Dimension Estimates for Parabolic Equations and Harmonic Maps of low Index

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Abstract

In the first part of this thesis we establish dimension bounds for spaces of polynomial growth solutions to divergence form parabolic partial differential equations with time-dependent coefficients as well as sharp bounds in the case of time-independent coefficients.
The second, main part of this thesis is devoted to smooth, harmonic maps of low index, in particular to the classification of such maps into round spheres.
Towards this we first study the change of the index of smooth, harmonic submersions into round spheres upon consideration as maps into higher-dimensional round spheres, that is after composition with the totally geodesic inclusion map into a higher-dimensional sphere. This is illustrated by the examples of the Hopf maps $S^3 \to S^2$ and $S^1 \to S^1$.
As an application we can describe for $n \geq 3$ a class of harmonic morphisms of low index from the round n-sphere into submanifolds of the round n-sphere arising from smooth maps of constant rank.
Finally we prove an El Soufi-type index bound for smooth, harmonic maps from simply connected Riemannian manifolds satisfying the Killing property into round spheres. This bound depends only on the domain manifold, in particular it does not depend on the dimension of the codomain sphere. Focussing thereafter on the case that the codomain is the round two-sphere, we can show that every smooth, harmonic map of low index with respect to this new index bound must be a harmonic morphism.

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Contents

Chapter 1

Introduction

Given smooth, compact Riemannian manifolds $(M,g),(N,h)$ without boundary, the Dirichlet energy of a map $u \in W^{1,2}(M,N)$ is given by

$$E(u) = \frac{1}{2}\int\limits_M |du|^2$$

with respect to the volume element of M. Formally, we can define $E(u)$ in this way also for non-compact domains M, however $E(u)$ may then not be finite.
Critical points u of the Dirichlet energy are referred to as (weakly) harmonic maps, and they are characterized by the vanishing of their tension field, that is

$$\tau(u) := \operatorname{tr}\left[\nabla du\right] = 0.$$

If the target manifold N is isometrically embedded in some Euclidean space $\mathbb{R}^l$, they also satisfy the so-called harmonic map equation

$$\Delta u + B(u)\,(du,du) = 0,$$

where B denotes the second fundamental form of the embedding $N \to \mathbb{R}^l$ and Δ is the Laplace-Beltrami operator of M. Moreover, for a (weakly) harmonic function $u : M \to \mathbb{R}$ this is consistent with the notion of harmonicity defined via the Laplacian $\Delta = \operatorname{div}(\nabla)$, i.e. that for any $\phi \in C_c^\infty(M,\mathbb{R})$ the function u satisfies

$$\int\limits_M \nabla u \nabla \phi = 0.$$

Considering general elliptic operators in divergence form

$$L := \operatorname{div}(A\,\nabla)\,, \tag{1.1}$$

where A is smooth in the time variable t, and for any fixed $t \in \mathbb{R}^-$ the coefficient $A(t,\cdot)$ is a smooth section of the bundle of symmetric automorphisms of the tangent bundle TM which fulfills for any $r > 0$, $q \in M$ and $X \in TM$

$$\|A\|_{L^\infty(Q_r(t,q))} \leq c \quad \text{as well as} \quad c_1\, g\,(X,X) \leq \; g\,(AX,X) \tag{1.2}$$

for universal constants $\frac{1}{2} < c_1 < +\infty$ and $0 < c < +\infty$ and with

$$Q_r(t,q) := \left]t - r^2, t\right] \times B_r(q)$$

the parabolic cylinder of radius r at (t,q), we can define L-harmonic maps. Namely, a map $u \in W^{1,2}(M,\mathbb{R})$ is called L-harmonic if we have for any $\phi \in C_c^\infty(M,\mathbb{R})$

$$\int_M A\nabla u \nabla\phi = 0.$$

It should be noted that the assumption $c_1 > \frac{1}{2}$ on the ellipticity constant c_1 holds up to re-scaling of A for any such divergence form elliptic operator L and can hence be made without loss of generality.
In particular, if A is the identity section, then L reduces to the Laplacian and L-harmonicity reduces to harmonicity.
Furthermore, a function $u : M \to \mathbb{R}$ is of polynomial growth of order at most $d \geq 0$ if there exists a constant $C_u > 0$ such that for some $p \in M$ and any $r > 0$ we get

$$\sup_{B_r(p)} |u(x)| \leq C_u \left(1+r\right)^d.$$

The set of all L-harmonic functions of polynomial growth of order at most d is denoted by $\mathcal{H}_d\left(M;L\right)$. In case $L = \Delta$ we also simply write $\mathcal{H}_d(M)$.
The spaces $\mathcal{H}_d(M)$ became a subject of interest after Yau conjectured in 1981 (e.g. [78]) that for M^n an open, non-compact Riemannian manifold with non-negative Ricci curvature $\mathcal{H}_d(M)$ is finite-dimensional. Li and Tam proved this for both the case of linear growth $d = 1$ ([51]) and the case of surfaces $n = 2$ ([52]). The latter was independently also shown by Donnelly and Fefferman in [24].
Eventually, in 1997, the conjecture was settled completely by Colding and Minicozzi in [19] by proving for $d \geq 1$

$$\dim\left(\mathcal{H}_d(M)\right) \leq C\, e^{\tilde{C}(d)}$$

for any non-compact Riemannian manifold M which has the volume doubling property and a scale-invariant Neumann-Poincaré inequality, both of which hold on Riemannian manifolds with non-negative Ricci curvature (cf. [9, 14]). Soon after they refined their proof and obtained in [20] the sharp bound

$$\dim\left(\mathcal{H}_d(M)\right) \leq C\, d^{C_V - 1}$$

for $d \geq 1$, assuming a scale-invariant Neumann-Poincaré inequality and the relative volume comparison property with constant C_V. The key to proving both these bounds and only place where harmonicity is explicitly used is a reverse Poincaré inequality, sometimes also called Caccioppoli inequality, for harmonic functions, which incidentally was already established by Yau in [77].
An alternative proof of the sharp dimension bound, using a mean value inequality for harmonic functions instead of a Neumann-Poincaré inequality and a reverse Poincaré inequality, was given by Li in [50]. This approach led to a vast number of results concerning more general elliptic operators, systems of equations or polynomial growth solutions on groups and graphs, for example [16, 39, 40, 41, 53, 54, 71, 79].
The natural next step is to consider dimension bounds for spaces of polynomial growth ancient solutions to parabolic partial differential equations, that is dimension bounds for $\mathcal{P}_d\left(M;L\right)$, $d \geq 0$, where for every $u : \mathbb{R}^- \times M \to \mathbb{R}$ in $\mathcal{P}_d\left(M;L\right)$ we have $\partial_t u = Lu$ and there exists a constant $C_u > 0$ so that for some $p \in M$ and any $r > 0$ we obtain

$$\sup_{Q_r(0,p)} |u(t,x)| \leq C_u \left(1+r\right)^d.$$

The first result in this direction was given by Lin and Zhang in [56] who proved, adapting Li's approach to the parabolic setting, under the assumptions of the volume doubling property with constant C_D and a mean value inequality for ancient solutions to $\partial_t u = \Delta u$ that

$$\dim\left(\mathcal{P}_d(M)\right) \leq C\, d^{C_D + 1}.$$

Using a reverse Poincaré inequality for ancient solutions of the heat equation, Colding and Minicozzi subsequently sharpened this bound in [21] for $d = 2k$, $k \geq 1$, to

$$\dim\left(\mathcal{P}_d(M)\right) \leq \; C\, d^{C_V},$$

assuming a scale-invariant Neumann-Poincaré inequality and the relative volume comparison property with constant C_V. In Chapter 2 we consider the spaces $\mathcal{P}_d\left(M;L\right)$, $d \geq 1$, with L defined as in (1.1). As a preparation for the first dimension bound we prove in section 2.2 a mean value inequality for solutions to $\partial_t u = Lu$. In section 2.3 we then adapt the reasoning of Lin and Zhang's proof for the Laplacian to our operator L and obtain the same dimension bound as they did for the heat operator.

Theorem 1.0.1. *Let M be a complete, non-compact Riemannian manifold on which the volume doubling property with constant C_D and a Neumann-Poincaré inequality with constant C_N hold. Then, there is a constant $0 < C < +\infty$, depending only on C_D and C_N, such that for any integer $q \geq 1$*

$$\dim\left(\mathcal{P}_q(M;L)\right) \leq C\, q^{C_D+1}.$$

Section 2.4 deals with the case when the coefficient A is independent of the time variable. In this situation we can transfer the arguments of [21] from Δ to L, giving us an improved dimension bound.

Theorem 1.0.2. *Suppose that M is a complete, non-compact Riemannian manifold which satisfies the relative volume comparison property with constant C_V and a Neumann-Poincaré inequality with constant C_N. Suppose that A is time-independent. Then, there exists a constant $0 < C < +\infty$ so that for any integer $k \geq 1$*

$$\dim\left(\mathcal{P}_{2k}\left(M;L\right)\right) \leq \; C\left(2k\right)^{C_V}.$$

Returning to harmonic maps between compact Riemannian manifolds, the second variation of the Dirichlet energy at a critical point $u:\ (M,g) \to \ (N,h)$ is given by

$$\left(\delta_u^2 E\right)(w) = \int\limits_M h\left(J^u(w), w\right),$$

where J^u is the linear, elliptic, self-adjoint operator called the Jacobi operator of u and w is a section of the pull-back bundle $u^{-1}TN \to \ M$ with

$$u^{-1}TN := \left\{ (x,v) \in M \times TN \,|\, v \in T_{u(x)}N \right\} \subseteq M \times TN.$$

The connection on $u^{-1}TN$ induced by the Levi-Civita connection on TN is called the pull-back connection and denoted by ∇^u. To be precise, ∇^u is defined for any section $s \in \Gamma\left(u^{-1}TN\right)$, $x \in M$ and $X \in T_xM$ by

$$\nabla^u_X s := \nabla_{du(X)}\sigma,$$

where ∇ is the Levi-Civita connection on TN and the vector field $\sigma \in \mathcal{X}\left(N\right)$ satisfies $s = u^*\sigma$. This gives rise to e.g. the pull-back Laplacian $\Delta^u := -\operatorname{tr}\left[\nabla^u_\cdot \nabla^u_\cdot\right]$ or the pull-back Riemann curvature tensor R^u. With this the Jacobi operator of u is given by

$$\begin{aligned} J^u : \Gamma\left(u^{-1}TN\right) &\to \Gamma\left(u^{-1}TN\right), \\ w &\mapsto -\operatorname{tr}\left[\nabla^u_\cdot \nabla^u_\cdot\right](w) - \operatorname{tr}\, R^N\left(w, du\cdot\right) du\cdot = \; \Delta^u w - \operatorname{tr}\, R^N\left(w, du\cdot\right) du\cdot, \end{aligned}$$

where R^N denotes the Riemann curvature tensor of N and $\Gamma\left(u^{-1}TN\right)$ the space of sections of $u^{-1}TN$. Then the index of a harmonic map $u:\ M \to \ N$ with respect to E, denoted

$\mathrm{ind}(u)$, is the dimension of the maximal subspace of $\Gamma\left(u^{-1}TN\right)$ on which the Hessian

$$\mathrm{Hess}_u(v,w) := \int_M h\left(J^u(v),w\right)$$

is negative definite. This is also equal to the sum of the dimensions of the eigenspaces of J^u corresponding to negative eigenvalues. In addition, the nullity of u, denoted $\mathrm{null}(u)$, is defined as the dimension of the kernel of J^u or equivalently as the dimension of the eigenspace of J^u corresponding to the eigenvalue zero.
If the domain manifold M is non-compact, we can define the index and nullity of a harmonic map on each relatively compact domain D in M.
Given a harmonic map $u : M \to N$ between compact Riemannian manifolds, we further call u (energy-)stable if $\mathrm{ind}(u) = 0$ and otherwise (energy-)unstable.
The stability of harmonic maps between Riemannian manifolds, respectively the non-existence of non-constant stable harmonic maps, is quite well-studied, cf. e.g. [13, 43, 49, 59, 62, 68, 73, 75, 76]. However, concerning unstable harmonic maps there mostly exist only bounds on their index (e.g. [5, 28, 29, 30, 63, 73]) and explicit computations of the index and nullity exist for just a few (simple) maps such as identity maps, the Hopf map $S^3 \to S^2$ ([73], note [60],Ex. 5.6) or the standard inclusion maps between spheres ([57]). One of these index bounds was established by El Soufi, saying that the index of a non-constant, smooth, harmonic map on a round sphere must be at least the index of the identity map of that round sphere.

Theorem 1.0.3. *([30])*
Any non-constant, smooth, harmonic map $\phi : (S^m, \mathrm{can}) \to (N,h)$, $m \geq 3$, into a Riemannian manifold N satisfies

$$\mathrm{ind}(\phi) \geq \mathrm{ind}(\mathrm{id}_{S^m}) = m+1.$$

We are particularly interested in the equality case, that is in non-constant, smooth, harmonic maps $S^m \to N$, $m \geq 3$, of index $m+1$. For brevity we will refer to such maps as harmonic maps of low index. When studying harmonic maps of low index, an important class of harmonic maps to be considered is the class of harmonic morphisms. A harmonic morphism $\phi : M \to N$ is a smooth map such that for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset V of N with $\phi^{-1}(V) \neq \emptyset$ the composition $f \circ \phi : \phi^{-1}(V) \to \mathbb{R}$ is again a harmonic function. They appeared first in an 1848 paper by Jacobi ([45]), where he gave a method to construct complex-valued such maps on domains in $\mathbb{R}^3$.
After playing a central role as the morphisms of Brelot harmonic spaces in potential theory, the theory of harmonic morphisms between Riemannian manifolds was initiated by three independent works in the years 1978/79 by Fuglede ([32]), Ishihara ([44]) as well as Bernard, Campbell and Davie ([8]). In the first two the following characterization of harmonic morphisms was shown.

Theorem 1.0.4. *([32, 44])*
A smooth map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if ϕ is both harmonic and horizontally weakly conformal.

Recall for this that $\phi : M \to N$ horizontally weakly conformal means that for every $x \in M$ either $d\phi_x = 0$ or $d\phi_x$ maps $(\ker d\phi_x)^\perp$ conformally onto $T_{\phi(x)}N$. In the special case that ϕ is a submersion and its conformality factor is constant ϕ is referred to as a Riemannian submersion up to scale. Further, a Riemannian submersion up to scale with conformality factor 1 is simply called a Riemannian submersion.
From this characterization of harmonic morphisms as harmonic, horizontally weakly conformal maps it can be seen that they are solutions to overdetermined, non-linear systems of equations. Contrary to the theory of harmonic maps, thus the existence of harmonic morphisms between given Riemannian manifolds is not even locally guaranteed. As it turns out, existence depends on the topology of the manifolds and Riemannian structures on them. Examples of simple harmonic morphisms are constant maps, isometries and hence in particular identity maps or the Hopf maps $S^1 \to S^1$, $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$.

Whenever we need properties of and results on harmonic morphisms we will in most cases cite for simplicity the monograph by Baird and Wood ([4]). A list of works on the topic of harmonic morphisms is maintained by Sigmundur Gudmundsson ([35]).
Our interest in harmonic morphisms arises from a classification of low index smooth harmonic maps from the round three-sphere into the round two-sphere established by Rivière ([65]), where he shows that all such maps must already be harmonic morphisms.
This classification result sparks a number of questions such as

(1) Is any low index smooth harmonic map necessarily a harmonic morphism?

(2) Conversely what can be said about the index of a given harmonic morphism?

(3) What is the interplay between low index and maximal rank, i.e. can maps into lower-dimensional submanifolds contribute to the set of low index maps with respect to a given codomain?

(4) Can we prove El Soufi-type index bounds for domain manifolds which are not round spheres?

Throughout Chapter 3 we will address these questions in specific cases. Starting off, the focus of section 3.1 will lie on comparing the index and nullity of a given smooth harmonic submersion into a round sphere with the index and nullity of the composition of said submersion with the canonical totally geodesic inclusion map into a higher-dimensional round sphere. Seeing that the conformality and total geodesicity of the inclusion map imply that eigenvalues and eigensections of the Jacobi operator are preserved under composition with the inclusion map, we first show in 3.1.2, based on the approach of Loubeau and Oniciuc in [58], that both index and nullity of the composition are forced to increase in comparison to the index and nullity of the submersion.

Theorem 1.0.5. *Let M^n be a compact, connected, smooth Riemannian manifold without boundary and $\psi : M^n \to S^m$, $n \geq m \geq 1$, be a smooth, harmonic submersion. Let $\mathrm{i} : S^m \to S^p$ for some $p > m$ be the canonical inclusion map of round spheres and set $\phi := \mathrm{i} \circ \psi : M^n \to S^p$. Then we obtain*

$$\operatorname{ind}(\phi) \geq \operatorname{ind}(\psi) \quad \textit{and} \quad \operatorname{null}(\phi) \geq \operatorname{null}(\psi).$$

Restricting to harmonic Riemannian submersions up to scale allows us to determine (counting with multiplicity) the additional eigenvalues of the Jacobi operator after composition with i, hence improving the bounds on the index and nullity of the composition.

Corollary 1.0.6. *Let M^n be a compact, connected, smooth Riemannian manifold without boundary and $\psi : M^n \to S^m$, $n \geq m \geq 1$, be a smooth, harmonic Riemannian submersion up to scale. Moreover for $p > m$ let $\mathrm{i} : S^m \to S^p$ be the canonical inclusion map and set $\phi := \mathrm{i} \circ \psi : M^n \to S^p$.*
For $k \geq 0$ denote by λ_k the k-th eigenvalue of $\Delta : C^\infty(M) \to C^\infty(M)$.
Then, in addition (counting with multiplicity) to the eigenvalues of J^ψ, J^ϕ has eigenvalues $\left\{ \lambda_k - |d\psi|^2 \,\middle|\, k \geq 0 \right\}$ of multiplicities $(p-m)\operatorname{mult}(\lambda_k)$, where $\operatorname{mult}(\lambda_k)$ is the multiplicity of λ_k as an eigenvalue of Δ. In particular $\operatorname{ind}(\phi) > \operatorname{ind}(\psi)$. Furthermore, $\operatorname{null}(\phi) > \operatorname{null}(\psi)$ if and only if $|d\psi|^2$ is in the spectrum of Δ. In that case the nullity of ϕ is given by

$$\operatorname{null}(\phi) = \operatorname{null}(\psi) + (p-m)\operatorname{mult}\left(\lambda_{\tilde{k}}\right),$$

where $\tilde{k} \geq 1$ is such that $\lambda_{\tilde{k}} = |d\psi|^2$.

In 3.1.3 we apply these bounds to the (second) Hopf map $\psi : S^3 \to S^2$, which is a harmonic Riemannian submersion up to scale. Making use of the well-known spectrum of the Jacobi operator of ψ and thus index and nullity of ψ, we not only can provide bounds for the index and nullity of ϕ but calculate them explicitly.

Theorem 1.0.7. *Let $\psi : S^3 \to S^2$ be the Hopf map, $\mathrm{i} : S^2 \to S^p$, $p \geq 3$, the canonical inclusion map and $\phi := \mathrm{i} \circ \psi : S^3 \to S^p$.*
Then, we obtain $\mathrm{ind}(\phi) = 5p - 6$ *as well as* $\mathrm{null}(\phi) = 9p - 10$.

We also add for comparison a proof of this in case $p = 3$ by explicitly computing all of the eigenvalues based on the approach of Loubeau and Oniciuc in [58], that is without using the results of 3.1.2. Adapting this explicit approach, we then calculate in 3.1.4 the Jacobi operator of the composition of the Hopf map $\psi : S^3 \to S^2$ with an arbitrary conformal transformation v of S^2. This is of interest as, due to a characterization by Baird and Wood, every non-constant submersive harmonic morphism $S^3 \to S^2$ is, up to an isometry of S^3, given this way. To be precise, we provide for an orthonormal frame $\{X_2, X_3\}$ of $(\ker d\psi)^{\perp}$ which will be defined in Lemma 3.1.9 a complete description of the Jacobi operator J^{ξ} of the composition $\xi := v \circ \psi$.

Lemma 1.0.8. *Let $\psi : S^3 \to S^2$ be the Hopf map, $v : S^2 \to S^2$ be a conformal transformation of S^2 and $\xi := v \circ \psi : S^3 \to S^2$. Then, the Jacobi operator of ξ is determined for any $f_2, f_3 \in C^{\infty}\left(S^3\right)$ by*

$$\begin{aligned}
J^{\xi}\left(f_2\, d\xi\left(X_2\right)\right) = & \, dv\left(J^{\psi}\left(f_2\, d\psi\left(X_2\right)\right)\right) - 2\left(X_2 f_2\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right) \\
& - 2\left(X_3 f_2\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right), \\
J^{\xi}\left(f_3\, d\xi\left(X_3\right)\right) = & \, dv\left(J^{\psi}\left(f_3\, d\psi\left(X_3\right)\right)\right) - 2\left(X_2 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right) \\
& + 2\left(X_3 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right).
\end{aligned}$$

In the special case that ξ is horizontally homothetic, that is when the gradient of the conformality factor of ξ lies in the kernel of $d\xi$, these expressions simplify, yielding in particular that composition of ξ with the inclusion map i results in a strict increase in index.

Corollary 1.0.9. *Let $\xi : S^3 \to S^2$ be a globally defined, submersive, horizontally homothetic harmonic morphism. Then, $\xi = v \circ \psi \circ \Phi$, where $\Phi : S^3 \to S^3$ and $v : S^2 \to S^2$ are isometries and $\psi : S^3 \to S^2$ is the Hopf map. Moreover, the Jacobi operator of ξ is determined by*

$$J^{\xi}\left(f_j\, d\xi\left(X_j\right)\right) = dv\left(J^{\psi}\left(f_j\, d\psi\left(X_j\right)\right)\right)$$

for $j \in \{2, 3\}$ and any function $f_j \in C^{\infty}\left(S^3\right)$. Furthermore, the spectra of J^{ξ} and J^{ψ} coincide. In particular, letting $\mathrm{i} : S^2 \to S^p$, $p \geq 3$, denote the canonical inclusion map, we find

$$\mathrm{ind}(\mathrm{i} \circ \xi) > \mathrm{ind}(\xi) = \mathrm{ind}(\psi) = 4.$$

Finally, in 3.1.5, we briefly treat the case of smooth, harmonic submersions between one-dimensional, compact, connected, smooth Riemannian manifolds and compute their indices and nullities in the same way as for the (second) Hopf map $S^3 \to S^2$. An example of such a harmonic submersion is the (first) Hopf map $S^1 \to S^1$.

Theorem 1.0.10. *Let (M, g) and (N, h) be one-dimensional, compact, connected, smooth Riemannian manifolds so that M admits a non-vanishing unit vector field X and let $\psi : M \to N$ be any smooth, harmonic submersion. Then the Jacobi operator of ψ is determined by*

$$J^{\psi}\left(f\, d\psi\left(X\right)\right) = \left(\Delta f\right) d\psi\left(X\right), \; f \in C^{\infty}\left(M\right).$$

Consequently, the spectrum of J^{ψ} coincides with the spectrum of $\Delta : C^{\infty}(M) \to C^{\infty}(M)$. In particular, $\mathrm{ind}(\psi) = 0$ *and* $\mathrm{null}(\psi) = 1$.

In section 3.2 we shortly turn to submersive harmonic morphisms from the round n-sphere into immersed submanifolds of the round n-sphere for $n \geq 3$ which arise from smooth maps $S^n \to S^n$ of constant rank. For $n = 3$ we get a full description of such maps by rank.

Theorem 1.0.11. *Let $u : \left(S^3, \mathrm{can}\right) \to \left(S^3, \mathrm{can}\right)$ be a non-constant, smooth map of constant rank k. Suppose that its image $N := u\left(S^3\right)$ is an immersed submanifold of S^3 and endow it with the inherited metric g. Assume further that $u : \left(S^3, \mathrm{can}\right) \to (N, g)$ is a harmonic morphism. Then, $k \in \{2, 3\}$ and u is given as follows*

(a) *For* $k = 2$ *there exist an isometry* $\Phi : S^3 \to S^3$ *and a conformal, surjective map* $v : S^2 \to N^2$ *so that* $u = v \circ \psi \circ \Phi$, *where* $\psi : S^3 \to S^2$ *is the Hopf map. Moreover,* N^2 *is conformally equivalent to* S^2 *or* $\mathbb{R}P^2$, *and we obtain* $\operatorname{ind}(u) \geq 4$.

(b) *For* $k = 3$ *we find* $N^3 = S^3$ *and that* $u : S^3 \to S^3$ *is an isometry. In particular,* $\operatorname{ind}(u) = 4$.

Restricting to images N which are totally geodesic submanifolds of S^n and maps with constant conformality factors, we can determine, using Corollary 1.0.6, for any $n \geq 3$ those harmonic morphisms of constant rank that are of low index.

Theorem 1.0.12. *Let* $u : (S^n, \mathrm{can}) \to (S^n, \mathrm{can})$, $n \geq 3$, *be a non-constant, smooth map of constant rank* k. *Let* N *denote the image* $u(S^n)$, *suppose that* N *is a totally geodesic immersed submanifold of* S^n *and endow it with the inherited metric* g.
Assume that $u : (S^n, \mathrm{can}) \to (N^k, g)$ *is a harmonic Riemannian submersion up to scale with* $\operatorname{ind}(\mathrm{i} \circ u) \leq n+1$, *where* $\mathrm{i} : N^k \to S^n$ *denotes the canonical inclusion map.*
Then, we have $\operatorname{ind}(\mathrm{i} \circ u) = n+1$ *and* u *is an isometry of* S^n.

Finally, in section 3.3, we consider non-constant, smooth, harmonic maps from simply connected Riemannian manifolds M admitting a frame consisting of Killing vector fields into round spheres S^p, $p \geq 2$.
Recall for this that a vector field X on a Riemannian manifold (M, g) is called Killing if $\mathcal{L}_X g = 0$, where $\mathcal{L}$ denotes the Lie derivative, or equivalently if for all vector fields Y, Z on M we find

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0.$$

So, for a simply connected Riemannian manifold M admitting a frame of such vector fields, denoting by $M = M_0 \times \ldots \times M_m$ the Riemannian product decomposition of M with M_0 a Euclidean space and M_μ, $\mu \in \{1, \ldots, m\}$, simply connected, compact, irreducible Riemannian manifolds, we first establish in the subsections 3.3.2-3.3.5, adapting the approach used by Rivière in [65], constructively an El Soufi-type index bound in the case where the Riemannian product decomposition of M is trivial, i.e. consists only of one factor.
For this we take an orthogonal eigenbasis $f^1, \ldots, f^{\mathcal{N}}$ of the eigenspace of the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$ corresponding to its first non-zero eigenvalue $\lambda_1(M)$. Taking the gradients of the basis elements we obtain gradient vector fields $X^1, \ldots, X^{\mathcal{N}}$ on M. Then we find for any non-constant, smooth, harmonic map $u : M \to S^p$ that, under some technical constraints specified below, the $du\left(X^j\right)$'s are linearly independent eigensections of the Jacobi operator of u with negative eigenvalues. Therefore we obtain a lower index bound for u that only depends on M.

Theorem 1.0.13. *Let* M *be a simply connected, complete Riemannian manifold of dimension at least three satisfying the Killing property, harmonically unstable if compact and with trivial Riemannian product decomposition. Let* $u : M \to S^p$, $p \geq 2$, *be a non-constant, globally defined, smooth, harmonic map of finite Dirichlet energy over* M. *Then,* M *is compact and either* S^7 *with its round metric or a simple Lie group of type* A_l, $l \geq 1$, *or* C_l, $l \geq 2$, *with a bi-invariant metric. Suppose in addition*

(1) *in case* $M = A_l$ *that* $u : M \to S^p$ *is locally injective;*

(2) *in case* $M = C_l$ *that* $u : M \to S^p$ *is locally injective and satisfies for some orthonormal frame* $\{\nu_k\}_{k=1}^{2l^2-l}$ *of the normal bundle* NM *for every* $k \in \{1, \ldots, 2l^2 - l\}$

$$\operatorname{tr}\left[\nabla du\left(\cdot, \nabla_{\cdot} \nu_k\right)\right] = 0,$$

where $\nabla_X \nu_k$ *denotes the projection of* $(d\nu_k)(X)$ *onto* TM *for any* $X \in \mathcal{X}(M)$.

Then we find in either case

$$\operatorname{ind}(u) \geq \mathcal{N},$$

where $\mathcal{N}$ *denotes the multiplicity of the first non-zero eigenvalue of* $\Delta : C^\infty(M) \to C^\infty(M)$, *i.e.* $\mathcal{N} = 8$ *for* $M = S^7$, $\mathcal{N} = (l+1)^2$ *for* $M = A_l$ *and* $\mathcal{N} = 4l^2$ *for* $M = C_l$.

Thereafter we use in 3.3.6 these lower bounds to prove index bounds for product maps of sphere-valued harmonic maps on domains with trivial Riemannian product decomposition as described in Theorem 1.0.13.

Corollary 1.0.14. *Let M be a simply connected, complete Riemannian manifold of dimension at least three satisfying the Killing property and with corresponding Riemannian product decomposition*

$$M = M_0 \times \ldots \times M_m$$

so that all of the M_μ, $\mu = 1, \ldots, m$, are harmonically unstable.
Let further $u_\mu : M_\mu \to S^{p_\mu}$, $p_\mu \geq 2$, for all $\mu = 0, \ldots, m$ be as in Theorem 1.0.13.
Then, M is compact and harmonically unstable, the product map $u := (u_1, \ldots, u_m) : M \to S^{p_1} \times \ldots \times S^{p_m}$ is non-constant, globally defined, smooth, harmonic and satisfies the index bound

$$\mathrm{ind}(u) \geq \sum_{\mu=1}^{m} \mathrm{ind}\,(u_\mu) \geq \sum_{\mu=1}^{m} \mathcal{N}_\mu,$$

where $\mathcal{N}_\mu$ denotes the multiplicity of the first non-zero eigenvalue of $\Delta : C^\infty(M_\mu) \to C^\infty(M_\mu)$.

Continuing to adapt the ideas of [65], we focus in 3.3.7 on the case $p = 2$, even further on non-constant, smooth, harmonic maps $u : M \to S^2$ of index $\mathcal{N}$ satisfying the constraints of Theorem 1.0.13, where M is a simply connected, compact, harmonically unstable Riemannian manifold admitting a Killing frame with trivial Riemannian product decomposition and $\mathcal{N}$ is the multiplicity of $\lambda_1(M)$ as an eigenvalue of $\Delta : C^\infty(M) \to C^\infty(M)$. We prove that any such map must be horizontally weakly conformal and hence a harmonic morphism.

Theorem 1.0.15. *Consider the setting of Theorem 1.0.13 with $p = 2$ and let $\mathcal{N}$ denote the multiplicity of the first non-zero eigenvalue of the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$.*
Assume also that $u : M \to S^2$ is of index at most $\mathcal{N}$.
Then, the index of u is equal to $\mathcal{N}$ and u is a harmonic morphism.

Remark. Due to the duality between the theory of harmonic morphisms to a surface and the theory of conformal minimal immersions from a surface ([3]), Theorem 1.0.15 in particular yields that given a non-constant, smooth, harmonic map of low index $u : M \to S^2$ as in 1.0.15 its fibres $u^{-1}(x)$, $x \in S^2$, are minimal submanifolds of M of codimension two.

Chapter 2

Dimension Estimates for Spaces of Polynomial Growth Solutions

Given an n-dimensional, complete, non-compact Riemannian manifold (M, g), we prove in this chapter dimension bounds for the spaces $\mathcal{P}_d(M; L)$ of polynomial growth ancient solutions to $\partial_t u = Lu$.
To this end we prove in section 2.2 an L^2-mean value inequality for solutions to $\partial_t u = Lu$, which we use in section 2.3, where we adapt the reasoning of Lin and Zhang for the heat operator $\partial_t - \Delta$ from [56] to our operator $\partial_t - L$. In 2.4 we see that in the case of time-independent coefficients A, thus time-independent L, we can improve this bound using the same approach as Colding and Minicozzi in [21] for the heat equation.

2.1 Preliminaries

In the following we introduce some notions and results that will be needed in order to prove the dimension bounds.
First, a Riemannian manifold (M, g) is said to have the weak volume comparison property with constant C_V if for universal constants $C, C_V < +\infty$ and any $r > s > 0$ and $p \in M$ we have

$$r^{-C_V} \operatorname{Vol}(B_r(p)) \leq C\, s^{-C_V} \operatorname{Vol}(B_s(p)).$$

In the case where $C = 1$ this is referred to as the relative volume comparison property. Moreover, if $C = 1$, and this is only satisfied for $r = 2s$, $s > 0$, then (M, g) is said to possess the volume doubling property.
In fact, the volume doubling property with comparison constant C_D is actually equivalent to the weak volume comparison property with the same comparison constant, i.e. $C_V = C_D$ (e.g. [50]). Due to this the volume doubling property and thus also the weak volume comparison property imply that (M, g) has polynomial volume growth, that is there are constants $d_V, V > 0$ so that for any $r > 0$ and $p \in M$ we obtain

$$\operatorname{Vol}(B_r(p)) \leq V(1 + r)^{d_V}.$$

Moreover, let us recall the definition of a scale-invariant Neumann-Poincaré inequality on M.

Definition 2.1.1. A Riemannian manifold M has a scale-invariant Neumann-Poincaré inequality if there exists a constant $C_N < +\infty$ so that for all $r > 0$, $p \in M$ and $f \in H^1_{loc}(M)$ we find

$$\int\limits_{B_r(p)} (f - \mathcal{A})^2 \leq C_N\, r^2 \int\limits_{B_r(p)} |\nabla f|^2, \quad \text{where} \quad \mathcal{A} := \frac{1}{\operatorname{Vol}(B_r(p))} \int\limits_{B_r(p)} f.$$

It is a well-known fact that the volume doubling property and Neumann-Poincaré inequality yield a local Sobolev inequality.

Lemma 2.1.2. *([66])*
Suppose that M satisfies a Neumann-Poincaré inequality and the volume doubling property. Then, there exist constants $\nu > 2$ and $S > 0$ such that for all $r > 0$, $p \in M$ and $f \in C_0^\infty(B_r(p))$ we get

$$\left(\int\limits_{B_r(p)} |f|^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \leq S \operatorname{Vol}(B_r(p))^{-\frac{2}{\nu}} r^2 \int\limits_{B_r(p)} \left(|\nabla f|^2 + r^{-2} |f|^2 \right).$$

In particular $\nu = \max\{3, C_D\}$, where C_D denotes the volume doubling constant.

2.2 Mean Value Inequality

Towards the first dimension bound we establish an L^2-mean value inequality for solutions to $\partial_t u = Lu$ with L as defined in (1.1) in the introduction using the methods of [67]. For this we note that, due to the smoothness of the coefficient A, standard regularity theory gives us that any weak solution to $\partial_t u = Lu$ on $Q_r(s,x)$ must already be smooth.

Lemma 2.2.1. *Suppose that M satisfies a Sobolev inequality and take $r \geq 1$, $(s,x) \in \mathbb{R}^- \times M$. Then, any solution u to $\partial_t u = Lu$ on the parabolic cylinder $Q_r(s,x)$ satisfies*

$$\sup_{\left]s-\frac{r^2}{2},s\right[\times B_{\frac{r}{2}}(x)} \{u^2\} \leq \frac{C_M}{r^2 \operatorname{Vol}(B_r(x))} \int\limits_{Q_r(s,x)} u^2$$

for a constant $0 < C_M < +\infty$ independent of u, s and $B_r(x)$.

Due to the smoothness of u on $Q_r(s,x)$ we can conclude from this the mean value inequality.

Corollary 2.2.2. *Let M be as in Lemma 2.2.1, $s \in \mathbb{R}^-$, $x \in M$, $r \geq 1$ and u be a solution to $\partial_t u = Lu$. Then,*

$$u^2(s,x) \leq \frac{C_M}{r^2 \operatorname{Vol}(B_r(x))} \int\limits_{Q_r(s,x)} u^2.$$

Since we have seen that the volume doubling property and the Neumann-Poincaré inequality together give us a Sobolev inequality, we can replace the latter by the former in the assumptions for the mean value inequality, that is the volume doubling property and a Neumann-Poincaré inequality imply a mean value inequality for solutions to $\partial_t u = Lu$.
We will split the proof of the mean value inequality into two parts. First we derive for the solutions an estimate for the right-hand side of the Sobolev inequality. In the second part we find that those together give us Lemma 2.2.1 and thus a mean value inequality.

Derivation of the estimate

Let u be a solution to $\partial_t u = Lu$ on $Q_r(s,x)$ and let $p \geq 1$. Then u^p is a solution to

$$(\partial_t - L)(u^p) = -p(p-1)u^{p-2} A\nabla u \nabla u,$$

that means for any $\phi \in H_0^1(B_r(x))$ we have

$$\int\limits_{B_r(x)} \left(\phi\, \partial_t(u^p) + A\nabla(u^p)\nabla\phi\right) = -p(p-1) \int\limits_{B_r(x)} \phi\, u^{p-2} A\nabla u \nabla u. \tag{2.1}$$

Then we take $\psi \in C_0^\infty(B_r(x))$ to be a cutoff function on $B_r(x)$ such that

$$0 \le \psi \le 1,\ \mathrm{supp}(\psi) \subseteq B_{(1-\sigma)r}(x),\ \psi|_{B_{(1-\sigma')r}(x)} \equiv 1,\ |\nabla\psi| \le \frac{1}{\tau\, r},$$

where $0 < \sigma < \sigma' < 1$ are arbitrary but fixed and $\tau := \sigma' - \sigma$.
Hence, setting $\phi = \psi^2 u^p$ in (2.1) and multiplying (2.1) by two gives us at any fixed $t \in]s - r^2, s[$

$$\begin{aligned}
&2p \int\limits_{B_r(x)} \psi^2 u^{2p-1}\partial_t u + \int\limits_{B_r(x)} A\nabla(\psi\, u^p)\nabla(\psi\, u^p) \\
&= -2 \int\limits_{B_r(x)} A\nabla(u^p)\nabla\left(\psi^2 u^p\right) - 2p(p-1)\int\limits_{B_r(x)} \psi^2 u^{2p-2} A\nabla u\nabla u + \int\limits_{B_r(x)} A\nabla(\psi\, u^p)\nabla(\psi\, u^p) \\
&= -p^2 \int\limits_{B_r(x)} \psi^2 u^{2p-2} A\nabla u\nabla u - 2p\int\limits_{B_r(x)} \psi\, u^{2p-1} A\nabla u\nabla\psi - 2p(p-1)\int\limits_{B_r(x)} \psi^2 u^{2p-2} A\nabla u\nabla u \\
&\quad + \int\limits_{B_r(x)} u^{2p}\, A\nabla\psi\nabla\psi \\
&\le -c_1\, p^2 \int\limits_{B_r(x)} \psi^2 u^{2p-2}\,|\nabla u|^2 + 2c\,p \int\limits_{B_r(x)} \psi\,|u|^{2p-1}\,|\nabla u|\,|\nabla\psi| \\
&\quad - 2c_1\, p(p-1)\int\limits_{B_r(x)} \psi^2 u^{2p-2}\,|\nabla u|^2 + c\int\limits_{B_r(x)} u^{2p}\,|\nabla\psi|^2,
\end{aligned}$$

having used our assumptions (1.2) on the coefficient A of L. As by definition of our cutoff function $\mathrm{supp}(\psi) \subseteq B_{(1-\sigma)r}(x)$, we obtain, estimating the second term on the right-hand side with Young's inequality and noting that the third term on the right-hand side is negative,

$$\begin{aligned}
&2p \int\limits_{B_r(x)} \psi^2 u^{2p-1}\partial_t u + \int\limits_{B_r(x)} A\nabla(\psi\, u^p)\nabla(\psi\, u^p) \\
&\qquad \le \left(\frac{1}{2} - c_1\right) p^2 \int\limits_{B_r(x)} \psi^2 u^{2p-2}\,|\nabla u|^2 + \left(2c^2 + c\right)\int\limits_{B_r(x)} u^{2p}\,|\nabla\psi|^2 \\
&\qquad\quad - 2c_1 p(p-1)\int\limits_{B_r(x)} \psi^2 u^{2p-2}\,|\nabla u|^2 \\
&\qquad \le \left(\frac{1}{2} - c_1\right) p^2 \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p-2}\,|\nabla u|^2 + \left(2c^2 + c\right)\int\limits_{B_{(1-\sigma)r}(x)} u^{2p}\,|\nabla\psi|^2.
\end{aligned}$$

Since $c_1 > \frac{1}{2}$ by assumption the first term on the right-hand side is non-positive as well and thus we can bound it from above by zero, so that we have

$$\begin{aligned}
&2p \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p-1}\partial_t u + \int\limits_{B_{(1-\sigma)r}(x)} A\nabla(\psi u^p)\nabla(\psi u^p) \\
&\qquad = 2p \int\limits_{B_r(x)} \psi^2 u^{2p-1}\partial_t u + \int\limits_{B_r(x)} A\nabla(\psi u^p)\nabla(\psi u^p) \\
&\qquad \le \frac{2c^2 + c}{\tau^2 r^2} \int\limits_{B_{(1-\sigma)r}(x)} u^{2p},
\end{aligned} \tag{2.2}$$

where in the last inequality we also used the gradient bound for the cutoff function ψ. Next, we also need a cutoff function for the time variable. Therefore, let $\chi \in C_0^\infty(\mathbb{R})$ be a cutoff function on $\mathbb{R}$, which satisfies

$$0 \le \chi \le 1,\ \chi|_{]-\infty, s-(1-\sigma)r^2[} \equiv 0,\ \chi|_{]s-(1-\sigma')r^2, +\infty[} \equiv 1,\ |\chi'| \le \frac{1}{\tau\, r^2}.$$

With this and (2.2) we obtain

$$\begin{aligned}
\partial_t \int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p} + \chi^2 \int\limits_{B_{(1-\sigma)r}(x)} A\nabla\left(\psi\, u^p\right)\nabla\left(\psi\, u^p\right)& \\
= 2\,\chi\,\chi' \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} + 2p\,\chi^2 \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p-1}\partial_t u& \\
+ \chi^2 \int\limits_{B_{(1-\sigma)r}(x)} A\nabla\left(\psi\, u^p\right)\nabla\left(\psi\, u^p\right)& \\
\le \frac{2}{\tau\, r^2}\,\chi \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} + \frac{2c^2+c}{\tau^2 r^2}\,\chi^2 \int\limits_{B_{(1-\sigma)r}(x)} u^{2p}& \\
\le \frac{\chi}{\tau^2 r^2}\left(2 + \chi\left(2c^2+c\right)\right) \int\limits_{B_{(1-\sigma)r}(x)} u^{2p},&
\end{aligned} \tag{2.3}$$

using $\tau < 1$ in the last inequality. Set $I_{\sigma'} :=]s-(1-\sigma')r^2, s[$ and fix some $t \in I_{\sigma'}$. Note that restricted to $I_{\sigma'}$ the cutoff function χ is constantly equal to one. Moreover, since we took $\sigma < \sigma'$ we have $s-(1-\sigma)r^2 < t$.
By the mean value theorem there exists some $s' \in [s-r^2, s-(1-\sigma)r^2]$ so that

$$\left.\int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p}\right|_{s'} = \frac{1}{\sigma\, r^2} \int\limits_{s-r^2}^{s-(1-\sigma)r^2} \int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p} \le \frac{\tilde{C}}{\tau^2 r^2} \int\limits_{s-r^2}^{t} \int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p}$$

for some constant $\tilde{C} = \tilde{C}\left(\sigma, \sigma'\right) > 0$.
Therefore, integrating (2.3) in time from s' to t yields, together with the properties of ψ and χ as well as the definition of s',

$$\begin{aligned}
&\left.\int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p}\right|_{t} + \int\limits_{s'}^{t} \int\limits_{B_{(1-\sigma)r}(x)} \chi^2 A\nabla\left(\psi\, u^p\right)\nabla\left(\psi\, u^p\right) \\
&\le (\tau\, r)^{-2} \int\limits_{s'}^{t} \chi\left(2 + \chi\left(2c^2+c\right)\right) \int\limits_{B_{(1-\sigma)r}(x)} u^{2p} + \left.\int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p}\right|_{s'} \\
&\le (\tau\, r)^{-2} \int\limits_{s-(1-\sigma)r^2}^{t} \chi\left(2 + \chi\left(2c^2+c\right)\right) \int\limits_{B_{(1-\sigma)r}(x)} u^{2p} + \frac{\tilde{C}}{\tau^2 r^2} \int\limits_{s-(1-\sigma)r^2}^{t} \chi^2 \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} \\
&\le \frac{2+2c^2+c+\tilde{C}}{\tau^2 r^2} \int\limits_{s-(1-\sigma)r^2}^{t} \int\limits_{B_{(1-\sigma)r}(x)} u^{2p}
\end{aligned}$$

$$\leq \frac{2+2c^2+c+\tilde{C}}{\tau^2 r^2} \int\limits_{s-(1-\sigma)r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} u^{2p}. \tag{2.4}$$

In the last inequality we also used that by definition of $I_{\sigma'}$ we have $t < s$ for any $t \in I_{\sigma'}$. As the right-hand side of (2.4) is independent of the choice of $t \in I_{\sigma'}$, taking the supremum over $t \in I_{\sigma'}$ and using that χ is equal to one on $I_{\sigma'}$ as well as $s' \leq s-(1-\sigma)r^2 < s-(1-\sigma')r^2$ give us

$$\begin{aligned}
&\sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} \right) + \int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} A\nabla\left(\psi u^p\right)\nabla\left(\psi u^p\right) \\
&\quad = \sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p} \right) + \int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} \chi^2 A\nabla\left(\psi u^p\right)\nabla\left(\psi u^p\right) \\
&\quad \leq \sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \chi^2\psi^2 u^{2p} \right) + \int\limits_{s'}^{s} \int\limits_{B_{(1-\sigma)r}(x)} \chi^2 A\nabla\left(\psi u^p\right)\nabla\left(\psi u^p\right) \\
&\quad \leq \frac{2+2c^2+c+\tilde{C}}{\tau^2 r^2} \int\limits_{s-(1-\sigma)r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} u^{2p}.
\end{aligned} \tag{2.5}$$

This finishes the first part of the proof of the mean value inequality.

Concluding the mean value inequality

In the following we denote by $E(B_r)$ the Sobolev constant for the ball $B_r(x)$, that is $E(B_r) = S\ \mathrm{Vol}\left(B_r(x)\right)^{-\frac{2}{\nu}} r^2$, and set for brevity $q := \frac{\nu}{\nu-2}$.
Using these notations, we get from Hölder and Sobolev inequality for any $w \in C_0^\infty(B_r(x))$

$$\begin{aligned}
\int\limits_{B_r(x)} w^{2\left(1+\frac{2}{\nu}\right)} &\leq \left(\int\limits_{B_r(x)} w^{2q}\right)^{\frac{1}{q}} \left(\int\limits_{B_r(x)} w^2\right)^{\frac{2}{\nu}} \leq \left(\int\limits_{B_r(x)} w^2\right)^{\frac{2}{\nu}} E(B_r) \int\limits_{B_r(x)} \left(|\nabla w|^2 + r^{-2}w^2\right) \\
&\leq \left(\int\limits_{B_r(x)} w^2\right)^{\frac{2}{\nu}} 2\,E(B_r) \int\limits_{B_r(x)} \left(A\nabla w\nabla w + r^{-2}w^2\right)
\end{aligned}$$

as $g\left(X,X\right) \leq \frac{1}{c_1}\, g\left(AX,X\right) <\ 2\, g\left(AX,X\right)$ by assumption (1.2) on A. Hence, using this with $w = \psi\, u^p$ and integrating in time from $s-(1-\sigma')r^2$ to s yields with the notation $\theta := 1+\frac{2}{\nu}$

$$\begin{aligned}
\int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma')r}(x)} u^{2p\theta} &\leq \int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} \left(\psi u^p\right)^{2\theta} \\
&\leq 2\,E(B_r) \left(\sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} \right)\right)^{\theta-1}
\end{aligned}$$

$$\times \int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} \left(A\nabla\left(\psi u^p\right)\nabla\left(\psi u^p\right) + r^{-2}\psi^2 u^{2p}\right). \tag{2.6}$$

Note that we have

$$\int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} < r^2 \sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} \right).$$

Therefore we obtain from (2.6)

$$\begin{aligned}\int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma')r}(x)} u^{2p\theta} &\le 2\, E(B_r) \left(\sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} \right)\right)^{\theta-1} \\ &\times \left(\int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} A\nabla\left(\psi u^p\right)\nabla\left(\psi u^p\right) + \sup_{t\in I_{\sigma'}} \left(\int\limits_{B_{(1-\sigma)r}(x)} \psi^2 u^{2p} \right)\right).\end{aligned} \tag{2.7}$$

Now we can estimate both factors on the right-hand side of (2.7) by the inequality (2.5) and get as a result

$$\begin{aligned}\int\limits_{s-(1-\sigma')r^2}^{s} \int\limits_{B_{(1-\sigma')r}(x)} u^{2p\theta} &\le 2\, E(B_r) \left(\left(2 + 2c^2 + c + \tilde{C}\right)(\tau\, r)^{-2} \int\limits_{s-(1-\sigma)r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} u^{2p} \right)^{\theta} \\ &\le E(B_r) \left(\frac{K}{\tau^2 r^2} \int\limits_{s-(1-\sigma)r^2}^{s} \int\limits_{B_{(1-\sigma)r}(x)} u^{2p} \right)^{\theta}\end{aligned} \tag{2.8}$$

for a constant $K \ge 1$, which may in the following change from line to line. Now we take $\tau_i := 2^{-(i+1)}$, so $\sum_{i=1}^{\infty} \tau_i = \frac{1}{2}$, and moreover we set

$$\sigma_0 := 0,\ \sigma_i := \sigma_{i-1} + \tau_i = \sum_{j=1}^{i} \tau_j,\ p_i := \theta^i.$$

Thus, taking in (2.8) $p = p_i$, $\sigma = \sigma_i$ and $\sigma' = \sigma_{i+1}$ yields $\tau = \tau_i$ and

$$\begin{aligned}\int\limits_{s-(1-\sigma_{i+1})r^2}^{s} \int\limits_{B_{(1-\sigma_{i+1})r}(x)} u^{2\theta^{i+1}} &\le E(B_r) \left(K\, 4^{i+1} r^{-2} \int\limits_{s-(1-\sigma_i)r^2}^{s} \int\limits_{B_{(1-\sigma_i)r}(x)} u^{2\theta^i} \right)^{\theta} \\ &\le E(B_r) \left(K^{i+1} r^{-2} \int\limits_{s-(1-\sigma_i)r^2}^{s} \int\limits_{B_{(1-\sigma_i)r}(x)} u^{2\theta^i} \right)^{\theta}.\end{aligned} \tag{2.9}$$

Hence, potentiating (2.9) with $\theta^{-(i+1)}$ gives us

$$\begin{aligned}
&\left(\int_{s-(1-\sigma_{i+1})r^2}^{s}\int_{B_{(1-\sigma_{i+1})r}(x)} u^{2\theta^{i+1}}\right)^{\theta^{-(i+1)}}\\
&\leq E(B_r)^{\theta^{-(i+1)}}\left(K^{i+1}r^{-2}\int_{s-(1-\sigma_i)r^2}^{s}\int_{B_{(1-\sigma_i)r}(x)} u^{2\theta^i}\right)^{\theta^{-i}}\\
&\leq E(B_r)^{\sum_{j=0}^{i}\theta^{-(j+1)}}K^{\sum_{j=0}^{i}(j+1)\theta^{-j}}r^{-2\sum_{j=0}^{i}\theta^{-j}}\int_{s-r^2}^{s}\int_{B_r(x)} u^2.
\end{aligned}\tag{2.10}$$

So, taking the limit $i \to +\infty$ yields

$$\sum_{j=0}^{\infty}\theta^{-j} = \frac{\nu+2}{2}, \sum_{j=0}^{\infty}\theta^{-(j+1)} = \frac{\nu}{2}, \sum_{j=0}^{\infty}(j+1)\theta^{-j} = \frac{(\nu+2)^2}{4}$$

and therefore taking $i \to +\infty$ in the estimate (2.10) implies, noting that $\left\|u^2\right\|_{L^{\theta^{i+1}}} \to \left\|u^2\right\|_{L^\infty}$ as $i \to +\infty$,

$$\sup_{\left]s-\frac{r^2}{2},s\right[\times B_{\frac{r}{2}}(x)}\left\{u^2\right\} \leq K\,E(B_r)^{\frac{\nu}{2}}r^{-(\nu+2)}\int_{s-r^2}^{s}\int_{B_r(x)} u^2 = \frac{K\,S^{\frac{\nu}{2}}}{r^2\,\mathrm{Vol}\,(B_r(x))}\int_{Q_r(s,x)} u^2,$$

which proves Lemma 2.2.1 and as a result also the mean value inequality.

2.3 Proof of the dimension bound for time-dependent coefficients

As announced in the introduction, we adapt the proof of Lin and Zhang's dimension bound for polynomial growth ancient solutions of the heat equation to polynomial growth ancient solutions of $\partial_t u = Lu$. Note that taking the coefficient A to be the identity section in the operator L reduces $\partial_t - L$ to the heat operator $\partial_t - \Delta$.
Since Lin and Zhang's proof uses truncated paraboloids $P_r(t,x)$ instead of parabolic cylinders $Q_r(t,x)$, we recall that for any $r > 0$ and $(t,x) \in \mathbb{R}^- \times M$, the truncated paraboloid is defined as

$$P_r(t,x) := \{\,(s,y)|\, d_p((t,x),(s,y)) \leq r,\, s \leq t\}\,,$$

where, denoting by d the Riemannian distance function on M, the parabolic distance function d_p on $\mathbb{R}^- \times M$ is given for any $(t,x),(s,y) \in \mathbb{R}^- \times M$ by

$$d_p((t,x),(s,y)) := \sqrt{|t-s|} + d(x,y).$$

For their dimension bound Lin and Zhang assume the volume doubling property as well as an L^1-mean value inequality, in their proof however they only use an L^2-mean value inequality, which follows from the L^1-mean value inequality by squaring it and applying the Cauchy-Schwarz inequality. As we have shown in the previous section, an L^2-mean value inequality for $\partial_t - L$ follows from the volume doubling property together with a Neumann-Poincaré inequality, thus we will assume those throughout this section.
The main idea of the proof is, given an orthonormal basis $\{u_i\}_{i=1}^k$ of an arbitrary k-dimensional subspace K of $\mathcal{P}_q\,(M;L)$, to prove an upper and a lower bound on
$\sum_{i=1}^k \int_{P_R(t_0,x_0)} u_i^2$, so that combination of these yields an upper bound for k independent of the choice of K.

As in [56] the heat operator $\partial_t - \Delta$ only enters through the mean value inequality its solutions are assumed to satisfy, it suffices for us, up to minor modification, to insert our mean value inequality into their proof to obtain the same dimension bound. For completeness and convenience of the reader however we add the proofs of [56] in the appendix A.1. We begin with stating the upper bound, which is Lemma 3.1 in [56] for $L = \Delta$.

Lemma 2.3.1. *Let $q \geq 1$, K be a k-dimensional subspace of $\mathcal{P}_q(M;L)$ and $\{u_i\}_{i=1}^k$ be any basis of K. Given $(t_0, x_0) \in \mathbb{R}^- \times M$, $R \geq 1$ and $\varepsilon \in]0,1]$, we find*

$$\sum_{i=1}^{k} \int_{P_R(t_0,x_0)} u_i^2 \leq C(C_D, C_M)\, \varepsilon^{-(C_D+1)} \sup_{u \in <\Lambda, U>} \int_{P_{(2+\varepsilon)R}(t_0,x_0)} u^2,$$

where $<\Lambda, U> = \left\{ v = \sum_{i=1}^k \lambda_i u_i \,\middle|\, \sum_{i=1}^k \lambda_i^2 = 1,\, \lambda_i \in [0,1] \right\}$.

In order to get the lower bound, we first need an inner product on the finite-dimensional subspaces K of $\mathcal{P}_q\,(M;L)$.

Lemma 2.3.2. *(cf. [56], Lem. 3.2 for $L = \Delta$)*
Let $q \geq 1$, K be a finite-dimensional subspace of $\mathcal{P}_q(M;L)$. There exists a constant $R_0 = R_0(K)$ such that for all $R \geq R_0$ and all $(t_0,x_0) \in \mathbb{R}^- \times M$

$$< u, v > := \int_{P_R(t_0,x_0)} u\, v$$

is an inner product on K.

Since in [56] only the proof of the upper bound uses the mean value inequality and the proof of the lower bound only needs the volume doubling property as well as the polynomial growth of the solutions, this transfers without any changes to our setting.

Lemma 2.3.3. *(cf. [56],p. 2019 for $L = \Delta$)*
Let $q \geq 1$, K be a k-dimensional subspace of $\mathcal{P}_q(M;L)$ and $\{u_i\}_{i=1}^k$ be an orthonormal basis of K with respect to the inner product

$$A_{\beta R}(u,v) = \int_{P_{\beta R}(t_0,x_0)} u\, v$$

for some $(t_0,x_0) \in \mathbb{R}^- \times M$ and $\beta > 1$. Then, for any $\delta > 0$ and $R_0 \geq 1$ there exists some $R > R_0$ such that

$$\sum_{i=1}^{k} \int_{P_R(t_0,x_0)} u_i^2 \geq k\, \beta^{-(2q + C_D + 2 + \delta)}.$$

Therefore, combination of the lower bound with $\beta = 2 + \varepsilon$ and the upper bound with $\varepsilon = \frac{1}{q}$, then taking $\delta \to 0$ implies the dimension bound for $\mathcal{P}_q\,(M;L)$.

Theorem 2.3.4. *Let M be a complete, non-compact Riemannian manifold on which the volume doubling property with constant C_D and a Neumann-Poincaré inequality with constant C_N hold. Then, there is a constant $0 < C < +\infty$, depending only on C_D and C_N, such that for any integer $q \geq 1$*

$$\dim\left(\mathcal{P}_q(M;L)\right) \leq C\, q^{C_D+1}.$$

2.4 Dimension bound for time-independent coefficients

In the special case that the coefficient A of L is independent of the time-variable, i.e. $A = A(x)$, we can improve the dimension bound for $\mathcal{P}_d\,(M;L)$ we have just shown in the same

way as Colding and Minicozzi improved in [21] the bound by Lin and Zhang for the heat operator. For this we take in the following $p \in M$ to be arbitrary but fixed and denote for brevity $B_r := B_r(p)$ and $Q_r := Q_r(0,p)$ for any $r > 0$.
Following the proof in [21], the main ingredient for proving the dimension bound will be a reverse Poincaré inequality for solutions to $\partial_t u = Lu$.

Lemma 2.4.1. *Let $r > 0$ and u be a solution to $\partial_t u = Lu$ on Q_r.*
Then there exists a constant $0 < K < +\infty$ such that

$$r^2 \int\limits_{Q_{\frac{r}{2}}} |\nabla u|^2 + r^4 \int\limits_{Q_{\frac{r}{2}}} |\partial_t u|^2 \leq K \int\limits_{Q_r} u^2.$$

Proof. Let $r > 0$ and u be a solution to $\partial_t u = Lu$ on Q_r.
Let $\phi \in C_0^\infty(M)$ be a cutoff function on M, which satisfies for some constant $\alpha \in \left[\frac{4}{5}, 1\right]$ to be fixed later

$$0 \leq \phi \leq 1, \ \mathrm{supp}(\phi) \subseteq B_{\alpha r}, \ \phi|_{B_{\frac{r}{2}}} \equiv 1 \text{ and } |\nabla \phi|^2 \leq \frac{100}{9r^2}.$$

Then we obtain by testing with $\phi^2 u$ and using the assumptions (1.2) on A as well as Young's inequality and the gradient bound on ϕ

$$\begin{aligned} \partial_t \int\limits_{B_{\alpha r}} \phi^2 u^2 &= 2 \int\limits_{B_{\alpha r}} \phi^2 u \,(\partial_t u) = -2 \int\limits_{B_{\alpha r}} \phi^2 A \nabla u \nabla u - 4 \int\limits_{B_{\alpha r}} \phi \, u \, A \nabla u \nabla \phi \\ &\leq -\,(2c_1 - 1) \int\limits_{B_{\alpha r}} \phi^2 |\nabla u|^2 + \frac{400c^2}{9r^2} \int\limits_{B_r} u^2. \end{aligned}$$

So integrating in time over $\left[-\alpha^2 r^2, 0\right]$ yields

$$\int\limits_{Q_{\alpha r}} \phi^2 |\nabla u|^2 \leq \frac{1}{2c_1 - 1} \int\limits_{B_r \times \{t = -\alpha^2 r^2\}} u^2 + \frac{400c^2}{9(2c_1 - 1)} r^{-2} \int\limits_{Q_r} u^2. \tag{2.11}$$

Now we fix $\alpha \in \left[\frac{4}{5}, 1\right]$ by applying the mean value theorem so that

$$\int\limits_{B_r \times \{t = -\alpha^2 r^2\}} u^2 \leq \frac{25}{9r^2} \int\limits_{Q_r} u^2.$$

Thus, putting this into (2.11) we find as ϕ was taken to be equal to one on $B_{\frac{r}{2}}$

$$r^2 \int\limits_{Q_{\frac{r}{2}}} |\nabla u|^2 \leq r^2 \int\limits_{Q_{\alpha r}} \phi^2 |\nabla u|^2 \leq \frac{400c^2 + 25}{9(2c_1 - 1)} \int\limits_{Q_r} u^2 =: \kappa \int\limits_{Q_r} u^2. \tag{2.12}$$

Now it remains to prove a corresponding bound for the time derivative. For this we take $\psi \in C_0^\infty(M)$ to be another cutoff function on M so that for a constant $\beta \in \left[\frac{3}{5}, \frac{4}{5}\right]$, which will be determined later, we have

$$0 \leq \psi \leq 1, \ \mathrm{supp}(\psi) \subseteq B_{\beta r}, \ \psi|_{B_{\frac{r}{2}}} \equiv 1 \text{ and } |\nabla \psi|^2 \leq \frac{100}{r^2}.$$

The weak formulation for u with test function $\phi^2 \psi^2 \partial_t u$, Young's inequality and our bounds (1.2) on A give us

$$\partial_t \int\limits_{B_{\beta r}} \phi^2 \psi^2 A \nabla u \nabla u = 2 \int\limits_{B_{\beta r}} A \nabla u \nabla \left(\phi^2 \psi^2 \partial_t u\right) - 4 \int\limits_{B_{\beta r}} \phi \, \psi^2 \,(\partial_t u)\, A \nabla u \nabla \phi$$

$$
\begin{aligned}
&-4\int_{B_{\beta r}} \psi\,\phi^2\,(\partial_t u)\,A\nabla u\nabla\psi\\
&=-2\int_{B_{\beta r}} \phi^2\psi^2\,|\partial_t u|^2-4\int_{B_{\beta r}} \phi\,\psi^2\,(\partial_t u)\,A\nabla u\nabla\phi\\
&\quad-4\int_{B_{\beta r}} \psi\,\phi^2\,(\partial_t u)\,A\nabla u\nabla\psi\\
&\le-\int_{B_{\beta r}} \phi^2\psi^2\,|\partial_t u|^2+\frac{800c^2}{9r^2}\int_{B_{\alpha r}} \psi^2\,|\nabla u|^2+\frac{800c^2}{r^2}\int_{B_{\alpha r}} \phi^2\,|\nabla u|^2,
\end{aligned}
$$

where in the last inequality we also used the gradient bounds for our cutoff functions ϕ and ψ. Now integration in time from $-\beta^2r^2$ to 0 yields, as $\beta\le\alpha$,

$$
\begin{aligned}
\int_{Q_{\beta r}} \phi^2\psi^2\,|\partial_t u|^2 &\le \int_{B_{\beta r}\times\{t=-\beta^2r^2\}} \phi^2\psi^2 A\nabla u\nabla u+\frac{800c^2}{9r^2}\int_{Q_{\alpha r}} \psi^2\,|\nabla u|^2+\frac{800c^2}{r^2}\int_{Q_{\alpha r}} \phi^2\,|\nabla u|^2\\
&\le c\int_{B_{\frac{4}{5}r}\times\{t=-\beta^2r^2\}} \phi^2\,|\nabla u|^2+\frac{800c^2}{9r^2}\int_{Q_{\alpha r}} \psi^2\,|\nabla u|^2+\frac{800c^2}{r^2}\int_{Q_{\alpha r}} \phi^2\,|\nabla u|^2.
\end{aligned}
\tag{2.13}
$$

We pick now $\beta\in\left[\frac{3}{5},\frac{4}{5}\right]$, using again the mean value theorem, such that

$$
\int_{B_{\frac{4}{5}r}\times\{t=-\beta^2r^2\}} \phi^2\,|\nabla u|^2\le\frac{25}{7r^2}\int_{Q_{\frac{4}{5}r}} \phi^2\,|\nabla u|^2\le\frac{25}{7r^2}\int_{Q_{\alpha r}} \phi^2\,|\nabla u|^2\le\frac{25\kappa}{7r^4}\int_{Q_r} u^2.
$$

Proceeding as for the gradient bound (2.12) with cutoff function ϕ, we can also show, now with cutoff function ψ,

$$
\partial_t\int_{B_r} \psi^2u^2=\partial_t\int_{B_{\beta r}} \psi^2u^2\le-(2c_1-1)\int_{B_{\beta r}} \psi^2\,|\nabla u|^2+\frac{400c^2}{r^2}\int_{B_r} u^2.
$$

Thus, since we chose $\beta\le\alpha$, we obtain after integrating from $-\alpha^2r^2$ to 0

$$
\int_{Q_{\alpha r}} \psi^2\,|\nabla u|^2\le\frac{1}{2c_1-1}\int_{B_r\times\{t=-\alpha^2r^2\}} u^2+\frac{400c^2}{(2c_1-1)r^2}\int_{Q_r} u^2\le\frac{\tilde\kappa}{r^2}\int_{Q_r} u^2,
\tag{2.14}
$$

where $\tilde\kappa:=\frac{1}{2c_1-1}\left(400c^2+\frac{25}{9}\right)$. Hence, combining (2.13) with (2.12) and (2.14) as well as using that ϕ and ψ were chosen so that they are equal to one on $B_{\frac{r}{2}}$, we find

$$
r^4\int_{Q_{\frac{r}{2}}} |\partial_t u|^2\le r^4\int_{Q_{\beta r}} \phi^2\psi^2\,|\partial_t u|^2\le\left(\frac{25\kappa\,c}{7}+\frac{800c^2\tilde\kappa}{9}+800c^2\kappa\right)\int_{Q_r} u^2=:\bar\kappa\int_{Q_r} u^2.
\tag{2.15}
$$

Therefore, summing (2.12) and (2.15), we conclude

$$
r^2\int_{Q_{\frac{r}{2}}} |\nabla u|^2+r^4\int_{Q_{\frac{r}{2}}} |\partial_t u|^2\le(\kappa+\bar\kappa)\int_{Q_r} u^2=:K\int_{Q_r} u^2.
$$

□

Due to the supposed time-independence of A and thus of L we now find that the operators

∂_t and $\partial_t - L$ commute, implying that the time derivatives of a solution u to $\partial_t u = Lu$ are again solutions, i.e. for any integer $k \geq 1$ we find

$$\partial_t \left(\partial_t^k u\right) = \partial_t^k \left(\partial_t u\right) = \partial_t^k \left(Lu\right) = L\left(\partial_t^k u\right).$$

Thus the reverse Poincaré inequality applies also to the time derivatives of solutions, giving us for any solution u on Q_r and integer $k \geq 1$

$$r^2 \int\limits_{Q_{\frac{r}{2}}} \left|\nabla \left(\partial_t^k u\right)\right|^2 + r^4 \int\limits_{Q_{\frac{r}{2}}} \left|\partial_t^{k+1} u\right|^2 \leq C \int\limits_{Q_r} \left|\partial_t^k u\right|^2.$$

Iterated application of this reverse Poincaré inequality yields that for any integer $k \geq 1$ there exists a constant $0 < c_k < +\infty$ so that for any solution u to $\partial_t u = Lu$ on Q_r we get

$$r^{4k} \int\limits_{Q_{\frac{r}{2^k}}} \left|\partial_t^k u\right|^2 \leq c_k \int\limits_{Q_r} u^2.$$

Hence, if u is of polynomial growth of order at most $d \geq 1$ and M has polynomial volume growth of order at most $d_V > 0$, we find

$$\int\limits_{Q_{\frac{r}{2^k}}} \left|\partial_t^k u\right|^2 \leq c_k\, r^{-4k} r^2 \operatorname{Vol}(B_r) \sup_{Q_r} \left\{u^2\right\} \leq c_k\, V\, C_u^2 r^{2-4k} \left(1+r\right)^{d_V + 2d}.$$

Therefore, choosing k so that $4k > 2d + d_V + 2$ and taking the limit $r \to +\infty$ gives us $\partial_t^k u = 0$. So we have shown

Corollary 2.4.2. *Suppose that M has polynomial volume growth of order at most d_V. Then, given $u \in \mathcal{P}_d(M; L)$ and an integer $k \geq 1$ so that $4k > 2d + d_V + 2$, we obtain $\partial_t^k u = 0$.*

Now, $\partial_t^k u = 0$ implies, as in [56] and [21] for the heat equation, that there exist functions $p_1, \ldots, p_k$ on M such that for any $t \in \mathbb{R}^-$ and $x \in M$ the solution u is of the form

$$u(t,x) = \sum_{l=0}^{k} t^l p_l(x).$$

Arguing as in Lemma 1.24 in [21], we obtain for each of the functions p_j that they are of polynomial growth of order at most $2(k-j)$. Furthermore, as A is time-independent, the dimension-counting argument in the proof of Theorem 0.3 in [21] remains valid upon replacing the Laplacian Δ with our elliptic operator L, so we can conclude the same dimension bound for $\mathcal{P}_d\left(M; L\right)$ as for $\mathcal{P}_d\left(M; \Delta\right)$.

Theorem 2.4.3. *Suppose that M has polynomial volume growth of order at most $d_V > 0$ and let $k \geq 1$ be an integer. Then*

$$\dim\left(\mathcal{P}_{2k}(M; L)\right) \leq \sum_{i=0}^{k} \dim\left(\mathcal{H}_{2i}(M; L)\right) \leq (k+1) \dim\left(\mathcal{H}_{2k}(M; L)\right).$$

Remark. For completeness let us recall the key steps in proving this bound.
First, we conclude from Corollary 2.4.2 that each $u \in \mathcal{P}_{2k}\left(M; L\right)$ can be written as a polynomial in time, i.e. there exist functions $p_0, \ldots, p_k$ on M so that $u(t,x) = \sum_{l=0}^{k} t^l p_l(x)$. Next, we obtain that each p_j has polynomial growth of order at most $2(k-j)$. Furthermore, as u is a solution to $\partial_t u = Lu$ and A is independent of the time variable, we have by comparing coefficients $Lp_k = 0$ and $Lp_j = (j+1)p_{j+1}$ for each $j \in \{0, \ldots, k-1\}$. Hence, we get a linear map $\mathcal{P}_{2k}(M; L) \to \mathcal{H}_0(M; L)$, $u \mapsto p_k$. This yields that the dimension of $\mathcal{P}_{2k}$ is bounded by the sum of the dimension of $\mathcal{H}_0$ and the dimension of the kernel. The latter can be estimated by considering the map from the kernel into $\mathcal{H}_2$ given by $u \mapsto p_{k-1}$.

Continuing like this yields the first inequality. The second inequality follows directly from the fact that $\mathcal{H}_{d_1}(M;L) \subseteq \mathcal{H}_{d_2}(M;L)$ for any $d_1 \leq d_2$.

As a result, to obtain an explicit dimension bound for $\mathcal{P}_{2k}(M;L)$ it suffices to bound the dimension of $\mathcal{H}_{2k}(M;L)$. In fact, for the latter, under some conditions on the geometry of M, there is a dimension bound already known (cf. [19, 20] or [50]). As this bound was only explicitly proven in the case of $M = \mathbb{R}^n$ and, except for the reverse Poincaré inequality for solutions, the proof can be executed exactly as for harmonic maps, we briefly prove said reverse Poincaré inequality to justify stating the dimension bound for the spaces $\mathcal{H}_{2k}(M;L)$.

Lemma 2.4.4. *Let $r, s > 0$ and u be a solution to $Lu = 0$ on $B_{(1+s)r}$. Then, there exists a constant $0 < \tilde{K} < +\infty$ such that*

$$r^2 \int_{B_r} |\nabla u|^2 \leq \frac{\tilde{K}}{s^2} \int_{B_{(1+s)r}} u^2.$$

Proof. Let $\psi \in C_0^\infty(M)$ be a cutoff function on M, which satisfies the properties

$$0 \leq \psi \leq 1, \, \mathrm{supp}\,(\psi) \subseteq B_{(1+s)r}, \; \psi|_{B_r} \equiv 1 \,\text{and}\, |\nabla \psi|^2 \leq \frac{1}{s^2 r^2}.$$

Then we obtain, using that u is a solution to $Lu = 0$, our assumptions (1.2) on A and Young's inequality,

$$\begin{aligned}
0 &= - \int_{B_{(1+s)r}} \psi^2 A \nabla u \nabla u - 2 \int_{B_{(1+s)r}} \psi \, u \, A \nabla u \nabla \psi \\
&\leq - \left(c_1 - \frac{1}{2}\right) \int_{B_{(1+s)r}} \psi^2 |\nabla u|^2 + 2c^2 \int_{B_{(1+s)r}} u^2 |\nabla \psi|^2 \\
&\leq - \left(c_1 - \frac{1}{2}\right) \int_{B_{(1+s)r}} \psi^2 |\nabla u|^2 + \frac{2c^2}{s^2 r^2} \int_{B_{(1+s)r}} u^2.
\end{aligned}$$

Thus, given the choice of cutoff function ψ and $c_1 > \frac{1}{2}$, this yields after multiplication by r^2

$$r^2 \int_{B_r} |\nabla u|^2 \leq r^2 \int_{B_{(1+s)r}} \psi^2 |\nabla u|^2 \leq \frac{2c^2}{c_1 - \frac{1}{2}} s^{-2} \int_{B_{(1+s)r}} u^2 =: \frac{\tilde{K}}{s^2} \int_{B_{(1+s)r}} u^2,$$

which is the claimed reverse Poincaré inequality. $\square$

This justifies the dimension bound on $\mathcal{H}_d(M;L)$.

Theorem 2.4.5. *([20])*
Suppose that M is a complete, non-compact Riemannian manifold which has a scale-invariant Neumann-Poincaré inequality with constant C_N and the relative volume comparison property with constant C_V. Then there exists a constant $0 < \bar{K} < +\infty$ such that for $d \geq 1$

$$\dim\left(\mathcal{H}_d(M;L)\right) \leq \bar{K} d^{C_V - 1}.$$

As the relative volume comparison property with constant C_V implies the volume doubling property with $C_D = C_V$ and thus also polynomial volume growth of M with $d_V = C_V$, we get an explicit dimension bound for $\mathcal{P}_{2k}(M;L)$ by combining the Theorems 2.4.3 and 2.4.5.

Theorem 2.4.6. *Suppose that M is a complete, non-compact Riemannian manifold which satisfies the relative volume comparison property with constant C_V and a Neumann-Poincaré inequality with constant C_N. Suppose that A is time-independent. Then, there exists a* *constant* $0 < C < +\infty$ *so that for any integer* $k \geq 1$

$$\dim\left(\mathcal{P}_{2k}\left(M;L\right)\right) \leq \; C\left(2k\right)^{C_V} .$$

Proof. Noting that the relative volume comparison property implies the volume doubling property with constant $C_D = C_V$, we get from combining the dimension bound on $\mathcal{P}_{2k}\left(M;L\right)$ in terms of the dimension of $\mathcal{H}_{2k}\left(M;L\right)$ with the dimension bound for the latter, given any integer $k \geq 1$,

$$\dim\left(\mathcal{P}_{2k}\left(M;L\right)\right) \leq (k+1)\dim\left(\mathcal{H}_{2k}\left(M;L\right)\right) \leq \; \bar{K}(k+1)\left(2k\right)^{C_V-1} \leq \; \bar{K}\left(2k\right)^{C_V} .$$

□

Remark. To end this chapter with an example, it is well-known that a manifold M with non-negative Ricci curvature satisfies the volume doubling property with constant $n = \dim(M)$ (e.g. [9]) and also a Neumann-Poincaré inequality with $C_N = C_N(n)$ (e.g. [14]). So we obtain in this case a constant $0 < C < +\infty$, depending only on n, such that for any integer $k \geq 1$

$$\dim\left(\mathcal{P}_{2k}\left(M;L\right)\right) \leq \; C\left(2k\right)^{n} .$$

Chapter 3

Harmonic Maps of low Index

As we outlined in the introduction, in this chapter we consider various aspects connected to smooth, harmonic maps of low index taking values in a sphere equipped with its round Riemannian metric. Contrary to Chapter 2 we will denote throughout this Chapter by Δ the negative Laplace operator(s), that is for the Laplacian $\Delta = -\operatorname{div}(\nabla)$, respectively $\Delta = -\operatorname{tr}[\nabla.\nabla.]$.

3.1 Index bounds for harmonic submersions

The first question we will address is the contribution of maps into lower-dimensional submanifolds to the set of harmonic maps of low index as maps into a given codomain.
For this we take smooth, harmonic submersions $\psi : (M, g) \to (S^m, \mathrm{can})$ and compare their indices and nullities to the indices and nullities of the composition $\phi := \mathrm{i} \circ \psi : (M, g) \to (S^p, \mathrm{can})$, $p > m$, where $\mathrm{i} : S^m \to S^p$ is the canonical, totally geodesic inclusion map, that is for any $x = (x_1, \ldots, x_{m+1}) \in S^m$ we have $\mathrm{i}(x) = (x_1, \ldots, x_{m+1}, 0, \ldots, 0) \in S^p$.
To illustrate how this gives us information on the role of maps into submanifolds, suppose for now that $M^n = S^n$ and $g = \mathrm{can}$. Then we know from El Soufi's index bound that $\operatorname{ind}(\psi) \geq n+1$ as well as $\operatorname{ind}(\phi) \geq n+1$ independent of the choice of $p > m$. So, if we are interested in low index maps $S^n \to S^p$ for fixed n and p, we aim to find harmonic maps $S^n \to S^p$ of index $n+1$. Thus, if we have for $m < p$ harmonic maps $\psi : S^n \to S^m$ and $\phi := \mathrm{i} \circ \psi : S^n \to S^p$ such that $\operatorname{ind}(\phi) > \operatorname{ind}(\psi)$, then ϕ cannot be of index $n+1$ as this would mean that ψ is of index at most n, contradicting El Soufi's index bound. Consequently, such maps ψ cannot contribute to low index maps $S^n \to S^p$ through i.
In 3.1.4 we also approach the question of the indices of harmonic morphisms, i.e. whether harmonic morphisms are necessarily of lowest possible index for the case of submersive harmonic morphisms from the round three-sphere into the round two-sphere by using their characterization established by Baird and Wood.

3.1.1 Notation and Preliminaries

For a given submersion $\psi : (M, g) \to (N, h)$ between Riemannian manifolds M and N we denote at each point $x \in M$ its vertical space $\ker(d\psi_x)$ by $\mathcal{V}_x$ and its horizontal space $(\ker(d\psi_x))^{\perp}$ by $\mathcal{H}_x$.
Moreover, for any tangent vector $v \in T_xM$ we let $v^{\mathcal{V}}$, respectively $v^{\mathcal{H}}$, denote its vertical, respectively horizontal, part so that $v = v^{\mathcal{V}} + v^{\mathcal{H}}$. In the case $v^{\mathcal{V}} = 0$, respectively $v^{\mathcal{H}} = 0$, we say that v is horizontal, respectively vertical.
In light of this, we can also decompose the metric g on M as $g = g^{\mathcal{V}} + g^{\mathcal{H}}$ with respect to the submersion ψ, that is for any $x \in M$ and $X, Y \in T_xM$

$$g(X, Y) = g\left(X^{\mathcal{V}}, Y^{\mathcal{V}}\right) + g\left(X^{\mathcal{H}}, Y^{\mathcal{H}}\right) =: g^{\mathcal{V}}(X, Y) + g^{\mathcal{H}}(X, Y).$$

Furthermore, we also need the notion of the vertical Laplacian associated to a Riemannian submersion.

Definition 3.1.1. ([7])
Let $\psi : (M, g) \to (N, h)$ be a Riemannian submersion. The vertical Laplacian Δ^V associated to ψ is the differential operator defined on $C^\infty(M, g)$ by

$$\left(\Delta^V f\right)(p) := \left(\Delta^{F_p}\left(f|_{F_p}\right)\right)(p),\ f \in C^\infty(M),$$

where Δ^{F_p} is the Laplace operator of the fibre $F_p = \psi^{-1}(\psi(p))$.

In [7] it is shown that when the fibres are totally geodesic, the Laplacian Δ on $C^\infty(M)$ and the vertical Laplacian Δ^V commute, which implies

Theorem 3.1.2. *([7])*
$L^2(M)$ admits a Hilbert basis consisting of simultaneous eigenfunctions for Δ and Δ^V.

3.1.2 The general case

Let throughout this subsection M denote an n-dimensional, compact, connected, smooth Riemannian manifold without boundary and $\psi : M^n \to S^m$, $n \geq m \geq 1$, be a smooth, harmonic submersion from M into the round m-sphere. Taking for any point $x \in M$ an orthonormal basis $\{v_1(x), \ldots, v_m(x)\}$ of the tangent space $T_{\psi(x)}S^m$, there must exist for each $j \in \{1, \ldots, m\}$ some $e_j(x) \in T_xM$ so that $v_j(x) = d\psi_x(e_j(x))$. Thus the v_j's are orthonormal sections of the pull-back bundle $\psi^{-1}TS^m$, which yields a decomposition of the space of sections of $\psi^{-1}TS^m$ as

$$\Gamma\left(\psi^{-1}TS^m\right) = \bigoplus_{j=1}^{m} \{ f_j\, d\psi(e_j) |\, f_j \in C^\infty(M)\}.$$

Consider now for any $p > m$ the canonical inclusion map $\mathrm{i} : S^m \to S^p$ and denote

$$\phi := \mathrm{i} \circ \psi : M^n \to S^p.$$

Moreover, take $\eta_{m+1}, \ldots, \eta_p$ unit sections of the normal bundle of $\mathrm{i}(S^m)$ in S^p such that for every $x \in M$ the set $\{\eta_{m+1}(\phi(x)), \ldots, \eta_p(\phi(x))\}$ gives us an orthonormal basis of the normal space $N_{\phi(x)}\mathrm{i}(S^m)$.
Due to the conformality of the inclusion map i, the space of sections of $\phi^{-1}TS^p$ inherits a decomposition from the decomposition of $\Gamma\left(\psi^{-1}TS^m\right)$, namely

$$\Gamma\left(\phi^{-1}TS^p\right) = \bigoplus_{j=1}^{m} \{ f_j\, d\phi(e_j) |\, f_j \in C^\infty(M)\} \oplus \bigoplus_{l=m+1}^{p} \{ f_l\, \eta_l \circ \phi |\, f_l \in C^\infty(M)\}.$$

Therefore, in order to compare the indices of ψ and ϕ, it suffices to compute J^ψ on the span of the $d\psi(e_j)$'s and J^ϕ on the spans of the $d\phi(e_j)$'s and the $\eta_l \circ \phi$'s.
Since the inclusion map i is totally geodesic, we find for any $j \in \{1, \ldots, m\}$

$$\nabla^\phi_{\cdot}(d\phi(e_j)) = \nabla \mathrm{di}(d\psi\cdot, d\psi(e_j)) + \mathrm{di}\left(\nabla^\psi_{\cdot}(d\psi(e_j))\right) = \mathrm{di}\left(\nabla^\psi_{\cdot}(d\psi(e_j))\right)$$

and consequently

$$\begin{aligned}\Delta^\phi(d\phi(e_j)) &= -\operatorname{tr}\left[\nabla^\phi_{\cdot}\nabla^\phi_{\cdot}(d\phi(e_j))\right] = -\operatorname{tr}\left[\nabla^\phi_{\cdot}\left(\mathrm{di}\left(\nabla^\psi_{\cdot}(d\psi(e_j))\right)\right)\right] \\ &= -\operatorname{tr}\left[\nabla \mathrm{di}\left(d\psi\cdot, \nabla^\psi_{\cdot}(d\psi(e_j))\right) + \mathrm{di}\left(\nabla^\psi_{\cdot}\nabla^\psi_{\cdot}(d\psi(e_j))\right)\right] \\ &= \mathrm{di}\left(\Delta^\psi(d\psi(e_j))\right).\end{aligned}$$

So, for any function $f_j \in C^\infty(M)$, we compute with this

$$\begin{aligned}\Delta^\phi\left(f_j\, d\phi\left(e_j\right)\right) &= \left(\Delta f_j\right) d\phi\left(e_j\right) + f_j \Delta^\phi\left(d\phi\left(e_j\right)\right) - 2\operatorname{tr}\left[\left(\nabla_\cdot f_j\right)\nabla^\phi_\cdot\left(d\phi\left(e_j\right)\right)\right]\\ &= \left(\Delta f_j\right) d\phi\left(e_j\right) + f_j \operatorname{di}\left(\Delta^\psi\left(d\psi\left(e_j\right)\right)\right) - 2\operatorname{tr}\left[\left(\nabla_\cdot f_j\right)\operatorname{di}\left(\nabla^\psi_\cdot\left(d\psi\left(e_j\right)\right)\right)\right]\\ &= \operatorname{di}\left(\left(\Delta f_j\right) d\psi\left(e_j\right) + f_j \Delta^\psi\left(d\psi\left(e_j\right)\right) - 2\operatorname{tr}\left[\left(\nabla_\cdot f_j\right)\nabla^\psi_\cdot\left(d\psi\left(e_j\right)\right)\right]\right)\\ &= \operatorname{di}\left(\Delta^\psi\left(f_j\, d\psi\left(e_j\right)\right)\right).\end{aligned}$$

Moreover, as round spheres are space forms, the curvature term in the Jacobi operators J^ϕ and J^ψ simplifies so that we obtain, for any $j \in \{1,\ldots,m\}$ and any orthonormal frame $\{X_1,\ldots,X_n\}$ on M^n, denoting by $(\cdot,\cdot)$ the round metrics on the spheres S^p and S^m,

$$\begin{aligned}\operatorname{tr} R^{S^p}\left(d\phi\left(e_j\right), d\phi\cdot\right) d\phi\cdot &= \sum_{k=1}^n \left[\left|d\phi\left(X_k\right)\right|^2 d\phi\left(e_j\right) - \left(d\phi\left(e_j\right), d\phi\left(X_k\right)\right) d\phi\left(X_k\right)\right]\\ &= \left|d\phi\right|^2 d\phi\left(e_j\right) - \sum_{k=1}^n \left(d\phi\left(e_j\right), d\phi\left(X_k\right)\right) d\phi\left(X_k\right)\\ &= \left|d\psi\right|^2 d\phi\left(e_j\right) - \sum_{k=1}^n \left(d\psi\left(e_j\right), d\psi\left(X_k\right)\right) d\phi\left(X_k\right)\\ &= \operatorname{di}\left(\left|d\psi\right|^2 d\psi\left(e_j\right) - \sum_{k=1}^n \left(d\psi\left(e_j\right), d\psi\left(X_k\right)\right) d\psi\left(X_k\right)\right)\\ &= \operatorname{di}\left(\operatorname{tr} R^{S^m}\left(d\psi\left(e_j\right), d\psi\cdot\right) d\psi\cdot\right).\end{aligned}$$

Here we also used that i is isometric as well as the linearity of di. Therefore, we can compare the Jacobi operators of ψ and ϕ for any $j \in \{1,\ldots,m\}$, to be precise for any $f_j \in C^\infty(M)$

$$\begin{aligned}J^\phi\left(f_j\, d\phi\left(e_j\right)\right) &= \Delta^\phi\left(f_j\, d\phi\left(e_j\right)\right) - \operatorname{tr} R^{S^p}\left(f_j\, d\phi\left(e_j\right), d\phi\cdot\right) d\phi\cdot\\ &= \operatorname{di}\left(\Delta^\psi\left(f_j\, d\psi\left(e_j\right)\right) - \operatorname{tr} R^{S^m}\left(f_j\, d\psi\left(e_j\right), d\psi\cdot\right) d\psi\cdot\right)\\ &= \operatorname{di}\left(J^\psi\left(f_j\, d\psi\left(e_j\right)\right)\right).\end{aligned} \tag{3.1}$$

As a result, given an eigensection V of J^ψ with eigenvalue λ we know from the decomposition of $\Gamma\left(\psi^{-1}TS^m\right)$ that $V = \sum_{j=1}^m f_j\, d\psi\left(e_j\right)$ for functions $f_1,\ldots,f_m \in C^\infty(M)$ such that this yields

$$\begin{aligned}\lambda\sum_{j=1}^m f_j\, d\phi\left(e_j\right) = \lambda\operatorname{di}(V) = \operatorname{di}\left(J^\psi(V)\right) &= \sum_{j=1}^m \operatorname{di}\left(J^\psi\left(f_j\, d\psi\left(e_j\right)\right)\right)\\ &= \sum_{j=1}^m J^\phi\left(f_j\, d\phi\left(e_j\right)\right) = J^\phi\left(\sum_{j=1}^m f_j\, d\phi\left(e_j\right)\right).\end{aligned}$$

Hence $\operatorname{di}(V)$ is an eigensection of J^ϕ with eigenvalue λ. As this does not depend on the choice of eigensection or eigenvalue of J^ψ, we obtain that every eigenvalue of J^ψ is an eigenvalue of J^ϕ with at least the same multiplicity. In particular this means that the index, respectively nullity, of ϕ cannot be smaller than the index, respectively nullity, of ψ.
In addition, let W be an eigensection of J^ϕ restricted to $\bigoplus_{j=1}^m\{f_j\, d\phi(e_j)|\, f_j \in C^\infty(M)\}$ with eigenvalue μ. Then $W = \sum_{j=1}^m f_j\, d\phi(e_j)$ for some functions $f_j \in C^\infty(M)$. Furthermore we find, using (3.1),

$$\mu\operatorname{di}\left(\sum_{j=1}^m f_j\, d\psi\left(e_j\right)\right) = \mu\sum_{j=1}^m f_j\, d\phi\left(e_j\right) = J^\phi(W) = \sum_{j=1}^m J^\phi\left(f_j\, d\phi\left(e_j\right)\right)$$

$$= \sum_{j=1}^{m} \mathrm{di}\left(J^{\psi}\left(f_j\, d\psi\left(e_j\right)\right)\right) = \mathrm{di}\left(J^{\psi}\left(\sum_{j=1}^{m} f_j\, d\psi\left(e_j\right)\right)\right),$$

which yields that $V := \sum_{j=1}^{m} f_j\, d\psi\left(e_j\right)$ is an eigensection of J^{ψ} with eigenvalue μ since i is an immersion.
From this we can conclude that J^{ψ} and J^{ϕ} restricted to $\bigoplus_{j=1}^{m}\{\, f_j\, d\phi(e_j) |\, f_j \in C^{\infty}(M)\}$ possess the same eigenvalues with the same multiplicities. In particular, any additional eigenvalue of J^{ϕ} or additional multiplicity of an eigenvalue of J^{ψ} must arise from a section of the normal bundle. Summarizing all of this, we have just shown

Theorem 3.1.3. *Let M^n be a compact, connected, smooth Riemannian manifold without boundary and $\psi : M^n \to S^m$, $n \geq m \geq 1$, be a smooth, harmonic submersion. Let* i $: S^m \to S^p$ *for some $p > m$ be the canonical inclusion map of round spheres and set $\phi :=$ i $\circ\, \psi : M^n \to S^p$. Then we obtain*

$$\mathrm{ind}\left(\phi\right) \geq \mathrm{ind}\left(\psi\right) \quad \textit{and} \quad \mathrm{null}\left(\phi\right) \geq \mathrm{null}\left(\psi\right).$$

Naturally, we are interested in the case where we obtain strict inequalities at least for the index. For this, as we have seen that (counting with multiplicity) any additional eigenvalue of J^{ϕ} must arise from a linear combination of the $\eta_l \circ \phi$'s, we turn to finding $J^{\phi}\left(f_l\, \eta_l \circ \phi\right)$ for arbitrary $f_l \in C^{\infty}\left(M\right)$.
As a preparation, note that at every point $x \in \mathrm{i}\left(S^m\right)$ the normal space with respect to S^p is given by

$$N_x \mathrm{i}\left(S^m\right) = \mathrm{span}\left\{\varepsilon_{m+2}, \ldots, \varepsilon_{p+1}\right\} \setminus \{0\},$$

where $\{\varepsilon_1, \ldots, \varepsilon_{p+1}\}$ denotes the standard basis of $\mathbb{R}^{p+1}$.
Therefore we can assume without loss of generality that for any $l \in \{m+1, \ldots, p\}$ and $x \in \mathrm{i}\left(S^m\right)$ the η_l's are given by $\eta_l(x) = \varepsilon_{l+1}$. Then we have for every $l \in \{m+1, \ldots, p\}$, $x \in M$ and $X(x) \in T_x M$, denoting by $P_{\phi(x)}$ the projection onto $T_{\phi(x)} S^p$,

$$\begin{aligned} \nabla^{\phi}_{X(x)} \eta_l\left(\phi(x)\right) &= \nabla_{d\phi_x(X(x))} \eta_l\left(\phi(x)\right) = P_{\phi(x)}\left[\left(d\left(\eta_l\left(\phi(x)\right)\right)\right)\left(d\phi_x\left(X(x)\right)\right)\right] \\ &= P_{\phi(x)}\left[\left(d\varepsilon_{l+1}\right)\left(d\phi_x\left(X(x)\right)\right)\right] = P_{\phi(x)}[0] = 0. \end{aligned}$$

Thus, also $\Delta^{\phi}\left(\eta_l \circ \phi\right) = -\,\mathrm{tr}\left[\nabla^{\phi}_{\cdot}\nabla^{\phi}_{\cdot}\left(\eta_l \circ \phi\right)\right] = 0$ for any $l \in \{m+1, \ldots, p\}$.
So, given any $f_l \in C^{\infty}\left(M\right)$, we get

$$\Delta^{\phi}\left(f_l\, \eta_l \circ \phi\right) = \left(\Delta f_l\right) \eta_l \circ \phi + f_l\, \Delta^{\phi}\left(\eta_l \circ \phi\right) - 2\,\mathrm{tr}\left[\nabla_{\cdot} f_l\, \nabla^{\phi}_{\cdot}\left(\eta_l \circ \phi\right)\right] = \left(\Delta f_l\right) \eta_l \circ \phi. \quad (3.2)$$

For the curvature term in J^{ϕ} we find, since by definition the $\eta_l \circ \phi$'s are orthogonal to the $d\phi\left(X_j\right)$'s with respect to the round metric $(\cdot, \cdot)$ on S^p for any orthonormal frame $\{X_1, \ldots, X_n\}$ on M,

$$\begin{aligned} \mathrm{tr}\ R^{S^p}\left(\eta_l \circ \phi, d\phi\cdot\right) d\phi\cdot &= \sum_{k=1}^{n}\left[\left|d\phi\left(X_k\right)\right|^2 \eta_l \circ \phi - \left(\eta_l \circ \phi, d\phi\left(X_k\right)\right) d\phi\left(X_k\right)\right] = \left|d\phi\right|^2 \eta_l \circ \phi \\ &= \left|d\psi\right|^2 \eta_l \circ \phi. \end{aligned} \quad (3.3)$$

Combining (3.2) and (3.3) we have for J^{ϕ} on the span of the $\eta_l \circ \phi$'s

$$J^{\phi}\left(f_l\, \eta_l \circ \phi\right) = \Delta^{\phi}\left(f_l\, \eta_l \circ \phi\right) - \mathrm{tr}\ R^{S^p}\left(f_l\, \eta_l \circ \phi, d\phi\cdot\right) d\phi\cdot = \left[\left(\Delta f_l\right) - \left|d\psi\right|^2 f_l\right] \eta_l \circ \phi.$$

In light of this we suppose from now on that ψ is a Riemannian submersion up to scale, that is a conformal map with constant dilation. Then we consider the subspaces

$$S^{\phi}_{\lambda_k} := \bigoplus_{l=m+1}^{p} \{ f_l\, \eta_l \circ \phi |\, \Delta f_l = \lambda_k\, f_l \} \subset \Gamma \left(\phi^{-1} TS^p \right)$$

for any integer $k \geq 0$, where λ_k denotes the k-th eigenvalue of the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$. From the computation of the $J^\phi(f_l\, \eta_l \circ \phi)$'s and the conformality of ψ it follows that any eigensection of J^ϕ restricted to $\bigoplus_{l=m+1}^{p} \{ f_l\, \eta_l \circ \phi |\, f_l \in C^\infty(M)\}$ must be contained in some $S^{\phi}_{\lambda_k}$. So, let $V = \sum_{l=m+1}^{p} f_l\, \eta_l \circ \phi \in S^{\phi}_{\lambda_k}$ for some $k \geq 0$. Then we have

$$\begin{aligned} J^\phi(V) &= \sum_{l=m+1}^{p} J^\phi(f_l\, \eta_l \circ \phi) = \sum_{l=m+1}^{p} \left[(\Delta f_l) - |d\psi|^2 f_l \right] \eta_l \circ \phi \\ &= \left(\lambda_k - |d\psi|^2 \right) \sum_{l=m+1}^{p} f_l\, \eta_l \circ \phi = \left(\lambda_k - |d\psi|^2 \right) V. \end{aligned}$$

This means that, counting with multiplicity, the additional eigenvalues of J^ϕ are given by $\lambda_k - |d\psi|^2$, $k \geq 0$, and have multiplicity $(p-m)\,\mathrm{mult}(\lambda_k)$, where $\mathrm{mult}(\lambda_k)$ denotes the multiplicity of λ_k as an eigenvalue of Δ. In particular, as M is assumed to be compact and connected, we know that $\lambda_0 = 0$ with multiplicity $\mathrm{mult}(\lambda_0) = 1$. This yields that one of these eigenvalues is $\lambda_0 - |d\psi|^2 = -|d\psi|^2 < 0$, which has multiplicity $(p-m)\,\mathrm{mult}(\lambda_0) = p - m$. Consequently, the index of ϕ must be strictly larger than the index of ψ, to be precise

$$\mathrm{ind}(\phi) \geq \mathrm{ind}(\psi) + p - m > \mathrm{ind}(\psi).$$

Regarding the nullity we find $0 \in \left\{ \lambda_k - |d\psi|^2 \,\middle|\, k \geq 0 \right\}$ if and only if there exists some $\tilde{k} \geq 1$ such that $\lambda_{\tilde{k}} = |d\psi|^2$. In this case the nullity of ϕ is given by

$$\mathrm{null}(\phi) = \mathrm{null}(\psi) + (p-m)\,\mathrm{mult}\left(\lambda_{\tilde{k}}\right).$$

Otherwise the nullities of ϕ and ψ must coincide. Now, all in all we have shown

Corollary 3.1.4. *Let M^n be a compact, connected, smooth Riemannian manifold without boundary and $\psi : M^n \to S^m$, $n \geq m \geq 1$, be a smooth, harmonic Riemannian submersion up to scale. Moreover for $p > m$ let $\mathrm{i} : S^m \to S^p$ be the canonical inclusion map and set $\phi := \mathrm{i} \circ \psi : M^n \to S^p$.*
For $k \geq 0$ denote by λ_k the k-th eigenvalue of $\Delta : C^\infty(M) \to C^\infty(M)$.
Then, in addition (counting with multiplicity) to the eigenvalues of J^ψ, J^ϕ has eigenvalues $\left\{ \lambda_k - |d\psi|^2 \,\middle|\, k \geq 0 \right\}$ of multiplicities $(p-m)\,\mathrm{mult}(\lambda_k)$, where $\mathrm{mult}(\lambda_k)$ is the multiplicity of λ_k as an eigenvalue of Δ. In particular $\mathrm{ind}(\phi) > \mathrm{ind}(\psi)$. Furthermore, $\mathrm{null}(\phi) > \mathrm{null}(\psi)$ if and only if $|d\psi|^2$ is in the spectrum of Δ. In that case the nullity of ϕ is given by

$$\mathrm{null}(\phi) = \mathrm{null}(\psi) + (p-m)\,\mathrm{mult}\left(\lambda_{\tilde{k}}\right),$$

where $\tilde{k} \geq 1$ is such that $\lambda_{\tilde{k}} = |d\psi|^2$.

Even though the proof of these strict bounds required ψ to have constant dilation and so cannot be transferred to submersive harmonic morphisms in general, we can get strict bounds up to a change of metric on the domain manifold.
From [4], Cor. 4.6.12 we know that given a submersive harmonic morphism $\psi : (M^n, g) \to (N^m, h)$ with $n > m$ and dilation $\lambda : M \to]0, +\infty[$, defining

$$\tilde{g} := \lambda^2 g^{\mathcal{H}} + \lambda^{\frac{4-2m}{n-m}} g^{\mathcal{V}},$$

the map $\psi : (M, \tilde{g}) \to (N, h)$ is a harmonic Riemannian submersion. In particular we can apply the strict bounds we have just proven to ψ.

Corollary 3.1.5. *Let (M^n, g) be a compact, connected, smooth Riemannian manifold without boundary and $\psi : (M^n, g) \to (S^m, \mathrm{can})$, $n > m \geq 1$, be a submersive harmonic morphism with dilation $\lambda : M \to]0, +\infty[$. Set*

$$\tilde{g} := \lambda^2 g^{\mathcal{H}} + \lambda^{\frac{4-2m}{n-m}} g^{\mathcal{V}}$$

and let $\phi := \mathrm{i} \circ \psi : (M^n, \tilde{g}) \to (S^p, \mathrm{can})$ be the composition of $\psi : (M, \tilde{g}) \to (S^m, \mathrm{can})$ with the canonical inclusion map i for some $p > m$.
Furthermore, let λ_k denote the k-th eigenvalue of $\Delta : C^\infty(M) \to C^\infty(M)$ with respect to $\tilde{g}$. Then J^ϕ has, in addition (counting with multiplicity) to the eigenvalues of J^ψ, eigenvalues $\left\{ \lambda_k - |d\psi|^2 \middle| k \geq 0 \right\}$ of multiplicities $(p-m)\,\mathrm{mult}(\lambda_k)$. In particular $\mathrm{ind}(\phi) > \mathrm{ind}(\psi)$.
Moreover, $\mathrm{null}(\phi) > \mathrm{null}(\psi)$ if and only if $|d\psi|^2$ is in the spectrum of Δ. In that case the nullity of ϕ is given by

$$\mathrm{null}(\phi) = \mathrm{null}(\psi) + (p-m)\,\mathrm{mult}\left(\lambda_{\tilde{k}}\right),$$

where $\lambda_{\tilde{k}} = |d\psi|^2$.

As a final remark in this general setting, in the codimension one case, that is $m = n-1$, for $n \geq 5$ every harmonic, horizontally weakly conformal map $M^n \to S^{n-1}$ has no critical point, i.e. is already a submersion, cf. [4], Thm. 5.7.3. Furthermore, we note that there are no non-constant harmonic morphisms $S^n \to S^{n-1}$ for $n \geq 5$ (cf. [4], Cor. 5.7.4), so in the case of $M = S^n$ and codimension one there are very few submersive harmonic morphisms.

3.1.3 Application to the Hopf map $S^3 \to S^2$

In this subsection we apply the index bounds obtained in 3.1.2 to the Hopf map $\psi : S^3 \to S^2$. Recall that ψ is defined at any $x = (x_1, x_2, x_3, x_4) \in S^3$ as

$$\psi(x) = \left(2\left(x_1x_3 + x_2x_4\right), 2\left(x_2x_3 - x_1x_4\right), x_1^2 + x_2^2 - x_3^2 - x_4^2\right).$$

Furthermore, it is well-known that ψ is a harmonic Riemannian submersion up to scale with dilation 2 so that we can apply our strict index bound to ψ.
Moreover, for any $n \geq 1$, the spectrum of Δ on $C^\infty(S^n)$ is known to be (e.g. [27, 69])

$$\{\lambda_k = k(k+n-1) |\, k \geq 0\}$$

with multiplicities

$$\mathrm{mult}(\lambda_k) = \binom{n+k}{n} - \binom{n+k-2}{n}.$$

So in this case Corollary 3.1.4 reads

Corollary 3.1.6. *Let $\psi : S^n \to S^m$, $n \geq m \geq 1$, be a smooth, harmonic Riemannian submersion up to scale, $\mathrm{i} : S^m \to S^p$ for some $p > m$ the canonical inclusion map and $\phi := \mathrm{i} \circ \psi : S^n \to S^p$. Denote for any $k \geq 0$ by $\lambda_k = k(k+n-1)$ the k-th eigenvalue of $\Delta : C^\infty(S^n) \to C^\infty(S^n)$.*
Then, in addition (counting with multiplicity) to the eigenvalues of J^ψ, J^ϕ has eigenvalues $\left\{ k(k+n-1) - |d\psi|^2 \middle| k \geq 0 \right\}$ of multiplicities $(p-m)\,\mathrm{mult}(\lambda_k)$, where

$$\mathrm{mult}(\lambda_k) = \binom{n+k}{n} - \binom{n+k-2}{n}.$$

In particular $\mathrm{ind}(\phi) > \mathrm{ind}(\psi)$. Furthermore, $\mathrm{null}(\phi) > \mathrm{null}(\psi)$ if and only if $|d\psi|^2 = \tilde{k}\left(\tilde{k} + n - 1\right)$ for some $\tilde{k} \geq 1$. In that case the nullity of ϕ is given by

$$\mathrm{null}(\phi) = \mathrm{null}(\psi) + (p-m)\,\mathrm{mult}\left(\lambda_{\tilde{k}}\right).$$

As the index and nullity of the Hopf map $S^3 \to S^2$ were already computed to be 4 and 8 (e.g. [4, 58]), we can find the index and nullity of its composition with i for any $p \geq 3$ explicitly from this by counting the negative and zero eigenvalues arising from the normal sections.

Theorem 3.1.7. *Let $\psi : S^3 \to S^2$ be the Hopf map, $\mathrm{i} : S^2 \to S^p$, $p \geq 3$, the canonical inclusion map and $\phi := \mathrm{i} \circ \psi : S^3 \to S^p$.*
Then, we obtain $\operatorname{ind}(\phi) = 5p - 6$ as well as $\operatorname{null}(\phi) = 9p - 10$.

Proof. Using Corollary 3.1.6 with $n = 3$ and $m = 2$ yields

$$\operatorname{ind}(\phi) > \operatorname{ind}(\psi) = 4,$$

so we can get the exact index of ϕ by calculating the eigenvalues generated by the action of J^ϕ on the sections of the form $f_l\, \eta_l \circ \phi$, where $f_l \in C^\infty\left(S^3\right)$. For those we have seen in the general case that

$$J^\phi\left(f_l\, \eta_l \circ \phi\right) = \left(\lambda_k - |d\psi|^2\right) \eta_l \circ \phi = \left(\lambda_k - 8\right) \eta_l \circ \phi$$

whenever $\Delta f_l = \lambda_k\, f_l$ for any $k \geq 0$. Hence, we restrict J^ϕ to $\bigoplus_{l=3}^p \{ f_l\, \eta_l \circ \phi | \, \Delta f_l = \lambda_k\, f_l \}$ and run over $k \geq 0$ to get the eigenvalues. Recall that for $n = 3$ the eigenvalues of the Laplacian on smooth functions are given by $\lambda_k = k(k+2)$.
Starting with $k = 0$, we have $\lambda_0 = 0$ of multiplicity one, so we obtain (counting with multiplicity) an additional eigenvalue -8 of multiplicity $p - 2$.
For $k = 1$, we get $\lambda_1 = 3$ of multiplicity four, hence each $\eta_l \circ \phi$ gives us (counting with multiplicity) an additional eigenvalue -5 of multiplicity $4p - 8$. At this point we already have the index of ϕ bounded from below by $4 + p - 2 + 4p - 8 = 5p - 6$.
As for $k = 2$ we have $\lambda_2 = 8 = |d\psi|^2$ of multiplicity nine, we get as eigenvalue 0. Since $\lambda_k > 8$ for $k \geq 3$, the Jacobi operator J^ϕ restricted to $\bigoplus_{l=3}^p \{ f_l\, \eta_l \circ \phi | \, \Delta f_l = \lambda_k\, f_l \}$ gives us only positive eigenvalues. Therefore, summing up yields that the index of ϕ is equal to $5p - 6$ as claimed.
Moreover, as we have seen, we are in the situation $|d\psi|^2 = \lambda_2$, so the nullity increases strictly as well, namely

$$\operatorname{null}(\phi) = \operatorname{null}(\psi) + (p-2) \operatorname{mult}(\lambda_2) = 8 + 9p - 18 = 9p - 10.$$

□

In particular, if we consider $p = 3$ we get

Corollary 3.1.8. *Let $\psi : S^3 \to S^2$ be the Hopf map, $\mathrm{i} : S^2 \to S^3$ the canonical inclusion map and $\phi := \mathrm{i} \circ \psi : S^3 \to S^3$. Then we have $\operatorname{ind}(\phi) = 9$ as well as $\operatorname{null}(\phi) = 17$.*

Of course we can also avoid using the known index and nullity of ψ by determining its Jacobi operator explicitly and calculating with this and the comparison to J^ϕ the eigenvalues of J^ϕ explicitly. For the sake of comparison we will give this proof by explicit calculation in the remainder of this subsection. We will follow the approach of Loubeau and Oniciuc in [58], where they computed the index and nullity of the scaled Hopf map $S^3\left(\sqrt{2}\right) \to S^2\left(\frac{1}{\sqrt{2}}\right)$ and the biharmonic index of its composition with the inclusion $S^2\left(\frac{1}{\sqrt{2}}\right) \to S^3$.
First we make use of the fact that S^3 admits a global Killing frame, that is a global frame consisting of Killing vector fields.

Lemma 3.1.9. *For each $x = (x_1, x_2, x_3, x_4) \in S^3$ we define*

$$X_1(x) := (-x_2, x_1, -x_4, x_3), X_2(x) := (-x_3, x_4, x_1, -x_2),$$
$$X_3(x) := (-x_4, -x_3, x_2, x_1).$$

Then, $\{X_1, X_2, X_3\}$ *is a global Killing frame on* S^3*, which satisfies*

$$\begin{aligned}
&\nabla_{X_1} X_1 = \nabla_{X_2} X_2 = \nabla_{X_3} X_3 = 0;\ \nabla_{X_1} X_2 = -\nabla_{X_2} X_1 = -X_3;\\
&\nabla_{X_1} X_3 = -\nabla_{X_3} X_1 = X_2;\ \nabla_{X_2} X_3 = -\nabla_{X_3} X_2 = -X_1;\\
&[X_1, X_2] = -2\, X_3;\ [X_2, X_3] = -2\, X_1;\ [X_3, X_1] = -2\, X_2.
\end{aligned}$$

Moreover, we have

$$X_1 = X_1^{\mathcal{V}},\ X_2 = X_2^{\mathcal{H}},\ X_3 = X_3^{\mathcal{H}}.$$

Proof. The fact that $\{X_1, X_2, X_3\}$ is a global Killing frame is shown e.g. in [22]. For such a frame we know, also from [22], $\nabla_{X_j} X_k = -\nabla_{X_k} X_j$. This gives us immediately $\nabla_{X_j} X_j = 0$ for any $j \in \{1, 2, 3\}$ and, since ∇ is torsion-free, also $[X_j, X_k] = 2\,\nabla_{X_j} X_k$. Hence it suffices to compute $\nabla_{X_1} X_2, \nabla_{X_1} X_3$ and $\nabla_{X_2} X_3$ to complete the table. For this we first find from the Killing frame condition for each $j, k \in \{1, 2, 3\}$

$$\left(\nabla_{X_j} X_k, X_j\right) = -\left(X_j, \nabla_{X_j} X_k\right),$$

yielding $\left(\nabla_{X_j} X_k, X_j\right) = 0$, as well as

$$\left(\nabla_{X_j} X_k, X_k\right) = -\left(X_j, \nabla_{X_k} X_k\right) = 0.$$

This means that e.g. $\nabla_{X_1} X_2$ must be parallel to X_3. Moreover, applying again the Killing frame condition, we get

$$\left(\nabla_{X_1} X_2, X_3\right) = -\left(\nabla_{X_1} X_3, X_2\right) = \left(\nabla_{X_2} X_3, X_1\right).$$

Hence, it suffices to compute e.g. $\left(\nabla_{X_1} X_2, X_3\right) =: c$. Now, by [22], p. 404 we know

$$const. = \left(\nabla_{X_1} X_2, \nabla_{X_1} X_2\right) = c^2 \left(X_3, X_3\right) = c^2$$

as well as

$$\left(\nabla_{X_1} X_2, \nabla_{X_1} X_2\right) = \left(R^{S^3}\left(X_1, X_2\right) X_2, X_1\right) = \left(X_1, X_1\right) = 1,$$

which yields $c^2 = 1$, therefore, as for manifolds admitting a Killing frame of dimension at most six c must be constant (cf. [22]), $c \in \{\pm 1\}$. Hence, to find the correct sign for our choice of X_j's, it is enough to determine $\nabla_{X_1} X_2$ at an arbitrary point $x \in S^3$. So, taking for any $x \in S^3$ normal coordinates at x, we calculate

$$\begin{aligned}
\nabla_{X_1(x)} X_2(x) &= -\, dx_3\left(X_1(x)\right) \partial_{x_1} + dx_4\left(X_1(x)\right) \partial_{x_2} + dx_1\left(X_1(x)\right) \partial_{x_1} - dx_2\left(X_1(x)\right) \partial_{x_4}\\
&= x_4 \partial_{x_1} + x_3 \partial_{x_2} - x_2 \partial_{x_3} - x_1 \partial_{x_4} = -X_3(x).
\end{aligned}$$

As a consequence we have as claimed

$$\nabla_{X_1} X_2 = -X_3,\ \nabla_{X_1} X_3 = X_2 \quad \text{and} \quad \nabla_{X_2} X_3 = -X_1.$$

Furthermore, we obtain for every $x \in S^3$

$$\begin{aligned}
&d\psi_x\left(X_1(x)\right) = 0,\\
&d\psi_x\left(X_2(x)\right) = 2\left(x_1^2 + x_4^2 - x_2^2 - x_3^3,\ 2\left(x_1 x_2 + x_3 x_4\right),\ 2\left(x_2 x_4 - x_1 x_3\right)\right),\\
&d\psi_x\left(X_3(x)\right) = 2\left(2\left(x_1 x_2 - x_3 x_4\right),\ x_2^2 + x_4^2 - x_1^2 - x_3^2,\ -2\left(x_1 x_4 + x_2 x_3\right)\right).
\end{aligned}$$

This implies, together with $\{X_1, X_2, X_3\}$ being an orthonormal frame on S^3, that X_1 is a vertical vector field and that X_2 and X_3 are horizontal vector fields with respect to ψ. $\square$

Remark. Note that we have also shown that the Killing frame on S^3 is unique up to sign since the only place where we used the explicit definition of the X_j's was to determine whether $c = +1$ or $c = -1$.

As a result of Lemma 3.1.9, at every $x \in S^3$ the vertical space with respect to ψ is spanned by X_1 and the horizontal space is spanned by X_2 and X_3.

Remark 3.1.10. We remark that, given a (conformal) submersion $\psi : M \to N$, the existence of a Killing frame on M which consists only of horizontal and vertical vector fields with respect to ψ is in general not to be expected.
However, if M and N are equidimensional, then the submersion $\psi : M \to N$ is also an immersion, implying for the vertical space $\mathcal{V}_x = \{0\}$ for any $x \in M$. So in this case if M admits a Killing frame $\{X_1, \ldots, X_n\}$, then this frame consists of horizontal vector fields.

We state now some properties of Killing vector fields that we will need later on.

Lemma 3.1.11. *([58])*
Let M be a compact Riemannian manifold, X a Killing vector field on M and $f, \tilde{f} \in C^\infty(M)$. Then we have

$$\int_M (Xf)\,\tilde{f} = -\int_M f\left(X\tilde{f}\right)$$

as well as

$$\Delta (Xf) = X(\Delta f).$$

Moreover, the second fundamental form of the Hopf map is with respect to $\{X_1, X_2, X_3\}$ particularly simple.

Corollary 3.1.12. *Let $\psi : S^3 \to S^2$ be the Hopf map and $\{X_1, X_2, X_3\}$ be the Killing frame defined in Lemma 3.1.9. Then we have*

$$\begin{aligned}
\nabla d\psi (X_1, X_1) &= \nabla d\psi (X_2, X_2) = \nabla d\psi (X_3, X_3) = 0;\\
\nabla d\psi (X_2, X_3) &= \nabla d\psi (X_3, X_2) = 0;\\
\nabla d\psi (X_1, X_2) &= \nabla d\psi (X_2, X_1) = -d\psi (X_3);\\
\nabla d\psi (X_1, X_3) &= \nabla d\psi (X_3, X_1) = d\psi (X_2).
\end{aligned}$$

Proof. We first note that Lemma A.2.1 in the appendix applies to our Killing frame since we have seen in Lemma 3.1.9 that X_1 is a vertical vector field and both X_2 and X_3 are horizontal vector fields with respect to ψ. As ψ is of constant dilation 2, it is in particular horizontally homothetic, so by (1) of Lemma A.2.1 its second fundamental form vanishes on horizontal vector fields, i.e.

$$\nabla d\psi (X_2, X_2) = \nabla d\psi (X_3, X_3) = \nabla d\psi (X_2, X_3) = \nabla d\psi (X_3, X_2) = 0.$$

Next, we know that ψ has round S^1 as fibres, which are totally geodesic, thus we also get from (2) of Lemma A.2.1

$$\nabla d\psi (X_1, X_1) = 0.$$

For the remaining terms, due to the symmetry of the second fundamental form, it suffices to compute $\nabla d\psi (X_2, X_1)$ and $\nabla d\psi (X_3, X_1)$. For those we get, using part (c) of Lemma A.2.1 and the properties of our Killing frame,

$$\begin{aligned}
\nabla d\psi (X_2, X_1) &= -d\psi (\nabla_{X_2} X_1) = -d\psi (X_3),\\
\nabla d\psi (X_3, X_1) &= -d\psi (\nabla_{X_3} X_1) = d\psi (X_2).
\end{aligned}$$

□

As a consequence of the Killing frame consisting of horizontal and vertical vector fields we obtain a decomposition of the space of sections of $\psi^{-1}TS^2$ as in the general case, namely

$$\Gamma\left(\psi^{-1}TS^2\right) = \left\{ f_2\, d\psi (X_2) \middle|\, f_2 \in C^\infty\left(S^3\right)\right\} \oplus \left\{ f_3\, d\psi (X_3) \middle|\, f_3 \in C^\infty\left(S^3\right)\right\}.$$

Therefore, to determine J^ψ completely it suffices to compute $J^\psi (f_2\, d\psi (X_2))$ and $J^\psi (f_3\, d\psi (X_3))$ for arbitrary functions $f_2, f_3 \in C^\infty\left(S^3\right)$.

Using the properties of the Killing frame as well as the computation of the second fundamental form of ψ, we obtain

$$\begin{aligned}
\nabla^{\psi}_{X_1}\left(d\psi\left(X_2\right)\right) &= \nabla d\psi\left(X_1, X_2\right) + d\psi\left(\nabla_{X_1} X_2\right) = -2\, d\psi\left(X_3\right), \\
\nabla^{\psi}_{X_2}\left(d\psi\left(X_2\right)\right) &= \nabla d\psi\left(X_2, X_2\right) + d\psi\left(\nabla_{X_2} X_2\right) = 0, \\
\nabla^{\psi}_{X_3}\left(d\psi\left(X_2\right)\right) &= \nabla d\psi\left(X_3, X_2\right) + d\psi\left(\nabla_{X_3} X_2\right) = d\psi\left(X_1\right) = 0.
\end{aligned}$$

Thus, we find for the pull-back Laplacian

$$\begin{aligned}
\Delta^{\psi}\left(d\psi\left(X_2\right)\right) &= -\sum_{j=1}^{3} \nabla^{\psi}_{X_j} \nabla^{\psi}_{X_j}\left(d\psi\left(X_2\right)\right) = 2\,\nabla^{\psi}_{X_1}\left(d\psi\left(X_3\right)\right) \\
&= 2\left(\nabla d\psi\left(X_1, X_3\right) + d\psi\left(\nabla_{X_1} X_3\right)\right) = 4\, d\psi\left(X_2\right).
\end{aligned}$$

For X_3 we have just seen $\nabla^{\psi}_{X_1}\left(d\psi\left(X_3\right)\right) = 2\, d\psi\left(X_2\right)$. The two remaining covariant derivatives are given by

$$\begin{aligned}
\nabla^{\psi}_{X_2}\left(d\psi\left(X_3\right)\right) &= \nabla d\psi\left(X_2, X_3\right) + d\psi\left(\nabla_{X_2} X_3\right) = -d\psi\left(X_1\right) = 0, \\
\nabla^{\psi}_{X_3}\left(d\psi\left(X_3\right)\right) &= \nabla d\psi\left(X_3, X_3\right) + d\psi\left(\nabla_{X_3} X_3\right) = 0.
\end{aligned}$$

So we get for the pull-back Laplacian of $d\psi\left(X_3\right)$

$$\Delta^{\psi}\left(d\psi\left(X_3\right)\right) = -2\,\nabla^{\psi}_{X_1}\left(d\psi\left(X_2\right)\right) = 4\, d\psi\left(X_3\right).$$

Now take any function $f_2 \in C^{\infty}\left(S^3\right)$. Then we find

$$\begin{aligned}
\Delta^{\psi}\left(f_2\, d\psi\left(X_2\right)\right) &= \left(\Delta f_2\right) d\psi\left(X_2\right) + f_2\, \Delta^{\psi}\left(d\psi\left(X_2\right)\right) - 2\ \mathrm{tr}\left[\left(\nabla_{\cdot} f_2\right) \nabla^{\psi}_{\cdot}\left(d\psi\left(X_2\right)\right)\right] \\
&= \left[\left(\Delta f_2\right) + 4\, f_2\right] d\psi\left(X_2\right) - 2\left(X_1 f_2\right) \nabla^{\psi}_{X_1}\left(d\psi\left(X_2\right)\right) \\
&\quad - 2\left(X_2 f_2\right) \nabla^{\psi}_{X_2}\left(d\psi\left(X_2\right)\right) - 2\left(X_3 f_2\right) \nabla^{\psi}_{X_3}\left(d\psi\left(X_2\right)\right) \\
&= \left[\left(\Delta f_2\right) + 4\, f_2\right] d\psi\left(X_2\right) + 4\left(X_1 f_2\right) d\psi\left(X_3\right).
\end{aligned}$$

Similarly we find for X_3, given any $f_3 \in C^{\infty}\left(S^3\right)$,

$$\begin{aligned}
\Delta^{\psi}\left(f_3\, d\psi\left(X_3\right)\right) &= \left(\Delta f_3\right) d\psi\left(X_3\right) + f_3\, \Delta^{\psi}\left(d\psi\left(X_3\right)\right) - 2\ \mathrm{tr}\left[\left(\nabla_{\cdot} f_3\right) \nabla^{\psi}_{\cdot}\left(d\psi\left(X_3\right)\right)\right] \\
&= \left[\left(\Delta f_3\right) + 4\, f_3\right] d\psi\left(X_3\right) - 2\left(X_1 f_3\right) \nabla^{\psi}_{X_1}\left(d\psi\left(X_3\right)\right) \\
&\quad - 2\left(X_2 f_3\right) \nabla^{\psi}_{X_2}\left(d\psi\left(X_3\right)\right) - 2\left(X_3 f_3\right) \nabla^{\psi}_{X_3}\left(d\psi\left(X_3\right)\right) \\
&= \left[\left(\Delta f_3\right) + 4\, f_3\right] d\psi\left(X_3\right) - 4\left(X_1 f_3\right) d\psi\left(X_2\right).
\end{aligned}$$

It then remains to calculate the curvature term of the Jacobi operator. As ψ is horizontally conformal, it simplifies even further (e.g. [4], (4.8.3)) so that we obtain for any section $V \in \Gamma\left(\psi^{-1} T S^2\right)$

$$\mathrm{tr}\ R^{S^2}\left(V, d\psi\cdot\right) d\psi\cdot = 4\,\mathrm{Ric}^{S^2}\left(V\right) = 4\, V.$$

Thus, we conclude for any $f_2, f_3 \in C^{\infty}\left(S^3\right)$

$$\begin{aligned}
J^{\psi}\left(f_2\, d\psi\left(X_2\right)\right) &= \Delta^{\psi}\left(f_2\, d\psi\left(X_2\right)\right) - \mathrm{tr}\ R^{S^2}\left(f_2\, d\psi\left(X_2\right), d\psi\cdot\right) d\psi\cdot \\
&= \left(\Delta f_2\right) d\psi\left(X_2\right) + 4\left(X_1 f_2\right) d\psi\left(X_3\right), \\
J^{\psi}\left(f_3\, d\psi\left(X_3\right)\right) &= \Delta^{\psi}\left(f_3\, d\psi\left(X_3\right)\right) - \mathrm{tr}\ R^{S^2}\left(f_3\, d\psi\left(X_3\right), d\psi\cdot\right) d\psi\cdot \\
&= \left(\Delta f_3\right) d\psi\left(X_3\right) - 4\left(X_1 f_3\right) d\psi\left(X_2\right).
\end{aligned}$$

Summarizing, we have proven

Lemma 3.1.13. *The Jacobi operator* $J^\psi : \Gamma\left(\psi^{-1}TS^2\right) \to \Gamma\left(\psi^{-1}TS^2\right)$ *is determined by*

$$\begin{aligned} J^\psi\left(f_2\, d\psi\left(X_2\right)\right) &= \left(\Delta f_2\right) d\psi\left(X_2\right) + 4\left(X_1 f_2\right) d\psi\left(X_3\right), \\ J^\psi\left(f_3\, d\psi\left(X_3\right)\right) &= \left(\Delta f_3\right) d\psi\left(X_3\right) - 4\left(X_1 f_3\right) d\psi\left(X_2\right), \end{aligned}$$

where $f_2, f_3 \in C^\infty\left(S^3\right)$ *are arbitrary functions.*

As in the general case we obtain, due to the conformality of i and the decomposition of $\Gamma\left(\psi^{-1}TS^2\right)$, a decomposition for the space of sections of $\phi^{-1}TS^3$, namely

$$\begin{aligned} \Gamma\left(\phi^{-1}TS^3\right) = &\left\{ f_2\, d\phi\left(X_2\right) \middle|\, f_2 \in C^\infty\left(S^3\right)\right\} \oplus \left\{ f_3\, d\phi\left(X_3\right) \middle|\, f_3 \in C^\infty\left(S^3\right)\right\} \\ &\oplus \left\{ f\, \eta \circ \phi \middle|\, f \in C^\infty\left(S^3\right)\right\}, \end{aligned}$$

where η is a unit section of the normal bundle of $\mathrm{i}\left(S^2\right)$ as a submanifold of S^3. In the same way as in the general case of 3.1.2 we then get for $j \in \{2,3\}$ and any $f_j \in C^\infty\left(S^3\right)$

$$J^\phi\left(f_j\, d\phi\left(X_j\right)\right) = \mathrm{di}\left(J^\psi\left(f_j\, d\psi\left(X_j\right)\right)\right).$$

Furthermore, also as in the general case, we can assume without loss of generality that $\eta(x) = \varepsilon_4 \in \mathbb{R}^4$ for any $x \in \mathrm{i}\left(S^2\right)$ so that for any $f \in C^\infty\left(S^3\right)$ we obtain

$$J^\phi\left(f\, \eta \circ \phi\right) = \left[\left(\Delta f\right) - \left|d\psi\right|^2 f\right] \eta \circ \phi = \left[\left(\Delta f\right) - 8\, f\right] \eta \circ \phi.$$

Consequently J^ϕ is completely determined as well.

Lemma 3.1.14. *The Jacobi operator* $J^\phi : \Gamma\left(\phi^{-1}TS^3\right) \to \Gamma\left(\phi^{-1}TS^3\right)$ *is given by*

$$\begin{aligned} J^\phi\left(f_2\, d\phi\left(X_2\right)\right) &= \left(\Delta f_2\right) d\phi\left(X_2\right) + 4\left(X_1 f_2\right) d\phi\left(X_3\right), \\ J^\phi\left(f_3\, d\phi\left(X_3\right)\right) &= \left(\Delta f_3\right) d\phi\left(X_3\right) - 4\left(X_1 f_3\right) d\phi\left(X_2\right), \\ J^\phi\left(f\, \eta \circ \phi\right) &= \left[\left(\Delta f\right) - 8\, f\right] \eta \circ \phi. \end{aligned}$$

For the calculation of the eigenvalues we consider the subspaces

$$S^\phi_{\lambda_k} := \left\{ f_2\, d\phi\left(X_2\right) \middle|\, \Delta f_2 = \lambda_k\, f_2\right\} \oplus \left\{ f_3\, d\phi\left(X_3\right) \middle|\, \Delta f_3 = \lambda_k\, f_3\right\} \oplus \left\{ f\, \eta \circ \phi \middle|\, \Delta f = \lambda_k\, f\right\}$$

for any $k \geq 0$, where $\lambda_k = k(k+2)$ is the k-th eigenvalue of $\Delta : C^\infty\left(S^3\right) \to C^\infty\left(S^3\right)$.
Since Killing vector fields preserve eigenspaces of Δ, the Jacobi operator J^ϕ leaves each $S^\phi_{\lambda_k}$ invariant and we may restrict J^ϕ to $\cup_{k\geq 0} S^\phi_{\lambda_k}$ to find its eigenvalues.
However, before explicitly computing the eigenvalues, observe that for the Hopf map the vertical Laplacian is given by $\Delta^V = -X_1 X_1$. Hence, as X_1 is Killing, the spectrum of Δ^V can be determined by considering simultaneous eigenfunctions of Δ and Δ^V, i.e. functions f so that

$$\Delta f = \lambda_k\, f \quad \text{and} \quad \Delta^V f = c_l\, f.$$

This was done by Loubeau and Oniciuc in [58], pp. 5244f. They obtained for each $k \geq 0$ eigenvalues

$$c_l = (k-2l)^2,\; l \in \{0, \ldots, k\},$$

of multiplicity $2(k+1)$ except for $c_{\frac{k}{2}} = 0$ for even k, which has multiplicity $k+1$.
With this we finally have everything to give the proof by explicit computation for the index and nullity of ϕ in case $p = 3$.

Proof. Let us shortly outline the structure of the proof.
We restrict J^ϕ to $S^\phi_{\lambda_k}$, recalling that J^ϕ preserves $S^\phi_{\lambda_k}$. Running over $k \geq 0$, we define an L^2-orthonormal basis of $S^\phi_{\lambda_k}$, determine the action of J^ϕ on it and compute the eigenvalues.

Starting with $\lambda_0 = 0$ we can define an L^2-orthonormal basis of $S^{\phi}_{\lambda_0}$ by

$$B^{\phi}_{\lambda_0} := \left\{ \frac{1}{2c}\, d\phi\left(X_2\right), \frac{1}{2c}\, d\phi\left(X_3\right), \frac{1}{c}\,\eta \circ \phi \right\},$$

where $c^2 := \mathrm{Vol}\left(S^3\right) = 2\pi^2$. The action of J^{ϕ} on $B^{\phi}_{\lambda_0}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix}.$$

So we have eigenvalues 0 of multiplicity two and -8 of multiplicity one.
Next for $\lambda_1 = 3$ we get, setting $a^2 := \frac{\pi^2}{2}$, an L^2-orthonormal basis of $S^{\phi}_{\lambda_1}$ by defining

$$B^{\phi}_{\lambda_1} := \left\{ \frac{1}{2} f^1_i d\phi\left(X_2\right) \middle| \, i = 1, \ldots, 4 \right\} \cup \left\{ \frac{1}{2} f^1_j d\phi\left(X_3\right) \middle| \, j = 1, \ldots, 4 \right\} \cup \left\{ f^1_k \eta \circ \phi \middle| \, k = 1, \ldots, 4 \right\},$$

where $f^1_i(x) := \frac{x_i}{a}$. The vector field X_1 acts on the f^1_i's in the following way

$$X_1 f^1_1 = -f^1_2,\; X_1 f^1_2 = f^1_1,\; X_1 f^1_3 = -f^1_4,\; X_1 f^1_4 = f^1_3$$

Hence, the action of J^{ϕ} on $B^{\phi}_{\lambda_1}$ is given by

$$\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5
\end{pmatrix}.$$

The eigenvalues are $-5, -1, 7$ each of multiplicity four. Note that at this point we already have that the index is bounded from below by 9. So, in the rest of the proof we show that we do not obtain further negative eigenvalues.
As for the case of $\lambda_2 = 8$, we can define an L^2-orthonormal basis of $S^{\phi}_{\lambda_2}$ by

$$B^{\phi}_{\lambda_2} := \left\{ \frac{1}{2} f^2_i d\phi\left(X_2\right) \middle| \, i = 1, \ldots, 9 \right\} \cup \left\{ \frac{1}{2} f^2_j d\phi\left(X_3\right) \middle| \, j = 1, \ldots, 9 \right\} \cup \left\{ f^2_k \eta \circ \phi \middle| \, k = 1, \ldots, 9 \right\},$$

where for $b^2 := \frac{\pi^2}{6}$

$$\begin{aligned}
& f^2_1(x) := \frac{1}{b}\left(x_1 x_2 + x_3 x_4\right),\; f^2_2(x) := \frac{1}{b}\left(x_1 x_2 - x_3 x_4\right),\; f^2_3(x) := \frac{1}{b}\left(x_1 x_3 + x_2 x_4\right), \\
& f^2_4(x) := \frac{1}{b}\left(x_1 x_3 - x_2 x_4\right),\; f^2_5(x) := \frac{1}{b}\left(x_1 x_4 + x_2 x_3\right),\; f^2_6(x) := \frac{1}{b}\left(x_1 x_4 - x_2 x_3\right), \\
& f^2_7(x) := \frac{1}{2b}\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right),\; f^2_8(x) := \frac{1}{2b}\left(x_1^2 - x_2^2 + x_3^2 - x_4^2\right), \\
& f^2_9(x) := \frac{1}{2b}\left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right).
\end{aligned}$$

Applying X_1 to the functions f_i^2 yields

$$X_1 f_1^2 = 2f_8^2,\ X_1 f_2^2 = 2f_9^2,\ X_1 f_3^2 = 0,\ X_1 f_4^2 = -2f_5^2,\ X_1 f_5^2 = 2f_4^2,$$
$$X_1 f_6^2 = 0,\ X_1 f_7^2 = 0,\ X_1 f_8^2 = -2f_1^2,\ X_1 f_9^2 = -2f_2^2.$$

Consider the subspaces

$$\begin{aligned}
B_1 &:= \left\{ \frac{1}{2} f_1^2 d\phi\left(X_2\right), \frac{1}{2} f_6^2 d\phi\left(X_2\right), \frac{1}{2} f_8^2 d\phi\left(X_2\right) \right\} \\
&\quad \cup \left\{ \frac{1}{2} f_1^2 d\phi\left(X_3\right), \frac{1}{2} f_6^2 d\phi\left(X_3\right), \frac{1}{2} f_8^2 d\phi\left(X_3\right) \right\} \cup \left\{ f_1^2 \eta \circ \phi, f_6^2 \eta \circ \phi, f_8^2 \eta \circ \phi \right\}, \\
B_2 &:= \left\{ -\frac{1}{2} f_9^2 d\phi\left(X_2\right), \frac{1}{2} f_3^2 d\phi\left(X_2\right), \frac{1}{2} f_2^2 d\phi\left(X_2\right) \right\} \\
&\quad \cup \left\{ -\frac{1}{2} f_9^2 d\phi\left(X_3\right), \frac{1}{2} f_3^2 d\phi\left(X_3\right), \frac{1}{2} f_2^2 d\phi\left(X_3\right) \right\} \cup \left\{ -f_9^2 \eta \circ \phi, f_3^2 \eta \circ \phi, f_2^2 \eta \circ \phi \right\}, \\
B_3 &:= \left\{ \frac{1}{2} f_4^2 d\phi\left(X_2\right), \frac{1}{2} f_7^2 d\phi\left(X_2\right), -\frac{1}{2} f_5^2 d\phi\left(X_2\right) \right\} \\
&\quad \cup \left\{ \frac{1}{2} f_4^2 d\phi\left(X_3\right), \frac{1}{2} f_7^2 d\phi\left(X_3\right), -\frac{1}{2} f_5^2 d\phi\left(X_3\right) \right\} \cup \left\{ f_4^2 \eta \circ \phi, f_7^2 \eta \circ \phi, -f_5^2 \eta \circ \phi \right\}.
\end{aligned}$$

The action of J^ϕ spans for each of the three B_k's the same matrix, namely

$$\begin{pmatrix}
8 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

with eigenvalues 0 of multiplicity five, 8 of multiplicity two and 16 of multiplicity two. Finally, we turn to the case $k \geq 3$. We want to show that in this case all eigenvalues of J^ϕ restricted to $S_{\lambda_k}^\phi$ are strictly positive. This can be done very similar to the case $k \geq 3$ for J^ψ in [58]. For this let $V := f_2\, d\phi\left(X_2\right) + f_3\, d\phi\left(X_3\right) + f\eta \circ \phi$, where $\Delta f_2 = \lambda_k f_2$, $\Delta f_3 = \lambda_k f_3$ and $\Delta f = \lambda_k f$, that is let $V \in S_{\lambda_k}^\phi$ be arbitrary. Then we obtain, using the expressions for the action of J^ϕ from Lemma 3.1.14, the orthogonality of $d\phi\left(X_2\right), d\phi\left(X_3\right), \eta \circ \phi$ and $\left|d\phi\left(X_j\right)\right|^2 = 4$ for $j \in \{2, 3\}$,

$$\begin{aligned}
\left(J^\phi\left(V\right), V\right) &= \int_{S^3} \left(\lambda_k f_2\, d\phi\left(X_2\right) + 4\left(X_1 f_2\right) d\phi\left(X_3\right), f_2\, d\phi\left(X_2\right) + f_3\, d\phi\left(X_3\right) + f\eta \circ \phi\right) \\
&\quad + \int_{S^3} \left(\lambda_k f_3\, d\phi\left(X_3\right) - 4\left(X_1 f_3\right) d\phi\left(X_2\right), f_2\, d\phi\left(X_2\right) + f_3\, d\phi\left(X_3\right) + f\eta \circ \phi\right) \\
&\quad + \int_{S^3} \left(\left(\lambda_k - 8\right) f\eta \circ \phi, f_2\, d\phi\left(X_2\right) + f_3\, d\phi\left(X_3\right) + f\eta \circ \phi\right) \\
&= \int_{S^3} \left|d\phi\left(X_2\right)\right|^2 \left(\lambda_k \left|f_2\right|^2 - 4\left(X_1 f_3\right) f_2\right) + \int_{S^3} \left|d\phi\left(X_3\right)\right|^2 \left(\lambda_k \left|f_3\right|^2 + 4\left(X_1 f_2\right) f_3\right) \\
&\quad + \left(\lambda_k - 8\right) \int_{S^3} \left|\eta \circ \phi\right|^2 \left|f\right|^2
\end{aligned}$$

$$= 4\,\lambda_k \int_{S^3} \left(|f_2|^2 + |f_3|^2\right) + (\lambda_k - 8) \int_{S^3} |f|^2 + 16 \int_{S^3} (X_1 f_2)\, f_3 - 16 \int_{S^3} (X_1 f_3)\, f_2.$$

Applying now that for a Killing vector field X and functions $f, \tilde{f}$ we have

$$\int_{S^3} (Xf)\,\tilde{f} = -\int_{S^3} f\,\left(X\tilde{f}\right)$$

as well as that for $k \geq 3$ the eigenvalue λ_k satisfies $\lambda_k > 8$, we can estimate this

$$\begin{aligned} \left(J^\phi(V), V\right) &= 4\,\lambda_k \int_{S^3} \left(|f_2|^2 + |f_3|^2\right) + (\lambda_k - 8) \int_{S^3} |f|^2 - 32 \int_{S^3} (X_1 f_3)\, f_2 \\ &\geq 4\,\lambda_k \int_{S^3} \left(|f_2|^2 + |f_3|^2\right) - 32 \left| \int_{S^3} (X_1 f_3)\, f_2 \right|. \end{aligned}$$

As we will show at the end of the proof, we obtain, denoting by c the maximum over the eigenvalues of Δ^V,

$$\left| \int_{S^3} (X_1 f_3)\, f_2 \right| \leq \frac{\sqrt{c}}{2} \int_{S^3} \left(|f_2|^2 + |f_3|^2\right). \tag{3.4}$$

With this we conclude

$$\left(J^\phi(V), V\right) \geq \left(4\,\lambda_k - 16\sqrt{c}\right) \int_{S^3} \left(|f_2|^2 + |f_3|^2\right).$$

Now, recall $\lambda_k = k(k+2)$ and $c_l = (k-2l)^2 < k^2$. Using this we find $\sqrt{c} < k$ so that

$$4\,\lambda_k - 16\sqrt{c} \geq 4k^2 - 8k = 4k(k-2) > 0$$

for any $k \geq 3$. Consequently, restricted to $S^\phi_{\lambda_k}$ with $k \geq 3$, the Jacobi operator J^ϕ can only have positive eigenvalues, which shows the claim.
It remains to prove (3.4). For this let $\left\{f_1^k, \ldots, f_{m_{\lambda_k}}^k\right\}$ be an L^2-orthonormal basis of the k-th eigenspace of Δ so that $\Delta^V f_i^k = c_i f_i^k$ for any $i = 1, \ldots, m_{\lambda_k}$. Then there exist coefficients $a_1, \ldots, a_{m_{\lambda_k}}$ such that $f_3 = \sum_{i=1}^{m_{\lambda_k}} a_i f_i^k$. Hence, we have $\Delta^V f_3 = \sum_{i=1}^{m_{\lambda_k}} a_i c_i f_i^k$. Upon recalling that for the Hopf map the vertical Laplacian is given by $\Delta^V = -X_1 X_1$ we get, expanding f_3 in the f_i^k,

$$\begin{aligned} \int_{S^3} (X_1 f_3)^2 &= \int_{S^3} \left(\Delta^V f_3\right) f_3 = \sum_{i,j=1}^{m_{\lambda_k}} \int_{S^3} a_i c_i a_j f_i^k f_j^k = \sum_{i,j=1}^{m_{\lambda_k}} a_i c_i a_j \delta_{ij} \\ &= \sum_{i=1}^{m_{\lambda_k}} c_i\,(a_i)^2 \leq \max\{c_1, \ldots, c_{m_{\lambda_k}}\} \sum_{i=1}^{m_{\lambda_k}} (a_i)^2 \leq c \int_{S^3} |f_3|^2. \end{aligned}$$

This and the Cauchy-Schwarz inequality then imply

$$\begin{aligned} \left| \int_{S^3} (X_1 f_3)\, f_2 \right| &\leq \left(\int_{S^3} (X_1 f_3)^2 \right)^{\frac{1}{2}} \left(\int_{S^3} |f_2|^2 \right)^{\frac{1}{2}} \leq \sqrt{c} \left(\int_{S^3} |f_3|^2 \right)^{\frac{1}{2}} \left(\int_{S^3} |f_2|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{c}}{2} \int_{S^3} \left(|f_2|^2 + |f_3|^2\right), \end{aligned}$$

which shows the claimed estimate (3.4) and finishes the proof. □

3.1.4 Composition with conformal transformation of S^2

Continuing to denote by $\psi : S^3 \to S^2$ the Hopf map, as defined at the beginning of 3.1.3, we will now see that, with the same approach as for computing the Jacobi operator of the composition of ψ with the canonical inclusion map i as well as employing ideas from [57], we can also calculate the Jacobi operator of the composition of ψ with a conformal transformation of S^2.
This is of particular interest as Baird and Wood showed (e.g. [4], Thm. 6.7.7) that any non-constant, globally defined harmonic morphism $S^3 \to S^2$ is given, up to isometry of S^3, by the composition of the Hopf map with a weakly conformal map $S^2 \to S^2$. This means that being able to deduce the index of ψ composed with a conformal transformation of S^2 from the index of the Hopf map ψ would enable us to compute the index of any submersive harmonic morphism from the round S^3 into the round S^2.
For this, let $v : S^2 \to S^2$ denote an arbitrary but fixed conformal map and $\xi := v \circ \psi : S^3 \to S^2$. Furthermore, we continue to let $\{X_1, X_2, X_3\}$ denote the Killing frame on S^3 defined in Lemma 3.1.9 such that the space of sections of $\psi^{-1}TS^2$ decomposes as

$$\Gamma\left(\psi^{-1}TS^2\right) = \left\{ f_2\, d\psi\left(X_2\right) \middle| f_2 \in C^\infty\left(S^3\right)\right\} \oplus \left\{ f_3\, d\psi\left(X_3\right) \middle| f_3 \in C^\infty\left(S^3\right)\right\}.$$

Then, the conformality of v yields a decomposition for the sections of $\xi^{-1}TS^2$ as

$$\Gamma\left(\xi^{-1}TS^2\right) = \left\{ f_2\, d\xi\left(X_2\right) \middle| f_2 \in C^\infty\left(S^3\right)\right\} \oplus \left\{ f_3\, d\xi\left(X_3\right) \middle| f_3 \in C^\infty\left(S^3\right)\right\}.$$

As a result, to determine $J^\xi : \Gamma\left(\xi^{-1}TS^2\right) \to \Gamma\left(\xi^{-1}TS^2\right)$ it suffices to compute $J^\xi\left(f_2\, d\xi\left(X_2\right)\right)$ and $J^\xi\left(f_3\, d\xi\left(X_3\right)\right)$ for arbitrary $f_2, f_3 \in C^\infty\left(S^3\right)$. So let $j \in \{2,3\}$ and $k \in \{1,2,3\}$. Then we find

$$\nabla^\xi_{X_k}\left(d\xi\left(X_j\right)\right) = \nabla dv\left(d\psi\left(X_k\right), d\psi\left(X_j\right)\right) + dv\left(\nabla^\psi_{X_k}\left(d\psi\left(X_j\right)\right)\right).$$

Differentiating again, the second covariant derivative is given by

$$\begin{aligned}
\nabla^\xi_{X_k}\nabla^\xi_{X_k}\left(d\xi\left(X_j\right)\right) &= \nabla^\xi_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right), d\psi\left(X_j\right)\right)\right] + \nabla^\xi_{X_k}\left[dv\left(\nabla^\psi_{X_k}\left(d\psi\left(X_j\right)\right)\right)\right] \\
&= \nabla^\xi_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right), d\psi\left(X_j\right)\right)\right] + \nabla dv\left(d\psi\left(X_k\right), \nabla^\psi_{X_k}\left(d\psi\left(X_j\right)\right)\right) \\
&\quad + dv\left(\nabla^\psi_{X_k}\nabla^\psi_{X_k}\left(d\psi\left(X_j\right)\right)\right) \\
&= \nabla^\xi_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right), d\psi\left(X_j\right)\right)\right] + \nabla dv\left(d\psi\left(X_k\right), \nabla d\psi\left(X_k, X_j\right)\right) \\
&\quad + \nabla dv\left(d\psi\left(X_k\right), d\psi\left(\nabla_{X_k}X_j\right)\right) + dv\left(\nabla^\psi_{X_k}\nabla^\psi_{X_k}\left(d\psi\left(X_j\right)\right)\right).
\end{aligned}$$

Therefore the pull-back Laplacian of $d\xi\left(X_j\right)$ can be expressed as

$$\begin{aligned}
\Delta^\xi\left(d\xi\left(X_j\right)\right) &= -\sum_{k=1}^{3}\nabla^\xi_{X_k}\nabla^\xi_{X_k}\left(d\xi\left(X_j\right)\right) \\
&= -\sum_{k=1}^{3}\left[\nabla^\xi_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right), d\psi\left(X_j\right)\right)\right] + \nabla dv\left(d\psi\left(X_k\right), \nabla d\psi\left(X_k, X_j\right)\right)\right] \\
&\quad - \sum_{k=1}^{3}\nabla dv\left(d\psi\left(X_k\right), d\psi\left(\nabla_{X_k}X_j\right)\right) + dv\left(\Delta^\psi\left(d\psi\left(X_j\right)\right)\right).
\end{aligned} \tag{3.5}$$

Let us continue the calculation with $j = 2$, the case $j = 3$ can be done analogously.
First we have, using the properties of the Killing frame as well as the computations for the

second fundamental form of ψ with respect to the X_k's,

$$\begin{aligned}\sum_{k=1}^{3}\nabla dv\left(d\psi\left(X_k\right),\nabla d\psi\left(X_k,X_2\right)\right) =&\ \nabla dv\left(d\psi\left(X_1\right),\nabla d\psi\left(X_1,X_2\right)\right)\\ &+\ \nabla dv\left(d\psi\left(X_2\right),\nabla d\psi\left(X_2,X_2\right)\right)+\nabla dv\left(d\psi\left(X_3\right),\nabla d\psi\left(X_3,X_2\right)\right)=\ 0.\end{aligned}\tag{3.6}$$

Second we get

$$\begin{aligned}\sum_{k=1}^{3}\nabla dv\left(d\psi\left(X_k\right),d\psi\left(\nabla_{X_k}X_2\right)\right) &= \nabla dv\left(d\psi\left(X_1\right),d\psi\left(\nabla_{X_1}X_2\right)\right)\\ &\quad+\nabla dv\left(d\psi\left(X_2\right),d\psi\left(\nabla_{X_2}X_2\right)\right)+\nabla dv\left(d\psi\left(X_3\right),d\psi\left(\nabla_{X_3}X_2\right)\right)\\ &= \nabla dv\left(d\psi\left(X_3\right),d\psi\left(X_1\right)\right)=0.\end{aligned}\tag{3.7}$$

Moreover, for ψ we have already seen $\Delta^{\psi}\left(d\psi\left(X_j\right)\right)=4\,d\psi\left(X_j\right)$ in 3.1.3, so it remains to compute the first term on the right-hand side of (3.5).
Since the second fundamental form is symmetric, we obtain

$$\begin{aligned}\nabla^{\xi}_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right),d\psi\left(X_2\right)\right)\right] &= \nabla^{\xi}_{X_k}\left[\nabla dv\left(d\psi\left(X_2\right),d\psi\left(X_k\right)\right)\right]\\ &= \nabla^{\xi}_{X_k}\left[\nabla^{\xi}_{X_2}\left(d\xi\left(X_k\right)\right)-dv\left(\nabla^{\psi}_{X_2}\left(d\psi\left(X_k\right)\right)\right)\right]\\ &= \nabla^{\xi}_{X_k}\nabla^{\xi}_{X_2}\left(d\xi\left(X_k\right)\right)-\nabla dv\left(d\psi\left(X_k\right),\nabla^{\psi}_{X_2}\left(d\psi\left(X_k\right)\right)\right)-dv\left(\nabla^{\psi}_{X_k}\nabla^{\psi}_{X_2}\left(d\psi\left(X_k\right)\right)\right)\\ &= \nabla^{\xi}_{X_k}\nabla^{\xi}_{X_2}\left(d\xi\left(X_k\right)\right)-\nabla dv\left(d\psi\left(X_k\right),\nabla d\psi\left(X_2,X_k\right)\right)-\nabla dv\left(d\psi\left(X_k\right),d\psi\left(\nabla_{X_2}X_k\right)\right)\\ &\quad-dv\left(\nabla^{\psi}_{X_k}\nabla^{\psi}_{X_2}\left(d\psi\left(X_k\right)\right)\right).\end{aligned}$$

Using again the symmetry of $\nabla d\psi$ as well as $\nabla_{X_j}X_k=-\nabla_{X_k}X_j$, we have already calculated in (3.6) and (3.7) that the second and third term on the right-hand side vanish for each $k\in\{1,2,3\}$. Consequently, we find

$$\begin{aligned}\nabla^{\xi}_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right),d\psi\left(X_2\right)\right)\right] &= \nabla^{\xi}_{X_k}\nabla^{\xi}_{X_2}\left(d\xi\left(X_k\right)\right)-dv\left(\nabla^{\psi}_{X_k}\nabla^{\psi}_{X_2}\left(d\psi\left(X_k\right)\right)\right)\\ &= \nabla^{\xi}_{X_2}\nabla^{\xi}_{X_k}\left(d\xi\left(X_k\right)\right)+\nabla^{\xi}_{[X_k,X_2]}\left(d\xi\left(X_k\right)\right)+R^{\xi}\left(X_k,X_2\right)d\xi\left(X_k\right)\\ &\quad-dv\left(\nabla^{\psi}_{X_2}\nabla^{\psi}_{X_k}\left(d\psi\left(X_k\right)\right)+\nabla^{\psi}_{[X_k,X_2]}\left(d\psi\left(X_k\right)\right)+R^{\psi}\left(X_k,X_2\right)d\psi\left(X_k\right)\right)\\ &= \nabla^{\xi}_{X_2}\left[\nabla d\xi\left(X_k,X_k\right)\right]-dv\left(\nabla^{\psi}_{X_2}\left[\nabla d\psi\left(X_k,X_k\right)\right]\right)+\nabla dv\left(d\psi\left([X_k,X_2]\right),d\psi\left(X_k\right)\right)\\ &\quad+R^{\xi}\left(X_k,X_2\right)d\xi\left(X_k\right)-dv\left(R^{\psi}\left(X_k,X_2\right)d\psi\left(X_k\right)\right).\end{aligned}\tag{3.8}$$

Since v is conformal we can compare the curvature terms for ξ and ψ, using that ξ and ψ take values in a round sphere,

$$\begin{aligned}R^{\xi}\left(X_k,X_2\right)d\xi\left(X_k\right) &= \left(d\xi\left(X_2\right),d\xi\left(X_k\right)\right)d\xi\left(X_k\right)-\left|d\xi\left(X_k\right)\right|^2 d\xi\left(X_2\right)\\ &= \Lambda\,\left(d\psi\left(X_2\right),d\psi\left(X_k\right)\right)d\xi\left(X_k\right)-\Lambda\left|d\psi\left(X_k\right)\right|^2 d\xi\left(X_2\right)\\ &= \Lambda\,dv\left(\left(d\psi\left(X_2\right),d\psi\left(X_k\right)\right)d\psi\left(X_k\right)-\left|d\psi\left(X_k\right)\right|^2 d\psi\left(X_2\right)\right)\\ &= \Lambda\,dv\left(R^{\psi}\left(X_k,X_2\right)d\psi\left(X_k\right)\right),\end{aligned}\tag{3.9}$$

where Λ denotes the square dilation, i.e. the squared conformality factor, of v. Putting this back into (3.8) yields

$$\begin{aligned}\nabla^{\xi}_{X_k}\left[\nabla dv\left(d\psi\left(X_k\right),d\psi\left(X_2\right)\right)\right] &= \nabla^{\xi}_{X_2}\left[\nabla d\xi\left(X_k,X_k\right)\right]-dv\left(\nabla^{\psi}_{X_2}\left[\nabla d\psi\left(X_k,X_k\right)\right]\right)\\ &\quad+\nabla dv\left(d\psi\left([X_k,X_2]\right),d\psi\left(X_k\right)\right)\end{aligned}$$

$$+ (\Lambda - 1)\, dv \left(R^{\psi}(X_k, X_2)\, d\psi(X_k)\right).$$

As both ξ and ψ are harmonic, ξ due to the composition of harmonic morphisms being again a harmonic morphism (e.g. [4], Prop. 4.1.3 (i)), summing over k gives us, using the description of the curvature tensor R^{ψ} in (3.9),

$$\begin{aligned}
\sum_{k=1}^{3} \nabla^{\xi}_{X_k} \left[\nabla dv \left(d\psi(X_k), d\psi(X_2)\right)\right] =& \ \nabla^{\xi}_{X_2} [\mathrm{tr}\ \nabla d\xi] - dv \left(\nabla^{\psi}_{X_2} [\mathrm{tr}\ \nabla d\psi]\right) \\
&+ \nabla dv \left(d\psi([X_1, X_2]), d\psi(X_1)\right) + \nabla dv \left(d\psi([X_2, X_2]), d\psi(X_2)\right) \\
&+ \nabla dv \left(d\psi([X_3, X_2]), d\psi(X_3)\right) + (\Lambda - 1)\, dv \left(R^{\psi}(X_1, X_2)\, d\psi(X_1)\right) \\
&+ (\Lambda - 1)\, dv \left(R^{\psi}(X_2, X_2)\, d\psi(X_2)\right) + (\Lambda - 1)\, dv \left(R^{\psi}(X_3, X_2)\, d\psi(X_3)\right) \\
=& \, 2\, \nabla dv \left(d\psi(X_1), d\psi(X_3)\right) + (1 - \Lambda)\ dv \left(|d\psi(X_3)|^2\, d\psi(X_2)\right) \\
=& \, 4(1 - \Lambda)\ d\xi(X_2).
\end{aligned} \tag{3.10}$$

Inserting (3.6),(3.7) and (3.10) into the expression (3.5) for the pull-back Laplacian, we end up with

$$\Delta^{\xi}\left(d\xi(X_2)\right) = 4(\Lambda - 1)\ d\xi(X_2) + 4\, dv\left(d\psi(X_2)\right) = 4\,\Lambda\, d\xi(X_2). \tag{3.11}$$

Note that this is not surprising as this merely means that the induced Laplacian scales under composition with a conformal map. A very similar calculation for $j = 3$ therefore also results in

$$\Delta^{\xi}\left(d\xi(X_3)\right) = 4\,\Lambda\, d\xi(X_3). \tag{3.12}$$

Turning now to the calculation of Δ^{ξ} on the span of $d\xi(X_2)$, we take an arbitrary function $f_2 \in C^{\infty}\left(S^3\right)$. Then, we get

$$\begin{aligned}
\Delta^{\xi}\left(f_2\, d\xi(X_2)\right) &= (\Delta f_2)\, d\xi(X_2) + f_2 \Delta^{\xi}\left(d\xi(X_2)\right) - 2\ \mathrm{tr}\left[(\nabla_{\cdot} f_2)\, \nabla^{\xi}_{\cdot}\left(d\xi(X_2)\right)\right] \\
&= [\Delta f_2 + 4\,\Lambda\, f_2]\, d\xi(X_2) - 2\,(X_1 f_2)\, \nabla^{\xi}_{X_1}\left(d\xi(X_2)\right) \\
&\quad - 2\,(X_2 f_2)\, \nabla^{\xi}_{X_2}\left(d\xi(X_2)\right) - 2\,(X_3 f_2)\, \nabla^{\xi}_{X_3}\left(d\xi(X_2)\right).
\end{aligned}$$

Recall that we have computed for an arbitrary $k \in \{1, 2, 3\}$

$$\begin{aligned}
\nabla^{\xi}_{X_k}\left(d\xi(X_2)\right) &= \nabla dv\left(d\psi(X_k), d\psi(X_2)\right) + dv\left(\nabla^{\psi}_{X_k}\left(d\psi(X_2)\right)\right) \\
&= \nabla dv\left(d\psi(X_k), d\psi(X_2)\right) + dv\left(\nabla d\psi(X_k, X_2)\right) + d\xi\left(\nabla_{X_k} X_2\right).
\end{aligned}$$

So, the individual values for $k \in \{1, 2, 3\}$ are

(1) $\nabla^{\xi}_{X_1}\left(d\xi(X_2)\right) = -2\, d\xi(X_3)$,

(2) $\nabla^{\xi}_{X_2}\left(d\xi(X_2)\right) = \nabla dv\left(d\psi(X_2), d\psi(X_2)\right)$,

(3) $\nabla^{\xi}_{X_3}\left(d\xi(X_2)\right) = \nabla dv\left(d\psi(X_3), d\psi(X_2)\right)$.

Thus, we conclude that the pull-back Laplacian of $f_2\, d\xi(X_2)$ is given by

$$\begin{aligned}
\Delta^{\xi}\left(f_2\, d\xi(X_2)\right) = &\ [\Delta f_2 + 4\,\Lambda\, f_2]\, d\xi(X_2) + 4\,(X_1 f_2)\, d\xi(X_3) \\
&- 2\,(X_2 f_2)\, \nabla dv\left(d\psi(X_2), d\psi(X_2)\right) - 2\,(X_3 f_2)\, \nabla dv\left(d\psi(X_2), d\psi(X_3)\right).
\end{aligned} \tag{3.13}$$

As for the case $j = 3$, we find

(1) $\nabla^{\xi}_{X_1}\left(d\xi(X_3)\right) = 2\, d\xi(X_2)$,

(2) $\nabla^{\xi}_{X_2}\left(d\xi(X_3)\right) = \nabla dv\left(d\psi(X_2), d\psi(X_3)\right)$,

(3) $\nabla^{\xi}_{X_3}\left(d\xi(X_3)\right) = \nabla dv\left(d\psi(X_3), d\psi(X_3)\right)$

so that we obtain in a similar fashion as for $j=2$ for any function $f_3 \in C^\infty\left(S^3\right)$

$$\begin{aligned}
\Delta^\xi\left(f_3\, d\xi\left(X_3\right)\right) = & \left[\Delta f_3 + 4\,\Lambda\, f_3\right] d\xi\left(X_3\right) - 4\left(X_1 f_3\right) d\xi\left(X_2\right) \\
& - 2\left(X_2 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right) - 2\left(X_3 f_3\right) \nabla dv\left(d\psi\left(X_3\right), d\psi\left(X_3\right)\right) \\
= & \left[\Delta f_3 + 4\,\Lambda\, f_3\right] d\xi\left(X_3\right) - 4\left(X_1 f_3\right) d\xi\left(X_2\right) \\
& - 2\left(X_2 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right) + 2\left(X_3 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right),
\end{aligned} \tag{3.14}$$

where in the second equality we used the harmonicity of v as well as the surjectivity of ψ. In order to determine J^ξ it therefore remains to compute the curvature term appearing in the Jacobi operator. Since ξ is horizontally conformal with square dilation $4\,\Lambda$, this term simplifies so that we have for both $j \in \{2,3\}$

$$\operatorname{tr} R^{S^2}\left(d\xi\left(X_j\right), d\xi\cdot\right) d\xi\cdot = 4\,\Lambda \operatorname{Ric}^{S^2}\left(d\xi\left(X_j\right)\right) = 4\,\Lambda\, d\xi\left(X_j\right). \tag{3.15}$$

Hence, combining (3.15) with (3.13) and (3.14), we find J^ξ to be given by

$$\begin{aligned}
J^\xi\left(f_2\, d\xi\left(X_2\right)\right) = & \left(\Delta f_2\right) d\xi\left(X_2\right) + 4\left(X_1 f_2\right) d\xi\left(X_3\right) \\
& - 2\left(X_2 f_2\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right) - 2\left(X_3 f_2\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right), \\
J^\xi\left(f_3\, d\xi\left(X_3\right)\right) = & \left(\Delta f_3\right) d\xi\left(X_3\right) - 4\left(X_1 f_3\right) d\xi\left(X_2\right) \\
& - 2\left(X_2 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right) + 2\left(X_3 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right).
\end{aligned}$$

All in all, recalling the Jacobi operator J^ψ of the Hopf map from Lemma 3.1.13, we have shown

Lemma 3.1.15. *Let $\psi : S^3 \to S^2$ be the Hopf map, $v : S^2 \to S^2$ be a conformal transformation of S^2 and $\xi := v \circ \psi : S^3 \to S^2$. Then, the Jacobi operator of ξ is determined for any $f_2, f_3 \in C^\infty\left(S^3\right)$ by*

$$\begin{aligned}
J^\xi\left(f_2\, d\xi\left(X_2\right)\right) = & \, dv\left(J^\psi\left(f_2\, d\psi\left(X_2\right)\right)\right) - 2\left(X_2 f_2\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right) \\
& - 2\left(X_3 f_2\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right), \\
J^\xi\left(f_3\, d\xi\left(X_3\right)\right) = & \, dv\left(J^\psi\left(f_3\, d\psi\left(X_3\right)\right)\right) - 2\left(X_2 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_3\right)\right) \\
& + 2\left(X_3 f_3\right) \nabla dv\left(d\psi\left(X_2\right), d\psi\left(X_2\right)\right).
\end{aligned}$$

Due to the additional terms containing the second fundamental form of v we unfortunately cannot easily determine the eigenvalues of J^ξ in this generality. In light of Lemma A.2.1, part (1) these additional terms vanish in the case that v is horizontally homothetic since the $d\psi\left(X_j\right)$, $j \in \{2,3\}$, are horizontal vector fields with respect to v. However, we have $\ker dv = \{0\}$ so that v being horizontally homothetic is equivalent to v being a homothety of S^2, that is v having constant dilation. Furthermore, every homothety of S^2 is already an isometry (e.g. [42]).
From all of this as well as Baird and Wood's classification of harmonic morphisms $S^3 \to S^2$ we can conclude that, up to isometries of S^3 and S^2, the only globally defined, submersive, horizontally homothetic harmonic morphism $S^3 \to S^2$ is the Hopf map $\psi : S^3 \to S^2$. Note that this is a special case of Thm. 2.5 in [64]. In particular the index of any such harmonic morphism behaves as the index of ψ under composition with $\mathrm{i} : S^2 \to S^p$.
Therefore we can summarize this special case as follows

Corollary 3.1.16. *Let $\xi : S^3 \to S^2$ be a globally defined, submersive, horizontally homothetic harmonic morphism. Then, $\xi = v \circ \psi \circ \Phi$, where $\Phi : S^3 \to S^3$ and $v : S^2 \to S^2$ are isometries and $\psi : S^3 \to S^2$ is the Hopf map. Moreover, the Jacobi operator of ξ is determined by*

$$J^\xi\left(f_j\, d\xi\left(X_j\right)\right) = dv\left(J^\psi\left(f_j\, d\psi\left(X_j\right)\right)\right)$$

for $j \in \{2,3\}$ and any function $f_j \in C^\infty\left(S^3\right)$. Furthermore, the spectra of J^ξ and J^ψ coincide. In particular, letting $\mathrm{i} : S^2 \to S^p$, $p \geq 3$, denote the canonical inclusion map, we *find*

$$\mathrm{ind}\,(\mathrm{i}\circ\xi) > \mathrm{ind}\,(\xi) = \mathrm{ind}\,(\psi) = 4.$$

Remark 3.1.17. Since $d\psi\,(X_2)$ and $d\psi\,(X_3)$ are horizontal vector fields with respect to v on $\psi\left(S^3\right) = S^2$, we can give a more explicit description of the second fundamental form terms using Lemma A.2.1 part (a), namely

$$\begin{aligned}
\nabla dv\,(d\psi\,(X_2)\,, d\psi\,(X_2)) &= 2\,\left(d\psi\,(X_2)\right)(\ln\lambda)\,d\xi\,(X_2) - |d\psi\,(X_2)|^2\,dv\,(\nabla\ln\lambda)\\
&= \left(d\psi\,(X_2)\right)(\ln\lambda)\,d\xi\,(X_2) - \left(d\psi\,(X_3)\right)(\ln\lambda)\,d\xi\,(X_3)\,,\\
\nabla dv\,(d\psi\,(X_2)\,, d\psi\,(X_3)) &= \left(d\psi\,(X_2)\right)(\ln\lambda)\,d\xi\,(X_3) + \left(d\psi\,(X_3)\right)(\ln\lambda)\,d\xi\,(X_2)\,,
\end{aligned}$$

where λ denotes the conformality factor of v.
Therefore the determining expressions for the Jacobi operator J^ξ now read

$$\begin{aligned}
J^\xi\,(f_2\,d\xi\,(X_2)) = {} & [\Delta f_2 - 2\,(d\psi\,(X_2))\,(\ln\lambda)\,(X_2 f_2) - 2\,(d\psi\,(X_3))\,(\ln\lambda)\,(X_3 f_2)]\,d\xi\,(X_2)\\
& + [4\,(X_1 f_2) + 2\,(d\psi\,(X_3))\,(\ln\lambda)\,(X_2 f_2) - 2\,(d\psi\,(X_2))\,(\ln\lambda)\,(X_3 f_2)]\,d\xi\,(X_3)\,,\\
J^\xi\,(f_3\,d\xi\,(X_3)) = {} & [\Delta f_3 - 2\,(d\psi\,(X_2))\,(\ln\lambda)\,(X_2 f_3) - 2\,(d\psi\,(X_3))\,(\ln\lambda)\,(X_3 f_3)]\,d\xi\,(X_3)\\
& - [4\,(X_1 f_3) + 2\,(d\psi\,(X_3))\,(\ln\lambda)\,(X_2 f_3) - 2\,(d\psi\,(X_2))\,(\ln\lambda)\,(X_3 f_3)]\,d\xi\,(X_2)\,.
\end{aligned}$$

3.1.5 Index and nullity of harmonic submersions $M^1 \to N^1$

Among the round spheres only S^1, S^3 and S^7 admit a global Killing frame (e.g. [22]) and we have already discussed the case of the Hopf map $S^3 \to S^2$, so now we consider the Hopf map $\psi:\, S^1 \to S^1$, which is for any $x = (x_1, x_2) \in S^1$ given by

$$\psi(x) = \left(x_1^2 - x_2^2, 2x_1x_2\right).$$

It is easy to see that ψ is a harmonic Riemannian submersion up to scale with dilation 2.
In the following we can however consider any smooth, harmonic submersion $\psi:\,\left(M^1, g\right) \to \left(N^1, h\right)$ between one-dimensional, compact, connected, smooth Riemannian manifolds M and N. As a submersion between equidimensional manifolds ψ is also an immersion. In particular we have at every point $x \in M$

$$\mathcal{V}_x = \{0\} \quad \text{and} \quad \mathcal{H}_x = T_xM \setminus \{0\}.$$

As we did for S^3 we introduce a global Killing frame on M, which is then immediately horizontal.

Lemma 3.1.18. *Let M^1 be as above and suppose in addition that it admits a non-vanishing unit vector field X.
Then $\{X\}$ is a global Killing frame on M and in particular $\nabla_X X = 0$.*

Proof. First, as X is taken to be non-vanishing and M is one-dimensional, $\{X\}$ is a frame on M. Further, due to X being of unit length we find

$$0 = \nabla_X\,(g\,(X, X)) = 2\,g\,(\nabla_X X, X)\,.$$

Since $\{X\}$ is a frame on M this immediately yields $\nabla_X X = 0$. Finally, for the Killing condition we take arbitrary vector fields Y, Z on M. Then there exist functions f_Y, f_Z on M so that $Y = f_Y X$ and $Z = f_Z X$. Therefore, we obtain

$$g\,(\nabla_Y X, Z) + g\,(Y, \nabla_Z X) = 2\,f_Y f_Z g\,(X, \nabla_X X) = 0,$$

which verifies the Killing condition for X, i.e. shows that X is a Killing vector field. This together with $\{X\}$ being a frame on M implies that $\{X\}$ is indeed a Killing frame on M. $\square$

As the vertical spaces with respect to ψ consist only of $\{0\}$ we find for the sections of $\psi^{-1}TN$

$$\Gamma\left(\psi^{-1}TN^1\right) = \left\{ f\,d\psi\,(X) \,\middle|\, f \in C^\infty\left(M^1\right)\right\}.$$

Now since ψ is harmonic we have

$$\nabla_X^{\psi}\left(d\psi\left(X\right)\right) = \nabla d\psi\left(X, X\right) + d\psi\left(\nabla_X X\right) = 0.$$

Hence, for the induced Laplacian of $d\psi\left(X\right)$ we get

$$\Delta^{\psi}\left(d\psi\left(X\right)\right) = -\nabla_X^{\psi}\nabla_X^{\psi}\left(d\psi\left(X\right)\right) = 0.$$

So, given any function $f \in C^{\infty}\left(M\right)$, we obtain

$$\Delta^{\psi}\left(f\, d\psi\left(X\right)\right) = \left(\Delta f\right) d\psi\left(X\right) + f\, \Delta^{\psi}\left(d\psi\left(X\right)\right) - 2\left(Xf\right)\nabla_X^{\psi}\left(d\psi\left(X\right)\right) = \left(\Delta f\right) d\psi\left(X\right).$$

As any one-dimensional Riemannian manifold is flat, the curvature term vanishes so that we conclude

$$J^{\psi}\left(f\, d\psi\left(X\right)\right) = \Delta^{\psi}\left(f\, d\psi\left(X\right)\right) - R^{N^1}\left(f\, d\psi\left(X\right), d\psi\left(X\right)\right) d\psi\left(X\right) = \left(\Delta f\right) d\psi\left(X\right).$$

From this it is evident that any eigensection of J^{ψ} is contained in $\cup_{k\geq 0} S_{\lambda_k}$, where

$$S_{\lambda_k} := \left\{ f\, d\psi\left(X\right) \middle|\, \Delta f = \lambda_k\, f \right\}.$$

As M was taken to be compact and connected we have $\lambda_0 = 0$ with multiplicity one as well as $\lambda_k > 0$ for any $k \geq 1$. So, we have proven

Theorem 3.1.19. *Let (M, g) and (N, h) be one-dimensional, compact, connected, smooth Riemannian manifolds so that M admits a non-vanishing unit vector field X and let $\psi : M \to N$ be any smooth, harmonic submersion. Then the Jacobi operator of ψ is determined by*

$$J^{\psi}\left(f\, d\psi\left(X\right)\right) = \left(\Delta f\right) d\psi\left(X\right),\; f \in C^{\infty}\left(M\right).$$

Consequently, the spectrum of J^{ψ} coincides with the spectrum of $\Delta : C^{\infty}\left(M\right) \to C^{\infty}\left(M\right)$. In particular, $\operatorname{ind}\left(\psi\right) = 0$ *and* $\operatorname{null}\left(\psi\right) = 1$.

Returning to the example of the first Hopf map $S^1 \to S^1$, we note that we can take as the non-vanishing unit vector field $X(x) := (x_2, -x_1)$ as well as that the spectrum of Δ on $C^{\infty}\left(S^1\right)$ is known, namely $\lambda_k = k^2$ for $k \geq 0$, so that we get in this case for each $k \geq 0$ and for any $V \in S_{\lambda_k}$

$$J^{\psi}\left(V\right) = \lambda_k\, V = k^2 V.$$

Regarding the result of the general case Corollary 3.1.4, let us remark that for $\mathrm{i} : S^1 \to S^p$, $p \geq 2$, the canonical inclusion map and $\phi := \mathrm{i} \circ \psi : S^1 \to S^p$ we obtain for J^{ϕ} (counting with multiplicity) additional eigenvalues $\left\{ k^2 - 4 \middle|\, k \geq 0 \right\}$ of multiplicity $(p-1)\operatorname{mult}\left(k^2\right)$. Therefore we can conclude for index and nullity that $\operatorname{ind}\left(\phi\right) = 3p - 3$ and $\operatorname{null}\left(\phi\right) = 2p - 1$.

3.2 Classification of harmonic morphisms

In this section we address by means of an example both the question of the possible indices of harmonic morphisms and the question of the relevance of maps into lower-dimensional submanifolds.
The goal of this section is to describe for $n \geq 3$ harmonic morphisms from round S^n into immersed submanifolds of round S^n arising from non-constant, smooth maps $u : (S^n, \mathrm{can}) \to (S^n, \mathrm{can})$ of constant rank, in particular those of low index, that is of index $n+1$.
From this we see that harmonic morphisms on round spheres do not necessarily need to be of lowest possible index.

Preliminaries and Notation

We start by elaborating on the construction of harmonic morphisms from smooth maps of constant rank.

Given a non-constant, globally defined, smooth map $u : (S^n, \mathrm{can}) \to (S^n, \mathrm{can})$, $n \geq 3$, from the round n-sphere into itself of constant rank k, it can be shown that as a consequence of the constant rank theorem and smooth immersions being local embeddings (e.g. [48] for both) $k \leq n$ and the image $N := u(S^n)$ is also the image of a smooth immersion $i : Z \to S^n$. In case this smooth immersion $i : Z \to S^n$ with image $i(Z) = N = u(S^n)$ is also injective, N is an immersed submanifold of S^n and thus inherits a Riemannian metric g from restriction of the round metric on S^n since for any $x \in N$ we then have $T_x N \subseteq T_x S^n$.
This is for example the case when u is injective since then by the global rank theorem (e.g. [48], Thm. 4.14) u is itself a smooth, injective immersion. In that particular case the only possibility for constant rank is rank n.
With that much as motivation we assume from now on that N is an immersed submanifold of S^n and that it is equipped with the inherited metric g that we just described. Moreover, we also suppose from now on that $u : (S^n, \mathrm{can}) \to \left(N^k, g\right)$ is a harmonic morphism. Note that, due to u being non-constant and of constant rank, u does not have any critical point so that u is actually horizontally conformal.
For brevity we will throughout this section refer to non-constant, smooth maps $u : S^n \to S^n$, $n \geq 3$, of constant rank k with image $N := u(S^n)$ such that $u : (S^n, \mathrm{can}) \to \left(N^k, g\right)$ is a harmonic morphism, where N is an immersed submanifold of S^n endowed with the inherited metric g, as harmonic morphisms $u : S^n \to N^k \subseteq S^n$ of constant rank k.
To describe these maps we treat the cases of different constant rank separately.

Case $k = 1$

First we consider the case of rank one. As any smooth map into an one-dimensional Riemannian manifold is horizontally weakly conformal (e.g. [4], Ex. 2.4.9), every smooth, harmonic map $u : S^n \to N^1 \subset S^n$ of constant rank one is a harmonic morphism of constant rank one. So we can equivalently treat smooth, harmonic maps $u : S^n \to N^1 \subset S^n$ of constant rank one, i.e. in the construction above we can replace the assumption that $u : (S^n, \mathrm{can}) \to \left(N^1, g\right)$ is a harmonic morphism by assuming just that it is harmonic.
However, as we will see now, such maps do not exist and therefore this case cannot occur. To this end we consider, using the method of [18], the Bochner identity (e.g. [26]) for an arbitrary smooth, harmonic map $u : S^n \to N^1$ with respect to any orthonormal frame $\{e_1, \ldots, e_n\}$, namely

$$\begin{aligned}\frac{1}{2}\Delta\left(|du|^2\right) = &-|\nabla du|^2 + \sum_{i,j=1}^{n} g\left(R^N\left(du\left(e_i\right), du\left(e_j\right)\right) du\left(e_i\right), du\left(e_j\right)\right) \\ &- \sum_{i=1}^{n} g\left(du\left(\mathrm{Ric}^{S^n}\left(e_i\right)\right), du\left(e_i\right)\right).\end{aligned}$$

Due to the flatness of one-dimensional Riemannian manifolds the second term on the right-hand side vanishes, and as round spheres S^n are well-known to be Einstein manifolds with Einstein constant $n-1$ the third term on the right-hand side reduces to $-(n-1)|du|^2$. Integrating the resulting

$$\frac{1}{2}\Delta\left(|du|^2\right) = -|\nabla du|^2 - (n-1)\,|du|^2$$

over S^n yields, as $\Delta = -\operatorname{div}(\nabla)$ and S^n is without boundary,

$$\int_{S^n} |\nabla du|^2 = -(n-1)\int_{S^n} |du|^2\,,$$

which for $n \geq 3$ can only be true if $|\nabla du|^2 = |du|^2 = 0$ at every point $x \in S^n$ and so if u is constant. This shows that there cannot be smooth, harmonic maps $u : S^n \to N^1 \subset S^n$ of constant rank one.

Case $k = n$

Now for the maximal rank case we first prove that any harmonic morphism $u : S^n \to N^n \subseteq S^n$ of constant rank n is a diffeomorphism of S^n. In fact, this holds true for any smooth immersion between simply connected, equidimensional, compact, smooth Riemannian manifolds.

Lemma 3.2.1. *Let $f : M \to N$ be a smooth immersion between smooth, n-dimensional, compact, simply connected Riemannian manifolds for any $n \geq 2$. Then f is a diffeomorphism.*

Proof. Since f is an immersion, we have that df_x is injective for any $x \in M$ and non-zero. As $\dim(M) = \dim(N) = n$, this yields that df_x is also surjective. Since immersions are also locally injective and M and N are equidimensional, the invariance of domain theorem ([12]) gives us that f must also be an open map. This implies that f is a local diffeomorphism. Thus, to obtain that f is a global diffeomorphism it remains to show that f is bijective.
For this, we first look at surjectivity. Since we assumed M to be compact and f to be smooth, the image $f(M)$ is compact as well and, as N is Hausdorff, also closed in N. As we have seen, f is an open map, therefore $f(M)$ must be open as well. Hence, as N is connected and $f(M) \neq \emptyset$, we must have $f(M) = N$, which proves surjectivity.
Moreover, due to the continuity of f and the compactness of M, we have that f is proper, i.e. for every compact $K \subseteq N$ the preimage $f^{-1}(K) \subseteq M$ is compact.
This together with f being surjective and a local homeomorphism between compact Hausdorff spaces implies that f is a covering map ([37], Lem. 2). As such the simple connectedness of N yields that f is also globally injective (e.g. [47], Cor. 11.33).
Combining all of the above then gives us that $f : M^n \to N^n$ is a diffeomorphism. $\square$

As no harmonicity or conformality was assumed in Lemma 3.2.1 and in our discussion $u : S^n \to S^n$ was taken to be a smooth map of constant rank n, which implies that $u : S^n \to S^n$ is a smooth immersion, this applies to our harmonic morphisms of constant rank n. As a result it remains to determine the submersive harmonic morphisms from round S^n onto itself.
This is now an easy task since Fuglede and Ishihara characterized harmonic morphisms between equidimensional manifolds.

Proposition 3.2.2. *([32, 44])*
A smooth map $\phi : M^m \to N^m$ between Riemannian manifolds of equal dimension strictly larger than two is a harmonic morphism if and only if it is constant or homothetic.

In the case of manifolds which are not locally flat, i.e. which have non-vanishing Riemann curvature tensor, the class of harmonic morphisms narrows down even further.

Lemma 3.2.3. *([42])*
Let M be a complete and connected Riemannian manifold, which is not locally flat. Then every homothety of M is an isometry of M.

Summarizing, we have found that any harmonic morphism $u : S^n \to N^n \subseteq S^n$ of constant rank n must be an isometry of S^n.

Cases $1 < k < n$

For the remaining cases $1 < k < n$ we unfortunately cannot give a description in this generality. However, for $n = 3$ we can describe the case $k = 2$ due to the following Bernstein theorem by Baird and Wood.

Theorem 3.2.4. *([4], Thm. 6.7.7)*
Let $u : S^3 \to N^2$ be a globally defined, non-constant harmonic morphism from the Euclidean 3-sphere to a conformal surface. Then, up to isometry of S^3, $u = v \circ \psi$, where $\psi : S^3 \to S^2$ is the Hopf fibration and $v : S^2 \to N^2$ is a weakly conformal map.

Remark 3.2.5. ([4], Rem. 6.7.8)
The weakly conformal map v is surjective and N^2 is conformally equivalent to S^2 or $\mathbb{R}P^2$. If N^2 is oriented, then it is conformally equivalent to the Riemann sphere S^2 and we can take v to be holomorphic.

As for the harmonic morphisms $u : S^3 \to N^2 \subset S^3$ of constant rank two the image N^2 admits conformal structure determined by the metric g, we can apply the result of Baird and Wood directly. Even more so, as in our case $u : S^3 \to N^2$ is of constant rank two by assumption, u has no critical point. But this means that neither can $v : S^2 \to N^2$ in the decomposition of u have a critical point, so that it is in fact a conformal map and not just a weakly conformal map.
In the case of general $n \geq 3$ we can consider the case where $N^k = S^k$, that is where the image N is a k-sphere for $1 < k < n$. Then we can use the results of 3.1.2 to exclude these cases as candidates for low index maps $S^n \to S^n$.
To be precise, as sketched at the beginning of section 3.1, suppose that $u : S^n \to S^k$ is a harmonic Riemannian submersion up to scale. Then, Corollary 3.1.4 implies $\operatorname{ind}(\mathrm{i} \circ u) > \operatorname{ind}(u)$ with $\mathrm{i} : S^k \to S^n$ the canonical inclusion map. Further, from El Soufi's index bound we know both $\operatorname{ind}(\mathrm{i} \circ u) \geq n+1$ and $\operatorname{ind}(u) \geq n+1$. So, if $\mathrm{i} \circ u$ were to be of low index, that is of index $n+1$, then by 3.1.4 we would need u to be of index at most n, which contradicts the El Soufi bound.
Consequently, if we restrict to totally geodesic images N^k and harmonic Riemannian submersions up to scale $u : S^n \to N^k \subseteq S^n$, then the cases $k \in \{2, \dots, n-1\}$ do not lead to low index maps $S^n \to S^n$.

Classification of harmonic morphisms

Having discussed the different cases of constant rank, we can now summarize what we have found. First, for $n = 3$ we obtained a complete description of all harmonic morphisms $u : S^3 \to N^k \subseteq S^3$ of constant rank.

Theorem 3.2.6. *Let $u : \left(S^3, \mathrm{can}\right) \to \left(S^3, \mathrm{can}\right)$ be a non-constant, smooth map of constant rank k. Suppose that its image $N := u\left(S^3\right)$ is an immersed submanifold of S^3 and endow it with the inherited metric g. Assume further that $u : \left(S^3, \mathrm{can}\right) \to (N, g)$ is a harmonic morphism. Then, $k \in \{2, 3\}$ and u is given as follows*

(a) For $k = 2$ there exist an isometry $\Phi : S^3 \to S^3$ and a conformal, surjective map $v : S^2 \to N^2$ so that $u = v \circ \psi \circ \Phi$, where $\psi : S^3 \to S^2$ is the Hopf map. Moreover, N^2 is conformally equivalent to S^2 or $\mathbb{R}P^2$, and we obtain $\operatorname{ind}(u) \geq 4$.

(b) For $k = 3$ we find $N^3 = S^3$ and that $u : S^3 \to S^3$ is an isometry. In particular, $\operatorname{ind}(u) = 4$.

Proof. The only assertions in 3.2.6 we haven't proven yet are those on the index of u. In the case $k = 2$ this is simply El Soufi's bound, and in the case $k = 3$ this follows from the fact that isometries preserve the index so that the index of u is equal to the index of the identity map of S^3, which is equal to four. □

Second, for general $n \geq 3$ we can determine the harmonic morphisms $u : S^n \to N^k \subseteq S^n$ of constant rank with constant conformality factor and totally geodesic N that are of low index by applying Corollary 3.1.4.

Theorem 3.2.7. *Let $u : (S^n, \mathrm{can}) \to (S^n, \mathrm{can})$, $n \geq 3$, be a non-constant, smooth map of constant rank k. Let N denote the image $u(S^n)$, suppose that N is a totally geodesic immersed submanifold of S^n and endow it with the inherited metric g.*
Assume that $u : (S^n, \mathrm{can}) \to \left(N^k, g\right)$ is a harmonic Riemannian submersion up to scale with $\operatorname{ind}(\mathrm{i} \circ u) \leq n+1$, where $\mathrm{i} : N^k \to S^n$ denotes the canonical inclusion map.
Then, we have $\operatorname{ind}(\mathrm{i} \circ u) = n+1$ and u is an isometry of S^n.

Proof. The assertion $\operatorname{ind}(\mathrm{i} \circ u) = n+1$ follows immediately from El Soufi's index bound and the assumption that $\mathrm{i} \circ u$ is of index at most $n+1$.
Furthermore we have seen that for $k = 1$ there are no harmonic morphisms $S^n \to N^1 \subset S^n$ of constant rank one. Moreover, in the discussion of the cases $1 < k < n$ we found that these cases do not contribute to low index maps $S^n \to S^n$. Hence, the only possible case is $k = n$. There we saw that u must be an isometry of S^n, in particular i is then just the identity. As the index is invariant under composition with isometries, we have

$$\operatorname{ind}(\mathrm{i} \circ u) = \operatorname{ind}(u) = \operatorname{ind}(\mathrm{id}_{S^n}) = n+1.$$

Therefore, any such harmonic morphism of constant rank with low index is an isometry of S^n. □

3.3 Harmonic maps on manifolds with the Killing property

In the first part of this section we consider the question regarding El Soufi-type index bounds for smooth, harmonic maps on manifolds which are not round spheres. In fact, we provide such index bounds for smooth, harmonic maps from simply connected Riemannian manifolds admitting a Killing frame into round spheres of dimension at least two.
In the second part of the section, restricting to the two-sphere as codomain, we can give a positive answer to whether any low index harmonic map must be a harmonic morphism in this particular setting. Here, low index will refer to the El Soufi-type bound proven in the first part of this section.
For this we generalize the approach of Rivière ([65]), however contrary to [65] we cannot give an explicit description of the low index harmonic maps. In [65] this description was possible due to the characterization of harmonic morphisms $S^3 \to S^2$ by Baird and Wood. In our case there does not exist such a characterization of harmonic morphisms from Riemannian manifolds admitting a Killing frame into the round S^2.

3.3.1 Preliminaries

First, we recall some results on Killing frames as well as on Riemannian manifolds admitting a global Killing frame.
In accordance with D'Atri and Nickerson (cf. [22]) we say that a Riemannian manifold (M^n, g) has the Killing property if on a neighborhood of every $x \in M$ there exists an orthonormal frame consisting of Killing vector fields, called a local Killing frame. In this section such a frame will always be denoted by $\{e_1, \ldots, e_n\}$.
Supposing that (M, g) has the Killing property gives us constraints on the geometry of M as it implies (cf. [22]) that M is locally Riemannian symmetric. Thus, there exists for each $x \in M$ an open neighborhood which is isometric to an open neighborhood of a simply connected Riemannian symmetric space $\tilde{M}$.
Then, as argued in [22], $\tilde{M}$ has the global Killing property, that is $\tilde{M}$ admits a global Killing frame.
As we want to study in the following sphere-valued smooth, harmonic maps on Riemannian manifolds with the Killing property and harmonicity is a local property, we only consider simply connected domain manifolds with the Killing property.
To describe such manifolds further we recall the Riemannian product decomposition of a simply connected Riemannian symmetric space mentioned in the introduction.

Theorem 3.3.1. *([74], Thm. 8.3.8)*
Let M be a simply connected Riemannian symmetric space. There exists a Riemannian product decomposition

$$M = M_0 \times M_1 \times \ldots \times M_m,$$

which is unique up to permutation of the $M_1, \ldots, M_m$, where M_0 is a Euclidean space and *each $M_\mu, \mu \in \{1, \ldots, m\}$, a simply connected, irreducible Riemannian symmetric space. If M is compact, then there is no Euclidean factor M_0.*

Concerning the Killing property D'Atri and Nickerson found that it suffices to treat each factor in the Riemannian product decomposition separately.

Theorem 3.3.2. *([22])*
A simply connected Riemannian symmetric space has the Killing property if and only if each factor in any Riemannian product decomposition has the Killing property.

Using this, Berestovskii and Nikonorov classified all simply connected Riemannian manifolds with the Killing property by determining all possible factors M_μ in their Riemannian product decompositions.

Theorem 3.3.3. *([6], Thm. 7.5.6)*
A simply connected, complete Riemannian manifold (M, g) has the Killing property if and only if it is isometric to a direct metric product of a Euclidean space, compact simply connected simple Lie groups with bi-invariant metrics and round spheres S^7.

As already noted at the beginning of 3.1.5, among round spheres S^1 and S^3 also possess the Killing property. However, S^1 is not simply connected and $S^3 \cong \mathrm{SU}(2)$ falls under the case of simply connected, compact, simple Lie groups.
To reduce the number of subsequent round brackets, we let throughout this section $< \cdot, \cdot >$ denote the natural duality pairing, i.e. for any one-form α and vector field X we have $\langle \alpha, X \rangle = \alpha(X)$. Recall also that the covariant derivative of a given one-form α on M is given by

$$\langle \nabla_X \alpha, Y \rangle = -\langle \alpha, \nabla_X Y \rangle + (d \langle \alpha, Y \rangle)(X)$$

for arbitrary vector fields X, Y on M.
In this section we let, unless otherwise stated, Δ denote the negative Laplace-Beltrami operator of M, that is so that $\Delta : C^\infty(M) \to C^\infty(M)$ has non-negative eigenvalues. Moreover, given a smooth map $u : M \to S^p$, $p \geq 2$, we let $P_u : M \to TS^p$ at every point $x \in M$ be the orthogonal projection onto $T_{u(x)} S^p$. Now, let us collect the properties of a Killing frame which we will be using throughout the entire section without further notice.

Lemma 3.3.4. *Any Killing frame $\{e_1, \ldots, e_n\}$ on (M^n, g) satisfies the following properties: For all $i, j = 1, \ldots, n$ we have*

$$\nabla_{e_i} e_j = \frac{1}{2} [e_i, e_j] \perp \{e_i, e_j\}, \; \nabla_{e_i} e_j^* = (\nabla_{e_i} e_j)^*.$$

Here, $\{e_1^*, \ldots, e_n^*\}$ denotes a dual frame to $\{e_1, \ldots, e_n\}$. In particular, for any $i = 1, \ldots, n$ we find

$$d\left(e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\right) = 0.$$

As a result we obtain for any one-form α and any vector field $X \in \mathcal{X}(M)$

$$\Delta \langle \alpha, X \rangle = - * d * d \langle \alpha, X \rangle = -\sum_{i=1}^{n} \left(\langle \nabla_{e_i} \nabla_{e_i} \alpha, X \rangle + 2 \langle \nabla_{e_i} \alpha, \nabla_{e_i} X \rangle + \langle \alpha, \nabla_{e_i} \nabla_{e_i} X \rangle \right). \tag{3.16}$$

Proof. Since $\{e_1, \ldots, e_n\}$ is a Killing frame, we have for arbitrary indices i, j, k that

$$g\left(\nabla_{e_j} e_k, e_i\right) = -g\left(e_j, \nabla_{e_i} e_k\right).$$

This yields in particular that $\nabla_{e_i} e_j = -\nabla_{e_j} e_i$ and hence, as the Levi-Civita connection is torsion free, $\nabla_{e_i} e_j = \frac{1}{2}[e_i, e_j]$ as already noted in the proof of Lemma 3.1.9. Concerning the orthogonality, note that we have by this and the fact that the e_i's form a Killing frame

$$\frac{1}{2} g\left(e_k, [e_k, e_l]\right) = g\left(e_k, \nabla_{e_k} e_l\right) = -g\left(\nabla_{e_k} e_k, e_l\right) = 0$$

as well as

$$\frac{1}{2}\, g\left(e_l, [e_k, e_l]\right) = g\left(e_l, \nabla_{e_k} e_l\right) = -g\left(e_l, \nabla_{e_l} e_k\right) = g\left(\nabla_{e_l} e_l, e_k\right) = 0.$$

These imply that $[e_k, e_l]$ is orthogonal to both e_k and e_l with respect to g.
To show the claim $\nabla_{e_i} e_j^* = \left(\nabla_{e_i} e_j\right)^*$, we observe that using the definition of $\nabla_X \alpha$ as well as the properties of our Killing frame we find

$$\begin{aligned}\left\langle \nabla_{e_i} e_j^*, e_k \right\rangle &= -\left\langle e_j^*, \nabla_{e_i} e_k \right\rangle + \left(d\left\langle e_j^*, e_k \right\rangle\right)(e_i) = -\left\langle e_j^*, \nabla_{e_i} e_k \right\rangle + \left(d\delta_{jk}\right)(e_i)\\ &= -\left\langle e_j^*, \nabla_{e_i} e_k \right\rangle = \left\langle e_k^*, \nabla_{e_i} e_j \right\rangle.\end{aligned}$$

Thus, expanding $\nabla_{e_i} e_j^*$ in the Killing frame yields

$$\nabla_{e_i} e_j^* = \sum_{k=1}^n \left\langle \nabla_{e_i} e_j^*, e_k \right\rangle e_k^* = \sum_{k=1}^n \left\langle e_k^*, \nabla_{e_i} e_j \right\rangle e_k^* = \sum_{k=1}^n \left\langle \left(\nabla_{e_i} e_j\right)^*, e_k \right\rangle e_k^* = \left(\nabla_{e_i} e_j\right)^*.$$

Now we can use what we obtained so far to see that

$$de_i^* = \sum_{j=1}^n e_j^* \wedge \nabla_{e_j} e_i^* = \sum_{j;j\neq i} e_j^* \wedge \nabla_{e_j} e_i^* = \sum_{j;j\neq i} e_j^* \wedge \left(\nabla_{e_j} e_i\right)^*.$$

Consider now for any fixed i for example

$$e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+2} \wedge de^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}.$$

This can only be non-vanishing if the expansion of de^*_{i+1} in the e_i^*-frame contains a multiple of the term $e^*_{i+1} \wedge e^*_i$. However, from the above description of de^*_i we conclude, together with the orthogonality of $\nabla_{e_j} e_i$ and e_i, that the expansion of de^*_{i+1} cannot contain any term of the form $e^*_{i+1} \wedge e^*_j$. This means that the above wedge product must vanish.
Arguing in the same way for each of the other terms resulting from applying the product rule to $d\left(e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\right)$ then yields

$$d\left(e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\right) = 0.$$

Now, to prove the last assertion, we obtain for any one-form α and any vector field X for the Laplacian on the duality pairing $\langle \alpha, X \rangle$, letting d denote the exterior derivative and $*$ denote the Hodge star operator,

$$\begin{aligned}&\Delta \langle \alpha, X \rangle = - * d * d \langle \alpha, X \rangle = - * d * \sum_{i=1}^n \left(\left\langle \nabla_{e_i} \alpha, X \right\rangle + \left\langle \alpha, \nabla_{e_i} X \right\rangle\right) e_i^*\\
&= - * d \sum_{i=1}^n \left(\left\langle \nabla_{e_i} \alpha, X \right\rangle + \left\langle \alpha, \nabla_{e_i} X \right\rangle\right) e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\\
&= - * \sum_{i,j=1}^n \left(\left\langle \nabla_{e_j} \nabla_{e_i} \alpha, X \right\rangle + \left\langle \nabla_{e_i} \alpha, \nabla_{e_j} X \right\rangle\right) e_j^* \wedge e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\\
&\quad - * \sum_{i,j=1}^n \left(\left\langle \nabla_{e_j} \alpha, \nabla_{e_i} X \right\rangle + \left\langle \alpha, \nabla_{e_j} \nabla_{e_i} X \right\rangle\right) e_j^* \wedge e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\\
&= - * \sum_{i=1}^n \left(\left\langle \nabla_{e_i} \nabla_{e_i} \alpha, X \right\rangle + 2\left\langle \nabla_{e_i} \alpha, \nabla_{e_i} X \right\rangle\right) e_i^* \wedge e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\\
&\quad - * \sum_{i=1}^n \left\langle \alpha, \nabla_{e_i} \nabla_{e_i} X \right\rangle e_i^* \wedge e^*_{i+\lfloor \frac{n}{2} \rfloor} \wedge \ldots \wedge e^*_{i+1} \wedge e^*_{i-1} \wedge \ldots \wedge e^*_{i-\lfloor \frac{n}{2} \rfloor}\end{aligned}$$

$$= -\sum_{i=1}^{n} \left(\langle \nabla_{e_i} \nabla_{e_i} \alpha, X \rangle + 2 \langle \nabla_{e_i} \alpha, \nabla_{e_i} X \rangle + \langle \alpha, \nabla_{e_i} \nabla_{e_i} X \rangle \right).$$

This shows the last of the claims and closes the proof. □

As a final preliminary remark, recall, for example from [55], Prop. 1.6.1, that for sphere-valued smooth harmonic maps the second variation of the Dirichlet energy at $u : M \to S^p$ simplifies to

$$\left(\delta_u^2 E\right)(w) = \int_M \left(|dw|^2 - |du|^2 |w|^2 \right) = \int_M \left(L_u(w), w \right) \tag{3.17}$$

for any section $w \in \Gamma\left(u^{-1}TS^p\right)$, where $L_u : \Gamma\left(u^{-1}TS^p\right) \to \Gamma\left(u^{-1}TS^p\right)$ is given by

$$L_u(w) = P_u \Delta\, w - |du|^2\, w. \tag{3.18}$$

Moreover, the harmonic map equation for u takes a simpler form as well, namely (e.g. [25], (4.14))

$$-\Delta u = -|du|^2\, u.$$

3.3.2 Derivation of lower index bound

To give an outline of the next subsections, we will in the following prove a lower index bound for any non-constant, smooth, harmonic map $u : M \to S^p$, where M is a simply connected Riemannian manifold with the Killing property and S^p is the round unit p-sphere, under some constraints on M and u.
Followed by this, we treat each of the factors occurring in the Riemannian product decomposition due to Berestovskii and Nikonorov separately to see whether in these cases the said constraints are satisfied.
In proving the lower index bound we will be using the ideas of Rivière in [65] who re-proved El Soufi's index bound for smooth, harmonic maps $S^3 \to S^2$ constructively as a first part to classifying such smooth, harmonic maps of low index.
Before considering solely unstable harmonic maps u into round spheres of dimension at least two, we shortly discuss the cases of codomain S^1 and of stable harmonic maps.
First, as we have already seen in section 3.2, smooth maps into one-dimensional Riemannian manifolds are automatically horizontally weakly conformal so that Fuglede and Ishihara's characterization of harmonic morphisms yields that any smooth, harmonic map $u : M \to S^1$ is a harmonic morphism. Furthermore, in case M is an Einstein manifold with strictly positive Einstein constant and without boundary, we can adapt the arguments of the case $k = 1$ in section 3.2 to show that any smooth, harmonic map $u : M \to S^1$ must already be constant. As we will see in the upcoming subsections, this is the case for all of the possible compact factors appearing in the result of Berestovskii and Nikonorov.
Second, the case of stable harmonic maps into round spheres has been studied well, providing for example the following results.

Theorem 3.3.5. *([17])*
Any stable harmonic map from any compact Riemannian manifold into the round S^2 is already a harmonic morphism.

Theorem 3.3.6. *([49])*
Any stable harmonic map from a compact Riemannian manifold without boundary into round S^p, $p \geq 3$, is constant.

Theorem 3.3.7. *([68])*
Assume $k \geq 3$ and $2 \leq n \leq \bar{d}(k)$. Every smooth stable harmonic map from $\mathbb{R}^n$ to S^k is constant, where $\bar{d}(3) = 2$ and $\bar{d}(k) = \min\left\{\lceil \frac{k}{2} \rceil, 4\right\}$.

As a consequence, from now on we turn to unstable maps into round spheres of dimension at least two. For this we start by stating the conditions that will be needed to derive the lower index bound.

Assumption 3.3.8. Take $\{f^1, \ldots, f^{\mathcal{N}}\}$ to be an orthogonal basis of the eigenspace corresponding to the first non-zero eigenvalue $\lambda_1(M)$ of the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$ and denote $X^j := \nabla f^j$ for any $j \in \{1, \ldots, \mathcal{N}\}$. Let also $u : (M, g) \to (S^p, \mathrm{can})$, $p \geq 2$, be any non-constant, smooth, harmonic map. We suppose

(a) (M, g) is an Einstein manifold with Einstein constant λ so that $2\lambda > \lambda_1(M)$,

(b) For any $j \in \{1, \ldots, \mathcal{N}\}$ we have

$$P_u \sum_{i=1}^{n} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^j \right\rangle = 0,$$

(c) The $\langle du, X^1 \rangle, \ldots, \langle du, X^{\mathcal{N}} \rangle$ are linearly independent.

Throughout this section we will refer to these assumptions as assumptions (a), (b) and (c). Assume for now that all of those are satisfied. As a first step we see that the gradient vector fields X^j are eigenfields of the Laplacian $\Delta : \mathcal{X}(M) \to \mathcal{X}(M)$.

Lemma 3.3.9. *([23])*
Let (M^n, g) be an Einstein manifold of constant scalar curvature S. Let $f \in C^\infty(M)$ such that $\Delta f = \lambda f$. Then the gradient of f satisfies $\Delta(\nabla f) = \mu(\nabla f)$, where μ is given by $n(\lambda - \mu) = S$.

As a result we find for any $j \in \{1, \ldots, \mathcal{N}\}$

$$\Delta X^j = (\lambda_1(M) - \lambda) X^j, \tag{3.19}$$

that is the X^j's are eigenfields of Δ with eigenvalue $\lambda_1(M) - \lambda$.
The next step towards the index bound is to show that the $\langle du, X^j \rangle$'s generate index, that is they are eigensections of L_u with some negative eigenvalue. Then we can conclude the lower index bound from their assumed linear independence. Note that since we constructed the X^j's from an eigenbasis of the first non-zero eigenvalue of Δ, the lower index bound must be given by the multiplicity of the first non-zero eigenvalue.

Lemma 3.3.10. *Let $p \geq 2$ and $u : M \to S^p$ be a non-constant, smooth, harmonic map. Then, for any $j = 1, \ldots, \mathcal{N}$,*

$$\Delta\left(\langle du, X^j \rangle\right) = \left(\lambda_1(M) - 2\lambda + |du|^2\right) \langle du, X^j \rangle - 2 \sum_{i=1}^{n} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^j \right\rangle$$

and consequently

$$L_u\left(\langle du, X^j \rangle\right) = (\lambda_1(M) - 2\lambda) \langle du, X^j \rangle .$$

In particular, each $\langle du, X^j \rangle$ is an eigensection of J^u with negative eigenvalue $\lambda_1(M) - 2\lambda$.

Proof. Using (3.16) with $\alpha = du$ and $X = X^j$ for arbitrary $j \in \{1, \ldots, \mathcal{N}\}$, (3.19) and our assumptions on X^j, we obtain

$$\begin{aligned} \Delta\left(\langle du, X^j \rangle\right) &= -\sum_{i=1}^{n} \left(\left\langle \nabla_{e_i} \nabla_{e_i} du, X^j \right\rangle + 2 \left\langle \nabla_{e_i} du, \nabla_{e_i} X^j \right\rangle + \left\langle du, \nabla_{e_i} \nabla_{e_i} X^j \right\rangle \right) \\ &= -\sum_{i=1}^{n} \left\langle \nabla_{e_i} \nabla_{e_i} du, X^j \right\rangle - 2 \sum_{i=1}^{n} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^j \right\rangle + (\lambda_1(M) - \lambda) \langle du, X^j \rangle . \end{aligned} \tag{3.20}$$

Applying the Weitzenböck formula (e.g. [25], (2.17)) to the one-form du and the harmonic map equation for u, we obtain for the first term on the right-hand side, denoting by d^* the codifferential,

$$\begin{aligned}\sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} du &= -\left(dd^* + d^* d\right)(du) - \sum_{i,k=1}^{n} e_i^* \wedge \left(R^M\left(e_i, e_k\right) du\right)\left(e_k\right) \\ &= -d\left(d^* du\right) - \sum_{i,k=1}^{n} e_i^* \wedge \sum_{l=1}^{n} \left\langle du, e_l\right\rangle \left(R^M\left(e_i, e_k\right) e_l^*\right)\left(e_k\right) \\ &= d\left(-\Delta u\right) - \sum_{i=1}^{n} e_i^* \wedge \sum_{l=1}^{n} \left\langle du, e_l\right\rangle \operatorname{Ric}^M\left(e_i, e_l\right) \\ &= -du\left|du\right|^2 - u\, d\left|du\right|^2 + \lambda \sum_{i=1}^{n} \left\langle du, e_i\right\rangle e_i^* = \left(\lambda - \left|du\right|^2\right) du,\end{aligned}$$

where we also used in the second to last equality that g is an Einstein metric with Einstein constant λ. Putting this back in (3.20) implies

$$\begin{aligned}\Delta\left(\left\langle du, X^j\right\rangle\right) &= \left(\left|du\right|^2 - \lambda\right)\left\langle du, X^j\right\rangle - 2\sum_{i=1}^{n}\left\langle \nabla_{e_i} du, \nabla_{e_i} X^j\right\rangle + \left(\lambda_1\left(M\right) - \lambda\right)\left\langle du, X^j\right\rangle \\ &= \left(\lambda_1\left(M\right) - 2\lambda + \left|du\right|^2\right)\left\langle du, X^j\right\rangle - 2\sum_{i=1}^{n}\left\langle \nabla_{e_i} du, \nabla_{e_i} X^j\right\rangle.\end{aligned} \tag{3.21}$$

Now for L_u we obtain from (3.21) by (3.18) and assumption (b)

$$\begin{aligned}L_u\left(\left\langle du, X^j\right\rangle\right) &= P_u \Delta\left(\left\langle du, X^j\right\rangle\right) - \left|du\right|^2\left\langle du, X^j\right\rangle \\ &= \left(\lambda_1\left(M\right) - 2\lambda\right)\left\langle du, X^j\right\rangle - 2\, P_u \sum_{i=1}^{n}\left\langle \nabla_{e_i} du, \nabla_{e_i} X^j\right\rangle \qquad (3.22) \\ &= \left(\lambda_1\left(M\right) - 2\lambda\right)\left\langle du, X^j\right\rangle.\end{aligned}$$

Therefore, $\left\langle du, X^j\right\rangle$ is an eigensection of L_u with eigenvalue $\lambda_1\left(M\right) - 2\lambda$ for each $j \in \{1, \ldots, \mathcal{N}\}$. By assumption (a) this eigenvalue is negative. □

As this did not depend on the choice of harmonic map, we find that for any non-constant, smooth, harmonic map $u : M \to S^p$ the index is bounded from below by $\mathcal{N}$.
With the index bound established in this generality, we now turn to checking the assumptions (a)-(c) for each of the possible factors in the Riemannian product decomposition of M separately.

3.3.3 Case $M = S^7$

The first case we consider is the 7-sphere equipped with its round metric, denoted in the following by g. Here, we can explicitly write down a Killing frame.

Lemma 3.3.11. *([22])*
A Killing frame on the round S^7 is given at any $x = (x_1, \ldots, x_8) \in S^7$ by

$$\begin{aligned}
e_1(x) &:= (x_4, x_3, -x_2, -x_1, -x_8, -x_7, x_6, x_5), & e_5(x) &:= (x_7, -x_8, x_5, -x_6, -x_3, x_4, -x_1, x_2),\\
e_2(x) &:= (x_3, -x_4, -x_1, x_2, x_7, -x_8, -x_5, x_6), & e_6(x) &:= (-x_6, -x_5, -x_8, -x_7, x_2, x_1, x_4, x_3),\\
e_3(x) &:= (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7), & e_7(x) &:= (-x_5, x_6, x_7, -x_8, x_1, -x_2, -x_3, x_4).\\
e_4(x) &:= (-x_8, -x_7, x_6, x_5, -x_4, -x_3, x_2, x_1), & &
\end{aligned}$$

It is a well-known fact that round spheres S^n are Einstein manifolds with Einstein constant $\lambda = n - 1$. Moreover, it is known that the first eigenvalue of $\Delta : C^\infty(S^n) \to C^\infty(S^n)$ is $\lambda_1(S^n) = n$ (e.g. [69], Ex. 2.12 for both). So, for any $n \geq 3$ we have $\lambda_1(S^n) < 2\lambda$, in

particular for our case $n = 7$. Hence, assumption (a) is fulfilled.
Next, we determine the eigenfunctions and gradient vector fields corresponding to $\lambda_1\left(S^7\right)$ in order to verify the assumptions (b) and (c).

Lemma 3.3.12. *Let $\{\varepsilon_1, \ldots, \varepsilon_8\}$ denote the standard basis of $\mathbb{R}^8$. Then, an orthogonal basis of the eigenspace of $\Delta : C^\infty\left(S^7\right) \to C^\infty\left(S^7\right)$ corresponding to its first non-zero eigenvalue is given by*

$$f^j : S^7 \to \mathbb{R},\, x \mapsto g\left(\varepsilon_j, x\right) =: x_j,\; j = 1, \ldots, 8.$$

The associated gradient vector fields $X^j := \nabla f^j$ are then defined for any $x \in S^7$ as

$$X^j(x) = \varepsilon_j - x_j x.$$

In particular, X^j is the tangential component of ε_j for each $j \in \{1, \ldots, 8\}$.

Proof. We know (e.g. [27, 69]) that the f^j's as defined above are eigenfunctions of the Laplacian on $C^\infty\left(S^7\right)$ with eigenvalue $7 = \lambda_1\left(S^7\right)$. Their orthogonality follows directly from

$$\int_{S^7} x_i x_j = 0 \quad \text{for any}\, i \neq j.$$

Concerning the gradient vector fields X^j, we find for any $i \in \{1, \ldots, 7\}$ and $x \in S^7$

$$\begin{aligned} g\left(X^j(x), e_i(x)\right) &= g\left(\nabla f^j(x), e_i(x)\right) = \left(df^j(x)\right)\left(e_i(x)\right) \\ &= g\left(\left(d\varepsilon_j\right)\left(e_i(x)\right), x\right) + g\left(\varepsilon_j, (dx)\left(e_i(x)\right)\right) = g\left(\varepsilon_j, e_i(x)\right). \end{aligned}$$

Thus we obtain

$$X^j(x) = \sum_{i=1}^{7} g\left(X^j(x), e_i(x)\right) e_i(x) = \sum_{i=1}^{7} g\left(\varepsilon_j, e_i(x)\right) e_i(x),$$

which yields that X^j is the tangential component of ε_j, i.e. for every $x \in S^7$

$$X^j(x) = \varepsilon_j - \nu_x\left(\varepsilon_j\right),$$

where $\nu_x\left(\varepsilon_j\right) \in N_x S^7$ denotes the projection of ε_j onto the normal space of S^7 (in $\mathbb{R}^8$) at x. Now, for round spheres S^n it is well-known that the normal spaces at x, considering S^n as a submanifold of $\mathbb{R}^{n+1}$, are spanned by x. Therefore we have in our case for any $x \in S^7$

$$\nu_x\left(\varepsilon_j\right) = g\left(\varepsilon_j, x\right) x = x_j x.$$

So we conclude for the gradient vector fields at any $x \in S^7$

$$X^j(x) = \varepsilon_j - x_j x.$$

□

Moreover, from Smith ([69]), we also know that the X^j's must be conformal vector fields as gradients of linear forms on S^7. Therefore, since gradient vector fields are closed, the covariant derivatives of X^j are given for any $j \in \{1, \ldots, 8\}$ by (cf. e.g. [23])

$$\nabla_X X^j = \tilde{f}^j X \quad \text{for any}\, X \in \mathcal{X}\left(S^7\right),$$

where $\tilde{f}^j$ denotes the potential function of X^j, i.e. $\mathcal{L}_{X^j} g = 2\tilde{f}^j X^j$, where $\mathcal{L}$ denotes the Lie derivative. Here, we find $\tilde{f}^j(x) = -f^j(x) = -x_j$ for any $x \in S^7$. Indeed, letting P_x denote the projection onto $T_x S^7$, we obtain

$$\begin{aligned} \nabla_{X(x)} X^j(x) &= P_x\left[\left(dX^j(x)\right)\left(X(x)\right)\right] \\ &= P_x\left[\left(d\varepsilon_j\right)\left(X(x)\right)\right] - \left(dx_j\right)\left(X(x)\right) P_x[x] - x_j\, P_x\left[(dx)\left(X(x)\right)\right] = -x_j\, X(x). \end{aligned}$$

So, given any non-constant, smooth, harmonic map $u : S^7 \to S^p$, we obtain for each $j \in \{1, \dots, 8\}$

$$\begin{aligned}\sum_{i=1}^{7} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^j \right\rangle &= -\, x_j \sum_{i=1}^{7} \left\langle \nabla_{e_i} du, e_i \right\rangle = -x_j \sum_{i,l=1}^{7} \left(\nabla_{e_i} \nabla_{e_l} u \left\langle e_l^*, e_i \right\rangle + \nabla_{e_l} u \left\langle \nabla_{e_i} e_l^*, e_i \right\rangle \right) \\ &= -\, x_j \sum_{i=1}^{7} \nabla_{e_i} \nabla_{e_i} u = x_j \, \Delta u = x_j \, |du|^2 \, u,\end{aligned} \tag{3.23}$$

where we used in the last equality the harmonic map equation for u. Consequently, as for any $x \in M$ we have $u(x) \in N_{u(x)} S^7$, taking P_u of (3.23) yields

$$P_u \sum_{i=1}^{7} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^j \right\rangle = 0$$

independent of the choices of u and j, thus verifying assumption (b).
Applying this to the expressions for Δ, i.e. (3.21), and for L_u, i.e. (3.22), of $\left\langle du, X^j \right\rangle$, we have

Corollary 3.3.13. *Let $p \geq 2$ and $u : S^7 \to S^p$ be a non-constant, smooth, harmonic map. Then, for any $j = 1, \dots, 8$,*

$$\Delta \left(\left\langle du, X^j \right\rangle \right) = \left(|du|^2 - 5 \right) \left\langle du, X^j \right\rangle - 2 x_j u \, |du|^2$$

and consequently

$$L_u \left(\left\langle du, X^j \right\rangle \right) = -5 \left\langle du, X^j \right\rangle .$$

Then, towards the index bound it remains to prove (c), i.e. linear independence of the $\left\langle du, X^j \right\rangle$'s. For this we first find the flows of the vector fields X^j.

Lemma 3.3.14. *For any $j \in \{1, \dots, 8\}$ the flow $\phi_t^j(x)$ of X^j is given by*

$$\phi_t^j(x) = \frac{e^t - e^{-t}}{e^t + e^{-t}} \, \varepsilon_j + \frac{2e^t}{\left(e^{2t} + 1\right)\left(1 + x_j\right) - 2x_j} \left(x - \left(x_j - \frac{2x_j e^t}{e^{2t} + 1} \right) \varepsilon_j \right),$$

where $t \in [0, +\infty[$ and $x \in S^7$.

Proof. While here we only prove that the ϕ^j's are the flows of the X^j's by verifying that they satisfy the corresponding flow equation and initial condition, in appendix A.3 we give an explicit construction of the flows.
First note that evaluating ϕ_t^j at $t = 0$ gives us for any $x \in S^7$

$$\phi_0^j(x) = \frac{1 - 1}{1 + 1} \, \varepsilon_j + \frac{2 \cdot 1}{(1 + 1)(1 + x_j) - 2x_j} \left(x - \left(x_j - \frac{2x_j \cdot 1}{1 + 1} \right) \varepsilon_j \right) = x,$$

so $\phi_0^j = \mathrm{id}_{S^7}$ as desired. To verify the flow equation we compute both sides of the equation separately to see that they are equal. We start with the time derivative of $\phi_t^j(x)$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} \phi_t^j(x) = &\frac{\left(e^t + e^{-t}\right)^2 - \left(e^t - e^{-t}\right)^2}{\left(e^t + e^{-t}\right)^2} \, \varepsilon_j \\ &+ \left[\frac{2e^t}{(e^{2t} + 1)(1 + x_j) - 2x_j} - \frac{4e^{3t}(1 + x_j)}{[(e^{2t} + 1)(1 + x_j) - 2x_j]^2} \right] \left(x - \left(x_j - \frac{2x_j e^t}{e^{2t} + 1} \right) \varepsilon_j \right) \\ &+ \frac{2e^t}{(e^{2t} + 1)(1 + x_j) - 2x_j} \left(\frac{2x_j e^t}{e^{2t} + 1} - \frac{4x_j e^{3t}}{(e^{2t} + 1)^2} \right) \varepsilon_j\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^2}\right) \varepsilon_j \\
&\quad + \left[\left(e^{2t}+1\right)(1+x_j) - 2x_j\right]^{-2} \left(2e^t \left(e^{2t}+1\right)(1+x_j) - 4x_j e^t - 4e^{3t}(1+x_j)\right) x \\
&\quad + \left[\left(e^{2t}+1\right)(1+x_j) - 2x_j\right]^{-2} \left(e^{2t}+1\right)^{-2} \times \\
&\quad [\left(2e^t(e^{2t}+1)(1+x_j) - 4x_j e^t - 4e^{3t}(1+x_j)\right)\left(2x_j e^{3t} + 2x_j e^t - x_j e^{4t} - x_j - 2x_j e^{2t}\right) \\
&\quad + \left(2e^t(e^{2t}+1)(1+x_j) - 4x_j e^t\right)\left(2x_j e^t - 2x_j e^{3t}\right)] \varepsilon_j \\
&=: I + II + III.
\end{aligned}
\tag{3.24}
$$

For the term II we find

$$
\begin{aligned}
2e^t \left(e^{2t}+1\right)(1+x_j) - 4x_j e^t - 4e^{3t}(1+x_j) &= 2e^t \left(1 + e^{2t} + x_j + x_j e^{2t} - 2x_j\right) \\
&\quad - 2e^t \left(2e^{2t} + 2x_j e^{2t}\right) \\
&= 2e^t \left(1 - x_j - e^{2t} - x_j e^{2t}\right).
\end{aligned}
\tag{3.25}
$$

For III we have first

$$
\begin{aligned}
&\left(2e^t \left(e^{2t}+1\right)(1+x_j) - 4x_j e^t - 4e^{3t}(1+x_j)\right)\left(2x_j e^{3t} + 2x_j e^t - x_j e^{4t} - x_j - 2x_j e^{2t}\right) \\
&= 2e^t \left(1 - x_j - e^{2t} - x_j e^{2t}\right)\left(2x_j e^{3t} + 2x_j e^t - x_j e^{4t} - x_j - 2x_j e^{2t}\right) \\
&= 2e^t [-x_j + 2x_j e^t - x_j e^{2t} + x_j e^{4t} - 2x_j e^{5t} + x_j e^{6t} + x_j^2 - 2x_j^2 e^t + 3x_j^2 e^{2t} - 4x_j^2 e^{3t} \\
&\quad + 3x_j^2 e^{4t} - 2x_j^2 e^{5t} + x_j^2 e^{6t}]
\end{aligned}
\tag{3.26}
$$

and second

$$
\begin{aligned}
&\left(2e^t \left(e^{2t}+1\right)(1+x_j) - 4x_j e^t\right)\left(2x_j - 2x_j e^{3t}\right) \\
&\qquad = 2e^t \left(1 - x_j + e^{2t} + x_j e^{2t}\right)\left(2x_j e^t - 2x_j e^{3t}\right) \\
&\qquad = 2e^t \left[2x_j e^t - 2x_j^2 e^t + 4x_j^2 e^{3t} - 2x_j e^{5t} - 2x_j^2 e^{5t}\right].
\end{aligned}
\tag{3.27}
$$

Putting (3.25)-(3.27) back into (3.24) yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} \phi_t^j(x) &= \left(1 - \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^2}\right) \varepsilon_j + \frac{2e^t \left(1 - x_j - e^{2t} - x_j e^{2t}\right)}{[(e^{2t}+1)(1+x_j) - 2x_j]^2} x \\
&\quad + \frac{2e^t}{[(e^{2t}+1)(1+x_j) - 2x_j]^2 (e^{2t}+1)^2} [x_j^2 - x_j + 4x_j e^t - 4x_j^2 e^t - x_j e^{2t} + 3x_j^2 e^{2t} \\
&\quad + x_j e^{4t} + 3x_j^2 e^{4t} - 4x_j e^{5t} - 4x_j^2 e^{5t} + x_j e^{6t} + x_j^2 e^{6t}].
\end{aligned}
\tag{3.28}
$$

Now we calculate $X^j\left(\phi_t^j(x)\right)$:

$$
\begin{aligned}
X^j\left(\phi_t^j(x)\right) &= \varepsilon_j - \left(\frac{e^t - e^{-t}}{e^t + e^{-t}} + \frac{2e^t}{(e^{2t}+1)(1+x_j) - 2x_j} \frac{2x_j e^t}{e^{2t}+1}\right) \times \\
&\quad \left[\frac{e^t - e^{-t}}{e^t + e^{-t}} \varepsilon_j + \frac{2e^t}{(e^{2t}+1)(1+x_j) - 2x_j} \left(x - \left(x_j - \frac{2x_j e^t}{e^{2t}+1}\right) \varepsilon_j\right)\right] \\
&= \left(1 - \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^2}\right) \varepsilon_j \\
&\quad - \frac{2e^t}{(e^{2t}+1)(1+x_j) - 2x_j} \left(\frac{e^t - e^{-t}}{e^t + e^{-t}} + \frac{2x_j e^t}{e^{2t}+1} \frac{2e^t}{(e^{2t}+1)(1+x_j) - 2x_j}\right) x
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2x_j e^t}{e^{2t}+1} \frac{2e^t}{(e^{2t}+1)(1+x_j)-2x_j} \frac{e^t - e^{-t}}{e^t + e^{-t}} \varepsilon_j \\
&+ \frac{e^t - e^{-t}}{e^t + e^{-t}} \frac{2e^t}{(e^{2t}+1)(1+x_j)-2x_j} \left(x_j - \frac{2x_j e^t}{e^{2t}+1} \right) \varepsilon_j \\
&+ \frac{4e^{2t}}{[(e^{2t}+1)(1+x_j)-2x_j]^2} \frac{2x_j e^t}{e^{2t}+1} \left(x_j - \frac{2x_j e^t}{e^{2t}+1} \right) \varepsilon_j \\
= &\left(1 - \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^2} \right) \varepsilon_j + \left[\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j \right]^{-2} \left(e^{2t}+1\right)^{-1} \times \\
&\left[\left(1-e^{2t}\right) \left(\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j \right) 2e^t - 2x_j e^t \, 2e^t \, 2e^t \right] x \\
&+ \left[\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j \right]^{-2} \left(e^{2t}+1\right)^{-2} \times \\
&[2x_j e^t (1-e^{2t}) 2e^t ((e^{2t}+1)(1+x_j)-2x_j) \\
&+ (e^{2t}-1) 2e^t ((e^{2t}+1)(1+x_j)-2x_j)(x_j + x_j e^{2t} - 2x_j e^t) \\
&+ 4e^{2t} \, 2x_j e^t (x_j + x_j e^{2t} - 2x_j e^t)] \, \varepsilon_j \\
=: &\, I + II' + III'.
\end{aligned} \tag{3.29}
$$

For II' we find

$$
\begin{aligned}
2e^t \left(1-e^{2t}\right) &\left(\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j \right) - 2e^t \, 2e^t \, 2x_j e^t \\
&= 2e^t \left[\left(1-e^{2t}\right)\left(1 - x_j + e^{2t} + x_j e^{2t}\right) - 4x_j e^{2t} \right] \\
&= 2e^t \left(1 - x_j - e^{4t} - x_j e^{4t} - 2x_j e^{2t}\right) \\
&= 2e^t \left(e^{2t}+1\right)\left(1 - x_j - e^{2t} - x_j e^{2t}\right).
\end{aligned} \tag{3.30}
$$

And for III' we get first

$$
\begin{aligned}
2e^t \, 2x_j e^t \left(1-e^{2t}\right) &\left(\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j \right) \\
&= 2e^t \left(2x_j e^t - 2x_j e^{3t}\right)\left(1 - x_j + e^{2t} + x_j e^{2t}\right) \\
&= 2e^t \left[2x_j e^t - 2x_j^2 e^t + 4x_j^2 e^{3t} - 2x_j e^{5t} - 2x_j^2 e^{5t} \right],
\end{aligned} \tag{3.31}
$$

second

$$
\begin{aligned}
2e^t &\left(e^{2t}-1\right) \left(\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j \right) \left(x_j + x_j e^{2t} - 2x_j e^t\right) \\
&= 2e^t \left(1 - x_j + e^{2t} + x_j e^{2t}\right)\left(x_j e^{4t} - 2x_j e^{3t} - x_j + 2x_j e^t\right) \\
&= 2e^t [x_j^2 - x_j + 2x_j e^t - 2x_j^2 e^t - x_j e^{2t} - x_j^2 e^{2t} + 4x_j^2 e^{3t} + x_j e^{4t} - x_j^2 e^{4t} \\
&\quad - 2x_j e^{5t} - 2x_j^2 e^{5t} + x_j e^{6t} + x_j^2 e^{6t}],
\end{aligned} \tag{3.32}
$$

and finally

$$
\begin{aligned}
4e^{2t} \, 2x_j e^t \left(x_j + x_j e^{2t} - 2x_j e^t\right) &= 2e^t \, 4x_j e^{2t} \left(x_j + x_j e^{2t} - 2x_j e^t\right) \\
&= 2e^t \left[4x_j^2 e^{2t} + 4x_j^2 e^{4t} - 8x_j^2 e^{3t} \right].
\end{aligned} \tag{3.33}
$$

Therefore, inserting (3.30)-(3.33) back into (3.29) gives us

$$
\begin{aligned}
X^j \left(\phi_t^j(x)\right) = &\left(1 - \frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^2} \right) \varepsilon_j + \frac{2e^t \left(1 - x_j - e^{2t} - x_j e^{2t}\right)}{[(e^{2t}+1)(1+x_j)-2x_j]^2} x \\
&+ \frac{2e^t}{[(e^{2t}+1)(1+x_j)-2x_j]^2 (e^{2t}+1)^2} [x_j^2 - x_j + 4x_j e^t - 4x_j^2 e^t - x_j e^{2t} \\
&+ 3x_j^2 e^{2t} + x_j e^{4t} + 3x_j^2 e^{4t} - 4x_j e^{5t} - 4x_j^2 e^{5t} + x_j e^{6t} + x_j^2 e^{6t}].
\end{aligned} \tag{3.34}
$$

Comparing the expressions (3.28) and (3.34) implies

$$\frac{\mathrm{d}}{\mathrm{dt}}\phi_t^j(x) = X^j\left(\phi_t^j(x)\right),$$

so, as we have also verified the initial condition, we can conclude that $\phi_t^j(x)$ is the flow of X^j for any $j \in \{1, \dots, 8\}$ as claimed. $\square$

Remark 3.3.15. Observing that at the limit $t \to +\infty$ we have

$$\frac{e^t - e^{-t}}{e^t + e^{-t}} \to 1, \ \frac{2e^t}{\left(e^{2t}+1\right)\left(1+x_j\right) - 2x_j} \to 0 \quad \text{and} \quad \frac{2x_j e^t}{e^{2t}+1} \to 0,$$

yields for the flow $\phi_t^j(x) \to \varepsilon_j$ as $t \to +\infty$ for every $x \in S^7$. Moreover, we can compute for every $x \in S^7$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x_j}\phi_t^j(x) = & \frac{2e^t\left(1-e^{2t}\right)}{\left(\left(e^{2t}+1\right)\left(1+x_j\right)-2x_j\right)^2}\left(x - \left(x_j - \frac{2x_j e^t}{e^{2t}+1}\right)\varepsilon_j\right) \\ & + \frac{4e^{2t}}{\left(\left(e^{2t}+1\right)\left(1+x_j\right)-2x_j\right)\left(e^{2t}+1\right)}\varepsilon_j \\ \to & 0 \quad \text{as} \quad t \to +\infty\end{aligned}$$

as well as for $l \neq j$

$$\frac{\mathrm{d}}{\mathrm{d}x_l}\phi_t^j(x) = \frac{2e^t}{\left(e^{2t}+1\right)\left(1+x_j\right)-2x_j}\varepsilon_l \to 0 \,\text{as}\, t \to +\infty.$$

Thus, we can conclude $\frac{\mathrm{d}}{\mathrm{d}x}\phi_t^j(x) \to 0$ as $t \to +\infty$.
Therefore the flow $\phi_t^j(x)$ converges to ε_j in $W^{1,2}$ as $t \to +\infty$.
With this as a preparation we can prove now linear independence of the $\left\langle du, X^j\right\rangle$'s.

Corollary 3.3.16. *The $\left\langle du, X^1\right\rangle, \dots, \left\langle du, X^8\right\rangle$ are linearly independent. In particular, the index of u is bounded from below by* 8.

Proof. Suppose for contradiction that $\left\langle du, X^1\right\rangle, \dots, \left\langle du, X^8\right\rangle$ are linearly dependent. That means there exists some $X \in \operatorname{span}\left\{X^1, \dots, X^8\right\} \setminus \{0\}$ such that $\langle du, X\rangle = 0$. Up to rotation we can assume that $X = X^1$.
Using the $W^{1,2}$-convergence of the flow ϕ^1 we just established, we find that the map $x \mapsto \left(u \circ \phi_t^1\right)(x)$ converges in $W^{1,2}$ to $u\left(\varepsilon_1\right)$ as $t \to +\infty$. Moreover, the flow property of ϕ^1 yields for any fixed $x \in S^7$

$$\frac{d}{dt}\left(u \circ \phi_t^1\right) = du\left(\phi_t^1\right)\frac{d}{dt}\phi_t^1 = \left\langle du, X^1\right\rangle\left(\phi_t^1\right) = 0.$$

Thus, all maps $\left(u \circ \phi_t^1\right)(x)$ are constant in time and must therefore be equal to $u\left(\varepsilon_1\right)$, their limit as $t \to +\infty$.
Further, as u was assumed to be non-constant, there exist points $x, y \in S^7$ so that $u(x) \neq u(y)$. But then combining all of the above gives us

$$u\left(\varepsilon_1\right) = \left(u \circ \phi_t^1\right)(x) = \left(u \circ \phi_0^1\right)(x) = u(x) \neq u(y) = \left(u \circ \phi_0^1\right)(y) = \left(u \circ \phi_t^1\right)(y) = u\left(\varepsilon_1\right),$$

which is a contradiction. So, this implies that we must have linear independence of the $\left\langle du, X^j\right\rangle$'s. The second claim follows immediately from this since we have seen that the $\left\langle du, X^j\right\rangle$'s are eigensections of L_u with eigenvalue -5. $\square$

Note that even though we can treat $S^3 \cong \mathrm{SU}(2)$ with the simply connected, compact, simple Lie groups, we can also restrict here in the S^7-case e_1, e_2, e_3 to the set $\{x_5 = x_6 = x_7 = x_8 = 0\}$, use $\lambda = 2$ and construct $X^1, \dots, X^4$ just as the X^j's above. Then we re-obtain the proof of Rivière in [65] and thus that the index of any non-constant, smooth, harmonic map $u : S^3 \to S^p, p \geq 2$, is bounded from below by four.

3.3.4 Case $M = G$

Continuing with the case of a simply connected, compact, simple Lie group G with a bi-invariant metric g, we recall in appendix A.4.1 some definitions and basic results on Lie groups, especially compact matrix Lie groups. Recall (e.g. [1], Prop. 2.24) that any simply connected, compact, simple Lie group G admits a bi-invariant metric.
Furthermore, such a Riemannian metric g is induced by any negative multiple of the Killing form K on the Lie algebra $\mathfrak{g}$ of G, and is an Einstein metric on G (e.g. [1], Prop. 2.48). In particular, taking $g = -K$ we have as Einstein constant $\lambda = \frac{1}{4}$.
Fix in the following $g := -K$ so that (G, g) is Einstein with constant $\lambda = \frac{1}{4}$. To verify assumption (a) it then remains to show $\lambda_1(G) < 2\lambda = \frac{1}{2}$. However this is not satisfied in this generality, that is for any simply connected, compact, simple Lie group, so we must restrict to a subclass of Lie groups. For this we introduce the notion of a harmonically unstable compact Riemannian manifold.

Definition 3.3.17. ([62])
Let M be a compact Riemannian manifold. We call M harmonically unstable if and only if there exists no non-constant, stable, harmonic map from M to any Riemannian manifold and no non-constant, stable, harmonic map from any compact Riemannian manifold without boundary to M.

Remark. As a consequence of [62], Thm. 4, we have for an n-dimensional compact irreducible symmetric space M of scalar curvature c that M is harmonically unstable if and only if $\lambda_1(M) < \frac{2c}{n}$. Since in our case (G, g) is Einstein with $\lambda = \frac{1}{4}$, this is equivalent to $\lambda_1(G) < \frac{1}{2}$.
Let us further note that an example of a harmonically unstable Riemannian manifold is the round S^n for $n \geq 3$, which follows from the combined works of Xin ([75]) and Leung ([49]), so in particular also the round S^7 is harmonically unstable. A characterization of all harmonically unstable, compact symmetric spaces can be found in [62], Thm. 1.
In particular, harmonic instability implies that there do not exist any non-constant, smooth, stable, harmonic maps $G \to S^p$.

Therefore, we can determine the harmonically unstable, simply connected, compact, simple Lie groups based on their classification up to isomorphism by type.

Theorem 3.3.18. *([72])*
Let G be a simply connected, compact, simple Lie group with bi-invariant metric $g = -K$. Then, (G, g) is harmonically unstable if and only if the type of G is one of

(1) $A_l \cong \mathrm{SU}(l+1), l \geq 1$*;*

(2) $B_2 \cong \mathrm{Spin}(5)$*;*

(3) $C_l \cong \mathrm{Sp}(l), l \geq 3$.

Furthermore, the first non-zero eigenvalue $\lambda_1(G)$ of the Laplacian acting on smooth functions on G is given for each of these cases by

(1) $\lambda_1(A_l) = \frac{l(l+2)}{2(l+1)^2}$ *of multiplicity* $(l+1)^2$*;*

(2) $\lambda_1(B_2) = \frac{5}{12}$ *of multiplicity* 16*;*

(3) $\lambda_1(C_l) = \frac{2l+1}{4l+4}$ *of multiplicity* $4l^2$.

Noting also that B_2 is isomorphic to C_2, it suffices to treat Lie groups of type A_l for $l \geq 1$ and C_l for $l \geq 2$. So, let us suppose from now on that G is of this type and denote by $\mathcal{N}$ the corresponding multiplicity of $\lambda_1(G)$. In particular, assumption (a) is then fully satisfied.

Before defining a Killing frame on G, we observe that we can in fact give an explicit expression for the Killing form K and thus also for our bi-invariant metric g. Indeed, we have for any $X, Y \in T_eG \cong \mathfrak{g}$ (cf. Lem. A.4.19)

$$g(X, Y) = -c \operatorname{tr}(XY). \tag{3.35}$$

Note that this is in fact the restriction of the inner product on $\mathbb{C}^{\sqrt{N}\times\sqrt{N}}$ defined as

$$g(X, Y) = c \operatorname{tr}\left(X^{\dagger}Y\right) \tag{3.36}$$

since for any $X \in \mathfrak{g}$ we have $X^{\dagger} = -X$. As the tangent spaces of G at $x \neq e$ are given by left translation of the Lie algebra by x (cf. Prop. A.4.3), that is for any $x \in G$ we have

$$T_xG = \{ xv | \, v \in \mathfrak{g} \},$$

and for each $x \in G$ we get $x^{\dagger} = x^{-1}$, we obtain for any $xv, xw \in T_xG$

$$g(xv, xw) = c \operatorname{tr}\left(v^{\dagger}x^{\dagger}xw\right) = -c \operatorname{tr}(vw) = g(v, w),$$

i.e. the inner product on $\mathfrak{g}$ transfers to any other tangent space via (3.36). In particular, g is left-invariant in the sense that for any $x \in G$ and any $xv, xw \in T_x \mathbb{C}^{\sqrt{N}\times\sqrt{N}}$ we have as just seen

$$g_x(xv, xw) = g_e(v, w).$$

After these preparations we can introduce a Killing frame on (G, g).

Lemma 3.3.19. *Let G be a simply connected, compact, simple, harmonically unstable Lie group supplied with the bi-invariant metric $g = -K$. Let $e \in G$ denote the identity element of G and $\{v_1, \ldots, v_n\}$ be any orthonormal basis of $T_eG \cong \mathfrak{g}$. Define for all $i = 1, \ldots, n$ and any $x \in G$*

$$\begin{aligned} e_i(x) &:= (dL_x(e))(v_i) = x\, v_i = \sum_{\alpha,\beta=1}^{\sqrt{N}} \sum_{m=1}^{\sqrt{N}} x_{(\alpha,m)} v_i^{(m,\beta)} E_{(\alpha,\beta)} \\ &= \frac{1}{c} \sum_{\alpha,\beta=1}^{\sqrt{N}} g\left(E_{(\alpha,\beta)}, e_i(x)\right) E_{(\alpha,\beta)}, \end{aligned}$$

where L_x denotes left multiplication by x and $E_{(\alpha,\beta)} \in \mathbb{C}^{\sqrt{N}\times\sqrt{N}}$ is the matrix with entries

$$\left(E_{(\alpha,\beta)}\right)_{(\mu,\nu)} = \delta_{\alpha\mu}\delta_{\beta\nu}.$$

Then $\{e_1, \ldots, e_n\}$ is a Killing frame on G.

Proof. First we note that $e_i(x) := (dL_x(e))(v_i)$ defines for each $i \in \{1, \ldots, n\}$ the unique left-invariant vector field on G, which takes the value v_i at e . Thus, as on (G, g) all left-invariant vector fields are Killing vector fields of constant length (cf. Thm. A.4.8 in the appendix), the e_i's are Killing vector fields of constant length on (G, g). Moreover, on (G, g) the left multiplication maps L_x are isometries as g is bi-invariant and thus in particular left-invariant. Hence, the $e_i(x)$'s inherit the orthonormality from the v_i's at every point $x \in G$. Therefore $\{e_1, \ldots, e_n\}$ is a Killing frame on G.
The expressions for $e_i(x)$ with respect to the $E_{(\alpha,\beta)}$'s follow from executing the matrix product $x\, v_i$ and using the choice of g in (3.36). ☐

Considering now the eigenfunctions of Δ corresponding to the eigenvalue $\lambda_1(G)$, we need some basic notions and results from representation theory of (matrix) Lie groups, which are provided in the appendices A.4.2-A.4.4.
In appendix A.4.5 we derive an orthogonal basis for the eigenspace of Δ corresponding to $\lambda_1(A_l)$ and $\lambda_1(C_l)$. For readability however we give here a short digression, assuming the

contents of the subsections A.4.2-A.4.4.
By e.g. [70] it is known that on a compact, connected Lie group G the matrix coefficients of irreducible, unitary highest-weight representations of G with highest weight Λ are eigenfunctions of Δ with eigenvalue $(\Lambda, \Lambda + 2\eta)$, where $(\cdot,\cdot)$ is induced by an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ and 2η is the sum of the positive roots. These eigenfunctions are already pairwise orthogonal. Thus it suffices to find the irreducible, unitary highest-weight representation of $\mathfrak{g}$ so that $\lambda_1(G) = (\Lambda, \Lambda + 2\eta)$. Moreover in this case of the first eigenvalue it is, by [72], enough to minimize over the fundamental weights of $\mathfrak{g}$, that is $\lambda_1(G) = \min_i \{(w_i, w_i + 2\eta)\}$ with the w_i's denoting the fundamental weights. The corresponding fundamental representation of $\mathfrak{g}$ lifts uniquely to a representation of G, whose matrix coefficients then form an orthogonal basis of the first eigenspace. The fundamental representations for the classical Lie algebras are derived for example in [11]. The one realizing $\lambda_1(A_l)$, respectively $\lambda_1(C_l)$, turns out to be the identity representation of $\mathfrak{a}_l$, respectively a subrepresentation of the identity representation of $\mathfrak{c}_l$.
From all of this we can conclude, changing in this case for the gradient vector fields notation from X^j to $X^{(\alpha,\beta)}$ to reflect G being a matrix Lie group,

Lemma 3.3.20. *Let G be a simply connected, compact, simple, harmonically unstable Lie group with bi-invariant metric $g = -K$. Let $\lambda_1(G)$ denote the first non-zero eigenvalue of $\Delta : C^\infty(G) \to C^\infty(G)$ and $\mathcal{N}$ its multiplicity as given in Theorem 3.3.18.*
Then, the functions $f^{(\alpha,\beta)} : G \to \mathbb{C}$, $x \mapsto x_{(\alpha,\beta)} = \frac{1}{c} g\left(E_{(\alpha,\beta)}, x\right)$, where $\alpha, \beta = 1, \dots, \sqrt{\mathcal{N}}$, define an orthogonal eigenbasis of the eigenspace corresponding to $\lambda_1(G)$. Moreover, the corresponding gradient vector fields $X^{(\alpha,\beta)} := \nabla f^{(\alpha,\beta)}$ are given by

$$X^{(\alpha,\beta)}(x) = \frac{1}{c}\left(E_{(\alpha,\beta)} - \nu_x\left(E_{(\alpha,\beta)}\right)\right),$$

where $\nu_x\left(E_{(\alpha,\beta)}\right)$ denotes the normal component of $E_{(\alpha,\beta)}$ at x, i.e. $\nu_x\left(E_{(\alpha,\beta)}\right) \in N_xG \subseteq \mathbb{C}^{\sqrt{\mathcal{N}}\times\sqrt{\mathcal{N}}}$ and c is the constant from (3.36).

Proof. As we see in subsection A.4.5 and have summarized above, the eigenfunctions for such a Lie group G are given by the matrix coefficients of the unique lift of the identity representation of $\mathfrak{g}$, which is the identity representation of G. Indeed, since G is a compact and connected Lie group, the exponential map is surjective, so for every $x \in G$ there exists $X \in \mathfrak{g}$ such that $x = \exp(X)$. Now, as G is moreover simply connected, every representation of $\mathfrak{g}$ exponentiates to a unique representation of G as discussed after Theorem A.4.15 in the appendix. This means that given a representation ρ of $\mathfrak{g}$, we obtain a unique representation U of G by setting for any $x = \exp(X) \in G$

$$U(x) = U(\exp(X)) := \exp(\rho(X)).$$

Taking ρ to be the identity on the Lie algebra, this yields that U is the identity on the Lie group. Thus, for all $\alpha, \beta = 1, \dots, \sqrt{\mathcal{N}}$ we obtain an eigenfunction $f^{(\alpha,\beta)} : G \to \mathbb{C}$, $x \mapsto x_{(\alpha,\beta)}$, where $x_{(\alpha,\beta)}$ denotes the (α,β)-element of the matrix $x \in G$. The orthogonality of the $f^{(\alpha,\beta)}$'s follows from Theorem A.4.17 in the appendix.
Now, proceeding similarly to the S^7-case, since $x_{(\alpha,\beta)} = \frac{1}{c} g\left(E_{(\alpha,\beta)}, x\right)$, we can calculate for any $x \in G$ and $i \in \{1, \dots, n\}$

$$\begin{aligned} g\left(X^{(\alpha,\beta)}(x), e_i(x)\right) &= g\left(\nabla f^{(\alpha,\beta)}(x), e_i(x)\right) = \left(df^{(\alpha,\beta)}(x)\right)(e_i(x)) \\ &= \frac{1}{c}\left(g\left(\left(dE_{(\alpha,\beta)}\right)(e_i(x)), x\right) + g\left(E_{(\alpha,\beta)}, (dx)(e_i(x))\right)\right) \\ &= \frac{1}{c} g\left(E_{(\alpha,\beta)}, e_i(x)\right). \end{aligned}$$

Thus, for any $x \in G$ we find

$$X^{(\alpha,\beta)}(x) = \sum_{i=1}^{n} g\left(X^{(\alpha,\beta)}(x), e_i(x)\right) e_i(x) = \frac{1}{c}\sum_{i=1}^{n} g\left(E_{(\alpha,\beta)}, e_i(x)\right) e_i(x),$$

i.e. $X^{(\alpha,\beta)}$ is, up to a factor $\frac{1}{c}$, the tangential projection of $E_{(\alpha,\beta)}$ as claimed. Therefore, $X^{(\alpha,\beta)}$ is given by

$$X^{(\alpha,\beta)}(x) = \frac{1}{c}\left(E_{(\alpha,\beta)} - \nu_x\left(E_{(\alpha,\beta)}\right)\right)$$

for every $\alpha, \beta \in \left\{1, \ldots, \sqrt{\mathcal{N}}\right\}$ and $x \in G$. □

Remark. We remark that in the case $G = C_l$ we have suppressed that the eigenfunctions arise from only a subrepresentation of the identity representation of G, resulting in them and thus also their gradients being only defined on a subset $D \subset G$. This is because we will consider only globally defined, smooth, harmonic maps $u : G \to S^p$, so the $X^{(\alpha,\beta)}$ not being defined on all of G poses no issues for the construction of the eigensections $\left\langle du, X^{(\alpha,\beta)}\right\rangle$.

Remark 3.3.21. Furthermore, we can write our gradient vector fields $X^1, \ldots, X^{\mathcal{N}}$ in terms of our Killing frame $\{e_1, \ldots, e_n\}$ on G and find for all $\alpha, \beta \in \left\{1, \ldots, \sqrt{\mathcal{N}}\right\}$ and $x \in G$

$$\begin{aligned}
X^{(\alpha,\beta)}(x) &= \frac{1}{c}\left(E_{(\alpha,\beta)} - \nu_x\left(E_{(\alpha,\beta)}\right)\right)\\
&= \frac{1}{c}\sum_{i=1}^{n}\left(g\left(E_{(\alpha,\beta)}, e_i(x)\right) - g\left(\nu_x\left(E_{(\alpha,\beta)}\right), e_i(x)\right)\right) e_i(x)\\
&= \frac{1}{c}\sum_{i=1}^{n} g\left(E_{(\alpha,\beta)}, e_i(x)\right) e_i(x) = \sum_{i=1}^{n}\sum_{m=1}^{\sqrt{\mathcal{N}}} x_{(\alpha,m)} v_i^{(m,\beta)} e_i(x).
\end{aligned}$$

Now, let us study whether the gradient vector fields $X^{(\alpha,\beta)}$ satisfy the assumptions (b) and (c). From Proposition A.4.3 we know that the tangent spaces of G at x are given by left translation by x of the Lie algebra. Thus, given an orthonormal basis $\{w_1, \ldots, w_m\}$ of the normal space N_eG of G at e, considering G as a submanifold of $\mathbb{C}^{\sqrt{\mathcal{N}}\times\sqrt{\mathcal{N}}}$, with respect to the extension of g defined in (3.36), we can define for any $k \in \{1, \ldots, m\}$ and $x \in G$

$$\nu_k(x) := (dL_x)(w_k) := x\, w_k.$$

Then we find for g as in (3.36) for any $i \in \{1, \ldots, n\}$, $k, k_1, k_2 \in \{1, \ldots, m\}$ and $x \in G$

$$g\left(e_i(x), \nu_k(x)\right) = g\left((dL_x)(v_i), (dL_x)(w_k)\right) = g\left(v_i, w_k\right) = 0$$

as well as

$$g\left(\nu_{k_1}(x), \nu_{k_2}(x)\right) = g\left((dL_x)(w_{k_1}), (dL_x)(w_{k_2})\right) = g\left(w_{k_1}, w_{k_2}\right) = \delta_{k_1 k_2}.$$

Thus, for any choice of $x \in G$, we have an orthonormal basis of N_xG given by $\nu_1(x), \ldots, \nu_m(x)$. With this notation we can write the gradient vector fields $X^{(\alpha,\beta)}$ as

$$X^{(\alpha,\beta)}(x) = \frac{1}{c}\left(E_{(\alpha,\beta)} - \nu_x\left(E_{(\alpha,\beta)}\right)\right) = \frac{1}{c}\left(E_{(\alpha,\beta)} - \sum_{k=1}^{m} g\left(E_{(\alpha,\beta)}, \nu_k(x)\right)\nu_k(x)\right).$$

Therefore, we obtain for the covariant derivatives of $X^{(\alpha,\beta)}$, denoting by P_x the projection onto T_xG for any $x \in G$, as $E_{(\alpha,\beta)}$ is a constant matrix,

$$\nabla_{e_i(x)} X^{(\alpha,\beta)}(x) = P_x\left[\left(dX^{(\alpha,\beta)}(x)\right)(e_i(x))\right]$$

$$
\begin{aligned}
&= \frac{1}{c} P_x \left[\left(dE_{(\alpha,\beta)} \right) (e_i(x)) \right] - \frac{1}{c} \sum_{k=1}^{m} \left(d \left(g \left(E_{(\alpha,\beta)}, \nu_k(x) \right) \right) \right) (e_i(x)) \, P_x \left[\nu_k(x) \right] \\
&\quad - \frac{1}{c} \sum_{k=1}^{m} g \left(E_{(\alpha,\beta)}, \nu_k(x) \right) P_x \left[(d\nu_k(x)) \, (e_i(x)) \right] \\
&= -\frac{1}{c} \sum_{k=1}^{m} g \left(E_{(\alpha,\beta)}, \nu_k(x) \right) P_x \left[(d\nu_k(x)) \, (e_i(x)) \right] \\
&=: -\frac{1}{c} \sum_{k=1}^{m} g \left(E_{(\alpha,\beta)}, \nu_k(x) \right) \nabla_{e_i(x)} \nu_k(x).
\end{aligned}
\tag{3.37}
$$

Moreover, we observe that given a non-constant, globally defined, smooth, harmonic map $u : G \to S^p$ and arbitrary vector fields X, Y on G we have

$$
P_u \left\langle \nabla_X du, Y \right\rangle = \nabla du \left(X, Y \right). \tag{3.38}
$$

So, taking $X = e_i$ and $Y = \nabla_{e_i} X^{(\alpha,\beta)}$ in (3.38) gives us, together with (3.37),

$$
\begin{aligned}
P_u \left\langle \nabla_{e_i} du, \nabla_{e_i} X^{(\alpha,\beta)} \right\rangle &= \nabla du \left(e_i, \nabla_{e_i} X^{(\alpha,\beta)} \right) \\
&= -\frac{1}{c} \sum_{k=1}^{m} g \left(E_{(\alpha,\beta)}, \nu_k \right) \nabla du \left(e_i, \nabla_{e_i} \nu_k \right).
\end{aligned}
$$

Now taking the sum over i implies

$$
\begin{aligned}
P_u \sum_{i=1}^{n} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^{(\alpha,\beta)} \right\rangle &= \sum_{i=1}^{n} P_u \left\langle \nabla_{e_i} du, \nabla_{e_i} X^{(\alpha,\beta)} \right\rangle \\
&= -\frac{1}{c} \sum_{k=1}^{m} g \left(E_{(\alpha,\beta)}, \nu_k \right) \sum_{i=1}^{n} \nabla du \left(e_i, \nabla_{e_i} \nu_k \right) \\
&= -\frac{1}{c} \sum_{k=1}^{m} g \left(E_{(\alpha,\beta)}, \nu_k \right) \operatorname{tr} \left[\nabla du \left(\cdot, \nabla_{\cdot} \nu_k \right) \right].
\end{aligned}
\tag{3.39}
$$

Let us consider first the case $G = A_l \cong \mathrm{SU}(l+1)$. In this case we note that $m = 1$, that is the normal space $N_x G$ at any $x \in G$ is one-dimensional. In particular, to obtain an orthonormal basis of $N_e G$ it suffices to normalize an arbitrary non-zero element of $N_e G$.
For this we observe for any $v \in T_e \mathrm{SU}(l+1)$ that, since $\operatorname{tr}(v) = 0$, we get

$$
g(v, e) = -2(l+1) \operatorname{tr}(ve) = -2(l+1) \operatorname{tr}(v) = 0.
$$

Thus $e \in N_e \mathrm{SU}(l+1)$. Hence we can set for any $x \in \mathrm{SU}(l+1)$

$$
\nu(x) := \frac{1}{\sqrt{2}(l+1)} \, x.
$$

With this we find for any $i \in \{1, \ldots, n\}$

$$
\nabla_{e_i(x)} \nu(x) = \frac{1}{\sqrt{2}(l+1)} P_x \left[(dx) \left(e_i(x) \right) \right] = \frac{1}{\sqrt{2}(l+1)} P_x \left[e_i(x) \right] = \frac{1}{\sqrt{2}(l+1)} e_i(x).
$$

Therefore, by inserting this into (3.39), we find

$$
P_u \sum_{i=1}^{l^2+2l} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^{(\alpha,\beta)} \right\rangle = -\frac{1}{2(l+1)} \, g \left(E_{(\alpha,\beta)}, \nu \right) \sum_{i=1}^{l^2+2l} \nabla du \left(e_i, \nabla_{e_i} \nu \right)
$$

$$= -\frac{1}{2\sqrt{2}(l+1)^2}\, g\left(E_{(\alpha,\beta)}, \nu\right) \sum_{i=1}^{l^2+2l} \nabla du\,(e_i, e_i)$$
$$= -\frac{1}{2\sqrt{2}(l+1)^2}\, g\left(E_{(\alpha,\beta)}, \nu\right) \operatorname{tr}\left[\nabla du\right] = 0$$

as u is harmonic. This verifies assumption (b) for the case $G = A_l$.
Hence it remains to prove assumption (b) in the case $G = C_l \cong \mathrm{Sp}(l)$ for $l \geq 2$.
Even though in this case we can also take $\nu_1(x) := \frac{1}{\sqrt{8l(l+1)}}\, x$ and obtain consequently $\operatorname{tr}\left[\nabla du\,(\cdot, \nabla.\nu_1)\right] = 0$ as in the A_l-case, here we have $\dim\left(N_x\mathrm{Sp}(l)\right) = 2l^2 - l \geq 6$ for any $x \in \mathrm{Sp}(l)$ and it is not clear that $\operatorname{tr}\left[\nabla du\,(\cdot, \nabla.\nu_k)\right] = 0$ for $k \in \left\{2, \ldots, 2l^2 - l\right\}$ as well.
Hence, for $G = C_l$, $l \geq 2$, we suppose from now on for some $\{\nu_k\}_{k=1}^{2l^2-l}$ that for any $k \in \left\{1, \ldots, 2l^2 - l\right\}$ and $x \in C_l$

$$\operatorname{tr}\left[\nabla du_x\,(\cdot, \nabla.\nu_k(x))\right] = 0.$$

Then, we can conclude from (3.39)

$$P_u \sum_{i=1}^{2l^2+l} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^{(\alpha,\beta)} \right\rangle = -\frac{1}{4(l+1)} \sum_{k=1}^{2l^2-l} g\left(E_{(\alpha,\beta)}, \nu_k\right) \operatorname{tr}\left[\nabla du\,(\cdot, \nabla.\nu_k)\right] = 0,$$

giving us assumption (b) also for $G = C_l$.
With this we can now show that the $\left\langle du, X^{(\alpha,\beta)} \right\rangle$'s are eigensections of L_u with negative eigenvalue.

Corollary 3.3.22. *Let $p \geq 2$ and $u : G \to S^p$ be a non-constant, globally defined, smooth, harmonic map. In case $G = C_l$ suppose further that for some orthonormal frame $\{\nu_k\}_k$ of the normal bundle NG we have for each k*

$$\operatorname{tr}\left[\nabla du\,(\cdot, \nabla.\nu_k)\right] = 0,$$

where $\nabla_X \nu_k$ denotes the projection of $(d\nu_k)(X)$ onto TG for any vector field X.
Then, for any $\alpha, \beta = 1, \ldots, \sqrt{\mathcal{N}}$,

$$\Delta\left(\left\langle du, X^{(\alpha,\beta)} \right\rangle\right) = \left(\lambda_1(G) - \frac{1}{2} + |du|^2\right) \left\langle du, X^{(\alpha,\beta)} \right\rangle - 2\sum_{i=1}^{n} \left\langle \nabla_{e_i} du, \nabla_{e_i} X^{(\alpha,\beta)} \right\rangle$$

and therefore

$$L_u\left(\left\langle du, X^{(\alpha,\beta)} \right\rangle\right) = \left(\lambda_1(G) - \frac{1}{2}\right) \left\langle du, X^{(\alpha,\beta)} \right\rangle.$$

Similar to the S^7-case, in order to prove linear independence of the eigensections $\left\langle du, X^{(\alpha,\beta)} \right\rangle$, we first need to take a look at the flows of the gradient vector fields $X^{(\alpha,\beta)}$.
Recall that the flow $\phi^{(\alpha,\beta)}$ of $X^{(\alpha,\beta)}$ is the collection of integral curves starting at each $x \in G$, that is $\phi^{(\alpha,\beta)} : [0, +\infty[\times G \to G$ must satisfy

(1) $\phi_0^{(\alpha,\beta)}(x) = x$ for any $x \in G$,

(2) For every $t \in [0, +\infty[$ and every $x \in G$ we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \phi_t^{(\alpha,\beta)}(x) = X^{(\alpha,\beta)}\left(\phi_t^{(\alpha,\beta)}(x)\right).$$

As discussed in the appendix (cf. A.4.9-A.4.13), we know that for $x = e$ the integral curve must be of the form

$$\phi_t^{(\alpha,\beta)}(e) = \exp(tS) \quad \text{for some} \quad S \in \mathfrak{g}.$$

Using this in (2) yields

$$S\,\exp(tS) = \frac{\mathrm{d}}{\mathrm{dt}}\phi_t^{(\alpha,\beta)}(e) = X^{(\alpha,\beta)}\left(\phi_t^{(\alpha,\beta)}(e)\right). \tag{3.40}$$

Hence, evaluating (3.40) at $t = 0$ and applying then (1) gives us

$$S = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \phi_t^{(\alpha,\beta)}(e) = X^{(\alpha,\beta)}\left(\phi_0^{(\alpha,\beta)}(e)\right) = X^{(\alpha,\beta)}(e).$$

So, we find for the integral curve at e

$$\phi_t^{(\alpha,\beta)}(e) = \exp\left(tX^{(\alpha,\beta)}(e)\right). \tag{3.41}$$

As $\phi_t^{(\alpha,\beta)}(x)$ must give back (3.41) when being evaluated at $x = e$ while still fulfilling (1) and (2), we conclude that, in light of the surjectivity of the exponential map $\exp : \mathfrak{g} \to G$, the flow has to be of the form

$$\phi_t^{(\alpha,\beta)}(x) = x\,\exp\left(Z^{(\alpha,\beta)}(t,x)\right), \tag{3.42}$$

where we find from (1) and (2), using that the derivative of the matrix exponential is given by (e.g. [36], Thm. 5.4)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\exp(X(t)) &= \exp(X(t))\,\frac{1-\exp(-\mathrm{ad}_X)}{\mathrm{ad}_X}\frac{\mathrm{d}X(t)}{\mathrm{dt}} \\ &= \exp(X(t))\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{ad}_X\right)^k\frac{\mathrm{d}X(t)}{\mathrm{dt}},\end{aligned}$$

that the matrix-valued map $Z^{(\alpha,\beta)} : [0,+\infty[\, \times G \to \mathfrak{g}$ has to satisfy

(A) $Z^{(\alpha,\beta)}(0,x) = 0$ for any $x \in G$,

(B) For every $(t,x) \in [0,+\infty[\, \times G$

$$\begin{aligned} x\,\exp\left(Z^{(\alpha,\beta)}(t,x)\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\left(\mathrm{ad}_{Z^{(\alpha,\beta)}(t,x)}\right)^k\left(\frac{\mathrm{d}Z^{(\alpha,\beta)}(t,x)}{\mathrm{dt}}\right) \\ = X^{(\alpha,\beta)}\left(x\,\exp\left(Z^{(\alpha,\beta)}(t,x)\right)\right).\end{aligned}$$

In particular, at $x = e$ we obtain $Z^{(\alpha,\beta)}(t,e) = tX^{(\alpha,\beta)}(e)$ for any $t \in [0,+\infty[$ and hence $Z^{(\alpha,\beta)}(t,x) = tX^{(\alpha,\beta)}(e) + \tilde{Z}^{(\alpha,\beta)}(t,x)$ with $\tilde{Z}^{(\alpha,\beta)}(t,e) = 0$.
Using this we can prove linear independence of the $\langle du, X^{(\alpha,\beta)}\rangle$'s.

Corollary 3.3.23. *For $u : G \to S^p$ also locally injective the $\langle du, X^{(\alpha,\beta)}\rangle$'s are linearly independent. In particular the index of $u : G \to S^p$ is at least $\mathcal{N}$, where $\mathcal{N} = (l+1)^2$ for $G = A_l$ and $\mathcal{N} = 4l^2$ for $G = C_l$.*

Proof. Suppose for contradiction that the $\langle du, X^{(\alpha,\beta)}\rangle$'s are linearly dependent. Then there exists some $X \in \operatorname{span}\left\{X^{(\alpha,\beta)} \,\middle|\, \alpha,\beta = 1,\ldots,\sqrt{\mathcal{N}}\right\} \setminus \{0\}$ such that $\langle du, X\rangle = 0$. Up to rotation we can assume that $X = X^{(\alpha,\beta)}$ for some $\alpha,\beta \in \left\{1,\ldots,\sqrt{\mathcal{N}}\right\}$.
Then we obtain, letting $\phi^{(\alpha,\beta)}$ denote the flow of $X^{(\alpha,\beta)}$ as before, for any fixed $x \in G$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(u\circ\phi_t^{(\alpha,\beta)}\right) = du\left(\phi_t^{(\alpha,\beta)}\right)\frac{\mathrm{d}}{\mathrm{dt}}\phi_t^{(\alpha,\beta)} = \left\langle du, X^{(\alpha,\beta)}\right\rangle\left(\phi_t^{(\alpha,\beta)}\right) = 0.$$

Therefore, the maps $x \mapsto \left(u\circ\phi_t^{(\alpha,\beta)}\right)(x)$ are constant in time. This yields in particular that for any time $t \in]0, +\infty[$ and $x \in G$

$$u\left(\phi_t^{(\alpha,\beta)}(x)\right) = u\left(\phi_0^{(\alpha,\beta)}(x)\right) = u(x)$$

since by definition $\phi_0^{(\alpha,\beta)} = \mathrm{id}_G$. Recalling (3.42), this means

$$u(x) = u\left(x \exp\left(Z^{(\alpha,\beta)}(t,x)\right)\right) \tag{3.43}$$

for every $t \in]0, +\infty[$ and $x \in G$.
Now, as we assumed u to be locally injective, we can find for any fixed $x \in G$ a subset $x \in U \subset G$ so that $u|_U$ is injective. Then, given any $y \in U$ we take $\kappa_y > 0$ such that for any $t \in [0, \kappa_y[$ we have $\phi_t^{(\alpha,\beta)}(y) \in U$. Then (3.43) yields for any $y \in U$ and $t \in [0, \kappa_y[$ that $y = \phi_t^{(\alpha,\beta)}(y) = y \exp\left(Z^{(\alpha,\beta)}(t,y)\right)$, which in turn implies $e = \exp\left(Z^{(\alpha,\beta)}(t,y)\right)$. As the exponential map is a local homeomorphism around the origin this gives us $Z^{(\alpha,\beta)}(t,y) = 0$ for any $(t,y) \in [0, \kappa_y[\times U$. This in turn yields that $\phi^{(\alpha,\beta)}(y)$ is constant, which implies $X^{(\alpha,\beta)}(y) = 0$. Since this holds in particular at the fixed $x \in G$ and did not depend on the choice of $x \in G$, we must find $X^{(\alpha,\beta)} = 0$, giving us a contradiction.
So the $\left\langle du, X^{(\alpha,\beta)}\right\rangle$'s must be linearly independent, verifying thus assumption (c). The claim on the index follows from this as we have seen that the $\left\langle du, X^{(\alpha,\beta)}\right\rangle$'s are eigensections of L_u with eigenvalue $\lambda_1(G) - 2\lambda$, which is negative due to the harmonic instability of G. □

3.3.5 Case $M = \mathbb{R}^n$

Regarding the case of Euclidean space with its canonical metric, being flat it is Einstein with $\lambda = 0$. However, on any relatively compact domain $D \subset \mathbb{R}^n$ we have $\lambda_1(D) > 0$, which means that $\lambda_1 < 2\lambda$ can never be satisfied. Hence in this case the construction described in the general case 3.3.2 does not yield eigensections with negative eigenvalues.
In light of this, we therefore aim to exclude this case from consideration. Having remarked that for compact manifolds with the Killing property the Riemannian product decomposition does not contain an Euclidean factor, we can exclude this case by restricting to compact manifolds. However, note that we have the following Liouville-type theorem.

Theorem 3.3.24. *([34])*
Each harmonic map $u : \mathbb{R}^n \to S^m, n \geq 3$, with finite Dirichlet energy on $\mathbb{R}^n$ has to be constant.

Hence, by restricting to domain manifolds of dimension at least three and non-constant, smooth, harmonic maps of finite Dirichlet energy, the case $M = \mathbb{R}^n$ can be excluded. Moreover, this does not pose any constraints on the compact cases as in these all possible factors are of dimension at least three and harmonic maps on compact manifolds have finite Dirichlet energy.

3.3.6 Index bounds for products

To summarize, we have shown so far that for M_μ a compact, harmonically unstable, simply connected, complete Riemannian manifold of dimension at least three with the Killing property and with trivial Riemannian product decomposition the index of any non-constant, globally defined, smooth, harmonic map of finite energy $u : M_\mu \to S^p$, $p \geq 2$, satisfying in the case of M_μ a simple Lie group the additional constraints of 3.3.22 and 3.3.23, is bounded from below by $\mathcal{N}_\mu := \mathrm{mult}\left(\lambda_1\left(M_\mu\right)\right)$, that is together with the Liouville-type theorem 3.3.24 we have proven Theorem 1.0.13.
Before considering low index maps we prove index bounds for product maps based on these bounds. Let us first look at the harmonic instability of compact manifolds M with the Killing property which possess a non-trivial Riemannian product decomposition

$$M = M_1 \times \ldots \times M_m$$

in the sense of the classification result by Berestovskii and Nikonorov. In that case it is known ([38], Rem. 5.5) that M is harmonically unstable if each of the M_μ in the Riemannian product decomposition is harmonically unstable.
Moreover, Fardoun and Ratto proved an index bound for product maps in terms of the indices of the factor maps.

Proposition 3.3.25. *([31])*
Let $f_i : (M_i, g_i) \to (N_i, h_i)$, $i = 1, 2$, be smooth, harmonic maps between compact, oriented Riemannian manifolds and let $f : (M_1 \times M_2, g_1 \times g_2) \to (N_1 \times N_2, h_1 \times h_2)$ be the product map taking (x, y) to $(f_1(x), f_2(y))$. Then, f is harmonic and we have

(1) If f_1 and f_2 are stable, then so is f;

(2) $\operatorname{ind}(f) \geq \operatorname{ind}(f_1) + \operatorname{ind}(f_2)$;

(3) $\operatorname{null}(f) \geq \operatorname{null}(f_1) + \operatorname{null}(f_2)$.

Now we can establish lower index bounds for harmonic maps from compact manifolds M with the Killing property arising as product maps of harmonic maps from the model cases S^7, A_l, C_l, $\mathbb{R}^n$ into round spheres. This is achieved through iterative application of Fardoun and Ratto's index bound as well as using the index bounds 3.3.16, 3.3.23 and Theorem 3.3.24 proven in the preceding subsections or equivalently using Theorem 1.0.13.

Corollary 3.3.26. *Let M be a simply connected, complete Riemannian manifold of dimension at least three satisfying the Killing property and with corresponding Riemannian product decomposition*

$$M = M_0 \times \ldots \times M_m$$

so that all of the M_μ, $\mu = 1, \ldots, m$, are harmonically unstable.
Let further $u_\mu : M_\mu \to S^{p_\mu}$, $p_\mu \geq 2$, for all $\mu = 0, \ldots, m$ be as in Theorem 1.0.13.
Then, M is compact and harmonically unstable, the product map $u := (u_1, \ldots, u_m) : M \to S^{p_1} \times \ldots \times S^{p_m}$ is non-constant, globally defined, smooth, harmonic and satisfies the index bound

$$\operatorname{ind}(u) \geq \sum_{\mu=1}^{m} \operatorname{ind}(u_\mu) \geq \sum_{\mu=1}^{m} \mathcal{N}_\mu,$$

where $\mathcal{N}_\mu$ denotes the multiplicity of the first non-zero eigenvalue of $\Delta : C^\infty(M_\mu) \to C^\infty(M_\mu)$.

3.3.7 Characterization of low index maps

With the lower index bound fully established, we can now turn to the second goal of this section. So in the remainder of this section we set $p = 2$ and prove a characterization of low index smooth harmonic maps $u : M_\mu \to S^2$, where low index refers here to the index bound we have just shown, i.e. we consider the case $\operatorname{ind}(u) = \mathcal{N}_\mu$.
To be precise, from now on M_μ is either the round S^7 or a simply connected, harmonically unstable, compact, simple Lie group G with bi-invariant metric $g = -K$ and $u : M_\mu \to S^2$ is any non-constant, globally defined, smooth, harmonic map of index at most $\mathcal{N}_\mu = \operatorname{mult}(\lambda_1(M_\mu))$ so that for $M_\mu \in \{A_l\,(l \geq 1), C_l\,(l \geq 2)\}$ the map u is also locally injective and further for $M_\mu = C_l$ we have $\operatorname{tr}[\nabla du(\cdot, \nabla_\cdot \nu_k)] = 0$ for some orthonormal frame $\{\nu_k\}_k$ of NM_μ. Then, from the index bound 1.0.13, it is evident that $\operatorname{ind}(u) = \mathcal{N}_\mu$ and that $\langle du, X^1 \rangle, \ldots, \langle du, X^{\mathcal{N}_\mu} \rangle$ form a basis of the space of eigensections of L_u with negative eigenvalues.
As for the index bound we follow the ideas of Rivière in [65] to show that such maps u must be harmonic morphisms. We start off by deriving an auxiliary inequality used later in the proof of the characterization theorem.
Recall that the first eigenvalue $\lambda_1(L_u)$ of L_u is defined as

$$\lambda_1(L_u) = \inf_{\substack{w \in \Gamma(u^{-1}TS^p) \\ w \neq 0}} \left\{ \frac{\left(\delta_u^2 E\right)(w)}{\|w\|^2_{L^2(M_\mu)}} \right\}.$$

Thus we have for any non-trivial section $w \in \Gamma\left(u^{-1}TS^p\right)$ that

$$\left(\delta_u^2 E\right)(w) \geq \lambda_1\left(L_u\right)\|w\|^2_{L^2(M_\mu)}.$$

In light of the form of the second variation of E, given by (3.17) due to u being sphere-valued, this means that

$$\int\limits_{M_\mu}\left(|dw|^2 - |du|^2|w|^2\right) \geq \lambda_1\left(L_u\right)\int\limits_{M_\mu}|w|^2$$

for any $w \in \Gamma\left(u^{-1}TS^p\right)$ or equivalently

$$\int\limits_{M_\mu}\left(|dw|^2 - \left(|du|^2 + \lambda_1\left(L_u\right)\right)|w|^2\right) \geq 0. \tag{3.44}$$

Now, with the assumption that u is of low index we can conclude from the just established fact that the $\left\langle du, X^j\right\rangle$'s are $\mathcal{N}_\mu$ linearly independent eigensections with eigenvalue $\lambda_1\left(M_\mu\right) - 2\lambda < 0$ that we must find $\lambda_1\left(L_u\right) = \lambda_1\left(M_\mu\right) - 2\lambda$.
Therefore we can conclude from (3.44) for any section $w \in \Gamma\left(u^{-1}TS^p\right)$

$$\int\limits_{M_\mu}\left[|dw|^2 - \left(|du|^2 - \left(2\lambda - \lambda_1\left(M_\mu\right)\right)\right)|w|^2\right] \geq 0. \tag{3.45}$$

Remark 3.3.27. Let us shortly remark that we have that the first eigenvalue of the Jacobi operator J^u of u is bounded from above by $\lambda_1\left(M_\mu\right) - 2\lambda$ independent of the codomain of u. To be precise, we show the following relation between the first eigenvalue of the Jacobi operator of u, denoted $\lambda_1\left(J^u\right)$, and the first eigenvalue of the Jacobi operator of id_{M_μ}, denoted $\lambda_1\left(J^{\mathrm{id}_{M_\mu}}\right)$, namely

$$\lambda_1\left(J^u\right) \leq \lambda_1\left(J^{\mathrm{id}_{M_\mu}}\right) = \lambda_1\left(M_\mu\right) - 2\lambda.$$

Indeed, first the expression for the first eigenvalue of the Jacobi operator of the identity map on M_μ follows from the conformality of the identity map, that M_μ is an Einstein manifold with constant λ and the definition of the Jacobi operator. Now for the inequality we consider $M_\mu = S^7$ and $M_\mu = G$ compact, simply connected, simple Lie group separately.
When proving the instability of harmonic maps $\phi : S^n \to N, n \geq 3$, Xin showed in [75] that for such maps $\lambda_1\left(J^\phi\right) \leq 2 - n$. Knowing that the Einstein constant for S^n is $\lambda = n-1$ and the first non-zero eigenvalue of the Laplacian is given by $\lambda_1\left(S^n\right) = n$ (e.g. [27, 69]), we have

$$\lambda_1\left(J^\phi\right) \leq 2 - n = \lambda_1\left(S^n\right) - 2\lambda.$$

In particular, we have the claimed inequality for S^7.
For harmonically unstable Lie groups this is not as straightforward. We know, given our bi-invariant metric $g = -K$, that we have $\lambda = \frac{1}{4}$, the first non-zero eigenvalues of the Laplacian are stated explicitly in the Lie group case. In [62] Ohnita proved, using the first standard minimal immersion of $\left(G^n, \frac{\lambda_1(G)}{n}g\right)$ into $S^{\mathcal{N}}$, where $\mathcal{N}$ denotes the multiplicity of $\lambda_1(G)$, that for any non-constant, harmonic map $\tilde{\phi} : \left(G, \frac{\lambda_1(G)}{n}g\right) \to N$ we obtain $\lambda_1\left(J^{\tilde{\phi}}\right) \leq n\left(1 - \frac{1}{2\lambda_1(G)}\right)$. Thus, for non-constant, harmonic maps $\phi : (G, g) \to N$ we get as claimed

$$\lambda_1\left(J^\phi\right) \leq \lambda_1(G) - \frac{1}{2} = \lambda_1(G) - 2\lambda.$$

Finally we note that this upper bound for $\lambda_1\left(J^u\right)$ holds for any non-constant, smooth, harmonic map $u : M_\mu \to N$ while for the lower bound and thus equality for $\lambda_1\left(L_u\right)$ we need u to be of low index and $N = S^p$ for some $p \geq 2$.

Let us remark that so far there is no need to restrict to $p = 2$, that is (3.45) holds for all non-constant, globally defined, smooth, harmonic maps $u : M_\mu \to S^p, p \geq 2$, of low index as specified above.
Before proving the characterization of low index maps we need one more auxiliary result.

Lemma 3.3.28. *Let α and β be one-forms on M_μ. Then it holds*

$$\alpha \cdot \beta = \sum_{j=1}^{\mathcal{N}_\mu} \left\langle \alpha, X^j \right\rangle \left\langle \beta, X^j \right\rangle .$$

Proof. For each of our compact M_μ we found that the X^j's are the tangential components of the ε_j's and the $X^{(\alpha,\beta)}$'s are the tangential components of the $E_{(\alpha,\beta)}$'s (up to a constant factor). So there exists a projection $\pi_\mu : \mathbb{R}^{\mathcal{N}_\mu} \to M_\mu$, respectively $\pi_\mu : \mathbb{C}^{\sqrt{\mathcal{N}_\mu} \times \sqrt{\mathcal{N}_\mu}} \to M_\mu$, such that they satisfy $X^j = (\pi_\mu)_* \partial_{x_j}$, respectively $X^{(\alpha,\beta)} = (\pi_\mu)_* \partial_{x_{(\alpha,\beta)}}$. Indeed, for S^7 this projection is given by $x \mapsto \frac{x}{|x|}$. For the case of Lie groups it suffices to construct a projection onto the Lie algebra, as composition with the exponential map gives a projection onto the Lie group. For $\mathfrak{a}_l$ this projection is $X \mapsto \frac{i}{4(l+1)} \left(X + X^\dagger\right) - \frac{i}{4(l+1)^2} \operatorname{tr} \left(X + X^\dagger\right) e$ and for $\mathfrak{c}_l$ it is $X \mapsto \frac{i}{16(l+1)} \left(X + X^\dagger + \Omega X^t \Omega + \Omega \bar{X} \Omega\right)$ with

$$\Omega := \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix},$$

where I_l denotes the $l \times l$ identity matrix. With this we get

$$\alpha \cdot \beta = \pi_\mu^* \alpha \cdot \pi_\mu^* \beta = \sum_{j=1}^{\mathcal{N}_\mu} \left\langle \pi_\mu^* \alpha, \partial_{x_j} \right\rangle \left\langle \pi_\mu^* \beta, \partial_{x_j} \right\rangle = \sum_{j=1}^{\mathcal{N}_\mu} \left\langle \alpha, X^j \right\rangle \left\langle \beta, X^j \right\rangle$$

and analogously for the $X^{(\alpha,\beta)}$'s. □

Finally, we can state and prove the characterization of low index maps from Riemannian manifolds with the Killing property into the round S^2 as harmonic morphisms.

Theorem 3.3.29. *Consider the setting of Theorem 1.0.13 with $p = 2$ and let $\mathcal{N}$ denote the multiplicity of the first non-zero eigenvalue of the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$.*
Assume also that $u : M \to S^2$ is of index at most $\mathcal{N}$.
Then, the index of u is equal to $\mathcal{N}$ and u is a harmonic morphism.

Proof. Take $l \in \{1, \dots, \mathcal{N}_\mu\}$. For simplicity we will in the following denote the $X^{(\alpha,\beta)}$'s from the Lie group case by X^l as well. Moreover, we let $\times$ denote the vector product in $\mathbb{R}^3$. Then we can compute, recalling the now for any M_μ verified expression for $\Delta \left\langle du, X^l \right\rangle$ from (3.21) and using harmonicity of u,

$$\begin{aligned}
\Delta \left(u \times \left\langle du, X^l \right\rangle\right) &= -\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} u \times \left\langle du, X^l \right\rangle - 2 \sum_{i=1}^n \nabla_{e_i} u \times \nabla_{e_i} \left\langle du, X^l \right\rangle \\
&\quad - \sum_{i=1}^n u \times \nabla_{e_i} \nabla_{e_i} \left\langle du, X^l \right\rangle \\
&= |du|^2 u \times \left\langle du, X^l \right\rangle - 2 \sum_{i=1}^n \nabla_{e_i} u \times \nabla_{e_i} \left\langle du, X^l \right\rangle \\
&\quad + \left(\lambda_1(M_\mu) - 2\lambda + |du|^2\right) u \times \left\langle du, X^l \right\rangle - 2 \sum_{i=1}^n u \times \left\langle \nabla_{e_i} du, \nabla_{e_i} X^l \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left(\lambda_1(M_\mu) - 2\lambda + 2\,|du|^2\right) u \times \left\langle du, X^l \right\rangle - 2 \sum_{i=1}^{n} \nabla_{e_i} u \times \nabla_{e_i} \left\langle du, X^l \right\rangle \\
&\quad - 2 \sum_{i=1}^{n} u \times \left\langle \nabla_{e_i} du, \nabla_{e_i} X^l \right\rangle .
\end{aligned} \tag{3.46}
$$

Focussing on the second term on the right-hand side we have

$$
\sum_{i=1}^{n} \nabla_{e_i} u \times \nabla_{e_i} \left\langle du, X^l \right\rangle = \sum_{i=1}^{n} \left(\nabla_{e_i} u \times \left\langle \nabla_{e_i} du, X^l \right\rangle + \nabla_{e_i} u \times \left\langle du, \nabla_{e_i} X^l \right\rangle \right)
$$

Taking the orthogonal projection P_u of this yields, using that since u is S^2-valued the vector product of two tangent vectors lies in the corresponding normal space,

$$
P_u \sum_{i=1}^{n} \nabla_{e_i} u \times \nabla_{e_i} \left\langle du, X^l \right\rangle = P_u \sum_{i=1}^{n} \nabla_{e_i} u \times \left\langle \nabla_{e_i} du, X^l \right\rangle .
$$

For the remaining term we expand X^l and the covariant derivative of du in terms of the Killing frame, respectively its dual frame, to get

$$
\begin{aligned}
&\sum_{i=1}^{n} \nabla_{e_i} u \times \left\langle \nabla_{e_i} du, X^l \right\rangle = \sum_{i,j=1}^{n} g\left(X^l, e_j\right) \nabla_{e_i} u \times \left\langle \nabla_{e_i} du, e_j \right\rangle \\
&\quad = \sum_{j=1}^{n} g\left(X^l, e_j\right) \sum_{i=1}^{n} \nabla_{e_i} u \times \left\langle \nabla_{e_i} du, e_j \right\rangle \\
&\quad = \sum_{j=1}^{n} g\left(X^l, e_j\right) \sum_{i,k=1}^{n} \nabla_{e_i} u \times \nabla_{e_i} \nabla_{e_k} u \left\langle e_k^*, e_j \right\rangle \\
&\qquad + \sum_{j=1}^{n} g\left(X^l, e_j\right) \sum_{i,k=1}^{n} \nabla_{e_i} u \times \nabla_{e_k} u \left\langle \nabla_{e_i} e_k^*, e_j \right\rangle \\
&\quad = \sum_{j=1}^{n} g\left(X^l, e_j\right) \sum_{i=1}^{n} \nabla_{e_i} u \times \nabla_{e_i} \nabla_{e_j} u + \sum_{j=1}^{n} g\left(X^l, e_j\right) \sum_{i=1}^{n} \sum_{k \notin \{i,j\}} \nabla_{e_i} u \times \nabla_{e_k} u \left\langle \nabla_{e_i} e_k^*, e_j \right\rangle .
\end{aligned} \tag{3.47}
$$

Hence, applying P_u to (3.47) gives us, as u takes values in S^2 so that $P_u\left(\nabla_{e_i} u \times \nabla_{e_k} u\right) = 0$ for all indices i, k, and $N_u S^2$ is spanned by $u \parallel \Delta u$,

$$
\begin{aligned}
P_u \sum_{i=1}^{n} \nabla_{e_i} u \times \left\langle \nabla_{e_i} du, X^l \right\rangle &= P_u \left(\sum_{j=1}^{n} g\left(X^l, e_j\right) \sum_{i=1}^{n} \nabla_{e_i} u \times \nabla_{e_i} \nabla_{e_j} u \right) \\
&= -\sum_{i=1}^{n} \left(\nabla_{e_i} u \times u\right) g\left(X^l, e_i\right) \left(\nabla_{e_i} u\right)^2 \\
&= -\sum_{i=1}^{n} \left(\nabla_{e_i} u \times u\right) g\left(\nabla_{e_i} u, \left\langle \left(\nabla_{e_i} u\right) e_i^*, X^l \right\rangle\right) \\
&= \sum_{i=1}^{n} \left(u \times \nabla_{e_i} u\right) g\left(\nabla_{e_i} u, \left\langle \left(\nabla_{e_i} u\right) e_i^*, X^l \right\rangle\right) \\
&= \left(u \times du\right) \cdot g\left(du, \left\langle du, X^l \right\rangle\right),
\end{aligned} \tag{3.48}
$$

where we also used the harmonic map equation for u. Note that in the above it was crucial that our codomain is the two-dimensional round sphere.

Therefore, taking the projection P_u of (3.46) yields, applying Lemma 3.3.28 as well as (3.48) and assumption (b), the latter to conclude that the third term on the right-hand side of (3.46) vanishes after applying P_u since $P_u\left(u \times \sum_i \left\langle \nabla_{e_i} du, \nabla_{e_i} X^l \right\rangle\right) = u \times P_u \sum_i \left\langle \nabla_{e_i} du, \nabla_{e_i} X^l \right\rangle$,

$$\begin{aligned} P_u \Delta\left(u \times \left\langle du, X^l \right\rangle\right) &= \left(2\left|du\right|^2 + \lambda_1(M_\mu) - 2\lambda\right) u \times \left\langle du, X^l \right\rangle - 2\left(u \times du\right) \cdot g\left(du, \left\langle du, X^l \right\rangle\right) \\ &= \left(2\left|du\right|^2 + \lambda_1(M_\mu) - 2\lambda\right) u \times \left\langle du, X^l \right\rangle \\ &\quad - 2\sum_{k=1}^{\mathcal{N}_\mu} u \times \left\langle du, X^k \right\rangle \left\langle du, X^k \right\rangle \cdot \left\langle du, X^l \right\rangle. \end{aligned}$$

Now multiplying this with $u \times \left\langle du, X^l \right\rangle$ and summing over $l \in \{1, \dots, \mathcal{N}_\mu\}$ gives us

$$\begin{aligned} \sum_{l=1}^{\mathcal{N}_\mu} P_u \Delta\left(u \times \left\langle du, X^l \right\rangle\right) \cdot \left(u \times \left\langle du, X^l \right\rangle\right) &= \left(2\left|du\right|^2 + \lambda_1(M_\mu) - 2\lambda\right) \sum_{l=1}^{\mathcal{N}_\mu} \left|u \times \left\langle du, X^l \right\rangle\right|^2 \\ &\quad - 2\sum_{k,l=1}^{\mathcal{N}_\mu} \left(u \times \left\langle du, X^k \right\rangle\right) \cdot \left(u \times \left\langle du, X^l \right\rangle\right) \left\langle du, X^k \right\rangle \cdot \left\langle du, X^l \right\rangle. \end{aligned} \tag{3.49}$$

We obtain by integrating (3.49) over M_μ

$$\begin{aligned} &\sum_{l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left|d\left(u \times \left\langle du, X^l \right\rangle\right)\right|^2 - \sum_{l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left(\left|du\right|^2 + \lambda_1(M_\mu) - 2\lambda\right) \left|u \times \left\langle du, X^l \right\rangle\right|^2 \\ &\qquad = \sum_{l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left|du\right|^2 \left|u \times \left\langle du, X^l \right\rangle\right|^2 \\ &\qquad\quad - 2\sum_{k,l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left(u \times \left\langle du, X^k \right\rangle\right) \cdot \left(u \times \left\langle du, X^l \right\rangle\right) \left\langle du, X^k \right\rangle \cdot \left\langle du, X^l \right\rangle \\ &\qquad =: I_1 + I_2. \end{aligned} \tag{3.50}$$

Computing first I_1 we get, applying Lemma 3.3.28 again, since u is sphere-valued

$$\begin{aligned} I_1 &= \sum_{l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left|du\right|^2 \left|\left\langle du, X^l \right\rangle\right|^2 = \sum_{k,l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left|\left\langle du, X^k \right\rangle\right|^2 \left|\left\langle du, X^l \right\rangle\right|^2 \\ &= \sum_{k=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left|\left\langle du, X^k \right\rangle\right|^4 + 2\sum_{k<l} \int_{M_\mu} \left|\left\langle du, X^k \right\rangle\right|^2 \left|\left\langle du, X^l \right\rangle\right|^2. \end{aligned} \tag{3.51}$$

Turning to I_2 we obtain, using again that our codomain is the round S^2,

$$\begin{aligned} I_2 &= -2\sum_{k,l=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left(\left\langle du, X^k \right\rangle \cdot \left\langle du, X^l \right\rangle\right)^2 \\ &= -2\sum_{k=1}^{\mathcal{N}_\mu} \int_{M_\mu} \left|\left\langle du, X^k \right\rangle\right|^4 - 4\sum_{k<l} \int_{M_\mu} \left(\left\langle du, X^k \right\rangle \cdot \left\langle du, X^l \right\rangle\right)^2. \end{aligned} \tag{3.52}$$

Therefore, inserting (3.51) and (3.52) back into (3.50) yields

$$
\begin{aligned}
&\sum_{l=1}^{\mathcal{N}_\mu} \int\limits_{M_\mu} \left|d\left(u \times \langle du, X^l\rangle\right)\right|^2 - \sum_{l=1}^{\mathcal{N}_\mu} \int\limits_{M_\mu} \Big(|du|^2 + \lambda_1(M_\mu) - 2\lambda\Big) \left|u \times \langle du, X^l\rangle\right|^2 \\
&\quad = - \sum_{k=1}^{\mathcal{N}_\mu} \int\limits_{M_\mu} \left|\langle du, X^k\rangle\right|^4 + 2 \sum_{k<l} \int\limits_{M_\mu} \Big(\left|\langle du, X^k\rangle\right|^2 \left|\langle du, X^l\rangle\right|^2 - 2\left(\langle du, X^k\rangle \cdot \langle du, X^l\rangle\right)^2 \Big) \\
&\quad = \int\limits_{M_\mu} |du|^4 - 2\left|du \dot{\otimes} du\right|^2 .
\end{aligned}
\tag{3.53}
$$

As we have shown at the beginning of this subsection (cf. (3.45)), we get for any section $w \in \Gamma\left(u^{-1}TS^2\right)$ the estimate

$$
\int\limits_{M_\mu} |dw|^2 - \Big(|du|^2 + \lambda_1(M_\mu) - 2\lambda\Big) |w|^2 \geq 0.
$$

Thus, in light of (3.53), we have, taking $w = u \times \langle du, X^l\rangle$ for each $l \in \{1, \ldots, \mathcal{N}_\mu\}$ and summing over l,

$$
\int\limits_{M_\mu} |du|^4 - 2\left|du \dot{\otimes} du\right|^2 \geq 0. \tag{3.54}
$$

Now, note that since u has S^2 as codomain we have at each $x \in M_\mu$ that $\operatorname{rank}(du_x) \leq 2$. If $\operatorname{rank}(du_x) = 0$, then $du_x \equiv 0$. Next we consider points $x \in M_\mu$ with $\operatorname{rank}(du_x) = 2$. In this case we can take an orthonormal basis $\{f_1, f_2\}$ of the horizontal space $\mathcal{H}_x = \ker(du_x)^\perp$ and evaluate the integrand of (3.54) with respect to this basis

$$
\begin{aligned}
|du_x|^4 - 2\left|du_x \dot{\otimes} du_x\right|^2 &= |\partial_{f_1} u|^4 + |\partial_{f_2} u|^4 + 2\,|\partial_{f_1} u|^2 |\partial_{f_2} u|^2 - 2\,|\partial_{f_1} u|^4 - 2\,|\partial_{f_2} u|^4 \\
&\quad - 4\,|(\partial_{f_1} u) \cdot (\partial_{f_2} u)|^2 \\
&= - |\partial_{f_1} u|^4 - |\partial_{f_2} u|^4 + 2\,|\partial_{f_1} u|^2 |\partial_{f_2} u|^2 - 4\,|(\partial_{f_1} u) \cdot (\partial_{f_2} u)|^2 \\
&= - \Big(|\partial_{f_1} u|^2 - |\partial_{f_2} u|^2\Big)^2 - 4\,|(\partial_{f_1} u) \cdot (\partial_{f_2} u)|^2 \leq 0.
\end{aligned}
\tag{3.55}
$$

Thus, for all $x \in M_\mu$ with $\operatorname{rank}(du_x) \in \{0, 2\}$ we find

$$
|du_x|^4 - 2\left|du_x \dot{\otimes} du_x\right|^2 \leq 0.
$$

Finally, take $x \in M_\mu$ with $\operatorname{rank}(du_x) = 1$. Then we have $|du_x|^4 = \left|du_x \dot{\otimes} du_x\right|^2$, hence in particular we get

$$
|du_x|^4 - 2\left|du_x \dot{\otimes} du_x\right|^2 < 0.
$$

But since at all other points $x \in M_\mu$ we have $|du_x|^4 - 2\left|du_x \dot{\otimes} du_x\right|^2 \leq 0$ and (3.54) holds, this case cannot occur. This means that at each point $x \in M_\mu$ the rank of du_x is either zero or two and we must also have, combining (3.54) and (3.55),

$$
|du_x|^4 - 2\left|du_x \dot{\otimes} du_x\right|^2 = 0.
$$

In the case of rank two this implies, recalling (3.55),

$$|\partial_{f_1} u|^2 = |\partial_{f_2} u|^2 \text{ and } (\partial_{f_1} u) \cdot (\partial_{f_2} u) = 0,$$

which yields that du_x restricted to $\mathcal{H}_x$ is conformal. Thus, together with $du_x \equiv 0$ at all other points, this implies that u is horizontally weakly conformal. By Fuglede and Ishihara's characterization of harmonic morphisms (e.g. [32, 44]) this is, together with the assumed harmonicity, equivalent to u being a harmonic morphism, which is what we claimed. $\square$

At several points in the proof it is crucial that the codomain of u is S^2 and not a higher-dimensional S^p as in this way we can e.g. conclude that the vector product of two vectors in $T_u S^2$ must lie in the normal space $N_u S^2$, yielding that the tangential projection must vanish. Hence it is not clear how this line of reasoning can be extended to higher-dimensional spheres as codomain.
Furthermore, we also can not easily replace the round 2-sphere by an arbitrary surface N^2 as this would alter, among others, the harmonic map equation, changing therefore the computation significantly. However, post-composition with a biholomorphic map into some surface N^2 is possible and preserves being a harmonic morphism.

Remark. Observe that for a conformal surface N^2, which is conformally equivalent to S^2 with corresponding biholomorphic map $\psi : N^2 \to S^2$, given a map $u : M_\mu \to S^2$ as specified in Theorem 3.3.29, we also obtain a harmonic morphism into N^2, namely $\psi^{-1} \circ u : M_\mu \to N^2$. Indeed, 3.3.29 shows that u is a harmonic morphism, and we know (e.g. [4]) that ψ is a bijective harmonic morphism, thus ψ^{-1} is a harmonic morphism as well (e.g. [4], p. 111). So $\psi^{-1} \circ u$ is, as the composition of two harmonic morphisms, a harmonic morphism as well (e.g. [4], Prop. 4.1.3 (i)).

Appendix A

Appendices

A.1 Additional material to dimension estimates

In this section we give, as announced in section 2.3, the proofs by Lin and Zhang of the upper and lower bound on $\sum_{i=1}^{k}\int_{P_R(t_0,x_0)} u_i^2$ using our L^2-mean value inequality for $\partial_t - L$. As the proof of Lemma 2.3.2 is a simple contradiction argument, it will be omitted here. Let us emphasize that since the specific operator whose solutions are considered only enters through the L^2-mean value inequality said solutions satisfy, the proofs that we will recapitulate now are exactly those of [56], Lem. 3.1 and of the claim on page 2019 in [56].

Proof of Lemma 2.3.1

Proof. Here we need to modify the proof of Lin and Zhang slightly as we have proven our mean value inequality only for radii $r \geq 1$ instead of $r > 0$ as supposed in [56].
Let $(t,x) \in P_R(t_0,x_0)$ be arbitrary but fixed. Then we can find coefficients $\lambda_1,\dots,\lambda_k \in [0,1]$ satisfying $\sum_{i=1}^{k}\lambda_i^2 = 1$ so that for $v := \sum_{i=1}^{k}\lambda_i u_i \in \langle \Lambda, U\rangle$ we have

$$v^2(t,x) = \sum_{i=1}^{k} u_i^2(t,x).$$

Define $r(t,x) := (2+\varepsilon)R - d_p\left((t,x),(t_0,x_0)\right) \geq 1$. Then, by the choice of (t,x) and since d_p is a distance function, we get $P_{r(t,x)}(t,x) \subseteq P_{(2+\varepsilon)R}(t_0,x_0)$.
As we have seen, under the assumptions of a Neumann-Poincaré inequality and the volume doubling property, we obtain an L^2-mean value inequality for solutions to $\partial_t u = Lu$ on parabolic cylinders of radii $r \geq 1$, so that we get here

$$v^2(t,x) \leq \frac{C_M}{r^2 \operatorname{Vol}\left(B_r(x)\right)} \int\limits_{Q_r(t,x)} v^2(s,y).$$

To transfer this to a mean value inequality on truncated paraboloids, we need a comparison of the volumes of parabolic cylinders and truncated paraboloids, i.e. we need to show

$$2^{-C_D-2} \leq \frac{\operatorname{Vol}\left(P_r(t,x)\right)}{\operatorname{Vol}\left(Q_r(t,x)\right)} \leq 1. \tag{A.1}$$

For this, note first that the definition of the paraboloid gives us

$$\operatorname{Vol}\left(P_r(t,x)\right) = \int\limits_{B_r(x)} \int\limits_{t-(r-d(x,y))^2}^{t} 1 = \int\limits_{B_r(x)} \left(r - d(x,y)\right)^2 \leq r^2 \int\limits_{B_r(x)} 1$$

$$= \int_{B_r(x)} \int_{t-r^2}^{t} 1 = \mathrm{Vol}\left(Q_r(t,x)\right),$$

which is the second inequality in (A.1). Moreover, starting in the same way we get, assuming the volume doubling property with constant C_D,

$$\begin{aligned}\mathrm{Vol}\left(P_r(t,x)\right) &\geq \int_{B_{\frac{r}{2}}(x)} \left(r - d(x,y)\right)^2 \geq \frac{r^2}{4} \int_{B_{\frac{r}{2}}(x)} 1 = \frac{r^2}{4} \mathrm{Vol}\left(B_{\frac{r}{2}}(x)\right) \\ &\geq \frac{r^2}{4} 2^{-C_D} \mathrm{Vol}\left(B_r(x)\right) = 2^{-C_D-2} \mathrm{Vol}\left(Q_r(t,x)\right).\end{aligned}$$

This proves the first inequality in (A.1), finishing the proof of (A.1).
Using this volume comparison, we can conclude that we also have an L^2-mean value inequality on paraboloids with possibly a changed constant, that is

$$v^2(t,x) \leq \frac{C}{\mathrm{Vol}\left(P_r(t,x)\right)} \int_{P_r(t,x)} v^2(s,y). \tag{A.2}$$

Furthermore, the volume comparison (A.1) implies together with our volume doubling property for balls a volume comparison between truncated paraboloids, i.e. for every $r_2 > r_1 > 0$ we get

$$\frac{\mathrm{Vol}\left(P_{r_2}(t,x)\right)}{r_2^{C_D+2}} \leq \frac{\mathrm{Vol}\left(B_{r_2}(t,x)\right)}{r_2^{C_D}} \leq K \frac{\mathrm{Vol}\left(B_{r_1}(x)\right)}{r_1^{C_D}} \leq K \frac{\mathrm{Vol}\left(P_{r_1}(t,x)\right)}{r_1^{C_D+2}}. \tag{A.3}$$

Taking $r_2 = 2R$ and $r_1 = r(t,x)$, this volume comparison yields

$$\begin{aligned}\mathrm{Vol}\left(P_r(t,x)\right) &\geq K^{-1} \mathrm{Vol}\left(P_{2R}(t,x)\right) \left(\frac{(2+\varepsilon)R - (d(x,x_0) + \sqrt{|t-t_0|})}{2R}\right)^{C_D+2} \\ &\geq K^{-1} R^{-(C_D+2)} \mathrm{Vol}\left(P_R(t_0,x_0)\right) \left((2+\varepsilon)R - (d(x,x_0) + \sqrt{|t-t_0|})\right)^{C_D+2},\end{aligned} \tag{A.4}$$

where we have used that applying that due to $(t,x) \in P_R(t_0,x_0)$ we get $B_{\frac{R}{2}}(x_0) \subseteq B_{\frac{3R}{2}}(x)$, the volume doubling property and (A.1) we can estimate

$$\begin{aligned}\mathrm{Vol}\left(P_{2R}(t,x)\right) &\geq \int_{B_{\frac{3R}{2}}(x)} \left(2R - d(x,y)\right)^2 \geq \frac{R^2}{4} \int_{B_{\frac{3R}{2}}(x)} 1 = \frac{R^2}{4} \mathrm{Vol}\left(B_{\frac{3R}{2}}(x)\right) \\ &\geq \frac{R^2}{4} \mathrm{Vol}\left(B_{\frac{R}{2}}(x_0)\right) \geq 2^{-C_D-2} R^2 \mathrm{Vol}\left(B_R(x_0)\right) \\ &= 2^{-C_D-2} \mathrm{Vol}\left(Q_R(t_0,x_0)\right) \geq 2^{-C_D-2} \mathrm{Vol}\left(P_R(t_0,x_0)\right).\end{aligned}$$

Applying (A.4), (A.1) and the inclusion $P_{r(t,x)}(t,x) \subseteq P_{(2+\varepsilon)R}(t_0,x_0)$ to our mean value inequality (A.2) gives us

$$\begin{aligned}v^2(t,x) &\leq K \left((2+\varepsilon)R - (d(x,x_0) + \sqrt{|t-t_0|})\right)^{-(C_D+2)} \frac{R^{C_D}}{\mathrm{Vol}\left(B_R(x_0)\right)} \int_{P_r(t,x)} v^2(s,y) \\ &\leq K \left((2+\varepsilon)R - (d(x,x_0) + \sqrt{|t-t_0|})\right)^{-(C_D+2)} \frac{R^{C_D}}{\mathrm{Vol}\left(B_R(x_0)\right)} \int_{P_{(2+\varepsilon)R}(t_0,x_0)} v^2(s,y).\end{aligned}$$

Integrating now over $P_R(t_0,x_0)$, using $v^2(t,x)=\sum_i u_i^2(t,x)$ and estimating the integral of v^2 on the right-hand side by taking the supremum over $v\in\langle\Lambda,U\rangle$, we obtain

$$\begin{aligned}\sum_{i=1}^k \int\limits_{P_R(t_0,x_0)} u_i^2(t,x) &\le \frac{K\,R^{C_D}}{\mathrm{Vol}\,(B_R(x_0))}\int\limits_{P_R(t_0,x_0)} \Big((2+\varepsilon)R-(d(x,x_0)+\sqrt{|t-t_0|})\Big)^{-C_D-2} \\ &\quad\times \sup_{v\in\langle\Lambda,U\rangle}\int\limits_{P_{(2+\varepsilon)R}(t_0,x_0)} v^2(s,y) \\ &=: \frac{K\,R^{C_D}}{\mathrm{Vol}\,(B_R(x_0))}\, I\times \sup_{v\in\langle\Lambda,U\rangle}\int\limits_{P_{(2+\varepsilon)R}(t_0,x_0)} v^2(s,y).\end{aligned}$$

To finish the proof of the upper bound it remains to estimate the integral I on the right-hand side. Using the definition of the truncated paraboloid, setting then $A(x):=R-d(x,x_0)$ and substituting $s=\sqrt{t}$, we find

$$\begin{aligned}I &= \int\limits_{B_R(x_0)}\int\limits_0^{(R-d(x,x_0))^2}\Big((2+\varepsilon)R-(d(x,x_0)+\sqrt{t})\Big)^{-C_D-2} \\ &= \int\limits_{B_R(x_0)}\int\limits_0^{A(x)}\frac{2s}{((1+\varepsilon)R+A(x)-s)^{C_D+2}} \le \int\limits_{B_R(x_0)} 2A(x)\int\limits_0^{A(x)}(R\varepsilon+A(x)-s)^{-C_D-2} \\ &\le \frac{2}{C_D+1}\int\limits_{B_R(x_0)} A(x)(R\varepsilon)^{-C_D-1} \le \frac{2}{C_D+1}\,R^{-C_D}\varepsilon^{-C_D-1}\,\mathrm{Vol}\,(B_R(x_0))\,.\end{aligned}$$

Hence, $I\le K\,\varepsilon^{-C_D-1}R^{-C_D}\,\mathrm{Vol}\,(B_R(x_0))$. Therefore, we finally conclude

$$\sum_{i=1}^k\int\limits_{P_R(t_0,x_0)} u_i^2(t,x)\le C\,\varepsilon^{-C_D-1}\sup_{v\in\langle\Lambda,U\rangle}\int\limits_{P_{(2+\varepsilon)R}(t_0,x_0)} v^2.$$

□

Proof of Lemma 2.3.3

Proof. Since the proof of the lower bound only uses the volume doubling property and the polynomial growth of the solutions, it transfers without changes from [56]. However we include it for completeness.
We will denote by $\mathrm{tr}_{R'}A_R$, respectively $\det_{R'}A_R$, the trace, respectively determinant, of the inner product A_R with respect to $A_{R'}$, i.e. the trace, respectively determinant, of the matrix $(A_R(v_i,v_j))_{i,j=1,\dots,k}$ for an orthonormal basis $\{v_i\}_i$ of K with respect to $A_{R'}$.
Let us suppose, for contradiction, that the claim does not hold. Then, for given $\beta>1$ and $\delta>0$ there exist $(t_0,x_0)\in\mathbb{R}^-\times M$ and $R_0\ge 1$ so that for all $R\ge R_0$

$$\mathrm{tr}_{\beta R}A_R=\sum_{i=1}^k\int\limits_{P_R(t_0,x_0)} u_i^2(t,x)<k\,\beta^{-(2q+C_D+2+\delta)},$$

where $\{u_i\}_i$ is an orthonormal basis of K with respect to $A_{\beta R}$. This implies

$$\Big(\det_{\beta R}A_R\Big)^{\frac{1}{k}}\le\frac{\mathrm{tr}_{\beta R}A_R}{k}<\beta^{-(2q+C_D+2+\delta)}.$$

Since $\det_{\beta R} A_R = \left(\det_R A_{\beta R}\right)^{-1}$, this gives

$$\det_R A_{\beta R} > \beta^{k(2q+C_D+2+\delta)}.$$

By iterating this for $R, \beta R, \ldots, \beta^j R$, we deduce

$$\det_R A_{\beta^j R} > \beta^{jk(2q+C_D+2+\delta)}. \tag{A.5}$$

However, from the polynomial growth assumption on the u_i's and the volume bound (A.3) on the paraboloids, we obtain

$$\det_R A_{\beta^j R} \leq k!\, C^k \left(\beta^j R\right)^{2q+C_D+2} \operatorname{Vol}\left(B_1(x_0)\right). \tag{A.6}$$

For large j combining (A.5) and (A.6) gives us the desired contradiction and therefore prove the lower bound. $\square$

A.2 Second fundamental form of conformal submersions

One of the advantages of having horizontal and vertical vector fields with respect to a horizontally conformal submersion is that we can give explicit expressions for the second fundamental form of the submersion with respect to these vector fields. Thus, with a frame consisting only of horizontal and vertical vector fields we obtain expressions for the second fundamental form with respect to any vector fields.

Lemma A.2.1. *([4], Lem. 4.5.1)*
Suppose that $\phi : (M, g) \to (N, h)$ is a horizontally conformal submersion with dilation λ. Then, for any horizontal vector fields X, Y and vertical vector fields V, W,

(a) $\nabla d\phi\,(X, Y) = X\,(\ln\lambda)\, d\phi\,(Y) + Y\,(\ln\lambda)\, d\phi\,(X) - g\,(X, Y)\, d\phi\,(\nabla\,(\ln\lambda))$*;*

(b) $\nabla d\phi\,(V, W) = -d\phi\,(\mathcal{H}\,(\nabla_V W))$ *with $\mathcal{H}(\cdot)$ the horizontal projection;*

(c) $\nabla d\phi\,(X, V) = -d\phi\,(\nabla_X V)$*.*

In particular,

(1) $\nabla d\phi\,(X, Y) = 0$ *for all horizontal vector fields X, Y if ϕ is horizontally homothetic, i.e.* $\mathcal{H}\,(\nabla\lambda) = 0$*;*

(2) $\nabla d\phi\,(V, W) = 0$ *for all vertical vector fields V, W if and only if the fibres of ϕ are totally geodesic;*

(3) $\nabla d\phi\,(X, V) = 0$ *for all horizontal vector fields X and all vertical vector fields V if and only if the foliation associated to ϕ is Riemannian and has integrable horizontal distribution.*

A.3 Construction of the flows in the S^7-case

While in subsection 3.3.3 we simply verified the flows for the gradient vector fields arising from a basis of the eigenspace of the Laplacian corresponding to its first non-zero eigenvalue, in the S^7-case, by showing that the maps actually fulfill the flow equation, we will give here the construction of these flows. We consider in the following without loss of generality the case $j = 1$. We will see that the construction for any other $j \in \{2, \ldots, 8\}$ is the same, up to minor changes.
Since we have seen that the vector field X^1 on S^7 is conformal, its flow ϕ^1 must consist of

conformal transformations of S^7. Hence, by the Liouville theorem for conformal maps, for every time $t \in [0, +\infty[$, the map ϕ_t^1 must be a Möbius transformation of S^7, i.e. of the form

$$\phi_t^1(x) = b(t) + \frac{\alpha(t)A(t)\,(x - a(t))}{|x - a(t)|^\varepsilon},\ x \in S^7,$$

where $a(t), b(t) \in \mathbb{R}^8$, $\alpha(t) \in \mathbb{R}$, $A(t) \in \mathrm{O}(8)$ and $\varepsilon \in \{0, 2\}$.
As in our case the approach with $\varepsilon = 2$ leads to a contradiction, we must have $\varepsilon = 0$, that is

$$\phi_t^1(x) = b(t) + \alpha(t)A(t)\,(x - a(t))\,,\ x \in S^7.$$

Using this expression in the flow equation for X^1, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}b(t) + \left(\frac{\mathrm{d}}{\mathrm{dt}}\alpha(t)\right)A(t)\,(x-a(t)) + \alpha(t)\left(\frac{\mathrm{d}}{\mathrm{dt}}A(t)\right)(x-a(t)) - \alpha(t)A(t)\left(\frac{\mathrm{d}}{\mathrm{dt}}a(t)\right)\\
&= \varepsilon_1 - g\left(b(t), \varepsilon_1\right)b(t) - \alpha(t)\,g\left(A(t)\,(x-a(t))\,, \varepsilon_1\right)b(t)\\
&\quad - \alpha(t)\,g\left(b(t), \varepsilon_1\right)A(t)\,(x-a(t)) - \alpha^2(t)\,g\left(A(t)\,(x-a(t))\,, \varepsilon_1\right)A(t)\,(x-a(t))
\end{aligned}$$

with initial values for the coefficients

$$b(0) = 0,\ \alpha(0) = 1,\ a(0) = 0, A(0) = \mathrm{Id}.$$

Suppose, for simplicity, that the coefficients are of the form

$$\begin{aligned}
\alpha(t) &= 1 + \kappa(t) \quad \text{with} \quad \kappa(0) = 0,\\
A(t) &= (1 + \mu(t))\,\mathrm{Id} \quad \text{with} \quad \mu(0) = 0,\\
a(t) &= \xi(t)\,\varepsilon_1 \quad \text{with} \quad \xi(0) = 0,\\
b(t) &= \lambda(t)\,\varepsilon_1 \quad \text{with} \quad \lambda(0) = 0.
\end{aligned}$$

Then, upon abbreviating $\Lambda(t) := (1 + \kappa(t))\,(1 + \mu(t))$, the flow equation turns into

$$\begin{aligned}
&\left[\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t) + \lambda(t)\Lambda(t) + \Lambda^2(t)\,(x_1 - \xi(t))\right]x\\
&= \left[-\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) + \left(\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t)\right)\xi(t) + \Lambda(t)\left(\frac{\mathrm{d}}{\mathrm{dt}}\xi(t)\right) + 1 - \lambda^2(t) - \Lambda(t)\lambda(t)\,(x_1 - \xi(t))\right]\varepsilon_1\\
&+ \left[\Lambda(t)\lambda(t)\xi(t) + \Lambda^2(t)\xi(t)\,(x_1 - \xi(t))\right]\varepsilon_1.
\end{aligned}$$

Since we can write x as $x = \sum_{k=1}^{8} x_k\varepsilon_k$ with $x_k := g\left(x, \varepsilon_k\right)$, we find

$$\begin{aligned}
&\left[\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t) + \lambda(t)\Lambda(t) + \Lambda^2(t)\,(x_1 - \xi(t))\right]\sum_{k=2}^{8} x_k\varepsilon_k\\
&= \left[-\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) + \left(\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t)\right)(\xi(t) - x_1) + \Lambda(t)\left(\frac{\mathrm{d}}{\mathrm{dt}}\xi(t)\right) + 1 - \lambda^2(t)\right]\varepsilon_1\\
&- \left[2\Lambda(t)\lambda(t)\,(x_1 - \xi(t)) + \Lambda^2(t)\,(\xi(t) - x_1)^2\right]\varepsilon_1.
\end{aligned}$$

Because of the linear independence of the ε_k's, this implies that the coefficient of each of the ε_k's must vanish. Consequently, it suffices to solve the equations

$$\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t) = -\lambda(t)\Lambda(t) + (\xi(t) - x_1)\,\Lambda^2(t), \tag{A.7}$$

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) + \left(\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t)\right)(\xi(t) - x_1) + \Lambda(t)\left(\frac{\mathrm{d}}{\mathrm{dt}}\xi(t)\right) &= \lambda^2(t) - 1 + 2\lambda(t)\Lambda(t)\,(x_1 - \xi(t)) \\
&\quad + \Lambda^2(t)\,(x_1 - \xi(t))^2\,.
\end{aligned}
\tag{A.8}
$$

Using (A.7) in (A.8) gives us

$$
-\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) + \Lambda(t)\left(\frac{\mathrm{d}}{\mathrm{dt}}\xi(t)\right) = \lambda^2(t) - 1 + \lambda(t)\Lambda(t)\,(x_1 - \xi(t))\,. \tag{A.9}
$$

Now consider

$$
\lambda(t) := \frac{e^t - e^{-t}}{e^t + e^{-t}} = \tanh(t)\,.
$$

Then, λ solves the Riccati equation

$$
\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) = 1 - \lambda^2(t) \quad \text{with} \quad \lambda(0) = 0.
$$

Inserting therefore λ into (A.9) yields

$$
\Lambda(t)\left[\frac{\mathrm{d}}{\mathrm{dt}}\xi(t) + \frac{e^t - e^{-t}}{e^t + e^{-t}}\,(\xi(t) - x_1)\right] = 0.
$$

So, since $\Lambda(t) \neq 0$, ξ must solve

$$
\frac{\mathrm{d}}{\mathrm{dt}}\xi(t) = \frac{e^{-t} - e^t}{e^t + e^{-t}}\,(\xi(t) - x_1) \quad \text{with} \quad \xi(0) = 0. \tag{A.10}
$$

As $\frac{\mathrm{d}}{\mathrm{dt}}\xi(t) = \frac{\mathrm{d}}{\mathrm{dt}}(\xi(t) - x_1)$, a solution to (A.10) is given by

$$
\xi(t) = x_1\left(1 - \frac{2e^t}{e^{2t} + 1}\right).
$$

Now it only remains to determine the coefficient Λ. For this, inserting the solutions for λ and ξ we just obtained into (A.7), we get

$$
\frac{\mathrm{d}}{\mathrm{dt}}\Lambda(t) = \frac{e^{-t} - e^t}{e^t + e^{-t}}\,\Lambda(t) - \frac{2x_1 e^t}{e^{2t} + 1}\,\Lambda^2(t). \tag{A.11}
$$

Substituting $w := \frac{1}{\Lambda}$ in (A.11), we obtain the linear equation

$$
\frac{\mathrm{d}}{\mathrm{dt}}w(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}\,w(t) + \frac{2x_1 e^t}{e^{2t} + 1} \quad \text{with} \quad w(0) = 1. \tag{A.12}
$$

Now, (A.12) can be solved via variation of constants to give

$$
w(t) = \frac{1}{2}\left[\left(e^t + e^{-t}\right)(1 + x_1) - 2x_1\right].
$$

Therefore, we have

$$
\Lambda(t) = \frac{2e^t}{(e^{2t} + 1)(1 + x_1) - 2x_1}.
$$

So, collecting the solutions found for the coefficients, we can conclude for the flow

$$
\begin{aligned}
\phi_t^1(x) &= b(t) + \alpha(t)A(t)\,(x - a(t)) = \lambda(t)\,\varepsilon_1 + (1 + \kappa(t))\,(1 + \mu(t))\,(x - \xi(t)\,\varepsilon_1) \\
&= \lambda(t)\,\varepsilon_1 + \Lambda(t)\,(x - \xi(t)\,\varepsilon_1)
\end{aligned}
$$

$$= \frac{e^t - e^{-t}}{e^t + e^{-t}} \varepsilon_1 + \frac{2e^t}{(e^{2t}+1)(1+x_1) - 2x_1} \left(x - \left(x_1 - \frac{2x_1 e^t}{e^{2t}+1} \right) \varepsilon_1 \right).$$

As remarked at the beginning of the construction, replacing 1 by any other $j \in \{2, \ldots, 8\}$ and thus ε_1 by ε_j as well as x_1 by x_j, we can construct ϕ^j exactly as we did for ϕ^1.

A.4 Representation theory of Lie groups

In the following we will give a very brief introduction to the representation theory of complex Lie groups and Lie algebras without any proofs. All of this, including proofs of the results, can be found, except otherwise stated, in the book by Kirillov Jr. ([46]). We start off by recalling some definitions and results on (matrix) Lie groups and continue with establishing the basic notions of representation theory and results.

A.4.1 Some basics on (matrix) Lie groups

For completeness we begin by recalling the definition of a Lie group.

Definition A.4.1. ([61], p. 446)
A Lie group G is a smooth manifold that is also a group with smooth operations, i.e. the maps

$$\mu : G \times G \to G, \, (a,b) \mapsto ab$$

and

$$\zeta : G \to G, \, a \mapsto a^{-1}$$

are both smooth. A Lie subgroup H of a Lie group G is an abstract subgroup of G, which is also an immersed submanifold of G.
In case G is a Lie subgroup of the general linear group $\mathrm{GL}(n;\mathbb{C})$, we call G a matrix Lie group.

In the following we will always denote the identity element of a Lie group G by e.
Furthermore, for any element $x \in G$ we let L_x, respectively R_x, denote the left, respectively right, multiplication by x with respect to the group action μ, that is

$$\begin{aligned} L_x &: G \to G, \, y \mapsto xy, \quad \text{respectively} \\ R_x &: G \to G, \, y \mapsto yx. \end{aligned}$$

In section 3.3, in particular in subsection 3.3.4, we need a specific Riemannian metric on G, namely a bi-invariant metric.

Definition A.4.2. ([1], Def.s 2.22, 2.23)
A Riemannian metric g on a Lie group G is left-invariant if L_x is an isometry for any $x \in G$, i.e. if for all $x, y \in G$ and $V, W \in T_yG$ we have

$$g_{xy}\left((dL_x(y))\, V, (dL_x(y))\, W \right) = g_y(V, W).$$

Similarly g is right-invariant if the R_x, $x \in G$, are isometries.
In the case that g is both left- and right-invariant it is said to be bi-invariant.

Next, we note that any Lie group possesses a trivial tangent bundle, which yields that the tangent space at the identity element determines any other tangent space.

Proposition A.4.3. *([2], Prop. 1.5)*
Any Lie group G is parallelizable, that is $TG \cong G \times T_eG$. The isomorphism is given by

$$T_xG \ni X(x) \mapsto \left(x, (dL_{x^{-1}}(x))\, X(x) \right).$$

In particular, $T_xG \cong (dL_x(e))\,(T_eG)$.

As a consequence, since every tangent bundle is orientable, any Lie group G must be orientable. Moreover if G is connected, there are exactly two orientations on G.
Further we recall the definition of a left-invariant vector field.

Definition A.4.4. ([61], Def. B.6)
A vector field X on a Lie group G is left-invariant provided that for all $x, y \in G$

$$(dL_x(y))\, X(y) = \, X(xy).$$

In particular, in light of the triviality of the tangent bundle, a left-invariant vector field is determined by its value at e, that is for any $x \in G$

$$X(x) = \, (dL_x(e))\, X(e).$$

Let us also recall the definition of a complex Lie algebra.

Definition A.4.5. ([61])
A Lie algebra over $\mathbb{C}$ is a real vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called its Lie bracket, such that for all $X, Y, Z \in \mathfrak{g}$ we have

(1) skew-symmetry: $[X, Y] = -[Y, X]$;

(2) Jacobi identity: $[[X, Y]Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

In the case of a matrix Lie algebra the Lie bracket is just the usual commutator $[X, Y] = XY - YX$, where XY denotes matrix multiplication of X and Y.
Given a Lie group G we have the notion of a Lie algebra associated with G.

Lemma A.4.6. *([61])*
Let G be a Lie group and let $\mathfrak{g}$ denote the set of all left-invariant vector fields on G. Equipping $\mathfrak{g}$ with the usual addition of vector fields and scalar multiplication by real numbers, $\mathfrak{g}$ is a Lie algebra, called the Lie algebra of G.
Moreover, the function $\mathfrak{g} \to T_eG$, $X \mapsto \, X(e)$ is a linear isomorphism. Therefore we usually identify $\mathfrak{g}$ with T_eG and use the two notions interchangeably.

Now, on matrix Lie groups we can say more about the differential of the multiplication operators L_x, R_x introduced above.

Lemma A.4.7. *([1])*
Let G be a matrix Lie group. Then, for any $x \in G$, the differentials of the left and right multiplication operators are given by

$$\begin{aligned} dL_x(e) &: \mathfrak{g} \ni v \mapsto \, xv, \\ dR_x(e) &: \mathfrak{g} \ni v \mapsto \, vx. \end{aligned}$$

In particular, the tangent space of a matrix Lie group G at any $x \in G$ is given by left multiplication of the Lie algebra $\mathfrak{g}$ with x.
On a compact Lie group equipped with a bi-invariant metric the set of left-invariant vector fields coincides with the set of Killing vector fields.

Theorem A.4.8. *([15], p. 323)*
Let G be a compact Lie group with bi-invariant metric g. Then the left-invariant vector fields on G are the Killing vector fields of constant length on G.

As we want to determine flows of vector fields on Lie groups, we first need the notion of a one-parameter subgroup on G.

Definition A.4.9. ([61], Def. B.9)
A one-parameter subgroup in a Lie group G is a smooth homomorphism $\alpha : \, (\mathbb{R}, +) \to \, G$.

Then the integral curves of elements of $\mathfrak{g}$ are determined.

Proposition A.4.10. *([61], Prop. B.10)*
The one-parameter subgroups of a Lie group G are exactly the maximal integral curves starting at e of elements of its Lie algebra $\mathfrak{g}$.

With this in mind we can define an exponential map on the Lie algebra of a Lie group.

Definition A.4.11. ([61], Def. B.11)
Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. The Lie exponential map is defined as

$$\exp :\ \mathfrak{g} \to G,\, X \mapsto \alpha_X(1),$$

where α_X is the one-parameter subgroup of $X \in \mathfrak{g}$.

However we can in the case of matrix Lie groups also define an exponential map on $\mathfrak{gl}(n;\mathbb{C})$ by generalizing the series expansion of the exponential function.

Definition A.4.12. The matrix exponential map $\exp :\ \mathfrak{gl}(n;\mathbb{C}) \to \mathrm{GL}(n;\mathbb{C})$ is defined as

$$\exp(X) := \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

In the case of Lie groups with bi-invariant metrics we consider in 3.3.4 this ambiguity resolves as the two notions of exponential maps coincide.

Theorem A.4.13. *([33])*
The Lie exponential map and the matrix exponential map at identity agree for Lie groups endowed with bi-invariant metrics.

A.4.2 Basic notions of representation theory

Given a Lie group G by a representation of G we mean a vector space V together with a group morphism $\rho :\ G \to \mathrm{End}(V)$. In the case of V being finite-dimensional we also require for the map

$$G \times V \to V, \quad (g, v) \mapsto \rho(g)\, v$$

to be smooth. Similarly, a representation of a Lie algebra $\mathfrak{g}$ is a vector space V together with a morphism $\rho :\ \mathfrak{g} \to \mathfrak{gl}(V)$.
Then a subrepresentation of a representation (V, ρ) of G, respectively $\mathfrak{g}$, is a vector subspace $W \subseteq V$, which is stable under the action, that is

$$\rho(g)\, W \subseteq W \quad \text{for all} \quad g \in G, \quad \text{respectively} \quad \rho(x)\, W \subseteq W \quad \text{for all} \quad x \in \mathfrak{g}.$$

We will call a representation (V, ρ) irreducible if it admits no other subrepresentations than $\{0\}$ and V. Moreover, if (V, ρ) is a complex representation of a Lie group G, we call it unitary if there exists a G-invariant inner product $(\cdot, \cdot)$, that is for every $g \in G$ and every $v, w \in V$

$$(\rho(g)\, v, w) = (v, w).$$

A representation (V, ρ) of a Lie algebra $\mathfrak{g}$ is said to be unitary if there is a $\mathfrak{g}$-invariant inner product $(\cdot, \cdot)$, i.e. for each $x \in \mathfrak{g}$ and $v, w \in V$

$$(\rho(x)\, v, w) + (v, \rho(x)\, w) = 0.$$

In the case of compact Lie groups however this is automatically satisfied for finite-dimensional representations.

Theorem A.4.14. *Any finite-dimensional representation of a compact Lie group is unitary.*

Further, we also need to introduce maps between representations. For this, given two representations $(V, \rho_V), (W, \rho_W)$ of a Lie group G, a morphism between (V, ρ_V) and (W, ρ_W)

is defined to be a linear map $f : V \to W$ so that for all $g \in G$ we have

$$f\,\rho_V(g) = \rho_W(g)\,f.$$

The space of such morphisms is denoted by $\mathrm{Hom}_G(V, W)$. The space of morphisms between representations (V, ρ_V) and (W, ρ_W) of a Lie algebra $\mathfrak{g}$, denoted by $\mathrm{Hom}_{\mathfrak{g}}(V, W)$, is defined analogously. As we will see now, there is a close connection between representations of Lie groups and representations of their Lie algebras, and thus between the corresponding spaces of morphisms.

Theorem A.4.15. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

(a) Every representation $\rho : G \to \mathrm{End}(V)$ of G defines a representation $(d\rho)(e) : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$, and every morphism of representations of G is automatically a morphism of representations of $\mathfrak{g}$.

(b) If G is a simply connected Lie group, then the map $\rho \mapsto (d\rho)(e)$ gives an equivalence of categories of representations of G and representations of $\mathfrak{g}$. In particular, every representation of $\mathfrak{g}$ can be uniquely lifted to a representation of G, and $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

In the case of a compact, connected matrix Lie group G equipped with a bi-invariant metric g, the surjectivity of the matrix exponential (e.g. [36], Cor. 11.10)

$$\exp : \mathfrak{gl}(n; \mathbb{C}) \to \mathrm{GL}(n; \mathbb{C})$$

yields that the unique lift of a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ to a representation $U : G \to \mathrm{GL}(V)$ is given by

$$U(x) = U(\exp(X)) := \exp(\rho(X)),$$

where $X \in \mathfrak{g}$ is such that $\exp(X) = x \in G$. Moreover, we introduce a Haar measure on compact Lie groups.

Theorem A.4.16. *Let G be a compact Lie group. Then it has a unique right Haar measure* dg*, which is given by some positive density. In addition, this measure is also left-invariant and invariant under $g \mapsto g^{-1}$.*

In the following we refer to this measure whenever we write dg.
Now, for a given representation (V, ρ) of G, let $\{v_i\}_i$ be a basis of V. Writing for every $g \in G$ the endomorphism $\rho(g) : V \to V$ in the basis $\{v_i\}_i$, we can consider ρ as a matrix-valued map on G. Considering each entry of the matrix as a scalar-valued function on G, we obtain the so-called matrix coefficients ρ_{ij} of the representation (V, ρ). Those matrix coefficients possess a natural orthogonality.

Theorem A.4.17. *(1) Let $(V, \rho^V), (W, \rho^W)$ be non-isomorphic, irreducible representations of G. Choose bases $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_m\}$ of W. Then, for any indices $i, j \in \{1, \ldots, n\}$ and $a, b \in \{1, \ldots, m\}$, the matrix coefficients ρ^V_{ij}, ρ^W_{ab} are orthogonal with respect to the inner product on $C^\infty(G, \mathbb{C})$ given by*

$$(f_1, f_2) := \int_G f_1(g)\overline{f_2(g)}\mathrm{d}g. \tag{A.13}$$

(2) Let (V, ρ^V) be an irreducible representation of G and let $\{v_i\}_i$ be an orthonormal basis of V with respect to a G-invariant inner product. Then, the matrix coefficients ρ^V_{ij} are pairwise orthogonal with respect to (A.13) and each has norm squared equal to $\frac{1}{\dim(V)}$.

The bilinear form on a given finite-dimensional Lie algebra $\mathfrak{g}$ we will mostly be considering is the Killing form K, sometimes also called Cartan-Killing form, which is defined for every $x, y \in \mathfrak{g}$ by

$$K(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y).$$

Proposition A.4.18. *([1], Prop. 2.32)*
The Killing form K on the Lie algebra $\mathfrak{g}$ of a Lie group G is $\mathrm{Ad}(G)$-invariant, that is

$$K\left(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y\right) = K(x, y)$$

for any $g \in G$ and $x, y \in \mathfrak{g}$.

In the specific cases of (compact) matrix Lie groups we are considering in 3.3.4 the expression for the Killing form simplifies.

Lemma A.4.19. *([61], Rem. 11.7; [33, 80])*
Consider a Lie algebra $\mathfrak{g} \in \{\mathfrak{o}(n), \mathfrak{u}(n), \mathfrak{sp}(n)\}$. Then we obtain

(a) $\mathrm{tr}\,(xy)$ *is real for all $x, y \in \mathfrak{g}$.*

(b) The Killing form on $\mathfrak{g}$ is given by $K(x,y) = c\ \mathrm{tr}(xy)$ with $c \neq 0$ if $\dim(\mathfrak{g}) > 1$.

In particular, for $\mathfrak{su}(n)$ we get $c = 2n$ and for $\mathfrak{sp}(n)$ we have $c = 4n + 4$.

Finally, since we restrict in the remaining subsections to semisimple Lie algebras, we also recall the notion of a semisimple Lie algebra.

Definition A.4.20. Let $\mathfrak{g}$ be a Lie algebra. We define a series of ideals of $\mathfrak{g}$ called the derived series $D^i\mathfrak{g}$ by

$$D^0\mathfrak{g} := \mathfrak{g}, \quad D^{i+1}\mathfrak{g} := \left[D^i\mathfrak{g}, D^i\mathfrak{g}\right].$$

Then we call $\mathfrak{g}$ solvable if $D^n\mathfrak{g} = 0$ for large enough n.

Definition A.4.21. A Lie algebra $\mathfrak{g}$ is called semisimple if it contains no non-zero solvable ideals, and $\mathfrak{g}$ is called simple if it is not abelian and contains no ideals other than 0 and $\mathfrak{g}$. Correspondingly, a Lie group G is called (semi-)simple if its associated Lie algebra $\mathfrak{g}$ is (semi-)simple.

A.4.3 Root systems

With the basics established we introduce two notions of root systems for complex semisimple Lie algebras, which we then see to be actually equivalent. For this, we denote from now on by $\mathfrak{g}$ a complex semisimple Lie algebra.
We call a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ toral if $\mathfrak{h}$ is commutative and consists of semisimple elements, where $x \in \mathfrak{h}$ is called semisimple if $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is a semisimple operator, that is if $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.
A toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which coincides with the set $\{x | \, [x, \mathfrak{h}] = 0\}$ is called a Cartan subalgebra of $\mathfrak{g}$.
Fixing a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we can state the root decomposition of $\mathfrak{g}$.

Theorem A.4.22. ***(Root decomposition)***

(1) We have for $\mathfrak{g}$ the so-called root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha := \{x | \, [h, x] = <\alpha, h> x \text{ for all } h \in \mathfrak{h}\}$ and $R := \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \, \mathfrak{g}_\alpha \neq 0\}$. The set R is called the root system of $\mathfrak{g}$ and the spaces $\mathfrak{g}_\alpha$ are called the root subspaces.

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *with $\mathfrak{g}_0 := \mathfrak{h}$.*

(3) If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are orthogonal with respect to the Killing form.

(4) For any α, the Killing form K defines a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$. In particular, the restriction of K to $\mathfrak{h}$ is non-degenerate.

For comparison with the second notion of a root system to be defined below we also state some properties of the root decomposition.

Theorem A.4.23. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$.*

(1) R spans $\mathfrak{h}^*$ as a vector space, and elements $h_\alpha := \frac{2\,H_\alpha}{K(\alpha,\alpha)}$, with H_α the dual element to $\alpha \in \mathfrak{h}^*$, span $\mathfrak{h}$ as a vector space.

(2) For each $\alpha \in R$ the root subspace $\mathfrak{g}_\alpha$ is one-dimensional.

(3) For any two roots α, β the number

$$< h_\alpha, \beta >= \frac{2\,K\,(\alpha, \beta)}{K\,(\alpha, \alpha)}$$

is integer.

(4) For $\alpha \in R$ define the reflection operator $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha\,(\lambda) := \lambda - < h_\alpha, \lambda > \alpha = \lambda - \frac{2\,K\,(\alpha, \lambda)}{K\,(\alpha, \alpha)}\,\alpha.$$

Then for roots α, β the image $s_\alpha\,(\beta)$ is also a root. In particular $s_\alpha\,(\alpha) = -\alpha \in R$.

(5) For any root α, the only multiples of α which are also roots are $\pm\alpha$.

(6) For roots α and $\beta \neq \pm\alpha$ the subspace $V = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is an irreducible representation of $\mathfrak{sl}\,(2, \mathbb{C})_\alpha$.

(7) If α, β are roots so that $\alpha + \beta$ is also a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

We can also define the notion of an abstract root system. We will see later that for a complex semisimple Lie algebra the two notions of root systems are in fact equivalent.

Definition A.4.24. (Abstract root system)
An abstract root system is a finite set of elements $R \subset E \backslash \{0\}$, where E is a real vector space with a positive definite inner product $(\cdot, \cdot)$, so that the following properties are satisfied

(R1) R generates E as a vector space.

(R2) For any two roots $\alpha, \beta \in R$ the number $n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is integer.

(R3) Define the maps $s_\alpha : E \to E$, $\lambda \mapsto \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\,\alpha$. If $\alpha, \beta \in R$, then also $s_\alpha\,(\beta) \in R$.

We say that $r := \dim(E)$ is the rank of R.
R is called a reduced root system if it satisfies in addition to $(R1) - (R3)$ also

(R4) If α, $c\alpha$ are both roots, then $c \in \{\pm 1\}$.

Complementary to any root $\alpha \in R$ of a root system R we have a coroot $\alpha^V \in E^*$, which is given by

$$\left\langle \alpha^V, \lambda \right\rangle = \frac{2\ (\alpha, \lambda)}{(\alpha, \alpha)},\ \lambda \in E.$$

From now on we take R to be a reduced root system, that is we suppose that $(R1) - (R4)$ hold. Let $t \in E$ be such that for any root α we have $(t, \alpha) \neq 0$. Then the set of roots admits a decomposition $R = R_+ \sqcup R_-$ into positive roots

$$R_+ := \{\,\alpha \in R|\,(t, \alpha) > 0\}$$

and negative roots

$$R_- := \{\,\alpha \in R|\,(t, \alpha) < 0\}\,.$$

Such a decomposition is referred to as a polarization of R and depends a priori on the choice of $t \in E$.
Given a polarization of R, a positive root that cannot be written as the sum of two positive roots is called a simple root. We denote the set of simple roots by $\Pi \subset R_+$.

Theorem A.4.25. *Let $R = R_+ \sqcup R_- \subset E$ be a root system.*
Then the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ form a basis of the vector space E. In particular, every $\alpha \in R$ can be uniquely written as

$$\alpha = \sum_{i=1}^{r} n_i \, \alpha_i \quad \textit{with} \quad n_i \in \mathbb{Z}$$

such that for $\alpha \in R_+$ we have $n_i \geq 0$ for all i and for $\alpha \in R_-$ we have $n_i \leq 0$ for all i.

Furthermore, R gives rise to the so-called root lattice

$$Q := \{\text{abelian group generated by}\, \alpha \in R\} \subset E$$

as well as the coroot lattice

$$Q^V := \left\{\text{abelian group generated by}\, \alpha^V, \alpha \in R\right\} \subset E^*.$$

In addition, the weight lattice $P \subset E$ is the dual lattice to Q^V, that is P is defined as

$$P := \left\{\, \lambda \in E | \left\langle \alpha^V, \lambda \right\rangle \in \mathbb{Z}\,\text{for all}\, \alpha \in R\right\} = \left\{\, \lambda \in E | \left\langle \alpha^V, \lambda \right\rangle \in \mathbb{Z}\,\text{for all}\, \alpha^V \in Q^V \right\}.$$

The elements of P are called integral weights. Observe that it suffices to run over the simple roots, i.e. it is enough to verify $\left\langle \alpha_i^V, \lambda \right\rangle \in \mathbb{Z}$ for all $\alpha_i \in \Pi$. In light of this we define the fundamental weights $w_i \in E$ by

$$\left\langle \alpha_j^V, w_i \right\rangle := \delta_{ij}.$$

Then, the w_i's form a basis of E and $P = \bigoplus_i \mathbb{Z} w_i$.
Now, the set of simple roots for a root system with a given polarization already contains all the information, that is

Lemma A.4.26. *The root system R can be recovered from the set of simple roots* Π.

We say that a root system R is reducible if it is of the form $R = R_1 \sqcup R_2$ with $R_1 \perp R_2$. Otherwise, R is called irreducible.
To establish the equivalence between the two notions of root systems, we first note that from the properties of the root system of a semisimple Lie algebra $\mathfrak{g}$ we have seen that this root system satisfies the properties $(R1) - (R4)$ of a reduced abstract root system, which is irreducible if and only if the Lie algebra $\mathfrak{g}$ is simple. Moreover, an abstract root system generates a semisimple Lie algebra whose root system gives back the abstract root system we started with.

Theorem A.4.27. *Let R be a reduced, irreducible root system with polarization $R = R_+ \sqcup R_-$ and set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Let $\mathfrak{g}(R)$ be the complex Lie algebra with generators e_i, f_i, h_i and Serre relations, i.e.*

$$\begin{aligned}&[h_i, h_j] = 0,\, [h_i, e_j] = a_{ij} e_j,\, [h_i, f_j] = -a_{ij} f_j,\, [e_i, f_j] = \delta_{ij} h_i,\\ &\left(\mathrm{ad}_{e_i}\right)^{1-a_{ij}} e_j = 0,\, \left(\mathrm{ad}_{f_i}\right)^{1-a_{ij}} f_j = 0,\end{aligned}$$

where $a_{ij} = \left\langle \alpha_i^V, \alpha_j \right\rangle$. Then, $\mathfrak{g}(R)$ is a finite-dimensional, semisimple Lie algebra with root system R.

This gives us the equivalence between the two notions of root systems for a complex, semisimple Lie algebra.

Corollary A.4.28. *(a) If $\mathfrak{g}$ is a semisimple Lie algebra with root system R, then there is a natural isomorphism $\mathfrak{g} \cong \mathfrak{g}(R)$.*

(b) *There is a natural bijection between the set of isomorphism classes of reduced root systems and the set of isomorphism classes of finite-dimensional, complex, semisimple Lie algebras.*

As a consequence, to classify finite-dimensional, complex, simple Lie algebras it suffices to classify irreducible, reduced root systems, which gives us the following result.

Theorem A.4.29. *([33], Thm. 21.11)*
Every complex, simple Lie algebra is isomorphic to one of the following

$$\begin{aligned}
&\mathfrak{a}_n = \mathfrak{sl}_{n+1}\,\mathbb{C},\ n \geq 1; && \mathfrak{e}_7;\\
&\mathfrak{b}_n = \mathfrak{so}_{2n+1}\,\mathbb{C},\ n \geq 2; && \mathfrak{e}_8;\\
&\mathfrak{c}_n = \mathfrak{sp}_n\,\mathbb{C},\ n \geq 3; && \mathfrak{f}_4;\\
&\mathfrak{d}_n = \mathfrak{so}_{2n}\,\mathbb{C},\ n \geq 4; && \mathfrak{g}_2.\\
&\mathfrak{e}_6; &&
\end{aligned}$$

We obtain from this a classification of simply connected, compact, simple Lie groups.

Corollary A.4.30. *Every simply connected, compact, simple Lie group is isomorphic to one of the following*

$$\begin{aligned}
&A_n = \mathrm{SU}(n+1),\ n \geq 1; && E_7;\\
&B_n = \mathrm{Spin}(2n+1),\ n \geq 2; && E_8;\\
&C_n = \mathrm{Sp}(n),\ n \geq 3; && F_4;\\
&D_n = \mathrm{Spin}(2n),\ n \geq 4; && G_2.\\
&E_6; &&
\end{aligned}$$

A.4.4 Highest-weight representations

In this subsection we turn to a specific kind of representations of Lie algebras, namely highest-weight representations.
For this we fix a finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and root decomposition as introduced in subsection A.4.3.
Then, given a representation (V, ρ) of $\mathfrak{g}$, a vector $v \in V$ is a vector of weight $\lambda \in \mathfrak{h}^*$ if for any $h \in \mathfrak{h}$ we have $\rho(h)v = \langle \lambda, h\rangle\, v$. The space of all such vectors for a fixed λ is called a weight space and denoted by $V[\lambda]$. Every $\lambda \in \mathfrak{h}^*$ such that $V[\lambda] \neq 0$ is called a weight of V and the set of all weights of V is denoted by $P(V)$.
Note that vectors of different weights are linearly independent and that for a finite-dimensional representation V the set $P(V)$ is finite. This gives rise to the so-called weight decomposition of a finite-dimensional representation of $\mathfrak{g}$.

Theorem A.4.31. *Every finite-dimensional representation V of $\mathfrak{g}$ admits a weight decomposition*

$$V = \bigoplus_{\lambda \in P(V)} V[\lambda].$$

Moreover, every weight is integral, that is $P(V) \subset P$.

Now, let us define the highest-weight representations of $\mathfrak{g}$.

Definition A.4.32. A non-zero representation V of $\mathfrak{g}$ is called highest-weight representation of highest weight λ if it is generated by a vector $v \in V[\lambda]$ such that $\rho(x)v = 0$ for all $x \in \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$. Then, v is called the highest-weight vector of V.
In addition, if the highest weight λ is a fundamental weight of $\mathfrak{g}$, then V is called fundamental representation of $\mathfrak{g}$.

The existence of such representations follows directly from the following

Theorem A.4.33. *Every irreducible, finite-dimensional representation of $\mathfrak{g}$ is a highest-weight representation.*

However, up to isomorphism we have uniqueness.

Theorem A.4.34. *For any $\lambda \in \mathfrak{h}^*$, there exists, up to isomorphism, a unique irreducible highest-weight representation with highest weight λ. We will denote this representation by L_λ. In particular, every irreducible, finite-dimensional representation of $\mathfrak{g}$ must be isomorphic to one of the L_λ's.*

We will call a weight $\lambda \in \mathfrak{h}^*$ of a representation V of $\mathfrak{g}$ dominant integral if we have, for all positive roots α, that $\langle \alpha^V, \lambda \rangle \in \mathbb{Z}_+$. The set of dominant integral weights is denoted by P_+. As $P(V) \subset P$ we also have $P_+ \subset P$.
Further, instead of checking this for all $\alpha \in R_+$, it suffices to consider only the simple roots. Then, the dominant integral weights give us a condition for the finite dimensionality of the highest-weight representations L_λ.

Theorem A.4.35. *The irreducible highest-weight representation L_λ is finite-dimensional if and only if the highest weight λ is dominant integral.*

A.4.5 Eigenfunctions of the Laplacian

With the necessary background on representation theory established, we can now determine the eigenfunctions of the Laplacian on $C^\infty(G)$ corresponding to its first non-zero eigenvalue. First we need the notion of the Casimir operator.

Proposition A.4.36. *Let $\mathfrak{g}$ be a semisimple Lie algebra and B be a non-degenerate, invariant, symmetric bilinear form on $\mathfrak{g}$. Let $\{v_i\}_i$ be a basis of $\mathfrak{g}$ and $\{v_i^*\}_i$ be its dual basis with respect to B. Then the Casimir operator determined by B is defined as*

$$\Delta_B := -\sum_i v_i \otimes v_i^*.$$

Moreover, Δ_B does not depend on the choice of basis $\{v_i\}_i$.
In particular, if $\mathfrak{g}$ is semisimple and $B = K$, i.e. B is the Killing form on $\mathfrak{g}$, then $\Delta := \Delta_K$ is just referred to as the Casimir operator.

It is a well-known result (e.g. [70]) that eigenfunctions of Δ are given by matrix coefficients of highest-weight representations.

Lemma A.4.37. *Let U^λ be an irreducible, unitary highest-weight representation of a compact, connected Lie group G with its highest weight λ being dominant integral. Fix an $\mathrm{Ad}(G)$-invariant inner producton $\mathfrak{g}$ and denote the induced inner product on $\mathfrak{g}^*$ by $(\cdot,\cdot)$.*

(a) Let dU^λ be the differential of U^λ. Then we have

$$dU^\lambda(\Delta) = (\lambda, \lambda + 2\eta)\,1,$$

where 2η is the sum of all positive roots of U^λ and 1 denotes the identity operator.

(b) The matrix coefficients u_{ij}^λ of U^λ are eigenfunctions of the Casimir operator Δ regarded as a differential operator on G, i.e. for each $i, j \in \{1, \dots, \dim(U^\lambda)\}$

$$\Delta\, u_{ij}^\lambda = (\lambda, \lambda + 2\eta)\, u_{ij}^\lambda.$$

Observe that the second part of Lemma A.4.37 also implies that the multiplicity of $(\lambda, \lambda + 2\eta)$ as an eigenvalue of Δ is given by the square of the dimension of the representation U^λ due to the orthogonality of the matrix coefficients.
As the notation suggests, the Casimir operator regarded as a differential operator on G coincides with the Laplacian on $C^\infty(G)$. In the case of $\lambda_1(G)$ we are interested in, we moreover find that we only have to minimize over the fundamental weights of $\mathfrak{g}$ instead of over all dominant integral weights.

Lemma A.4.38. *([72])*
Let G be a simply connected, compact, simple Lie group with Lie algebra $\mathfrak{g}$ and bi-invariant metric $g = -K$. Then, the first non-zero eigenvalue of $\Delta : C^\infty(G) \to C^\infty(G)$ is given by

$$\lambda_1(G) = \min_{i \in \{1,\dots,l\}} \{(w_i, w_i + 2\eta)\},$$

where $\{w_1, \dots, w_l\}$ is the set of fundamental weights of $\mathfrak{g}$ and $(\cdot,\cdot)$ is the inner product on $\mathfrak{g}^*$ induced by an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$.

As a result, an orthogonal basis of the eigenspace of $\Delta : C^\infty(G) \to C^\infty(G)$ corresponding to the first eigenvalue $\lambda_1(G)$ is given by the matrix coefficients of the unique lift of the fundamental representation of $\mathfrak{g}$ whose fundamental weight w_i realizes $\min_j \{(w_j, w_j + 2\eta)\}$. Since we are only interested in harmonically unstable Lie groups, we discuss in the following fundamental weights and representations for $G \in \{A_l(l \geq 1), C_l(l \geq 2)\}$ and determine those eigenfunctions that give rise to the eigenvalue $\lambda_1(G)$. However, we will only give a very cropped derivation, the detailed derivation of the Cartan subalgebra, root system, fundamental weights and fundamental representations can be found e.g. in [10, 11].

Type $A_l\,(l \geq 1)$, cf. [10, 11]

As we are considering in 3.3.4 the Killing form on $\mathfrak{a}_l$ and we have seen in Lemma A.4.19 that it is given by

$$K(x,y) = (2l+2)\,\mathrm{tr}\,(xy),\; x, y \in \mathfrak{a}_l,$$

the induced inner product on $\mathfrak{a}_l^*$ has to be defined as

$$(x,y) = \frac{1}{2l+2}\,\mathrm{tr}\,(xy),\; x, y \in \mathfrak{a}_l^*.$$

Let $E := \left\{ x = (x_1, \dots, x_{l+1}) \in \mathbb{R}^{l+1} \,\middle|\, \sum_{i=1}^{l+1} x_i = 0 \right\}$ and denote by ε_j, $j = 1, \dots, l+1$, the j-th standard basis vector of $\mathbb{R}^{l+1}$. Then the set of roots is given by

$$R = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i \leq l,\, 1 \leq j \leq l+1\}$$

and the subset of simple roots is

$$\Pi = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \,|\, 1 \leq i \leq l\}.$$

Then we obtain for the fundamental weights w_i corresponding to the α_i's

$$w_i = \sum_{k=1}^{i} \varepsilon_k - \frac{i}{l+1} \sum_{j=1}^{l+1} \varepsilon_j.$$

Computing also the sum 2η of the positive roots to be

$$2\eta = \sum_{j=1}^{l+1} (l - 2(j-1))\,\varepsilon_j,$$

we find

$$\begin{aligned} w_i + 2\eta = &\sum_{k=1}^{i} (l+1)^{-1} \left(l^2 - 2kl + 4l - 2k + 3 - i\right) \varepsilon_k \\ &+ \sum_{j=i+1}^{l+1} (l+1)^{-1} \left(l^2 - 2jl + 3l - 2j + 2 - i\right) \varepsilon_j. \end{aligned}$$

and so

$$\begin{aligned}
2(l+1)\,(w_i, w_i+2\eta) = \; & (l+1)^{-2} \sum_{k=1}^{i} (l+1-i)\left(l^2-2kl+4l-2k+3-i\right) \\
& + (l+1)^{-2} \sum_{j=i+1}^{l+1} (-i)\left(l^2-2jl+3l-2j+2-i\right) \\
= & \,(l+1)^{-2} i \left(l^3+5l^2+7l-il^2-5il+i^2-4i+3\right) \\
& + (l+1)^{-2}\left(-2l^2-4l-2+2il+2i\right) \sum_{k=1}^{i} k \\
& + (l+1)^{-2}\left(i^2-il^2-3il-2i\right)(l+1-i) + (l+1)^{-2}(2il+2i) \sum_{j=i+1}^{l+1} j \\
= & \,(l+1)^{-2}\left(il^2+2il-i^2l-i^2+i\right) + (l+1)^{-2}\left(-l^2-2l-1+il+i\right)\left(i^2+i\right) \\
& + (l+1)^{-2}(il+i)(l+i+2)(l+1-i) \\
= & \,(l+1)^{-2}\left(il^3-i^2l^2+4il^2-3i^2l+5il-2i^2+2i\right)
\end{aligned}$$

of multiplicity $\binom{l+1}{i}^2$. Thus, taking the minimum over i yields for the first eigenvalue of the Laplacian

$$\lambda_1(A_l) = (w_1, w_1+2\eta) = \frac{l(l+2)}{2(l+1)^2}$$

with multiplicity equal to $(l+1)^2$.
Denoting by σ the identity representation of the Lie algebra $\mathfrak{a}_l$ of A_l, it is known that for every $i \in \{1,\ldots,l\}$ the fundamental representation corresponding to w_i is given by $\wedge^i\sigma$. Thus, the representation giving rise to the eigenvalue $\lambda_1(A_l)$ is just the identity representation σ. Note that even though we also obtain $(w_l, w_l+2\eta) = \lambda_1(A_l)$ the first eigenvalue is not of multiplicity $2(l+1)^2$ since the fundamental representation corresponding to w_l must be isomorphic to the identity representation as otherwise there would be two non-isomorphic, irreducible highest-weight representations with the same highest weight, which is not possible by Theorem A.4.34.

Type $C_l\,(l \geq 2)$, cf. [10, 11]

Here we have seen for the Killing form on $\mathfrak{c}_l$ in Lemma A.4.19 that it is for any $x, y \in \mathfrak{c}_l$

$$K(x,y) = (4l+4)\,\mathrm{tr}\,(xy)\,,$$

so the induced inner product on $\mathfrak{c}_l^*$ is given for any $x, y \in \mathfrak{c}_l^*$ by

$$(x,y) = \frac{1}{4l+4}\,\mathrm{tr}\,(xy)\,.$$

In this case we can let $E = \mathbb{R}^l$ and ε_j, $j = 1,\ldots,l$, denote the j-th standard basis vector of $\mathbb{R}^l$. Then the set of roots R is given by

$$R = \{\pm 2\varepsilon_i \mid 1 \leq i \leq l\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l\}$$

with the set of simple roots

$$\Pi = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1},\, \alpha_l := 2\varepsilon_l \mid 1 \leq i \leq l-1\}\,.$$

For the corresponding fundamental weights we find

$$w_i = \sum_{k=1}^{i} \varepsilon_k,\ 1 \leq i \leq l.$$

Calculating the sum of all positive roots to be

$$2\eta = \sum_{j=1}^{l} (2l - 2(j-1))\, \varepsilon_j,$$

we obtain

$$w_i + 2\eta = \sum_{k=1}^{i} (2l - 2k + 3)\, \varepsilon_k + \sum_{j=i+1}^{l} (2l - 2j + 2)\, \varepsilon_j$$

and so

$$\begin{aligned} 4(l+1)\,(w_i, w_i + 2\eta) &= \sum_{k=1}^{i} (2l - 2k + 3) = (2l+3)i - 2\sum_{k=1}^{i} k \\ &= 2il + 3i - \left(i^2 + i\right) = -i^2 + 2i(l+1) \end{aligned}$$

of multiplicity $\left[\binom{2l}{i} - \binom{2l}{i-2}\right]^2$. Thus, taking the minimum over $i \in \{1, \ldots, l\}$, we conclude for the first eigenvalue of Δ

$$\lambda_1\,(C_l) = (w_1, w_1 + 2\eta) = \frac{2l+1}{4(l+1)}$$

with multiplicity equal to $4l^2$.

As elaborated in detail for example in [11], Ch. 8, §13.3, for each $i \in \{1, \ldots, l\}$ the fundamental representation corresponding to w_i is given by a subrepresentation of $\wedge^i \sigma$, where σ denotes the identity representation of $\mathfrak{c}_l$. Hence the representation giving rise to the eigenvalue $\lambda_1\,(C_l)$ is a subrepresentation of the identity representation σ.

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