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Tail processes and tail measures: An approach via Palm calculus

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Abstract

Using an intrinsic approach, we study some properties of random fields which appear as tail fields of regularly varying stationary random fields. The index set is allowed to be a general locally compact Hausdorff Abelian group $\mathbb{G}$. The values are taken in a measurable cone, equipped with a pseudo norm. We first discuss some Palm formulas for the exceedance random measure ξ associated with a stationary (measurable) random field $Y = (Y_s)_{s\in\mathbb{G}}$. It is important to allow the underlying stationary measure to be σ-finite. Then we proceed to a random field (defined on a probability space) which is spectrally decomposable, in a sense which is motivated by extreme value theory. We characterize mass-stationarity of the exceedance random measure in terms of a suitable version of the classical Mecke equation. We also show that the associated stationary measure is homogeneous, that is a tail measure. We then proceed with establishing and studying the spectral representation of stationary tail measures and with characterizing a moving shift representation. Finally we discuss anchoring maps and the candidate extremal index.

Keywords: tail process, exceedances, tail measure, spectral representation, random measure, Palm measure, stationarity, mass-stationarity, locally compact Abelian group, anchoring map, candidate extremal index

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1 Introduction

The tail process of regularly varying time series was introduced in [1]. It is a useful tool for describing and handling the extreme value behavior of such time series; see e.g. [3, 10, 20]. The recent paper [17] has made some interesting connections to Palm theory for point processes on $\mathbb{Z}^d$. In particular it has been observed there that the exceedance point process of the tail process is point-stationary in the sense of [21]; see also [14]. One aim of the present paper is to extend [17] to the case of a general locally compact Hausdorff Abelian group $\mathbb{G}$, for instance $\mathbb{G} = \mathbb{R}^d$. Even in the case $\mathbb{G} = \mathbb{Z}^d$ our approach will provide further insight into the results from [17]. Another aim is to extend the concept of a tail measure

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(as defined in [3, 20]) to spaces of functions on Abelian groups, to relate these measures to Palm calculus and to study their spectral representation.

Section 2 contains some basic definitions and facts from Palm theory. In Section 3 we will first provide a modest but useful generalization of [12, Theorem 4.1] on allocations and Palm measures. Then we summarize some facts on point- and mass-stationarity. In Section 4 we consider a field $Y = (Y_s)_{s\in\mathbb{G}}$ indexed by the group. The field takes its values in a measurable cone $\mathbb{H}$ equipped with a pseudo norm $|\cdot|$. A key example is $\mathbb{H} = \mathbb{R}^d$ with the Euclidean norm. We require Y to have natural measurability properties but do not impose continuity or separability assumptions. The exceedance random measure is defined by $\xi := \int \mathbf{1}\{s \in \cdot, |Y_s| > 1\}\lambda(ds)$, where λ is a Haar measure on $\mathbb{G}$. We briefly discuss stationarity, mass-stationarity and the Palm measure of ξ. For our purposes it is important to allow the underlying stationary measure $\mathbb{P}$ to be infinite (but σ-finite). The Palm measure of ξ is simply the restriction of $\mathbb{P}$ to the event $\{|Y_0| > 1\}$. Starting with Section 5 we shall work on a suitable canonical function space $(\mathbf{F}, \mathcal{F})$ with the field Y given as the identity on $\mathbf{F}$. At the cost of a more abstract setting, this could be generalized along the lines of Remark 5.9. In Section 5 we assume that Y is spectrally decomposable with index $\alpha > 0$ w.r.t. a probability measure $\mathbb{Q}$ on $(\mathbf{F}, \mathcal{F})$. This assumption is strongly motivated by [1] and means that $|Y_0|$ has a Pareto distribution (on $(1,\infty)$) with parameter α and is independent of $W := (|Y_0|^{-1}Y_s)_{s\in\mathbb{G}}$. Our Theorem 5.2 shows that the exceedance random measure ξ is mass-stationary in the sense of [14] if and only if W satisfies the space shift formula (5.2), a version of the classical Mecke equation from [15]. This generalizes the main result in [17] from $\mathbb{Z}^d$ to general locally compact Hausdorff Abelian groups. In establishing this result, we will not refer to a regularly varying field in the background. Under the assumptions of Theorem 5.2, general Palm theory essentially guarantees the existence of a stationary σ-finite measure ν such that $\mathbb{Q}$ is the Palm measure of ξ w.r.t. ν, that is $\mathbb{Q} = \nu(\cdot \cap \{|Y_0| > 1\})$. In Section 6 we shall prove among other things that ν is α-homogeneous, that is a tail measure. In Section 7 we shall prove with Theorem 7.3 that any stationary tail measure ν has a spectral representation. While the existence of such a representation can be derived from [5, Proposition 2.8] (see Remark 7.4), our result provides an explicit construction of the spectral measure in terms of the Palm measure $\mathbb{Q}$ of ξ along with further properties. Theorem 7.3 extends the stationary case of [3, Theorem 2.4] (dealing with $\mathbb{G} = \mathbb{Z}$) and [20, Theorem 2.3] (dealing with the case $\mathbb{G} = \mathbb{R}$) to general Abelian groups. We also characterize a moving shift representation. In the final Section 8 we study anchoring maps, as defined in [17, 20] for mass-stationary fields with the property $\mathbb{Q}(0 < \xi(\mathbb{G}) < \infty) = 1$. Proposition 8.1 extends [17, Proposition 3.2] to general Abelian groups. In the remainder of the section we assume Y to be spectrally decomposable. Motivated by [20, Section 2.3] we provide some information on the candidate extremal index.

In this paper we treat tail processes in an intrinsic way, namely as a spectrally decomposable random field $Y = (Y_s)_{s\in\mathbb{G}}$ such that ξ is mass-stationary. This is in line with the developments in [3, 10, 20] and in the recent preprints [2, 6].

2 Some Palm calculus

Assume that $\mathbb{G}$ is a locally compact Hausdorff group with Borel σ-field $\mathcal{G}$ and (non-trivial) Haar measure λ. Important special cases are $\mathbb{G} = \mathbb{Z}^d$ with λ being the counting measure and $\mathbb{G} = \mathbb{R}^d$ with λ being the Lebesgue measure. Let $\mathbf{M}$ denote the space of measures on $\mathbb{G}$ which are locally finite (that is, finite on compact sets) and let $\mathcal{M}$ be the smallest σ-field on $\mathbf{M}$ making the mappings $\mu \mapsto \mu(B)$ measurable for all $B \subset \mathbb{G}$. Let $\mathbf{N}$ be the measurable subset of $\mathbf{M}$ of those $\mu \in \mathbf{M}$ which are integer-valued on relatively compact Borel sets. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a σ-finite measure space. At the moment the reader might think of $\mathbb{P}$ as of a probability measure. However, for our later purposes it is important to allow for $\mathbb{P}(\Omega) = \infty$. Still we shall use a probabilistic language. A *random measure* (resp. *point process*) ξ on $\mathbb{G}$ is a measurable mapping $\xi\colon \Omega \to \mathbf{M}$ (resp. $\xi\colon \Omega \to \mathbf{N}$). We find it convenient to use this terminology even without reference to a (probability) measure on $(\Omega, \mathcal{A})$. We often use the kernel notation $\xi(\omega, B) := \xi(\omega)(B)$, $(\omega, B) \in \Omega \times \mathcal{G}$. A point process ξ is said to be *simple*, if $\xi(\omega, \{s\}) \le 1$ for all $(\omega, s) \in \Omega \times \mathbb{G}$.

Next we give a short but self-contained introduction into Palm calculus, using the setting from [16] and [14]. A more comprehensive summary can be found in [12]. Assume that $\mathbb{G}$ acts measurably on $(\Omega, \mathcal{A})$. This means that there is a family of measurable mappings $\theta_s\colon \Omega \to \Omega$, $s \in \mathbb{G}$, such that $(\omega, s) \mapsto \theta_s\omega$ is measurable, θ_0 is the identity on Ω and

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s, t \in \mathbb{G}, \tag{2.1}$$

where $\circ$ denotes composition. The family $\{\theta_s : s \in \mathbb{G}\}$ is said to be (measurable) *flow* on Ω. A random measure on $\mathbb{G}$ is said to be *invariant* (w.r.t. to the flow) or *flow-adapted* if

$$\xi(\omega, B + s) = \xi(\theta_s\omega, B), \quad \omega \in \Omega,\ s \in \mathbb{G}, B \in \mathcal{G}. \tag{2.2}$$

Let us illustrate these concepts with two examples.

Example 2.1. Assume that $(\Omega, \mathcal{F}) = (\mathbf{M}, \mathcal{M})$ and define $\theta_s\mu := \mu(\cdot + s)$, for $\mu \in \mathbf{M}$ and $s \in \mathbb{G}$. Then $\{\theta_s : s \in \mathbb{G}\}$ is a flow and the identity on $\mathbf{M}$ is invariant.

Example 2.2. Let $\mathbb{H}$ be a (non-empty) Polish space equipped with the Borel σ-field $\mathcal{H}$ and consider the space $\mathbb{H}^{\mathbb{G}}$ of all functions $\omega\colon \mathbb{G} \to \mathbb{H}$. For each $s \in \mathbb{G}$ we define the shift-operator $\theta_s\colon \mathbb{H}^{\mathbb{G}} \to \mathbb{H}^{\mathbb{G}}$ by $\theta_s\omega := \omega(\cdot + s)$. Assume now that $\mathbf{F}$ is shift-invariant subset of $\mathbb{H}^{\mathbb{G}}$ equipped with a σ-field $\mathcal{F}$ such that $(\omega, s) \mapsto (\theta_s\omega, \omega(0))$ is measurable with respect to $\mathcal{F} \otimes \mathcal{H}$. For instance we can take $\mathbb{G} = \mathbb{R}^d$, $\mathbb{H} = \mathbb{R}$, $\mathbf{F}$ as the *Skorohod space* of all càdlàg functions (see e.g. [8]) and $\mathcal{F}$ as the smallest σ-field rendering the mappings $\omega \mapsto \omega(t)$, $t \in \mathbb{G}$, measurable. Then even $(\omega, t) \mapsto \omega(t)$ is measurable and therefore also $(\omega, s) \mapsto \theta_s\omega$, as required. An example of an invariant random measure (defined on $\mathbf{F}$) is $\xi(\omega) := \int \mathbf{1}\{t \in \cdot\} f(\theta_t\omega)\, \lambda(dt)$, where $f\colon \mathbf{F} \to [0, \infty)$ is measurable and bounded.

In view of the preceding examples it is helpful to think of $\theta_s\omega$ as of ω *shifted* by s. A measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ is called *stationary* if it is invariant under the flow, i.e.

$$\mathbb{P} \circ \theta_s = \mathbb{P}, \quad s \in \mathbb{G},$$

where θ_s is interpreted as a mapping from $\mathcal{A}$ to $\mathcal{A}$ in the usual way:

$$\theta_s A := \{\theta_s\omega : \omega \in A\}, \quad A \in \mathcal{A},\ s \in \mathbb{G}.$$

Throughout the paper $\mathbb{P}$ will denote a σ-finite stationary measure on $(\Omega, \mathcal{A})$.

Let $B \in \mathcal{G}$ be a set with positive and finite Haar measure $\lambda(B)$ and ξ be an invariant random measure on $\mathbb{G}$. The measure

$$\mathbb{P}_\xi(A) := \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s\omega)\mathbf{1}_B(s)\,\xi(\omega, ds)\,\mathbb{P}(d\omega), \quad A \in \mathcal{A}, \tag{2.3}$$

is called the *Palm measure* of ξ (with respect to $\mathbb{P}$).

For discrete groups the previous definition becomes very simple:

Example 2.3. Assume that $\mathbb{G}$ is discrete. Then we can take $B := \{0\}$ and obtain that

$$\mathbb{P}_\xi(A) = \mathbb{E}\mathbf{1}_A\xi(\{0\}).$$

The *intensity* of ξ is the number $\gamma_\xi := \mathbb{E}[\xi(B)] = \mathbb{P}_\xi(\Omega)$. If this intensity is positive and finite then the normalized Palm measure

$$\mathbb{P}^0_\xi := \gamma_\xi^{-1}\mathbb{P}_\xi$$

is called *Palm probability measure* of ξ (w.r.t. $\mathbb{P}$). Note that $\mathbb{P}_\xi$ and $\mathbb{P}^0_\xi$ are defined on the underlying space $(\Omega, \mathcal{A})$. The *Palm distribution* of ξ is the distribution $\mathbb{P}^0_\xi(\xi \in \cdot)$ of ξ under $\mathbb{P}^0_\xi$. If ξ is a *simple* point process (that is $\xi(\{s\}) \le 1$ for all $s \in \mathbb{G}$), the number $\mathbb{P}^0_\xi(A)$ can be interpreted as the conditional probability of $A \in \mathcal{A}$ given that ξ has a point at $0 \in \mathbb{G}$.

In the general case the Palm measure $\mathbb{P}_\xi$ is σ-finite. Moreover, if $\gamma_\xi > 0$ and $A \in \mathcal{A}$ is flow-invariant (that is $\theta_s A = A$ for each $s \in \mathbb{G}$), then $\mathbb{P}(A) = 0$ iff $\mathbb{P}_\xi(A) = 0$. Since the definition (2.3) does not depend on B, we have the *refined Campbell theorem*

$$\iint f(\theta_s\omega, s)\,\xi(\omega, ds)\,\mathbb{P}(d\omega) = \iint f(\omega, s)\,\lambda(ds)\,\mathbb{P}_\xi(d\omega) \tag{2.4}$$

for all measurable $f\colon \Omega \times \mathbb{G} \to [0, \infty]$. We write this as

$$\mathbb{E}\left[\int f(\theta_s, s)\,\xi(ds)\right] = \mathbb{E}_{\mathbb{P}_\xi}\left[\int f(\theta_0, s)\,\lambda(ds)\right], \tag{2.5}$$

where $\mathbb{E}$ and $\mathbb{E}_{\mathbb{P}_\xi}$ denote integration with respect to $\mathbb{P}$ and $\mathbb{P}_\xi$, respectively. Note that

$$\mathbb{P}_\xi(\xi(\mathbb{G}) = 0) = 0. \tag{2.6}$$

If ξ is a point process, then $\mathbb{P}_\xi$ is concentrated on the event $\{\omega \in \Omega : \xi(\omega, \{0\}) \ge 1\}$.

Let ξ be an invariant random measure on $\mathbb{G}$ and let $\tilde{h}\colon \Omega \times \mathbb{G} \to [0, \infty]$ be a measurable function such $\int \tilde{h}(\theta_0, s)\,\xi(ds) = \mathbf{1}\{\xi(\mathbb{G}) > 0\}$ $\mathbb{P}$-a.e. Then the refined Campbell theorem implies the *inversion formula*

$$\mathbb{E}\mathbf{1}\{\xi(\mathbb{G}) > 0\}f = \mathbb{E}_{\mathbb{P}_\xi}\int f(\theta_{-s})\tilde{h}(\theta_{-s}, s)\,\lambda(ds), \tag{2.7}$$

for each measurable $f\colon \Omega \to [0, \infty]$; see also [15]. This shows that the restriction of the measure $\mathbb{P}$ to $\{\xi(\mathbb{G}) > 0\}$ is uniquely determined by $\mathbb{Q}$.

Let ξ and η be two invariant random measures on $\mathbb{G}$ and $g\colon \Omega \times \mathbb{G} \to [0, \infty]$ be measurable. Neveu's [16] *exchange formula* says that

$$\mathbb{E}_{\mathbb{P}_\xi}\left[\int g(\theta_0, s)\,\eta(ds)\right] = \mathbb{E}_{\mathbb{P}_\eta}\left[\int g(\theta_s, -s)\,\xi(ds)\right]. \tag{2.8}$$

3 Allocations, point- and mass-stationarity

As in Section 2 we consider a measurable space $(\Omega, \mathcal{A})$ equipped with a measurable flow $\{\theta_s : s \in \mathbb{G}\}$ and a stationary σ-finite measure $\mathbb{P}$.

A measurable function $\tau\colon \Omega \times \mathbb{G} \to \mathbb{G} \cup \{\infty\}$ (it is understood here that $\infty \notin \mathbb{G}$) is said to be an *allocation*, if it satisfies the covariance property

$$\tau(\theta_t \omega, s - t) = \tau(\omega, s) - t, \quad s, t \in \mathbb{G},\ \omega \in \Omega, \tag{3.1}$$

where $\infty - t := \infty$. Given such an allocation we define the (random) sets

$$C_\tau(s) := \{t \in \mathbb{G} : \tau(t) = s\}, \quad s \in \mathbb{G}, \tag{3.2}$$

where, as usual, $\tau(t) := \tau(\cdot, t)$.

The following result generalizes [12, Theorem 4.1]. The latter arises in the special case where $\mathbb{G} = \mathbb{R}^d$ and ξ equals Lebesgue measure. We denote by $\operatorname{supp}\mu$ the *support* of a measure μ on $\mathbb{G}$.

Proposition 3.1. *Suppose that ξ is an invariant random measure and that η is a simple invariant point process. Let τ be an allocation satisfying*

$$\tau(s) \in \operatorname{supp}\eta \cup \{\infty\}, \quad \xi\text{-a.e. } s \in \mathbb{G},\ \mathbb{P}\text{-a.e.}$$

Let $h\colon \Omega \times \Omega \to [0, \infty]$ be measurable. Then

$$\mathbb{E}_{\mathbb{P}_\xi} \mathbf{1}\{\tau(0) \neq \infty\} h(\theta_0, \theta_{\tau(0)}) = \mathbb{E}_{\mathbb{P}_\eta} \int_{C_\tau(0)} h(\theta_s, \theta_0)\, \xi(ds). \tag{3.3}$$

Proof. It follows from (2.2) and (3.1) that the event consisting of all $\omega \in \Omega$ satisfying

$$\xi(\omega, \{s \in \mathbb{G} : \tau(\omega, s) \notin (\operatorname{supp}\eta(\omega) \cup \{\infty\}\}) = 0$$

is shift-invariant. Therefore,

$$\mathbb{P}_\xi(\tau(0) \notin (\operatorname{supp}\eta \cup \{\infty\})) = 0. \tag{3.4}$$

The proof proceeds now as the one of [12, Theorem 4.1], applying the exchange formula (2.8) instead of the refined Campbell theorem. We apply (2.8) with the function $(\omega, s) \mapsto h(\omega, \theta_s\omega)\mathbf{1}\{\tau(\omega, 0) = s\}$. In view of (3.4), the left-hand coincides with the left-hand side of (3.3). The right-hand side equals

$$\mathbb{E}_{\mathbb{P}_\eta} \int h(\theta_s, \theta_0) \mathbf{1}\{\tau(\theta_s, 0) = -s\}\, \xi(ds).$$

Since $\tau(\theta_s, 0) = \tau(\theta_0, s) - s$, this equals the right-hand side of (3.3). □

Let ξ, η and τ be as in Proposition 3.1 and assume moreover that $\mathbb{P}$-a.e. $\xi(C_\tau(s)) = 1$ for all $s \in \operatorname{supp}\eta$. Then (3.3) implies the *shift-coupling*

$$\mathbb{E}_{\mathbb{P}_\xi} \mathbf{1}\{\tau(0) \neq \infty\} \mathbf{1}\{\theta_{\tau(0)} \in \cdot\} = \mathbb{P}_\eta. \tag{3.5}$$

The additional assumption on τ is equivalent to the *balancing* property

$$\int \mathbf{1}\{\tau(s) \neq \infty, \tau(s) \in \cdot\}\, \xi(ds) = \eta, \quad \mathbb{P}\text{-a.e.} \tag{3.6}$$

Since the above balancing event is easily seen to be flow-invariant, equation (3.6) does also hold $\mathbb{P}_\xi$-a.e. and $\mathbb{P}_\eta$-a.e. Of particular interest is the case $\xi = \eta$. Then (3.6) implies $\mathbb{P}$-a.e. that $\tau(s) \neq \infty$ for all $s \in \operatorname{supp} \xi$ and (3.6) means that $\tau(\omega, \cdot)$ induces for $\mathbb{P}$-a.e. ω a bijection between the points of $\operatorname{supp} \xi$. We say that τ is a *bijective point map* for ξ w.r.t. $\mathbb{P}$ (see [21, 7]) and use this terminology also for other measures $\mathbb{P}$.

Given an invariant simple point process ξ and a measure $\mathbb{Q}$ on Ω, we call ξ *point-stationary* if $\mathbb{Q}(0 \notin \operatorname{supp} \xi) = 0$ and $\mathbb{Q}(\theta_{\tau(0)} \in \cdot) = \mathbb{Q}$ holds for each bijective point map τ for ξ w.r.t. $\mathbb{Q}$. It was proved in [7] that a σ-finite measure $\mathbb{Q}$ on Ω is point-stationary iff it is the Palm measure of ξ with respect to some σ-finite stationary measure on Ω. A key ingredient of the proof is the following intrinsic characterization of general Palm measures; see [15, Satz 2.5]. Mecke proved his fundamental result in a canonical setting. As discussed in [14] his proof applies in our more general framework.

Theorem 3.2 (Mecke 1967)**.** *Let ξ be an invariant random measure on $\mathbb{G}$ and $\mathbb{Q}$ be a σ-finite measure on $(\Omega, \mathcal{A})$. Then $\mathbb{Q}$ is the Palm measure of ξ w.r.t. a σ-finite stationary measure on Ω iff $\mathbb{Q}(\xi(\mathbb{G}) = 0) = 0$ and*

$$\mathbb{E}_{\mathbb{Q}} \int g(\theta_s, -s)\, \xi(ds) = \mathbb{E}_{\mathbb{Q}} \int g(\theta_0, s)\, \xi(ds) \tag{3.7}$$

for all measurable $g \colon \Omega \times \mathbb{G} \to [0, \infty]$. Equation (3.7) *determines the stationary measure on $\{\xi(\mathbb{G}) > 0\}$.*

The final assertion of Theorem 3.2 follows from the inversion formula (2.7). .

Point stationarity was extended in [14] to *mass-stationarity* of an invariant random measure ξ w.r.t. a given σ-finite measure $\mathbb{Q}$ on Ω. Roughly speaking, mass-stationarity of ξ can be described as follows. Let $C \in \mathcal{G}$ be a relatively compact set with positive Haar measure whose boundary is not charged by λ. Let U be a random element of $\mathbb{G}$, independent of ξ and with distribution $\lambda(C)^{-1}\lambda(C \cap \cdot)$. Given (ξ, U) pick a random point V according to the normalized restriction of λ to $C - U$. If $\mathbb{Q}$ is the Palm measure of ξ w.r.t. some stationary measure, then

$$\mathbb{Q}((\xi \circ \theta_V, U + V) \in \cdot) = \mathbb{Q}((\xi, U) \in \cdot).$$

As shown by [14, Theorem 6.3], a version of this property (assumed to be true for all C as above) is equivalent to (3.7) and hence provides another intrinsic chracterization of Palm measures. Justified by this result we call ξ *mass-stationary* (w.r.t. $\mathbb{Q}$) if (3.7) holds. In this paper $\mathbb{Q}$ will always denote a probability measure while, as a rule, $\mathbb{P}$ is only σ-finite.

4 Exceedance random measures

Let $\mathbb{H}$ be a (non-empty) Polish space equipped with the Borel σ-field $\mathcal{H}$. Assume that $|\cdot| \colon \mathbb{H} \to [0, \infty)$ is a measurable mapping. One might think of $\mathbb{H} = \mathbb{R}^d$ equipped with the Euclidean norm.

In this and later sections we consider a measurable mapping $Y\colon \Omega\times\mathbb{G}\to\mathbb{H}$. For $s\in\mathbb{G}$ we write Y_s for the random variable $\omega\mapsto Y_s(\omega)$. Then Y can be considered as a (measurable) random field $(Y_s)_{s\in\mathbb{G}}$. We assume the shift-covariance

$$Y_s(\theta_t\omega)=Y_{s+t}(\omega),\quad (\omega,s,t)\in\Omega\times\mathbb{G}\times\mathbb{G}. \tag{4.1}$$

We call

$$\xi:=\int \mathbf{1}\{s\in\cdot,|Y_s|>1\}\,\lambda(ds) \tag{4.2}$$

the *exceedance measure* of Y. By (4.1) this is an invariant random measure. If $\mathbb{G}$ is discrete, ξ is a simple point process. The Palm measure of ξ takes a rather simple form:

Lemma 4.1. *Let $\mathbb{P}$ be a σ-finite stationary measure on Ω. Then*

$$\mathbb{P}_\xi=\mathbb{P}(\cdot\cap\{|Y_0|>1\}). \tag{4.3}$$

Proof. Let $B\in\mathcal{G}$ satisfy $\lambda(B)=1$. Then

$$\begin{aligned}\mathbb{P}_\xi&=\mathbb{E}_\mathbb{P}\int\mathbf{1}\{\theta_s\in\cdot\}\mathbf{1}\{s\in B\}\,\xi(ds)=\mathbb{E}_\mathbb{P}\int\mathbf{1}\{\theta_s\in\cdot\}\mathbf{1}\{s\in B,|Y_s|>1\}\,\lambda(ds)\\&=\int_B\mathbb{E}_\mathbb{P}[\mathbf{1}\{\theta_s\in\cdot,|Y_s|>1\}]\,\lambda(ds)=\int_B\mathbb{E}_\mathbb{P}[\mathbf{1}\{\theta_0\in\cdot,|Y_0|>1\}]\,\lambda(ds),\end{aligned}$$

where we have used stationarity of $\mathbb{P}$ to get the final equation. This proves the assertion. □

In this paper we will mostly be concerned with a probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{A})$ such that ξ is mass-stationary w.r.t. $\mathbb{Q}$. In this case Theorem 3.2 shows that there exists a unique σ-finite stationary measure $\mathbb{P}$ on Ω satisfying $\mathbb{P}_\xi=\mathbb{Q}$ and $\mathbb{P}(\xi(\mathbb{G})=0)=0$. If $\mathbb{Q}(\xi(\mathbb{G})<\infty)=1$ then $\mathbb{P}$ cannot be finite:

Remark 4.2. Let $\mathbb{Q}$ be as above and assume that $\mathbb{Q}(0<\xi(\mathbb{G})<\infty)=1$. Define a measure $\mathbb{P}$ on Ω by

$$\mathbb{P}:=\mathbb{E}_\mathbb{Q}\xi(\mathbb{G})^{-1}\int\mathbf{1}\{\theta_s\in\cdot\}\,\lambda(ds). \tag{4.4}$$

Since $\xi(\mathbb{G})$ is invariant under shifts, this measure is stationary. Let $f\colon\mathbf{M}\to[0,\infty]$ be measurable and $B\in\mathcal{G}$ with $\lambda(B)=1$. A simple calculation (using invariance of ξ and Fubini's theorem) shows that

$$\mathbb{E}_\mathbb{P}\int\mathbf{1}\{s\in B\}f\circ\theta_s\,\xi(ds)=\mathbb{E}_\mathbb{Q}\xi(\mathbb{G})^{-1}\int f\circ\theta_s\,\xi(ds).$$

Since we have assumed ξ to be mass-stationary, we can use the Mecke equation (3.7) to find that the latter expression equals $\mathbb{E}_\mathbb{Q}f$. Hence $\mathbb{Q}$ is the Palm measure of ξ. Note that $\mathbb{P}(\Omega)=\infty$, unless $\lambda(\mathbb{G})<\infty$.

5 Spectrally decomposable fields

In this and later sections we take $(\Omega, \mathcal{A})$ as the function space $(\mathbf{F}, \mathcal{F})$ satisfying the assumptions of Example 2.2. In addition we assume that $\mathbb{H}$ is a *measurable cone*, that is, there exists a measurable mapping $(u, x) \mapsto u \cdot x$ from $(0, \infty) \times \mathbb{H}$ to $\mathbb{H}$ such that $1 \cdot x = x$ and $u \cdot (v \cdot x) = (uv) \cdot x$ for all $x \in \mathbb{H}$ and $u, v \in (0, \infty)$. We mostly write ux instead of $u \cdot x$. The function $|\cdot|$ is assumed to be homogeneous, that is $|ux| = u|x|$ for all $u > 0$ and $x \in \mathbb{H}$. If $\omega \in \mathbb{H}^{\mathbb{G}}$ and $u \in (0, \infty)$, then, as usual, $u \cdot \omega \equiv u\omega$ is the function in $\mathbb{H}^{\mathbb{G}}$ given by $u\omega(s) := u \cdot \omega(s)$, $s \in \mathbb{G}$. We assume that $\mathbf{F}$ is closed under the action of $(0, \infty)$. The σ-field $\mathcal{F}$ has been assumed to render the mapping $(\omega, s) \mapsto (\theta_s\omega, \omega(0))$ to be measurable and we assume now in addition that the mapping $(\omega, u) \mapsto u \cdot \omega$ is measurable on $\mathbf{F} \times (0, \infty)$. If $\mathbb{G}$ is discrete, then we take $\mathbf{F} = \mathbb{H}^{\mathbb{G}}$ and equip it with the product σ-algebra. We write Y_s for the mapping $\omega \mapsto \omega(s)$, $s \in \mathbb{G}$, and note that $(\omega, s) \mapsto Y_s(\omega)$ is measurable. We also write $Y := (Y_s)_{s \in \mathbb{G}}$, which is simply the identity on $\mathbf{F}$. We define another random field W, by $W_s := |Y_0|^{-1} Y_s$ if $|Y_0| > 0$ and by $W_s := x_0$ otherwise, where x_0 is some fixed element of $\mathbb{H}$ with $|x_0| = 1$. Since we do not assume $\mathbb{H}$ to contain a zero element we make the general convention $|y|^{-1}x := x_0$ whenever $x, y \in \mathbb{H}$ and $|y| = 0$.

Remark 5.1. Assume that $\mathbf{F}' \subset \mathbb{H}^{\mathbb{G}}$ is shift-invariant and closed under the action of $(0, \infty)$. Assume that $\mathbf{F}'$ is equipped with the Kolmogorov product σ-field, that is, the smallest σ-field making the mappings $\omega \mapsto \omega(s)$ (from $\mathbf{F}'$ to $\mathbb{H}$) measurable for each $s \in \mathbb{G}$. Assume, moreover, that $(\omega, s) \mapsto \omega(s)$ is a (jointly) measurable mapping on $\mathbf{F}' \times \mathbb{G}$. Then it is easy to see that $(\omega, s, u) \mapsto (\theta_s\omega, u \cdot \omega)$ is a measurable function on $\mathbf{F}' \times \mathbb{G} \times (0, \infty)$. This shows that for a proper choice of $\mathbf{F}$, the product σ-field is a natural candidate for $\mathcal{F}$.

We often consider a probability measure $\mathbb{Q}$ on $\mathbf{F}$ with the following properties. The probability measure $\mathbb{Q}(|Y_0| \in \cdot)$ is a Pareto distribution on $(1, \infty)$ with parameter $\alpha > 0$ and W is independent of $|Y_0|$. To achieve this, we take a probability measure $\mathbb{Q}'$ on $\mathbf{F}$ such that $\mathbb{Q}'(|Y_0| = 1) = 1$ and define

$$\mathbb{Q} := \iint \mathbf{1}\{u\omega \in \cdot, u > 1\}\, \alpha u^{-\alpha-1}\, \mathbb{Q}'(d\omega)\, du. \tag{5.1}$$

For the special groups $\mathbb{G} = \mathbb{Z}^d$ and $\mathbb{G} = \mathbb{R}$ such processes occur in extreme value theory; see the seminal paper [1] (treating $\mathbb{G} = \mathbb{Z}$) and [10, 3, 20]. Note that W is a measurable function of Y, that $\mathbb{Q}(W \in \cdot) = \mathbb{Q}'$ and that the pair (W, Y_0) has the desired properties. We say that Y is *spectrally decomposable with index* α (w.r.t. $\mathbb{Q}$) or, synonomously, that $\mathbb{Q}$ is spectrally decomposable.

Define the exceedance random measure ξ by (4.2). If $\mathbb{Q}$ is given as in (5.1), it is natural to characterize mass-stationarity of ξ (w.r.t. $\mathbb{Q}$) in terms of suitable invariance properties of the field W. In the context of tail processes the following property (5.2) was proved in [1] in the case $\mathbb{G} = \mathbb{Z}$ (see also [3, 18]) and in the case $\mathbb{G} = \mathbb{R}$ in [20]. The fact that (5.2) implies mass-stationarity in the case $\mathbb{G} = \mathbb{Z}^d$ was derived in [17], exploiting the connection to regularly varying random fields. We use here an intrinsic non-asymptotic approach. It is worth noticing that [9] identifies (5.2) (in the case $\mathbb{G} = \mathbb{Z}$) as being characteristic for the tail processes introduced in [1].

Theorem 5.2. *Assume that Y is spectrally decomposable with index α. Then the exceedance random measure ξ is mass-stationary if and only if*

$$\mathbb{E}_{\mathbb{Q}} \int g(\theta_s W, -s) \mathbf{1}\{|W_s| > 0\} \, \lambda(ds) = \mathbb{E}_{\mathbb{Q}} \int g(|W_s|^{-1} W, s) |W_s|^{\alpha} \, \lambda(ds), \tag{5.2}$$

holds for all measurable $g \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$.

Proof. Let us first assume that ξ is mass-stationary. We generalize the arguments from the proof of Lemma 2.2 in [18]. Let $h \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ be measurable and $\varepsilon \in (0, 1]$. Then

$$\begin{aligned} I &:= \mathbb{E}_{\mathbb{Q}} \int h(\theta_s Y, -s) \mathbf{1}\{|Y_s| > \varepsilon\} \, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}} \iint h(u \theta_s W, -s) \mathbf{1}\{u |W_s| > \varepsilon\} \mathbf{1}\{u > 1\} \alpha u^{-\alpha - 1} \, du \, \lambda(ds) \\ &= \varepsilon^{-\alpha} \, \mathbb{E}_{\mathbb{Q}} \iint h(\varepsilon v \theta_s W, -s) \mathbf{1}\{v |W_s| > 1\} \mathbf{1}\{v > 1/\varepsilon\} \alpha v^{-\alpha - 1} \, dv \, \lambda(ds), \end{aligned}$$

where we have made the change of variables $v := u/\varepsilon$ to get the second identity. Since $1/\varepsilon > 1$ we obtain that

$$\begin{aligned} I &= \varepsilon^{-\alpha} \, \mathbb{E}_{\mathbb{Q}} \int h(\varepsilon \theta_s Y, -s) \mathbf{1}\{|Y_s| > 1\} \mathbf{1}\{|Y_0| > 1/\varepsilon\} \, \lambda(ds) \\ &= \varepsilon^{-\alpha} \, \mathbb{E}_{\mathbb{Q}} \int h(\varepsilon \theta_s Y, -s) \mathbf{1}\{|Y_0| > 1/\varepsilon\} \, \xi(ds). \end{aligned}$$

Using now the assumption (3.7) together with $Y_0 = (Y \circ \theta_s)_{-s}$ we arrive at

$$I = \varepsilon^{-\alpha} \, \mathbb{E}_{\mathbb{Q}} \int h(\varepsilon Y, s) \mathbf{1}\{|Y_s| > 1/\varepsilon\} \, \xi(ds),$$

that is

$$\mathbb{E}_{\mathbb{Q}} \int h(\theta_s Y, -s) \mathbf{1}\{|Y_s| > \varepsilon\} \, \lambda(ds) = \varepsilon^{-\alpha} \, \mathbb{E}_{\mathbb{Q}} \int h(\varepsilon Y, s) \mathbf{1}\{|Y_s| > 1/\varepsilon\} \, \lambda(ds). \tag{5.3}$$

We apply this with $h(Y, s) := g(|Y_s|^{-1} Y, s)$ for some measurable function $g \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$, noting that

$$h(\theta_s Y, -s) = g(|(\theta_s Y)_{-s}|^{-1} \theta_s Y, -s) = g(|Y_0|^{-1} \theta_s Y, -s) = g(\theta_s W, -s).$$

Using monotone convergence this yields

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} \int g(\theta_s W, -s) \mathbf{1}\{|W_s| > 0\} \, \lambda(ds) \\ &\qquad = \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} \, \mathbb{E}_{\mathbb{Q}} \int g(|W_s|^{-1} W, s) \mathbf{1}\{|Y_0| |W_s| > 1/\varepsilon\} \, \lambda(ds). \end{aligned}$$

We have that

$$
\begin{aligned}
&\varepsilon^{-\alpha}\,\mathbb{E}_{\mathbb{Q}}\int g(|W_s|^{-1}W,s)\mathbf{1}\{|Y_0||W_s|>1/\varepsilon\}\,\lambda(ds)\\
&\qquad=\varepsilon^{-\alpha}\,\mathbb{E}_{\mathbb{Q}}\int g(|W_s|^{-1}W,s)\mathbf{1}\{u|W_s|>1/\varepsilon\}\mathbf{1}\{u>1\}\alpha u^{-\alpha-1}\,du\,\lambda(ds)\\
&\qquad=\varepsilon^{-\alpha}\,\mathbb{E}_{\mathbb{Q}}\int g(|W_s|^{-1}W,s)\min\{|W_s|^{\alpha}\varepsilon^{\alpha},1\}\,\lambda(ds)\\
&\qquad=\mathbb{E}_{\mathbb{Q}}\int g(|W_s|^{-1}W,s)\min\{|W_s|^{\alpha},\varepsilon^{-\alpha}\}\,\lambda(ds).
\end{aligned}
$$

As $\varepsilon\to 0$ the latter term tends to $\mathbb{E}_{\mathbb{Q}}\int g(|W_s|^{-1}W,s)|W_s|^{\alpha}\,\lambda(ds)$, yielding (5.2).

To prove the converse implication we assume that (5.2) holds. We take a measurable $g\colon \mathbf{F}\times\mathbb{G}\to[0,\infty]$ and aim at establishing (3.7). We have that

$$
\begin{aligned}
I' &:= \mathbb{E}_{\mathbb{Q}}\int g(\theta_sY,-s)\,\xi(ds)=\mathbb{E}_{\mathbb{Q}}\int g(\theta_sY,-s)\mathbf{1}\{|Y_s|>1\}\,\lambda(ds)\\
&\qquad=\int\left[\mathbb{E}_{\mathbb{Q}}\int g(u\theta_sW,-s)\mathbf{1}\{u|W_s|>1\}\,\lambda(ds)\right]\mathbf{1}\{u>1\}\alpha u^{-\alpha-1}\,du.
\end{aligned}
$$

For each $u>1$ we can apply (5.2) with the function $\tilde h(\omega,s)=g(u\omega,s)\mathbf{1}\{u|\omega(0)|>1\}$. Then $\tilde h(\theta_sW,-s)=g(u\theta_sW,-s)\mathbf{1}\{u|W_s|>1\}$ and

$$
\tilde h(|W_s|^{-1}W,s)=g(u|W_s|^{-1}W,s)\mathbf{1}\{u|W_s|^{-1}>1\}
$$

Therefore

$$
I'=\mathbb{E}_{\mathbb{Q}}\iint g(u|W_s|^{-1}W,s)|W_s|^{\alpha}\mathbf{1}\{u|W_s|^{-1}>1,u>1\}\,\alpha u^{-\alpha-1}\,du\,\lambda(ds).
$$

In the above inner integral we can assume that $|W_s|>0$. After the change of variables $v:=|W_s|^{-1}u$ we obtain that

$$
\begin{aligned}
I' &= \mathbb{E}_{\mathbb{Q}}\iint g(vW,s)\mathbf{1}\{v>1,v|W_s|>1\}\,\alpha v^{-\alpha-1}\,dv\,\lambda(ds)\\
&= \mathbb{E}_{\mathbb{Q}}\iint g(Y,s)\mathbf{1}\{|Y_s|>1\}\,\lambda(ds),
\end{aligned}
$$

establishing (3.7). □

Remark 5.3. The equations (5.2) are clearly equivalent to

$$
\mathbb{E}_{\mathbb{Q}}h(\theta_{-s}W)\mathbf{1}\{|W_{-s}|>0\}=\mathbb{E}_{\mathbb{Q}}h(|W_s|^{-1}W)|W_s|^{\alpha},\quad \lambda\text{-a.e. } s\in\mathbb{G}, \tag{5.4}
$$

for all measurable $h\colon\mathbf{F}\to[0,\infty]$. They are also equivalent to the equations

$$
\mathbb{E}_{\mathbb{Q}}\int g(W,-s)\mathbf{1}\{|W_s|>0\}\,\lambda(ds)=\mathbb{E}_{\mathbb{Q}}\int g(|W_s|^{-1}\theta_sW,s)|W_s|^{\alpha}\,\lambda(ds). \tag{5.5}
$$

as well as to the equations

$$\mathbb{E}_{\mathbb{Q}} \int g(W,-s)|W_s|^{\alpha}\,\lambda(ds) = \mathbb{E}_{\mathbb{Q}} \int g(|W_s|^{-1}\theta_s W, s)\mathbf{1}\{|W_s|>0\}\,\lambda(ds). \tag{5.6}$$

To see the latter equivalence, we can use the function $\tilde{h}\colon \mathbf{F}\times\mathbb{G}\to[0,\infty)$ given by $\tilde{h}(\omega,s) := |\omega(-s)|$. If $|W_s|>0$ we have that $\tilde{h}(|W_s|^{-1}\theta_s W, s) = |W_s|^{-1}$. Applying (5.5) with $g\cdot\tilde{h}^{\alpha}$ instead of g yields (5.6).

Remark 5.4. Assume that $\mathbb{G}$ is discrete and that Y is spectrally decomposable with index α. Then ξ is mass-stationary iff

$$\mathbb{E}_{\mathbb{Q}} g(\theta_{-s}W)\mathbf{1}\{|W_{-s}|>0\} = \mathbb{E}_{\mathbb{Q}} g(|W_s|^{-1}W)|W_s|^{\alpha} \tag{5.7}$$

holds for all measurable $g\colon \mathbf{F}\to[0,\infty]$ and all $s\in\mathbb{G}$.

In the case $\mathbb{G}=\mathbb{Z}$ equation (5.2) (see also equation (5.7)) was called *time change formula*. In our general setting (and in particular for $\mathbb{G}=\mathbb{Z}^d$ or $\mathbb{G}=\mathbb{R}^d$) this terminology might be replaced by *space shift formula*. We can rewrite (5.2) as

$$\mathbb{E}_{\mathbb{Q}} \int g(\theta_s W,-s)\,\xi'(ds) = \mathbb{E}_{\mathbb{Q}} \int g(|W_s|^{-1}W, s)|W_s|^{\alpha}\,\xi'(ds), \tag{5.8}$$

where ξ' is the invariant random measure defined by

$$\xi' := \int \mathbf{1}\{s\in\cdot, |W_s|>0\}\,\lambda(ds). \tag{5.9}$$

This makes the intimate relationship between (3.7) and (5.2) even more transparent.

Remark 5.5. If $\mathbb{P}(\xi'(\mathbb{G})=0)=1$ then the equations (5.2) are empty. In the spectrally positive case it is, however, quite natural to assume that $\mathbb{P}(\xi'(\mathbb{G})=0)=0$. Indeed, if $\mathbb{G}$ is discrete or if Y has suitable continuity properties. then this follows from $\mathbb{P}(|Y_0|>0)=0$.

In the spectrally decomposable case the space-shift formula has the following equivalent version; see [18, 20].

Lemma 5.6. *Assume that Y is spectrally decomposable with index α. Then the equations* (5.2) *hold iff the following equations holds for all measurable $g\colon \mathbf{F}\times\mathbb{G}\to[0,\infty]$:*

$$\mathbb{E}_{\mathbb{Q}} \int g(Y,s)\mathbf{1}\{|Y_s|>r\}\,\lambda(ds) = r^{-\alpha}\,\mathbb{E}_{\mathbb{Q}} \int g(r\theta_{-s}Y,s)\mathbf{1}\{r|Y_{-s}|>1\}\,\lambda(ds), \quad r>0, \tag{5.10}$$

Proof. Assume that the equations (5.10) hold. Clearly they are equivalent with (5.3). We have already seen in the proof of Theorem 5.2 that (5.3) implies (5.2).

Assume, conversely, that (5.2) holds. We can assume that $\mathbb{P}(\xi'(\mathbb{G})>0)>0$. (Otherwise there is nothing to prove.) Define $\tilde{\mathbb{Q}} := \mathbb{Q}(\cdot\mid\xi'(\mathbb{G})>0)$ and $\mathbb{Q}'' := \mathbb{Q}'(\cdot\mid\xi'(\mathbb{G})>0)$. Then (5.1) holds with $(\mathbb{Q},\mathbb{Q}')$ replaced by $(\tilde{\mathbb{Q}},\mathbb{Q}'')$. Hence $\tilde{\mathbb{Q}}$ is spectrally decomposable. The measure $\mathbb{Q}$ satisfies (5.10) (resp. (5.2)) iff this is the case for $\tilde{\mathbb{Q}}$. Hence it is no loss of generality to assume that $\mathbb{Q}(\xi'(\mathbb{G})>0)=1$. Corollary 6.12 will show that there is a σ-finite stationary measure ν on $\mathbf{F}$ such that $\mathbb{Q}=\nu_{\xi}$. Equation (5.10) then follows easily from the homogeneity of ν, to be discussed in the next section; see Remark 6.14. □

In the following we denote σ-finite (stationary) measures on $\mathbf{F}$ with greek letters. This is at odds with Section 2 (and parts of point process literature), but in accordance with extreme value theory.

Remark 5.7. Assume that Y is spectrally decomposable and that the exceedance random measure ξ is mass-stationary and satisfies $\mathbb{Q}(\xi(\mathbb{G}) = 0) = 0$. By Theorem 3.2, there exists a unique σ-finite stationary measure ν such that $\nu(\xi(\mathbb{G}) = 0) = 0$ and $\nu_\xi = \mathbb{Q}$. Let $H\colon \mathbf{F} \times \mathbb{G} \to [0, \infty)$ be measurable such that

$$\int H(Y, s)\mathbf{1}\{|Y_s| > 1\}\,\lambda(ds) = 1, \quad \nu\text{-a.e.} \tag{5.11}$$

By the inversion formula (2.7) we have that

$$\nu = \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_{-s}Y \in \cdot\}H(\theta_{-s}Y, s)\,\lambda(ds).$$

Inserting here the spectral decomposition (5.1), yields

$$\nu = \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}\{u\theta_{-s}W \in \cdot\}H(u\theta_{-s}W, s)\mathbf{1}\{u > 1\}\,\alpha u^{-\alpha-1}\,du\,\lambda(ds). \tag{5.12}$$

Example 5.8. Consider the setting of Remark 5.7 and assume moreover that $\mathbb{G}$ is discrete. Let τ be an allocation such that

$$\sum_{s\in\mathbb{G}} \mathbf{1}\{\tau(Y, 0) = s, |Y_s| > 1\} = 1, \quad \nu\text{-a.e.} \tag{5.13}$$

Then we can apply (5.12) with $H(Y, s) := \mathbf{1}\{\tau(Y, 0) = s\}$. Since $\tau(\theta_{-s}Y, 0) = \tau(Y, -s) + s$ we can change variables $s := -s$ to obtain that the measure (5.12) is given by

$$\nu = \mathbb{E}_{\mathbb{Q}} \sum_{s\in\mathbb{G}} \int \mathbf{1}\{u\theta_s W \in \cdot, \tau(uW, s) = 0\}\mathbf{1}\{u > 1\}\,\alpha u^{-\alpha-1}\,du. \tag{5.14}$$

Remark 5.9. The preceding results can be generalized as follows. Let $(\Omega, \mathcal{A})$ be a measurable space and suppose that that there is measurable action $(u, \omega) \mapsto u\omega$ from $(0, \infty) \times \Omega$ to Ω. Let Y be a random element of $\mathbf{F}$ satisfying (4.1) and also

$$Y_s(u\omega) = uY_s(\omega), \quad (\omega, s, u) \in \Omega \times \mathbb{G} \times (0, \infty). \tag{5.15}$$

Let $\mathbb{Q}$ be a probability measure on Ω given by (5.1), where $\mathbb{Q}'$ is a probability measure on Ω such that $\mathbb{Q}'(|Y_0| = 1) = 1$. Then ξ is mass-stationary w.r.t. $\mathbb{Q}$ iff

$$\mathbb{E}_{\mathbb{Q}} \int g(\vartheta_s, -s)\mathbf{1}\{|W_s| > 0\}\,\lambda(ds) = \mathbb{E}_{\mathbb{Q}} \int g(|W_s|^{-1}\vartheta_0, s)|W_s|^\alpha\,\lambda(ds), \tag{5.16}$$

for each measurable $g\colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$, where $\vartheta_s\omega := |Y_0(\omega)|^{-1}\theta_s\omega$, $\omega \in \Omega$, whenever $|Y_0(\omega)| > 0$. Such a generalization is certainly useful when considering more randomness. For instance we may consider a second Polish space $\mathbb{H}'$ and a suitable subset of $(\mathbb{H}\times\mathbb{H}')^{\mathbb{G}}$. The shifts are defined as before, while multiplication acts only on the first component $Y(\omega)$ of an element $\omega \in (\mathbb{H} \times \mathbb{H}')^{\mathbb{G}}$. If $\mathbb{Q}(\xi(\mathbb{G}) = 0) = 0$, Palm theory would still guarantee the existence of stationary measure $\mathbb{P}$ (uniquely determined on $\{\xi(\mathbb{G}) > 0\}$) such that $\mathbb{P}_\xi = \mathbb{P}(\cdot \cap \{|Y_0| > 1\}) = \mathbb{Q}$.

6 Tail measures

In this section we let $(\mathbf{F}, \mathcal{F})$ be as in Section 5. Throughout we work with the exceedance random measure ξ (defined by (4.2)) and the random measure ξ', defined by (5.9). We say that a measure ν on $\mathbf{F}$ is a *tail measure* if

$$\int \mathbf{1}\{|Y_s| > 0\}\, \lambda(ds) > 0, \quad \nu\text{-a.e.}, \tag{6.1}$$

$$\mathbb{E}_\nu \xi(B) < \infty, \quad B \in \mathcal{G} \text{ compact}, \tag{6.2}$$

and if there exists an $\alpha > 0$ such that ν is *α-homogeneous*, that is

$$\nu(uY \in \cdot) = u^\alpha \nu(Y \in \cdot), \quad u > 0. \tag{6.3}$$

In accordance with the literature we call α the *index* of ν.

This definition extends the one in [3]. A rather general (but slightly different) definition of a tail measure has very recently been given in [2]. In this paper we are mostly interested in stationary tail measures. In this case (6.2) implies that $B \mapsto \mathbb{E}_\nu \xi(B)$ (the intensity measure of ξ) is a finite multiple of the Haar measure λ. In accordance with the literature we shall always assume then, that this multiple equals 1, that is

$$\mathbb{E}_\nu \int \mathbf{1}\{s \in B, |Y_s| > 1\}\, \lambda(ds) = \lambda(B), \quad B \in \mathcal{G}, \tag{6.4}$$

or, equivalently,

$$\nu(|Y_0| > 1) = 1. \tag{6.5}$$

Up to Remark 6.1 our definition of a stationary tail measure generalizes the one given [20] in the case $\mathbb{G} = \mathbb{R}$.

If $\mathbb{Q}$ is a probability measure on $\mathbf{F}$ such that $\mathbb{Q}(\xi(\mathbb{G}) = 0) = 0$ and ξ is mass-stationary w.r.t. $\mathbb{Q}$, then Theorem 3.2 shows that there exists a stationary measure ν on $\mathbf{F}$ (uniquely determined on $\{\xi(\mathbb{G}) > 0\}$) such that $\mathbb{Q} = \nu_\xi$ is the Palm measure of ξ w.r.t. ν. The main purpose of this section is to show that, if $\mathbb{Q}$ is spectrally decomposable, then ν is a tail measure.

Remark 6.1. Condition (6.1) means that $\nu(\xi'(\mathbb{G}) = 0) = 0$ and should be compared with the condition $\nu(Y \equiv 0) = 0$, made in [20]. Our assumption is (slightly) stronger, also in the stationary case. Without any topological structure of $\mathbf{F}$ such a stronger assumption appears to be appropriate. (The set $\{Y \equiv 0\}$ does not even need to be measurable.)

If ν is a σ-finite measure on $\mathbf{F}$ and η a random measure on $\mathbb{G}$ we define the *Campbell measure*

$$C_{\nu,\eta} := \mathbb{E}_\nu \int \mathbf{1}\{(Y, s) \in \cdot\}\, \eta(ds),$$

which is a measure on $\mathbf{F} \times \mathbb{G}$. It is well-known (and easy to prove) that $C_{\nu,\xi'}$ determines $\mathbb{P}$ on the event $\{\xi'(\mathbb{G}) > 0\}$. For tail measures this can be refined as follows.

Lemma 6.2. *Let ν be a tail measure on $\mathbf{F}$. Then ν is σ-finite and uniquely determined by $C_{\nu,\xi}$.*

Proof. It follows from (6.1) and (6.2) that

$$\mathbb{E}_\nu \int \mathbf{1}\{s \in B, |Y_s| > c\}\, \lambda(ds) < \infty$$

for each $c > 0$ and whenever $B \subset \mathbb{G}$ is compact. Take a sequence B_k, $k \in \mathbb{N}$, of compact sets increasing towards $\mathbb{G}$. Then ν is finite on the sets

$$U_k := \Big\{\omega \in \mathbf{F} : \int \mathbf{1}\{s \in B_k, |\omega(s)| \geq 1/k\}\, \lambda(ds) \geq 1/k\Big\}, \quad k \in \mathbb{N}, \tag{6.6}$$

which increase towards $\big\{\omega \in \mathbf{F} : \int \mathbf{1}\{|\omega(s)| > 0\}\, \lambda(ds) > 0\big\}$. In view of (6.1) we obtain that ν is σ-finite.

By homogeneity the Campbell measure $C_{\nu,\xi}$ determines the Campbell measures

$$\mathbb{E}_\nu \int \mathbf{1}\{(Y, s) \in \cdot\}\mathbf{1}\{|Y_s| > c\}\, \lambda(ds)$$

for each $c > 0$ and hence also $C_{\nu,\xi'}$. Take a measurable $\tilde{h}\colon \mathbf{F} \times \mathbb{G} \to [0, \infty)$ such that $\int \tilde{h}(\omega, s)\xi'(ds) = 1$, whenever $\xi'(\omega, \mathbb{G}) > 0$; see Remark 6.11. Then we obtain for each measurable $g\colon \mathbf{F} \to [0, \infty]$ that

$$\mathbb{E}_\nu \mathbf{1}\{\xi'(\mathbb{G}) > 0\}g(Y) = \mathbb{E}_\nu \int g(Y)\tilde{h}(Y, s)\, \xi'(ds) = \int g(\omega)\tilde{h}(\omega, s)\, C_{\nu,\xi'}(d(\omega, s)).$$

Since $\nu(\xi'(\mathbb{G}) = 0) = 0$, this proves the second assertion. □

Given a σ-finite stationary measure ν on $\mathbf{F}$, we recall that $\nu_\xi = \nu(\cdot \cap \{|Y_0| > 1\})$ is the Palm measure of ξ w.r.t. ν. If $\nu(\xi'(\mathbb{G}) = 0) = 0$ (e.g. if ν is a tail measure), then the definition (2.3) and the shift-invariance of the event $\{\xi' = 0\}$ show that

$$\nu_\xi(\xi' = 0) = 0. \tag{6.7}$$

Corollary 6.3. *A stationary tail measure ν is uniquely determined by ν_ξ.*

Proof. Let ν' be another stationary tail measure with $\nu_\xi = \nu'_\xi$. By the refined Campbell theorem (2.4) we obtain that $C_{\nu,\xi} = C_{\nu',\xi}$. Lemma 6.2 shows that $\nu = \nu'$, as asserted. □

Next we connect tail measures with spectrally decomposable fields. The first part of the following proposition is a classical result.

Proposition 6.4. *Let ν be a stationary tail measure with index $\alpha > 0$. Then there exists a probability measure $\mathbb{Q}'$ on $\mathbf{F}$ such that $\mathbb{Q}'(|Y_0| = 1) = 1$ and*

$$\nu(\cdot \cap \{|Y_0| > 0\}) = \iint \mathbf{1}\{u\omega \in \cdot\}\mathbf{1}\{u > 0\}\, \alpha u^{-\alpha-1}\, \mathbb{Q}'(d\omega)\, du. \tag{6.8}$$

Moreover, ξ is mass-stationary with respect to the probability measure

$$\mathbb{Q} := \iint \mathbf{1}\{u\omega \in \cdot\}\mathbf{1}\{u > 1\}\, \alpha u^{-\alpha-1}\, \mathbb{Q}'(d\omega)\, du. \tag{6.9}$$

Further we have $\nu_\xi = \mathbb{Q}$.

Proof. The first part follows by a classical argument; see also [5] for a general version. For the convenience of the reader we give the short proof. Define

$$\mathbb{Q}' := \nu(\{\omega \in \mathbf{F} : |\omega(0)| > 1, |\omega(0)|^{-1}\omega \in \cdot\}).$$

By (6.5) (and stationarity), this is a probability measure and we have that $\mathbb{Q}'(|Y_0| = 1) = 1$ by definition. Take $u > 0$ and $A \in \mathcal{F}$. By (6.3),

$$\nu(\{\omega \in \mathbf{F} : |\omega(0)| > u, |\omega(0)|^{-1}\omega \in A\} = u^{-\alpha}\,\mathbb{Q}'(A).$$

This implies (6.8).

To prove the second assertion we proceed similarly as in the first part of the proof of Theorem 5.2. Let us first note, that

$$\mathbb{Q} = \nu(\cdot \cap \{|Y_0| > 1\}). \tag{6.10}$$

Let $h\colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ be measurable and $\varepsilon \in (0, 1]$. Then

$$\begin{aligned}
I &:= \mathbb{E}_{\mathbb{Q}} \int h(\theta_s Y, -s)\mathbf{1}\{|Y_s| > \varepsilon\}\,\lambda(ds) \\
&= \mathbb{E}_{\mathbb{Q}'} \iint h(u\theta_s W, -s)\mathbf{1}\{u|W_s| > \varepsilon, u > 1\}\alpha u^{-\alpha-1}\,du\,\lambda(ds) \\
&= \varepsilon^{-\alpha}\,\mathbb{E}_{\mathbb{Q}'} \iint h(\varepsilon v\theta_s W, -s)\mathbf{1}\{v|W_s| > 1, v > \varepsilon^{-1}\}\alpha v^{-\alpha-1}\,dv\,\lambda(ds).
\end{aligned}$$

By (6.8) and $\mathbb{Q}'(|W_0| = 1) = 1$,

$$\begin{aligned}
I &= \varepsilon^{-\alpha}\,\mathbb{E}_{\nu} \iint h(\varepsilon\theta_s Y, -s)\mathbf{1}\{|Y_s| > 1, |Y_0| > \varepsilon^{-1}\}\alpha v^{-\alpha-1}\,dv\,\lambda(ds) \\
&= \varepsilon^{-\alpha}\,\mathbb{E}_{\nu} \iint h(\varepsilon Y, -s)\mathbf{1}\{|Y_0| > 1, |Y_{-s}| > \varepsilon^{-1}\}\alpha v^{-\alpha-1}\,dv\,\lambda(ds),
\end{aligned}$$

where we have used stationarity, to obtain the second equality. From here we can proceed as in the proof of Theorem 5.2 to obtain (5.2).

The final assertion $\nu_\xi = \mathbb{Q}$ follows from (6.10) and Lemma 4.1. □

Remark 6.5. Let ν be a tail measure. By Proposition 6.4, $\nu_\xi(W \in \cdot)$ $(= \mathbb{Q})$ determines ν_ξ and hence, by Corollary 6.3 also ν.

Generalizing [3, Theorem 2.9] (treating $\mathbb{G} = \mathbb{Z}$) and [20, Theorem 2.3] (treating $\mathbb{G} = \mathbb{R}$) we next provide a construction of a stationary tail measure ν, assuming the space shift formula (5.2) to hold for some given probability measure $\mathbb{Q}$. This measure ν satisfies $\nu_\xi = \mathbb{Q}$. A function h from $\mathbf{F}$ into some space is said to be 0-*homogeneous* if $h(u\omega) = h(\omega)$ for each $\omega \in \mathbf{F}$ and each $u > 0$.

Theorem 6.6. *Assume that $\mathbb{Q}$ is a spectrally decomposable probability measure on $\mathbf{F}$ such that the space shift formula (5.2) holds for some $\alpha > 0$. Assume that $H\colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ is a measurable function, 0-homogeneous in the first coordinate and such that*

$$\int H(\theta_t W, s - t)\mathbf{1}\{|W_s| > 0\}\,\lambda(ds) = 1, \quad \lambda\text{-a.e. } t,\ \mathbb{Q}\text{-a.s.} \tag{6.11}$$

Define a measure ν^H on $\mathbf{F}$ by

$$\nu^H = \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}\{u\theta_{-s}W \in \cdot\} H(\theta_{-s}W, s)\mathbf{1}\{u > 0\}\alpha u^{-\alpha-1}\, du\, \lambda(ds). \tag{6.12}$$

Then ν^H is a stationary tail measure satisfying $(\nu^H)_\xi = \mathbb{Q}$.

Proof. For the proof we generalize some of the arguments from [3, 20]. The fact that ν^H is α-homogeneous is an immediate consequence of the definition. Assumption (6.11) implies that $\mathbb{Q}(\xi'(\mathbb{G}) = 0) = 0$. Since $\{\xi'(\mathbb{G}) = 0\}$ is shift and scale invariant, we obtain again directly from the definition of ν^H that $\nu^H(\xi'(\mathbb{G}) = 0) = 0$, that is (6.1).

Let $f\colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ be measurable and set $\rho(du) := \mathbf{1}\{u > 0\}\alpha u^{-\alpha-1}du$. Then

$$\begin{aligned} I &:= \mathbb{E}_{\nu^H} \int f(\theta_t Y, t)\, \xi(dt) = \mathbb{E}_{\nu^H} \int f(\theta_t Y, t)\mathbf{1}\{|Y_t| > 1\}\, \lambda(dt) \\ &= \mathbb{E}_{\mathbb{Q}} \iiint f(u\theta_{t-s}W, t)\mathbf{1}\{u|W_{t-s}| > 1, H(\theta_{-s}W, s)\, \rho(du)\, \lambda(ds)\, \lambda(dt) \\ &= \mathbb{E}_{\mathbb{Q}} \iiint f(v|W_{t-s}|^{-1}\theta_{t-s}W, t)|W_{t-s}|^\alpha H(\theta_{-s}W, s)\}\mathbf{1}\{v > 1\}\, \lambda(ds)\, \lambda(dt)\, \rho(dv), \end{aligned}$$

where we have used the homogeneity of H and a change of variables. By the invariance properties of Haar measure (set $r := t - s$ in the inner integral),

$$I = \mathbb{E}_{\mathbb{Q}} \iiint f(v|W_r|^{-1}\theta_r W, t)|W_r|^\alpha H(\theta_{r-t}W, t - r)\}\mathbf{1}\{v > 1\}\, \lambda(dr)\, \lambda(dt)\, \rho(dv).$$

Now we can use assumption (5.2) (and again the homogeneity of H) to obtain that

$$I = \mathbb{E}_{\mathbb{Q}} \iiint f(vW, t)H(\theta_{-t}W, r + t)\}\mathbf{1}\{|W_r| > 0\}\mathbf{1}\{v > 1\}\, \lambda(dr)\, \lambda(dt)\, \rho(dv).$$

By assumption (6.11),

$$\mathbb{E}_{\nu^H} \int f(\theta_t Y, t)\, \xi(dt) = \mathbb{E}_{\mathbb{Q}} \iint f(vW, t)\mathbf{1}\{v > 1\}\, \lambda(dt)\, \rho(dv). \tag{6.13}$$

From (6.13) we conclude that (6.4) holds for $\nu = \nu^H$. The right-hand side of (6.13) does not depend on the specific choice of H. Take $r \in \mathbb{G}$ and apply (6.13) with H^r instead of H, where $H^r(\omega, s) := H(\theta_r\omega, s - r)$. Lemma 6.2 yields that $\nu^H = \nu^{H_r}$. On the other hand we obtain for each measurable $g\colon \mathbf{F} \to [0, \infty]$ that

$$\mathbb{E}_{\nu^{H^r}} g(\theta_r Y) = \mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}\{u\theta_{r-s}W \in \cdot\}H(\theta_{r-s}W, s - r)\mathbf{1}\{u > 0\}\alpha u^{-\alpha-1}\, du\, \lambda(ds),$$

which equals $\mathbb{E}_{\nu^H} g(Y)$. Hence ν^H is stationary and (6.13) shows that $(\nu^H)_\xi = \mathbb{Q}$. $\square$

Next we discuss some special cases of Theorem 6.6. Given a measurable function $G\colon \mathbb{G} \to [0, \infty]$ we define a measurable function $J_G\colon \mathbf{F} \to [0, \infty]$ by

$$J_G(\omega) := \int |\omega(s)|^\alpha G(s)\, \lambda(ds), \quad \omega \in \mathbf{F}. \tag{6.14}$$

and a measure $\mathbb{Q}^G$ on $\mathbf{F}$ by

$$\mathbb{Q}^G := \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{J_G(\theta_{-s}W)^{-1/\alpha}\theta_{-s}W \in \cdot\}G(s)\, \lambda(ds). \tag{6.15}$$

Corollary 6.7. *Let $\mathbb{Q}$ satisfy the assumptions of Theorem 6.6. Let $G\colon \mathbf{F} \to [0,\infty]$ be a measurable function satisfying*

$$0 < \int |W_s|^\alpha G(s+r)\,\lambda(ds) < \infty, \quad \lambda\text{-a.e. } r,\ \mathbb{Q}\text{-a.s.} \tag{6.16}$$

Then

$$\nu^G := \mathbb{E}_{\mathbb{Q}^G} \int \mathbf{1}\{uY \in \cdot, u > 0\}\alpha u^{-\alpha-1}\,du \tag{6.17}$$

is a stationary tail measure satisfying $(\nu^G)_\xi = \mathbb{Q}$.

Proof. We wish to apply Theorem 6.6 with the function

$$H(\omega,t) := J_G(\omega)^{-1}|\omega(t)|^\alpha G(t).$$

For each $t \in \mathbb{G}$ we have

$$\int H(\theta_t W, s-t)\mathbf{1}\{|W_s| > 0\}\,\lambda(ds) = J_G(\theta_t W)^{-1}\int |W_s|^\alpha G(s-t)\,\lambda(ds).$$

By assumption (6.16) this equals 1 for λ-a.e. t. Since $|(\theta_{-s}W)_s| = |W_0| = 1$ we obtain that ν^H is given by

$$\mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}\{v\theta_{-s}W \in \cdot, v > 0\}J_G(\theta_{-s}W)^{-1}G(s)\alpha v^{-\alpha-1}\,dv\,\lambda(ds).$$

Changing variables $u := J_G(\theta_{-s}W)v$ yields the assertion. □

Corollary 6.8. *Let $\mathbb{Q}$ be a spectrally decomposable probability measure on $\mathbf{F}$ such that $\mathbb{Q}(\xi'(\mathbb{G}) = 0) = 0$. Assume that the space shift formula (5.2) holds for some $\alpha > 0$. Let $G\colon \mathbf{F} \to (0,\infty)$ be measurable with $\int G d\lambda = 1$. Define a probability measure $\mathbb{Q}^G$ by (6.15). Then ν^G defined by (6.17) is a stationary tail measure satisfying $(\nu^G)_\xi = \mathbb{Q}$.*

Proof. We wish to apply Corollary 6.7. The first inequality in (6.16) follows from our assumptions $\mathbb{Q}(\xi'(\mathbb{G}) > 0) = 1$ and $G > 0$. Assumption (5.2) implies for each $r \in \mathbb{G}$ that

$$\mathbb{E}_{\mathbb{Q}} \int |W_s|^\alpha G(s+r)\,\lambda(ds) \le \int G(t+s)\,\lambda(ds) \le 1.$$

Hence the second inequality in (6.16) holds as well, proving the result. □

Remark 6.9. The assumption $\mathbb{Q}(\xi'(\mathbb{G}) > 0) = 1$, made in Corollary 6.8, is a probabilistic counterpart of (6.1). This assumption is very natural (see Remark 5.5) and cannot be avoided in our general setting.

Remark 6.10. Consider the assumptions of Theorem 6.6 and assume moreover that $\mathbb{Q}(|Y_s| > 0) = 1$ for λ-a.e. $s \in \mathbb{G}$. Then we can choose $G = \mathbf{1}_B$ for any $B \in \mathcal{G}$ with $0 < \lambda(B) < \infty$. If, for instance, $\mathbb{Q}$ is discrete, then we can take $B = \{0\}$ to obtain that $\mathbb{Q}^G = \mathbb{Q}$; see also [3, Remark 2.10].

Remark 6.11. We can follow [15] to construct a function H satisfying the assumptions of Theorem 6.6. Take a measurable partition $\{B_n : n \in \mathbb{N}\}$ of $\mathbb{G}$ into relatively compact Borel sets. Define $\tilde{H} \colon \mathbf{F} \times \mathbb{G} \to (0, \infty)$ by

$$\tilde{H}(\omega, s) := \sum_n 2^{-n} (\xi'(\omega, B_n) + 1)^{-1} \mathbf{1}\{s \in B_n\}.$$

Since $\xi'(\omega) = \xi'(W(\omega))$ we have that $\tilde{H}(\omega, s) = \tilde{H}(W(\omega), s)$. Define a random variable S by $S(\omega) := \int \tilde{H}(\omega, s)\, \xi'(\omega, ds)$, $\omega \in \Omega$. Then $S \leq 1$ and $S > 0$, whenever $\xi'(\mathbb{G}) > 0$. Define a function H by $H(\omega, s) := S^{-1}(\omega)\tilde{H}(\omega, s)$. By definition of ξ', $\tilde{H}$ and hence also H is 0-homogeneous in the first argument. Furthermore we have for $t \in \mathbb{G}$ that

$$\begin{aligned}
\int \tilde{H}(\theta_t W(\omega), s - t) \mathbf{1}\{|W_s(\omega)| > 0\}\, \lambda(ds) &= \int \tilde{H}(\theta_t W(\omega), s - t)\, \xi'(W(\omega), ds) \\
&= \sum_n 2^{-n} (\xi'(\theta_t W(\omega), B_n) + 1)^{-1} \mathbf{1}\{s - t \in B_n\}\, \xi'(W(\omega), ds) \\
&= \sum_n 2^{-n} (\xi'(W(\omega), B_n + t) + 1)^{-1} \xi'(W(\omega), B_n + t).
\end{aligned}$$

This is positive as soon as $\xi'(\omega, \mathbb{G}) > 0$. Since

$$S(\theta_t W(\omega)) = \int \tilde{H}(\theta_t W(\omega), s)\, \xi'(\theta_t W(\omega), ds) = \int \tilde{H}(\theta_t W(\omega), s - t)\, \xi'(W(\omega), ds),$$

we obtain (6.11), provided that $\mathbb{Q}(\xi'(\mathbb{G}) > 0) = 1$.

Corollary 6.12. *Suppose that $\mathbb{Q}$ is a spectrally decomposable probability measure on $\mathbf{F}$ such that $\mathbb{Q}(\xi'(\mathbb{G}) > 0) = 1$. Assume that ξ is mass-stationary w.r.t. $\mathbb{Q}$. Then there exists a unique stationary tail measure ν such that $\nu_\xi = \mathbb{Q}$. This tail measure is given by* (6.17) *and (under the hypothesis* (6.11)*) also by* (6.12)

Proof. Theorem 5.2, assumption $\mathbb{Q}(\xi'(\mathbb{G}) > 0) = 1$ and Remark 6.11 allow us to apply Theorem 6.6. Combining this with Corollary 6.3 shows that (6.12) is the unique tail measure ν with $\nu_\xi = \mathbb{Q}$. By Corollary 6.8, ν is also given by (6.17). □

By (2.6), Corollary 6.12 (or Corollary 6.8) has the following (quite natural) consequence.

Corollary 6.13. *Assume that Y is spectrally decomposable and that ξ is mass-stationary w.r.t. $\mathbb{Q}$. If $\mathbb{Q}(\xi'(\mathbb{G}) = 0) = 0$, then $\mathbb{Q}(\xi(\mathbb{G}) = 0) = 0$.*

Remark 6.14. Suppose that ν is a tail measure and write $\mathbb{Q} = \nu_\xi$. Let $g \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ be measurable and $r > 0$. Then

$$\mathbb{E}_{\mathbb{Q}} \int g(Y, s) \mathbf{1}\{|Y_s| > r\}\, \lambda(ds) = \mathbb{E}_\nu \int g(Y, s) \mathbf{1}\{|Y_0| > 1, |Y_s| > r\}\, \lambda(ds).$$

By homogeneity and stationarity of ν this equals

$$r^{-\alpha}\mathbb{E}_\nu \int g(rY,s)\mathbf{1}\{r|Y_0|>1,|Y_s|>1\}\,\lambda(ds)$$

$$= r^{-\alpha}\mathbb{E}_\nu \int g(r\theta_{-s}Y,s)\mathbf{1}\{r|Y_s|>1,|Y_0|>1\}\,\lambda(ds),$$

which yields (5.10). In view of Corollary 6.12 this completes the proof of Lemma 5.6.

Example 6.15. Assume that $\mathbb{G}$ is discrete, (5.7) holds and that $T\colon \mathbf{F}\to\mathbb{G}\cup\{\infty\}$ is a measurable and 0-homogeneous mapping satisfying

$$\sum_{s\in\mathbb{G}} \mathbf{1}\{T(\theta_t W)=s+t,|W_{s+t}|>0\}=1,\quad \mathbb{Q}\text{-a.s., } t\in\mathbb{G}, \tag{6.18}$$

Then we can apply Theorem 6.6 with $H(W,s)=\mathbf{1}\{T(W)=s\}$. The measure (6.12) takes the form

$$\nu^T := \mathbb{E}_{\mathbb{Q}}\sum_{s\in\mathbb{G}}\int \mathbf{1}\{u\theta_{-s}W\in\cdot,T(\theta_{-s}W)=s\}\mathbf{1}\{u>0\}\alpha u^{-\alpha-1}\,du, \tag{6.19}$$

providing a modest generalization of [3, Proposition 2.12]. Using the arguments in [3, Section 2.4] (and assuming $\mathbb{Q}(Y\equiv 0)=0$) it is possible to construct a mapping T with the preceding properties.

Remark 6.16. We can extend the mapping T from Example 6.15 to an allocation by setting $\tau(\omega,s):=T(\theta_s\omega,0)+s$. Then the formulas (5.14) and (6.19) look very similar. The crucial difference is that the allocation in the first formula picks a point from ξ while the one from (6.19) picks a point from ξ'. This explains the difference in the range of integration for the scaling variable u. A similar remark applies to Remark 5.7 and Theorem 6.6.

7 Spectral representation

Again we establish the canonical setting of Section 5. Let ν be a measure on $\mathbf{F}$. In accordance with the literature we say that ν has a *spectral representation*, if there exists a probability measure $\mathbb{Q}^*$ on $\mathbf{F}$ and an $\alpha>0$ satisfying

$$\nu = \mathbb{E}_{\mathbb{Q}^*}\int \mathbf{1}\{uY\in\cdot,u>0\}\alpha u^{-\alpha-1}\,du. \tag{7.1}$$

In this case we refer to $\mathbb{Q}^*$ as a *spectral measure* of ν and to α as the index of ν. Our previous results will show rather quickly that any stationary tail measure has a spectral representation. In a sense this section is dual to the previous one. We start with the non-probabilistic object ν and derive the probabilistic representation (7.1).

First we will state a few basic properties of a spectral representation, to be found (in special cases) in [3, 20] and in the recent preprint [2] dealing with more general fields. Recall that a stationary tail measure is assumed to be normalized as in (6.4).

Proposition 7.1. *Suppose that ν admits a spectral representation with spectral measure $\mathbb{Q}^*$ and index $\alpha > 0$. Assume that $\mathbb{Q}^*(\xi'(\mathbb{G}) > 0) = 1$. Then we have:*

(i) *ν is a tail measure iff*

$$\mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{s \in B\} |Y_s|^\alpha \, \lambda(ds) < \infty, \quad B \in \mathcal{G} \text{ compact.} \tag{7.2}$$

(ii) *Assume in addition that (7.2) holds. Then ν is stationary iff*

$$\mathbb{E}_{\mathbb{Q}^*} \int g(Y, s) |Y_s|^\alpha \, \lambda(ds) = \mathbb{E}_{\mathbb{Q}^*} \int g(\theta_{-s} Y, s) |Y_0|^\alpha \, \lambda(ds), \tag{7.3}$$

holds for all measurable $g \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ which are 0-homogeneous in the first argument. If these conditions hold, then ν is a stationary tail measure iff

$$\mathbb{E}_{\mathbb{Q}^*} |Y_s|^\alpha = 1, \quad \lambda\text{-a.e. } s \in \mathbb{G}. \tag{7.4}$$

Proof. For the proof we generalize the arguments in [3] (given for $\mathbb{G} = \mathbb{Z}$) in a straightforward manner. Clearly ν is α-homogeneous. By $\mathbb{Q}^*(\xi'(\mathbb{G}) > 0) = 1$ and (7.1), ν satisfies property (6.1).

(i) Let $B \in \mathcal{G}$. Then

$$\begin{aligned} \mathbb{E}_\nu \int \mathbf{1}\{s \in B, |Y_s| > 1\} \, \lambda(ds) &= \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{s \in B, u|Y_s| > 1\} \alpha u^{-\alpha-1} \, du \, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{s \in B\} |Y_s|^\alpha \, \lambda(ds). \end{aligned} \tag{7.5}$$

Hence (6.4) and (7.2) are equivalent.

(ii) Assume that ν is stationary and take a measurable $g \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ which is 0-homogeneous in the first argument. Then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^*} \int g(Y, s) |Y_s|^\alpha \, \lambda(ds) &= \mathbb{E}_{\mathbb{Q}^*} \iint g(uY, s) \mathbf{1}\{u|Y_s| > 1\} \alpha u^{-\alpha-1} \, du \, \lambda(ds) \\ &= \mathbb{E}_\nu \int g(Y, s) \mathbf{1}\{|Y_s| > 1\} \, \lambda(ds). \end{aligned}$$

By stationarity of ν this equals

$$\mathbb{E}_\nu \int g(\theta_{-s} Y, s) \mathbf{1}\{|Y_0| > 1\} \, \lambda(ds) = \mathbb{E}_{\mathbb{Q}^*} \iint g(\theta_{-s} Y, s) \mathbf{1}\{u|Y_0| > 1\} \alpha u^{-\alpha-1} \, du \, \lambda(ds).$$

This equals the right-hand side of (7.3).

Assume now that (7.3) holds. Take a measurable $f \colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ and $t \in \mathbb{G}$. Then

$$\begin{aligned} I &:= \mathbb{E}_\nu \int f(\theta_t Y, s) \mathbf{1}\{|Y_{s+t}| > 1\} \, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}^*} \iint f(u\theta_t Y, s) \mathbf{1}\{u|Y_{s+t}| > 1\} \alpha u^{-\alpha-1} \, du \, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}^*} \iint f(u\theta_t Y, s-t) \mathbf{1}\{u|Y_s| > 1\} \alpha u^{-\alpha-1} \, du \, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}^*} \iint f(|Y_s|^{-1} v \theta_t Y, s-t) |Y_s|^\alpha \mathbf{1}\{v > 1\} \alpha v^{-\alpha-1} \, dv \, \lambda(ds). \end{aligned}$$

By the homogeneity of $|Y_s|^{-1}Y$ and (7.3),

$$I = \mathbb{E}_{\mathbb{Q}^*} \iint f(|Y_0|^{-1}v\theta_{t-s}Y, s-t)|Y_0|^\alpha \mathbf{1}\{v > 1\}\alpha v^{-\alpha-1}\, dv\, \lambda(ds)$$

$$= \mathbb{E}_{\mathbb{Q}^*} \iint f(u\theta_{t-s}Y, s-t)\mathbf{1}\{u|Y_0| > 1\}\alpha u^{-\alpha-1}\, du\, \lambda(ds)$$

$$= \mathbb{E}_{\mathbb{Q}^*} \iint f(u\theta_{-s}Y, s)\mathbf{1}\{u|Y_0| > 1\}\alpha u^{-\alpha-1}\, du\, \lambda(ds).$$

This shows that the Campbell measures $C_{\nu\circ\theta_t,\xi'}$ do not depend on $t \in \mathbb{G}$. But $\nu \circ \theta_t$ does also satisfy (6.1) and (6.3) along with the assumption of (ii). Hence Lemma 6.2 implies that ν is stationary.

The final assertion follows from (7.5). □

Let us mention the following fact; cf. [20, (2.8)].

Corollary 7.2. *Suppose that ν is a stationary tail measure with index $\alpha > 0$. Let $\mathbb{Q}^*$ be a spectral measure of ν. Then*

$$\nu_\xi(W \in \cdot) = \mathbb{E}_{\mathbb{Q}^*}\mathbf{1}\{|Y_s|^{-1}\theta_s Y \in \cdot\}|Y_s|^\alpha, \quad \lambda\text{-a.e. } s.$$

Proof. In the proof of Proposition 7.1 we have seen that

$$\mathbb{E}_{\mathbb{Q}^*} \int g(Y, s)|Y_s|^\alpha\, \lambda(ds) = \mathbb{E}_\nu \int g(\theta_{-s}Y, s)\mathbf{1}\{|Y_0| > 1\}\, \lambda(ds).$$

holds, provided that g is 0-homogeneous in the first argument. The right-hand side equals $\mathbb{E}_{\nu_\xi} \int g(\theta_{-s}Y, s)\lambda(ds)$. Equivalently,

$$\mathbb{E}_{\mathbb{Q}^*} \int g(\theta_s Y, s)|Y_s|^\alpha\, \lambda(ds). = \mathbb{E}_{\nu_\xi} \int g(Y, s)\, \lambda(ds)$$

Applying this with $g(Y, s) := h(|Y_0|^{-1}Y, s)$ for a measurable $h\colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ yields the assertion. □

The following result extends the stationary case of [3, Theorem 2.4] (covering the case $\mathbb{G} = \mathbb{Z}$) and [20, Theorem 2.3] (dealing with the case $\mathbb{G} = \mathbb{R}$). A general non-stationary (and therefore less specific) version can be found as Lemma 3.10 in the recent preprint [2].

Theorem 7.3. *Suppose that ν is a stationary tail measure with index $\alpha > 0$. Then ν has a spectral representation with a spectral measure $\mathbb{Q}^*$ satisfying* (7.3) *and* (7.4).

Proof. By Theorem 3.2 the measure $\mathbb{Q} := \nu_\xi$ is mass-stationary. By Proposition 6.4, Theorem 5.2 and (6.7), $\mathbb{Q}$ satisfies the assumptions of Theorem 6.6. By Corollary 6.8 we can therefore define a tail measure ν' (with index α) by (6.17) for some given function G with the required properties. Then ν' admits a spectral representation with spectral measure $\mathbb{Q}^* := \mathbb{Q}^G$. By Corollary 6.8 we also have $\nu'_\xi = \mathbb{Q}$, that is $\nu'_\xi = \nu_\xi$. Corollary 6.3 shows that $\nu = \nu'$, proving the spectral representation (7.1). By Proposition 7.1, $\mathbb{Q}^*$ satisfies (7.4) and (7.3). □

Remark 7.4. The existence of a spectral representation of a tail measure ν can also be derived from Proposition 2.8 in [5]. Indeed, the sets U_k defined in (6.6) satisfy the assumptions of that proposition. However, Theorem 7.3 and its proof provide more detailed information on the spectral measure $\mathbb{Q}^*$. In fact, $\mathbb{Q}^*$ is explicitly given in terms of the Palm measure ν_ξ of ξ w.r.t. ν.

A spectral measure is not uniquely determined by the tail measure. Depending on the properties of ν_ξ, the proof of Theorem 7.3 provides several ways of constructing a spectral measure. The recent preprint [6] contains a systematic discussion of the relationships between random fields (on $\mathbb{R}^d$ or $\mathbb{Z}^d$) satisfying (7.3) and stationary tail measures.

Remark 7.5. Let ν be a stationary tail measure. Then ν_ξ is said to be the distribution of the *tail process* associated with ν; see [3, 20]. Under ν_ξ the process W is called a *spectral (tail) process* associated with ν; see again [3, 20]. By Corollary 6.3, ν is uniquely determined by $\nu_\xi(W \in \cdot)$. But in general, $\nu_\xi(W \in \cdot)$ is not a spectral measure of ν. This clash of terminology is a bit unfortunate.

A tail measure ν is said to admit a *moving shift representation* if there exists a probability measure $\mathbb{Q}^*$ on $\mathbf{F}$ such that

$$\nu = \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u\theta_s Y \in \cdot, u > 0\} \alpha u^{-\alpha-1}\, du\, \lambda(ds). \tag{7.6}$$

Theorem 7.6. *Suppose that ν is a stationary tail measure with index $\alpha > 0$. Then there exists a probability measure $\mathbb{Q}^*$ on $\mathbf{F}$ such that (7.6) holds iff*

$$\int |Y_s|^\alpha \lambda(ds) < \infty, \quad \nu\text{-a.e.} \tag{7.7}$$

Proof. Assume first, that (7.7) holds. As noticed in the proof of Theorem 7.3 the probability measure $\mathbb{Q} := \nu_\xi$ satisfies the assumptions of Theorem 6.6. Define the probability measure

$$\mathbb{Q}^* := \mathbb{Q}(Z^{-1/\alpha} W \in \cdot), \tag{7.8}$$

where $Z := \int |W_s|^\alpha \lambda(ds)$. Applying Corollary 6.7 with $G \equiv 1$ shows the right-hand side of (7.6) is a stationary tail measure ν' with $\nu'_\xi = \mathbb{Q}$. As in the proof of Theorem 7.3 we obtain $\nu = \nu'$.

Assume, conversely, that (7.6) holds. Then

$$1 = \nu(|Y_0| > 1) = \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u|Y_s| > 1, u > 0\} \alpha u^{-\alpha-1}\, du\, \lambda(ds) = \mathbb{E}_{\mathbb{Q}^*} \int |Y_s|^\alpha\, \lambda(ds).$$

Hence $\mathbb{Q}^*(A) = 0$, where $A := \{\int |Y_s|^\alpha \lambda(ds) = \infty\}$. Since A is invariant under translation and scaling, we obtain from (7.6) that $\nu(A) = 0$. □

We refer the reader to [3, 4, 20] for a more detailed analysis of moving shift representations for special groups $\mathbb{G}$ and under additional continuity assumptions on Y. Extending some of those results to general groups is an interesting task, beyond the scope of this paper.

8 Anchoring maps

In this section we let Y and ξ be as in Section 4 and suppose that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{A})$ such that ξ is mass-stationary w.r.t. $\mathbb{Q}$.

Following [17, 20] we say that a measurable mapping $T\colon \mathbf{F} \to \mathbb{G}$ is an *anchoring map* if

$$T(\theta_s \omega) = T(\omega) - s, \quad s \in \mathbb{G}, \text{ if } 0 < \xi(\omega, \mathbb{G}) < \infty. \tag{8.1}$$

In stochastic geometry such functions are known as *center functions*; see e.g. [13, Chapter 17].

In the following the number

$$\vartheta := \mathbb{E}_{\mathbb{Q}} \xi(\mathbb{G})^{-1}. \tag{8.2}$$

will play an important rule. If $\mathbb{Q}(\xi(\mathbb{G}) < \infty) > 0$, then $\vartheta > 0$.

Proposition 8.1. *Assume that ξ is mass-stationary w.r.t. $\mathbb{Q}$. Assume also that*

$$\mathbb{Q}(0 < \xi(\mathbb{G}) < \infty) = 1 \tag{8.3}$$

and $\vartheta < \infty$. Let T be an anchoring map and define the probabiliy measure

$$\mathbb{Q}_T := \vartheta^{-1} \mathbb{E}_{\mathbb{Q}} \xi(\mathbb{G})^{-1} \mathbf{1}\{\theta_T \in \cdot\} \tag{8.4}$$

Then we have for all measurable $g\colon \Omega \to [0, \infty]$ that

$$\mathbb{E}_{\mathbb{Q}} g = \vartheta \, \mathbb{E}_{\mathbb{Q}_T} \int g \circ \theta_s \, \xi(ds). \tag{8.5}$$

Proof. Let ν be the σ-finite stationary measure on Ω such that $\mathbb{Q} = \nu_\xi$ and $\nu(\xi(\mathbb{G}) = 0) = 0$. Define an allocation τ by $\tau(\omega, s) := T(\theta_s \omega) + s$. By assumption $\tau(\omega, s) = T(\omega)$ for each $s \in \mathbb{G}$, provided that $0 < \xi(\omega, \mathbb{G}) < \infty$. Moreover,

$$\eta := \mathbf{1}\{0 < \xi(\omega, \mathbb{G}) < \infty\} \mathbf{1}\{T \in \cdot\}$$

is an invariant simple point process. By Proposition 3.1 we have for each measurable $h\colon \Omega \times \Omega \to [0, \infty]$ that

$$\mathbb{E}_{\mathbb{Q}} h(\theta_0, \theta_{\tau(0)}) = \mathbb{E}_{\nu_\eta} \int \mathbf{1}\{\tau(s) = 0\} h(\theta_s, \theta_0) \, \xi(ds).$$

Since $\nu_\eta(\xi(\mathbb{G}) \in \{0, \infty\}) = 0$, this means

$$\mathbb{E}_{\mathbb{Q}} h(\theta_0, \theta_T) = \mathbb{E}_{\nu_\eta} \int \mathbf{1}\{T = 0\} h(\theta_s, \theta_0) \, \xi(ds).$$

It follows straight from the definition (2.3) that $\nu_\eta(T \neq 0) = 0$. Therefore

$$\mathbb{E}_{\mathbb{Q}} h(\theta_0, \theta_T) = \mathbb{E}_{\nu_\eta} \int h(\theta_s, \theta_0) \, \xi(ds). \tag{8.6}$$

Applying this to the function $(\omega, \omega') \mapsto \xi(\omega', \mathbb{G})^{-1} g(\omega')$ (for some measurable $g\colon \Omega \to [0, \infty]$) yields

$$\mathbb{E}_{\mathbb{Q}} \xi(\mathbb{G})^{-1} g(\theta_T) = \mathbb{E}_{\nu_\eta} g$$

and therefore $\mathbb{Q}_T = \vartheta^{-1} \nu_\eta$. Hence (8.5) follows from (8.6). □

Example 8.2. Assume that $\mathbb{G}$ is discrete as in Example 2.3, so that ξ is a simple point process on $\mathbb{G}$. Under assumption (8.3) we have $\vartheta \le 1$ with equality iff $\mathbb{Q}(\xi(\mathbb{G}) = 1) = 1$. From (8.5) we obtain for each measurable $f : \mathbf{F} \to [0, \infty]$ that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}} f \mathbf{1}\{T = 0\} &= \vartheta\, \mathbb{E}_{\mathbb{Q}_T} \int f \circ \theta_s \mathbf{1}\{T \circ \theta_s = 0\}\, \xi(ds) \\ &= \vartheta\, \mathbb{E}_{\mathbb{Q}_T} \int f \circ \theta_s \mathbf{1}\{T = s\}\, \xi(ds) = \mathbb{E}_{\mathbb{Q}_T} f \mathbf{1}\{|Y_0| > 1\},\end{aligned}$$

where we have used that $\mathbb{Q}_T(T = 0) = 1$. As in [17] it is natural to assume that

$$|Y_T| > 1. \tag{8.7}$$

Then $\mathbb{E}_{\mathbb{Q}} f \mathbf{1}\{T = 0\} = \mathbb{E}_{\mathbb{Q}_T} f$. Hence $\vartheta = \mathbb{Q}(T = 0)$ and $\mathbb{Q}_T = \mathbb{Q}(\cdot \mid T = 0)$. Therefore Proposition 8.1 extends [17, Proposition 3.2]. Further formulas in the spectrally decomposable case can be found in Subsection 3.3 of [17].

In the remainder of this section we establish the canonical setting of Section 5. We assume that $\mathbb{Q}$ is spectrally decomposable and satisfies $\mathbb{Q}(\xi'(\mathbb{G}) = 0) = 0$. By Corollary 6.8 we can associate with $\mathbb{Q}$ a unique stationary tail measure ν such that $\nu_\xi = \mathbb{Q}$. We assume moreover that

$$\int |Y_s|^\alpha \lambda(ds) < \infty, \quad \mathbb{Q}\text{-a.e.} \tag{8.8}$$

Note that

$$\xi(\mathbb{G}) \le \int \mathbf{1}\{|Y_s| > 1\} |Y_s|^\alpha\, \lambda(ds) \le \int |Y_s|^\alpha\, \lambda(ds),$$

so that $\mathbb{Q}(0 < \xi(\mathbb{G}) < \infty) = 1$; see also Corollary 6.13. Since (8.8) does also hold ν-a.e., we can and will use the moving shift representation (7.6) with $\mathbb{Q}^*$ given by (7.8).

For each $\omega \in \mathbf{F}$ the function $u \mapsto \int \mathbf{1}\{|\omega(s)| > u^{-1}\}\, \lambda(ds)$ from $(0, \infty)$ to $[0, \infty]$ is increasing and left-continuous. Therefore we can define a random variable κ by

$$\kappa := \inf\{u > 0 : \xi(uZ^{-1/\alpha} W, \mathbb{G}) > 0\}. \tag{8.9}$$

The following lemma gives an alternative expression for the number ϑ defined by (8.2).

Lemma 8.3. *Assume that $\mathbb{Q}$ is spectrally decomposable, that ξ is mass-stationary w.r.t. $\mathbb{Q}$ and that (8.8) holds. Then $\vartheta = \mathbb{E}_{\mathbb{Q}} \kappa^\alpha$.*

Proof. We start with a preliminary comment. Since $\mathbb{Q}(|Y_0| > 0) = 1$ we have $\mathbb{Q}$-a.e. $Z^{-1/\alpha} W = \tilde{Z}^{-1/\alpha} Y$, where

$$\tilde{Z} := \int |Y_s|^\alpha\, \lambda(ds).$$

In particular,

$$\kappa = \inf\{u > 0 : \xi(u\tilde{Z}^{-1/\alpha} Y, \mathbb{G}) > 0\}. \tag{8.10}$$

By our assumptions, $\mathbb{Q}(0 < \tilde{Z} < \infty) = 1$. Since for each $v > 0$

$$\int \mathbf{1}\{v|Y_s| > 1\}\, \lambda(ds) \le v^\alpha \int |Y_s|^\alpha\, \lambda(ds),$$

we obtain

$$\xi(u\tilde{Z}^{-1/\alpha}Y, \mathbb{G}) < \infty, \quad u > 0,\ \mathbb{Q}\text{-a.e.} \tag{8.11}$$

Since $\mathbb{Q}$ is the Palm measure of ξ w.r.t. ν we have $\vartheta = \mathbb{E}_\nu \mathbf{1}\{|Y_0| > 1\}\xi(\mathbb{G})^{-1}$. Hence we obtain from (7.6) and (7.8) that

$$\vartheta = \mathbb{E}_\mathbb{Q} \iint \mathbf{1}\{u\tilde{Z}^{-1/\alpha}|Y_s| > 1\}\xi(u\tilde{Z}^{-1/\alpha}Y, \mathbb{G})^{-1}\alpha u^{-\alpha-1}\, du\, \lambda(ds).$$

By Fubini's theorem and (8.11),

$$\vartheta = \mathbb{E}_\mathbb{Q} \int \mathbf{1}\{\xi(u\tilde{Z}^{-1/\alpha}Y, \mathbb{G}) > 0\}\alpha u^{-\alpha-1}\, du.$$

By (8.10) this yields $\vartheta = \mathbb{E}_\mathbb{Q} \int \mathbf{1}\{u > \kappa\}\alpha u^{-\alpha-1}\, du$ and hence the asserted formula. $\square$

Let τ be an allocation such that

$$\tau(u\omega, s) = \tau(u\omega, 0) \in \mathbb{G}, \quad \mathbb{Q} \otimes \lambda \otimes \lambda_+\text{-a.e. } (\omega, s, u), \tag{8.12}$$

where λ_+ denotes Lebesgue measure on $(0, \infty)$. In view of (8.11) we can interpret $\tau(0)$ as an almost every version of an anchoring map. Motivated by [20, Section 2.3] we collect some (preliminary) information on the distribution of $(\tau(Y, 0), Y)$. Though the principal calculations are similar, we cannot use the specific moving shift representation from [20, Theorem 2.9].

Lemma 8.4. *Assume that the assumptions of Lemma 8.3 hold. Let τ be an allocation satisfying* (8.12) *and suppose that $g\colon \mathbf{F} \times \mathbb{G} \to [0, \infty]$ is measurable and shift-invariant in the first coordinate. Then*

$$\mathbb{E}_\mathbb{Q} g(Y, \tau(0)) = \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u > \kappa\}\mathbf{1}\{u|Y_{s+\tau(uY,0)}| > 1\}g(uY, -s)\alpha u^{-\alpha-1}\, du\, \lambda(ds). \tag{8.13}$$

In particular $\mathbb{Q}(\tau(0) \in \cdot)$ has the λ-density

$$f_\tau(s) := \mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{u > \kappa, u|Y_{-s+\tau(uY,0)}| > 1\}\alpha u^{-\alpha-1}\, du, \quad s \in \mathbb{G}. \tag{8.14}$$

Proof. We have

$$\begin{aligned}\mathbb{E}_\mathbb{Q} g(Y, \tau(0)) &= \mathbb{E}_\nu \mathbf{1}\{|Y_0| > 1\}g(Y, \tau(0)) \\ &= \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u|Y_s| > 1\}g(u\theta_s Y, \tau(u\theta_s Y, 0))\alpha u^{-\alpha-1}\, du\, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u|Y_s| > 1\}g(uY, \tau(uY, 0) - s)\alpha u^{-\alpha-1}\, du\, \lambda(ds),\end{aligned}$$

where we have used that $\tau(u\theta_s Y, 0) = \tau(uY, s) - s = \tau(uY, 0) - s$ holds for $\lambda \otimes \lambda_+$-a.e. (s, u) and $\mathbb{Q}^*$-a.e. Changing variables gives

$$\mathbb{E}_{\mathbb{Q}} g(Y, \tau(0)) = \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u|Y_{s+\tau(uY,0)}| > 1\} g(uY, -s) \alpha u^{-\alpha-1} \, du \, \lambda(ds).$$

If $u < \kappa$ then $\int \mathbf{1}\{u|Y_{s+\tau(uY,0)}| > 1\} \, \lambda(ds) = 0$. Therefore (8.13) follows. The second assertion is an immediate consequence. □

In the remainder of the section we assume that τ is an allocation satisfying (8.12). Suppose that $h\colon \mathbf{F} \to [0, \infty)$ is measurable and shift invariant. Then we obtain from (8.13) that

$$\mathbb{E}[h(Y) \mid \tau(0) = s] = f_\tau(s)^{-1} \mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{u > \kappa\} \mathbf{1}\{u|Y_{s+\tau(uY,0)}| > 1\} h(uY) \alpha u^{-\alpha-1} \, du \tag{8.15}$$

holds for $\mathbb{Q}(\tau(0) \in \cdot)$-a.e. s. To discuss this formula we make the ad hoc assumption

$$\lim_{s \to 0} \mathbf{1}\{u|Y_{s+\tau(uY,0)}| > 1\} = 1, \quad u > \kappa, \ \lambda_+\text{-a.e. } u, \ \mathbb{Q}\text{-a.e.}, \tag{8.16}$$

see [20, (2.25)] for a similar hypothesis in the case $\mathbb{G} = \mathbb{R}$. This can be seen as a continuous space version of (8.7) and might be achieved under appropriate continuity assumptions on Y. If, in addition, $\vartheta = \mathbb{E}\kappa^\alpha < \infty$, then dominated convergence yields the existence of the limit

$$\lim_{s \to 0} f_\tau(s) = \vartheta. \tag{8.17}$$

If

$$\mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{u > \kappa\} h(uY) \alpha u^{-\alpha-1} \, du < \infty$$

then dominated convergence yields the existence of the limit

$$\lim_{s \to 0} \mathbb{E}[h(Y) \mid \tau(0) = s] = \vartheta^{-1} \mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{u > \kappa\} h(uY) \alpha u^{-\alpha-1} \, du. \tag{8.18}$$

In particular we may take $h(Y) = \xi(\mathbb{G})$. Indeed, we have that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*} \int \mathbf{1}\{u > \kappa\} \xi(uY, \mathbb{G}) \alpha u^{-\alpha-1} \, du &= \mathbb{E}_{\mathbb{Q}^*} \iint \mathbf{1}\{u|Y_s| > 1\} \alpha u^{-\alpha-1} \, du \, \lambda(ds) \\ &= \mathbb{E}_{\mathbb{Q}^*} \int |Y_s|^\alpha \, \lambda(ds) = \mathbb{E}_{\mathbb{Q}} \int Z^{-1} |W_s|^\alpha \, \lambda(ds) = 1.\end{aligned}$$

Therefore,

$$\lim_{s \to 0} \mathbb{E}[\xi(\mathbb{G}) \mid \tau(0) = s] = \vartheta^{-1}; \tag{8.19}$$

see [20, (2.26)] for the case $\mathbb{G} = \mathbb{R}$. In view of the discussion in [10, 20] we might call ϑ the *candidate extremal index* of Y.

The results of this section are certainly preliminary. But without continuity assumptions on the elements of $\mathbf{F}$ it seems difficult to make further progress. If $\mathbb{G} = \mathbb{R}^d$ and $\mathbf{F}$ is a Skorohod space (see Example 2.2), then it might be possible to establish an analog of [20, Theorem 2.9]. In particular the assumptions of Lemma 8.3 should then imply $\vartheta < \infty$.

9 Concluding remarks

The results from Sections 5-7 generalize to the setting described in Remark 5.9. This would mean, for instance, that tail measures are then defined on a more general space Ω and not just on the function space $\mathbf{F}$. To avoid an abstract (and potentially confusing) notation we have chosen to stick to the present more specific setting.

Given the results of this paper, one might define a tail process in an intrinsic way, namely as a spectrally decomposable random field $Y = (Y_s)_{s\in\mathbb{G}}$ such that the exceedance random measure is mass-stationary. It would be interesting to identify such processes as the tail processes of regularly varying stationary fields, beyond the known special cases. It would also be interesting to further study tail measures of such fields, as introduced in great generality in [19]. In particular it might be worthwhile exploring further relationships between tail measures, tail processes and Palm calculus.

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