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Locally Homogeneous C^0-Riemannian Manifolds

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Abstract
We show that locally homogeneous C^0-Riemannian manifolds are smooth.

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1 Introduction

In this paper we prove that if a C^0-Riemannian manifold is locally homogeneous, then it is indeed smooth, more precisely we obtain the following theorem:

Main Theorem (Local homogeneity implies smoothness). *Let (M, g_0) be a locally homogeneous C^0-Riemannian manifold and denote by d_{g_0} the induced metric, then (M, d_{g_0}) is isometric to a smooth Riemannian manifold.*

In fact we show that for any point there is a small neighborhood U, such that the set of local isometries on U, which will be denoted by U_G, forms a local Lie group with Lie algebra $\mathfrak{g}$ acting transitively on U. The isotropy local isometries determine a local Lie group U_H with Lie algebra $\mathfrak{h}$ and U is isometric to the coset space U_G/U_H carrying an invariant metric with respect to the left action of U_G (for definitions see [20, 23, 29, 30]). In particular all spaces appearing in the main theorem are determined by Lie algebras $\mathfrak{g} \supset \mathfrak{h}$ together with a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}/\mathfrak{h}$, which is skew symmetric with respect to the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ [30]. Thus, they are given by

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purely algebraic data. Moreover, this implies that M and its Riemannian metric are real analytic.

Our result in some sense generalizes the Myers-Steenrod theorems [22], which in particular assert that the isometry group of a smooth Riemannian manifold is a Lie group. Several theorems are known in that direction: Metric spaces with geometric assumptions such as curvature conditions imply regularity of the isometry group. For example isometry groups of Alexandrov spaces or RCD*(K, N) spaces are known to be Lie groups [10, 13, 33].

The question "When is a homogeneous/locally homogeneous space a smooth manifold?" has been investigated in [1, 25, 26].

In [1][Theorem 7] Berestovskii studied when a globally homogeneous inner metric space is isometric to a homogeneous Riemannian manifold. His findings show in particular that a homogeneous Alexandrov space is in fact a smooth Riemannian manifold. In contrast to that we obtain a theorem of a local nature. One can show (using [17, 18] for upper curvature bounds and [24] for lower bounds) that a locally homogeneous space with an upper or lower curvature bound in the sense of Alexandrov is a C^0-Riemannian manifold. Hence our main theorem implies:

Corollary *Let X be a locally homogeneous, locally compact, length space of finite Hausdorff dimension. If there exists a point together with a convex neighborhood admitting a curvature bound from either above or below in the sense of Alexandrov, then X is isometric to a smooth Riemannian manifold.*

It would be interesting to obtain a full description of locally homogeneous, locally compact length spaces similar to [1] without assuming any regularity on the metric and topology. This could be considered as a metric version of the Bing-Borsuk conjecture [14].

There exist different results in the local setting; however, they are making stronger assumptions on the regularity of the manifold. In [28] Singer showed: If a complete, simply connected Riemannian manifold is curvature homogeneous and the derivatives of the curvature tensor agree up to some order at all points, then the manifold is globally homogeneous. If the Riemannian metric is complete and sufficiently smooth, the conclusion of our main theorem follows from this result. While the proof is essentially local and completeness does not play a central role, it relies heavily on the existence of high order derivatives of the metric [23, 28].

Lately local versions with lower regularity have been obtained by Pediconi [25, 26] with different additional assumptions on the space and the group action.

Riemannian manifolds with low regularity do not satisfy classical results in Riemannian geometry: There is no meaningful notion of curvature and shortest curves do not need to solve a differential equation, they may branch and the injectivity radius may be zero [15]. Shortest curves do not even need to be C^1 [15]. We refer to [4] for some basic properties of C^0-Riemannian manifolds and to [5, 7, 8] for further results.

A metric space M is called locally homogeneous if for any two points of M there is a local isometry taking one to the other. One important problem and the difference to the non-local case (as considered by Berestovskii) is that the set of local isometries is a priori not known to form a local group. The technical tool to overcome this obstacle is to extend local isometries, defined on arbitrary small balls, to balls of fixed radius.

Once a local group structure is established one can apply structure theory of locally compact groups [12, 27] to show that it is a local Lie group. We then construct a local isometry between our metric space M and a local quotient of the local group equipped with an invariant Riemannian metric.

The paper is organized as follows: In Section 2 we fix notation, explain what a C^0-Riemannian manifold is and give definitions and notions regarding local groups. In Section 3 we prove that every local isometry can be extended to an isometry of fixed size. In Section 4 we explain how to obtain a local topological group and prove that some restriction is a local Lie group. After that we will explain how to obtain a left-invariant metric on the quotient, which is isometric to some open subset of M.

2 Preliminaries

2.1 C^0-Riemannian Manifolds

In this subsection we collect all definitions and results regarding C^0-Riemannian manifolds.

Definition 2.1 (C^0-Riemannian manifold) *A C^0-Riemannian manifold is a pair (M, g_0) consisting of a C^1-manifold M together with a continuous Riemannian metric g_0.*

The Riemannian metric g_0 induces a canonical length structure, which in turn induces an intrinsic metric d_{g_0} on M. This allows us to formulate local homogeneity in purely metric terms. We denote open (closed) balls with radius r around the point x by $B_r(x)$ ($\overline{B}_r(x)$).

Definition 2.2 (Local homogeneity) *A metric space M is called locally homogeneous if for every $x, y \in M$ there exists $r > 0$ and an isometry $f : B_r(x) \to B_r(y)$ satisfying $f(x) = y$. We call such a map f a pointed isometry.*

We want to make frequent use of the upcoming lemma, which is implied by the C^0-Riemannian manifold structure.

Lemma 2.3 (Maps are bilipshitz) *Let (M, g_0) be a C^0-Riemannian manifold then the coordinate maps of a C^1-atlas are locally bilipshitz.*

Proof Compare [19][Section 3.2]. □

2.2 Local Topological Groups

In this subsection we introduce, for the convenience of the reader, the basic definitions and notations regarding local groups. The exposition is mostly taken from [12].

Definition 2.4 (Local Group). *A local topological group $G = (G, \Omega, e, m, i)$ is a Hausdorff topological space G together with a neutral element $e \in G$, a partially defined but continuous multiplication operation $m : \Omega \to G$ for some open domain*

$\Omega \subset G \times G$, and continuous inversion operation $i : G \to G$ obeying the following axioms:

(1) Ω is an open neighborhood of $G \times \{1\} \cup \{1\} \times G$.
(2) If $g, h, k \in G$ satisfy $m(g, h), m(h, k) \in \Omega$ and $m(m(g, h), k), m(m(g, m(h, k))$ $\in \Omega$ then $m(m(g, h), k) = m(g, m(h, k))$.
(3) For all $g \in G$ one has $m(g, e) = g = m(e, g)$.
(4) If $g \in G$, then $m(g, i(g)) = e = m(i(g), g)$.

We will use the shorthand notation $g \cdot h$ for $m(g, h)$. We call an open neighborhood U of $e \in G$ symmetric if it satisfies $U = i(U)$. Note that if U is an arbitrary open neighborhood of e, then $U \cap i^{-1}(U)$ is an open symmetric neighborhood of e. If the local group G is additionally a smooth manifold and the local group operations are smooth, then we say that G is a local Lie group. The basic example of a local group is the restriction of a topological group.

Definition 2.5 (Restriction of a local group). *Let G be a local topological group and U a symmetric open neighborhood of the identity of G. We have a local group $G|_U$, it has the subspace U as underlying space, e_G as its neutral element, the restriction of inversion to U as its inversion, and the restriction of the product to*

$$\Omega_U := \{(x, y) \in \Omega \cap (U \times U) : m(x, y) \in U\}$$

as its product. Such a local group $G|_U$ is called a restriction of G.

We want to define the notion when local topological groups are equivalent.

Definition 2.6 (Locally isomorphic top. groups). *Let $G = (G, \Omega, e, m, i)$ and $G' = (G', \Omega', e', m', i')$ be local topological groups. A morphism from G to G' is a continuous function $f : G \to G'$ such that*

(1) $f(e) = e'$ and $(f \times f)(\Omega) \subset \Omega'$.
(2) $f(i(g)) = i'(f(g))$ for all $g \in G$.
(3) $f(m(g, h)) = m'(f(g), f(h))$ for all $(g, h) \in \Omega$.

We say G and G' are locally isomorphic if there exist open symmetric neighborhoods U and U' of e and e' in G and G' respectively, $f : U \to U'$ a homeomorphism and $f : G|_U \to G'|_{U'}$, $f^{-1} : G'|_{U'} \to G|_U$ are morphisms.

3 Isometry Extensions

The goal of this section is to obtain an extension property. That is local isometries defined on arbitrary balls can be extended to balls of fixed radius. More precisely we want to prove the following statement:

Proposition 3.1 (Extension Property). *Let (M, g_0) be a locally homogeneous C^0-Riemannian manifold. Then for any compact $K \subset M$ there exists $R > 0$ such that for all points $x, y \in K$ and all $r < R$ any isometry $f : B_r(x) \to B_r(y)$ satisfying $f(x) = y$ can be extended uniquely to an isometry $F : B_R(x) \to B_R(y)$.*

Once this Proposition is established, we can define a local group structure on the set of local isometries. This will be carried out in Section 4.

Moreover, a detailed analysis of our proof gives the following quantitative estimate on R: If $\rho > 0$ satisfies the condition that for any $x \in K$ every loop in $B_\rho(x)$ is contractible in $B_{3\rho}(x)$ and the closure $\overline{B}_{3\rho}(x)$ is compact then we have $R \geq \rho$. The existence of such ρ follows from the following fact: A small neighborhood of a point in a C^0-Riemannian manifold is $(1 \pm \varepsilon)$-bi-Lipschitz equivalent to a neighborhood in Euclidean space.

Now we turn to the proof of Proposition 3.1; it is subdivided into two steps executed in Sections 3.1 and 3.2 respectively.

The first subsection deals with small extensions of isometries, meaning that given an isometry $f : B_r(x) \to B_r(y)$ it can be extended to an isometry $F : B_{r+\varepsilon}(x) \to B_{r+\varepsilon}(y)$, where $\varepsilon \ll r$. The main ingredient is the Lipschitz version of the Hilbert-Smith conjecture, which shows that the isometry groups, we are dealing with, are actually Lie groups. This enables us to formulate statements about extensions of isometries in terms of Lie groups. The second subsection deals with large extensions of isometries, meaning that given an isometry $f : B_r(x) \to B_r(y)$ it can be extended to an isometry $F : B_R(x) \to B_R(y)$ where $r \ll R$. The main ingredient is to extend the isometry along paths using the results obtained in Section 3.1, thus proving Proposition 3.1.

Using Lemma 3.10 the proofs of Proposition 3.1 and Proposition 3.2 can be reduced to the special case where K is some compact ball, that is $K = \overline{B}_{r_0}(x_0)$.

From here on we make the standing assumption that all balls are small enough to have compact closure.

3.1 Small Isometry Extension

The goal of this subsection is to obtain:

Proposition 3.2 (Existence of local Extension). *Let (M, g_0) be a locally homogeneous C^0-Riemannian manifold. Then for any compact $K \subset M$ there is $R > 0$ such that for all but countably many $r < R$ there is $\varepsilon_r > 0$ such that every isometry $f : B_r(x) \to B_r(y)$ satisfying $f(x) = y$ can be extended to an isometry $F : B_{r+\varepsilon_r}(x) \to B_{r+\varepsilon_r}(y)$ for every pair of $x, y \in K$.*

The first step is to obtain a well-behaved subset of M on which we will define the local group of isometries. Observe that for arbitrary $x, y \in M$ and the isometry $f : B_r(x) \to B_r(y)$, coming from the local homogeneity condition, there can in general be no lower bound for $r > 0$. Using the Baire category theorem we want to find $B_{r_0}(x_0)$ such that for all $x, y \in B_{r_0}(x_0)$ one has a lower bound on $r > 0$.

Lemma 3.3 (Lower domain bound) *Denote by (M, g_0) a locally homogeneous C^0-Riemannian manifold. Then for every $x_0 \in M$ there exists $r_0, R > 0$ such that for all points $x, y \in B_{r_0}(x_0)$ there exists an isometry $f_{xy} : B_R(x) \to B_R(y)$ satisfying $f(x) = y$.*

Proof Fix some point $x_0 \in M$ and consider a closed compact ball $\overline{B}$ containing x_0, then by local homogeneity for every $x \in \overline{B}$ there exists maximal $r_x > 0$ and an isometry $f_x : B_{r_x}(x_0) \to B_{r_x}(x)$ satisfying $f(x_0) = x$. Define for $n \in \mathbb{N}$ the set

$$\mathcal{F}_{\frac{1}{n}} = \left\{ x \in \overline{B} : r_x \geq \frac{1}{n} \right\}.$$

By the above one has $\overline{B} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{\frac{1}{n}}$ and each $\mathcal{F}_{\frac{1}{n}}$ is closed. Therefore Baire's category theorem [21][Theorem 48.2] implies that for some $m \in \mathbb{N}$ the set $\mathcal{F}_{\frac{1}{m}}$ has non-empty interior. Thus, there are a point x_0 and a radius $r_0 > 0$ such that $B_{r_0}(x_0) \subset \mathcal{F}_{\frac{1}{m}}$. Now $f_{xy} := f_y \circ f_x^{-1}$ yields the desired map. □

Since the Hilbert-Smith theorem will be used frequently, we recall it for the convenience of the reader.

Theorem 3.4 (Hilbert-Smith theorem [27]). *If G is a locally compact group, which acts effectively by Lipschitz homeomorphisms on a Riemannian manifold, then G is a Lie group.*

Lemma 3.5 (Pointed isometries form a Lie group). *In the situation of Proposition 3.1 the group of pointed isometries*

$$\mathrm{Iso}_x(B_r(x)) := \{f : B_r(x) \to B_r(x) : f \text{ isometry}, f(x) = x\}$$

is a compact Lie group.

Proof By [16][Corollary 4.8] we have that $\mathrm{Iso}_x(B_R(x))$ is compact. Moreover, the isometry group $\mathrm{Iso}_x(B_R(x))$ of $B_R(x)$ acts effectively by Lipschitz homeomorphisms on an open set of $\mathbb{R}^n$; therefore, $\mathrm{Iso}_x(B_R(x))$ is a Lie group by the Hilbert-Smith theorem (Theorem 3.4). □

Lemma 3.6 (Uniqueness of Extensions). *Let (M, g_0) be a C^0-Riemannian manifold, $r > 0$ and $f : B_r(x) \to B_r(y)$ an isometry satisfying $f(x) = y$. If there exist $\varepsilon > 0$ and extensions $F, G : B_{r+\varepsilon}(x) \to B_{r+\varepsilon}(y)$ of f, then one has $F = G$.*

Proof Consider the isometry

$$H : B_{r+\varepsilon}(x) \to B_{r+\varepsilon}(x), z \mapsto G^{-1} \circ F(z).$$

□

The group $\overline{\langle H \rangle} \subset \mathrm{Iso}_x(B_{r+\varepsilon}(x))$ generated by H is a compact Lie Group, since it is a closed subgroup of a compact Lie group by Lemma 3.5. Observe that all elements of $\overline{\langle H \rangle}$ fix the open set $B_r(x)$. This is a contradiction to the Newmann Theorem [3][Theorem 9.5], since the fixed point set of a compact Lie group cannot contain an open set.

Lemma 3.7 (Local extensions of isometry groups). *Denote by $B_{r_0}(x_0)$ the set coming from Lemma 3.3, then for every $x \in B_{r_0}(x_0)$ and every sufficiently small $r > 0$ there exists $R > 0$ such that for all $\varepsilon_1 < \varepsilon_2 < R$ one has*

$$\mathrm{Iso}_x(B_{r-\varepsilon_1}(x)) = \mathrm{Iso}_x(B_{r-\varepsilon_2}(x)).$$

Proof By Lemma 3.6 we have a natural inclusion $\mathrm{Iso}_x(B_{r+\varepsilon}(x)) \hookrightarrow \mathrm{Iso}_x(B_r(x))$ just by restricting the maps to the smaller domain. Thus $L_i := \mathrm{Iso}_x(B_{r-\frac{1}{i}}(x))$ defines by Lemma 3.5 a sequence of compact Lie groups satisfying $L_{i+1} \subset L_i$. Hence the L_i must stabilize, meaning there exists $N \in \mathbb{N}$ such that $L_n = L_m$ for all $n, m \geq N$. This proves statement. □

Corollary 3.8 (Local extensions of isometry groups are homogeneous). *Denote by $B_{r_0}(x_0)$ the set coming from Lemma 3.3, then there is $R > 0$ such that for all but countably many $R > r > 0$ there exists $\varepsilon(r) > 0$ such that*

$$\mathrm{Iso}_x(B_r(x)) = \mathrm{Iso}_x(B_{r+\varepsilon(r)}(x))$$

for all $x \in B_{r_0}(x_0)$.

Proof Fix $x \in B_{r_0}(x)$. We show that for all but countably many $r > 0$ an isometry can be extended, i.e., $\mathrm{Iso}_x(B_r(x)) = \mathrm{Iso}_x(B_{r+\varepsilon(r)}(x))$ for some $\varepsilon(r) > 0$. This immediately follows from Lemma 3.7. Indeed without loss of generality assume $R = 1$. We want to show that the set

$$C := \{r \in (0,1) |\text{ For all } \varepsilon > 0 \text{ one has } \mathrm{Iso}_x(B_r(x)) \neq \mathrm{Iso}_x(B_{r+\varepsilon}(x))\}$$

is countable. By Lemma 3.7 we have

$$\alpha := \inf\{\tau \in [0,1] : C \cap [\tau, 1] \text{ is countable }\} < \infty,$$

since the set appearing in the definition is not empty. If we can show $\alpha = 0$, then the statement follows. Assume $\alpha > 0$, then by Lemma 3.7 $C \cap [\alpha - \varepsilon, 1]$ is countable for some $\varepsilon > 0$. This is a contradiction to the choice of α and the statement follows. The second statement is that this $r > 0$ does not depend on the point, i.e.,

$$\mathrm{Iso}_x(B_r(x)) = \mathrm{Iso}_x(B_{r+\varepsilon(r)}(x)) \Rightarrow \mathrm{Iso}_y(B_r(y)) = \mathrm{Iso}_y(B_{r+\varepsilon(r)}(y))$$

for all $x, y \in B_{r_0}(x_0)$.

Fix an element $g \in \mathrm{Iso}_y(B_r(y))$ and consider a pointed isometry $f : B_R(x) \to B_R(y)$ provided by Lemma 3.3. If $r \leq R$, then one has

$$f^{-1} \circ g \circ f|_{B_r(x)} \in \mathrm{Iso}_x(B_r(x)).$$

By assumption there exists an isometry

$$\overline{f^{-1} \circ g \circ f} : B_{r+\varepsilon(r)}(x) \to B_{r+\varepsilon(r)}(x)$$

extending $f^{-1} \circ g \circ f|_{B_r(x)}$. Now $f \circ \overline{f^{-1} \circ g \circ f} \circ f^{-1}$ is an extension of g and by the uniqueness result Lemma 3.6 the statement follows. □

Lemma 3.9 (Existence of local extensions). *Denote by $B_{r_0}(x_0)$ the set coming from Lemma 3.3, then there exists $R > 0$ such that for all but countably many $r < R$ there is $\varepsilon_r > 0$ such that every pointed isometry $f : B_r(x) \to B_r(y)$ can be extended to an isometry $F : B_{r+\varepsilon_r}(x) \to B_{r+\varepsilon_r}(y)$ for every pair of $x, y \in B_{r_0}(x_0)$.*

Proof Choose $r > 0$ according to Corollary 3.8 and an isometry $G : B_R(x) \to B_R(y)$ provided by Lemma 3.3, where x, y denote points in $B_{r_0}(x_0)$. We can assume without loss of generality $r + \varepsilon(r) < R$. Then $G^{-1} \circ f$ is an element of $\mathrm{Iso}_x(B_r(x))$ and thus has an extension

$$\overline{G^{-1} \circ f} : B_{r+\varepsilon(r)}(x) \to B_{r+\varepsilon(r)}(x).$$

Now $F := G \circ \overline{G^{-1} \circ f}$ is the desired extension of f. □

The next lemma gives the formal description how to extend results obtained for a ball to any compact set.

Lemma 3.10 *For any compact $K \subset M$ and any ball $B_{r_0}(x_0)$ there is $r_0 > R(K) > 0$ such that for all $x \in K$ there is a pointed isometry $f : B_{R(K)}(x) \to B_{R(K)}(x_0)$.*

Proof Consider the ball $B_{r_0}(x_0)$ obtained in Lemma 3.3. By local homogeneity for every $x \in K$ there exists maximal $r_x > 0$ and an isometry $f_x : B_{r_x}(x_0) \to B_{r_x}(x)$ satisfying $f(x_0) = x$. We can cover K with finitely many such neighborhoods $B_{r_{x_1}}(x_1), \ldots, B_{r_{x_N}}(x_N)$. Denote by r the Lebesgue number of this covering. By construction there is an isometry $g : B_r(x) \to B_r(x_0)$, which not necessary satisfies $g(x) = x_0$. However, by Lemma 3.3 we have a map $h : B_R(g(x)) \to B_R(x_0)$ satisfying $h(g(x)) = x_0$. Setting $f := h \circ g$ and $R(K) := \min\{r, R\}$ the claim follows. □

Now Proposition 3.2 follows from Lemma 3.9 and Lemma 3.10

3.2 Large Extension

The goal of this subsection is to prove Proposition 3.1 using Lemma 3.9. Consider an isometry $f : B_r(x) \to B_r(y)$, a point $z \in B_r(x)$ and a point $z' \notin B_r(x)$. The idea is to extend f along a path γ from z to z' by repeatedly applying Lemma 3.9. It remains to prove that this extension procedure is well-defined. This will take up most of the subsection. Once this is established, we will be able to construct a "large" extension.

Definition 3.11 (Isometry caterpillar). *Let (M, g_0) be a C^0-Riemannian manifold. Consider a path $\gamma : [a, b] \to M$ parametrized proportional to arc-length by constant speed C (i.e., $L(\gamma|_{[t_1,t_2]}) = C(t_2 - t_1)$ for all $t_1, t_2 \in [a, b]$ with $t_1 < t_2$). An r-isometry caterpillar along γ is a family of isometries $f_t : B_r(\gamma(t)) \to B_r(f_t(\gamma(t)))$ for $t \in [a, b]$, such that for all $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \frac{r}{10C}$ we have*

$$f_{t_1}|_{B_r(\gamma(t_1)) \cap B_r(\gamma(t_2))} = f_{t_2}|_{B_r(\gamma(t_1)) \cap B_r(\gamma(t_2))}.$$

We say an isometry caterpillar is fat, if every isometry $f_t : B_r(\gamma(t)) \to B_r(f_t(\gamma(t)))$ *can be extended to an isometry* $F_t : B_{10r}(\gamma(t)) \to B_{10r}(f_t(\gamma(t)))$.

Observe that the above definition is invariant under linear reparameterizations. The next Lemma shows: If two r-isometry caterpillars (for the same path γ) agree at the starting point, then they agree everywhere.

Lemma 3.12 (Caterpillar uniqueness). *Let* (M, g_0) *be a* C^0*-Riemannian manifold and* $\gamma : [a, b] \to M$ *a path parametrized proportional to arc-length. If* f_t^1, f_t^2 *are two* r*-isometry caterpillars along* γ *satisfying* $f_a^1 \equiv f_a^2$ *on* $B_r(\gamma(a))$*, then one has* $f_t^1 = f_t^2$ *on* $B_r(\gamma(t))$ *for all* $t \in [a, b]$.

Proof Parametrize γ by arc-length and consider $t \in [a, b]$ such that $|a - t| \leq r/2$. Then f_t^1 and f_t^2 agree on $B_{r/2}(\gamma(t))$ by triangle inequality. Now Lemma 3.6 implies $f_t^1 \equiv f_t^2$ on $B_r(\gamma(t))$. Thus, the claim follows inductively by subdividing γ into pieces of length smaller that $r/2$. □

The upcoming Lemma proves that fat caterpillars can be concatenated; observe that this statement strongly relies on the fatness condition. Let us give an example, where two caterpillars not having the fatness condition cannot be concatenated.

Consider the standard cylinder embedded in $\mathbb{R}^3$ with radius 1 and the curve

$$\gamma : [-0.002\pi, 0.002\pi] \to S^1 \times \mathbb{R}, t \mapsto (\cos(t), \sin(t), 0).$$

Set $\gamma_1 := \gamma|_{[-0.002\pi,0]}$ and $\gamma_2 := \gamma|_{[0,0.002\pi]}$. Since every open ball on the cylinder of radius π admits a local isometry (unfolding) to a ball in the plane, one can describe an r-isometry caterpillar with $r = 0.999\pi$ along geodesics of length ≤ 0.002 using unfolding maps. If f_t is the rotation around the angle t, then f_t will define an r-isometry caterpillar along γ_1 and γ_2. However, it will not define an r-isometry caterpillar along γ, since γ is to long and the balls $B_r(\gamma(-0.002\pi))$, $B_r(\gamma(0.002\pi))$ overlap but the maps f_t do not agree.

Lemma 3.13 (Concatenation of caterpillars). *Let* (M, g_0) *be a* C^0*-Riemannian manifold,* $\gamma_1 : [a, b] \to M, \gamma_2 : [b, c] \to M$ *paths parametrized by arc-length with* $\gamma_1(b) = \gamma_2(b)$ *and* f_t^1, f_t^2 *fat* r*-isometry caterpillars along* γ_1 *and* γ_2 *respectively satisfying* $f_b^1 \equiv f_b^2$ *on* $B_r(\gamma_1(b)) = B_r(\gamma_2(b))$.

Then the family of isometries $f_t : B_r(\gamma(t)) \to B_r(f(\gamma(t)))$ *for* $t \in [a, c]$ *is a fat isometry caterpillar along the concatenation* γ *of* γ_1 *and* γ_2.

Proof We need to check for $b - r/10 < t_1 < b < t_2 < b + r/10$ the condition

$$f_{t_1}|_{B_r(\gamma_1(t_1))\cap B_r(\gamma_2(t_2))} = f_{t_2}|_{B_r(\gamma_1(t_1))\cap B_r(\gamma_2(t_2))}.$$

Consider a point $z \in B_r(\gamma_1(t_1)) \cap B_r(\gamma_2(t_2))$ and the extensions

$$F_{t_i}^i : B_{10r}(\gamma_i(t_i)) \to B_{10r}(f_{t_i}^i(\gamma_i(t_i))) \text{ for } i = 1, 2$$

and

$$F_b^1 : B_{10r}(\gamma_1(b)) \to B_{10r}(f_{t_1}^1(\gamma_1(b)))$$

provided by the fatness condition of the caterpillar. By construction $F^2_{t_2}$ and F^1_b agree in a small neighborhood of $\gamma(t_2)$; therefore, they have also to satisfy $F^2_{t_2}(z) = F^1_b(z)$ by Lemma 3.6, otherwise we would obtain two different extensions fixing an open neighborhood of $\gamma(t_2)$. A similar argument gives $F^1_{t_1}(z) = F^1_b(z)$. Since these maps are just extensions of f_{t_1}, f_{t_2}, we have $f_{t_1}(z) = f_{t_2}(z)$. Hence the result. □

Lemma 3.14 (Existence of caterpillar isometries). *Let (M, g_0) be a locally homogeneous C^0-Riemannian manifold. Suppose that for $x, y \in M$ and $R > 0$ the balls $\overline{B}_R(x)$ and $\overline{B}_R(y)$ are compact. Then for any $0 < r < R$, every pointed isometry $f : B_r(x) \to B_r(y)$, every $z \in B_R(x)$ and any rectifiable path $\gamma : [a, b] \to \overline{B}_R(x)$ from x to z there exist $\rho \in (0, r)$ and a fat ρ-isometry caterpillar extending $f|_{B_\rho(x)}$.*

Proof First we apply Proposition 3.2 for $K = \overline{B}_R(x) \cup \overline{B}_R(y)$ and find $R(K)$. Now Proposition 3.2 provides $r' < \min\{r, R(K)\}$ and $\varepsilon' > 0$ such that for all $v, w \in \overline{B}_R(x) \cup \overline{B}_R(y)$ every pointed isometry $g : B_{r'}(v) \to B_{r'}(w)$ can be extended to an isometry $G : B_{r'+\varepsilon'}(v) \to B_{r'+\varepsilon'}(w)$. We can assume $r' \ll r$.

Consider the isometry $f : B_r(x) \to B_r(y)$ and a rectifiable path $\gamma : [a, b] \to \overline{B}_R(x)$ from x to z parameterized by arc-length coming from the assumption of Lemma 3.14. Subdivide γ into paths $\gamma_1, \ldots, \gamma_N$ of length less than $\varepsilon'/10$. Inductively we can construct an isometry caterpillar along γ.

For the induction start set $f_t|_{B_{r'/10}(\gamma(t))} := F|_{B_{r'/10}(\gamma(t))}$ for $t \in [a, a + \varepsilon'/10]$, where F denotes the extension $F : B_{r'+\varepsilon'}(x) \to B_{r'+\varepsilon'}(y)$ of $f : B_{r'}(x) \to B_{r'}(y)$ (this is possible since $B_{r'/10}(\gamma(t)) \subset B_{r'/10+\varepsilon'}(x)$). By definition the compatibility condition for f_t is satisfied and thus it is a fat isometry caterpillar for $0 < \rho < \varepsilon'/10$. Now by Lemma 3.13 we can extend this construction to the concatenation of γ_1 and γ_2. This way we obtain the result. □

Lemma 3.15 (Close path endpoint compatibility). *Let (M, g_0) be a C^0-Riemannian manifold and $\gamma_1, \gamma_2 : [a, b] \to M$ two paths parametrized proportional to arc-length satisfying $0.9 \le L(\gamma_1)/L(\gamma_2) \le 1.1$. Moreover, assume there is an $r > 0$ such that*

$$d(\gamma_1(t), \gamma_2(t)) < r/2 \text{ for all } t \in [a, b]$$

and γ_1, γ_2 admit fat r-isometry caterpillars f^1_t and f^2_t with the property

$$f^1_a|_{B_r(\gamma_1(a)) \cap B_r(\gamma_2(a))} = f^2_a|_{B_r(\gamma_1(a)) \cap B_r(\gamma_2(a))}.$$

Then we have

$$f^1_b|_{B_r(\gamma_1(b)) \cap B_r(\gamma_2(b))} = f^2_b|_{B_r(\gamma_1(b)) \cap B_r(\gamma_2(b))}.$$

Proof After reparametrization we can assume that γ_1 is parametrized by arc-length and γ_2 is parametrized by constant speed $C \in [1, 1.1]$. Find a subdivision $a = t_1, \ldots, t_N = b$ of $[a, b]$ such that the length of the curves $\gamma_1|_{[t_i, t_{i+1}]}, \gamma_2|_{[t_i, t_{i+1}]}$ is $\le \frac{r}{1.1 \cdot 10}$ for $i = 1, \ldots, N - 1$. We will show the stronger claim

$$f^1_t|_{B_r(\gamma_1(t)) \cap B_r(\gamma_2(t))} = f^2_t|_{B_r(\gamma_1(t)) \cap B_r(\gamma_2(t))} \text{ for all } t \in [a, b].$$

For $t \in [t_1, t_2]$ consider the extensions F_a^1, F_t^1, F_t^2. All these isometries agree in a neighborhood of $\gamma(a)$: F_a^1 and F_t^1 by the caterpillar condition and F_a^1, F_t^2 by assumption together with the caterpillar condition. By the triangle inequality and the fatness condition, all these extensions are defined on $B_r(\gamma_1(t)) \cap B_r(\gamma_2(t)) \neq \emptyset$ and they agree by Lemma 3.6. In particular, the restrictions agree as well. Thus, the claim is shown for $\gamma_1|_{[t_1,t_2]}, \gamma_2|_{[t_1,t_2]}$. Using the same argument, one gets by induction the statement for all of γ_1, γ_2. □

Lemma 3.16 (Existence of global extensions). *Let (M, g_0) be a locally homogeneous C^0-Riemannian manifold, $x, y \in M$ and $R > 0$ be such that $\overline{B}_{2R}(x), \overline{B}_{2R}(y)$ are compact and every loop in $B_R(x)$ is contractible in $B_{2R}(x)$. Then for all $r < R$ any isometry $f : B_r(x) \to B_r(y)$ satisfying $f(x) = y$ can be extended to a local isometry $F : B_R(x) \to B_R(y)$.*

Proof Construction of F:
Denote by R the bound coming from Lemma 3.14. Fix the pointed isometry $f : B_r(x) \to B_r(y)$, a point $z \in B_R(x)$ and a rectifiable path $\gamma : [a, b] \to B_R(x)$ from x to z (for example one randomly chosen shortest path from x to z). Then by Lemma 3.14 and Lemma 3.12 there exists a unique fat isometry caterpillar f_t, which coincides with f in a neighborhood of the point x. Define $F(z) := f_b(\gamma(b))$.

It remains to show that F is well-defined, i.e., the definition does not depend on the path γ. Let γ' be another rectifiable path from x to z. Since the argument is local, by Lemma 2.3. we know that there is a homotopy γ_t between γ and γ' keeping the endpoints fixed such that $L(\gamma_t)$ depend continuously on the parameter t. We can find $0 = t_1, \dots, t_N = 1$ such that γ_{t_i} and $\gamma_{t_{i+1}}$ satisfy the assumption of Lemma 3.15. This proves that $F(z)$ does not depend on the path γ.

It remains to show that F is a local isometry. Indeed consider a point $z \in B_R(x)$ and a fat isometry caterpillar f_t along some rectifiable path $\gamma : [a, b] \to B_R(x)$ from x to z. Then F coincides with the isometry $f_b : B_{r'}(\gamma(b)) \to B_{r'}(f_b(\gamma(b)))$ on $B_{r'}(\gamma(b))$. To see this, consider a point $v \in B_{r'}(z)$, we have shown above that the point $F(v)$ does not depend on the path. Therefore consider the path γ followed by a shortest path from z to v, then f_t defines a fat isometry caterpillar along that path. It follows that $f_b(v) = F(v)$. □

Proof of Proposition 3.1. In order to prove Proposition 3.1 (for the particular case when K is a ball) it remains to show that the restriction of the extension $F|_{B_{R/2}(x)}$ constructed in Lemma3.16 is an isometry, provided the additional assumption that every loop in $B_{R/2}(y)$ is contractible in $B_R(y)$. In view of Lemma 3.10 this is sufficient to prove Proposition 3.1 for an arbitrary compact K.

Observe that since F is a local isometry any homotopy between paths with initial point y inside $B_R(y)$ can be lifted; this implies that F is injective. Indeed, consider two points $p, q \in \overline{B}_{R/2}(x)$ satisfying $F(p) = F(q)$, then consider the shortest paths γ_1 from x to p and γ_2 from x to q. The paths $F(\gamma_1)$ and $F(\gamma_2)$ are homotopic inside $B_R(y)$, lifting this homotopy gives $p = q$. Now using injectivity obtained above, we show that $F|_{\overline{B}_{R/2}(x)}$ is distance-preserving. Consider two distinct points $v, w \in F(\overline{B}_{R/2}(x))$ and γ an arbitrary shortest path between them. Its lift $\tilde{\gamma}$ has the same length as γ, since F is a local isometry. Observe that the map F is 1-Lipschitz, hence $\tilde{\gamma}$ must be a shortest path. Therefore F is distance preserving.

It remains to show that $F|_{\overline{B}_{R/2}(x)}$ is surjective. Assume this is not the case, then there is a $z \in \overline{B}_{R/2}(y)$ not in the image of $F|_{\overline{B}_{R/2}(x)}$. Set $\rho := d_{g_0}(y, z)$ and consider the isometry $f_{yx} : \overline{B}_\rho(y) \to \overline{B}_\rho(x)$ existing by Lemma 3.3. Then $F|_{\overline{B}_{R/2}(x)} \circ f_{yx} : \overline{B}_\rho(y) \to \overline{B}_\rho(y)$ is a distance preserving map, mapping a compact subset to a proper subset of itself. This is a contradiction. Therefore $F|_{\overline{B}_{R/2}(x)}$ is a bijective distance-preserving map and thus an isometry. □

4 Local Lie Group Structure

The goal of this section is to show that an open subset of (M, d_{g_0}) is isometric to a smooth locally homogeneous space.

In Section 4.1 we define, using the local isometries and Proposition 3.1, a local group G acting transitively on an open subset of $O \subset M$ and prove that G is locally isomorphic to a Lie group. This makes it possible to write O as a local quotient of this Lie group by a local isotropy group.

In Section 4.2, we find a left invariant Riemannian metric on this quotient which turns the canonical homeomorphism into an isometry.

4.1 Extracting the smooth structure

We start by defining the local group.

Let us collect the data for the local group in the sense of Definition 2.4. Starting with our locally homogeneous C^0-Riemannnian manifold M regard some compact ball $\overline{B}_{r_0}(x_0)$ and let $R > 0$ be a constant coming from Proposition 3.1. We can assume $r_0 \leq R$.

In the following, we will consider distance preserving maps, which are not neccessary bijective. To simplify notation we will also call them isometries.

Denote by F_1 and F_2 the extensions of isometries $f_1, f_2 : B_{r_0/10}(x_0) \to M$, which exist due to Proposition 3.1.

(1) Endow the collection of maps

$$G := \left\{ f : B_{\frac{r_0}{10}}(x_0) \to M : f \text{ isometric}, \ f(B_{\frac{r_0}{10}}(x_0)) \cap B_{\frac{r_0}{10}}(x_0) \neq \emptyset \right\}.$$

with the compact open topology.

(2) On $\Omega := \{(f_2, f_1) \in G \times G : F_2 \circ f_1 \in G\}$ define the partial multiplication by $m(f_2, f_1) := F_2 \circ f_1$.

(3) The neutral element e of G is the identity map.

(4) For every $f \in G$ define the inversion operation: Given $f \in G$ there exists a point $y \in f(B_{\frac{r_0}{10}}(x_0)) \cap B_{\frac{r_0}{10}}(x_0)$ and $r > 0$ such that

$$B_r(y) \subset f(B_{\frac{r_0}{10}}(x_0)) \cap B_{\frac{r_0}{10}}(x_0).$$

Consider the restriction of f to $B_r(f^{-1}(y))$. Since f is an isometry, its inverse map $f^{-1} : B_r(y) \to B_r(f^{-1}(y))$ is also an isometry and thus has an extension $F^{-1} : B_R(y) \to B_R(f^{-1}(y))$. Now define $i(f)$ to be the restriction $F^{-1}|_{B_{\frac{r_0}{10}}(x_0)}$.

To unburden the notation, we will write G and mean G together with the partial group structure as specified above. With a slight abuse of notation, we will call G the local isometries of M.

Now it is straightforward to verify that G as defined above is indeed a local topological group, which is locally compact.

Proposition 4.1 (Local isometries form a local group). *Let M be a locally homogeneous C^0-Riemannian manifold and denote by G the local isometries, as defined above. Then G is a locally compact local topological group and the canonical action $G \times B_{\frac{r_0}{10}}(x_0) \mapsto M$; $(g, p) \mapsto g(p)$ is continuous.*

Proof The axioms (1)-(4) in Definition 2.4 follow immediately from our construction. Local compactness follows from an Arzela-Ascoli type argument. □

We want to show that the local group G of local isometries is locally isomorphic to a Lie group. In order to do this, we want to apply van den Dries-Goldbring globalization [32] together with the Gleason-Yamabe theorem. An important observation is that small subgroups that could appear in G are actually Lie groups. This fact is encoded in the next Lemma.

Lemma 4.2 *Let G be the local group defined in Proposition 4.1. Then there exists a neighborhood U of the identity, such that every locally compact subgroup H satisfying $H \subset U$ is a Lie group.*

Proof Denote by $U \subset G$ the local isometries satisfying $d_{g_0}(x_0, f(x_0)) < r_0/100$. Then for any locally compact subgroup $H \subset U$ one has

$$\sup_{f \in H} d_{g_0}(x_0, f(x_0)) < \frac{r_0}{100}.$$

This implies $H(B_{\frac{r_0}{100}}(x_0)) \subset B_{\frac{r_0}{50}}(x_0)$, meaning that for every $f \in H$ the restriction of f to $\cup_{h \in H} h(B_{\frac{r_0}{100}}(x_0))$ is defined.

Observe that $\cup_{h \in H} h(B_{\frac{r_0}{100}}(x_0))$ is an open set, which is invariant under all elements of H since H is a group. Moreover, H is a group acting effectively via Lipschitz homeomorphisms on an open set of $\mathbb{R}^n$. So by the Hilbert-Smith theorem (Theorem 3.4) it is a Lie group. □

Proposition 4.3 *The local group defined in Proposition 4.1 is locally isomorphic to a Lie group.*

Proof By Proposition 4.1 the local group G is locally compact; thus, applying van den Dries-Goldbring globalization theorem [32] produces a restriction $G|_U$ of G and a topological group $\hat{G}$ such that $G|_U$ is a restriction of $\hat{G}$ (compare Definition 2.5).

Applying the Gleason-Yamabe theorem [31][Theorem 1.1.17] to an open neighborhood U as in Lemma 4.2 yields an open subgroup G' of $\hat{G}$ and a compact normal subgroup K of G' such that G'/K is isomorphic to a Lie group. By Lemma 4.2 K is a Lie group and therefore G' is a Lie group as well by [11][Theorem 1]. The claim now follows. □

By the above proposition we can assume that a local Lie group, which we denote by U_G, is acting on a subset of M by local isometries. Consider the isotropy group H of G, by the above some restriction U_H of it is a local Lie group. One can define a local quotient U_G/U_H, such that U_G/U_H is a smooth manifold, U_G operates transitively on U_G/U_H and it is homeomorphic to an open subset of M. Local factor spaces of this type have appeared in [20, 30] and a rigorous definition has been written by Pediconi [26][Proposition 6.1].

4.2 Capitalizing the Smooth Structure

In this subsection we will use the smooth structure on U_G/U_H obtained in the previous Section 4.1. Let us denote the canonical homeomorphism by $h : O \to U_G/U_H$ where $O \subset M$ and U_G acts transitively on O.

First we find some left invariant Riemannian metric on the local quotient $M' = U_G/U_H$. In a second step we show: If we consider M' together with this left invariant metric, then the canonical homeomorphism $h : O \to M'$ is a Lipshitz map (this statement is a modification of [1][Lemma 1]). Then applying the Rademacher theorem we find a point p such that $dh_p : T_pO \to T_{h(p)}M'$ is an isomorphism of tangent spaces. Using this isomorphism we can push forward the continuous Riemannian metric of M at a point and extend it to a left-invariant (and thus smooth) Riemannian metric on M'. Finally it will be shown that this metric space is isometric to (O, d_{g_0}), which proves the main theorem.

Our first intermediate goal is to construct a U_G-invariant Riemannian metric on U_G/U_H. If G is a global Lie group and H is a closed subgroup using an averaging procedure such a metric is known to exist if G acts effectively on G/H and the closure of Ad_H is compact [6][Proposition 3.16].

Returning to our setting there exists a global, connected, simply connected Lie group $\hat{G}$, which is locally isomorphic to U_G. Denote by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of U_G, U_H respectively and set $\hat{H} := \langle \exp(\mathfrak{h}) \rangle$. In order to reproduce the averaging procedure mentioned above, it is sufficient to show the compactness of $Ad_{\hat{H}}$, this is carried out in the next Lemma.

Lemma 4.4 *Let $\hat{G}, \hat{H}$ be as described above and denote by $\mathfrak{g}, \mathfrak{h}$ their Lie algebras. Then $Ad_{\hat{H}}$ is compact.*

Proof Observe that the stabilizer H of x_0 is a compact global group and moreover by Theorem 3.4 is a Lie group. This means that the identity component H_0 is a compact, connected Lie group. Hence for any $h \in H_0$ there exist $h_1, \ldots, h_n \in U_H$ and $n \in \mathbb{N}$ such that $h = h_1 \cdots h_n$. Similarly, if $i : U_G \mapsto \hat{G}$ denotes the local isomorphism and $\hat{U}_H := i(U_H)$, then for every $\hat{h} \in \hat{H}$ there are $\hat{h}_1, \ldots, \hat{h}_n \in \hat{U}_H$ and $n \in \mathbb{N}$ such that $\hat{h} = \hat{h}_1 \cdots \hat{h}_n$.

Analogously to the proof of Lemma 4.2 one can define a neighborhood U_0 of the identity in U_G satisfying $hU_0h^{-1} \subset U_0$ such that all products between elements in U_0 are defined. Using the invariance property of U_0 one can prove inductively for $g \in U_0$, $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in U_0$ the identity

$$i(h_1 \cdots h_n \cdot g \cdot h_n^{-1} \cdots h_1^{-1}) = i(h_1) \cdots i(h_n) \cdot i(g) \cdot i(h_n)^{-1} \cdots i(h_1)^{-1}.$$

Observe that this property does not follow from the homomorphism property, since n-fold multiplication might not be defined. We have achieved the following: One can write conjugation with arbitrary elements in H_0, $\hat{H}$ in terms of elements of U_H and $\hat{U}_H$. This makes it possible to relate Ad_{H_0}, which is known to be compact, to $Ad_{\hat{H}}$. For $h \in H_0$ write $h = h_1 \cdots h_n$ and define the map

$$F_h : \hat{G} \to \hat{G}; g \mapsto i(h_1) \cdots i(h_n) \cdot i(g) \cdot i(h_n)^{-1} \cdots i(h_1)^{-1}.$$

By the formula above F_h is well-defined. Consider the map

$$f : H_0 \to Ad_{\hat{H}}, h \mapsto d_e(F_h).$$

It remains to show that f is continuous and surjective. Surjectivity follows from the formula above. For continuity of f observe that it is sufficient to prove continuity in e. Then we have $f = Ad \circ i$, which is a composition of continuous maps. This finishes the proof. □

With Lemma 4.4 and the remark preceding it we obtain:

Corollary 4.5 *There is a U_G-invariant Riemannian metric $\hat{g}$ on U_G/U_H.*

It follows from the next lemma that the canonical homeomorphism $h : (M, d_{g_0}) \to (U_G/U_H, d_{\hat{g}})$ is a Lipschitz map.

Lemma 4.6 (f is Lipschitz). *Let d^* be an intrinsic metric on U_G/U_H, which is U_G-invariant and let g be a smooth U_G-invariant Riemannian metric on U_G/U_H. Then for some constant $C > 0$ we have $d_g \leq Cd^*$.*

Proof It is possible to apply almost the same arguments as in [1][Lemma 1]. Consider the Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ of U_G and U_H and the Ad_H-invariant subspace (where H denotes the stabilizer of x_0 as in the proof of Lemma 4.4) $\mathfrak{M} \subset \mathfrak{g}$ complementary to $\mathfrak{h}$. Further we consider a map $\varphi(m) = \exp(m)U_H$ defined around the origin $\varphi : \mathfrak{M} \to U_G/U_H$, its differential at 0 is a linear isomorphism. Then we lift the scalar product g_{eH} to $\mathfrak{M}$ and consider the corresponding metric. Consider a small ball $B_{2r}(0) \subset \mathfrak{M}$ such that φ maps $B_{2r}(0)$ diffeomorphically onto its image.

Set $C = \inf_{y \in S_r(0)} d^*(eH, \exp(y)U_H) > 0$. For any $k \in \mathbb{N}$ by triangle inequality we have

$$d^*(eH, \exp(y/k)U_H) \geq \frac{1}{k} d^*(eH, \exp(y)U_H) \geq C/k \geq d_g(eH, \exp(y/k)U_H).$$

Then the inequality claimed in the assertion follows since both metrics are intrinsic and invariant under the group action of U_G. □

Denote by $\varphi_1 : \Omega_1 \subset \mathbb{R}^n \to O, \varphi_2 : \Omega_2 \subset \mathbb{R}^n \to U_G/U_H$ charts around x_0, eH respectively. In view of Lemma 2.3, Lemma 4.6 we have that the coordinate expression $F := \varphi_2^{-1} \circ f \circ \varphi_1$ is a Lipschitz map between open subsets of $\mathbb{R}^n$. By the Rademacher theorem F is differentiable almost everywhere and we have the area formula [9][3.3.2]

$$\int_{\Omega_1} \det(dF_x)dx = \int_{\mathbb{R}^n} \mathcal{H}^0(\Omega_1 \cap F^{-1}(z))\,dz = \int_{F(\Omega_1)} 1\,dz > 0,$$

where $\mathcal{H}^0$ denotes the counting measure. The second equality comes from the fact that F is a homeomorphism. This implies that there is a point such that dF is an isomorphism, without loss of generality at $\varphi_1^{-1}(x_0)$, then df is an isomorphism at x_0. Define a scalar product g^* at eH by the formula

$$g^*_{eH}(v, w) := g_0((df)^{-1}_{eH}v, (df)^{-1}_{eH}w)(x_0).$$

Observe that one can define a smooth Riemannian metric g^* by

$$g^*_{gH}(v', w') := g^*_{eH}(dL_{g^{-1}}v', dL_{g^{-1}}w').$$

The expression above is well-defined, since g^*_{eH} is Ad_H-invariant. This comes from the fact that it is defined in terms of a linear isomorphism, whose original map is adapted to the group action.

Lemma 4.7 (Equality condition for homogeneous Riemannian metrics). *Let* (Ω, g) (Ω', g') *be locally homogeneous* C^0*-Riemannian manifolds and* $h : \Omega \to \Omega'$ *be a homeomorphism satisfying the following conditions:*

(1) The map h *respects the local isometries of* Ω *in the following sense: For any* $x, y \in \Omega$ *there exist open neighborhoods* $x \in U_x, y \in U_y$ *and an isometry* $f_{xy} : U_x \to U_y$ *such that* $h \circ f_{xy} \circ h^{-1} : h(U_x) \to h(U_y)$ *is an isometry as well.*
(2) There exists a point $p \in \Omega$ *such that the map* h *is differentiable at* p *and* $d_p h$ *is an isometry.*

Then the map $h : (\Omega, d_g) \to (\Omega', d_{g'})$ *is an isometry, where* $d_g, d_{g'}$ *denote the metrics induced by* g, g' *respectively.*

Proof It is sufficient to show that the length of rectifiable curves is the same with regard to both Riemannian metrics. That means $L_{d_g}(\gamma) = L_{d_{g'}}(h \circ \gamma)$ for every rectifiable curve γ in M.

Using the C^0-Riemannian manifold structure we can construct for every $\varepsilon > 0$ an $(1 + \varepsilon)$-bilipschitz chart around p and $h(p)$ respectively. From this together with condition (2) we can establish for all x in a neighborhood of p the following estimate

$$|d_g(p, x) - d_{g'}(h(p), h(x))| < \varepsilon \cdot d_g(p, x). \tag{1}$$

More precisely: If h is a map between two open subsets of $\mathbb{R}^n$, which is differentiable at a fixed point p, then the above estimate is obvious. Since our $(1 + \varepsilon)$-bilipschitz charts almost not distort distances the estimate holds in the general situation as well. Denote by $\gamma : [0, 1] \to \Omega$ a rectifiable curve and by $0 = z_0 < z_1 < \cdots < z_N = 1$ a partition of [0, 1]. We will show

$$L_{d_g}(\gamma) \geq \sum_{i=0}^{N-1} d_{g'}(h \circ \gamma(z_i), h \circ \gamma(z_{i+1})),$$

which implies $L_{d_g}(\gamma) \geq L_{d_{g'}}(h \circ \gamma)$. Indeed from assumption (1) of Lemma 4.7 and inequality (1) we have for any $\varepsilon > 0$ and any $0 \leq i \leq N-1$ the inequality $d_{g'}(h \circ \gamma(z_i), h \circ \gamma(z_{i+1})) \leq (1+\varepsilon) \cdot d_g(\gamma(z_i), \gamma(z_{i+1}))$. Since ε was arbitrary summation implies the claim. We have established that $h \circ \gamma$ is rectifiable, therefore by a similar argument as above we obtain $L_{d_g}(\gamma) \geq L_{d_{g'}}(h \circ \gamma)$, which yields the result. □

We are now ready to prove the main theorem.

Proof of Main Theorem First we want to show that our construction implies that the map $h : (O, d_0) \to (U_G/U_H, d_{g*})$ is an isometry. This follows Lemma 4.7. We have now proven a local version of the main theorem. For the general statement we construct a smooth chart for some neighborhood of an arbitrary point using the local homogenity condition and the local statement above. Then we observe that transition functions are smooth since by the Meyers-Steenrod theorem isometries between smooth manifolds are actually smooth maps. □

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