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Poset Ramsey Number $R(P, Q_n)$. I. Complete Multipartite Posets

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Abstract
A poset $(P', \leq_{P'})$ contains a copy of some other poset $(P, \leq_P)$ if there is an injection $f\colon P' \to P$ where for every $X, Y \in P$, $X \leq_P Y$ if and only if $f(X) \leq_{P'} f(Y)$. For any posets P and Q, the poset Ramsey number $R(P, Q)$ is the smallest integer N such that any blue/red coloring of a Boolean lattice of dimension N contains either a copy of P with all elements blue or a copy of Q with all elements red. A complete ℓ-partite poset $K_{t_1,\dots,t_\ell}$ is a poset on $\sum_{i=1}^{\ell} t_i$ elements, which are partitioned into ℓ pairwise disjoint sets A^i with $|A^i| = t_i$, $1 \leq i \leq \ell$, such that for any two $X \in A^i$ and $Y \in A^j$, $X < Y$ if and only if $i < j$. In this paper we show that $R(K_{t_1,\dots,t_\ell}, Q_n) \leq n + \frac{(2+o_n(1))\ell n}{\log n}$.

Keywords Poset Ramsey · Boolean lattice · Complete multipartite poset · Induced subposet

1 Introduction

Ramsey theory is a field of combinatorics that asks whether in any coloring of the elements in a discrete host structure we find a particular monochromatic substructure. This question offers a lot of variations depending on the chosen sub- and host structure. While originating from a result of Ramsey [8] on uniform hypergraphs from 1930, the most well-known setting considers monochromatic subgraphs in edge-colorings of complete graphs. In contrast, this paper considers a Ramsey-type problem using partially ordered sets, or *posets* for short, as the host structure. A *poset* is a set P which is equipped with a relation $\leq_P$ on the elements of P that is transitive, reflexive, and antisymmetric. Whenever it is clear from the context we refer to such a poset $(P, \leq_P)$ just as P. Given a non-empty set $\mathcal{X}$, the poset consisting of all subsets of $\mathcal{X}$ equipped with the inclusion relation $\subseteq$ is the *Boolean lattice* $\mathcal{Q}(\mathcal{X})$ of *dimension* $|\mathcal{X}|$. We use Q_n to denote a Boolean lattice with an arbitrary n-element ground set.

We say that a poset P_1 is an *induced subposet* of another poset P_2 if $P_1 \subseteq P_2$ and for every two $X, Y \in P_1$,

$$X \leq_{P_1} Y \text{ if and only if } X \leq_{P_2} Y.$$

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A *copy* of P_1 in P_2 is an induced subposet P' of P_2 which is isomorphic to P_1. Here we consider color assignments of the elements of a poset P using the colors *blue* and *red*, i.e. mappings $c\colon P \to \{\text{blue}, \text{red}\}$, which we refer to as a *blue/red coloring* of P. A poset is colored *monochromatically* if all its elements have the same color. If a poset is colored monochromatically in blue [red], we say that it is a *blue* [*red*] *poset*. The elements of a poset P are usually referred to as *vertices*.

Axenovich and Walzer [1] were the first to consider the following Ramsey variant on posets. For posets P and Q, the *poset Ramsey number* of P versus Q is given by

$$R(P,Q) = \min\{N \in \mathbb{N}\colon \text{every blue/red coloring of } Q_N \text{ contains either a blue copy of } P \text{ or a red copy of } Q\}.$$

As a central focus of research in this area, bounds on the poset Ramsey number $R(Q_n, Q_n)$ were considered and gradually improved with the best currently known bounds being $2n+1 \le R(Q_n, Q_n) \le n^2 - n + 2$, see listed chronologically Walzer [9], Axenovich and Walzer [1], Cox and Stolee [4], Lu and Thompson [7], Bohman and Peng [3]. Falgas-Ravry, Markström, Treglown and Zhao [5] showed computationally that $R(Q_3, Q_3) = 7$. The related off-diagonal setting $R(Q_m, Q_n)$, $m < n$, also received considerable attention over the last years. A trivial lower bound in this setting is $R(Q_m, Q_n) \ge m + n$ obtained from a layered coloring. When both m and n are large, the best known upper bound is due to Lu and Thompson [7] who showed that $\left(m - 2 + \frac{5}{m}\right)n + m$. When m is fixed and n is large, an exact result is only known in the trivial case $m = 1$ where $R(Q_1, Q_n) = n + 1$. For $m = 2$, after earlier estimates by Axenovich and Walzer [1] as well as Lu and Thompson [7], the best known upper bound is due to Grósz, Methuku, and Tompkins [6], which is complemented by a lower bound shown recently by Axenovich and the present author [2]:

$$n\left(1 + \frac{1}{15\log n}\right) \le R(Q_2, Q_n) \le n\left(1 + \frac{2 + o(1)}{\log n}\right).$$

In this paper we generalize the upper bound of Grósz, Methuku and Tompkins [6] on $R(Q_2, Q_n)$ to a broader class of posets, namely we discuss the poset Ramsey number of a *complete multipartite poset* versus the Boolean lattice Q_n. A *complete ℓ-partite poset* $K_{t_1,\dots,t_\ell}$ is a poset on $\sum_{i=1}^{\ell} t_i$ vertices obtained as follows. Consider ℓ pairwise disjoint sets $A^1, \dots, A^\ell$ of vertices, where A^i consists of t_i distinct vertices. Now for any two indexes $i, j \in \{1, \dots, \ell\}$ and any vertices $X \in A^i$, $Y \in A^j$, let $X < Y$ if and only if $i < j$ (Fig. 1). Such a poset can be seen as a complete blow-up of a chain and in the literature is also referred to as a (*strict*) *weak order*. Note that $Q_2 = K_{1,2,1}$.

Theorem 1 *For $n \in \mathbb{N}$, let $\ell \in \mathbb{N}$ be an integer such that $\ell = o(\log n)$ and for $i \in \{1, \dots, \ell\}$, let $t_i \in \mathbb{N}$ be integers with $\sup_i t_i = n^{o(1)}$. Then*

$$R(K_{t_1,\dots,t_\ell}, Q_n) \le n\left(1 + \frac{2 + o(1)}{\log n}\right)^{\ell} \le n + \frac{\big(2 + o(1)\big)\ell n}{\log n}.$$

Here and throughout this paper, the O-notation is used exclusively depending on n, i.e. $f(n) = o(g(n))$ if and only if $\frac{f(n)}{g(n)} \to 0$ for $n \to \infty$. For parameters as above, this theorem implies that $R(K_{t_1,\dots,t_\ell}, Q_n) = n + o(n)$. Moreover, under the precondition that ℓ is fixed, our result provides the order of magnitude of the two leading additive terms: We say that a complete ℓ-partite poset $K = K_{t_1,\dots,t_\ell}$ is *non-trivial* if it is neither a chain nor an antichain, i.e. if $\ell \ge 2$ and $t_i \ge 2$ for some $i \in \{1, \dots, \ell\}$. Observe that such a non-trivial K contains

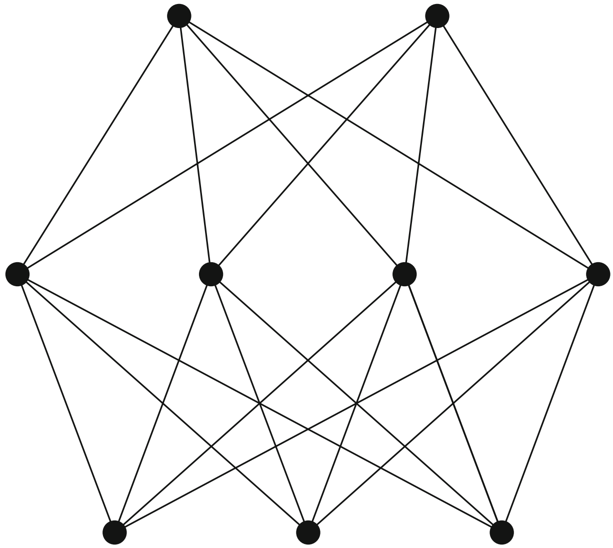

Fig. 1 Hasse diagram of the complete 3-partite poset $K_{3,4,2}$

a copy of $K_{1,2}$ or $K_{2,1}$, so Theorem 2 of [2] yields $R(K, Q_n) \geq n + \frac{n}{15 \log n}$. Thus, for non-trivial K, $R(K, Q_n) = n + \Theta\big(\frac{n}{\log n}\big)$. For trivial K, it is known that $R(K, Q_n) = n + \Theta(1)$. More precisely, if K is a chain on ℓ vertices, then $R(K, Q_n) = n + \ell - 1$, which is an easy consequence of Lemma 4 of Axenovich and Walzer [1]. If K is an antichain on t vertices, then a trivial lower bound, Lemma 3 of [1], and Sperner's Theorem imply $n \leq R(K, Q_n) \leq n + \alpha(t)$ where $\alpha(t)$ is the smallest integer such that $\binom{\alpha(t)}{\lfloor \alpha(t)/2 \rfloor} \geq t$. Ramsey bounds for an antichain versus a Boolean lattice are considered in detail in [10].

First we shall consider a special complete multipartite poset that we call a *spindle*. Given $r \geq 0$, $s \geq 1$ and $t \geq 0$, an *(r,s,t)-spindle* $S_{r,s,t}$ is defined as the complete multipartite poset $K_{t'_1,\dots,t'_{r+1+t}}$ where $t'_1, \dots, t'_r = 1$ and $t'_{r+1} = s$ and $t'_{r+2}, \dots, t'_{r+1+t} = 1$. In other words, this poset on $r + s + t$ vertices is constructed by combining an antichain A of size s and two chains C_r, C_t on r and t vertices, respectively, such that every vertex of A is larger than every vertex from C_r but smaller than every vertex from C_t (Fig. 2).

Theorem 2 *Let r, s, t be non-negative integers with $r + t = o(\sqrt{\log n})$ and $s = n^{o(1)}$ for $n \in \mathbb{N}$. Then*

$$R(S_{r,s,t}, Q_n) \leq n + \frac{\big(1 + o(1)\big)(r + t)n}{\log n}.$$

If $s \geq 2$, the lower bound $R(S_{r,s,t}, Q_n) \geq R(Q_2, Q_n) + c(r, t) \geq n\big(1 + \frac{1}{15 \log n}\big) + c(r, t)$ can be obtained by following the construction in [2] with some additional monochromatic blue layers at the exterior. It remains open whether there is a lower bound such that the second summand depends on r and t.

The spindle $S_{1,s,1}$ is known in the literature as an *s-diamond* D_s, while the poset $S_{1,s,0}$ is usually referred to as an *s-fork* V_s.

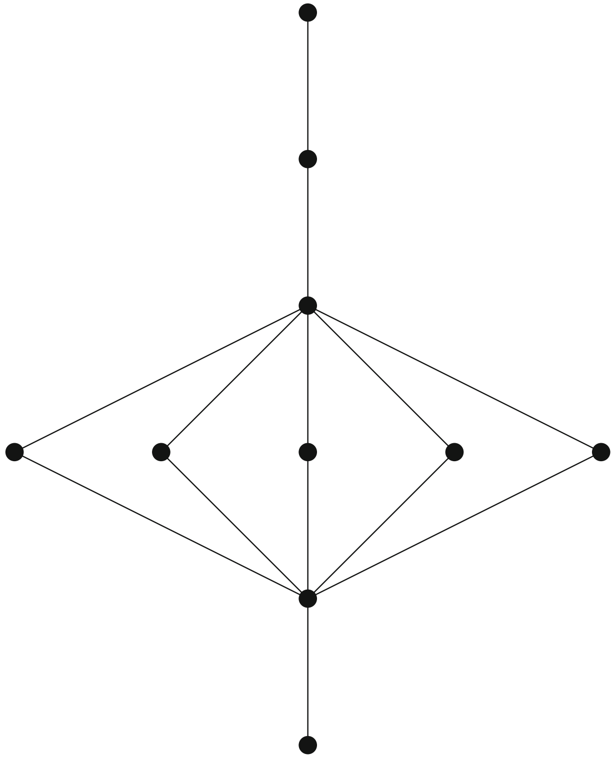

Fig. 2 Hasse diagram of the spindle $S_{2,5,3}$

Corollary 3 *Let* $s \in \mathbb{N}$ *with* $s = n^{o(1)}$ *for* $n \in \mathbb{N}$. *Then*

$$R(D_s, Q_n) \leq n + \frac{\big(2 + o(1)\big)n}{\log n} \qquad \textit{and} \qquad R(V_s, Q_n) \leq n + \frac{\big(1 + o(1)\big)n}{\log n}.$$

For a positive integer $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, \ldots, n\}$. Here 'log' always refers to the logarithm with base 2. We omit floors and ceilings where appropriate.

The structure of the paper is as follows. In Section 2 we introduce some notation and two preliminary lemmas. In Section 3 we show the bound for spindles and subsequently the generalization for general complete multipartite posets.

2 Preliminaries

2.1 Red Q_n Versus Blue Chain

Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets. Then the vertices of the Boolean lattice $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$, i.e. the subsets of $\mathcal{X} \cup \mathcal{Y}$, can be partitioned with respect to $\mathcal{X}$ and $\mathcal{Y}$ in the following manner. Every $Z \subseteq \mathcal{X} \cup \mathcal{Y}$ has an $\mathcal{X}$*-part* $X_Z = Z \cap \mathcal{X}$ and a $\mathcal{Y}$*-part* $Y_Z = Z \cap \mathcal{Y}$. In this setting, we refer to Z alternatively as the pair (X_Z, Y_Z). Conversely, for any $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, the pair (X, Y) corresponds uniquely to the vertex $X \cup Y \in \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$. One can think of such pairs as elements of the Cartesian product $2^{\mathcal{X}} \times 2^{\mathcal{Y}}$ which has a canonical bijection to $2^{\mathcal{X} \cup \mathcal{Y}} = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$.

Observe that for $X_i \subseteq \mathcal{X}$, $Y_i \subseteq \mathcal{Y}$, $i \in [2]$, we have $(X_1, Y_1) \subseteq (X_2, Y_2)$ if and only if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$.

We shall need the following lemma.

Lemma 4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$ for some $n, k \in \mathbb{N}$. Let $\mathcal{Q} = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ be a blue/red colored Boolean lattice. Fix some linear ordering $\pi = (y_1, \dots, y_k)$ of $\mathcal{Y}$ and define $Y(0), \dots, Y(k)$ by $Y(0) = \varnothing$ and $Y(i) = \{y_1, \dots, y_i\}$ for $i \in [k]$. Then there exists at least one of the following in $\mathcal{Q}$:*

(a) a red copy of Q_n, or
(b) a blue chain of length $k+1$ of the form $(X_0, Y(0)), \dots, (X_k, Y(k))$.

Note that a version of this lemma was used implicitly in a paper of Grósz, Methuku and Tompkins [6]. It was stated explicitly and reproved by Axenovich and the author, see Lemma 8 in [2].

2.2 Gluing Two Posets

By identifying vertices of two posets, they can be "glued together" creating a new poset. We will later construct complete multipartite posets by gluing spindles on top of each other using the following definition. Given a poset P_1 with a unique maximal vertex Z_1 and a poset P_2 disjoint from P_1 with a unique minimal vertex Z_2, let $P_1 \between P_2$ be the poset obtained by identifying Z_1 and Z_2. Formally speaking, $P_1 \between P_2$ is the poset $(P_1 \setminus \{Z_1\}) \cup (P_2 \setminus \{Z_2\}) \cup \{Z\}$ for a $Z \notin P_1 \cup P_2$ where for any two $X, Y \in P_1 \between P_2$, $X <_{P_1 \between P_2} Y$ if and only if one of the following five cases hold: $X, Y \in P_1$ and $X <_{P_1} Y$; $X, Y \in P_2$ and $X <_{P_2} Y$; $X \in P_1$ and $Y \in P_2$; $X \in P_1$ and $Y = Z$; or $X = Z$ and $Y \in P_2$ (Fig. 3).

Lemma 5 *Let P_1 be a poset with a unique maximal vertex and let P_2 be a poset with a unique minimal vertex. Then $R(P_1 \between P_2, Q_n) \le R(P_1, Q_{R(P_2, Q_n)})$.*

Proof Let $N = R(P_1, Q_{R(P_2, Q_n)})$. Consider a blue/red colored Boolean lattice $\mathcal{Q}$ of dimension N which contains no blue copy of $P_1 \between P_2$. We shall prove that there exists a red copy of Q_n in this coloring. We say that a blue vertex X in $\mathcal{Q}$ is $P_1 - clear$ if there is no blue copy of P_1 in $\mathcal{Q}$ containing X as its maximal vertex. Similarly, a blue vertex X is $P_2 - clear$ if there is no blue copy of P_2 in $\mathcal{Q}$ with minimal vertex X. Observe that every blue vertex is P_1-clear or P_2-clear (or both), since there is no blue copy of $P_1 \between P_2$.

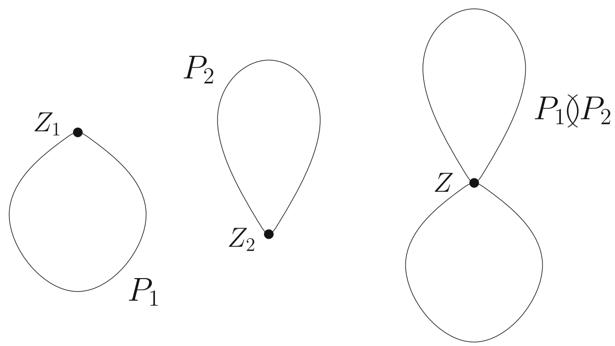

Fig. 3 Creating $P_1 \between P_2$ from P_1 and P_2

We introduce an auxiliary coloring of $\mathcal{Q}$ using colors green and yellow. Color all blue vertices which are P_1-clear in green and all other vertices in yellow. Then this coloring does not contain a monochromatic copy of P_1 with all vertices green, since otherwise the maximal vertex of such a copy is not P_1-clear. Recall that $N = R(P_1, Q_{R(P_2,Q_n)})$, thus $\mathcal{Q}$ contains a copy of $Q_{R(P_2,Q_n)}$ colored monochromatically yellow, which we refer to as $\mathcal{Q}'$.

Consider the original blue/red coloring of $\mathcal{Q}'$. Every blue vertex of $\mathcal{Q}'$ is yellow in the auxiliary coloring, i.e. not P_1-clear. Thus every blue vertex of $\mathcal{Q}'$ is P_2-clear. This coloring of $\mathcal{Q}'$ does not contain a blue copy of P_2, since otherwise the minimal vertex of such a copy is not P_2-clear. Note that the Boolean lattice $\mathcal{Q}'$ has dimension $R(P_2, Q_n)$, thus there exists a monochromatic red copy of Q_n in $\mathcal{Q}'$, hence also in $\mathcal{Q}$. □

Corollary 6 *Let P_1 be a poset with a unique maximal vertex and let P_2 be a poset with a unique minimal vertex. Suppose that there are functions $f_1, f_2 \colon \mathbb{N} \to \mathbb{R}$ with $R(P_1, Q_n) \leq f_1(n)n$ and $R(P_2, Q_n) \leq f_2(n)n$ for any $n \in \mathbb{N}$ and such that f_1 is monotonically non-increasing. Then for every $n \in \mathbb{N}$,*

$$R(P_1 \between P_2, Q_n) \leq f_1(n) f_2(n) n.$$

Proof For an arbitrary $n \in \mathbb{N}$, let $n' = f_2(n)n$. Note that for any poset P, $R(P, Q_n) \geq n$, so $n' \geq n$. Thus, $f_1(n') \leq f_1(n)$, and Lemma 5 provides

$$R(P_1 \between P_2, Q_n) \leq R(P_1, Q_{n'}) \leq f_1(n')n' \leq f_1(n) f_2(n) n.$$

□

3 Proofs of Theorem 2 and Theorem 1

Proof of Thereom 2 Let $\epsilon = \frac{\log s}{\log n}$, so $s = n^{\epsilon}$ and $\epsilon = o(1)$. We can suppose that n is large and hence $\epsilon < 1$. Then let $c = \frac{r+t+\delta}{1-\epsilon}$ where $\delta = \frac{2(r+1)}{\log n}(\log\log n + r + t)$. Since $r + t = o(\sqrt{\log n})$, $\delta = o(1)$. Let $k = \frac{cn}{\log n}$. We show for sufficently large n that $R(S_{r,s,t}, Q_n) \leq n+k$. If $s = 1$, then $S_{r,s,t}$ is a chain and $R(S_{r,s,t}, Q_n) \leq n + r + s \leq n + k$ by Lemma 4 of [1], so suppose $s \geq 2$.

Claim: For sufficiently large n, $k! > 2^{(r+t)(n+k)} \cdot (s-1)^{k+1}$.
Note that $k! > \left(\frac{k}{e}\right)^k = 2^{k(\log k - \log e)}$ and $(s-1)^{k+1} = 2^{(k+1)\log(s-1)}$. Thus, we shall prove that $k(\log k - \log e) > (r + t + \log(s-1))k + \log(s-1) + (r+t)n$. Using the fact that $k = \frac{cn}{\log n}$ and $s - 1 \leq n^{\epsilon}$, we obtain

$$\begin{aligned}
&k\big(\log k - \log(s-1)\big) - k\big(r + t + \log e\big) - \log(s-1) - \big(r+t\big)n \\
&\geq \frac{cn}{\log n}\big(\log c + \log n - \log\log n - \epsilon \log n\big) - \frac{cn}{\log n}\big(r + t + \log e\big) - \epsilon \log n - \big(r+t\big)n \\
&\geq cn\big(1-\epsilon\big) - \big(r+t\big)n - \frac{cn}{\log n}\big(\log\log n + r + t + \log e\big) - \epsilon \log n \\
&> \delta n - \frac{2(r+1)n}{\log n}\big(\log\log n + r + t\big) = 0,
\end{aligned}$$

where the last inequality holds for sufficiently large n. □

Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$. We consider a blue/red coloring of $\mathcal{Q} = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ with no red copy of Q_n. We shall show that there is a monochromatic blue copy of $S_{r,s,t}$ in $\mathcal{Q}$. For every linear ordering $\pi = (y_1^\pi, \dots, y_k^\pi)$ of $\mathcal{Y}$, Lemma 4 provides a blue chain C^π of the form $Z_0^\pi = (X_0^\pi, \varnothing)$, $Z_1^\pi = (X_1^\pi, \{y_1^\pi\}), \dots, Z_k^\pi = (X_k^\pi, \mathcal{Y})$, where $X_i^\pi \subseteq \mathcal{X}$.

For every ordering π of $\mathcal{Y}$, we consider the r smallest vertices $Z_0^\pi, \dots, Z_{r-1}^\pi$ and the t largest vertices $Z_{k-t+1}^\pi, \dots, Z_k^\pi$ of its corresponding chain C^π, so let $I = \{0, \dots, r-1\} \cup \{k-t+1, \dots, k\}$. Each Z_i^π is a vertex of $\mathcal{Q}$, so one of the 2^{n+k} distinct subsets of $\mathcal{X} \cup \mathcal{Y}$. Thus for a fixed π, there are at most $\left(2^{n+k}\right)^{r+t}$ distinct combinations of the Z_i^π, $i \in I$. Recall that $k! > 2^{(r+t)(n+k)} \cdot (s-1)^{k+1}$. By the pigeonhole principle, we find a collection $\pi_1, \dots, \pi_m$ of $m = (s-1)^{k+1} + 1$ distinct linear orderings of $\mathcal{Y}$ such that for any $j \in [m]$ and $i \in I$, $Z_i^{\pi_j} = Z_i$, where $Z_i \subseteq \mathcal{X} \cup \mathcal{Y}$ is a fixed vertex independent of j. In other words, we find many chains with the same r smallest vertices Z_i, $i \in \{0, \dots, r-1\}$, and the same t largest vertices Z_i, $i \in \{k-t+1, \dots, k\}$. Let $\mathcal{P}$ be the poset induced in $\mathcal{Q}$ by the chains C^{π_j}, $j \in [m]$.

If there is an antichain A of size s in $\mathcal{P}$, then none of the vertices Z_i, $i \in I$, is in A, because each of them is contained in every chain C^{π_j} and therefore comparable to all other vertices in $\mathcal{P}$. Note that here we used that $s \geq 2$. Now A together with the vertices Z_i, $i \in I$, form a copy of $S_{r,s,t}$ in $\mathcal{P}$. Recall that all vertices in every C^{π_j} are blue, i.e. $\mathcal{P}$ is monochromatic blue. Thus we obtain a blue copy of the spindle $S_{r,s,t}$ in $\mathcal{Q}$, so we are done. From now on, suppose that there is no antichain of size s in $\mathcal{P}$. By Dilworth's Theorem we obtain $s-1$ chains $\mathcal{C}_1, \dots, \mathcal{C}_{s-1}$ which cover all vertices of $\mathcal{P}$, i.e. all vertices of the C^{π_j}'s. Note that the chains $\mathcal{C}_i$ might consist of significantly more vertices than the $(k+1)$-element chains C^{π_j}.

Now we consider the restriction to $\mathcal{Y}$ of each vertex in $\mathcal{P}$, i.e. the sets $Z_i^\pi \cap \mathcal{Y}$, in order to apply the pigeonhole principle once again. Assume for a contradiction that for some $i \in [s-1]$ there are $Z, Z' \in \mathcal{C}_i$ with $|Z \cap \mathcal{Y}| = |Z' \cap \mathcal{Y}|$ but $Z \cap \mathcal{Y} \neq Z' \cap \mathcal{Y}$. This implies that $Z \cap \mathcal{Y} \nsubseteq Z' \cap \mathcal{Y}$ and $Z \cap \mathcal{Y} \nsupseteq Z' \cap \mathcal{Y}$, so Z and Z' are incomparable, a contradiction as they are both contained in the chain $\mathcal{C}_i$. Consequently, there is only at most one ℓ-element set $Y_i^\ell \subseteq \mathcal{Y}$, $\ell \in \{0, \dots, k\}$, for which there exists a $Z \in \mathcal{C}_i$ with $Z \cap \mathcal{Y} = Y_i^\ell$.

Note that for any $j \in [m]$ and for any $\ell \in \{0, \dots, k\}$, $|Z_\ell^{\pi_j} \cap \mathcal{Y}| = \ell$, i.e. $Z_\ell^{\pi_j} \cap \mathcal{Y} = Y_i^\ell$ for some $i \in [s-1]$. In other words, for fixed j, each of the $k+1$ sets $Z_\ell^{\pi_j} \cap \mathcal{Y}$, $\ell \in \{0, \dots, k\}$, is equal to one of at most $s-1$ Y_i^ℓ's. Recall that we have chosen $m = (s-1)^{k+1} + 1$ distinct linear orderings π_j of $\mathcal{Y}$. Using the pigeonhole principle we find two indexes j_1, j_2 such that $Z_\ell^{\pi_{j_1}} \cap \mathcal{Y} = Z_\ell^{\pi_{j_2}} \cap \mathcal{Y}$ for any $\ell \in \{0, \dots, k\}$. This implies that $y_\ell^{\pi_{j_1}} = y_\ell^{\pi_{j_2}}$, i.e. π_{j_1} and π_{j_2} are equal. But this is a contradiction to the fact that all orderings π_j are distinct.
Now we extend Theorem 2 to general complete multipartite posets by the use of Corollary 6.

***Proof of Thereom* 1** Let $t = \sup_i t_i$. Then Theorem 2 shows the existence of a function $\epsilon(n) = o(1)$ with $R(K_{1,t,1}, Q_n) \leq n\left(1 + \frac{2+\epsilon(n)}{\log n}\right)$. We can suppose that ϵ is monotonically non-increasing by replacing $\epsilon(n)$ with $\max_{N>n}\{\epsilon(N), 0\}$ where necessary. Note that this maximum exists since $\epsilon(N) \to 0$ for $N \to \infty$. In order to prove the theorem, we show a stronger statement using the auxiliary $(2\ell+1)$-partite poset $P = K_{1,t,1,t,\dots,1,t,1}$. Observe that $K_{t_1,\dots,t_\ell}$ is an induced subposet of P, thus $R(K_{t_1,\dots,t_\ell}, Q_n) \leq R(P, Q_n)$. In the following we verify that

$$R(P, Q_n) \leq n\left(1 + \frac{2+\epsilon(n)}{\log n}\right)^\ell.$$

We use induction on ℓ. If $\ell = 1$, then $P = K_{1,t,1}$, so $R(P, Q_n) \leq n\left(1 + \frac{2+\epsilon(n)}{\log n}\right)$. If $\ell \geq 2$, we "deconstruct" the poset into two parts. Consider $P_1 = K_{1,t,1}$ and the complete

$(2\ell - 1)$-partite poset $P_2 = K_{1,t,1,t,\dots,1,t,1}$. Then P_1 has a unique maximal vertex and P_2 has a unique minimal vertex. Observe that $P_1 \between P_2 = P$. Using the induction hypothesis

$$R(P_1, Q_n) \leq n\left(1 + \frac{2+\epsilon(n)}{\log n}\right) \text{ and } R(P_2, Q_n) \leq n\left(1 + \frac{2+\epsilon(n)}{\log n}\right)^{\ell-1}.$$

Now Corollary 6 provides the required bound.

4 Concluding Remarks

In this paper we considered $R(K, Q_n)$, where K is a complete multipartite poset. Although the presented bounds hold if the parameters of K depend on n, the original motivation for these results concerned the case where K is fixed, i.e. independent from n:

After $R(Q_2, Q_n)$ was bounded asymptotically sharply by Grósz, Methuku and Tompkins [6] and Axenovich and the present author [2], the examination of $R(Q_3, Q_n)$ is an obvious follow-up question. The best known upper bound is due to Lu and Thompson [7], while the best known lower bound can be deduced from a bound on $R(K_{1,2}, Q_n)$ in [2],

$$n + \tfrac{n}{15\log n} \leq R(K_{1,2}, Q_n) \leq R(Q_3, Q_n) \leq \tfrac{37}{16}n + \tfrac{39}{16}.$$

In order to find better upper bounds and answer the question as to whether or not $R(Q_3, Q_n) = n + o(n)$, the consideration of $R(P, Q_n)$ for small posets P might prove helpful. We have seen in Corollary 6 how small posets can be used as building blocks for more complex posets P' when bounding $R(P', Q_n)$. Going one step further, a potential generalization of Corollary 6 might allow for building the poset Q_3. For example, Q_3 can be partitioned into a copy of $K_{1,3}$ and a copy of $K_{3,1}$ which interact in a proper way. Both of these building blocks are complete 2-partite posets with, as shown here, Ramsey numbers bounded by

$$R(K_{1,3}, Q_n) = R(K_{3,1}, Q_n) = n + \Theta\left(\frac{n}{\log n}\right).$$

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