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Some Notes on Two Tests for Stability in Lossy Power Systems

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Abstract: Analysis of the small-signal stability of power system is commonly based on the swing equation model. Due to the special structure of the power grid swing equations, an equilibrium set corresponding to a so-called frequency equilibrium has to be analyzed. We present two new and short proofs for two tests by Skar concerning both the nonuniformly and the uniformly damped system. Based on the latter we derive a simple stability test for uniformly damped systems, which does not rely on eigenvalue computations. The nonhyperbolicity of the equilibria in original coordinates is tackled by the concept of normally hyperbolic invariant manifolds. The derivations are completely based on the theory of quadratic pencils.

Keywords: Power and Energy Systems, Power System Stability, Quadratic Matrix Pencils, Networked Systems

1. INTRODUCTION

The transient stability of electrical power systems has been studied for a long time. While it has been shown that in lossless grids, equilibria with so-called phase cohesivity are always stable, this is not true for the lossy case (Skar, 1980), i.e. for nonneglible transfer conductances. The case of significant transfer losses occurs especially in low-voltage grids (Engler and Soultanis, 2005).

A commonly used model for the analysis of the transient stability when a disturbance occurs, are the multi-machine swing equations, which describe power grid dynamics dominated by the behavior of synchronous generators. The continued use of this model for modern grids with higher proportions of inverter-based power generation is motivated by the widespread use of model-matching control techniques (virtual synchronous generators) emulating synchronous machines. Linearization of the swing equations leads to a second order linear ODE. While many results exist for this system class from the area of mechanics, there is still ongoing research into the topic of so-called nonconservative systems (see e.g. Kirillov (2013)), which include a skew-symmetric part in the stiffness matrix. In the linearization of power systems, transmission losses introduce the skew-symmetric part; a similar effect also results from the properties of phase-shifting transformers.

There are some sufficient conditions for the stability of nonconservative systems based on a Lyapunov approach (Kwatny et al., 1985), conditions that rely on commutativity and eigenvalue conditions (Bulatovic, 2020), but not all of them can be applied for our system class, or they require higher computational effort, or the assumptions of the conditions are not met in our application. However, under assumptions specific to the use-case, such as the topology of the network, the type of damping, or the kind of equilibrium, simple stability tests are possible.

The present paper offers shorter proofs of two stability theorems by Skar (1980). In preparation of the proofs, some results which are useful by themselves are formulated in lemmas. Moreover, a theorem for the case of uniform damping is strengthened by formulating it as a necessary and sufficient condition. The relationship between the number of zero eigenvalues of the second-order system and a coefficient matrix is addressed. Variants of stability tests for the uniformly damped system based on the Brauer eigenvalue shifting theorem and the Gershgorin circle theorem are proposed.

The plan of the paper is as follows: Section 2 describes the multi-machine swing equation model for transient analysis of power systems, the properties of the nonlinear system and its linearization about an equilibrium. In Section 3, a parameter-based stability test is derived. Section 4 considers systems with uniform damping ratio for all generators and provides new results for the stability of the equilibrium subspace.

2. ELECTRICAL POWER SYSTEMS

For modeling the transient behavior of the electrical power grid, we use the multi-machine swing equation

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i + P_{\mathrm{e}i}(\theta) - P_{\mathrm{m}i} = 0, \quad i = 1,\dots,n, \tag{1}$$

where the variable θ denotes the electrical angle at the generator nodes. The electrical power transmitted over the lines is given by

$$P_{\mathrm{e}i}(\theta) = \sum_{j=1}^{n} E_iE_j|Y_{ij}|\cos(\gamma_{ij} - \theta_i + \theta_j). \tag{2}$$

Table 1 gives the physical interpretation of the parameters. Such a model considering only the generator buses is commonly used, e.g. by Gholami and Sun (2020), and can be derived from a model including further buses with

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constant impedance loads via Kron reduction (Kron, 1939, Ch. 10).

In vector-matrix notation, (1) becomes

$$M\ddot{\theta} + D\dot{\theta} + P_{\mathrm{e}}(\theta) - P_{\mathrm{m}} = 0_n, \tag{3}$$

where M and D are diagonal matrices. Note that while the systems configuration space has dimension n, the state-space has dimension $2n$ with elements $(\theta, \dot{\theta}) = (\theta, \omega) \in \mathbb{R}^{2n}$.

The equilibrium conditions for system (3) are

$$P_{\mathrm{e}}(\theta) = P_{\mathrm{m}} \tag{4a}$$

$$\dot{\theta} = 0_n \tag{4b}$$

P_{e} is translationally invariant, i.e.

$$P_{\mathrm{e}}(\theta) = P_{\mathrm{e}}(\theta + c1_n) \text{ with } c \in \mathbb{R}. \tag{5}$$

Therefore, any equilibrium point $(\theta_{\mathrm{e}}, 0_n)$ is part of a one-dimensional equilibrium manifold

$$\mathcal{E}_{\mathrm{nl}} = \{(\theta_{\mathrm{e}} + c1_n, 0_n) \colon c \in \mathbb{R}\}. \tag{6}$$

Because of this, any equilibrium point of the system will be nonhyperbolic.

While it is possible to obtain a hyperbolic equilibrium by considering a state space of dimension $2n-1$, we prefer to operate in the original system, which has other advantages due to its structure as a second-order system.

We place the following assumptions on the nonlinear system:

- N1 All generators have positive inertia and damping, i.e. $m_i, d_i > 0$.
- N2 The undirected graph associated with the grid is connected.
- N3 The equilibrium set is phase cohesive, i.e. $0 < \gamma_{ij} - \theta_i + \theta_j < \pi$.

2.1 Linearization of the Swing Equation

Linearization of (3) about any point of the equilibrium manifold $\mathcal{E}_{\mathrm{nl}}$ leads to the system

$$M\ddot{x} + D\dot{x} + Lx = 0_n, \tag{7}$$

where $x = \Delta\theta$ and $L = \frac{\partial P_{\mathrm{e}}}{\partial \theta^{\top}}\big|_{\theta=\theta_{\mathrm{e}}}$ with entries

$$l_{ij} = \begin{cases} \displaystyle\sum_{k=1, k\neq i}^{n} E_i E_k |Y_{ik}| \sin(\gamma_{ik} - \theta_{\mathrm{e}i} + \theta_{\mathrm{e}k}), & i = j \\ -E_i E_j |Y_{ij}| \sin(\gamma_{ij} - \theta_{\mathrm{e}i} + \theta_{\mathrm{e}j}), & i \neq j. \end{cases} \tag{8}$$

Clearly, the matrix L has the row sum zero property

$$L1_n = 0_n. \tag{9}$$

From the state space representation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -M^{-1}L & -M^{-1}D \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

Table 1. System Parameters

Symbol	Parameter
m_i	inertia constant
d_i	damping constant
$P_{\mathrm{m}i}$	mechanical power
E_i	generator voltage amplitude
Y_{ij}	complex entries of nodal admittance matrix
$\gamma_{ij} = \arg Y_{ij}$	argument of admittance

it follows immediately that (7) has a nontrivial equilibrium subspace

$$\mathcal{E}_{\mathrm{lin}} \supseteq \mathrm{span}(1_n) \times \{0_n\}. \tag{10}$$

To conclude that equality holds in (10), it is necessary to show that the geometric multiplicity of the zero eigenvalue of L is one. As consequences of the connectedness of the network and the phase cohesiveness of the equilibrium follow some interesting matrix properties used later. They directly imply that zero is a simple eigenvalue of L.

2.2 Connectedness and Phase Cohesiveness

Firstly, we establish system properties based on the above assumptions.

Lemma 1. (Gholami and Sun (2020)). If the equilibrium is phase cohesive, then the directed graph associated with L is strongly connected with positive weights if and only if the undirected network graph is connected.

Lemma 2. (Lancaster and Tismenetsky (1985, 15.1.1)). A square matrix is irreducible if and only if its directed graph is strongly connected.

In the following we frequently use the fact that L is an M-matrix, i.e., a matrix that can be split into $A = sI_n - B$, $s > 0$, $B \geq 0$ with $s \geq \rho(B)$, the spectral radius of B (Ostrowski's definition).

Lemma 3. For a phase cohesive equilibrium of a network described by a connected undirected graph, L is a singular, irreducible M-matrix with $l_{ii} > 0$ and $l_{ij} \leq 0$, $i \neq j$ with a simple zero eigenvalue.

Proof. The strong phase cohesiveness assumption implies that L is a Z-matrix, i.e., $l_{ij} \leq 0$, $i \neq j$. Since the network graph is connected, there is at least one nonzero off-diagonal entry in each row. Consequently, $l_{ii} > 0$ due to the zero-sum property. A Z-matrix is an M-matrix if the real part of each nonzero eigenvalue is positive (Berman and Plemmons, 1994, thm. 6.4.6). By Gershgorin's circle theorem all eigenvalues lie in

$$G = \bigcup_{i=1}^{n} B(l_{ii}, l_{ii})$$
$$= B(l_{kk}, l_{kk}) \text{ with } l_{kk} \text{ as largest diagonal entry of } L,$$

and with $l_{kk} > 0$ all real parts of each nonzero eigenvalue of L are positive. Since the rows of L sum to zero, $(\lambda, v) = (0, 1_n)$ is a eigenpair of L. Consequently, L is singular. By Lemmas 1 and 2, it is irreducible. A singular, irreducible M-matrix of order n has rank $n-1$ (Berman and Plemmons, 1994, thm. 6.4.16). Using the results of Hershkowitz and Schneider (1985), especially their Proposition 3.1, L is diagonally semistable. Consequently, it can only have semisimple eigenvalues on the imaginary axis. Combining the geometric multiplicity of 1 for the zero eigenvalue due to the rank $n-1$ property, its algebraic multiplicity is also 1. □

By Lemma 3, zero is a simple eigenvalue of L and so it is proven that the equilibrium set is

$$\mathcal{E}_{\mathrm{lin}} = \mathrm{span}(1_n) \times \{0_n\}. \tag{11}$$

In the following, we show that this equilibrium subspace is a normally hyperbolic invariant manifold (Eldering, 2013). For this, the existence of additional zero eigenvalues

(Section 2.3) and of purely imaginary eigenvalues (Section 3) of the quadratic pencil has to be excluded.

2.3 Relations between eigenvalues of pencils and coefficient matrices

When considering matrix pencils, it is often desirable to conclude properties of the pencil from properties of the coefficient matrices. Essentially, one is interested in the location of eigenvalues with respect to the real and imaginary axes. There are many results based on the degree of instability (Poincaré instability degree, Kirillov (2013)), the Krein index, and the Krein signature (Kollár, 2011). We summarize only some results concerning the zero eigenvalue, supplemented by two results for uniform damping and note that Lemma 5 is another supplement for a special case.

Lemma 4. Let $\sigma(Q)$ denote the spectrum defined as multiset of the eigenvalues of $Q(\lambda)$. Between the zero eigenvalues of L and the ones of $Q(\lambda) = I_n\lambda^2 + D\lambda + L$ exist the following relations:

(i) $\operatorname{card}(\sigma(Q) = 0) \geq \operatorname{geom}(\lambda(L) = 0) = \dim\ker L$
(ii) $\operatorname{card}(\sigma(Q) = 0) = \operatorname{alg}(\lambda(L) = 0)$ if $D = \zeta I_n$, $\zeta \neq 0$.
(iii) $\operatorname{geom}(\sigma(Q) = 0) = \operatorname{geom}(\lambda(L) = 0)$ if $D = \zeta I_n$,$\zeta \neq 0$.
(iv) $\operatorname{card}(\sigma(Q) = 0) = \operatorname{geom}(\lambda(L) = 0)$ if and only if $\ker L \cap \ker \operatorname{her} D \cap \ker \operatorname{skh} D = \{0_n\}$, supposed L is a Hermitian matrix
(v) $\operatorname{card}(\sigma(Q) = 0) = 2\operatorname{geom}(\lambda(L) = 0)$ if $\ker L \subseteq \ker D$, supposed D, L Hermitian

Proof. (i) $\operatorname{card}(\sigma(Q) = 0) = \operatorname{alg}(\lambda(Q(0) = 0) \geq \operatorname{geom}(\lambda(Q(0)) = \dim\ker Q(0) = \dim\ker L$
(ii)
(iii) With $D = \zeta I_n$, $Q(\lambda) = \lambda(\lambda + \zeta)I_n + L$, resp. $\tilde{Q}(\mu) = \mu I_n - (-L)$. For $k = \operatorname{alg}(\lambda(L) = 0)$,

$$\chi_{\tilde{Q}}(\mu) = \det(\mu I_n + L) = \mu^n + \cdots + c_k\mu^k;\ c_k \neq 0$$
$$\begin{aligned}\chi_Q(\lambda) &= \det(Q(\lambda)) \\ &= (\lambda(\lambda+\zeta))^n + \cdots + c_k(\lambda(\lambda+\zeta))^n \\ &= \lambda^{2n} + \cdots + c_k\zeta^k\lambda^k \text{ since } c_k, \zeta \neq 0.\end{aligned}$$

(iv) $Q(\lambda)$ corresponds to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \dot{x}\end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -L & -D\end{bmatrix}\begin{bmatrix} x \\ \dot{x}\end{bmatrix},\ \text{compact } \dot{\xi} = A\xi.$$

Then $\operatorname{geom}(\sigma(Q) = 0) = \operatorname{corank} A = 2n - \operatorname{rank} A = n - \operatorname{rank} L = \operatorname{geom}(\lambda(L) = 0)$, where the full rank of the second block column was used for $\operatorname{rank} A = n + \operatorname{rank} L$.
(v) Bilir and Chicone (1998, prop. 3)
(vi) (Kollár, 2011, sec. 4) □

Note that the inequality in (i) can be strict. See

$$Q(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\lambda^2 + \begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}$$

where $Q(\lambda)$ has two zero eigenvalues while L has only one zero eigenvalue. Here, even

$$\operatorname{card}(\sigma(Q) = 0) > \operatorname{alg}(\lambda(L) = 0).$$

Moreover, (vi) provides a stronger statement in this case.

Remark: Despite the special cases, there is no general result for the relation between $\operatorname{card}(\sigma(Q) = 0)$ and $\operatorname{alg}(\lambda(L) = 0)$ available if D is an arbitrary singular matrix. Consider

$$Q(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\lambda^2 + \begin{bmatrix} 1 & 0 \\ 0 & 2\end{bmatrix}\lambda + \begin{bmatrix} 1 & -1 \\ 1 & -1\end{bmatrix},$$

where $Q(\lambda)$ has one and L has two zero eigenvalues and

$$Q(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\lambda^2 + \begin{bmatrix} 3 & -2 \\ -2 & 1\end{bmatrix}\lambda + \begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix},$$

where $Q(\lambda)$ has two and L has only one zero eigenvalue.

Since Lemma 4 in combination with the above remark yield no statement for our application, it is necessary to use the additional properties of L to derive a statement on the number of zero eigenvalues of the pencil.

Lemma 5. (Zero is a simple eigenvalue of the pencil). The pencil $Q(\lambda) = M\lambda^2 + D\lambda + L$ with positive diagonal matrices M, D and a singular, irreducible matrix L admits a simple eigenvalue $\lambda = 0$.

Proof. $Q(0)1_n = L1_n = 0$ shows the eigenpair $(0, 1_n)$. To clarify that $\lambda = 0$ is a simple eigenvalue, it is shown that the linear coefficient c_1 of the characteristic polynomial

$$\det Q(\lambda) = \lambda^{2n} + c_{2n-1}\lambda^{2n-1} + \cdots + c_1\lambda$$

does not vanish; $c_0 = 0$ because $\lambda = 0$. Using Jacobi's formula (Horn and Johnson, 2012, thm. 0.8.10), the linear coefficient is given by

$$\begin{aligned} c_1 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\det Q(\lambda)\Big|_{\lambda=0} &= \operatorname{tr}(Q'(\lambda)\operatorname{adj} Q(\lambda))|_{\lambda=0} \\ &= \operatorname{tr}(D\operatorname{adj} L).\end{aligned}$$

Since L is a singular, irreducible matrix by Lemma 3, each proper principal submatrix of L is a nonsingular M-matrix (Berman and Plemmons, 1994, thm. 6.4.16). Since all principle minor of a nonsingular M-matrices are positive (Berman and Plemmons, 1994, thm. 6.2.3), the cofactors in the main diagonal of the adjugate $\operatorname{adj} L$ are positive. Together with D positive follows $\operatorname{tr}(D\operatorname{adj} L) > 0$. □

We can now formulate that based on the assumptions N1–N3 on the nonlinear system (3), the linear system (7) fulfills:

L1 M and D are positive diagonal matrices.
L2 L is a singular, irreducible M-matrix with a simple zero eigenvalue associated with the eigenvector 1_n, positive diagonal and nonnegative off-diagonal elements.

3. PARAMETER-BASED STABILITY TEST FOR NONUNIFORM DAMPING

The test by Skar (1980, thm. 2.6.iii) is a simple formula containing the parameters of Table 1 and the equilibrium point. Since the test gives a sufficient condition for stability based on the linearized system, we formulate it in dependence of the matrices in (7). Notably, it does not require exact eigenvalue calculations since it is derived from the Gershgorin circle theorem. This theorem in its standard version for constant matrices has some generalizations to polynomial matrices (Michailidou and Psarrakos, 2018) and non-polynomial pencils (Bindel and Hood, 2015). Eigenvalues at infinity, unbounded regions, or unimodular pencils introduce some difficulties into these generalizations, but in our system class with a nonsingular matrix M, the generalization can easily be performed via an argument concerning λ-parametrized matrices.

For this, the quadratic eigenvalue problem $Q(\lambda)v = 0_n$ is rewritten as a generalized eigenvalue problem

$$\mu M v = (-D\lambda - L)v =: A(\lambda)v \tag{12}$$

with $\mu = \lambda^2$. Hence, each generalized eigenvalue μ of the λ-parametrized EVP lies within at least one of the Gershgorin discs

$$|m_i\mu - a_{ii}(\lambda)| \le r_i(A(\lambda)) = \sum_{j=1, j\ne i}^{n} |a_{ij}(\lambda)|. \tag{13}$$

Since μ as eigenvalue of (12) and λ as eigenvalue of $Q(\lambda)$ are coupled by $\mu = \lambda^2$, it follows from (13) that all eigenvalues λ of $Q(\lambda)$ are contained in the set

$$G_\mathrm{r} = \bigcup_{i=1}^{n} \{\lambda \in \mathbb{C}\colon |q_{ii}(\lambda)| \le r_i(Q(\lambda))\}. \tag{14}$$

Now we are ready to provide a new, short proof for Skar's test, which also differs significantly from the proof of a closely related result by Nieuwenhuis and Schoonbeek (1997, cor. 1).

Theorem 6. (Linear Skar test, Skar (1980, thm. 2.6.iii)). The equilibrium subspace $\mathcal{E}_\mathrm{lin}$ of the linear swing equation (7) under the linear assumptions is globally exponentially stable, if

$$l_{ii} \le \frac{d_i^2}{2m_i}, \quad \forall i \in \{1, \dots, n\}. \tag{15}$$

Proof. By the assumptions, L is a Z-matrix with $l_{ii} = \sum_{j\ne i} |l_{ij}|$. This translates (14) into

$$G_\mathrm{r} = \bigcup_{i=1}^{n} \left\{\lambda \in \mathbb{C}\colon |m_i\lambda^2 + d_i\lambda + l_{ii}| \le l_{ii}\right\}.$$

Obviously, the set G contains the roots of $m_i s^2 + d_i s + l_{ii} = 0$. They are in $\mathbb{C}^-$, i.e. in the open left half plane, by Hurwitz' criterion, since $m_i, d_i, l_{ii} > 0$. Hence, it is to clarify how large can l_{ii} be such that G admits only the origin as a common point with the imaginary axis. We check for additional common points with $s = \mathrm{i}\omega$ in

$$|m_i(\mathrm{i}\omega)^2 + d_i(\mathrm{i}\omega) + l_{ii}| = l_{ii}$$

$$m_i^2\omega^4 + (d_i^2 - 2m_i l_{ii})\omega^2 = 0,$$

which should have no real root except $\omega = 0$. This is the case if and only if $d_i^2 - 2m_i l_{ii} \ge 0$, as in (15). By Lemma 5, zero is a simple eigenvalue of the pencil, which corresponds to the equilibrium set. Hence, the $2n - 1$ remaining eigenvalues are in $\mathbb{C}^-$ by the above Gershgorin argument. Linearity ensures the global exponential stability. □

In order to remain in the second-order system setting, we do not work in a reduced state space of dimension $2n - 1$, which would lead to a statement of stability of an equilibrium point as in the work of Skar (1980). We instead consider the stability of the equilibrium set $\mathcal{E}_\mathrm{nl}$.

Theorem 7. (Nonlinear Skar test, Skar (1980, Thm. 3.6.v)). Under the nonlinear assumptions, the equilibrium set $\mathcal{E}_\mathrm{nl}$ of the swing equation is locally uniformly exponentially stable under the condition

$$\sum_{j\ne i} E_i E_j\,|Y_{ij}| \sin(\gamma_{ij} - \theta_i + \theta_j) \le \frac{d_i^2}{2m_i}. \tag{16}$$

Proof. As shown in the proof for the linear case, under condition (15), a simple eigenvalue at $\lambda = 0$ is the only eigenvalue on the imaginary axis. Moreover, this eigenvalue corresponds to the equilibrium set. Consequently, the transverse dynamics is locally determined only by all the other eigenvalues. As these are all located in $\mathbb{C}^-$, the equilibrium set is an exponentially stable normally hyperbolic manifold. Time-invariance of the system ensures uniformity. □

Remark: Although the test (16) is conservative, it attests stability for the IEEE standard test systems (Gholami and Sun, 2020). The applicability in inverter-based grids is a topic of current research.

4. CASE OF UNIFORM DAMPING

In conventional grids, the ratio between the inertia and damping for the synchronous machines is often nearly the same. This means that by left-muliplication with the inverse of the inertia matrix a scalar ratio matrix $M^{-1}D = dI_n$ with $d > 0$ results. Especially in modern inverter-based grids working with the principle of model-matching control, such a uniform damping ratio can be realized. This kind of damping is also relevant in mechanical systems, where it is referred to as mass-proportional damping, a special case of Rayleigh damping. In order to simplify the wording, we refer to dI_n as the uniform damping matrix. Since M is positive definite, the required left-multiplication with L results in a new matrix $M^{-1}L = \tilde{L}$. Once again in order to simplify the wording, we denote $\tilde{L}$ by L in the following because $\tilde{L}$ shares the matrix properties of L described in the previous section. Hence, we analyze the following system:

$$\ddot{x} + d\dot{x} + Lx = 0. \tag{17}$$

Applying Theorem 6 to (17), we obtain

$$\max_i l_{ii} \le \frac{d^2}{2}. \tag{18}$$

With a restricted structure $D = dI_n$ compared to D diagonal, better conditions for stability will be derived in the next section.

4.1 A stability criterion for uniform damping

We give a stronger version of (Skar, 1980, thm. 2.6.ii) for the case of uniform damping, which has a shorter proof thanks to application of the complex Routh-Hurwitz criterion.

Theorem 8. The number of zero eigenvalues of the quadratic pencil $Q(\lambda) = I_n\lambda^2 + dI_n\lambda + L$ with $d > 0$ and of L coincide and all nonzero eigenvalues of $Q(\lambda)$ have negative real parts if and only if for each nonzero eigenvalue $\nu_i = \alpha_i + \beta_i\mathrm{i}$, $i = 1, \dots, n$ of L,

$$\alpha_i d^2 > \beta_i^2 \tag{19}$$

or in other notation

$$d > d_\mathrm{crit} = \max_i \frac{|\mathrm{Im}\,\lambda_i(L)|}{\sqrt{\mathrm{Re}\,\lambda_i(L)}}. \tag{20}$$

Proof. The zero eigenvalues and nonzero eigenvalues of L are considered separately. The characteristic equation of the pencil $\det(Q(\lambda)) = \det(\lambda(\lambda + d)I_n + L) = 0$ yields the relation $-\nu_i = \lambda(\lambda + d)$, respectively

$$\lambda^2 + d\lambda + \nu_i = 0. \tag{21}$$

For each zero eigenvalue $\nu_i = 0$, $\lambda^2 + d\lambda = 0$ leads to one zero eigenvalue $\lambda = 0$ of the pencil and an additional stable eigenvalue at $\lambda = -d$. For each nonzero eigenvalue $\nu_i \in \mathbb{C}$, (21) is a complex polynomial. The complex Hurwitz theorem, known as the Bilharz-Schur theorem (Bilharz, 1944), establishes Hurwitz stability for the polynomials (21) if and only if (19) applies. □

The importance of Theorem 8 lies in the fact that the pencil eigenvalue problem can be reduced to a common eigenvalue problem for L. Using estimates for the location of the eigenvalues of L, sufficient criteria for stability without calculating the full spectrum of L can be provided.

4.2 Estimates for the eigenvalues of L

Instead of applying the Gershgorin circle theorem directly to L, we first apply the well-known Brauer eigenvalue shift theorem (Horn and Johnson, 2012, thm. 2.4.10.1). According to this theorem, the zero eigenvalue associated with the 1-vector can be shifted via

$$B = L + 1_n w^*, \tag{22}$$

where the arbitrary vector $w \in \mathbb{C}^n$ is chosen as a real vector since this is the best choice in the present real eigenvalue problem.

Example 9. Consider the matrix

$$L = \begin{bmatrix} 3 & -1 & -2 \\ -2 & 3 & -1 \\ -1 & -4 & 5 \end{bmatrix} \tag{23}$$

with eigenvalues $\lambda_1(L) = 0$, $\lambda_{2/3}(L) = 5.5 \pm \mathrm{i}\sqrt{3}/2$. By applying (22) with $w = [1\ 2\ 1]^\top$, we obtain

$$B = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 5 & 0 \\ 0 & -2 & 6 \end{bmatrix},$$

where $\lambda_1(B) = w^\top 1_n = 4$. The eigenvalues and row Gershgorin discs of both matrices are presented in Figure 1. The example shows that the shift has multiple effects:

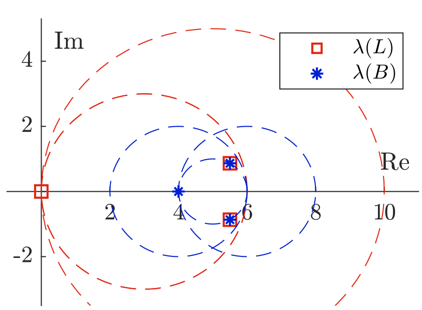

Fig. 1. Eigenvalues and row Gershgorin circles of the example matrices

the diagonal elements are increased and the Gershgorin row radii r_i are reduced, and the zero eigenvalue is shifted into the $\mathbb{C}^+$.

To estimate the location of the nontrivial eigenvalues $\nu_i \neq 0$ of L, we apply the row Gershgorin circle theorem to B, obtaining

$$|\nu_i - b_{ii}| \le r_i = \sum_{j \neq i} |b_{ij}|$$
$$|\nu_i - (l_{ii} + w_i)| \le \sum_{j \neq i} |l_{ij} + w_j|.$$

The boundaries of the Gershgorin discs are described by

$$\beta_i^2 = r_i^2 - (\alpha_i - b_{ii})^2.$$

Combining this with (19) results in

$$d^2 \alpha_i = r_i^2 - (\alpha_i - b_{ii})^2,$$

which has no real solutions if

$$\frac{d^2}{2} > b_{ii} - \sqrt{b_{ii}^2 - r_i^2}$$
$$\frac{d^2}{2} > l_{ii} + w_i - \sqrt{(l_{ii} + w_i)^2 - \Big(\sum_{j \neq i} |l_{ij} + w_j|\Big)^2}. \tag{24}$$

Hence, all nontrivial eigenvalues of the pencil are in $\mathbb{C}^-$ if (24) holds for all i.

Note that the case $w_i = 0$ for all i is a strict version of the inequality (18). Although (24) can be evaluated easily, it is not well-suited for the choice of appropriate w_i. For this purpose, we recall the well-known inequality

$$\sqrt{x^2 - y^2} > |x| - |y|, \quad |x| > |y|. \tag{25}$$

With (25), a weaker sufficient condition than (24) is obtained:

$$\frac{d^2}{2} > \sum_{j \neq i} |l_{ij} + w_j|, \tag{26}$$

supposed that w is chosen such that $l_{ii} \ge \sum_{j \neq i} |l_{ij} + w_j|$. Although (26) is weaker than (24), it is still better than (18), if for the critical index $l_{ii} > \sum_{j \neq i} |l_{ij} + w_j|$ holds. From (26), it is obvious that the w_i should be chosen positive, because l_{ij}, $i \neq j$ are negative, and they also contribute to increasing the main diagonal elements.

We will now discuss multiple estimates that can be used in applying Theorem 8:

- To guarantee that no row Gershgorin circles increases in size, i.e. $r_i(B) \le r_i(L)$ for all i,
$$w_j = \min_{i \neq j} |l_{ij}| \tag{27}$$
can be used, which ensures that no off-diagonal elements become positive.
- Another choice which guarantees that $r_i(B) \le r_i(L)$ for all i is
$$w_i = \min_k \operatorname*{med}_{j \neq k} |l_{kj}|, \tag{28}$$
the minimum of the rowwise medians of off-diagonal elements.
- The choice of the columnwise median of off-diagonal elements
$$w_j = \operatorname*{med}_{i \neq j} |l_{ij}| \tag{29}$$
minimizes the column Gershgorin radii $c_j(B)$. In contrast to the methods using the row Gershgorin radii, the resulting column radii can be calculated from the elements of L more directly: Let x_{kj}, $k = 1, \dots, n-1$ with $x_{1j} \ge x_{2j} \ge \cdots \ge x_{n-1,j}$ be the ordered off-diagonal elements $l_{1j}, \dots, l_{j-1,j}, l_{j+1,j}, \dots, l_{nj}$. For odd n, the median is $w_j = x_{(n-1)/2,j}$ and the radius
$$c_j(B) = \sum_{i=1}^{(n-3)/2} x_{ij} - \sum_{i=(n+1)/2}^{n-1} x_{ij}. \tag{30}$$

For even n, any value $x_{n/2-1,j} \geq w_j \geq x_{n/2,j}$ can be used for the not uniquely defined median, and the column Gershgorin radii are obtained as

$$c_j(B) = \sum_{i=1}^{n/2-1} x_{ij} - \sum_{i=n/2}^{n-1} x_{ij}. \quad (31)$$

In the case of a matrix L associated with a weight-balanced graph, i.e. $L^\top 1_n = 0_n$, we have $c_j(B) \leq b_{jj}$, and the Gershgorin circles are within $\overline{\mathbb{C}^+}$. Note however, that for L which are not weight balanced, there are some $c_j(L) > l_{jj}$, and there may be some $c_j(B) > b_{jj}$ even after this shift.

- An optimal choice for w is

$$w_{\text{opt}} = \underset{w \in \mathbb{R}_+^n}{\operatorname{argmin}} \max_{i \in \{1,\ldots,n\}} \quad (32)$$
$$l_{ii} + w_j - \sqrt{(l_{ii} + w_j)^2 - \Big(\sum_{j \neq i} |l_{ij} + w_j|\Big)^2},$$

which obtains the best approximation using the Gershgorin circle theorem. However, the added computational cost would be better spent on eigenvalue calculations.

Remark: Marsli and Hall (2020) proposed the method of Gershgorin disks of the second type for constant row-sum matrices. This method offers inclusion regions for the nontrivial eigenvalues. The estimates are generally better than with the classical Gershgorin disks, because smaller column radii are obtained. The comparison of the radius formula in Marsli and Hall (2020) and (30) and (31) shows that the shift using the columnwise medians offers a better estimate.

4.3 Comparison of the shifting approaches

We apply the methods proposed in the previous section to the matrix (23) and present the discovered minimal d which guarantee stability in Table 2. Note that the method using the columnwise median off-diagonal elements may still work even for matrices that are not weight-balanced.

Table 2. Minimal uniform damping for example (23) by different estimates

Method	w	$d_{\min}$
(18)	$(0, 0, 0)$	3.1623
(27)	$(1, 1, 1)$	1.2679
(28)	$(1.5, 1.5, 1.5)$	1.2114
(29) with $c_i(B)$	$(1.5, 2.5, 1.5)$	1.3343
(29) with $r_i(B)$	$(1.5, 2.5, 1.5)$	0.9684
(32)	$(1.3459, 2.4698, 2)$	0.7157
exact	N/A	0.3693

Although for all the proposed choices of w in Section 4.2 examples can be designed where a specific choice offers no improvement, our comprehensive studies show that in common electrical networks after a Kron reduction dense matrices L appear. For them, an improvement in nearly all cases was observed.

5. FURTHER RESEARCH

Motivated by the success of the shifting approach in Section 4, we want to try to derive eigenvalue estimations for the pencil directly for a new stability test. Another research direction is to use the stability margin measured with the spectral abscissa of the nontrivial eigenvalues to derive intervals around the uniform damping ratio.

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