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Interval colorings of graphs—Coordinated and unstable no-wait schedules

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Abstract

A proper edge-coloring of a graph is an interval coloring if the labels on the edges incident to any vertex form an interval, that is, form a set of consecutive integers. The interval coloring thickness $\theta(G)$ of a graph G is the smallest number of interval colorable graphs edge-decomposing G. We prove that $\theta(G) = o(n)$ for any graph G on n vertices. This improves the previously known bound of $2\lceil n/5 \rceil$, see Asratian, Casselgren, and Petrosyan. While we do not have a single example of a graph with an interval coloring thickness strictly greater than 2, we construct bipartite graphs whose interval coloring spectrum has arbitrarily many arbitrarily large gaps. Here, an interval coloring spectrum of a graph is the set of all integers t such that the graph has an interval coloring using t colors. Interval colorings of bipartite graphs naturally correspond to no-wait schedules, say for parent–teacher conferences, where a conversation between any teacher and any parent lasts the same amount of time. Our results imply that any such conference with n participants can be coordinated in $o(n)$ no-wait periods. In addition, we show that for any integers t and T, $t < T$, there is a set of pairs of parents and teachers wanting to talk to each other, such that any no-wait schedules are *unstable*—they could last t

hours and could last T hours, but there is no possible no-wait schedule lasting x hours if $t < x < T$.

KEYWORDS
edge-decomposition, interval coloring, no-wait, schedule

1 | INTRODUCTION

Asratian and Kamalian [5, 6] introduced the notion of interval colorability of graphs. We say that a graph $G = (V, E)$ is *interval colorable* if there is an edge-coloring $c : E \to \mathbb{Z}$ such that for any vertex x, the multiset of colors incidents to x, that is, $\{c(xy) : xy \in E\}$ forms a set of consecutive integers, in other words an *interval of integers*. The respective coloring is called an *interval coloring*. In particular, an interval coloring is a proper coloring, that is, there are no two adjacent edges having the same color.

Interval colorings are applied in scheduling—for example, in case of teacher–parent conferences or machine-jobs assignments. In the former case one wants to schedule meetings between a parent and a teacher for given parent–teacher pairs such that each such meeting lasts the same amount of time and there is no waiting time between the meetings for any of the parents and any of the teachers.

Interval colorable graphs include all trees. In addition, all regular bipartite graphs are interval colorable since by Kőnig's theorem they are edge decomposable into perfect matchings. On the other hand, any graph of Class 2 is not interval colorable, where a graph is of Class 2 if its edge-chromatic number is greater than its maximum degree, $\Delta(G)$. For example, a triangle is such a graph. Indeed, otherwise considering the labels in an interval coloring modulo $\Delta(G)$ gives a proper edge-coloring using at most $\Delta(G)$ colors. Interval colorings for special classes of graphs and related problems were considered, see, for example, [2–4, 6, 7, 9–13, 16–21, 24, 25].

Let $c(G) = \{c(e) : e \in E(G)\}$ be the set of colors used on G by a coloring c. It is easy to see that for a connected graph G, and an interval coloring c, $c(G)$ is a set of consecutive integers. Here, we shall assume that all considered graphs are connected. Moreover, we assume that all objects considered are finite.

If there is an interval coloring of a graph G such that $|c(G)| = t$, we say that G is *t-interval colorable*. Let the *interval coloring spectrum* of G, denoted by $S(G)$, be the set of all integers t such that G is t-interval colorable. Note that $S(G)$ might be empty. The *interval coloring thickness* of a graph G, denoted $\theta(G)$, is the smallest integer k such that the graph can be edge-decomposed into k interval colorable graphs. In the language of parent–teacher conferences, having an interval coloring thickness of the respective graph equal to x implies that one can schedule the conference in x days with no waiting time for anyone during any of these x days. Let $\theta(n)$ be the largest interval coloring thickness of an n-vertex graph. Interval coloring thickness was considered for several special classes of graphs and bounded in terms of various graph parameters, [2]. Most notably $\theta(G) \le \gamma(G)$, where $\gamma(G)$ is the arboricity of G, that is, the minimum number of forests edge-decomposing G. The following result gives best-known bounds on $\theta(n)$.

Theorem 1 (Asratian, Casselgren, and Petrosyan [2]). *For any integer $n \ge 3$, $2 \le \theta(n) \le 2\lceil n/5 \rceil$.*

Here, we improve the upper bound:

Theorem 2. $\theta(n) = o(n)$.

To prove this result we employ the Regularity Lemma of Szemerédi, a version presented in Diestel [15], and a result by Alon, Rödl, and Ruciński [1] showing an existence of dense regular subgraphs in ε-regular pairs.

In addition, we show that the spectrum could have large gaps of large sizes. Here, a *gap* of a set of integers S is a maximal nonempty set X of consecutive integers, such that $X \cap S = \emptyset$, $\min S < \min X$, and $\max S > \max X$. For example, a set $\{2, 3, 6, 7\}$ has one gap $\{4, 5\}$ of size 2.

Theorem 3. *For any natural numbers k and d there is a graph G such that the spectrum $S(G)$ has exactly k gaps of size at least d each.*

This theorem is proved by giving an explicit construction of such a graph that in turn is built of parts from a construction by Sevastianov [22]. In [22], see also a translation [23], a construction of a graph with a single but arbitrarily large gap in its interval coloring spectrum is given. In the language of parent–teacher conferences, this result implies for example that there could be such a set of parent–teacher pairs willing to talk to each other so that one can schedule an optimal no-wait conference lasting 5 h, but if the school secretary does not manage to find an optimal scheduling, the only other option for a no-wait conference would require at least 105 h.

We shall give necessary definitions and preliminary results for the upper bound on interval coloring thickness in Section 2 and for the gaps in the interval coloring spectrum in Section 3. The main results are proved in Section 4. For some results in this paper, see also a bachelor thesis of the second author, M. Zheng [25].

2 | DEFINITIONS AND PRELIMINARY RESULTS

For standard graph theoretic notions we refer the reader to the book by Diestel [15]. We shall denote the number of vertices and the number of edges in a graph G by $|G|$ and $\|G\|$, respectively. We shall need some standard terminology for using the Regularity Lemma. For a graph G, let X and Y be disjoint vertex sets and $\varepsilon > 0$. We define $G[X, Y]$ to be a bipartite graph with parts X and Y containing all edges of G with one endpoint in X and another in Y. Let $\|X, Y\|$ be the number of edges in $G[X, Y]$ and the density $d(X, Y)$ of (X, Y) to be $d(X, Y) = \frac{\|X, Y\|}{|X\|Y|}$. Let $\delta(X, Y) = \delta(G[X, Y])$, be the minimum degree of G. For a vertex x, we denote the neighborhood of x by $N(x)$ and the degree of x by $\deg(x)$.

A pair (X, Y) is an *ε-regular pair* in G or more precisely a (d, ε)-regular pair if X and Y are disjoint vertex sets in G and $|d - d(A, B)| \leq \varepsilon$ for all $A \subseteq X, B \subseteq Y$ with $|A| \geq \varepsilon|X|$, $|B| \geq \varepsilon|Y|$, and $d = d(X, Y)$. We call an ε-regular pair (X, Y) in G a *super ε-regular pair* in G if $|X| = |Y|$ and

$$\delta(X, Y) \geq (d(X, Y) - \varepsilon)|X|.$$

A bipartite graph G' with parts X and Y is called a *super (d, ε)-regular graph* if (X, Y) is super ε-regular pair in G' with density d.

A vertex-set partition $V_0 \cup \cdots \cup V_k = V$ for a graph $G = (V, E)$ is an *ε-regular partition* if

1. $|V_0| \leq \varepsilon|V|$,
2. $|V_1| = |V_2| = \cdots = |V_k|$,
3. All but at most εk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ε-regular.

Theorem 4 (Szemerédi's Regularity Lemma, Diestel [15]). *For every $\varepsilon > 0$ and every integer $m \in \mathbb{N}$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ε-regular partition $V_0 \cup \cdots \cup V_k$ with $m \leq k \leq M$.*

Lemma 5 (Alon, Rödl, and Ruciński [1]). *Let G' be a bipartite super (d, ε)-regular graph with parts of size n each and let $d > 2\varepsilon$. Then G' contains a spanning k-regular subgraph, where $k = \lceil (d - 2\varepsilon)n \rceil$.*

The following standard lemma shows that an ε-regular pair contains a large super ε-regular pair.

Lemma 6. *Let (X, Y) be a (d, ε)-regular pair in a graph G with $d > 4\varepsilon$ and $|X| = |Y| = n$. Then, there are sets $X' \subseteq X, Y' \subseteq Y$ such that $|X'| = |Y'| > (1 - \varepsilon)n$ and (X', Y') is a super 3ε-regular pair in G with density d' where $d' \geq d - \varepsilon$.*

Proof. Let

$$\tilde{X} = \{x \in X : |N(x) \cap Y| \geq (d - \varepsilon)|Y|\} \text{ and}$$
$$\tilde{Y} = \{y \in Y : |N(y) \cap X| \geq (d - \varepsilon)|X|\}.$$

Note that $|\tilde{X}| > (1 - \varepsilon)|X|$. Otherwise, let $X_2 = X \backslash \tilde{X}$ and observe that any vertex in X_2 has less than $(d - \varepsilon)$ neighbors in Y, thus $d(X_2, Y) < (d - \varepsilon)$, a contradiction to ε-regularity since $|X_2| \geq \varepsilon n$. A similar argument holds for $\tilde{Y}$.

Let $X' \subseteq \tilde{X}, Y' \subseteq \tilde{Y}$ such that $|X'| = |Y'| = \min\{|\tilde{X}|, |\tilde{Y}|\}$. Then $|X'| = |Y'| = n' > (1 - \varepsilon)|X| = (1 - \varepsilon)n$. Note that for $\varepsilon < 1/2$ we have $n' > \varepsilon n$. We shall show that (X', Y') satisfies the minimum degree and regularity conditions of a super-regular pair.

Let $d' = d(X', Y')$. By ε-regularity of (X, Y), we have $d' \leq d + \varepsilon$. Moreover, from the definition of X' we have $\delta(X', Y') \geq (d - \varepsilon)n - \varepsilon n = (d - 2\varepsilon)n \geq (d' - 3\varepsilon)n \geq (d' - 3\varepsilon)n'$. Now, consider $A \subseteq X', B \subseteq Y'$, such that $|A| \geq 3\varepsilon|X'|$ and $|B| \geq 3\varepsilon|Y'|$. Observe that $|A| \geq 3\varepsilon|X'| > 3\varepsilon(1 - \varepsilon)n > \varepsilon n$. Similarly, $|B| > \varepsilon n$. Then, by ε-regularity of (X, Y), we obtain that

$$|d(X', Y') - d(A, B)| \leq |d(X', Y') - d(X, Y)| + |d(A, B) - d(X, Y)| \leq 2\varepsilon \leq 3\varepsilon.$$

Thus, (X', Y') is a 3ε-regular pair with density $d - \varepsilon \leq d' \leq d + \varepsilon$ and minimum degree $\delta(X', Y') \geq (d' - 3\varepsilon)n$. □

Theorem 7. *For every γ, $1/2 > \gamma > 0$, there exists $M \in \mathbb{N}$ such that every graph G contains a subgraph G' with $\theta(G') \leq M$ and $||G|| - ||G'|| \leq \gamma|G|^2$.*

Proof. Let $1/2 > \gamma > 0$ be arbitrary. Choose $\varepsilon > 0$ sufficiently small and $m \in \mathbb{N}$ sufficiently large such that $\frac{1}{2m} + \left(11 + \frac{\varepsilon}{2}\right)\varepsilon \leq \gamma$. By Szemerédi's Regularity Lemma, see Theorem 4, there exists $M \in \mathbb{N}$ such that every graph of order at least m has an ε-regular partition $V_0 \cup \cdots \cup V_k$ with $m \leq k \leq M$. Now, let G be a graph of order $n \in \mathbb{N}$. If $n < m$, we have that $\theta(G) \leq n \leq M$ using an upper bound in Theorem 1. Thus, we may assume that $n \geq m$. By our choice of M, we know that G has an ε-regular partition $V_0 \cup \cdots \cup V_k$ with $m \leq k \leq M$. Let $\ell = |V_1|$. We shall define a subgraph G' of G corresponding to regular pairs of sufficiently high density.

For each pair $(V_i, V_j), 1 \leq i < j \leq k$, we shall define a graph $G_{i,j}$. Let $d_{i,j}$ be the density of (V_i, V_j). If $d_{i,j} > 7\varepsilon$ and (V_i, V_j) is an ε-regular pair, consider $G[V_i, V_j]$ and apply Lemmas 5 and 6 to it. Let $G_{i,j}$ be a subgraph of $G[V_i, V_j]$ that is $q_{i,j}$-regular on at least $2(1-\varepsilon)\ell$ vertices and $q_{i,j} \geq (d_{i,j} - \varepsilon - 2(3\varepsilon))n$. Note that since $G_{i,j}$ is bipartite and regular, it is interval colorable. Note also that $G_{i,j}$ contains most of the edges of $G[V_i, V_j]$. We shall make this statement more precise below. If a pair (V_i, V_j) is not ε-regular or has density at most 7ε, let $G_{i,j}$ be an empty graph. Let

$$G' = \bigcup_{1 \leq i < j \leq k} G_{i,j}.$$

We shall show first that $\theta(G') \leq M$. For that let c be a proper edge-coloring of a complete graph with vertex set $\{1, \ldots, k\}$ using colors from $\{1, \ldots, k\}$ and let, for $s \in \{1, \ldots, k\}$,

$$G_s = \bigcup_{1 \leq i < j \leq k, c(ij) = s} G_{i,j}.$$

Since G_s is the vertex-disjoint union of interval colorable graphs, G_s is itself interval colorable, for all $s \in \{1, \ldots, k\}$. Since $G' = \bigcup_{1 \leq s \leq k} G_s$, we have that $\theta(G') \leq k \leq M$.

We will bound the number of edges from G that are not in G'. We call a pair (V_i, V_j), $i \neq j$, *nontrivial* if $i \neq 0$ and $j \neq 0$. We have that $\|G\| - \|G'\| = x_1 + x_2 + x_3 + x_4$, where

- x_1 is the number of edges in non-ε-regular pairs or with exactly one endpoint in V_0,
- x_2 is the number of edges induced by V_i's for $0 \leq i \leq k$,
- x_3 is the number of edges in nontrivial ε-regular pairs with density at most 7ε, and
- x_4 is the number of edges in nontrivial ε-regular pairs with density greater than 7ε that are not in G'.

Note that $\ell = |V_1| = \cdots = |V_k| \leq n/k$ and $|V_0| \leq \varepsilon n$. Moreover, the maximum number of edges in $G[V_i, V_j]$ is at most $(n/k)^2$, for $1 \leq i < j \leq k$. Since there are at most εk^2 nontrivial pairs that are non-ε-regular and at most $|V_0||V(G) - V_0| \leq \varepsilon(1-\varepsilon)n^2$ edges with exactly one endpoint in V_0, we have

$$x_1 \leq \varepsilon k^2 \cdot \left(\frac{n}{k}\right)^2 + \varepsilon n(n - \varepsilon n) \leq 2\varepsilon n^2.$$

In addition,

$$x_2 \le \binom{\varepsilon n}{2} + k\binom{n/k}{2} \le \frac{(\varepsilon n)^2}{2} + \frac{n^2}{2k}$$

and

$$x_3 \le \binom{k}{2} 7\varepsilon (n/k)^2 \le \frac{7}{2}\varepsilon n^2.$$

Finally, for x_4, note that for a pair with parts of size ℓ each and with density $d > 7\varepsilon$, the number of edges that are not in G' is at most $d \cdot \ell^2 - (d - 7\varepsilon) \cdot (\ell(1-\varepsilon))^2 \le 10\varepsilon l^2 \le 10\varepsilon\left(\frac{n}{k}\right)^2$. Thus

$$x_4 \le \binom{k}{2} 10\varepsilon (n/k)^2 \le 5\varepsilon n^2.$$

Therefore,

$$\begin{aligned} \|G\| - \|G'\| &= x_1 + x_2 + x_3 + x_4 \\ &\le \left(2\varepsilon + \frac{\varepsilon^2}{2} + \frac{1}{2k} + \frac{7}{2}\varepsilon + 5\varepsilon\right) n^2 \\ &\le \left(\frac{1}{2m} + 11\varepsilon + \frac{\varepsilon^2}{2}\right) n^2 \\ &\le \gamma n^2. \end{aligned}$$

This concludes the proof. □

3 | CONSTRUCTION OF A GRAPH WITH A GIVEN INTERVAL COLORING SPECTRUM

3.1 | Construction and properties of the graph $F(b, T)$

For positive integers b, D, where $b \le D$, let $T = D + 25$ and the graph $F = F(b, T)$ be formed by a union of five complete bipartite graphs with pairs of parts $(\{v, v'\}, V_0)$, $(\{v, v_r\}, V_r)$, $(\{v, v_l\}, V_l)$, $(\{u, u_r\}, U_r)$, $(\{u\}, U_d)$, as well as additional vertices w_l, w_r and edges $w_l v'$, $w_r v'$, $w_l v_l$, $w_r v_r$, $w_l x$, $w_r y$, xu, where $y \in U_r$. Here the vertices $x, v, v', v_l, v_r, u, u_r, w_l$, and w_r are distinct and not contained in any of the pairwise disjoint sets V_0, V_l, V_r, U_l, U_r, and U_d. Moreover $|V_0| = D + 12$, $|V_l| = |V_r| = 7$, $|U_r| = D - b + 2$, and $|U_d| = b$. We refer to the edges incident to U_d as *pendant*. See Figure 1.

As a part of a larger construction, Sevastianov [22] proved the interval coloring properties of F. Here we include the arguments for completeness. The following lemma claims that the interval colorings of F are very rigid. Depending on the smallest label used on F, the colors of certain edges are fixed. For integers q, i, j, $i \le j$, we shall denote the interval of integers

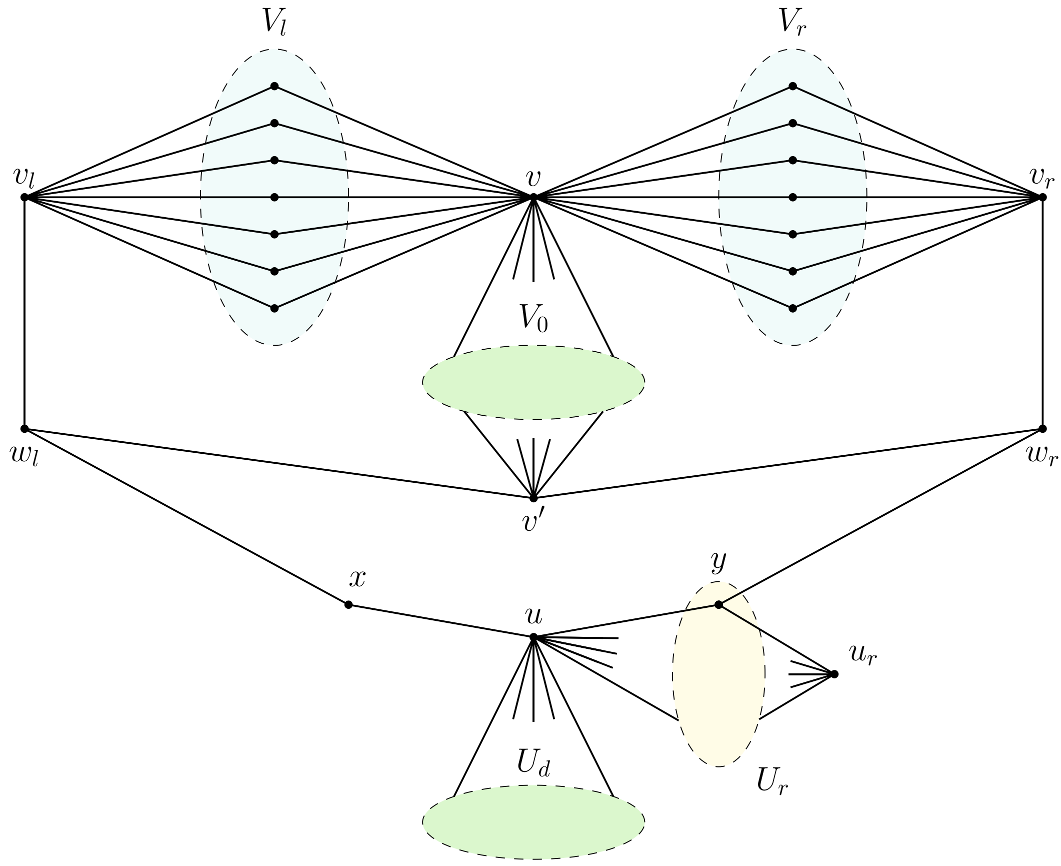

FIGURE 1 Graph $F(b, T)$. [Color figure can be viewed at wileyonlinelibrary.com]

$\{q + i, q + i + 1, \ldots, q + j\}$ as $q + [i, j]$ and call it a *shift* of an interval $[i, j]$. Moreover, for $i \leq j$ let $-[i, j] = [-j, -i]$.

Lemma 8 (Sevastianov [22])**.** *For any positive integers b and D, where $b \leq D$ and D is even, the graph $F = F(b, T)$ is planar, bipartite, and interval colorable for $T = D + 25$. Moreover, for any interval coloring c of F the following properties hold:*

1. $c(F) = c_1 + [0, T]$, *for some integer* c_1,
2. $c(w_l v_l) \in \{c_1 + 8, c_1 + T - 8\}$,
3. *if* $c(w_l v_l) = c_1 + 8$, *then the set of colors on the pendant edges is* $c_1 + 11 + [1, b]$*; if* $c(w_l v_l) = c_1 + T - 8$, *then the set of colors on the pendant edges is* $c_1 + T - 11 - [1, b]$.

Remark. The lemma implies that in any interval coloring of $F = F(b, T)$ the number of colors used is $T + 1$, the colors of the pendant edges form an interval of b numbers either starting with the first 13th number used on F or ending with the last 13th number used on F. For example, if $T = 37$, $b = 2$, and for an interval coloring c of F, $c(F) = \{3, \ldots, 40\}$, then the set of colors of the pendant edges is either $\{15, 16\}$ or $\{27, 28\}$.

Proof. To see that F is interval colorable, one can give an explicit coloring c as follows. We denote by $c(U, z)$ a set of colors on edges incident to a set of vertices U and a vertex z. Let $c(v_l w_l) = 9$, $c(w_l v') = 10$, $c(w_l x) = 11$, $c(xu) = 12$, $c(uy) = D + 14$, $c(yu_r) = D + 15$, $c(yw_r) = D + 16$, $c(v' w_r) = D + 17$, $c(v_r w_r) = D + 18$, and

$$\begin{aligned}
c(U_d, u) &= \{13, 14, \ldots, 12 + b\},\\
c(V_l, v) &= \{1, \ldots, 7\},\\
c(V_l, v_l) &= \{2, \ldots, 8\},\\
c(V_r, v) &= \{D + 26, D + 25, \ldots, D + 20\},\\
c(V_r, v_r) &= \{D + 25, D + 24, \ldots, D + 19\},\\
c(V_0, v) &= \{8, 9, 10, 11, 12, \ldots, D + 15, D + 16, D + 17, D + 18, D + 19\},\\
c(V_0, v') &= \{7, 8, 9, 12, 11, \ldots, D + 16, D + 15, D + 18, D + 19, D + 20\},\\
c(U_r, u) &= \{D + 14, D + 13, \ldots, 13 + b\},\\
c(U_r, u_r) &= \{D + 15, D + 14, \ldots, 14 + b\}.
\end{aligned}$$

Note that for this iterative pattern at V_0, one needs D to be even. The pattern for $c(V_0, v')$ is built based on $c(V_0, v)$ by splitting the corresponding ordered set into consecutive subsets of order $3, 2, 2, \ldots, 2$, and 3. The sets are ordered according to the order of appearance of the respective edges in Figure 1 from left to right. For the parts of order 2 in $c(V_0, v)$, the respective sets in $c(V_0, v')$ are obtained by flipping the elements. For example, 11, 12 corresponds to 12, 11. For the sets of order 3 in the beginning and the end of the list, a shift of labels is used. As a result, the labels 10 and $D + 17$ are missing from $c(V_0, v')$. However, they are present on other two edges incident to v'. The respective vertices in V_0 are incident to two edges labeled by consecutive integers: $(8, 7), (9, 8), (10, 9), (11, 12), (12, 11), \ldots, (D + 19, D + 20)$.

The main idea of the remaining proof is an observation that in an interval coloring of a graph the difference between the labels on two edges incident to a vertex z is less than the degree of z. Note that the degree of the vertex v in F is $D + 26$. Assume first that there is an edge e labeled 1 incident to v and that 1 is the smallest label at v. Then there is an edge e' incident to v and labeled $D + 26$.

Claim. Either e is incident to V_l and e' is incident to V_r or e' is incident to V_l and e is incident to V_r.

To prove the claim, note first that e and e' cannot both be incident to the same set V_0, V_l, or V_r. Indeed, otherwise we consider two edges e_1 and e_1' adjacent to e and e', respectively, and incident to a vertex r that is v', v_l, or v_r, respectively. Then the labels of e_1 and e_1' are at most 2 and at least $D + 25$, respectively, contradicting the fact that the degree of r is at most $D + 14$.

Now, assume that e is incident to V_0 and e' is incident to V_l. Consider a shortest path P joining noncommon endpoints of e' and e and avoiding v. It has length 4 and passes through vertices of degrees $2, 8, 3, D + 14$, and 2, respectively. The respective labels on the edges of P are at least $D + 25$, $D + 25 - 7, D + 25 - 7 - 2$, and $D + 25 - 7 - 2 - (D + 13)$, respectively. The label on the edge of P incident to e is at least 3, a contradiction since it must be at most 2. By a similar argument, it is impossible for e and e' to be incident to V_0 and V_r, to V_l and V_0, or to V_r and V_0, respectively. This proves the claim.

Assume first that e is incident to V_l and e' is incident to V_r.

To prove part 2 of the lemma, consider a path P' joining noncommon endpoints of e' and e and passing through u. Recall that $\deg(u) = D + 3$. The path P' has eight edges, with the second and next to last edges being $v_l w_l$ and $w_r v_r$, respectively. As before, the labels on consecutive edges of P' have labels at most $2, 9, 11, 12, 14 + D, 16 + D, 18 + D, 25 + D$, respectively. Since the label of the last edge is exactly $D + 25$, we see that all the edges of P' have exactly the labels listed: $2, 9, 11, 12, 14 + D, 16 + D, 18 + D, 25 + D$. So, $c(w_l v_l) = 9$ and $c(w_r v_r) = D + 18$.

To prove part 3 of the lemma, consider vertices x and y. Because of the properties of P' we see that $c(yu_r) = 15 + D$. Since $c(xu) = 12$, $c(uy) = 14 + D$, and $\deg(u) = D + 3$, all other labels on edges incident to u are greater than 12 and less than $14 + D$. As the labels on edges incident to u_r form an interval, the largest label on an edge incident to u_r is therefore $15 + D$, such that the interval is $15 + D, 14 + D, \ldots, 14 + b$. This implies that labels on edges incident to u and U_r form an interval of $D - b + 2$ integers with the largest one $14 + D$. Thus, edges incident to U_d get labels forming an interval of b integers with the smallest integer in the interval equal to 13.

Finally, we have seen that $c(F) = \{1, \ldots, D + 26\}$.

If e is incident to V_r, a similar to the above argument gives that the edges incident to U_d have labels forming an interval of b integers ending with $14 + D$. In this case we have the roles of w_r, v_r and w_l, v_l swapped, so $c(w_l v_l) = D + 18$ and $c(w_r v_r) = 9$. As before, $c(F) = \{1, \ldots, D + 26\}$. So, this establishes the lemma in the case when $c_1 = 1$.

If $\min c(V_l \cup V_r \cup V_0, v) = c_1 \neq 1$, consider an interval coloring c' defined by $c'(z) = c(z) - c_1 + 1$, $z \in V(F)$, that is, done by an appropriate label shift. Now, we have that $\min c'(V_l \cup V_r \cup V_0, v) = 1$ and we can apply the above considerations. $\square$

3.2 | Construction and properties of the graph $\mathbf{F}(k, d)$

For positive integers k, d where d is even and $d \geq 24$, let $F_0 = F(k, 3k^2 d + 1)$. Note that there are k pendant edges in F_0. Further, let $F_j = F(1, 2jdk + 1)$, $j = 1, \ldots, k$. Note that each F_j has a single pendant edge, $j = 1, \ldots, k$. Let $\mathbf{F}(k, d)$ be formed by first considering pairwise vertex disjoining copies of $F_0, F_1, \ldots, F_k$ and then identifying the jth pendant edge of F_0 with a pendant edge of F_j such that the vertex of degree one of the jth pendant edge of F_0 is identified with the vertex of degree greater than one in the pendant edge of F_j, $j = 1, \ldots, k$. See Figure 2 for an illustration.

Lemma 9. *For any positive integers k, d, $d \geq 24$, the graph $\mathbf{F} = \mathbf{F}(k, d)$ is interval colorable and has exactly k gaps, each of size at least d, in its interval coloring spectrum.*

Proof. Assume without loss of generality that d is even. Since the F_j's are interval colorable, one can create an interval coloring of each F_j, $j = 1, \ldots, k$, such that the pendant edge gets an arbitrary assigned value by shifting the labels appropriately. So, consider an arbitrary interval coloring of F_0 and then consider interval colorings of $F_1, \ldots, F_k$ such that the colors of the pendant edges equal the colors of the corresponding pendant edges of F_0. This gives an interval coloring of $\mathbf{F}$.

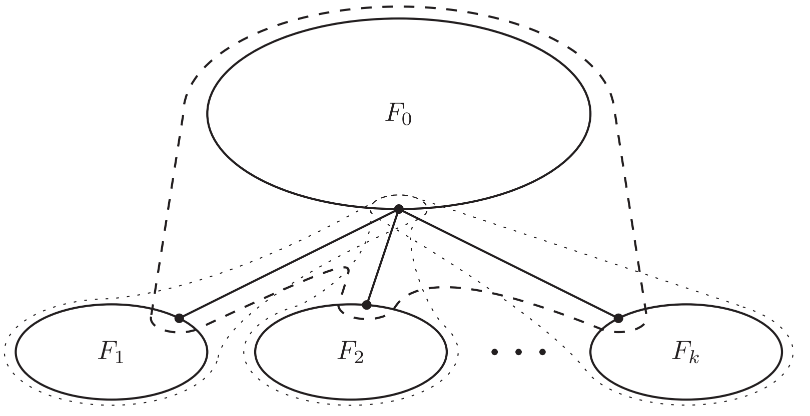

FIGURE 2 Graph $\mathbf{F}(k, d)$.

Let $T_0 + 1, T_1 + 1, ..., T_k + 1$ be the number of colors used in interval colorings of $F_0, ..., F_k$, respectively. Let c be an interval coloring of $\mathbf{F}$, assume without loss of generality that $c(F_0) = \{1, ..., T_0 + 1\}$. Note that c restricted to respective copies of $F_0, F_1, ..., F_k$ is an interval coloring. Instead of saying "a copy of F_j," we just say F_j for the rest of the proof. We know from Lemma 8 that there are k pendant edges in F_0 whose set of colors is either $\{13, 14, ..., 12 + k\}$ or $\{(T_0 + 1) - 12, (T_0 + 1) - 13, ..., (T_0 + 1) - 11 - k\}$.

Assume that the pendant edges of F_0 get the colors $13, 14, ..., 12 + k$. The situation when pendant edges in F_0 get colors $(T_0 + 1) - 12, (T_0 + 1) - 13, ..., (T_0 + 1) - 11 - k$ is completely symmetric resulting in the same number of colors used on $\mathbf{F}$ as in the respective configuration when the pendant edges of F_0 get the colors $13, 14, ..., 12 + k$. Lemma 8 implies that for each $j = 1, ..., k$, the pendant edge of F_j either gets the 13th color of $c(F_j)$ or the last 13th such color. We say that in the former case F_j is of *type 1* under c and in the latter case F_j is of *type 2* under c.

If F_j is of type 1 under c, $c(F_j) \subseteq c(F_0)$ since $T_0 > T_j + k$. If F_j is of type 2 under c, $\max c(F_j) \in c(F_0)$ and $\min c(F_j)$ must take one of the values on the interval $[-T_j + 25, -T_j + 24 + k]$, depending on whether the pendant edge of F_j is identified with the edge of color $13, 14, ...,$ or $12 + k$, respectively.

If $j \in [1, k]$ is the largest index for which F_j is of type 2, then any such interval coloring of $\mathbf{F}$ uses t colors for $t \in [T_0 + T_j - 22 - k, T_0 + T_j - 23]$. Moreover, for any t in this interval there is a corresponding interval coloring of $\mathbf{F}$. Observe that for any $j \in [1, k]$, there is an interval coloring of $\mathbf{F}$ such that F_j is of type 2 and each F_i is of type 1 for $i \in [1, k] \setminus \{j\}$. Thus, $S(\mathbf{F}) = \{T_0 + 1\} \cup \bigcup_{j=1}^{k} [T_0 + T_j - 22 - k, T_0 + T_j - 23]$. Since $T_j = 2jdk + 1$, $j \in [1, k]$, we see that the interval coloring spectrum has k gaps of sizes at least $2dk - k - 23 \geq d$. □

4 | PROOFS OF THE MAIN RESULTS

Proof of Theorem 2. Let $0 < \gamma < 1/2$ and M be the constant guaranteed by Theorem 7. Let G be a graph on n vertices. Then, by Theorem 7, G is a union of two graphs G' and G'', where

$\theta(G') \le M$ and $\|G''\| \le \gamma n^2$. By a result by Dean, Hutchinson, and Scheinermann [14], the arboricity of any graph is at most $\lceil \sqrt{e/2} \rceil$, where e is the number of edges in that graph. Since the interval coloring thickness is at most the arboricity, we have $\theta(G'') \le \sqrt{\gamma} n$. In particular, for large enough n, we have that $\theta(G) \le M + \sqrt{\gamma} n \le 2\sqrt{\gamma} n$. This implies in particular that $\theta(n) = o(n)$. □

Proof of Theorem 3. This theorem follows immediately from Lemma 9 using a construction of the graph $\mathbf{F}(k, d)$. □

Remark. After this paper has been accepted for publication, Axenovich et al. [8] proved that $\log(n)^{1/3-o(1)} \le \theta(n) \le n^{5/6+o(1)}$ and conjectured that $\theta(n) = n^{o(1)}$.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are openly available in ArXiv at https://arxiv.org/abs/2205.05947.

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