Dynamic Flows with Time-Dependent Capacities

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Abstract. Dynamic network flows, sometimes called flows over time, extend the notion of network flows to include a transit time for each edge. While Ford and Fulkerson showed that certain dynamic flow problems can be solved via a reduction to static flows, many advanced models considering congestion and time-dependent networks result in NP-hard problems. To increase understanding of these advanced dynamic flow settings we study the structural and computational complexity of the canonical extensions that have time-dependent capacities or time-dependent transit times.

If the considered time interval is finite, we show that already a single edge changing capacity or transit time once makes the dynamic flow problem weakly NP-hard. In case of infinite considered time, one change in transit time or two changes in capacity make the problem weakly NP-hard. For just one capacity change, we conjecture that the problem can be solved in polynomial time. Additionally, we show the structural property that dynamic cuts and flows can become exponentially complex in the above settings where the problem is NP-hard. We further show that, despite the duality between cuts and flows, their complexities can be exponentially far apart.

1 Introduction

Network flows are a well established way to model transportation of goods or data through systems representable as graphs. Dynamic flows (sometimes called flows over time) include the temporal component by considering the time to traverse an edge. They were introduced by Ford and Fulkerson [2], who showed that maximum dynamic flows in static networks can be found using *temporally repeated flows*, which send flow over paths of a static maximum flow as long as possible.

Since capacities in real-world networks tend to be more dynamic, several generalizations have been considered in the literature. One category here is congestion modeling networks, where transit times of edges can depend on the flow routed over them [6,7]. Other generalizations model changes in the network independently from the routed flow [4,11,9]. This makes it possible to model known physical changes to the network and allows for situations, where we have estimates of the overall congestion over time that is caused by external entities that are not part of the given flow problem. There are also efforts to include different objectives for the flow, e.g., for evacuation scenarios, it is beneficial for a flow to maximize arrival for all times, not just at the end of the considered time interval [1].

Most problems modeling congestion via flow-dependent transit times are NP-hard. If the transit time depends on the current load of the edge, the flow problems become strongly NP-hard and no ε approximation exists unless $\mathrm{P} = \mathrm{NP}$ [6]. If the transit time of an edge instead only depends on its inflow rate while flow that entered the edge earlier is ignored the flow problems are also strongly NP-hard [7]. When allowing to store flow at vertices, pseudo-polynomial algorithms are possible if there are time-dependent capacities [4] and if there additionally are time-dependent transit times [11,9]. In the above mentioned evacuation scenario, one aims at finding the so-called earliest arrival flow (EAF). It is also NP-hard in the sense that it is hard to find the average arrival time of such a flow [1]. Moreover, all known algorithms to find EAFs have worst case exponential output size for all known encodings [1].

In this paper, we study natural generalizations of dynamic flows that have received little attention so far, allowing time-dependent capacities or time-dependent transit times. We prove that finding dynamic flows with time-dependent capacities or time-dependent transit times is weakly NP-hard, even if the graph is acyclic and only a single edge experiences a capacity change at a single point in time. This shows that a single change in capacity already increases the complexity of the – otherwise polynomially solvable – dynamic flow problem. It also implies that the dynamic flow problem with time-dependent capacities is not FTP in the number of capacity changes. The above results hold in the setting where the considered time interval is finite. If we instead consider infinite time, the results remain the same for time-dependent transit times. For time-dependent capacities, two capacity changes make the problem weakly NP-hard. We conjecture that it can be solved in polynomial when there is only one change.

Beyond these results on the computational complexity, we provide several structural insights. For static flows, one is usually not only interested in the flow value but wants to output a maximum flow or a minimum cut. The concept of flows translates more or less directly to the dynamic setting [2], we need to consider time-dependent flows and cuts if we have time-dependent capacities or transit times. In this case, instead of having just one flow value per edge, the flow is a function over time. Similarly, in a dynamic cut, the assignment of vertices to one of two partitions changes over time. The cut–flow duality, stating that the capacity of the minimum cut is the same as the value of the maximum flow also holds in this and many related settings [5,8,11]. Note that the output complexity can potentially be large if the flow on an edge or the partition of a vertex in a cut changes often. For dynamic flows on static graphs (no changes in capacities or transit times) vertices start in the target vertices' partition and at some point change to the source partition, but never the other way [10], which shows that cuts have linear complexity in this setting.

In case of time-dependent capacities or transit times, we show that flow and cut complexity are sometimes required to be exponential. Specifically, for all cases where we show weak NP-hardness, we also give instances for which every maximum flow and minimum cut have exponential complexity. Thus, even a single edge changing capacity or transit time once can jump the output complexity from linear to exponential. Moreover, we give examples where the flow complexity is exponential while there exists a cut of low complexity and vice versa.

We note that the scenario of time-dependent capacities has been claimed to be strongly NP-complete [9] before. However, we suspect the proof to be flawed as one can see that this scenario can be solved in pseudo-polynomial time. Moreover, the above mentioned results on the solution complexity make it unclear whether the problem is actually in NP. In Appendix A, we point out the place where we believe the proof for strong NP-hardness is flawed.

2 Preliminaries

We consider dynamic networks $G = (V, E)$ with directed edges and designated source and target vertices $s, t \in V$. Edges $e = (v, w) \in E$ have a time-dependent non negative *capacity* $u_e \colon [0, T] \to \mathbb{R}_0^+$, specifying how much flow can enter e via v at each time. We allow u_e to be non-continuous but only for finitely many points in time. In addition, each edge $e = (v, w)$ also has a non negative *transit time* $\tau_e \in \mathbb{R}^+$, denoting how much time flow takes to move from v to w when traversing e. Note that the capacity is defined on $[0, T]$, i.e., time is considered from 0 up to a *time horizon* T.

Let f be a collection of measurable functions $f_e \colon [0, T - \tau_e] \to \mathbb{R}$, one for each edge $e \in E$, assigning every edge a flow value depending on the time. The restriction to the interval $[0, T - \tau_e]$ has the interpretation that no flow may be sent before time 0 and no flow should arrive after time T in a valid flow. To simplify notation, we allow time values beyond $[0, T - \tau_e]$ and implicitly assume

$f_e(\Theta) = 0$ for $\Theta \notin [0, T - \tau_e]$. We call f a *dynamic flow* if it satisfies the *capacity constraints* $f_e(\Theta) \leq u_e(\Theta)$ for all $e \in E$ and $\Theta \in [0, T - \tau_e]$, and *strong flow conservation*, which we define in the following.

The *excess flow* $\mathrm{ex}_f(v, \Theta)$ of a vertex v at time Θ is the difference between flow sent to v and the flow sent from v up to time Θ, i.e.,

$$\mathrm{ex}_f(v, \Theta) \coloneqq \int_0^{\Theta} \sum_{e=(u,v)\in E} f_e(\zeta - \tau_e) - \sum_{e=(v,u)\in E} f_e(\zeta) \,\mathrm{d}\zeta.$$

We have strong flow conservation if $\mathrm{ex}_f(v, \Theta) = 0$ for all $v \in V \setminus \{s, t\}$ and $\Theta \in [0, T]$.

The *value* of f is defined as the excess of the target vertex at the time horizon $|f| \coloneqq \mathrm{ex}_f(t, T) = -\,\mathrm{ex}_f(s, T)$. The *maximum dynamic flow problem with time-dependent capacities* is to find a flow of maximum value. We refer to its input as *dynamic flow network*.

A cut-flow duality similar to the one of the static maximum flow problem holds for the maximum dynamic flow problem with the following cut definition. A *dynamic cut* or *cut over time* is a partition of the vertices $(S, V \setminus S)$ for each point in time, where the source vertex s always belongs to S while the target t never belongs to S. Formally, each vertex $v \in V$ has a boolean function $S_v \colon [0, T] \to \{0, 1\}$ assigning v to S at time Θ if $S_v(\Theta) = 1$. As for the flow, we extend S_v beyond $[0, T]$ and set $S_v(\Theta) = 1$ for $\Theta > T$ for all $v \in V$ (including t). The *capacity* $\mathrm{cap}(S)$ of a dynamic cut S is the maximum flow that could be sent on edges from S to $V \setminus S$ during the considered time interval $[0, T]$, i.e.,

$$\mathrm{cap}(S) = \int_0^T \sum_{\substack{(v,w)\in E \\ S_v(\Theta)=1 \\ S_w(\Theta+\tau_e)=0}} u_{(v,w)}(\Theta) \,\mathrm{d}\Theta.$$

An edge (v, w) *contributes* to the cut S at time Θ if it contributes to the above sum, so $S_v(\Theta) = 1 \wedge S_w(\Theta + \tau_e) = 0$. Note that this is similar to the static case, but in the dynamic variant the delay of the transit time needs to be considered. Thus, for the edge $e = (v, w)$ we consider v at time Θ and w at time $\Theta + \tau_e$. Moreover, setting $S_v(\Theta) = 1$ for all vertices v if $\Theta > T$ makes sure that no point in time beyond the time horizon contributes to $\mathrm{cap}(S)$.

Theorem 1 (Min-Cut Max-Flow Theorem [8,11]). *For a maximum flow over time f and a minimum cut over time S it holds $|f| = cap(S)$.*

Proof. The theorem by Philpott [8, Theorem 1] is more general than the setting considered here. They in particular allow for time-dependent storage capacities of vertices. We obtain the here stated theorem by simply setting them to constant zero. The theorem by Tjandra [11, Theorem 3.4] is even more general and thus also covers the setting with time-dependent transit times. □

Though the general definition allows the capacity functions to be arbitrary, for our constructions it suffices to use piecewise constant capacities. We note that in this case, there always exists a maximum flow that is also piecewise constant, assigning flow values to a set of intervals of non-zero measure. The property that the intervals have non-zero measure lets us consider an individual point Θ in time and talk about the contribution of an edge to a cut or flow at time Θ, as Θ is guaranteed to be part of a non-empty interval with the same cut or flow. For the remainder of this paper, we assume that all flows have the above property.

We define the following additional useful notation. We use $S(\Theta) \coloneqq \{v \in V \mid S_v(\Theta) = 1\}$ and $\bar{S}(\Theta) \coloneqq \{v \in V \mid S_v(\Theta) = 0\}$ to denote the cut at time Θ. Moreover, a vertex v changes its partition at time Θ if $S_v(\Theta - \varepsilon) \neq S_v(\Theta + \varepsilon)$ for every sufficiently small $\varepsilon > 0$. We denote a change from S to $\bar{S}$ with $S_v \xrightarrow{\Theta} \bar{S}_v$ and a change in the other *direction* from $\bar{S}$ to S with $\bar{S}_v \xrightarrow{\Theta} S_v$. We denote the number of partition changes of a vertex v in a cut S with $\mathrm{ch}_v(S)$. Moreover the total number of changes in S is the *complexity* of the cut S. For a flow f, we define changes on edges as well as the complexity of f analogously.

In the above definition of the maximum dynamic flow problem we allow time-dependent capacities but assume constant transit times. Most of our results translate to the complementary scenario where transit times are time-dependent while capacities are constant. In this setting $\tau_e(\Theta)$ denotes how much time flow takes to traverse e, if it enters at time Θ. Similarly to the above definition, we allow τ_e to be non-continuous for finitely many points in time.

Additionally we look at the scenarios where infinite time ($\Theta \in (-\infty, \infty)$) is considered instead of only considering times in $[0, T]$. This removes structural effects caused by the boundaries of the considered time interval. Intuitively, because we are working with piecewise constant functions with finitely many incontinuities, there exists a point in time Θ that is sufficiently late that all effects of capacity changes no longer play a role. From that time on, one can assume the maximum flow and minimum cut to be constant. The same holds true for a sufficiently early point in time. Thus, to compare flow values it suffices to look at a finite interval I. Formally, f is a maximum dynamic flow with *infinite considered time* if it is constant outside of I and maximum on I, such that for any larger interval $J \supset I$ there exists a large enough interval $K \supset J$ so that a maximum flow with considered time interval K can be f during J. Minimum cuts with infinite considered time are defined analogously. Such maximum flows and minimum cuts always exist as temporally repeated flows provide optimal solutions to dynamic flows and we only allow finitely many changes to capacity or traversal time.

We will need the set of all integers up to k and denote it $[k] \coloneqq \{i \in \mathbb{N}^+ | i \leq k\}$.

3 Computational Complexity

In this section we study the computational complexity of the dynamic flow problem with time-dependent capacities or transit times. We consider finite and infinite time. For all cases except for a single capacity change with infinite considered time, we prove NP-hardness.

We start by showing hardness in the setting where we have time-dependent capacities with only one edge changing capacity once. Our construction directly translates to the setting of infinite considered time with one edge changing capacity twice. For the case of time-dependent transit times we prove hardness for one change even in the infinite considered time setting. This also implies hardness for one change when we have a finite time horizon.

We reduce from the *partition problem*, which is defined as follows. Given a set of positive integers $S = \{b_1, \ldots, b_k\}$ with $\sum_{i=1}^{k} b_i = 2L$, is there a subset $S' \subset S$ such that $\sum_{a \in S'} a = L$?

Theorem 2. *The dynamic flow problem with time-dependent capacities is weakly NP-hard, even for acyclic graphs with only one capacity change.*

Proof. Given an instance of the partition problem, we construct $G = (V, E)$ as shown in Figure 1 and show that a solution to partition is equivalent to a flow of value 1 in G.

Every $b_i \in S$ corresponds to a vertex x_i which can be reached by x_{i-1} with one edge of transit time b_i and one bypass edge of transit time zero. The last of these vertices x_k is connected to the target t with an edge only allowing flow to pass during $[L+1, L+2]$, where the lower border is ensured by the capacity change of (x_k, t) and the upper border is given by the time horizon $T = L+3$. The source s is connected to x_0 with an edge of low capacity $\frac{1}{L+1}$, so that the single flow unit that can enter this edge in $[0, T-2]$ can pass (x_k, t) during one time unit.

Since a solution to the partition problem is equivalent to a path of transit time L through the x_i, we additionally provide paths of transit time $0, 1, \ldots, L-1$ bypassing the b_i edges via the y_i so that a solution for partition exists, if and only if flow of value 1 can reach t. To provide the bypass paths, we set $\ell \in \mathbb{N}_0$ so that $L = 2^{\ell+1} + r,\ r \in \mathbb{N}_0, r < 2^{\ell+1}$ and define vertices $y_i, i \in \mathbb{N}_0, i \leq \ell$. They create a path of transit time $L-1$ where the edges' transit times are powers of two and one edge of transit time r and all edges can be bypassed by an edge with transit time zero. This allows all integer transit times smaller than $L-1$. All edges except for (s, x_0) have unit capacity when they are active.

Given a solution S' to the partition problem, we can route flow leaving s during $[0, 1]$ through the x_i along the non zero transit time edges if and only if the corresponding b_i is in S'. Flow leaving s in $[1, L+1]$ can trivially reach x_k during $[L+1, L+2]$ using the bypass paths, providing a maximum flow of 1.

Only one unit of flow can reach x_0 until $L+2$, considering the time horizon $T = L+3$ and the transit time of (x_k, t), the flow can have value at most 1. Given a flow that sends one unit of flow to t, we can see that the flow has to route all flow that can pass (s, x_0) during $[0, L+1]$ to t. Due to the integrality of transit times, the flow leaving s during $[0, 1]$ has to take exactly time L to traverse from x_0 to x_k. The bypass paths via y_0 are too short for this. As such, this time is the sum of edge transit times taken from the partition instance and zeroes from bypass edges, and there exists a solution S' to the partition problem that consists of the elements corresponding to the non zero transit time edges taken by this flow. □

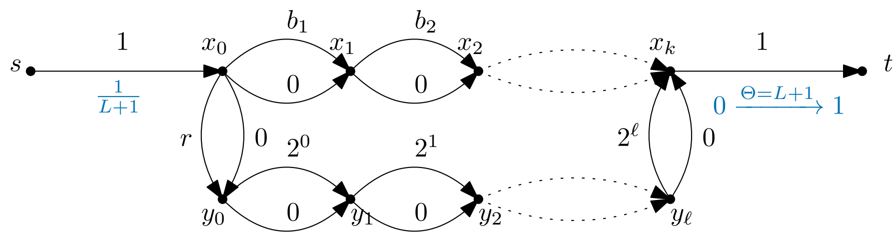


Fig. 1: Graph constructed for the reduction of the partition problem to dynamic flow with time-dependent capacities. Flow leaving s at time zero can only reach t if it takes exactly time L to traverse from x_0 to x_k, such choosing a partition. Black numbers are transit times, blue numbers indicate capacity, all unspecified capacities are 1, time horizon is $T = L + 3$.

Corollary 1. *The dynamic flow problem with time-dependent capacities and infinite considered time is weakly NP-hard, even for acyclic graphs with only two capacity changes.*

Proof. In the proof of Theorem 2, we restricted the flow on the edge from x_k to t to have non-zero capacity only at time $[L + 1, L + 2]$. For Theorem 2, we achieved the lower bound with one capacity change and the upper bound with the time horizon. Here, we can use the same construction but use a second capacity change for the upper bound. □

For the case of time-dependent transit times, we use a similar reduction. We start with the case of infinite considered time.

Theorem 3. *The dynamic flow problem with infinite considered time and time-dependent transit times is weakly NP-hard, even for acyclic graphs with only one transit time change.*

Proof. Similar to the proof of Theorem 2 we give a reduction of the partition problem. The constructed graph can be seen in Figure 2. We want to link the existence of a transit time L path to a maximum flow sending 1 flow per time from s. For this, we start the graph with an edge (s, x_0) whose transit time gets reduced from 1 to zero at time $\Theta = 0$. This results in 2 units of flow reaching x_0 at time $\Theta = 0$, while only a flow of 1 can traverse (x_0, t). This means that flow of 1 has to pass through the x_i and y_i. The paths through the x_i and the bypass paths through the y_i function like in the proof of Theorem 2, but here the bypass edges also provide paths of transit times $L + 1$ to $2L$. Because the edge (x_k, t) has capacity $u_{(x_k,t)} = \frac{1}{2L+1}$, the flow routed through x_k has to arrive at x_k using at least $2L+1$ paths with different transit times. The paths from x_0 to x_k have integer transit times between zero and $2L$, so to route the extra unit of flow arriving at x_0 at time $\Theta = 0$, flow needs to be routed through one path of each integer transit time between 0 and $2L$. The bypass paths do not offer a path of transit time L, so, like in the proof of Theorem 2, this flow unit can be completely routed through the network if and only if the partition problem has a solution. □

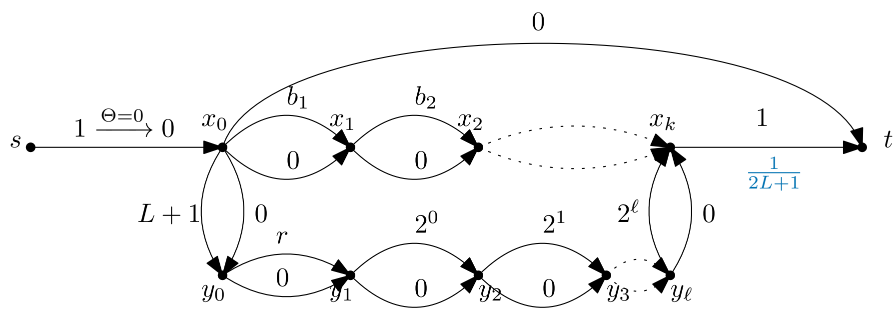


Fig. 2: Graph constructed for the reduction of the partition problem to dynamic flow with infinite considered time and time-dependent transit times. The flow units leaving s at times -1 and zero can only reach t if they take the direct (x_0, t) edge and $2L+1$ paths with different transit times to x_k. Black numbers are transit times, blue numbers specify capacity, all unspecified capacities are 1.

To translate this result to the case of a finite time horizon, note that we can use the above construction and choose the time horizon sufficiently large to obtain the following corollary.

Corollary 2. *The dynamic flow problem with time-dependent transit times is weakly NP-hard, even for acyclic graphs with only one transit time change.*

Proof. Using the construction of the proof for Theorem 3, we can restrict time to the interval $[-1, 2L+1]$, then a solution to the partition problem is equivalent to the existence of a flow of value $2L+2$. To get a considered time interval from zero to a time horizon, we let the transit time change of (s, x_0) occur at time $\Theta = 1$ instead and set $T = 2L+2$. □

This leaves one remaining case: infinite considered time and a single capacity change. For this case, we can show that there always exists a minimum dynamic cut, where each vertex changes partition at most once and all partition changes are of the same direction. Furthermore, for given partitions before and after the changes, a linear program can be used to find the optimal transition as long as no vertex changes partition more than once and all partition changes are of the same direction. This motivates the following conjecture.

Conjecture 1. The minimum cut problem in a dynamic flow network with only a single change in capacity and infinite considered time can be solved in polynomial time.

4 The Complexity of Maximum Flows and Minimum Cuts

We first construct a dynamic flow network such that all maximum flows and minimum cuts have exponential complexity. Afterwards, we show that there are also instances that require exponentially complex flows but allow for cuts of linear size and vice versa. These results are initially proven for a single change in capacity and are then shown to also hold in the setting with time dependent transit times, likewise with only one change in transit time required.

4.1 Exponentially Complex Flows and Cuts

We initially focus on the complexity of cuts and only later show that it transfers to flows. Before we start the construction, note that the example in Figure 3 shows how the partition change of two vertices a and b can force a single vertex v to change its partition back and forth. This type of enforced partition change of v is at the core of our construction.

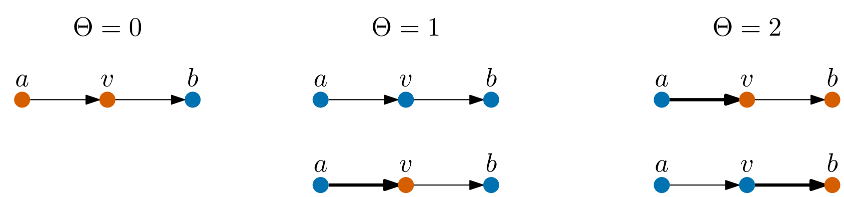


Fig. 3: Example where a changes from $\bar{S}$ (red) to S (blue) at time 1 and b changes from S to $\bar{S}$ at time 2. Only edges from S to $\bar{S}$ contribute to the cut (bold edges). Assuming v starts in $\bar{S}$ and $u_{(a,v)} < u_{(v,b)}$ as well as $\tau_{(a,v)} = \tau_{(v,b)} = 0$, v has to change to S at time 1 and back to $\bar{S}$ at time 2 in a minimum cut (top row). The bottom row illustrates the alternative (more expensive) behavior of v.

More specifically, we first give a structure with which we can force vertices to mimic the partition changes of other vertices, potentially with fixed time delay.

The *mimicking gadget* links two non terminal vertices $a, b \in V \setminus \{s, t\}$ using edges $(a, b), (b, t) \in E$ with capacities $u_{(a,b)} = \alpha, u_{(b,t)} = \beta$. The following lemma shows what properties α and β need to have such that the mimicking gadget does its name credit, i.e., that b mimics a with delay $\tau_{(a,b)}$. A visualization of the mimicking gadget is shown in Figure 4.

Lemma 1. *Let G be a graph that contains the mimicking gadget as a sub-graph, such that*

$$\alpha > \sum_{w|(b,w)\in E} u_{(b,w)} \quad \text{and} \quad \beta > \sum_{w|(w,b)\in E\setminus(a,b)} u_{(w,b)}.$$

Then, $S_b(\Theta) = S_a(\Theta - \tau_{(a,b)})$ for every minimum cut S and times $\Theta \in (\tau_{(a,b)}, T - \tau_{(b,t)})$.

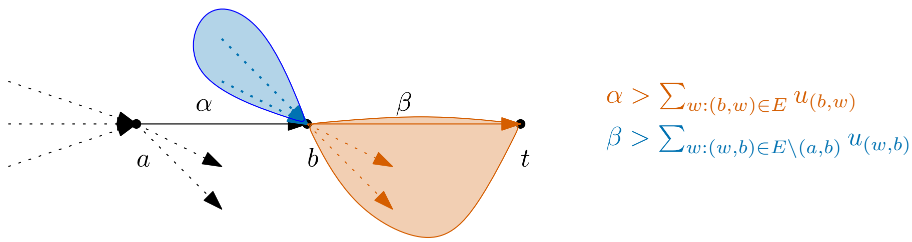


Fig. 4: Gadget linking the partitions of two vertices a and b, so that b mimics a with a delay of $\tau_{(a,b)}$; α, β are capacities.

Proof. We first show $a \in S(\Theta - \tau_{(a,b)}) \implies b \in S(\Theta)$. With the partition of a fixed, we look at possible contribution to S of edges incident to b at time Θ. For $b \in \bar{S}(\Theta)$ the contribution is at least α, because $\Theta \in (\tau_{(a,b)}, T - \tau_{(b,t)})$ ensures that (a, b) can contribute to S. For $b \in S(\Theta)$ the contribution is at most $\sum_{w|(b,w)\in E} u_{(b,w)} < \alpha$. Because S is a minimum cut, we obtain $b \in S(\Theta)$. The other direction $a \in \bar{S}(\Theta - \tau_{(a,b)}) \implies b \in \bar{S}(\Theta)$ holds for similar reasons. For $b \in S(\Theta)$ the contribution is at least β. For $b \in \bar{S}(\Theta)$ the contribution is at most $\sum_{w|(w,b)\in E} u_{(w,b)} < \beta$. $\square$

Note that Lemma 1 does not restrict the edges incident to a. Thus, we can use it rather flexibly to transfer partition changes from one vertex to another.

To enforce exponentially many partition changes, we next give a gadget that can double the number of partition changes of one vertex. To this end, we assume that, for every integer $i \in [k]$, we already have access to vertices a_i with *period* $p_i \coloneqq 2^i$, i.e., a_i changes partition every p_i units of time. Note that a_1 is the vertex with the most changes. With this, we construct the so-called binary counting gadget that produces a vertex v with period $p_0 = 1$, which results in it having twice as many changes as a_1. Roughly speaking, the binary counting gadget, shown in Figure 6, consists of the above mentioned vertices a_i together with additional vertices b_i such that b_i mimics a_i. Between the a_i and b_i lies the central vertex v with edges from the vertices a_i and edges to the b_i. Carefully chosen capacities and synchronization between the a_i and b_i results in v changing partition every step.

To iterate this process using v as vertex for the binary counting gadget of the next level, we need to ensure functionality with the additionally attached edges of the mimicking gadget.

The *binary counting gadget* H_k shown in Figure 6 is formally defined as follows. It contains the above mentioned vertices a_i, b_i for $i \in [k]$ and the vertex v. Additionally, it contains the source s and target t. On this vertex set, we have five types of edges. All of them have transit time 1 unless explicitly specified otherwise. The first two types are the edges (a_i, b_i) and (b_i, t) for $i \in [k]$, which form a mimicking gadget. We set $\tau_{(a_i,b_i)} = p_i + 1$ which makes b_i mimic the changes of a_i with delay $p_i + 1$. Moreover, we set $u_{(a_i,b_i)} = \alpha_i \coloneqq 2^{i-1} + 2\varepsilon$ and $u_{(b_i,t)} = \beta_i \coloneqq 2^{i-1} + \varepsilon$. We will see that these α_i and β_i satisfy the requirements of the mimicking gadget in Lemma 1. The third and fourth types of edges are (a_i, v) and (v, b_i) for $i \in [k]$ with capacities $u_{(a_i,v)} = u_{(v,b_i)} = 2^{i-1}$. These edges have the purpose to force the partition changes of v, similar to the simple example in Figure 3. Finally, we have the edge (s, v) with capacity $u_{(s,v)} = 1-\varepsilon$. It has the purpose to fix the initial partition of v and introduce some asymmetry to ensure functionality even if additional edges are attached to v.

Our plan is to prove that the binary counting gadget H_k works as desired by induction over k. We start by defining the desired properties that will serve as induction hypothesis.

Definition 1. *Let G be a graph. We say that H_k is a* valid binary counting gadget *in G if H_k is a subgraph of G and every minimum cut S has the following properties.*

- *For $i \in [k-1]$, the vertex a_i has period p_i. It changes its partition 2^{k-i} times starting with a change from S to $\bar{S}$ at time 0 and ending with a change at time $2^k - 2^i$.*
- *For $i = k$, a_k changes from S to $\bar{S}$ at time 0 and additionally back to S at time 2^k.*

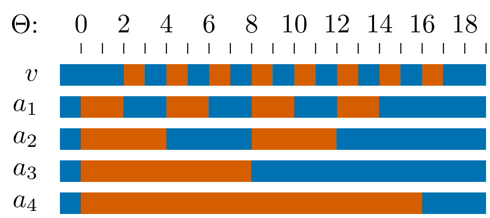


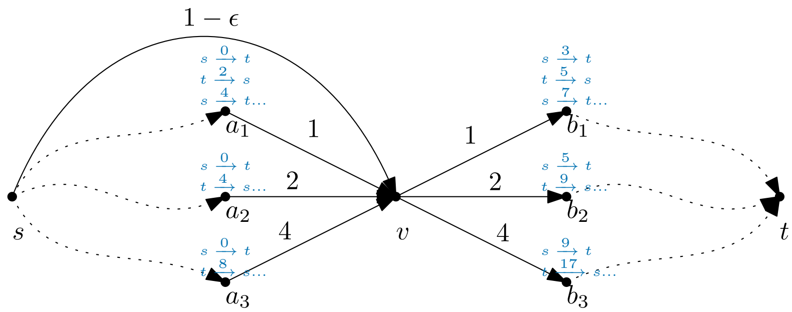
Fig. 5: Visualization of the partition change patterns of a valid binary counting gadget H_4. In blue sections the vertex is in S and in red sections it is in $\bar{S}$.

Note that in a valid binary counting gadget the a_i and v form a binary counter from 0 to $2^k - 1$ when regarding $\bar{S}$ as zero and S as 1, with v being the least significant bit (shifted back two time steps); see Figure 5.

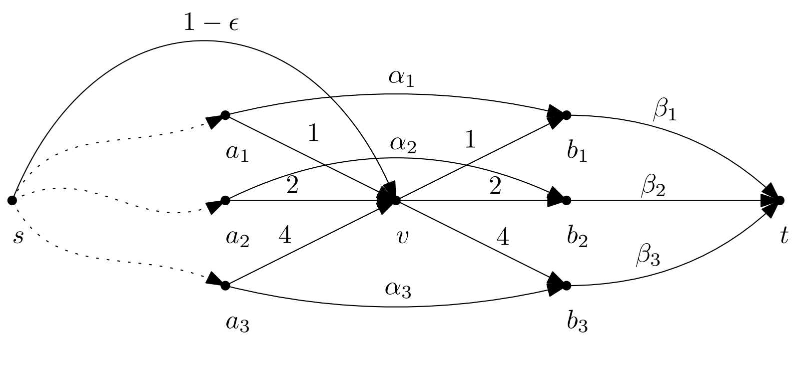
Lemma 2. *Let G be a graph containing the valid binary counting gadget H_k such that the central vertex v has no additional incoming edges, the sum of the capacities of additional outgoing edges of v is less than $1-\varepsilon$, and no additional edges are incident to the b_i. Then, for every minimum cut S, the central vertex v has period $p_0 = 1$ and changes 2^k times, starting with a change from S to $\bar{S}$ at time 2 and ending with a change at time $2^k + 1$.*

Proof. First note that the mimicking gadget allows causing an inverted counter behavior for the b_i, affecting v one time step before the corresponding a_i. For $\Theta \in \{0, \ldots, 2^k - 1\}$, the a_i change

$$S_{a_i} \xrightarrow{\Theta \equiv 0 \bmod 2^{i+1}} \bar{S}_{a_i} \bar{S}_{a_i} \xrightarrow{\Theta \equiv 2^i \bmod 2^{i+1}} S_{a_i} \bar{S}_{a_k} \xrightarrow{2^k} S_{a_k}.$$



(a) More intuitive visualization exemplary showing the partition changes of a_i, b_i in blue, omitting the mimicking gadgets. For improved readability the partitions are denoted by their terminal, i.e. s for S and t for $\bar{S}$.



(b) Full visualization with all edges, including the mimicking gadgets.

Fig. 6: Visualization of the binary counting gadget allowing to double the number of partition changes of a single vertex, ensuring 2^k changes of vertex v assuming that vertices $a_i, i < k$ are changing partition $\mathrm{ch}_{a_i} = 2^{k-i}$ times each, with $\mathrm{ch}_{a_k} = 2$ and correct timing; Black numbers are capacities.

The delay of $p_i + 1$ for inverting the counter is chosen because the activated states of the a_i and b_i are opposite, i.e. $S_{a_i}(\Theta - \tau_{(a_i,v)}) = 1$ allows contribution of $u_{(a_i,v)}$, but $S_{b_i}(\Theta + \tau_{(v,b_i)}) = 0$ allows contribution of $u_{(v,b_i)}$. The reset step at time 1, changing all b_i to S can be omitted, as all vertices start in S. So the necessary delay between a_i and b_i is $2^i + 1$. The desired change pattern therefore is

$$\bar{S}_{b_i} \xrightarrow{\Theta \equiv 1 \bmod 2^{i+1}} S_{b_i}\bar{S}_{b_k} \xrightarrow{2^{k+1}+1} S_{b_k}S_{b_i} \xrightarrow{\Theta \equiv 2^i+1 \bmod 2^{i+1}} \bar{S}_{b_i}$$

for $\Theta \in \{3, \ldots, 2^k + 1\}$. This pattern is achieved by the functionality of the mimicking gadget shown in Lemma 1 and the fact that G cannot have additional edges incident to any b_i.

To realize that the partition changes of v have to occur in the claimed way for any minimum cut S, we look at the edges incident to v. All other edges' contribution to any cut is already fixed. An edge (a_i, v) contributes 2^i to $\operatorname{cap}(S)$ if and only if $a_i \in S(\Theta - 1)$ and $v \in \bar{S}(\Theta)$ ($s \in S(\Theta)$ always holds, so (s, v) contributes $1 - \varepsilon$ if $v \in \bar{S}(\Theta)$) for some time Θ. Likewise the only way for (v, b_i) to contribute to $\operatorname{cap}(S)$ is $v \in S(\Theta)$ and $b_i \in \bar{S}(\Theta + 1)$ for some time Θ. Evaluating these contributions for the given partition changes of a_i, b_i we see that the counter of the b_i contributions is half a counting step – which corresponds to one time unit – ahead of the a_i contribution counter. For the last change of a_k, affecting v at time $2^k + 1$, the a_i counter gets larger than the b_i counter instead of equaling it. Formally for $\Theta \in [1, 2^k + 1)$ the contribution of (a_i, v) and (v, b_i) edges is

$$\sum_{i|S_{a_i}(\Theta-1)=1} u_{(a_i,v)} = \lfloor \frac{\Theta - 1}{2} \rfloor \sum_{i|S_{b_i}(\Theta+1)=0} u_{(v,b_i)} = \lfloor \frac{\Theta}{2} \rfloor$$

and for $\Theta \in [2^k + 1, 2^k + 2)$

$$\sum_{i|S_{a_i}(\Theta-1)=1} u_{(a_i,v)} = 2^k - 1 > 2^{k-1} = \sum_{i|S_{b_i}(\Theta+1)=0} u_{(v,b_i)}.$$

With the additional edge (s, v), the side with more potential to contribute to the cut changes every $\Theta \in \{2, \ldots, 2^k + 1\}$, forcing v to change its partition every time to ensure minimality of the cut. So the change pattern of v is

$$S_v \xrightarrow{\Theta \equiv 0 \bmod 2} \bar{S}_v\bar{S}_v \xrightarrow{\Theta \equiv 1 \bmod 2} S_v$$

for $\Theta \in \{2, \ldots, 2^k + 1\}$.

Edges leaving v can only contribute to $\operatorname{cap}(S)$ if $v \in S(\Theta)$, so whenever the gadget already ensures $v \in \bar{S}(\Theta)$, added outgoing edges cannot impede the gadgets behavior. When H_k ensures $v \in S(\Theta)$, we have

$$\sum_{i|S_{a_i}(\Theta-1)=1} u_{(a_i,v)} \geq \sum_{i|S_{b_i}(\Theta+1)=0} u_{(v,b_i)}.$$

To minimize $\operatorname{cap}(S)$, the assignment $v \in S(\Theta)$ remains necessary to minimize $\operatorname{cap}(S)$, because of the edge (s, v_k) with capacity $1 - \varepsilon$, which is larger than the sum over the capacities of all added edges. $\square$

Note that Lemma 2 provides the first part towards the induction step of constructing a valid H_{k+1} from a valid H_k. In the following, we show how to scale periods of the a_i such that a_i from H_k can serve as the a_{i+1} from H_{k+1} and v can serve as the new a_1. Afterwards, it remains to show two things. First, additional edges to actually build H_{k+1} from H_k can be introduced without losing validity. And secondly, we need the initial step of the induction, i.e., the existence of a valid H_1 even in the presence of only one capacity change.

We say that a minimum dynamic cut S *remains optimal under scaling and translation of time* if S is a minimum cut on graph $G = (V, E)$ with transit times τ_e, capacities $u_e(\Theta)$ and time interval $[0, T]$ if and only if $\hat{S}$ with $\hat{S}_v(r \cdot \Theta + T_0) \coloneqq S_v(\Theta)\ \forall \Theta \in [0, T]$ is a minimum cut on $\hat{G} = (V, E)$ with transit times $\hat{\tau}_e \coloneqq r \cdot \tau_e$, capacities $\hat{u}_e(r \cdot \Theta + T_0) \coloneqq u_e(\Theta)$ and time interval $[T_0, r \cdot T + T_0]$ for any $r \in \mathbb{R}^+, T_0 \in \mathbb{R}$.

Lemma 3. *Any dynamic cut remains optimal under scaling and translation of time. It also remains optimal under scaling of capacities.*

Proof. The capacity of a cut is unaffected by translation of time. Scaling time scales the capacity of any cut by the same factor. So the relative difference of the capacity of different cuts is not affected by scaling and translation of time.

Scaling capacities alters the capacity of every cut by the same factor, so the relative difference in capacity of cuts remains unchanged. □

With this, we can combine binary counting gadgets of different sizes to create a large binary counting gadget H_ℓ while only requiring a single capacity change.

Lemma 4. *For every $\ell \in \mathbb{N}^+$, there exists a polynomially sized, acyclic dynamic flow network with only one capacity change that contains a valid binary counting gadget H_ℓ.*

Proof. To create the necessary change patterns for the a_i of H_ℓ, we chain binary counting gadgets of increasing size, beginning with H_2 up to H_ℓ together creating the graph G_ℓ as shown in Figure 7. The coarse idea is to ensure the behavior of all $a_{k,i}$ by having them mimic the central vertex v_{k-i} of the correct smaller binary counting gadget.

Timewise, the binary counting gadgets H_k are scaled by $\Delta_k \coloneqq 2^{-k}$ and translated by $T_{0,k} \coloneqq 2 + 3(1 - 2\Delta_k) + \Delta_k$. So the first change from S to $\bar{S}$ of the $a_{k,i}$ in H_k should happen at time $T_{0,k}$ and the period between two successive changes of $a_{k,i}$ is $p_{k,i} \coloneqq \Delta_k \cdot 2^i = \Delta_{k-i}$, which is the period between two successive changes of v_{k-i}. To correctly synchronize the different H_k, the delay for the mimicking between the binary counting gadgets is set to $\tau_{(v_{k-i}, a_{k,i})} = 3 \cdot (\Delta_{k-i} - 2\Delta_k) + \Delta_k$. This is chosen so that $T_{0,k-i} + 2\Delta_{k-i} + \tau_{(v_{k-i}, a_{k,i})} = T_{0,k}$. We set $\varepsilon \coloneqq \frac{1}{6}$ for the capacity of (s, v_k) in H_k.

To ensure that the connecting mimicking gadgets do not exceed the permitted capacities leaving v_k of H_k, the capacities of all edges in the H_k are scaled by factor $\lambda_k \coloneqq \frac{1}{5^k}$. In accordance with this scaling and the capacity requirements of Lemma 1, the capacities of the mimicking gadget connecting v_{k-i} to $a_{k,i}$ are $u_{(v_{k-i}, a_{k,i})} = \lambda_k \gamma_i$ with $\gamma_i \coloneqq 2^i + 4\varepsilon$ and $u_{(a_{k,i}, t)} = \lambda_k \varepsilon$.

Ensuring the partition changes of v_0 can be done with one capacity change as shown in Figure 8, using a path s, v_0, t. Transit times for H_{start} are $\tau_{(s,v_0)} = 1$ and $\tau_{(v_0,t)} = T - 3$, capacities are $u_{(s,v_0)} = 2$ and $u_{(v_0,t)}(\Theta) = 1, \Theta < 2$ changing to $u_{(v_0,t)}(\Theta) = 3, \Theta \geq 2$. The time horizon is set to $T \coloneqq 6$.

G_ℓ clearly has polynomial size in regard to ℓ and there is only one capacity change. The previously shown functionality of the binary counting gadget in Lemma 2 and the mimicking gadget in Lemma 1 are the basis for showing, that the contained H_ℓ is valid. Lemma 3 provides that those gadgets' functionality is also given under the shifted and compressed time in which they are used for the construction of G_ℓ. To prove the correct behavior of the constructed graph, it needs to be shown that the mimicking gadgets adhere to the capacity restrictions established earlier, and that their attachment to smaller binary counting gadgets does not impede the behavior of those counting gadgets.

The correctness of the mimicking gadgets' capacities can easily be seen. The $a_{k,i}$ have no incoming edges outside of the mimicking gadget, so $u_{(a_{k,i},t)} > 0$ fulfills the requirement for $(a_{k,i}, t)$. There are two additional edges leaving $a_{k,i}$, one to $b_{k,i}$ and one to v_k. The combined capacity of all outgoing edges of $a_{k,i}$ is therefore $\lambda_k(\alpha_i + 2^{i-1} + \varepsilon) = \lambda_k(2^i + 3\varepsilon) < \lambda_k\gamma_i$, so the restriction for $(v_{k-i}, a_{k,i})$ holds.

To see that the chaining of binary counting gadgets does not impede the behavior of smaller binary counting gadgets, notice that the capacities of the edges leaving the v_k are chosen to not cross the established threshold:

$$\sum_{i\in\mathbb{N}^+, i\leq \ell-k} \lambda_{k+i}\gamma_i = \lambda_k \sum_{i\in\mathbb{N}^+, i\leq \ell-k} \left(\frac{2}{5}\right)^i + \frac{4}{6\cdot 5^i} < \lambda_k\left(\frac{2}{3} + \frac{1}{6}\right) = \lambda_k(1-\varepsilon)$$

Note that the behavior of v_0 is also unimpeded by the connections to the binary counting gadgets. This follows from the same argument for $k = 0$ as well as the observation, that the changes of v_0 are – ignoring connections to the binary counting gadgets – always ensured by a capacity difference of at least λ_0.

We use induction to show that the binary counting gadgets' $a_{k,i}$ change at the required times for any minimum cut S. More specifically we show

$$S_{v_k} \xrightarrow{\Theta\equiv T_{0,k} \bmod 2\Delta_k} \bar{S}_{v_k}\bar{S}_{v_k} \xrightarrow{\Theta\equiv T_{0,k}+\Delta_k \bmod 2\Delta_k} S_{v_k}$$

for all $\Theta \in \{T_{0,k} + 2\Delta_k, T_{0,k} + 3\Delta_k, \ldots, T_{0,k} + 1 + \Delta_k\}$.

The correct startup behavior of v_0 requires two partition changes. For now we ignore the edges from the attachment to the binary counting gadgets, since they do not affect behavior, as argued above. The partition change $S \xrightarrow{2} \bar{S}$ directly follows from the capacity change of (v_0, t) at time $\Theta = 2$ increasing the potential contribution of $v_0 \in S(2)$. The other partition change $\bar{S} \xrightarrow{3} S$ is a result of the approaching time horizon T, which reduces the potential contribution of $v_0 \in S(3)$ to zero.

Now assume, for a fixed $k \in \mathbb{N}$, gadgets H_{start} to H_{k-1} work correctly, producing the desired changes. Because of the functionality of the mimicking gadget,

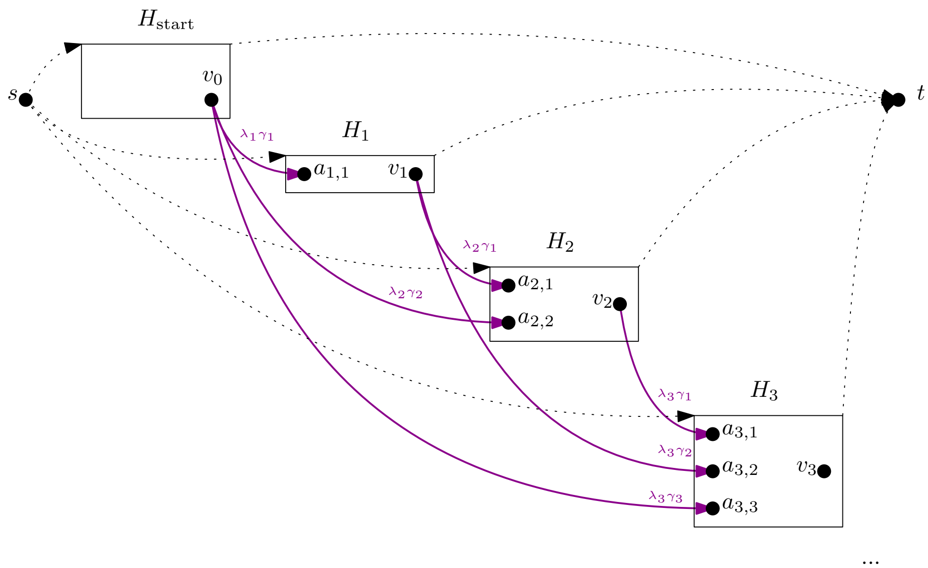


Fig. 7: Construction linking binary counting gadgets to ensure $\mathrm{ch}_{v_\ell} = 2^\ell$ partition changes at v_ℓ in a minimum cut; purple edges represent mimicking gadgets, numbers are capacities.

with the delay $\tau_{(v_{k-i},a_{k,i})}$, by induction $a_{k,i}$ experiences changes $S_{a_{k,i}} \xrightarrow{\Theta} \bar{S}_{a_{k_i}}$ at times

$$\Theta \equiv T_{0,k-i} + \tau_{(v_{k-i},a_{k,i})} \bmod 2\Delta_{k-i} \equiv T_{0,k} \bmod 2\Delta_{k-i}$$

and changes $\bar{S}_{a_{k,i}} \xrightarrow{\Theta} S_{a_{k_i}}$ at times

$$\Theta \equiv T_{0,k-i} + \Delta_{k-i} + \tau_{(v_{k-i},a_{k,i})} \bmod 2\Delta_{k-i} \equiv T_{0,k} + \Delta_{k-i} \bmod 2\Delta_{k-i}$$

beginning with $\Theta = T_{0,k-i} + 2\Delta_{k-i} + \tau_{(v_{k-i},a_{k,i})} = T_{0,k}$ up to $\Theta = T_{0,k-i} + 1 + \Delta_{k-i} + \tau_{(v_{k-i},a_{k,i})} = T_{0,k} + 1 - \Delta_{k-1}$. This means that beginning at $T_{0,k}$ the $a_{k,i}$ form a binary counter increasing every $2\Delta_k$ with the additional change $\bar{S}_{a_k} \xrightarrow{T_{0,k}+1} S_{a_k}$. Now Lemma 2 – multiplying the time with factor Δ_k and adding the initial offset of $T_{0,k}$ – provides the desired change timings for v_k.

The correct change pattern of the $a_{\ell,i}$ required for the validity of H_ℓ follow from the stronger induction hypothesis for the v_k, as seen above during the induction step. □

To be able to use the complexity of minimum cuts to show complexity of maximum flows, we need the following lemma.

Lemma 5. *Every edge contributing to the capacity of some minimum cut has to be saturated by every maximum flow during the time where it contributes to a*

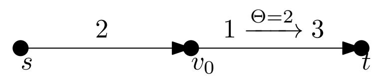


Fig. 8: Construction of H_{start}, providing the partition changes of v_0 needed for G_ℓ with one capacity change, numbers are capacities, $\tau_{(v_0,t)} = T - 3$.

cut. Moreover, every edge $e = (v, w)$ with $v \in \bar{S}(\Theta)$ and $w \in S(\Theta + \tau_e)$ for some minimum cut S may not route flow at time Θ for any maximum flow.

Proof. For any minimum cut S, flow is routed from $s \in S$ to $t \in \bar{S}$. This means that any path along which flow is routed has to contain at least one edge allowing flow to move from S to $\bar{S}$. All edges allowing flow to traverse from S to $\bar{S}$ contribute to $\text{cap}(S)$. As such, if one contributing edge was not saturated by a flow f, the value $|f|$ would be smaller than $\text{cap}(S)$. Then the cut-flow duality of Theorem 1 provides that f cannot be a maximum flow.

Given a minimum cut S, flow of f routed over an edge $e = (v, w)$ with $v \in \bar{S}(\Theta)$ and $w \in S(\Theta + \tau_e)$ has to cross multiple edges from S to $\bar{S}$ to reach t, one before e and one after it. So even if f saturates all edges contributing to $\text{cap}(S)$ as discussed above, less flow than $\text{cap}(S)$ can reach t. With this, Theorem 1 again provides that f is no maximum flow. □

To obtain the following theorem, it only remains to observe that the structure of the minimum cut in the construction of Lemma 4 also implies exponentially complex maximum flows, using Lemma 5. Further note that discretization of time is possible.

Theorem 4. *There exist dynamic flow networks with only one capacity change where every minimum cut and maximum flow has exponential complexity. This even holds for acyclic networks and discrete time.*

Proof. A valid binary counting gadget H_ℓ has a central vertex v_ℓ that experiences 2^k partition changes in any minimum cut S, as shown in Lemma 2. Since Lemma 4 provides the existence of a polynomially sized, acyclic dynamic flow network G_ℓ containing a valid H_ℓ, so any minimum cut in G_ℓ has exponential complexity. The construction of G_ℓ uses only transit times and change timings that are multiples of Δ_ℓ, so discretizing time to units of length Δ_ℓ provides a discrete time dynamic flow network with the same properties.

To see that this partition change pattern with exponentially many changes also implies that any maximum flow has to have exponential complexity, we need Lemma 5, which shows that the minimum cuts impose restrictions on maximum flows. The partition change pattern of $a_{\ell,1}$, changing every $2\Delta_\ell$ and v_ℓ, changing every Δ in S results in $(a_{\ell,1}, v_\ell)$ changing from an edge from S to $\bar{S}$ to an edge from $\bar{S}$ to S exponentially often. This implies exponentially many changes of $f_{(a_{\ell,1},v_\ell)}$ in any maximum flow f in G_ℓ. □

Note that the construction from Lemma 4 requires a specific time horizon T. In the case of infinite considered time, two capacity changes suffice to obtain the same result.

Corollary 3. *Theorem 4 also holds for infinite considered time with two capacity changes.*

Proof. The starting gadget H_{start} shown in Figure 8 can be modified to force the two changes $S \xrightarrow{2} \bar{S}$ and $\bar{S} \xrightarrow{3} S$ of v_0 with infinite considered time by additionally changing the capacity of (v_0, t) to zero at time $\Theta = 3$. The rest of the proof of Theorem 4 is not changed by the introduction of infinite considered time. □

Note that if Conjecture 1 holds, two capacity changes are necessary in this setting.

As mentioned in the introduction, the above complexity results transfer to the setting where we have time-dependent transit times instead of time-dependent capacities. The result of Corollary 3 can even be strengthened to only require a single transit time change.

Corollary 4. *Theorem 4 also holds in the setting of static capacities and time-dependent transit times with a single change, with finite time horizon and with infinite considered time.*

Proof. For infinite considered time, we give the starting gadget H_{start} presented in Figure 9. Here (v_0, t) is always saturated for any maximum flow f, except during $[2, 3]$, when no flow can be routed over it due to the increase in transit time by 1 of (s, v_0). The corresponding minimum cut S requires v_0 to be in S always except during $[2, 3]$, when v_0 has to be in $\bar{S}$. These are the changes H_{start} needs to provide the induction start for the proof of Theorem 2.

This clearly also works for only considering time in $[0, 6]$ as in the proof of Theorem 2. □

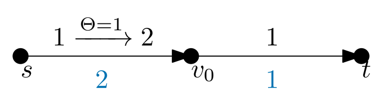


Fig. 9: Construction of H_{start}, providing the partition changes of v_0 needed for G_ℓ with one change in transit time, black numbers are transit times, blue numbers are capacities.

Note that the construction of Theorem 4 causes every minimum cut and every maximum flow to have exponential complexity. In the following we show that exponentially many changes in cut or flow can occur independently. Specifically, we provide constructions that require exponentially complex flows but allow for cuts of low complexity and vice versa.

4.2 Complex Flows and Simple Cuts (and Vice Versa)

All above constructions require all minimum cuts *and* all maximum flows to have exponential complexity. Here, we show that flows and cuts can be independent in the sense that their required complexity can be exponentially far apart (in both directions).

Theorem 5. *There exist acyclic dynamic flow networks with only one capacity change where every maximum flow has exponential complexity, while there exists a minimum cut of constant complexity. The same is true for static capacities and time-dependent transit times.*

Proof. Figure 10 shows a graph with these properties. This is achieved by only allowing flow to enter v_0 during $[0, 1]$, but it has to leave v_k during $[0, 2^k]$ due to the reduced capacity of (v_k, t) for a time horizon $T \geq 2^k$. Apart from v_k all v_i are connected to the next v_{i+1} with a pair of edges, with transit times 2^{k-i-1} and zero, all these edges have capacity 1. So all 2^k paths of different transit time through the v_i have to be used to route flow for a maximum flow. Every second of those paths has an even transit time, so flow has to traverse the edge with transit time zero between v_{k-1} and v_k every second integer time interval, which results in exponentially many changes in flow over that edge. However assigning all v_i to $\bar{S}$ for all time is a minimum cut without partition changes. This generalizes to time-dependent transit times, as we can block the edge (s, v_0) at time $\Theta = 1$ by increasing its transit time to T at that time. □

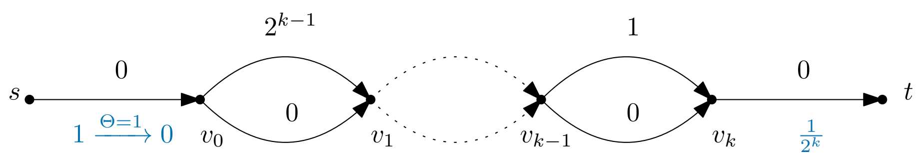


Fig. 10: Example of a graph where any maximum flow contains exponentially many changes, but there is a minimum cut with no changes, black numbers are transit times, blue numbers are capacities, unspecified capacities are 1.

Theorem 6. *There exist acyclic dynamic flow networks with only one capacity change where every minimum cut has exponential complexity, while there exists a maximum flow of linear complexity. The same is true for static capacities and time-dependent transit times.*

Proof. Figure 11 shows a graph where any minimum cut needs to contain exponentially many changes, but a maximum flow with linearly many changes exists. The idea of this construction is that only $\frac{1}{2^k+1}$ flow can enter s' at any time and flow can only leave from x_{2k} to t during one integer interval after 2^k and before the time horizon $T = 2^k + 1$. All other edges have capacity 1. There is a set of bypass paths through the y_i, that allows flow to be routed from s' to x_{2k} in any integer time up to $2^k - 1$, by connecting y_i to y_{i+1} with a pair of edges with transit times 2^{k-i-1} and zero. So $\frac{2^k}{2^k+1}$ flow can be routed through the graph. The section where exponential cuts will be necessary consists of the x_i. The initial x_0 can be reached from s' with transit time 1, internally each $x_i, i < k$ is connected with the next $x_i + 1$ with a pair of edges with transit times 2^{i+1} and zero. The later $x_i, i \geq k$ are connected to the next x_{i+1} with a pair of edges of transit time 2^{2k-i} and zero.

If the bypass paths are used no additional flow can move through the upper paths via the x_i because of the capacity of (s, s') and the lack of a transit time 2^k path through the x_i. There is no maximum flow that saturates any edge except for (s, s') at any time, so, because of the cut flow duality of Theorem 1, this is the only edge that can contribute to the capacity of a minimum cut. Since there are integer transit time paths from s' to x_{2k} for any transit time up to $2^k - 1$, we know that s' has to be in $\bar{S}$ during $[1, 2^k + 1)$ to prevent any other edges from contributing to the cut. This already results in the capacity of the cut being at least $\frac{2^k}{2^k+1}$, so s' has to be in S during $[0, 1)$. To prevent any other contributions to the capacity of the cut, x_k needs to change partition exponentially often. Observe that any path's transit time from any s' to x_k plus the 1 from (s', x_0) marks a timing where x_j has to be in the S partition. Likewise 2^k minus path transit times of paths from x_k to x_{2k} mark timings where x_k has to be in the $\bar{S}$ partition. So during any integer interval before 2^k starting with an odd time, x_k has to be in S and at any such interval starting at an even time, it has to be in $\bar{S}$.

The flow through this graph can easily be represented in linear time with at most one change in flow rate per edge. Using the bypass paths as described above, no flow gets routed through any x_i. The bypass edges are ordered to ensure that flow moves through any y_i during $[2^k - 2^{k-i} + 1, 2^k + 1)$ when using all different path transit times as required for this maximum flow. This means that each edge from y_i to y_{i+1} in the bypass edges sends $\frac{2^i}{2^k+1}$ during $[2^k - 2^{k-i} + 1, 2^k - 2^{k-i-1} + 1)$ for the 2^{k-i-1} transit time edge and during $[2^k - 2^{k-i-1} + 1, 2^k + 1)$ for the edge with transit time zero. Flow over the remaining edges is easily representable as well; (s, s') is saturated during $[1, T)$, (x_{2k}, t) sends $\frac{2^k}{2^k+1}$ during $[2^k, T)$ and $(s', y_0), (y_k, x_{2k})$ only exist to improve the visual representation, flow over them is given by $(s, s'), (x_{2k}, t)$.

This generalizes to time-dependent transit times, as we can activate the edge (x_{2k}, t) at time $\Theta = 2^k$ by decreasing its transit time from T to 0 at that time.

□

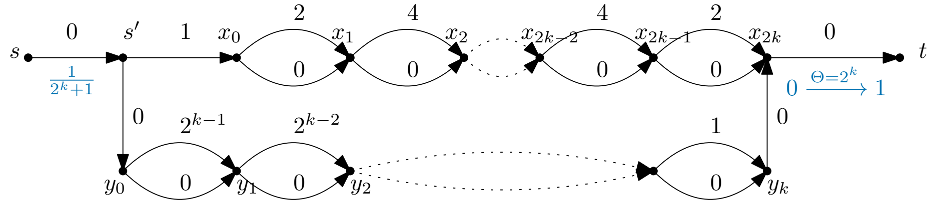


Fig. 11: Example of a graph where any minimum cut contains exponentially many changes, but there is a flow with only linearly many changes, black numbers are transit times, blue numbers are capacities, unspecified capacities are 1, time horizon is $T = 2^k + 1$.

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A On the Strong NP-Completeness of Dynamic Flows with Time-Dependent Capacities

As mentioned in the introduction, a poof for strong NP-completeness for the dynamic flow problem with time-dependent capacities was claimed by Sha Cai and Wong [9, Theorem 2]. The exponential complexity we proof in Theorem 4 does not disproof that the maximum dynamic flow problem is in NP, but it invalidates the use of the two canonical witnesses for verifying a solution in polynomial time. As such the claim that the maximum dynamic flow problem is obviously in NP is in doubt.

The reduction from the 3-Dimensional Matching Problem suffers from the issue that flow can take a path from s to t using edges belonging to different triplets of te 3DM. One example of this is visualized in Figure 12. The 3DM Problem has four possible triples $M = \{(w_1, x_1, y_1), (w_1, x_2, y_2), (w_2, x_2, y_3), (w_3, x_3, y_3)\}$ to hit each of the three elements of each set $\{w_1, w_2, w_3\}, \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}$ exactly once. This is impossible because the need to hit y_1 and w_3 necessitates the use of $(w_1, x_1, y_1), (w_3, x_3, y_3)$, but the remaining three elements w_2, x_2, y_2 cannot be covered by either of the remaining triples in M. However there are three edge disjoint paths $(w_1, x_1, y_1), (w_2, x_2, y_2), (w_3, x_3, y_3)$ in the induced graph, allowing a flow of 3 to pass from s to t, which should correspond to a solvable 3DM instance.

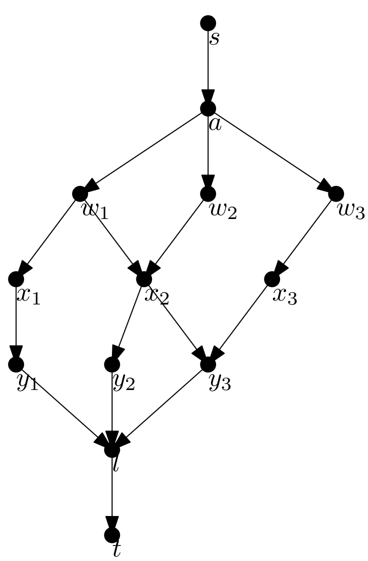

Fig. 12: Visualization of the counterexample to the reduction of 3DM to maximum dynamic flow, where the 3DM instance is not solvable, but the induced maximum dynamic flow instance achieves the required flow of 3, implying a solution to the 3DM problem.

Furthermore, it seems unlikely that the dynamic flow problem with time-dependent capacities, where capacities are piecewise constant functions, capacity changes happen at integer times and edge transit times are integer, is strongly NP hard. This case can be solved in pseudo-polynomial time using the time expanded graph, which has one vertex for every integer time and edges connecting instances with the correct time difference. In the time expanded graph, there are no more changing capacities, so it can be solved using temporally repeated flows. With this in mind a strong NP-hardness proof would show P = NP.

B Strong NP-Hardness for Simple Flow Paths

During our studies, we stumbled upon the following related NP-hardness reduction. However, it is somewhat beyond the scope of the paper and thus only mentioned here in the appendix.

Theorem 7. *The (maximum) dynamic flow problem restricted to simple flow paths with time-dependent capacities or time-dependent transit times is strongly NP-hard.*

Proof. This follows from the NP-hardness proof for the multi-commodity flow over time problem with simple flow paths and without storage presented by Hall, Hippler and Skutella [3, Theorem 7]. Their construction requires a traffic light gadget, but otherwise works with only a single commodity of flow. The traffic light gadget allows setting the capacity of an edge to zero except for a interval $[a, b)$ during which the edge offers usable capacity. Since the scenario considered here allows adjusting an edge's capacity based on time, creation of such a gadget is trivial for time-dependent capacities. For time-dependent transit times, we can construct a traffic light gadget, allowing flow over e only during $[a, b)$ by setting the transit time of e to T during $[0, a)$ and $[b, T)$. $\square$