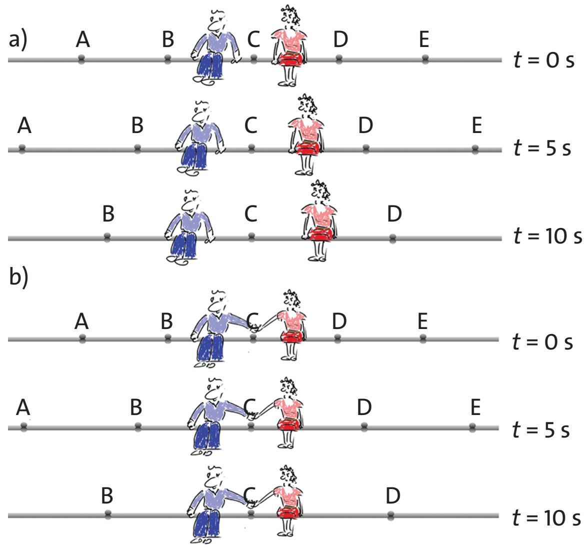


Fig. 10.2 (a) Willy and Lilly move away from each other at the expansion velocity. (b) The rope moves away under Willy and Lilly. Willy and Lilly move relative to the rope.

In this case, the proportionality factor is 0.25 s^{-1}. We therefore obtain

$$v_e = 0.25\text{ s}^{-1} \cdot d\,.$$

We apply the equation to the universe.

Also here, v_e is proportional to d. The proportionality factor in this case is denominated with H and referred to as *rate of expansion* of the universe:

$$v_e(d) = H \cdot d\,. \qquad (10.1)$$

We have

$$H = \frac{2.1\text{ m/s}}{100\text{ ly}}.$$

In words: each time we move forward by 100 light years, the expansion velocity increases by 2.1 m/s.

> Rate of expansion: $H = \dfrac{2.1\text{ m/s}}{100\text{ ly}}.$

We have to distinguish the real movement of the individual stars and galaxies in diverse directions from the expansion movement. But the farther a galaxy is away from us, the lower the relevance of this real movement in relation to the expansion movement.

The expansion of the universe has interesting consequences. We would like to address two of them.

Expansion velocity higher than limiting velocity

Fig. 10.3 shows the linear relationship between the expansion velocity and the distance. (See also equation (10.1).)

You might make a disturbing observation: for distances that are longer than $14 \cdot 10^9$ light years, the expansion velocity becomes higher than c, i.e. higher than the limiting velocity. Can that be correct? Yes, don't worry. c is only the limiting velocity for real movements, not for expansion movements during which new space is formed.

The Big Bang

From today's expansion we can „calculate back“ and it becomes evident that there must have been some sort of start. $13.8 \cdot 10^9$ years ago, all matter and all radiation was very concentrated. Mass (energy) density, pressure and temperature had gigantic values. The universe was formed out of that state.

The moment in which this occurred can be regarded as the beginning of time. This beginning of the expansion is referred to as *Big Bang*.

> The expansion of the universe began 13.8 billion years ago with the Big Bang.

Earlier we found that the universe is homogeneous. Everywhere it is just like where we are. This statement also holds true for the expansion: the rate of expansion is the same everywhere. This was also the case for our model universe in Fig. 10.1: all spacings between two neighboring knots increase equally fast.

But this does not yet tell us anything about the behavior of the rate of expansion as a function of time. For example, the spacings on our elastic rope could increase faster now and more slowly at a later time. Indeed, this is exactly the case for the real universe. Until approximately 5 billion years after the Big Bang, the expansion became increasingly slow over time and subsequently faster again.

Today, the question why the expansion becomes faster again has not yet been answered.

> Until 5 billion years after the Big Bang, the expansion was decelerating; then, it was accelerating again.

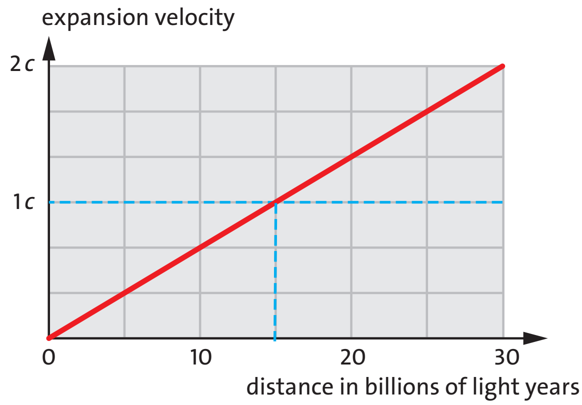


Fig. 10.3 Expansion velocity as a function of the distance: for $d > 14 \cdot 10^9$ light years, the rate of change becomes greater than the limiting velocity c.

Exercise

1. By how much will a distance of 1km in the universe increase in one year?

10.5 Looking back to the past

Almost everything we learn about the universe and almost everything we „see“ of the universe reaches us with electromagnetic radiation: especially with the normal „visible“ light, but also with a diversity of other electromagnetic radiations such as gamma radiation, Xrays, UV, infrared and microwave radiation.

So what can be „seen“ of the universe in this way? We could believe to see the universe as it is. Although this sounds natural, it is incorrect. We see the universe the way it was!

The light that reaches us on Earth and that creates the images of stars and galaxies had to travel a long way and therefore it needed time. The longer the way, the more time has passed between the emission (the creation) of the light by a star and its arrival at our place on Earth. Hence, what we see is not the star today, but the star at the time when its light was emitted; and this time can be in the remote past. The farther the object is away from us, the farther we look into the past.

> The farther we look into the distance, the farther we look into the past.

If we are now looking into the Sun, we can see the Sun the way it was approximately 8 minutes ago. Not much has certainly changed on the Sun during such a short time. Things are very different for a quasar that is located at a distance of 10^{10} light years and that we can see through a telescope. We can be sure that the quasar does not exist anymore. The supernova explosion that could be observed in 1987 had occurred 179 000 light years away from us. This means that it did not take place in 1987 but 179 000 years ago. The gravitational waves that reached us in 2016 had been created $1.3 \cdot 10^9$ years ago while two black holes were falling into one another at a distance of $1.3 \cdot 10^9$ light years.

If we assume the universe to be homogeneous, i.e. the same everywhere, and to have developed equally everywhere, we can conclude from what we see when looking at the distance (and into the past) how our closer environment looked like in the past. Hence, we see the universe in the different stages of its evolution.

As the universe is homogeneous, we can also see our own past in the distance.

We will examine later what has happened to these objects, i.e. the remote quasar and the remote black hole, in the meantime.

10.6 What we see of the universe

The answer to the question of what can be seen on Earth from our current location is easy (provided that the view is good): We can see the landscape up to the horizon.

Asking the corresponding question for the universe, the answer will be slightly more difficult.

1. The way we see a chronological sequence

We can look into the past with our telescopes. But we still do not see the world as it used to be back then; we see it as temporally distorted in a peculiar way. And this is due to the expansion of the universe.

What happens to the light that started traveling to us five or ten billion years ago (i.e. that flew in our direction by chance)? Light is an electromagnetic wave; the carrier of this wave is the space, just as water is the carrier of a water wave. Now the space expands, and this causes the wave to expand along with it. Therefore, its wavelength becomes longer and its frequency is reduced. This means that the wavelength of the radiation increases with a longer distance of travel. Blue light turns green, green light becomes yellowish, yellow light becomes increasingly red and red light turns into infrared. This change of the spectrum of light is called *redshift* (not the most suitable term, though).

However, not only the oscillations of light become slower on the way but also any other chronological sequence appears to us as temporally stretched. Does this remind you of anything?

In a supernova explosion, as much light as comes otherwise from an entire galaxy is formed. Therefore, supernovas can be seen from very long distances, i.e. of up to several billion light years. Due to the expansion of the universe, the light of a supernova is not only „redshifted“ (i.e. the light waves are stretched), but also the chronological sequence of the overall phenomenon of light, which takes several weeks, appears to us as temporally stretched. The longer the distance at which the supernova takes place, the longer its glowing effect.

Light that comes from a remote distance is „redshifted“.

Processes that we observe at a long distance appear to us as temporally stretched.

2. What we see

We only see the objects whose light reaches us, i.e. stars, galaxies, quasars,... (By light we mean electromagnetic radiation in this context, even if it is invisible for us.)

This sounds very natural at first. However, the statement is somewhat trappy. As the universe expands, the expansion velocity at a certain distance from us, i.e. at approximately 14.2 billion light years, is equal to the limiting velocity c. Everything that is located beyond this distance moves away at a velocity faster than c. We call this limit „c limit“.

But here is the difficulty. You will certainly assume that light emitted beyond the c limit has no chance to reach us. Things are just as in case of Willy when he walks on a moving walkway (the type we know from airports) against the direction of movement of the

Fig. 10.4 Willy will only reach Lilly if he moves to the left faster than the moving walkway moves to the right.

walkway, Fig. 10.4. When the moving walkway is faster than Willy, he moves backward instead of forward and will never reach Lilly. Fair enough ...

However, we have to consider that the expansion rate has decreased over time during the first 5 billion years. This means that light emitted behind the *c* limit has moved away from us at first. As the expansion velocity has decreased over time, however, the *c* limit moved outwards and our light was suddenly back ahead of the *c* limit so that it could eventually reach us.

In case this was too complicated, here is the corresponding situation with the moving walkway again, Fig. 10.4.

Willy walks on the moving walkway at his maximum velocity towards Lilly, i.e. against the movement direction of the walkway. The walkway moves faster than him; in spite of all his efforts, Willy moves away from Lilly. But now the walkway becomes increasingly slower. At some point, it is just as fast as Willy; now, Willy remains where he is. Then, the walkway becomes even slower and Willy moves forward towards Lilly and finally reaches her. Once again back to Willy's distance to Lilly: Willy always moves equally fast (as fast as he can); at the beginning, he moves away from Lilly, then the direction of his movement changes, he approaches Lilly again and finally reaches her.

The same applies for the light that was emitted in the young universe of galaxies and that we are receiving today, Fig. 10.5.

The blue line is the world line of a galaxy. The red one is the world line of the light that we are receiving today from this galaxy. It was emitted by the galaxy at a time when the universe was still very young, i.e. less than a billion years old. It was located at a place not very far away from „here". However, the expansion velocity at this place was higher than *c*. The light consequently moved away from „here" at first. But the rate of expansion has decreased over time, and so has the expansion velocity of our light. It finally became lower than *c* and the light could approach the „here" again.

And how has the galaxy in Fig. 10.5 evolved in the meantime? Provided that nothing has happened to it, it is now located at a distance of approximately 30 billion light years.

The fact that light was finally able to reach us after having moved away at first only applies for a small area behind the *c* limit, though. There is eventually a distance beyond the *c* limit from which the light is no longer able and has not been able to move towards us. We cannot see (not even in the future) what happens behind this limit. Like in the black hole, an event horizon exists here. It is located at 16.2 billion light years.

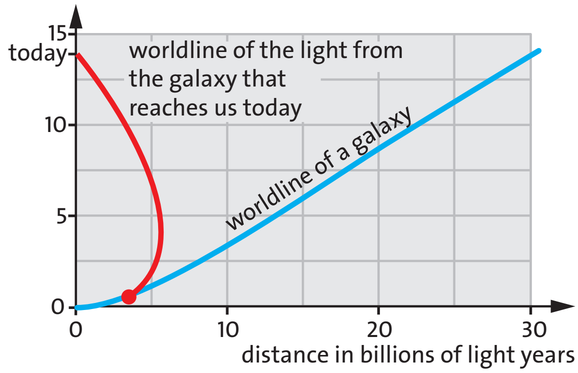


Fig. 10.5 World line of a galaxy and world line of the light from said galaxy that reaches us today distance (in billions of lightyears)

On the *c* limit (14.2 billion light years), the expansion velocity is equal to the limiting velocity *c*.

Neither in the future, we will be able to see what happens beyond the event horizon.

Exercises

1. Imagine the following universe: it was formed 14 billion years ago (we do not ask how), it is infinitely large and it does not expand. What would we see of this universe today?

10.7 The evolution of the universe – cosmic background radiation

When looking into space with telescopes, we see the evolution of the universe. As the universe looks the same everywhere, we do not only see the evolution of remote stars and galaxies but also how it used to be here, i.e. at the place where we are located today.

We could expect to see the evolution of the universe since the Big Bang. This is not quite correct, though. There is a time limit: until 400 000 years after the Big Bang, the universe was intransparent for all electromagnetic radiation so that we cannot see anything that dates back to this starting time with our telescopes.

But still, what happened in the first 400 000 years is not completely unknown because there are reliable theories (the theories of particle physics) that allow us to calculate what happened earlier.

Table 10.1 lists some stages of the evolution of the universe. It shows a progression in large powers of ten from one line to the next. The closer we come to the

time after the big bang	temperature	
10^{-35} s — 10^{-33} s	10^{27} K	expansion by a factor of 10^{50} (inflationary universe)
10^{-33} s	10^{25} K	beginning of the creation of quarks and gluons
10^{-6} s	10^{13} K	beginning of the creation of hadrons: protons, antiprotons, neutrons, antineutrons and others
Hadrons are the dominant particles of matter.		
10^{-4} s	10^{12} K	Protons react with antiprotons, neutrons with antineutrons. Only a small number protons and neutrons are left. The excess of matter over antimatter amounts to one billionth.
Leptons (electrons, antielectrons and others) are the dominant particles of matter		
1 s	10^{10} K	Electrons react with antielectrons. Only a small number of electrons are left over.
10 s	10^{9} K	creation of helium nuclei
400 000 years	3000 K	Electromagnetic raditation ceases to react with matter. The universe becomes transparent.
10^{9} years		creation of stars and galaxies
$13.7 \cdot 10^{9}$ years		today

Table 10.1 The universe since the big bang

beginning of time, the smaller will be the time intervals in which the characteristics of the universe change dramatically. At the beginning, the temperature had huge values. However, it has decreased steadily due to expansion.

The table contains different names of particles that we will only address later.

Prior to the formation of the large-scale structures, i.e. galaxies and stars, the universe had also been homogeneous on small length scales. The entire evolution was similar to a chemical reaction in a constantly increasing reaction space that is always in a chemical equilibrium.

Up to the time $t \approx 400\,000$ years, 75 % of the (mass of the) universe consisted of ionized hydrogen (i.e. protons) and 25 % of ionized helium as well as the corresponding electrons. (We have not included the so-called dark matter in our calculation.) At the beginning, it was still intransparent for all types of electromagnetic radiation.

After the temperature had dropped to approximately 3 000 K due to the expansion, hydrogen and helium atoms were formed out of the atomic nuclei and the electrons, and the universe became transparent. Hence, it then consisted of hydrogen, helium and radiation.

The radiation was the one of a body with a temperature of 3 000 K, i.e. approximately the same as that of an incandescent lamp's glowing wire. Due to the fast expansion, the temperature of the radiation decreased further while its wavelength grew. And this is how it has survived up to present. Its temperature is 2.7 K today, its wavelength amounts to several millimeters up to several centimeters. Hence, the radiation is extremely „redshifted". It is called *cosmic background radiation.*

It fills the entire universe and reaches us from all directions. It bears important information about the universe at the time of its formation. The atoms that emitted the radiation (or new atoms that were formed of them in the meantime) are located at a distance of 44 billion light years today.

> The cosmic background radiation was emitted 400 000 years after the Big Bang.