INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

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INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

Intl. Trans. in Op. Res. 0 (2023) 1–30
DOI: 10.1111/itor.13305

Comparison of different approaches to multistage lot sizing with uncertain demand

Viktor Bindewald, Fabian Dunke* and Stefan Nickel

Institute for Operations Research, Discrete Optimization and Logistics, Karlsruhe Institute of Technology, Kaiserstr. 12, 76131, Karlsruhe, Germany

E-mail: viktor.bindewald@partner.kit.edu [Bindewald]; fabian.dunke@kit.edu [Dunke]; stefan.nickel@kit.edu [Nickel]

Received 4 February 2022; received in revised form 17 March 2023; accepted 2 April 2023

Abstract

We study a new variant of the classical lot sizing problem with uncertain demand where neither the planning horizon nor demands are known exactly. This situation arises in practice when customer demands arriving over time are confirmed rather lately during the transportation process. In terms of planning, this setting necessitates a rolling horizon procedure where the overall multistage problem is dissolved into a series of coupled snapshot problems under uncertainty. Depending on the available data and risk disposition, different approaches from online optimization, stochastic programming, and robust optimization are viable to model and solve the snapshot problems. We evaluate the impact of the selected methodology on the overall solution quality using a methodology-agnostic framework for multistage decision-making under uncertainty. We provide computational results on lot sizing within a rolling horizon regarding different types of uncertainty, solution approaches, and the value of available information about upcoming demands.

Keywords: rolling horizon; demand uncertainty; online optimization; stochastic programming; robust optimization

1. Introduction

Operational decision-making in practice often suffers from uncertainty in the input data. This applies in particular when the uncertain information concerns future developments. In this case, decisions must be made at certain points in time even though there is no chance of knowing how the future will unfold. Many problems exhibiting these features arise in settings which are driven by the release process of customer orders, e.g., in production, supply chain management, and logistics. A well-known setting from the context of inventory management and production planning is lot sizing. In its basic form, a lot sizing model aims at determining a production plan over T time periods such that given customer demands $d_1, d_2, \dots, d_T$ for these periods are met and the sum of

*Corresponding author.

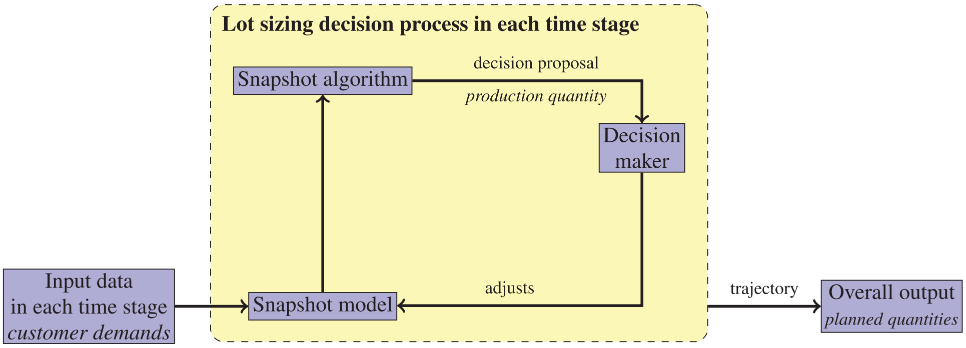
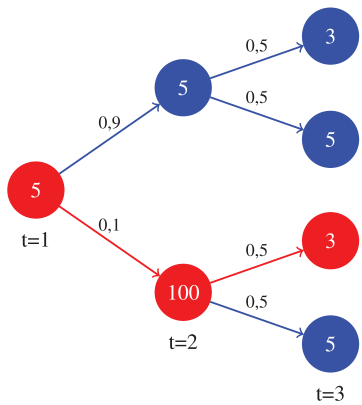


Fig. 1. Lot sizing in a rolling horizon procedure.

production and inventory holding costs is minimized. The problem in practice, however, is that usually neither the length of the planning horizon T nor all values $d_1, d_2, \ldots, d_T$ are known exactly. Existing research mainly focuses on additional features for the deterministic model or on specific uncertainty types. The approach in this paper is different insofar as assumptions on the planning horizon and information availability are relaxed and kept mutable. More precisely, we allow for an open planning horizon and different specifications of uncertainty information about customer demands. The recently proposed framework for multistage decision-making under uncertainty by Bindewald et al. (2020) has been designed exactly with the goal of enabling to analyze and quantify the effects of different types of data uncertainty and related solution techniques. In this paper, such an analysis will be carried out for multistage lot sizing considering uncertain customer demands as required by the disciplines of stochastic programming, robust optimization, and online optimization with lookahead. Hence, this paper aims at answering the natural question about how these different solution paradigms behave when deployed in a comparative setting with respect to both theoretical issues (e.g., modeling, analysis) as well as practical issues (e.g., solution quality, value of additional information, computational effort). According to Bakker et al. (2020), such an interdisciplinary comparison between different solution approaches is required by any comprehensive analysis of a specific problem setting involving uncertain data. Therefore, while practitioners from logistics, production, and supply chain management often refer to the practical notion of rolling horizon planning with inherent difficulties caused by uncertainty and time progression (Fleischmann et al., 2015) without emphasizing the methodological aspect of this solution outline, we explicitly address and investigate these aspects by means of different solution methods to be used in such a multistage setting with uncertainty.

In the framework for multistage decision-making under uncertainty (Bindewald et al., 2020), a sliding information window moves across the time axis regulating the information available for the decision-making process. Hence, the decision to be implemented in each stage results from solving a snapshot problem with the current data. The overall outline then consists of a series of snapshot problems to be solved, and decisions to be implemented until new data becomes known. Figure 1 displays such a rolling horizon procedure for the case of lot sizing. Each time a new customer demand becomes known, the interaction chain is started with the updated data. This requires a new snapshot lot sizing problem to be solved with a user-selected snapshot algorithm. The outcome of solving the previously specified snapshot problem with a suitable snapshot algorithm is a tentative production plan upon which a production decision suggestion is generated. This suggestion is then reported to the decision-maker and made available for evaluation. The cycle defining the multistage lot sizing decision process is repeated in an open time horizon as long as customer demand values for upcoming periods are reported from an external data source. The framework provides the user with freedom concerning the demand representation and the related choices of snapshot models and algorithms. For instance, demand data can consist of a deterministic lookahead, probabilistic information (stochastic representation), or uncertainty sets. In the latter case, typical ways of representing demand uncertainty are finite discrete sets, interval-based sets, or polytope-based sets. Due to the flexible nature of the framework, hybrid approaches are possible such as combining lookahead for the near future with uncertainty sets for the more remote future. Amenable approaches to solving the snapshot problems arising from these data representations then are naturally found in the disciplines of online optimization with lookahead, stochastic programming, and robust optimization. Decision-making is assessed in the evaluation module, and the user may decide to change the snapshot approach. Hence, in contrast to fixed solution methodologies, the overall outline combining a rolling horizon framework with the repeated solution of snapshot problems yields various additional degrees of freedom for controlling a multistage lot sizing application. At the same time, this freedom comes with the burden of deciding which snapshot problem would be most suitable for the lot sizing system under consideration. In particular, every choice for a specific modeling approach implies further issues to be addressed carefully. For instance, lot sizing in a rolling time horizon requires to address related topics such as targeted ending inventory levels (Van Den Heuvel and Wagelmans, 2005), uncertain demand representations (Coniglio et al., 2018), or strategies for dealing with uncertain demand (Bookbinder and Tan, 1988). According to Tavaghof-Gigloo and Minner (2021), research on the combination of rolling horizon schemes with demand uncertainty is rather limited. This paper focuses on different representations of uncertain demands and tackles multistage lot sizing in a rolling time horizon with snapshot problems referencing to the respective type of uncertainty information.

To convey the idea of comparing different solution paradigms in a rolling horizon setting with demand uncertainty, we exemplarily discuss the production plans obtained by solving the snapshot problems through online optimization (OO), stochastic programming (SP), and robust optimization (RO) as shown in Fig. 2. In the scenario tree, each path from the root to a leaf corresponds to a possible scenario with demand realization probabilities displayed as edge labels; the red path indicates the realized scenario. However, the decision-maker only has information about the current and subsequent stage at each decision moment, i.e., the lookahead horizon for OO is $\tau = 1$, and the uncertainty horizon for SP and RO is $\sigma = 1$. Production costs are 3, setup costs 20, storage costs 1, and shortage costs 100 monetary units; the initial stock is empty. For this typical realization, the three variants exhibit rather typical behavior when three stages are considered: OO with its access to the deterministic lookahead performs best; SP, on the other hand, aims to find a balanced solution taking all scenarios into account; RO finishes last as it produces in every stage (and solely for the current stage) hence neglecting available information. Finally, we illustrate the effect of increasing the information horizon, say the lookahead size in OO from $\tau = 1$ to $\tau = 2$. In case of an additional time stage $t = 4$ with realized demand $d_4 = 8$, the changed production plan favorably avoids to produce in $t = 4$ which is made possible only by knowing $d_4 = 8$ already in $t = 2$.



Variant	x_1	x_2	x_3	x_4	total costs
$\text{OO}_{\tau=1}$	5	103	0	-	367
$\text{SP}_{\sigma=1}$	10	98	0	-	372
$\text{RO}_{\sigma=1}$	5	100	3	-	384
$\text{OO}_{\tau=1}$	5	103	0	8	411
$\text{OO}_{\tau=2}$	5	111	0	0	407

Fig. 2. Comparison of different solution paradigms for rolling-horizon lot sizing.

In Sections 3.3 and 4, further (also counterintuitive) effects resulting from the rolling horizon are illustrated demonstrating the necessity for having different solution strategies such as OO, SP, and RO at one's disposal.

Based on a methodology-agnostic framework for multistage decision-making under uncertainty Bindewald et al. (2020), the paper contributes an extensive cross-methodological analysis of multistage lot sizing with different types of uncertainty representations and corresponding solution approaches. Hence, the definitions of the framework components—especially with respect to a clear separation between data and solution process—suitably specify the required level of modeling granularity to allow for a concrete specification in settings coined by recurring combinatorial optimization tasks as typically found, e.g., in production and logistics. To the best of our knowledge, the wide range array of analysis possibilities which can be derived from this combination of a generic overarching solution outline with a specific application context is the first of this kind. On the most granular level, solution trajectories are recorded during the rolling-horizon procedure such that effects of data uncertainty representation and snapshot methodologies on solution quality are tracked over time and compared to each other. As a result of the data collection over several sample settings, we derive domain-specific knowledge and insights for the further operational control of multistage lot sizing applications under uncertainty. The deployment of the general modeling framework in the case of multistage uncertain lot sizing thus provides guidance for similar kinds of analysis in other problem settings. On an abstract level, the realized mix between general and problem-specific analysis aspects serves as a blueprint for approaching a problem both from a general and a tailored perspective.

The paper implies several managerial insights: First, the experimental results provide helpful knowledge about value of information, interparadigmal comparison, and comparison between exact and heuristic methods. The required computational budget is well justified for a general logistics system analysis and can be reduced even further for daily operations by integrating heuristics or metamodels. Second, the framework yields comprehensive evaluation capabilities. Reports and charts on general quantities of interest (e.g., production horizon) can be generated as shown for multistage lot sizing at hand with uncertain demand data. Using the underlying flexible modeling framework, similar evaluations can be carried out for different problems with uncertain parameters. Finally, managers can zoom in on specific aspects at the solution trajectory level such as an instance-wise analysis of production horizon lengths.

The rest of the paper is organized as follows: Section 2 discusses existing research on lot sizing with a focus on different methodologies that can be used to tackle uncertainty. Section 3 then introduces the overall algorithmic approach, required mathematical programming (MP) formulations, and all algorithms representing the lot sizing instantiations of the general modeling framework. The computational experiments in Section 4 illustrate how different solution methods for snapshot problems can be utilized and which types of conclusions can be drawn from the availability of several options for dealing with uncertainty. Finally, Section 5 summarizes the findings from the application of the modeling framework to lot sizing and points towards future research directions filling remaining research gaps.

2. Related work

We first discuss existing research on multistage lot sizing on a general basis. To this end, we address concepts such as multistage solution strategies and the rolling horizon paradigm. We then scrutinize multistage lot sizing under uncertainty with the disciplines of stochastic programming, robust optimization, and online optimization. Hence, a comparison of the different uncertainty approaches in multistage lot sizing as discussed in this paper also becomes apparent in the literature review.

2.1. Multistage lot sizing

The basis for mathematical models on multistage lot sizing is the Wagner–Whitin model for inventory management and production planning under time-varying demands (Wagner and Whitin, 1958). Assuming a fixed time horizon, the related Wagner–Whitin algorithm is an exact DP-based offline algorithm which runs in time polynomial in the number of stages. In practice, the rolling horizon setting with an indefinite planning horizon is encountered frequently. In this setting, replanning decisions are carried out in each stage. For this reason, many forward-only heuristics (DeMatteis, 1968; Silver and Meal, 1973; Groff, 1979) were proposed which in numerical tests perform extremely well (Silver et al., 1998). Bookbinder and Tan (1988) structure rolling horizon approaches by distinguishing between static, dynamic, and static-dynamic strategies. The first fixes production quantities and timings in advance (recommendable under limited capacities), the second operates on a period-by-period basis (recommendable under demand variation), and the third only fixes production timings (recommendable when quantity adjustments are possible). The effectiveness of rolling schedules in production planning is emphasized both for the deterministic and the probabilistic setting (Bookbinder and H'ng, 1986a, 1986b). To allow for demand uncertainty also in the static setting, explicit and detailed consideration of different types of service levels are discussed and extended to the rolling horizon setting (see, e.g., Sereshti et al., 2020). Extensive overviews on surveys of lot sizing displaying the evolution of this field of research are given by Andriolo et al. (2014), Glock et al. (2014), and Bushuev et al. (2015). Even though there exist numerous extensions for the multistage case focusing on specific characteristics (e.g., multiple products and their scheduling, sustainability aspects like reverse logistics, or stochasticity), there is a lack of comparative research focusing on different models of uncertainty at the same time and relating implications of the uncertainty models to each other. Due to its fundamental character in production planning, lot sizing has undergone a development parallel to that of operations research over the past decades in terms of successively enriching models with constraints, uncertainty, and other aspects from related disciplines such as sustainability or pricing. A classification survey on single-item dynamic lot sizing is found in Brahimi et al. (2017) focusing on MP formulations and related issues such as complexity and solution approaches. Moreover, it yields a comprehensive classification for problem characteristics such as backlogging/lost sales, time windows, resource constraints, pricing, and perishability. The survey also devotes a section to lot sizing under uncertainty, classifying existing research according to different types of uncertain parameters and modeling approaches. Concerning uncertainty aspects, it is distinguished between uncertainty in demands, costs, yields, lead times, and capacities. By far the most attention has been received by demand uncertainty. A modern review capturing the outline and required taxonomy of rolling horizon planning systems as considered in this paper is given by Sahin et al. (2013). The paper addresses both the more general context of supply chain management and in particular the task of lot sizing due to its central role under recent trends like joint replenishment or collaboration between inventories. The authors find that exact approaches have limited applicability for use in modern environments and require adaptations accounting for coordination of long-term stochastic demands and short-term ordering processes. Finally, as required by the practical inter-relatedness between quantity and timing decisions in production (Dauzère-Pérès and Lasserre, 2002), several papers extend multistage lot sizing by scheduling (Hu and Hu, 2018; Curcio et al., 2018; Alem et al., 2018; Rehman et al., 2021). Due to the complexity caused by the multistage and uncertainty consideration, this then leads to analyses depending more specifically on the adopted solution methodology and process of information release, respectively.

2.2. *Stochastic programming for lot sizing*

Most formulations of the stochastic multistage setting rely on a scenario tree. Each tree vertex corresponds to a staged outcome and a root–leaf path corresponds to a scenario over all stages. In this framework, Guan et al. (2006) introduce valid (l, S)-inequalities to tighten the convex hull of the feasible set and hereby enhance bounding procedures. Lot sizing is used to illustrate how general cutting planes for multistage stochastic integer programs can be instantiated (Guan et al., 2009). Moreover, they relate MP and dynamic programming (DP): In Guan and Miller (2008), polynomial-time DP algorithms are proposed for the scenario tree representation based on an optimality condition (production path property), which implies that at each vertex exactly enough is produced to satisfy demands along the path from this vertex to one of its descendants. The optimality condition is used by Zhao and Guan (2014) to derive extended MP formulations (with and without backlogging) which are beneficial under integral order quantities. Further extensions (e.g., inventory bounds and order capacities) resulting in DP algorithms are considered by Guan and Liu (2010). For the multi-item case of combined lot sizing and scheduling, Alem et al. (2018) discuss a comparison between stochastic programming (with scenario-based uncertainty) and robust optimization (with budgeted uncertainty) to guide decision-makers in practice. Curcio et al. (2018) employ the same methodologies to obtain MP-based adaptation and approximation strategies.

Tailored robust optimization performs best in a rolling time horizon when short computation times are required. A hybrid robust-stochastic approach originates from Coniglio et al. (2018) for the case of storage losses (as typical for energy applications) where the objective is to minimize expected costs, but recourse actions ensure worst-case feasibility. Likewise, lot sizing and scheduling with machine eligibility and sequence-dependent setups are considered by Chen and Su (2022) through multistage stochastic programming and a related analysis of classical measures (expected value of perfect information, value of the stochastic solution). Multistage lot sizing embedded within a rolling horizon procedure of a material requirements planning setting is studied experimentally in Thevenin et al. (2021). A financial extension is discussed by Li and Thorstenson (2014) for combined lot sizing and pricing under capacity constraints and backlogging. Because of the nonlinear pricing component, the stochastic programming approach is relaxed into a multiphase heuristic decoupling lot sizing and pricing. Raa and Aghezzaf (2005) use stochastic programming along with decision rules for adapting the dynamic case to starting inventories to obtain robust production plans. A multiechelon extension with returns and lost sales is considered by Quezada et al. (2020), and a new family of valid inequalities (tree inequalities) is set up through combining scenario-specific valid inequalities leading to a new branch-and-cut algorithm. Computational tests are performed in a rolling time horizon simulation. Brandimarte (2006) also argues that the rolling time horizon is crucial in industrial settings. Therefore, a comparison is sought between a (plant location based) multistage mixed-integer stochastic programming model, the deterministic model with expected values for uncertain demands, and a fix-and-relax strategy. The analysis suggests that the advantage of the stochastic approach strongly relies on the criticality of the problem instance concerning reactive possibilities as given in a rolling horizon. Tavaghof-Gigloo and Minner (2021) discuss capacitated lot sizing with service level constraints. For the interrelated tasks of safety stock determination and lot sizing, they develop mixed integer programming models for a sequential outline, an integrated outline, and an integrated model with replanning like in rolling horizon planning. Under limited capacity, the integrated model exhibits clear advantages compared to the sequential approach. Yet, under sufficient capacities, the safety stock may be oversized such that replanning upon demand observations becomes crucial. In line with industrial requirements, Tempelmeier and Herpers (2011) consider a fill rate constraint ensuring a β-service level. Exact algorithms as well as heuristics for the dynamic case are developed and tested against standard heuristics. Service level considerations in combined lot sizing and scheduling with sequence-dependent changeover costs leading to a nonlinear model are presented by De Smet et al. (2020). Several linearization techniques are introduced, and a relax-and-fix heuristic is developed. Addressing the problem of frequent production plan revisions, Koca et al. (2018) generalize the classification of static and dynamic strategies with the concept of nervousness via promised production quantities. For the resulting model formulation, valid inequalities are derived and the problem is tackled by a branch-and-cut algorithm. A unified modeling framework amenable to penalty costs, service level constraints, backorders, or lost sales is introduced by Tunc et al. (2018).

2.3. Robust optimization for lot sizing

Budgeted uncertainty as defined by Bertsimas and Sim (2004) (also known as Γ-uncertainty) represents the most popular uncertainty concept in robust optimization. It assumes that at the same time at most Γ (a previously specified number) of the demands can deviate from their nominal values. Bertsimas and Thiele (2006) apply this concept to robust supply chain control, in particular lot sizing. Another approach to uncertainty in multistage robust optimization is adjustable robustness (Ben-Tal et al., 2004). Similar to two-stage stochastic programming, parts of the variables are adjustable, i.e., their values can be chosen upon demand realization. To ensure tractability, adjustments may be restricted, e.g., to affine functions of uncertain parameters which is exemplified for a single product inventory system. A comparison covering different uncertainty concepts and modeling methods for robust inventory control under demand interval uncertainty is given by Solyali et al. (2016). Based on a disaggregation between order quantity variables, a general formulation which is compatible with the rolling horizon setting is developed. Postek and Den Hertog (2016) propose a mixed integer linear model methodology for adjustable robust lot sizing where the uncertainty set is iteratively split up into subsets for scenario separation. The framework is applied to static decisions and linear decision rules. Influenced both by robust optimization and control theory, Wagner (2018) frame demand uncertainty through stochastic processes and develops closed forms of optimal ordering rate functions for the static and dynamic cases. Under seasonality trends, reoptimization capabilities pay off in significantly better objective values. Upper and lower bounds for robust lot sizing are derived by Santos et al. (2018). In particular, a lower bounding technique is set up analogous to the perfect information relaxation from stochastic programming through relaxing nonanticipativity. This results in a significant solution time speed-up. For multistage lot sizing with storage losses (as typical for applications in the energy sector) and nonlinear objective, Coniglio et al. (2016) propose a two-stage model under budgeted uncertainty with first-stage decisions on production and second-stage decisions on storage. Robust MP models for multistage inventory control are developed by Thorsen and Yao (2017) for budgeted uncertainty and based on the central limit theorem. A Benders' decomposition outline is developed and tested against sample average approximation results leading to stable solutions, even when sampled and realized distributions do not coincide. The need for robust lot sizing is also emphasized by Santos et al. (2020) in the case of perishable products. Their model is endowed with two recourse possibilities in each stage for an affine model of demand uncertainty: Total production corresponds to here-and-now decisions; inventories, backlogs, and spoilage are recourse decisions. To account for the infinite number of variables and constraints, a row-and-column generation algorithm relying on a finite master problem is established and used for analyzing the influence of product perishability. Uncertain demand intervals are considered by Zhang (2011) under minmax regret robust objective where regret amounts to opportunity loss. The two-stage model admits production decisions in the first stage and production quantity and inventory level decisions in the second stage. Moreover, a polynomial time algorithm is developed. For combined lot sizing and cutting stock problems (as typical for raw materials processing), Jose Alem and Morabito (2012) formulate robust MP models under budgeted uncertainty accounting for uncertain cost coefficients and demands. For the multi-item setting, Abdel-Aal (2019) gives an MP formulation under budgeted uncertainty along with a robust counterpart based on duality theory. A general framework for tackling robust optimization through decomposition into a master problem (containing the robust constraints for a set of scenarios) and adversarial problems (aiming at generating additional scenarios) is proposed by Agra et al. (2016). Using the concept of budgeted uncertainty, DP algorithms are devised for the adversarial problems. The methodology is exemplified for lot sizing. A similar decomposition is discussed by Attila et al.

(2021) for lot sizing with remanufacturing under budgeted uncertainty with uncertain demands and returns.

2.4. Online optimization with lookahead for lot sizing

Rolling horizon lot sizing resonates with online optimization due to not knowing the end of an input sequence. Nonetheless, the literature on online lot sizing is scarce compared to stochastic programming and robust optimization. This is due to the focus of online optimization literature on competitive analysis (Fiat and Woeginger, 1998; Borodin and El-Yaniv, 2005), a worst-case analysis for providing performance guarantees of online algorithms compared to an optimal offline algorithm. Lot sizing becomes amenable to online optimization when the available data is extended by lookahead data holding future demands which can be incorporated for decision-making. For lot sizing, Ahlroth et al. (2010) perform such a worst-case analysis for an online algorithm with access to lookahead over a fixed prospective time interval. The authors propose an algorithm which asymptotically (for increasing lookahead) yields the optimal worst-case performance. Competitive analysis is also conducted by Wagner (2011) for online lot sizing in the cases of perishable products with lost sales and durable products with backlogged demand, respectively. The analysis relies on fractional programming and duality theory when applied to MP formulations of the lot sizing problem. Closely related to the idea of rolling horizon procedures, Van den Heuvel and Wagelmans (2010) develop for rolling horizon lot sizing an online algorithm with a worst-case performance ratio of at least 2, and the authors show how to construct corresponding worst-case instances. The analysis is carried over to the case of lookahead. This paper also provides a review of worst-case performance ratios of other well-known heuristics as discussed in Section 2.1. Dunke and Nickel (2021) consider online lot sizing with gradual lookahead as an extension of deterministic lookahead with demand intervals narrowing down to singletons as time advances. In a rolling time horizon, different heuristic snapshot algorithms as well as an exact reoptimization approach based on MP are tested. The authors also employ numerical experiments on lot sizing with gradual lookahead to examine the value of better data accuracy, more frequent input data updates, longer lookahead horizon, and solving snapshot problems to optimality rather than heuristically. Finally, Dunke and Nickel (2020) deal with the automated generation of rule-based online algorithms. For the lot sizing application, it is shown that (meta-) heuristically generated data-driven rules are in a position to yield competitive results.

3. Multistage lot sizing under demand uncertainty

In classical deterministic lot sizing, all future demands are known within a closed time horizon and the goal is to balance storage and setup costs. In our setting, however, the decision-maker does not know when the process will end and there is uncertainty about future demands. Hence, we assume an open (infinite) time horizon, and we have to decide repeatedly on lot sizes without knowing the complete future at the time the decisions have to be made. We model our decision-making problem as a multistage decision process within the recently introduced framework for multistage decision-making under incomplete and uncertain information (Bindewald et al., 2020).

Before presenting the formal definition, we give a brief description of the setting: We are operating in an online manner, i.e., we receive information about the latest and the upcoming demands periodically. However, this information covers only a small portion of the overall time horizon. This means that the information about the future demands is incomplete and may additionally be uncertain. Thus, we are not obliged to meet the demands in each stage, but not meeting them incurs shortage costs. Different approaches to handling unavailability and uncertainty in the data are possible, and we refer to these approaches as *paradigms*. They reflect the decision-maker's attitude towards uncertainty (e.g., to optimize the average or hedge against the worst case) and can have a big impact on the resulting decisions. We make decisions using all available information in accordance with the selected paradigm. Once a decision is made, it is implemented immediately and cannot be corrected later. Since we are primarily interested in the impact of different representations of the future demands, we consider time-independent costs. To sum up, the goal is to balance storage, setup, and shortage costs within a rolling horizon procedure with the additional difficulty of not knowing all the upcoming demands exactly. The notation for the upcoming models is given in Table 1.

We now present the formal definition of the decision-making problem under consideration as a multistage decision process (see Bindewald et al. (2020, Def. 2)).

Definition 1 (MSLSUD). *Multistage lot sizing with uncertain demands (MSLSUD) and open time horizon T is specified by the following six components.*

1. *Initial data* $D_0 = (c^p, c^f, c^h, c^s, s_0)$.
2. *Time-dynamic data* $D_t = (d_t, \mathbb{F}_t, \texttt{paradigm}_t)$ *at stage* t, $t \in \{1, \dots, T\}$ *consisting of latest demand* d_t*, future data* $\mathbb{F}_t$ *(see 3.), and solution paradigm* $\texttt{paradigm}_t$ *(see 4.).*
3. *Future data* $\mathbb{F}_t = (\mathbb{L}_t, \mathbb{U}_t)$, $t \in \{1, \dots, T\}$*: information about the upcoming demands consisting of deterministic lookahead* $\mathbb{L}_t$ *for the next* $\tau \in \mathbb{N}_0$ *stages and uncertain data* $\mathbb{U}_t$ *for the subsequent* $\sigma \in \mathbb{N}_0$ *stages.*
4. *Stage task: decide on production output* x_t *with solution method* $\texttt{paradigm}_t$ *describing how to consider* $\mathbb{F}_t$ *in the decision-making process (see below for examples) with associated cost-per-stage:* $c_t(x_t) = c^p x_t + c^f$ *(if* $x_t > 0$*)* $+ c^h s_t + c^s \max\{d_t - x_t - s_{t-1}, 0\}$.
5. *Stage transition description: the next stock level is* $s_t = \max\{x_t + s_{t-1} - d_t, 0\}$.
6. *Decision criterion: minimize overall costs.*

The set $\mathbb{F}_t$ contains deterministic as well as uncertain information about the near future. The considered time period covers $\tau + \sigma$ stages and is referred to as the *information time horizon* at stage t. For some applications, it is reasonable to assume that τ and σ depend on t. This extension can be easily integrated but is omitted here for the sake of clarity. Note that the overall costs cannot be computed in advance due to the open overall horizon. This is the reason why we specify a decision criterion rather than an objective function. The decision criterion is usually evaluated in retrospective, e.g., via $C_t := \sum_{s=1}^{t} c_s(x_s)$.

The transition from stage t to its successor stage $t + 1$ is illustrated in Fig. 3. The new stage data (d_{t+1}, s_t) implicitly reflects all previously made decisions $x_{t'}$ for $t' \leq t$ because they influence the new stock level s_t. Because the decision x_t depends on the feedback from the environment, the framework defines a closed-loop control system. Borrowing from the terminology of control theory which aims at influencing the behavior of dynamical systems, s_t represents a state variable whereas

Table 1
Notation for multistage lot sizing with uncertain demands (MSLSUD).

Symbol	Explanation
c^p, c^f, c^h, c^s	costs for production/setup/storage/shortage
T	time horizon, revealed to the decision-maker only upon reaching stage T
s_t	stock level at the end of stage t
D_0	initial data $D_0 := (c^p, c^f, c^h, c^s, s_0)$
d_t	customer demand for stage t
d_t^k	kth possible customer demand for stage t
p_t^k	probability of kth possible customer demand for stage t
τ, σ	length of lookahead/uncertainty horizon
$\mathbb{L}_t$	deterministic lookahead demands for stages $t+1, \ldots, t+\tau$
$\mathbb{U}_t$	uncertain demands for stages $t+1, \ldots, t+\sigma$; given as • list of sets of possible demand realizations with corresponding probabilities (in the case of stochastic programming) • list of sets of possible demand realizations (in case of robust optimization)
$\mathbb{F}_t$	future demand information $\mathbb{F}_t := (\mathbb{L}_t, \mathbb{U}_t)$
$\texttt{paradigm}_t$	solution approach for snapshot problem at stage t
D_t	time-dynamic data $D_t := (d_t, \mathbb{F}_t, \texttt{paradigm}_t)$ at stage t
$D_{\leq T}$	sequence $D_{\leq T} := (D_1, D_2, \ldots, D_T)$ of time-dynamic data at stages $t = 1, 2, \ldots, T$
$\mathcal{I}$	problem instance containing all data $\mathcal{I} = (T, D_0, D_{\leq T})$
x_t	(planned) production quantity at stage t
c_t	cost-per-stage at stage t
C_t	sum of costs-per-stage until stage t
P_t, I_t, Y_t, L_t	decision variables for production quantity/stock level/ordering/ lost quantities at stage t in the snapshot problem
H	information time horizon of the snapshot problem at stage t
κ	planned stock at the end of the information time horizon
ξ_t	demand realization at stage t
ξ	scenario, i.e., sequence of demand realizations
$\mathcal{U}$	set of all scenarios

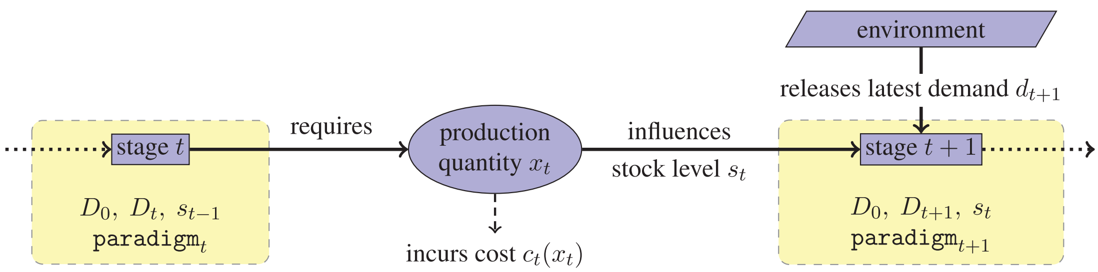


Fig. 3. Illustration of the transition process from stage t to stage $t+1$.

x_t corresponds to a control (or action) variable. Using these two types of variables, Definition 1 yields the complete representation of both the lot sizing system dynamics and the related accumulation of costs over time. In particular, observe that due to the definition of the cost-per-stage as $c_t(x_t) = c^p x_t + c^f$ (if $x_t > 0$) $+ c^h s_t + c^s \max\{d_t - x_t - s_{t-1}, 0\}$, there is no need to explicitly introduce further types of variables for setups and shortages at this point. In order to obtain a decision x_t on the production output in stage t, we need to formulate a *snapshot problem*, i.e., an optimization problem using all the data available within the information time horizon and complying with the stage paradigm and the decision criterion. In an abstract sense, solving a snapshot problem can be seen as solving a temporally local optimization problem within the overall solution process. Before we discuss different formulations for snapshot problems in Section 3.2, we explain different settings regarding availability and uncertainty captured by $\mathbb{F}_t$ and $\texttt{paradigm}_t$.

Different requirements of the decision-maker on dealing with uncertain problem parameters are prescribed by $\texttt{paradigm}_t$. Each $\texttt{paradigm}_t$ has to be accompanied by suitable data represented in the future data $\mathbb{F}_t = (\mathbb{L}_t, \mathbb{U}_t)$. Depending on $\texttt{paradigm}_t$, $\mathbb{F}_t$ is capable of holding both deterministic data (in $\mathbb{L}_t$) as well as nondeterministic data (in $\mathbb{U}_t$) in different data representations. We select these representations to be amenable to techniques from online optimization, stochastic programming, and robust optimization. Hence, we first set $\texttt{paradigm}_t$ to OO, SP, and RO, respectively. This results in three variants of our basic problem MSLSUD, which are denoted by MSLSUD-OO, MSLSUD-SP, and MSLSUD-RO. Concrete definitions of $\mathbb{F}_t = (\mathbb{L}_t, \mathbb{U}_t)$ as required by $\texttt{paradigm}_t$ are given next. The most simple one is for $\texttt{paradigm}_t = \text{OO}$. Here, we assume that all information about upcoming demands is deterministic and stored in the lookahead, i.e., $\mathbb{L}_t = (d_{t+1}, \dots, d_{t+\tau})$ and $\mathbb{U}_t = \emptyset$. For $\texttt{paradigm}_t = \text{SP}$, we need to have information about possible demand realizations and their probabilities amenable to techniques from stochastic programming in the form $\mathbb{U}_t = (\{(d^1_{t+1}, p^1_{t+1}), \dots, (d^k_{t+1}, p^k_{t+1})\}, \dots, \{(d^1_{t+\sigma}, p^1_{t+\sigma}), \dots, (d^k_{t+\sigma}, p^k_{t+\sigma})\})$. Information about probability distributions are also possible, but hard to obtain in practice. In the last variant $\texttt{paradigm}_t = \text{RO}$, probabilities for demand realizations are omitted in the previous definition of $\mathbb{U}_t$. Concerning SP and RO, there are two possibilities regarding the lookahead: (1) no deterministic information is available, i.e., $\mathbb{L}_t = \emptyset$; 2) some upcoming demands are known in addition to the information provided in $\mathbb{U}_t$. We call the second case a *hybrid* setting. An example of a hybrid $\mathbb{F}_t$ for RO with information time horizon $\tau + \sigma$ is $\mathbb{L}_t = (d_{t+1}, \dots, d_{t+\tau})$ and $\mathbb{U}_t = (\{d^1_{t+\tau+1}, \dots, d^k_{t+\tau+1}\}, \dots, \{d^1_{t+\tau+\sigma}, \dots, d^k_{t+\tau+\sigma}\})$. Similar hybrid extensions of SP are possible. Further options for $\texttt{paradigm}_t$ like `stochastic programming with value at risk objective` or `adjustable robustness` are also possible, but not considered in this work. Depending on the solution process, it may also be reasonable to change $\texttt{paradigm}_t$ over the course of time, e.g., by starting with RO and switching to SP once enough historic data is available to obtain reasonable forecasts.

3.1. Algorithmic outline

To formalize the input for an algorithm for MSLSUD, we need to adapt the general definition of a multistage process instance (cf. Bindewald et al. (2020, Def. 4)) to lot sizing. In practice, such an instance becomes known to the decision-maker successively. Therefore, it is best understood as an instance seen in retrospective. Nonetheless, for any computational experiment and the related

Algorithm 1. `solveLotSizing`

Input: MSLSUD-instance $\mathcal{I} = (T, D_0 = (c^p, c^f, c^h, c^s, s_0), D_{\leq T})$
Output: Solution $x = (x_1, x_2, \ldots, x_T) \in \text{Sol}(\mathcal{I})$, total cost $C_T(x)$
1: $C_T \leftarrow 0$
2: $t \leftarrow 1$
3: **while** $t \leq T$ **do**
4:   **if** $s_{t-1} \geq d_t$ **then**
5:     $x_t \leftarrow 0$
6:   **else**
7:     $x_t \leftarrow$ `solveSnapshotProblem`$(D_0, D_t = (d_t, \mathbb{F}_t, \texttt{paradigm}_t), s_{t-1})$
8:   **end if**
9:   $s_t \leftarrow \max\{x_t + s_{t-1} - d_t, 0\}$
10:   $C_T \leftarrow C_T + c_t(x_t)$
11:   $t \leftarrow t + 1$
12: **end while**

evaluation of achieved solution qualities, we will have to impose a (sufficiently large) finite value for the overall time horizon T. This allows us to compare the costs of different solutions over the (arbitrarily chosen) time horizon T. However, recalling that T is unknown to the decision-maker until stage T has elapsed, it follows that in fulfilling the stage task at any stage $t \in \{1, \ldots, T\}$ (i.e., in determining the production quantity x_t) any solution method must operate independently from expecting an end of the overall time horizon.

Definition 2 (MSLSUD instance)**.** *An MSLSUD instance is a collection of concrete values for all parameters of MSLSUD and can be described as a tuple* $\mathcal{I} = (T, D_0, D_{\leq T})$ *where* $D_{\leq T} := (D_1, D_2, \ldots, D_T)$ *and* $D_t := (d_t, \mathbb{F}_t, \texttt{paradigm}_t)$ *is the time-dynamic data available in stage* $t \in \{1, \ldots, T\}$.

Note that without imposing a finite value on T the size of an MSLSUD instance $\mathcal{I}$ would be infinite. Depending on the definition of $\mathbb{F}_t$ and $\texttt{paradigm}_t$ we will also speak of instances for MSLSUD-OO etc. The solution to an MSLSUD instance is defined next.

Definition 3 (Solution, solution set)**.** *For a given MSLSUD instance* $\mathcal{I} = (T, D_0, D_{\leq T})$*, a solution to* $\mathcal{I}$ *is a sequence of decisions* $x = (x_1, \ldots, x_T)$ *for every stage* $t \in \{1, \ldots, T\}$*. We denote the set of all feasible solutions by* $\text{Sol}(\mathcal{I})$*. Solution costs are defined as* $C_T(x) := \sum_{t=1}^{T} c_t(x_t)$.

A generic algorithm to solve an MSLSUD instance is presented in Algorithm 1. Observe that this outline implements the rolling horizon procedure illustrated by Fig. 1 and works fully independent of the value of T. From a practical perspective, the value determined for the control variable x_t on the production quantity is implemented in stage t (due to line 7) leading to the new value of the state variable s_t on the stock level (due to line 9). Cost-related implications of this action implementation are accounted for in line 10 by the cost-per-stage $c_t(x_t) = c^p x_t + c^f$ (if $x_t > 0$) $+ c^h s_t + c^s \max\{d_t - x_t - s_{t-1}, 0\}$. Note that the cost-per-stage implicitly accounts for setup changes and shortages which are a direct consequence of the production quantity decisions x_t. In the subsequent Section 3.2 on the snapshot problems, the determination of the production quantities will be based explicitly on balancing the different cost types.

Algorithm 2. `solveSnapshotProblem`

Input: D_0, $D_{t'} = (d_{t'}, \mathbb{F}_{t'}, \texttt{paradigm}_{t'})$, $s_{t'-1}$
Output: decision $x_{t'}$ for stage t'
1: Construct $\texttt{paradigm}_t$-complying MIP model M for the information time horizon $\tau + \sigma$
2: *quantities* $\leftarrow$ optimal values for the production quantity decision variables in M
3: $x_{t'} \leftarrow$ first entry of *quantities*

Snapshot problems for different MSLSUD variants are discussed in Section 3.2. In Step 7, we can use both exact and heuristic methods. A comparison between these approaches is studied in Section 4 as part of the computational analysis. Steps 9 and 10 address the consequences of the irrevocable implementation of decision x_t on the stock level and accrued costs, respectively.

3.2. *Snapshot problem formulations*

Consider an MSLSUD instance $\mathcal{I} = (T, D_0, D_{\leq T})$. At any fixed stage $t' \in \{1, \ldots, T\}$ during the solution process, an algorithm for MSLSUD has access to the parameters in the stage-independent data $D_0 = (c^p, c^f, c^h, c^s, s_0)$ and the time-dynamic data $D_{t'} = (d_{t'}, \mathbb{F}_{t'}, \texttt{paradigm}_{t'})$. The algorithm utilizes this data in a snapshot problem that complies with $\texttt{paradigm}_{t'}$ in Step 7. We now present three mixed-integer-programming-(MIP)-based snapshot formulations for the three main variants of MSLSUD introduced previously. Since these MIP-based models determine optimal decisions with respect to the objective function given as the total costs for production, setups, storage, and shortage, decision variables have to be introduced to explicitly address these aspects: P_t for production quantities, Y_t for setups, I_t for stock levels, and L_t for shortage quantities with $t \in H$. For transferring the lot sizing system from stage t' to stage $t' + 1$ according to Definition 1, it then suffices to pass on the first value of the snapshot decision variables P_t on the production quantities in order to update the global production quantity variable $x_{t'}$, the stock level state variable $s_{t'}$, and the costs from production, setup, storage, and shortage incurred in stage t'. The detailed steps to obtain the stage decision $x_{t'}$ from a MIP-based model are summarized in Algorithm 2.

3.2.1. *Snapshot problem for MSLSUD-OO*

The available demand information consists of the latest demand $d_{t'}$ and the upcoming demands in the lookahead $\mathbb{L}_{t'} = \{d_{t'+1}, \ldots, d_{t'+\tau}\}$ and thus is deterministic. Hence, we will use the usual MIP formulation for classical deterministic lot sizing, and we do not allow shortages. The information time horizon is τ (recap that because $\mathbb{U}_t = \emptyset$, we have $\sigma = 0$), and we assume $\tau \geq 1$. Otherwise, we have only the trivial option to set $x_{t'} = d_{t'}$ which corresponds to the algorithmically rather uninteresting pure online setting. The information time horizon is represented by periods $H := \{t', t' + 1, \ldots, t' + \tau\}$, and the variables are the following: P_t indicates the production output in period t, Y_t is a binary variable which describes whether production happens in period t, and I_t specifies the stock level at the end of period t. The resulting formulation is presented in model MSLSUD-OO.

MSLSUD-OO:

$$\min \quad \sum_{t=t'}^{t'+\tau} c^p P_t + c^f Y_t + c^h I_t, \tag{1a}$$

$$\text{s.t.} \quad P_t + I_{t-1} - I_t = d_t \quad t \in H, \tag{1b}$$

$$P_t \leq MY_t \quad t \in H, \tag{1c}$$

$$I_{t'-1} = s_{t'-1}, \tag{1d}$$

$$I_{t'+\tau} \leq \kappa, \tag{1e}$$

$$Y_t \in \{0, 1\} \quad t \in H, \tag{1f}$$

$$P_t \in \mathbb{R}_{\geq 0} \quad t \in H, \tag{1g}$$

$$I_t \in \mathbb{R}_{\geq 0} \quad t \in H. \tag{1h}$$

The objective function (1a) minimizes the total cost. Constraints (1b) describe the inventory balance equations. Constraints (1c) relate the binary setup variables Y_t to the production variables P_t. The parameter κ governs the stock level after the information time horizon τ in stage $t' + \tau + 1$. It is specified by the decision-maker, e.g., according to the typical settings as discussed below. The value of M can be set to $\sum_{t\in H} d_t + \kappa$. Constraints (1d) and (1e) determine the stock levels at the beginning and the end of the planning horizon. Constraints (1f)–(1h) define the variable domains.

In the rolling horizon approach, it may not be the best choice to enforce an empty inventory at the end of the information horizon $t' + \tau$. Typical choices are $\kappa \in \{0, \frac{1}{\tau+1} \sum_{t\in H} d_t, \max\{d_t : t \in H\}\}$, and we abbreviate them by 0, AVG, and MAX, respectively. Setting κ to a specific threshold value is also possible, but not used here. Different choices for κ allow to add flexibility to the model. Especially in the presence of uncertain data (as shown for the RO and SP variants in the following subsections), it can happen that for $\kappa = 0$, production takes place too often because it is easier this way to achieve an empty inventory at the end of the planning horizon H. In particular, when there is a rather small demand value $d_{t'+\tau+1}$ which is just not seen in stage t', a new production batch must be employed in stage $t' + \tau + 1$ even though it would have been cheaper to include $d_{t'+\tau+1}$ into the batch from stage t'. The ending stock condition could also be $I_{t'+\tau} \geq \kappa$, but this would artificially increase the stock levels, partly independent of available demand information. From the solution vector $(P_{t'}, \ldots, P_{t'+\tau})$ to model MSLSUD-OO, we only consider the first entry $P_{t'}$ as the decision $x_{t'}$ in the current stage t' (see Algorithm 2).

3.2.2. Snapshot problem for MSLSUD-SP

In this setting, we assume an empty lookahead $\mathbb{L}_{t'}$, but instead we have stochastic information about the forthcoming σ stages given as $\mathbb{U}_{t'} = (\{(d_1^1, p_1^1), \ldots, (d_1^k, p_1^k)\}, \ldots, \{(d_\sigma^1, p_\sigma^1), \ldots, (d_\sigma^k, p_\sigma^k)\})$ where d_t^l specifies the lth possible demand value in stage $t' + t$ and p_t^l is the probability for realization of the lth possible demand value in stage $t' + t$. Hybrid settings can be integrated by setting $k = 1$ in $\mathbb{U}_{t'}$. Hence, this snapshot problem translates into a stochastic lot sizing problem with time periods $H := \{t', t' + 1, \ldots, t' + \sigma\}$. We call a sequence of demand realizations a scenario; it is denoted by $\xi = (\xi_{t'}, \xi_{t'+1}, \ldots, \xi_{t'+\sigma})$. All possible scenarios comprise the scenario set $\mathcal{U}$ where

$|\mathcal{U}| = k^\sigma$. Note that the first period t' is deterministic, hence $\xi_{t'} = d_{t'}$, for each $\xi \in \mathcal{U}$. The probability of a scenario $\xi \in \mathcal{U}$ is the product of the probabilities for each of the scenario's realizations. This means that for a scenario $\xi = (d_{t'}, d^1_1, d^1_2, \dots, d^1_\sigma) \in \mathcal{U}$ the probability is $\Pr(\xi) = \prod_{i=1}^{\sigma} p^1_i$. We will formulate the snapshot problem as a MIP using the deterministic equivalent formulation. The key ideas are explained next. For more details on deterministic equivalent formulations for stochastic multistage problems, we refer the reader to Shapiro et al. (2014, Section 3.1). For each scenario $\xi \in \mathcal{U}$, we introduce a new set of variables $P_t(\xi)$, $Y_t(\xi)$, $I_t(\xi)$, $t \in H$ which have to satisfy the constraints of model MSLSUD-OO. One can also interpret the variables P_t, Y_t, and I_t as adjustable to data once new data has been revealed. In order to ensure that each $P_t(\xi)$, $Y_t(\xi)$, $I_t(\xi)$ only depends on data already known at period t we introduce *nonanticipativity constraints*. We denote elements of a scenario ξ revealed up to period $t \in H$ by $\xi_{[t]} := (\xi_{t'}, \dots, \xi_t)$, i.e., $\xi_{[t']} = \xi_{t'}$ and $\xi_{[t'+\sigma]} = \xi$. The nonanticipativity constraints for the selected production quantities P_t then read $P_t(\xi) = P_t(\xi')$ for $\xi, \xi' \in \mathcal{U},\ \xi_{[t]} = \xi'_{[t]},\ t \in H$.

Uncertainty in the demand data requires flexibility in the model, and we allow shortages. There are several possibilities for resolving shortages such as backlogging (demand has to be met in later periods) or lost sales (demand once unmet is ignored for good which incurs additional costs (Aksen et al., 2003; Absi and Kedad-Sidhoum, 2008)). Here, we consider shortages through incorporating lost sales into the model. To this end, we introduce variables $L_t(\xi)$ that indicate the number of stockouts. The complete MIP formulation for the stochastic snapshot problem is given by model MSLSUD-SP.

MSLSUD – SP:

$$\min \quad \sum_{\xi \in \mathcal{U}} \Pr(\xi) \Big(\sum_{t \in H} \big(c^p P_t(\xi) + c^f Y_t(\xi) + c^h I_t(\xi) + c^s L_t(\xi) \big) \Big), \tag{2a}$$

$$\text{s.t.} \quad P_t(\xi) + I_{t-1}(\xi) - I_t(\xi) + L_t(\xi) = d_t(\xi) \quad \xi \in \mathcal{U},\ t \in H, \tag{2b}$$

$$L_t(\xi) \le d_t(\xi) \quad \xi \in \mathcal{U},\ t \in H, \tag{2c}$$

$$P_t(\xi) \le M Y_t(\xi) \quad \xi \in \mathcal{U},\ t \in H, \tag{2d}$$

$$I_{t'-1}(\xi) = s_{t'-1} \quad \xi \in \mathcal{U}, \tag{2e}$$

$$I_{t'+\tau}(\xi) \le \kappa(\xi) \quad \xi \in \mathcal{U}, \tag{2f}$$

$$P_t(\xi) = P_t(\xi') \quad \xi, \xi' \in \mathcal{U},\ \xi_{[t]} = \xi'_{[t]},\ t \in H, \tag{2g}$$

$$Y_t(\xi) = Y_t(\xi') \quad \xi, \xi' \in \mathcal{U},\ \xi_{[t]} = \xi'_{[t]},\ t \in H, \tag{2h}$$

$$I_t(\xi) = I_t(\xi') \quad \xi, \xi' \in \mathcal{U},\ \xi_{[t]} = \xi'_{[t]},\ t \in H, \tag{2i}$$

$$L_t(\xi) = L_t(\xi') \quad \xi, \xi' \in \mathcal{U},\ \xi_{[t]} = \xi'_{[t]},\ t \in H, \tag{2j}$$

$$P_t(\xi),\ L_t(\xi) \in \mathbb{R}_{\ge 0} \quad \xi \in \mathcal{U},\ t \in H, \tag{2k}$$

$$Y_t(\xi) \in \{0, 1\} \quad \xi \in \mathcal{U},\ t \in H, \tag{2l}$$

$$I_t(\xi) \in \mathbb{R}_{\ge 0} \quad \xi \in \mathcal{U},\ t \in H. \tag{2m}$$

Most of the constraints are the same as in model MSLSUD-OO, and we explain only the differences here. The objective (2a) minimizes the expected total costs. The constraints (2c) ensure that lost sales do not exceed the demand. Constraints (2g)–(2j) enforce nonanticipativity. The parameter $\kappa(\xi)$ governs the ending condition. Possible choices are $\kappa(\xi) \in \{0, \frac{1}{\sigma+1}\sum_{t\in H} d_t(\xi), \max\{d_t(\xi) : t \in H\}\}$ or a specific threshold value. Typically, $\kappa =$ AVG or MAX is a more natural choice than $\kappa = 0$. From the solution set $\{x_t(\xi) : t \in H, \ \xi \in \mathcal{U}\}$ to model MSLSUD-SP, we only implement $P_{t'}(\xi)$ (which is the same for all scenarios due to nonanticipativity constraints) in the current stage t'. Note that in any optimal solution, if $L_t(\xi) > 0$ then we have $L_t(\xi) = d_t(\xi) - (P_t(\xi) + I_{t-1}(\xi))$. Otherwise $L_t(\xi) = 0$, because the shortage costs c^s are chosen sufficiently high.

Model MSLSUD-SP can be easily extended to handle hybrid settings. Any deterministic demand values in the lookahead $\mathbb{L}_{t'}$ can be integrated into all scenarios $\xi \in \mathcal{U}$. This increases the horizon H by τ stages but allows to reuse the MIP formulation in model MSLSUD-SP.

3.2.3. Snapshot problem for MSLSUD-RO

This snapshot problem is very similar to the one of MSLSUD-SP. The main difference is that while MSLSUD-SP is aiming at the average case, MSLSUD-RO is aiming at the worst case. Therefore, no probability information is required for MSLSUD-RO, and the uncertain information about the upcoming demands $\mathbb{U}_{t'}$ is given as

$$\mathbb{U}_{t'} = \left(\{d_1^1, \dots, d_1^k\}, \dots, \{d_\sigma^1, \dots, d_\sigma^k\}\right),$$

where $\{d_i^1, \dots, d_i^k\}$, $i \in \{1, \dots, \sigma\}$ are the possible demand realizations in stage $t' + i$. To switch from average case to worst case orientation, we replace the expected cost objective in MSLSUD-SP by the min max objective function widely used in robust optimization (see Equation (3a)). Since the multistage structure of the problem remains, we keep all the constraints of MSLSUD-SP resulting in model MSLSUD-RO.

MSLSUD-RO:

$$\min \quad \max_{\xi\in\mathcal{U}} \sum_{t\in H}\left(c^p P_t(\xi) + c^f Y_t(\xi) + c^h I_t(\xi) + c^s L_t(\xi)\right), \tag{3a}$$

$$\text{s.t.} \quad P_t(\xi),\ Y_t(\xi),\ I_t(\xi), L_t(\xi) \text{ satisfy constraints (2b)–(2m).} \tag{3b}$$

We remark that models MSLSUD-OO, MSLSUD-SP, and MSLSUD-RO can be easily extended by a constant lead time τ through introducing an offset in the indices of the production quantity variables $P_{t-\tau}$ and $P_{t-\tau}(\xi)$ in the inventory balance constraints, respectively, and adapting the index range of t accordingly.

3.3. Illustrative examples

We present two example instances to illustrate the differences between stochastic programming, robust, and online optimization variants of MSLSUD. Moreover, we discuss the peculiarities of each paradigm. Besides showing the application of the modeling framework used, we particularly clarify the trade-offs between these different variants.

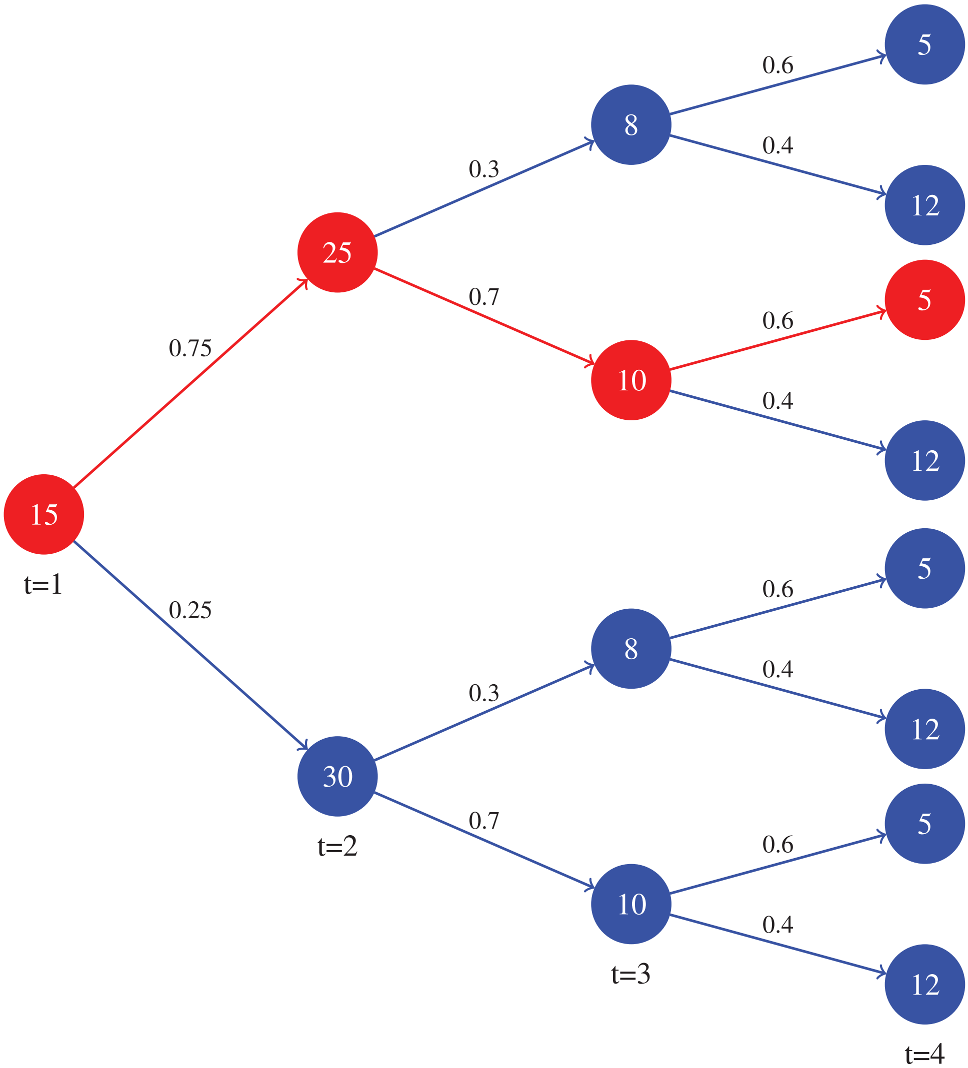


Fig. 4. Scenario Tree - Example A.

3.3.1. Example A

This example demonstrates the potential benefits that larger τ/σ values entail for the OO and SP variants, respectively. We choose $c^p = 2$, $c^f = 20$, $c^h = 1$, $c^s = 100$, $s_0 = 0$, $\kappa =$ MAX. Moreover, we set $\tau = 1$ whereas $\sigma = 2$. Figure 4 exhibits the problem's scenario tree, the results obtained can be found in Table 2. Results for SP with $\sigma = 1$ are reported for the sake of completeness and are not further discussed.

We notice the following: If $\kappa =$ AVG or MAX and the variation in demand is small in the last periods, then production of the corresponding demands is bundled in earlier periods. Production usually takes place for the demand realization with the highest value. In case of a different realization, the fact that residual stock exists at the end of the planning horizon is accepted. SP performs

Table 2
Results obtained for the different examples

Example A						Example B				
Variant	x_1	x_2	x_3	x_4	z^*	Variant	x_1	x_2	x_3	z^*
$OO_{\tau=1}$	15	35	0	5	180	$OO_{\tau=1}$	12	0	2	88
$SP_{\sigma=1}$	15	35	0	5	180	$SP_{\sigma=1}$	14	0	0	72
$SP_{\sigma=2}$	15	40	0	0	170	$RO_{\sigma=1}$	13, 98	0	0	73, 88

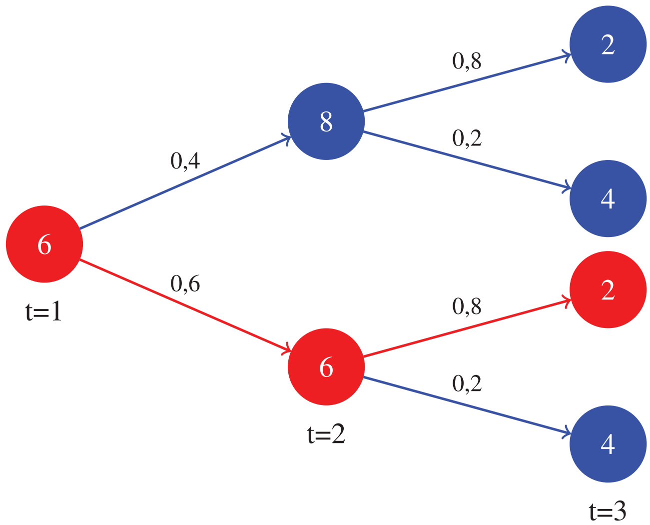


Fig. 5. Scenario tree: Example B.

better than OO as it is given information about an additional future period (in case $\tau = \sigma = 1$ both OO and SP lead to the same results). Despite the fact that the information SP has at its disposal is uncertain, it saves setup costs by producing in advance. In order to decide on the quantities to produce, it utilizes the probabilities of the different demands being realized. Note that in this example the scenario that is realized is the one with the highest overall probability. This makes it particularly easy for SP to produce competitive results.

3.3.2. Example B

To conclude the examples, we present another example to illustrate that SP can yield a better solution than OO despite the fact that $\tau = \sigma = 1$. This behavior may be observed when the problem has a special structure which will lead SP to make a decision based on the uncertain data that, ex post, turns out better than the decision made by OO using deterministic information about the future. Parameter values $c^p = 3,\ c^f = 20, c^h = 1,\ c^p = 100,\ s_0 = 0,\ \kappa = \text{MAX}$, and $\tau = \sigma = 1$ are selected. A simple three-period problem with a scenario tree depicted in Fig. 5 is sufficient to exhibit the desired behavior.

The results are given in Table 2. Stochastic programming provides a better solution than online optimization in this case despite having effectively less information about the future. SP prepares for a demand of 8 units in period 2 and already produces this quantity in period 1 in order to not bear setup costs again in period 2. In fact, less quantity is demanded in period 2, so the remaining quantity is stocked for period 3 where it turns out to be sufficient for satisfying the corresponding demand d_3. OO knows for sure that $d_2 = 8$ will not be realized and therefore does not prepare for it. In fact, it ends up having a disadvantage because of its certain information. In comparison, SP's excessive hedging also prepares for period 3 without possessing any information about it in period 1. For the sake of completeness, the result of MSLSUD-RO is listed despite not being the focus here. Note that RO produces a nonintegral quantity despite integral demands. The reason is that it is cheaper in terms of total costs not to produce the full demand of 14 (as in the case of SP) and pay penalty costs for the missing 0.2 units of demand because of RO's limited flexibility due to its worst-case orientation. The overall costs of RO are higher than SP's but less in comparison to OO.

We conclude this section with the discussion of a well-known property in lot sizing problems, the zero inventory ordering property. This property does not hold for MSLSUD. As demand quantities are not known in advance for the entire time horizon, it may occur that production takes place in a period despite the inventory not being empty, but insufficient to satisfy the period's demand. We can, however, establish a weaker form of the zero inventory ordering property stating that production only takes place if the current demand exceeds the current stock. This property follows directly from Step 5 in Algorithm 1 where the production quantity x_t for the current stage t is set to 0 if the current demand d_t can be met from the stock.

4. Computational experiments

4.1. Experimental setup

All models and algorithms are implemented in C++ and evaluated on a Ubuntu Linux 20.04 machine (Intel Core i7-5930K CPU @ 3.50GHz × 12 with 64 GB of RAM). MIPs are solved using CPLEX 20.1. We use a MIP gap of 0.0, integer tolerance of 0.0, and a single thread. Other CPLEX settings remain at the defaults.

All instances are generated in a holistic way. In general, it is not obvious how to produce adequately comparable instances for different paradigms due to paradigms' intrinsic differences. The main goal of the generation process is to obtain instances that have the same demand realizations per stage and hence are as comparable as possible. For each parameter set, the generation process looks as follows. In the first step, we create possible demand realizations and the corresponding realization probabilities based on a fixed random number generator (RNG) seed. We draw demand values according to a uniform distribution with requested upper and lower bounds. In the second step, we choose one of the possible realizations as the actual demand according to the realization probabilities. This is the raw data used subsequently (see Table 3 for an example).

Note that this approach fixes one possible trajectory for the transition between stages. From these data, it is possible to combine single entries to specific instances of a certain paradigm and a prescribed information horizon. SP uses all information and picks a suitable subset of the realization set according to the prescribed size of uncertainty set $\mathbb{U}_t$. RO drops the probabilities, and OO utilizes only the realized demands of each stage (see Table 4). This approach evidently extends to

Table 3
An example raw instance for $T = 2$ and information horizon of 1

Period	Possible realizations	Chosen realization
$t = 1$	[21, 0.15; 13, 0.85]	21
$t = 2$	[40, 0.39; 53, 0.61]	53
$t = 3$	[26, 0.42; 39, 0.58]	39

Table 4
Example instances for $T = 2$ and information horizon of 1 based on the raw instance from Table 3

`paradigm`	Information available at $t = 1$	Information available at $t = 2$
OO	$d_1 = 21, \mathbb{L}_1 = [53]$	$d_2 = 53, \mathbb{L}_2 = [39]$
SP	$d_1 = 21, \mathbb{U}_1 = [40, 0.39; 53, 0.61]$	$d_2 = 53, \mathbb{U}_2 = [26, 0.42; 39, 0.58]$
RO	$d_1 = 21, \mathbb{U}_1 = [40; 53]$	$d_2 = 53, \mathbb{U}_2 = [26; 39]$

hybrid settings. Instances with a different number of realizations per stage use different RNG seeds and hence are not comparable.

The solution process terminates after a predefined number of stages that are prescribed in the parameter T of an instance. In order to make different information horizon settings comparable, we cut off all information exceeding stage T. Otherwise larger $\mathbb{F}_t$ may be discriminated because much more production may be planned in stages close to T as for smaller $\mathbb{F}_t$.

4.2. Influence of parameters and modeling choices on solution quality

Unless stated otherwise, we use a planning horizon of $T = 100$ and consider averaged values over $N = 100$ instances. The demand is drawn uniformly from the interval [10, 100]. The value of T is unknown to the decision-maker as long as stage T has not been finished. For RO and SP instances, we consider two possible realizations per stage in order to keep the computational effort of the overall experiment set under control. This restriction significantly reduces the computational effort without changing the overall qualitative statement obtained from the computations. Clearly, excessive experimentation based on solving MIP formulations often induces large running times. Therefore, as a glimpse, we provide some rough numbers: Experiments for one specific setting using the OO paradigm in Fig. 6 take 1–3 minutes. RO and SP experiments as presented in Fig. 7 consume 3–6 times more time each, namely 3–18 minutes. Solving a single snapshot problem takes seconds. We consider running times of up to 18 minutes of an experiment with $T = 100$ and 100 repetitions as negligible in the sense that they are small enough for practical implementation and do not provide further details. Note also that by relaxing the MIP gap from 0.0 (see Section 4.1) to less strict, but still practically acceptable levels or by using heuristics (see Section 4.2.3) the running times can be kept low even for much larger instances.

To facilitate the reading and interpretation of the plots, we indicate OO by solid lines, RO by dotted lines, and SP by dashed lines.

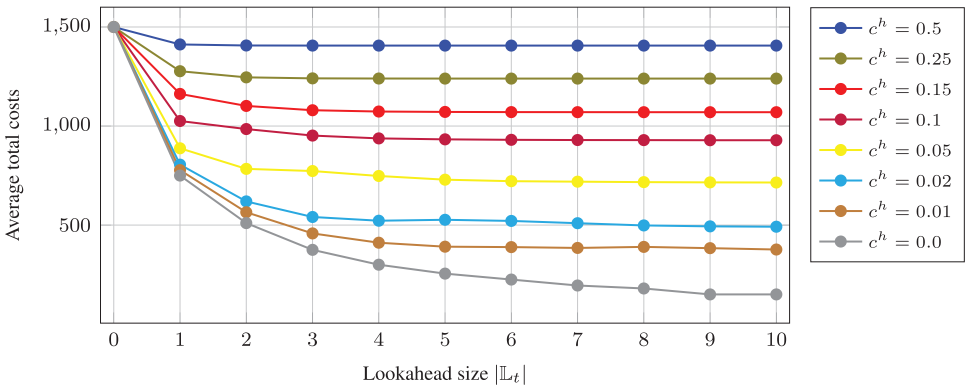


Fig. 6. Value of lookahead in online optimization ($\kappa = 0$).

4.2.1. *Value of lookahead in OO and value of forecasts in RO and SP*

This set of experiments studies the benefits of increasing the size of lookahead $\mathbb{L}_t$ for MSLSUD-OO (Fig. 6) and the size of forecasts $\mathbb{U}_t$ for MSLSUD-RO and SP (Fig. 7). We use fixed cost parameters $c^p = 0,\ c^f = 15,\ c^s = 5$, and storage costs c^h varying from 0 to 0.5 in every setting. Unarguably, the value of lookahead/forecasts highly depends on the storage costs c^h. Even in the utopian setting $c^h = 0$ (gray line) which we use as the baseline, the value of additional information in $\mathbb{L}_t$ and $\mathbb{U}_t$ is rapidly decreasing with increasing amount of future information. For OO, there is no difference concerning the ending condition κ. As expected, RO produces higher costs than SP. Additionally, RO seems to struggle with $\kappa = 0$ being able to perform only slightly better for $|\mathbb{U}_t| = 1$ compared to $|\mathbb{U}_t| = 0$. This effect disappears if κ is changed to AVG which indicates that AVG is a more natural choice for κ since it gives RO enough flexibility to find a robust solution for all possible scenarios. One reason for RO's higher costs is lost sales which occur more often than for SP where they are very scarce.

In order to be able to study the value of information in more detail, we will take a look at the actual production activities of MSLSUD-OO next. For RO and SP, the results are similar. First, we introduce the notion of a *production horizon*. This quantity describes the number of time periods (beyond the current one) for which the demand can be met out of stock. In the computation, we only look at stages where production actually happens. As before, the costs are fixed as follows: $c^p = 0,\ c^f = 15$. In Fig. 8, we can see directly when the production horizon stagnates depending on the holding costs c^h.

4.2.2. *Interparadigmal comparison*

Here we compare different paradigms and hybrids thereof. We combine lookahead and forecast information, and we use the combined information horizon $|\mathbb{L}_t| + |\mathbb{U}_t|$ as the benchmark. The costs are fixed ($c^p = 0,\ c^f = 15,\ c^s = 5$) except for c^h which varies from 0 to 0.15. The ending condition κ is set to AVG, and OO (black solid line) is used as a baseline. In the utopian setting $c^h = 0$ (Fig. 9), all approaches perform similarly with minor advantages for SP over RO and deterministic over nondeterministic information (i.e., $\mathbb{L}_t$ over $\mathbb{U}_t$). This picture repeats itself with higher storage costs but with more distance between the approaches (see Figs. 10 and 11). Deterministic lookahead

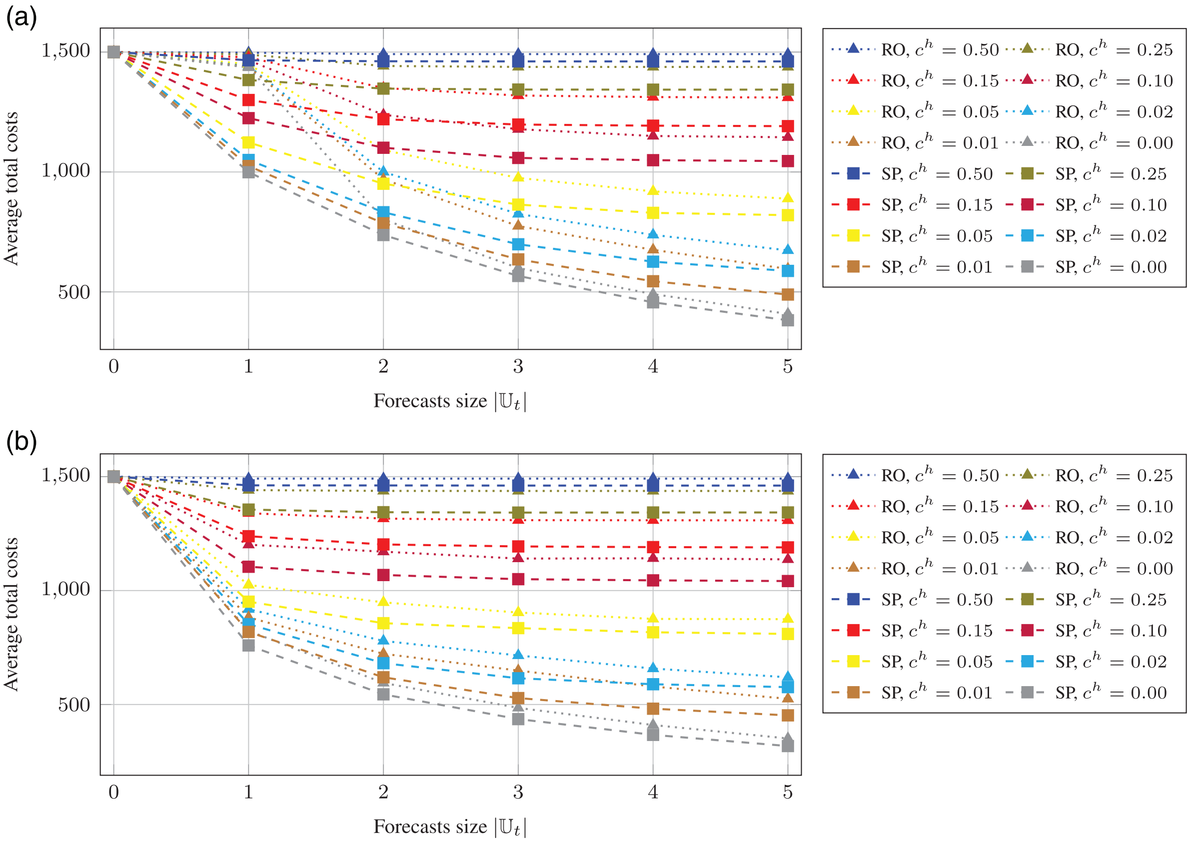


Fig. 7. Value of forecasts in RO and SP (top: $\kappa = 0$, bottom: $\kappa = \text{AVG}$).

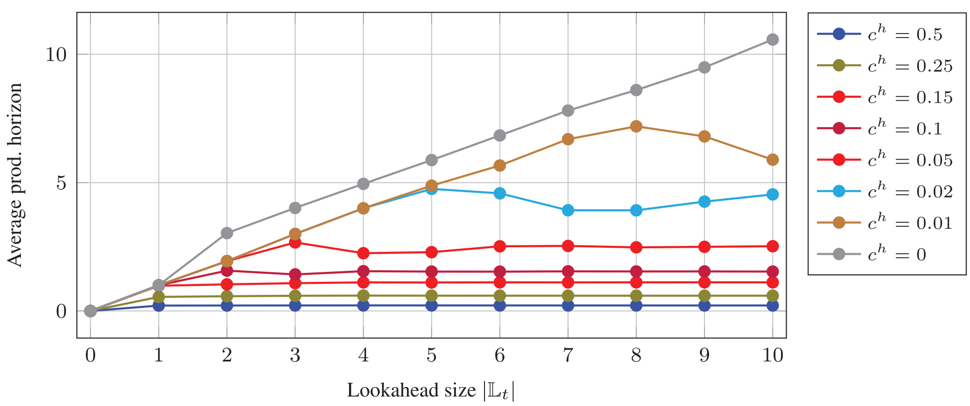


Fig. 8. Production horizon for OO ($\kappa = \text{AVG}$).

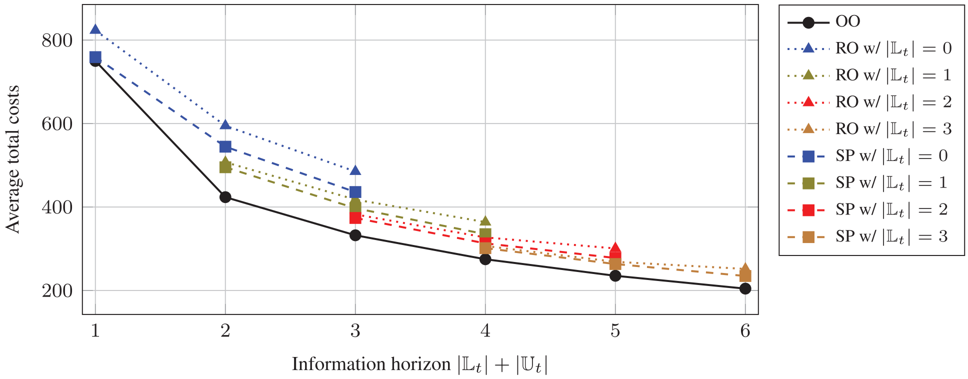


Fig. 9. Comparison of hybrid uncertainty models ($c^h = 0$).

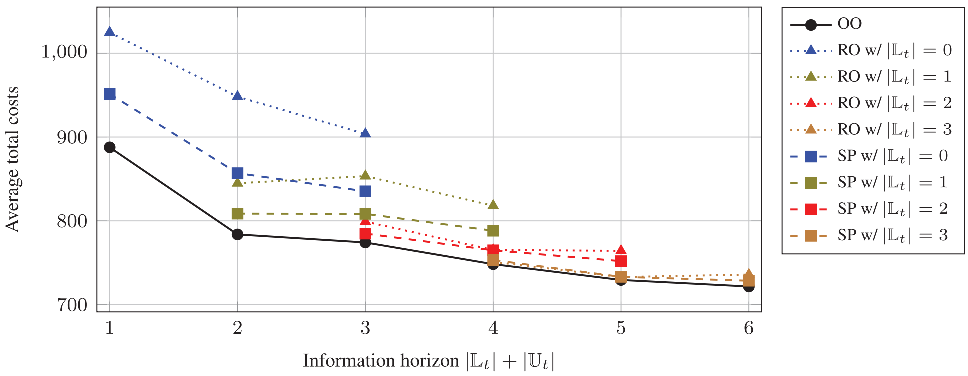


Fig. 10. Comparison of hybrid uncertainty models ($c^h = 0.05$).

information is in tendency more valuable, e.g., RO with $|\mathbb{L}_t| = 1$ and $|\mathbb{U}_t| = 1$ slightly outperforms SP with $|\mathbb{L}_t| = 0$ and $|\mathbb{U}_t| = 2$ for $c^h = 0.05$ (Fig. 10). With increasing information horizon and storage costs all variants behave almost the same because the value of future information drops regardless of its (non-)deterministic nature (Fig. 11).

4.2.3. Solving snapshot problems within MSLSUD-OO with inexact methods

Here we study the influence of the use of exact methods compared to heuristics on the snapshot problem level, i.e., instead of solving the MIP-model MSLSUD-OO exactly in Step (7) of Algorithm 1 as in the previous experiments, we employ a heuristic. The well-known and commonly used Silver–Meal heuristic (Silver and Meal, 1973) is chosen to represent inexact methods. We omit the Silver–Meal heuristic's functionality details here since they are not important. As in Algorithm 2 we only use the first entry of the heuristic solution and return it to the rolling horizon procedure as the production decision x_t for the current stage. We use $c^p = 0,\ c^f = 15$ as fixed cost parameters, while the storage costs c^h vary from 0 to 0.1. The clear conclusion from Fig. 12 is that inexact and exact methods lead to very comparable if not identical results. We also conducted experiments with higher storage costs c^h (as in the above experiments). In these cases, the overall costs remain the

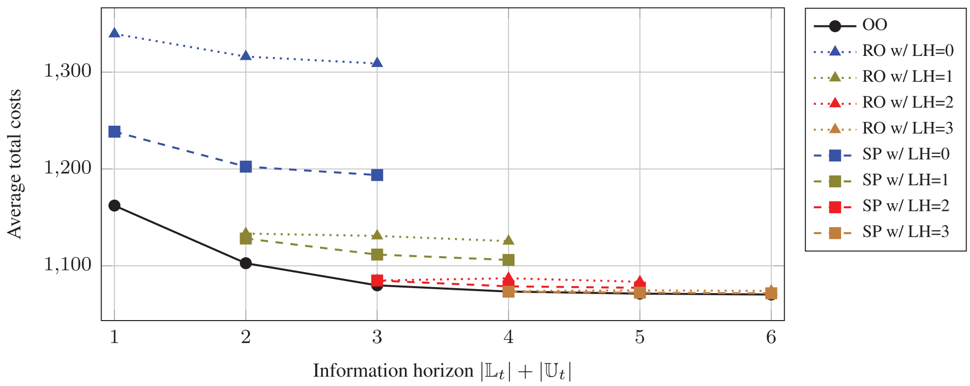


Fig. 11. Comparison of hybrid uncertainty models ($c^h = 0.15$).

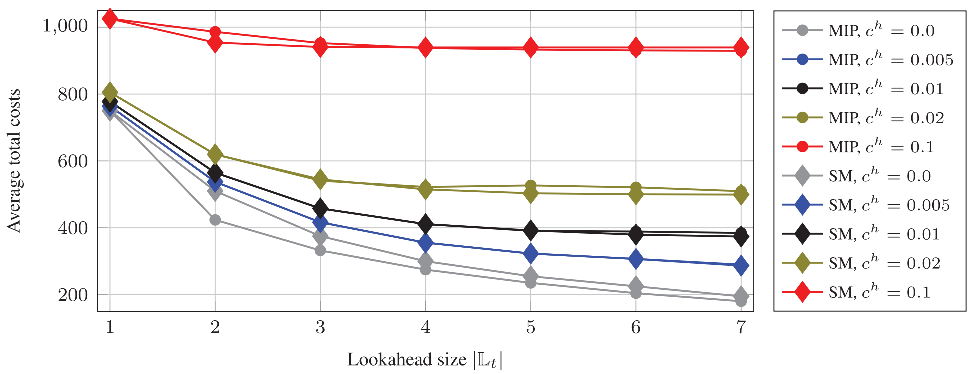


Fig. 12. Solving snapshot problems exactly and via the Silver–Meal heuristic (κ = AVG).

same regardless of whether the snapshot problems were solved to optimality or heuristically and thus we omitted the presentation of these results in Fig. 12. Hence, we conclude that the selection of the snapshot algorithms should not be based on snapshot solution quality but rather on other metrics like running time.

4.2.4. Influence of demand deviation on RO and SP

This set of experiments studies whether RO and SP exhibit differing sensitivity upon demand deviation. Unlike in the experiments before, the interval the demand is drawn from uniformly varies from [10,20] to [10,100] which is used in all other experiments. The storage costs c^h vary but the remaining costs are fixed to $c^p = 0$, $c^f = 15$, $c^s = 5$. For holding costs $c^h = 0.25$ (Fig. 13), the results are inconclusive because not much is produced in advance anyway. But for $c^h = 0.05$ (Fig. 14), SP and RO are approaching each other as the demand interval is getting smaller implying that RO partly loses its conservatism compared to SP. However, SP consistently yields much better results in terms of total costs.

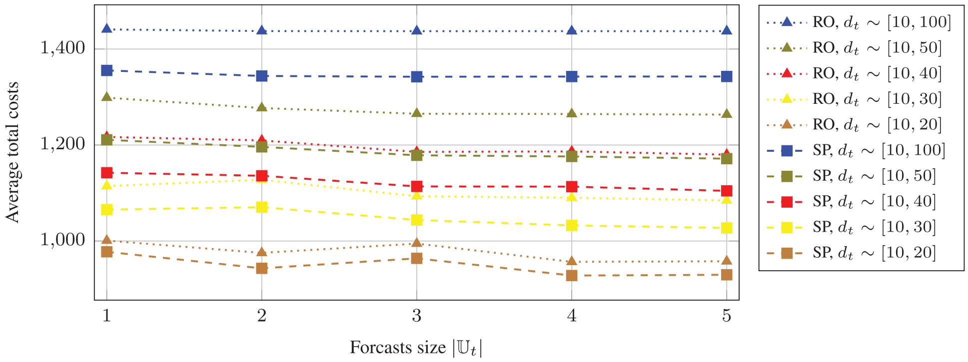


Fig. 13. Influence of demand deviation on RO and SP (κ = AVG, $c^h = 0.25$).

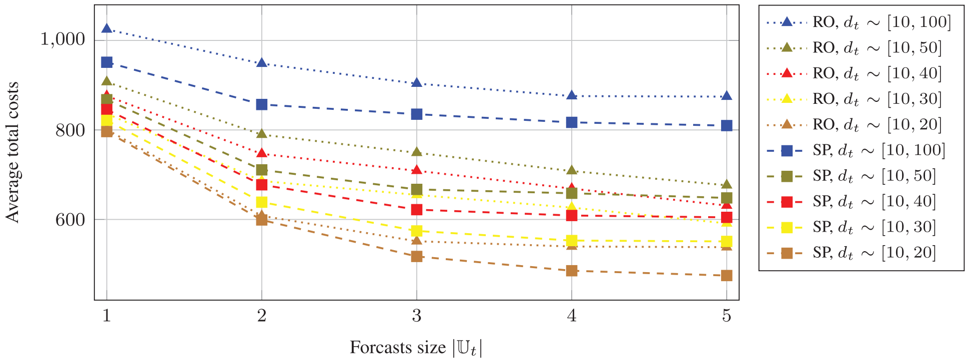


Fig. 14. Influence of demand deviation on RO and SP (κ = AVG, $c^h = 0.05$).

4.2.5. Conclusion from computational experiments

The overall outcome of the presented computational experiments is twofold: First, the modeling framework for decision-making under uncertainty introduced in Bindewald et al. (2020) can be applied to design and carry out a wide range of experiments demonstrating the framework's flexibility. Second, regarding the concrete application of lot sizing important questions like “What do we gain from expanding the forecasts” or “How much do we lose by employing heuristics?” can be answered by helping practitioners in improving the quality of their decision-making routines with respect to the specific degrees of freedom available in controlling the production quantities to be implemented.

5. Conclusion and outlook

The variety of research questions addressed in Section 4 for examining the specific problem of multistage lot sizing under uncertainty through instantiation of a generic modeling framework shows that there is a need for supporting decision-makers both systematically and methodologically in their task of establishing promising algorithmic solutions for time-dynamic optimization problems under uncertainty. Time and uncertainty are the two main factors accounting for the increased difficulty both with respect to representing and evaluating solution trajectories of different optimization paradigms. Manual output analysis and comparison between solution trajectories are rather tedious, necessitating the development of an automated process for the output analysis in time-dynamic optimization under uncertainty. Clearly, such a framework then defines the kind of questions which can be answered within reasonable computational time required for experimentation as has been demonstrated with an investigation of several research matters to be accounted for in multistage lot sizing with uncertain demands. Finally, the computational studies here are the first of their kind and allow for different subsequent investigations. For instance, solution trajectories can be influenced through data-dependent dynamic switching between snapshot algorithms (online optimization, robust optimization, stochastic programming) or through altering parameters of the snapshot algorithms (such as the size of the lookahead/forecasts, ending condition on inventory level in MP formulations). On the application level, the setting can be developed further by additional features such as the integration of lead times. Depending on whether these represent another source of randomness, the framework must be adapted to handle several factors of uncertainty simultaneously. Concerning the methodological perspective, the applicability of the multistage approach could be further studied with respect to different aspects: To improve running times, tailored decomposition methods can be incorporated; to tackle the conservatism of solutions resulting from the robust optimization paradigm, one could resort to adjustable robustness.

Acknowledgments

This work is supported by the German Research Foundation (DFG) [Grants DU 1755/1-1 and NI 521/9-1, project no. 354864080]. This support is gratefully acknowledged.

Open access funding enabled and organized by Projekt DEAL.

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