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Wellposedness and regularity for linear Maxwell equations with surface current

Benjamin Dörich and Konstantin Zerulla

Abstract. We study linear time-dependent Maxwell equations on a cuboid consisting of two homogeneous subcuboids. At the interface, we allow for nonzero surface charge density and surface current. This model is a first step towards a detailed mathematical analysis of the interaction of single-layer materials with electromagnetic fields. The main results of this paper provide several wellposedness and regularity statements for the solutions of the Maxwell system. To prove the statements, we employ extension arguments using interpolation theory, as well as semigroup theory and regularity theory for elliptic transmission problems.

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1. Introduction

In the past few years, there has been an extensive study of the properties of single-layer materials (or 2D materials) such as the semimetal graphene and the semiconductors called transition-metal dichalcogenides (TMDCs). This is mainly due to the large area of possible application such as optoelectronics, spintronics, energy storage, lubrication, and catalysis. We refer to the reviews [1,36] on graphene and TMDCs, respectively, for a detailed discussion.

In order to study the optical properties of such materials, the sheet is placed on top of a thin dielectric and a metal plate and ultrafast optical pulses are sent towards the material. The behavior of the light pulses is described by Maxwell equations which interact with an interface induced by the 2d material. In the general model, the 2d material itself has a dynamic often described by a quantum mechanical model [7,23,34], influencing the electromagnetic waves via induced surface currents, see for example [6] and Section 3 in [25]. In graphene models, the surface current satisfies in frequency domain

$$j_{\text{surf}}(\omega) = \sigma(\omega)[\![\mathbf{n}_{\text{int}} \times \mathbf{E}(\omega)]\!]$$

where σ is the surface conductivity associated with the 2d material, $[\![\cdot]\!]$ denotes the jump at the interface, and $\mathbf{E}$ is the electric field and $\mathbf{n}_{\text{int}}$ denotes the unit normal vector associated with the interface, see, for example, Chapter 1 in [13,20,24,40]. This is a special case of the linear response theory which makes the ansatz

$$j_{\text{surf}}(\mathbf{k},\omega) = \sigma(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega)$$

corresponding to Ohm's law, see [19], Chapter 6 in [31].

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In order to understand the physical model better, there is also put a large amount of work in the numerical treatment of Maxwell equations with inhomogeneous interface conditions, which are usually referred to as current sheets. For the finite volume method, this was considered in [28], for the discontinuous Galerkin method in [38–40] and for a finite difference method in [5,30].

As a first step towards the full model, we consider in this work linear time-dependent Maxwell equations on a cuboid Q composed of two homogeneous cuboids Q_1 and Q_2 with an interface $F_{\mathrm{int}} = \overline{Q}_1 \cap \overline{Q}_2$, see Fig. 1. On this we assume an abstract external surface current j_{surf} only depending on space and time which replaces the dynamics of the 2d material. We note that in the full Maxwell–Schrödinger model one usually has, combining the approach in [6,25] with the minimal coupling discussed in Chapter 5 of [32], the dependency

$$j_{\mathrm{surf}} = j_{\mathrm{surf}}(t, \psi, A),$$

where A is the magnetic vector potential and ψ is the wave function on the 2d material. We consider the Maxwell equations

$$\partial_t \mathbf{E}^{(i)} = \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \mathbf{H}^{(i)} - \frac{1}{\varepsilon^{(i)}} \mathbf{J}_\Sigma^{(i)}, \qquad \partial_t \mathbf{H}^{(i)} = -\frac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{E}^{(i)}, \tag{1.1a}$$

$$\operatorname{div}\big(\varepsilon^{(i)} \mathbf{E}^{(i)}\big) = \rho^{(i)}, \qquad \operatorname{div}\big(\mu^{(i)} \mathbf{H}^{(i)}\big) = 0, \tag{1.1b}$$

on Q_i, $i \in \{1, 2\}$, for $t \geq 0$. Throughout we denote by $f^{(i)}$ the restriction of a function $f \in L^2(Q)$ to the subcuboid Q_i. The vector $\mathbf{E} = \mathbf{E}(t, x) \in \mathbb{R}^3$ is the electric field, $\mathbf{H} = \mathbf{H}(t, x) \in \mathbb{R}^3$ the magnetic field, $\mathbf{J}_\Sigma = \mathbf{J}_\Sigma(t, x) \in \mathbb{R}^3$ models an external current, and $\rho = \rho(t, x)$ denotes the volume charge. The material is described by the scalar electric permittivity $\varepsilon > 0$ and the scalar magnetic permeability $\mu > 0$.

We accompany the Maxwell system with perfectly conducting boundary conditions

$$\mathbf{E} \times \nu = 0, \quad \mu \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q$$

and interface conditions

$$[\![\mu \mathbf{H} \cdot \mathbf{n}_{\mathrm{int}}]\!] = 0, \quad [\![\mathbf{E} \times \mathbf{n}_{\mathrm{int}}]\!] = 0, \quad [\![\varepsilon \mathbf{E} \cdot \mathbf{n}_{\mathrm{int}}]\!] = \rho_{\mathrm{surf}}, \quad [\![\mathbf{H} \times \mathbf{n}_{\mathrm{int}}]\!] = j_{\mathrm{surf}} \quad \text{on } F_{\mathrm{int}}, \tag{1.2}$$

involving the surface charge ρ_{surf} and the surface current j_{surf}, see Section 4.12 in [35], Section 1.1.3 in [4] and Section I.5 in [22]. Note that ν denotes the exterior unit normal vector on the boundary ∂Q, and that the inner normal vector $\mathbf{n}_{\mathrm{int}}$ points from Q_1 to Q_2. For the jump $[\![\cdot]\!]$, we use the convention

$$[\![f]\!] := f^{(2)}|_{F_{\mathrm{int}}} - f^{(1)}|_{F_{\mathrm{int}}},$$

whenever the traces of f are well-defined at the interface. The parameters ε and μ are assumed to be positive and constant on each subcuboid, modeling a piecewise homogeneous medium. After coordinate transformation, we can assume the identities

$$Q = (-1, 1) \times (0, 1)^2, \quad Q_1 = (-1, 0) \times (0, 1)^2, \quad Q_2 = (0, 1)^3.$$

To the best of our knowledge, there is so far no detailed regularity and wellposedness theory available for our model problem (1.1)–(1.2). On the one hand, Maxwell equations with discontinuous material parameters on more general and complicated polyhedral Lipschitz domains are studied in [2,3,8–10,12] for instance. These papers, however, assume homogeneous transmission conditions, meaning the surface charge density and the surface current are zero. Note that one of the authors analyzes a similar model problem with nontrivial surface charge density but zero surface current, see [41,42]. On the other hand, a nonlinear time-dependent Maxwell system is recently studied in [33] with discontinuous material parameters. On the interface, the analysis allows nonzero surface charge density and surface current. Note, however, that the boundary of the domain in [33] is regular, and that the interface between the two submedia has a positive distance to the boundary.

Recently, a nonlinear transmission problem on the full space is analyzed with homogeneous transmission conditions at the interface $\{x_1 = 0\}$, see [14]. The results are also applicable to time-dependent

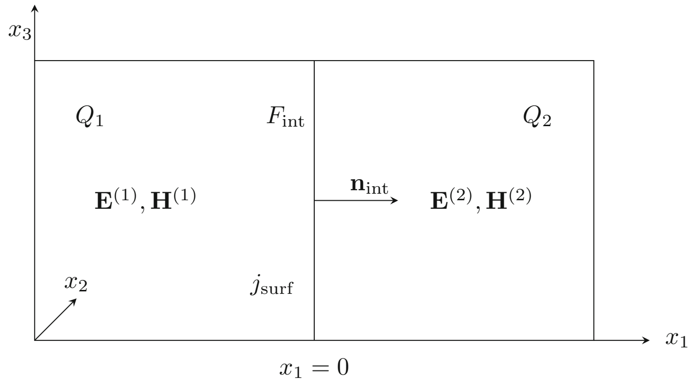

FIG. 1. Sketch of the domain Q

Maxwell equations. In [15], transverse magnetic wave packets are studied and approximated for a nonlinear time dependent Maxwell system on the full space with discontinuous material parameters at the interface $\{x_1 = 0\}$. The surface current is assumed to be zero, and the surface charge is nonzero and time independent.

Structure of the paper

We present our main results in Sect. 2: Depending on the regularity of the initial data for (1.1) as well as the regularity of the surface current j_{surf} and the external current $\mathbf{J}_\Sigma$, we establish three wellposedness and regularity statements for the solution $(\mathbf{E}, \mathbf{H})$ of (1.1). Among others, the initial data and the surface current have to satisfy certain compatibility conditions at $t = 0$, see also [33]. In 3.1, we additionally show that the assumed compatibility conditions are in fact necessary. An overview of the involved function spaces is given in Fig. 2.

To achieve our results, we first transform the Maxwell system (1.1) with interface conditions (1.2) into an evolution equation with zero surface current j_{surf}, see (3.5). To be more precise, we construct suitable regular extensions $\mathbf{J}_\mathbf{H}$ and $\mathbf{J}_\mathbf{E}$ of j_{surf} and $\partial_t j_{\mathrm{surf}}$ in Sect. 4 by means of interpolation theory involving analytic semigroups. This turns out to be quite sophisticated. Here we also use ideas and techniques from [16,41].

For the Cauchy problem (3.5), we can employ semigroup theory on appropriate function spaces, see Sect. 3.3. To conclude the desired piecewise Sobolev regularity of the solutions of (3.5) (and thus eventually of (1.1)), we show that elements of the arising function spaces are piecewise H^1- and H^2-regular. Here we use regularity theory for elliptic transmission problems, see Sect. 3.2 and compare [11,12,16,21,41,42].

2. Framework and results

In this section, we present our main results, i.e., wellposedness and regularity analysis of the system (1.1), under precise assumptions on the regularity of the initial data and the surface current.

Spaces

We first introduce the relevant spaces which are necessary to state our main results. For a Lipschitz domain $\mathcal{O}$, we denote for $k \in \mathbb{N}$ by $H^k(\mathcal{O})$ the space of functions with weak derivatives up to order k in

$L^2(\mathcal{O})$. By $H_0^1(\mathcal{O})$ we mean the closure of test functions $C_c^\infty(\mathcal{O})$ in $H^1(\mathcal{O})$. To define Sobolev spaces of fractional order, we use real interpolation spaces, see Section 1.1 in [27] for instance. For $s \geq 0$, we define the fractional Sobolev spaces

$$H^s(\mathcal{O}) := \big(L^2(\mathcal{O}), H^2(\mathcal{O})\big)_{s/2,2}, \quad H_{00}^{1/2}(\mathcal{O}) := \big(L^2(\mathcal{O}), H_0^1(\mathcal{O})\big)_{1/2,2},$$
$$H_0^{3/2}(\mathcal{O}) := \big(H_0^1(\mathcal{O}), H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})\big)_{1/2,2},$$

see [26,37]. For the electric volume charge, we also need the space

$$H_{00}^1(Q_i) := \{\varphi \in H^1(Q_i) \mid \operatorname{tr}_F \varphi \in H_{00}^{1/2}(F) \text{ for each face } F \text{ of } Q_i\},$$
$$\|\varphi\|^2_{H_{00}^1(Q_i)} := \|\varphi\|^2_{H^1(Q_i)} + \sum_{\substack{F \text{ face} \\ \text{of } Q_i}} \|\varphi|_F\|^2_{H_{00}^{1/2}(F)},$$

for $i \in \{1,2\}$. For the disjoint union $Q = Q_1 \cup Q_2$, we define the piecewise Sobolev space of order $s \geq 0$ by

$$PH^s(Q) = \{u \in L^2(Q) \mid u^{(i)} \in H^s(Q_i),\ i = 1,2\},$$

and use the notation

$$X_0 := L^2(Q)^6 = PH^0(Q)^6.$$

The space X_0 is equipped with the weighted inner product

$$\big((\mathbf{E},\mathbf{H}),(\tilde{\mathbf{E}},\tilde{\mathbf{H}})\big) := \int_Q \varepsilon \mathbf{E}\cdot\tilde{\mathbf{E}} + \mu \mathbf{H}\cdot\tilde{\mathbf{H}} \mathrm{d}x,$$

inducing the norm $\|\cdot\|$. (Note that this norm is equivalent to the standard L^2-norm, due to the assumption on ε and μ.) In addition, we make use of the maximal domains of the rotation curl and divergence div

$$H(\operatorname{curl},Q) = \{\varphi \in L^2(Q)^3 \mid \operatorname{curl}\varphi^{(i)} \in L^2(Q_i)^3,\ i=1,2, \qquad [\![\varphi \times \mathbf{n}_{\mathrm{int}}]\!] = 0\},$$
$$H(\operatorname{div},Q) = \{\varphi \in L^2(Q)^3 \mid \operatorname{div}\varphi^{(i)} \in L^2(Q_i),\ i=1,2, \qquad [\![\varphi \cdot \mathbf{n}_{\mathrm{int}}]\!] = 0\},$$

and the corresponding spaces $H_0(\operatorname{curl},Q)$ and $H_0(\operatorname{div},Q)$ with vanishing tangential or normal boundary traces, respectively. With this, we define the extended Maxwell operator $\widetilde{\mathbf{M}}$

$$\widetilde{\mathbf{M}} = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\operatorname{curl} \\ -\frac{1}{\mu}\operatorname{curl} & 0 \end{pmatrix}$$

on its domain

$$\begin{aligned}\mathcal{D}(\widetilde{\mathbf{M}}) &:= \{(\mathbf{E},\mathbf{H}) \in X_0 \mid \operatorname{curl}\mathbf{E}^{(i)}, \operatorname{curl}\mathbf{H}^{(i)} \in L^2(Q_i),\ [\![\mathbf{E}\times\mathbf{n}_{\mathrm{int}}]\!] = 0,\ \mathbf{E}\times\nu = 0 \text{ on } \partial Q, \quad i=1,2\} \\ &= H_0(\operatorname{curl},Q) \times \{\mathbf{H} \in L^2(Q)^3 \mid \operatorname{curl}\mathbf{H}^{(i)} \in L^2(Q_i)^3,\ i=1,2\},\end{aligned}$$

which neglects the magnetic transmission conditions. In particular, the curl in the second component is only applied piecewise on the subcuboids Q_i. Including the magnetic conditions, we obtain the usual Maxwell operator $\mathbf{M}$ on

$$\mathcal{D}(\mathbf{M}) = \mathcal{D}(\widetilde{\mathbf{M}}) \cap \{(\mathbf{E},\mathbf{H}) \in X_0 \mid [\![\mathbf{H}\times\mathbf{n}_{\mathrm{int}}]\!] = 0\} = H_0(\operatorname{curl},Q) \times H(\operatorname{curl},Q)$$

via the restriction

$$\mathbf{M} = \widetilde{\mathbf{M}}\big|_{\mathcal{D}(\mathbf{M})}. \tag{2.1}$$

Remark 2.1. We note that $\widetilde{\mathbf{M}}$ cannot be the generator of a strongly continuous semigroup on $\mathcal{D}(\widetilde{\mathbf{M}})$ since this would imply that for some $\rho > 0$ the map $\rho I - \widetilde{\mathbf{M}} \colon \mathcal{D}(\widetilde{\mathbf{M}}) \to X_0$ is bijective. However, by (2.1) and the generator property of $\mathbf{M}$, also $\rho I - \widetilde{\mathbf{M}} \colon \mathcal{D}(\mathbf{M}) \to X_0$ is bijective. This is a contradiction though, since $\mathcal{D}(\mathbf{M})$ is a strict subspace of $\mathcal{D}(\widetilde{\mathbf{M}})$. ♢

Additionally, we require more structure in order to prove regularity statements for (1.1) in piecewise Sobolev spaces. Inspired by [16,41], we consider the Hilbert space

$$\widehat{X}_0 := \{(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in L^2(Q)^6 \mid \operatorname{div}(\widetilde{\mathbf{E}}^{(i)}) \in L^2(Q_i),\ \mu\widetilde{\mathbf{H}} \in H_0(\operatorname{div}, Q),\ [\![\varepsilon\widetilde{\mathbf{E}} \cdot \mathbf{n}_{\mathrm{int}}]\!] \in H^{1/2}_{00}(F_{\mathrm{int}})\},$$
$$\|(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})\|^2_{\widehat{X}_0} := \|(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})\|^2 + \|\operatorname{div}(\mu\widetilde{\mathbf{H}})\|^2_{L^2(Q)} + \|[\![\varepsilon\widetilde{\mathbf{E}} \cdot \mathbf{n}_{\mathrm{int}}]\!]\|^2_{H^{1/2}_{00}(F_{\mathrm{int}})}, \tag{2.2}$$

which is intersected below with domains of certain powers of the Maxwell operator. For the surface current, we identify F_{int} with the square $S = (0,1)^2$ and use the two negative Laplacians

$$\mathcal{D}(-\Delta_2) := \{u \in H^2(S) \mid u(0,\cdot) = u(1,\cdot) = 0,\ \partial_2 u(\cdot,0) = \partial_2 u(\cdot,1) = 0\}, \tag{2.3a}$$
$$\mathcal{D}(-\Delta_3) := \{u \in H^2(S) \mid u(\cdot,0) = u(\cdot,1) = 0,\ \partial_1 u(0,\cdot) = \partial_1 u(1,\cdot) = 0\}, \tag{2.3b}$$

which are both positive definite and self-adjoint on $L^2(S)$. We can thus define the fractional powers $(-\Delta_j)^{\gamma/2}$ on domains

$$\mathcal{X}_j^\gamma := \mathcal{D}(-\Delta_j)^{\gamma/2} \tag{2.4}$$

for $\gamma \in \mathbb{R}$. These spaces are used to formulate the assumptions on the surface current. We elaborate further on the fractional domains in the following remark.

Remark 2.2. The domain of $(-\Delta_2)^{1/2}$ can be represented via

$$\mathcal{D}(-\Delta_2)^{1/2} = \{u \in H^1(S) \mid u(0,\cdot) = u(1,\cdot) = 0\}.$$

Using the trace method in interpolation theory, see Section 1.3.2 in [26], we can express the other arising fractional domains of $-\Delta_2$ as images of the trace operator $\mathrm{tr}_{F_{\mathrm{int}}}$ on F_{int} (with equivalence of norms). We have for $\Gamma_j = \{x \in \partial Q \mid x_j \in \{0,1\}\}$, $j \in \{2,3\}$,

$$\begin{aligned}
\mathcal{D}(-\Delta_2)^{1/4} &= \mathrm{tr}_{F_{\mathrm{int}}}\Big(\{u \in H^1(Q_2) \mid u = 0 \text{ on } \Gamma_2\}\Big),\\
\mathcal{D}(-\Delta_2)^{3/4} &= \mathrm{tr}_{F_{\mathrm{int}}}\Big(\{u \in H^1(Q_2) \mid u = 0 = \partial_1 u \text{ on } \Gamma_2,\ \partial_3 u = 0 \text{ on } \Gamma_3,\\
&\qquad \int_0^1 \big(\|\Delta u(x_1,\cdot)\|^2_{L^2(S)} + \|\partial_1 u(x_1,\cdot)\|^2_{H^1(S)}\big)\mathrm{d}x_1 < \infty\}\Big),\\
\mathcal{D}(-\Delta_2)^{5/4} &= \mathrm{tr}_{F_{\mathrm{int}}}\Big(\{u \in H^1(Q_2) \mid u = 0 = \partial_1 u = \Delta u \text{ on } \Gamma_2,\ \partial_3 u = \partial_1\partial_3 u = 0 \text{ on } \Gamma_3,\\
&\qquad \int_0^1 \big(\|\Delta u(x_1,\cdot)\|^2_{H^1(S)} + \|\Delta\partial_1 u(x_1,\cdot)\|^2_{L^2(S)}\big)\mathrm{d}x_1 < \infty\}\Big).
\end{aligned}$$

(The arising traces in the spaces on the right-hand side of the above equations are understood in the following sense: In the first line, $u(x_1,\cdot) = 0$ on $\{0,1\} \times [0,1]$ for almost all $x_1 \in (0,1)$. Analogous interpretations hold for the other lines.)

There is also a less precise but easier way to interpret the arising fractional domains. Avoiding technicalities regarding the regularity issues in taking traces, we have for $\epsilon > 0$

$$\{u \in H^{1/2+\epsilon}(S) \mid u(0,\cdot) = u(1,\cdot) = 0\} \subseteq \mathcal{D}(-\Delta_2)^{1/4} \subseteq H^{1/2}(S)$$

and similarly for the Neumann traces

$$\{u \in H^{3/2+\epsilon}(S) \mid u(0,\cdot) = u(1,\cdot) = 0, \partial_2 u(\cdot,0) = \partial_2 u(\cdot,1) = 0\} \subseteq \mathcal{D}(-\Delta_2)^{3/4} \subseteq H^{3/2}(S).$$

Since we only work on a convex Lipschitz domain, we may only conclude

$$\{u \in H^{5/2+\epsilon}(S) \ \mid u(0,\cdot) = u(1,\cdot) = 0, \Delta_2 u(0,\cdot) = \Delta_2 u(1,\cdot) = 0, \\ \partial_2 u(\cdot,0) = \partial_2 u(\cdot,1) = 0\} \subseteq \mathcal{D}(-\Delta_2)^{5/4} \subseteq H^2(S).$$

For Δ_3, we only change the boundary conditions and obtain the same observations. ◊

2.1. First-order regularity result

In order to state the wellposedness and regularity statements for (1.1), we introduce the necessary spaces for the surface current given by

$$V_{T,j} := \bigcap_{l=0}^{1} C^{1+l}([0,T], \mathcal{X}_j^{3/2-l}), \qquad j \in \{2,3\},\ T > 0, \tag{2.5}$$

and several state spaces for the solution, which we collect in Fig. 2. For the lowest order, we introduce

$$\widetilde{X}_1 := \mathcal{D}(\widetilde{\mathbf{M}}) \cap \widehat{X}_0, \qquad X_1 := \widetilde{X}_1 \cap \mathcal{D}(\mathbf{M}), \qquad \|\cdot\|_{\widetilde{X}_1}^2 := \|\cdot\|_{X_1}^2 := \|\cdot\|_{\widehat{X}_0}^2 + \|\widetilde{\mathbf{M}}\cdot\|_{L^2(Q)}^2. \tag{2.6}$$

This enables us to formulate our first main result which shows existence and uniqueness of piecewise H^1-regular solutions. We elaborate on the appearing compatibility conditions between the initial values in Sect. 3.

Theorem 2.3. *Let the initial data satisfy* $(\mathbf{E}_0, \mathbf{H}_0) \in \widetilde{X}_1$, $[\![\mathbf{H}_0 \times \mathbf{n}_{int}]\!] = j_{surf}(0)$, *and assume for the currents that* $(j_{surf}^2, j_{surf}^3) \in V_{T,3} \times V_{T,2}$, $\big(\frac{1}{\varepsilon}\mathbf{J}_\Sigma, 0\big) \in C^1([0,T], \widehat{X}_0) + C([0,T], X_1)$. *Then there is a unique solution* $(\mathbf{E}, \mathbf{H}) \in C^1\big([0,T], L^2(Q)\big) \cap C\big([0,T], PH^1(Q)\big)$ *of (1.1) with*

$$\sum_{j=0}^{1} \|(\mathbf{E}, \mathbf{H})\|_{C^j([0,T], PH^{1-j}(Q))} \le C\, \|(\mathbf{E}_0, \mathbf{H}_0)\|_{\widetilde{X}_1} \\ + C(1+T)\Big(\big\|\big(\tfrac{1}{\varepsilon}\mathbf{J}_\Sigma, 0\big)\big\|_{C^1([0,T],\widehat{X}_0) + C([0,T],X_1)} + \big\|(j_{surf}^2, j_{surf}^3)\big\|_{V_{T,3}\times V_{T,2}}\Big)$$

involving a uniform constant $C = C(\varepsilon, \mu, Q) > 0$.

Proof. (1) By Corollary 4.3, there is a function $\mathbf{J}_\mathbf{H}$ with $[\![\mathbf{J}_\mathbf{H}(t) \times \mathbf{n}_{\text{int}}]\!] = j_{\text{surf}}(t)$, $\mathbf{J}_\mathbf{H}(t) \cdot \nu = 0$ on ∂Q, $t \in [0,T]$, and

$$\sum_{l=0}^{1} \|\mathbf{J}_\mathbf{H}\|_{C^{1+l}([0,T], PH^{2-l}(Q))} \le C \big\|(j_{\text{surf}}^2, j_{\text{surf}}^3)\big\|_{V_{T,3}\times V_{T,2}}. \tag{2.7}$$

Choose $\mathbf{J}_{\mathbf{E}} = 0$, and define $\check{\mathbf{J}}$ as in (3.4a). By Corollary 4.3 and the precondition on $\mathbf{J}_\Sigma$, $\check{\mathbf{J}} \in C^1([0,T], \widehat{X}_0) + C([0,T], X_1)$ with

$$\begin{aligned}
\|\check{\mathbf{J}}\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)} &\leq C\Big(\|(\tfrac{1}{\varepsilon}\mathbf{J}_\Sigma, 0)\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)} + \sum_{i=1}^2 \Big\|\tfrac{1}{\varepsilon}\operatorname{curl}\mathbf{J}_{\mathbf{H}}^{(i)}\Big\|_{C^1([0,T],L^2(Q_i))} \\
&\quad + \|\partial_t \mathbf{J}_{\mathbf{H}}\|_{C^1([0,T],L^2(Q))} + \|[\![\operatorname{curl}\mathbf{J}_{\mathbf{H}}\cdot\mathbf{n}_{\mathrm{int}}]\!]\|_{C^1([0,T],H_{00}^{1/2}(F_{\mathrm{int}}))} \\
&\quad + \|\partial_t \operatorname{div}\mathbf{J}_{\mathbf{H}}\|_{C^1([0,T],L^2(Q))} + \sum_{i=1}^2 \Big(\Big\|\operatorname{curl}\tfrac{1}{\varepsilon}\operatorname{curl}\mathbf{J}_{\mathbf{H}}^{(i)}\Big\|_{C([0,T],L^2(Q_i))} \\
&\quad + \Big\|\operatorname{curl}\partial_t\mathbf{J}_{\mathbf{H}}^{(i)}\Big\|_{C([0,T],L^2(Q_i))}\Big)\Big) \\
&\leq C\Big(\|(\tfrac{1}{\varepsilon}\mathbf{J}_\Sigma, 0)\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)} + \|\mathbf{J}_{\mathbf{H}}\|_{C^2([0,T],PH^1(Q))} \\
&\quad + \|\mathbf{J}_{\mathbf{H}}\|_{C^1([0,T],PH^2(Q))}\Big).
\end{aligned}$$

Taking (2.7) into account, we obtain

$$\|\check{\mathbf{J}}\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)} \leq C\Big(\|(\tfrac{1}{\varepsilon}\mathbf{J}_\Sigma, 0)\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)} + \|(j_{\mathrm{surf}}^2, j_{\mathrm{surf}}^3)\|_{V_{T,3}\times V_{T,2}}\Big). \tag{2.8}$$

We then put $\widetilde{\mathbf{E}}_0 := \mathbf{E}_0 - \mathbf{J}_{\mathbf{E}}(0)$ and $\widetilde{\mathbf{H}}_0 := \mathbf{H}_0 - \mathbf{J}_{\mathbf{H}}(0)$, see (3.2). By construction $(\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0) \in X_1$. Proposition 3.6 then provides a unique solution $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in C^1([0,T], \widehat{X}_0) \cap C([0,T], X_1)$ of (3.3) with

$$\sum_{j=0}^1 \Big\|(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})\Big\|_{C^j([0,T],X_{1-j})} \leq C\Big(\Big\|(\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0)\Big\|_{X_1} + (1+T)\,\|\check{\mathbf{J}}\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)}\Big). \tag{2.9}$$

(2) Set $\mathbf{E} := \widetilde{\mathbf{E}} + \mathbf{J}_{\mathbf{E}}$ and $\mathbf{H} := \tilde{\mathbf{H}} + \mathbf{J}_{\mathbf{H}}$. Employing Lemma 3.1, $(\mathbf{E}, \mathbf{H}) \in C^1([0,T], L^2(Q)) \cap C([0,T], PH^1(Q))$, and by construction $(\mathbf{E}, \mathbf{H})$ solves (1.1), see also Sect. 3.1. Combining (2.7), (2.8), and (2.9), the asserted estimate finally follows. □

Remark 2.4. The regularity of j_{surf} is used below to construct regular extensions from F_{int} to Q, see Sect. 4.2. Certain combinations of spatial and temporal derivatives and the extensions then have to satisfy similar regularity requirements as the external (volume) current $\frac{1}{\varepsilon}\mathbf{J}_\Sigma$. This leaves several degrees of freedom in the choices for the regularity of j_{surf}. However, for the sake of presentation, we only elaborate on the notionally most convenient variant. ♢

2.2. Second-order regularity result

In the second result, we aim for solutions which are piecewise H^2-regular. We emphasize that due to the Lipschitz regularity of the boundary, in general one cannot expect the existence of higher order spatial derivatives. In order to derive such a result, we assume more regular surface currents, in particular in the spaces

$$W_{T,j} := \bigcap_{l=0}^2 C^{1+l}([0,T], \mathcal{X}_j^{5/2-l}), \tag{2.10}$$

for $T > 0$ fixed and $j \in \{2,3\}$. In addition, we need the spaces

$$\widehat{X}_1 := \{(\mathbf{E},\mathbf{H}) \in X_1 \mid \operatorname{div}(\mathbf{E}^{(i)}) \in H_{00}^1(Q_i),\ [\![\varepsilon\mathbf{E}\cdot\mathbf{n}_{\mathrm{int}}]\!] \in H_0^{3/2}(F_{\mathrm{int}}),\ \operatorname{div}(\mu\mathbf{H}) \in PH^1(Q)\}, \tag{2.11}$$

$$\widetilde{X}_2 := \{(\mathbf{E},\mathbf{H}) \in \mathcal{D}(\widetilde{\mathbf{M}}^2) \cap \widehat{X}_0 \mid \operatorname{div}(\mathbf{E}^{(i)}) \in H_{00}^1(Q_i),\ [\![\varepsilon\mathbf{E}\cdot\mathbf{n}_{\mathrm{int}}]\!] \in H_0^{3/2}(F_{\mathrm{int}}),\ \operatorname{div}(\mu\mathbf{H}) \in PH^1(Q)\},$$

$$X_2 := \widehat{X}_1 \cap \mathcal{D}(\mathbf{M}^2), \tag{2.12}$$

where the latter one allows for an embedding into $PH^2(Q)$, see Proposition 3.4. The spaces $\widehat{X}_1$ and X_2 are equipped with the norms

$$\|(\mathbf{E},\mathbf{H})\|_{\widehat{X}_1}^2 := \|(\mathbf{E},\mathbf{H})\|_{X_1}^2 + \sum_{i=1}^{2} \big(\|\operatorname{div}(\mathbf{E}^{(i)})\|_{H^1_{00}(Q_i)}^2 + \|\operatorname{div}(\mu\mathbf{H})\|_{H^1(Q_i)}^2\big) + \|[\![\varepsilon\mathbf{E}\cdot\mathbf{n}_{\text{int}}]\!]\|_{H_0^{3/2}(F_{\text{int}})}^2,$$
$$\|(\mathbf{E},\mathbf{H})\|_{X_2}^2 := \|(\mathbf{E},\mathbf{H})\|_{\widehat{X}_1}^2 + \|\mathbf{M}^2(\mathbf{E},\mathbf{H})\|^2.$$

The norm on $\widetilde{X}_2$ is defined in an analogous way. With this, we state our second main result.

Theorem 2.5. *Let the initial data satisfy*

$$(\mathbf{E}_0,\mathbf{H}_0)\in\widetilde{X}_2, \quad [\![\mathbf{H}_0\times\mathbf{n}_{int}]\!] = j_{surf}(0), \quad [\![\tfrac{1}{\mu}\operatorname{curl}\mathbf{E}_0\times\mathbf{n}_{int}]\!] = -\partial_t j_{surf}(0),$$

and assume for the currents that
$(j_{surf}^2, j_{surf}^3)\in W_{T,3}\times W_{T,2}$, $\big(\frac{1}{\varepsilon}\mathbf{J}_\Sigma, 0\big)\in C^1\big([0,T],\widehat{X}_1\big)$. *Then there is a unique solution* $(\mathbf{E},\mathbf{H})$ *of (1.1) with*

$$(\mathbf{E},\mathbf{H})\in C^2\big([0,T],L^2(Q)\big)\cap C^1\big([0,T],PH^1(Q)\big)\cap C\big([0,T],PH^2(Q)\big)$$

and it holds

$$\sum_{j=0}^{2}\|(\mathbf{E},\mathbf{H})\|_{C^j([0,T],PH^{2-j}(Q))} \le C\,\|(\mathbf{E}_0,\mathbf{H}_0)\|_{\widetilde{X}_2}$$
$$+ C(1+T)\Big(\big\|\big(\tfrac{1}{\varepsilon}\mathbf{J}_\Sigma,0\big)\big\|_{C^1([0,T],\widehat{X}_1)} + \big\|(j_{surf}^2, j_{surf}^3)\big\|_{W_{T,3}\times W_{T,2}}\Big)$$

with a uniform constant $C = C(\varepsilon,\mu,Q) > 0$.

Proof. We combine Lemma 3.1, Proposition 3.4, Proposition 3.7 and Corollary 4.4. □

As explained in Remark 2.4, we do not state all admissible settings for j_{surf} and $\mathbf{J}_\Sigma$ which lead to the same result as above.

2.3. Higher-order regularity result

Our last main result is motivated from the error analysis of second-order time integration schemes for Maxwell equations. Here, it is necessary to control derivatives up to order three in space or time. For the surface current, we introduce the spaces

$$Y_{T,j} := \bigcap_{l=0}^{3} C^{1+l}([0,T],\mathcal{X}_j^{5/2-l}), \qquad Z_{T,1} := \bigcap_{l=0}^{2} C^{1+l}([0,T],\mathcal{X}_3^{1/2-l}\times\mathcal{X}_2^{1/2-l}), \tag{2.13a}$$

$$Z_{T,2} := \bigcap_{l=0}^{2} C^{1+l}([0,T],\mathcal{X}_2^{1/2-l}\times\mathcal{X}_3^{1/2-l}), \tag{2.13b}$$

for $T>0$ and $j\in\{2,3\}$. As state spaces for the electric and magnetic fields, we employ

$$\widetilde{X}_3 = \mathcal{D}(\widetilde{\mathbf{M}}^3)\cap\widetilde{X}_2, \qquad X_3 = \widetilde{X}_3\cap\mathcal{D}(\mathbf{M}^3). \tag{2.14}$$

Additionally, we use the sum

$$\{\{f\}\} := f^{(1)}|_{F_{\text{int}}} + f^{(2)}|_{F_{\text{int}}}, \tag{2.15}$$

for a function f with well-defined traces at the interface F_{int}. We are now in the position to state the final main result.

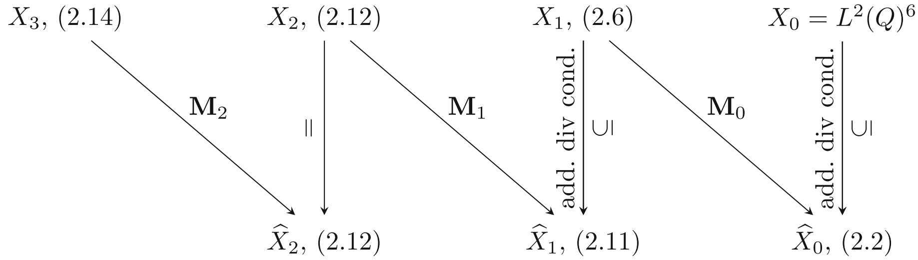

FIG. 2. Connection of spaces. Vertical lines are restrictions and diagonal lines indicate the domain of the operator, i.e., for $\mathbf{M}_i$ on ground spaces $\widehat{X}_i$ it holds $\mathcal{D}(\mathbf{M}_i) = X_{i+1}$, see Lemma 3.5

Theorem 2.6. *Let the initial data satisfy*

$$(\mathbf{E}_0, \mathbf{H}_0) \in \widetilde{X}_3, \qquad [\![\mathbf{H}_0 \times \mathbf{n}_{int}]\!] = j_{surf}(0),$$
$$[\![\tfrac{1}{\mu} \operatorname{curl} \mathbf{E}_0 \times \mathbf{n}_{int}]\!] = -\partial_t j_{surf}(0), \qquad [\![\tfrac{1}{\mu} \operatorname{curl} \tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{H}_0 \times \mathbf{n}_{int}]\!] = -\partial_t^2 j_{surf}(0)$$

and assume for the currents that $(j^2_{surf}, j^3_{surf}) \in Y_{T,3} \times Y_{T,2}$, $\big(\frac{1}{\varepsilon}\mathbf{J}_\Sigma, 0\big) \in C^1\big([0,T], X_2\big) \cap C^2\big([0,T], \widehat{X}_0\big)$. *In addition, let* $\operatorname{div} j_{surf} \in C^1([0,T], H_0^{3/2}(F_{int}))$ *and further*

$$g := \begin{pmatrix} \frac{1}{2}\{\{\frac{1}{\varepsilon\mu}\}\}\partial_2 \operatorname{div} j_{surf} - \frac{1}{2}\{\{\frac{\varepsilon}{\mu}\}\}\Delta j^3_{surf} - \partial_t^2 j^2_{surf} \\ \frac{1}{2}\{\{\frac{1}{\varepsilon\mu}\}\}\partial_3 \operatorname{div} j_{surf} - \frac{1}{2}\{\{\frac{\varepsilon}{\mu}\}\}\Delta j^2_{surf} - \partial_t^2 j^3_{surf} \end{pmatrix} \in Z_{T,1}, \quad \tilde{g} := \{\{\tfrac{1}{\varepsilon}\}\} \begin{pmatrix} -(-\Delta_2)^{1/2}\partial_t j^3_{surf} \\ (-\Delta_3)^{1/2}\partial_t j^2_{surf} \end{pmatrix} \in Z_{T,2}.$$

Then there is a unique solution $(\mathbf{E}, \mathbf{H})$ *of (1.1) with*

$$(\mathbf{E}, \mathbf{H}) \in C^3\big([0,T], L^2(Q)\big) \cap C^2\big([0,T], PH^1(Q)\big) \cap C^1\big([0,T], PH^2(Q)\big) \cap C\big([0,T], \widetilde{X}_3\big)$$

and it holds

$$\begin{aligned} &\sum_{j=0}^{2} \|(\mathbf{E}, \mathbf{H})\|_{C^{1+j}([0,T], PH^{2-j}(Q))} + \|(\mathbf{E}, \mathbf{H})\|_{C([0,T], \widetilde{X}_3)} \\ &\quad \le C\,\|(\mathbf{E}_0, \mathbf{H}_0)\|_{\widetilde{X}_3} + C(1+T)\Big(\big\|(\tfrac{1}{\varepsilon}\mathbf{J}_\Sigma, 0)\big\|_{C^1([0,T],X_2) \cap C^2([0,T],\widehat{X}_0)} \\ &\quad + \|\operatorname{div} j_{surf}\|_{C^1([0,T], H_0^{3/2}(F_{int}))} + \big\|(j^2_{surf}, j^3_{surf})\big\|_{Y_{T,3} \times Y_{T,2}} + \|g\|_{Z_{T,1}} + \|\tilde{g}\|_{Z_{T,2}}\Big) \end{aligned}$$

with a uniform constant $C = C(\varepsilon, \mu, Q) > 0$.

Proof. We combine Lemma 3.1, Proposition 3.4, Proposition 3.8, and Corollary 4.5. □

3. Transformation and analytical framework

In this section, we replace the original system (1.1) by a (nonphysical) shifted version for which we can show wellposedness and regularity results by means of semigroup theory. To motivate the shifts, we consider the interface condition in (1.2) for a smooth solution of (1.1). We differentiate in time and obtain

$$\partial_t j_{\mathrm{surf}}(t) = [\![\partial_t \mathbf{H} \times \mathbf{n}_{\mathrm{int}}]\!] = -[\![\tfrac{1}{\mu} \operatorname{curl} \mathbf{E} \times \mathbf{n}_{\mathrm{int}}]\!], \tag{3.1a}$$
$$\partial_t^2 j_{\mathrm{surf}}(t) = -[\![\tfrac{1}{\mu} \operatorname{curl} \partial_t \mathbf{E} \times \mathbf{n}_{\mathrm{int}}]\!] = -[\![\tfrac{1}{\mu} \operatorname{curl} \tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{H} \times \mathbf{n}_{\mathrm{int}}]\!], \tag{3.1b}$$

where we used the continuity conditions of $\frac{1}{\varepsilon}\mathbf{J}_\Sigma$ across the interface. The shifts are chosen in such a way that we can work with homogeneous interface conditions, i.e., the modified fields satisfy (3.1) with zero left-hand side, which means in particular using the state spaces $\widehat{X}_i$ instead of $\widetilde{X}_i$, see Sect. 2. Eventually, this enables us to conclude wellposedness of the system (1.1) in piecewise Sobolev spaces.

3.1. Transformation to homogeneous interface conditions

Let $(\mathbf{E}, \mathbf{H})$ be a solution of (1.1), and define the modified fields $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$

$$\widetilde{\mathbf{H}} = \mathbf{H} - \mathbf{J}_{\mathbf{H}}, \qquad \widetilde{\mathbf{E}} = \mathbf{E} - \mathbf{J}_{\mathbf{E}}, \tag{3.2}$$

with piecewise sufficiently regular currents $\mathbf{J}_{\mathbf{E}}$, $\mathbf{J}_{\mathbf{H}}$. The latter are chosen such that in particular

$$[\![\widetilde{\mathbf{H}} \times \mathbf{n}_{\text{int}}]\!] = 0 = [\![\widetilde{\mathbf{E}} \times \mathbf{n}_{\text{int}}]\!]$$

on F_{int}, meaning that $(\widetilde{\mathbf{E}}(t), \widetilde{\mathbf{H}}(t))$ satisfies the transmission conditions in $\mathcal{D}(\mathbf{M})$ for $t \geq 0$. We formally derive the evolution equations for the modified fields in the following and make these calculations rigorous in Sect. 4. We insert the modified fields into (1.1) and obtain

$$\partial_t \widetilde{\mathbf{E}}^{(i)} = \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \widetilde{\mathbf{H}}^{(i)} - \check{\mathbf{J}}^{1,(i)}, \qquad \partial_t \widetilde{\mathbf{H}}^{(i)} = -\frac{1}{\mu^{(i)}} \operatorname{curl} \widetilde{\mathbf{E}}^{(i)} - \check{\mathbf{J}}^{2,(i)}, \tag{3.3a}$$

$$\operatorname{div}\big(\varepsilon \widetilde{\mathbf{E}}^{(i)}\big) = \widetilde{\rho}_{\mathbf{E}}^{(i)}, \qquad \operatorname{div}\big(\mu \widetilde{\mathbf{H}}^{(i)}\big) = \widetilde{\rho}_{\mathbf{H}}^{(i)}, \tag{3.3b}$$

on Q_i with currents $\check{\mathbf{J}}$ and charges $\widetilde{\rho}$ given by

$$\check{\mathbf{J}}^{1,(i)} = \frac{1}{\varepsilon^{(i)}} \mathbf{J}_{\Sigma}^{(i)} - \frac{1}{\varepsilon^{(i)}} \operatorname{curl} \mathbf{J}_{\mathbf{H}}^{(i)} + \partial_t \mathbf{J}_{\mathbf{E}}^{(i)}, \qquad \check{\mathbf{J}}^{2,(i)} = \partial_t \mathbf{J}_{\mathbf{H}}^{(i)} + \frac{1}{\mu^{(i)}} \operatorname{curl} \mathbf{J}_{\mathbf{E}}^{(i)}, \tag{3.4a}$$

$$\widetilde{\rho}_{\mathbf{E}}^{(i)} = \rho^{(i)} - \operatorname{div} \mathbf{J}_{\mathbf{E}}^{(i)}, \qquad \widetilde{\rho}_{\mathbf{H}}^{(i)} = -\operatorname{div} \mathbf{J}_{\mathbf{H}}^{(i)}. \tag{3.4b}$$

Further, the modified solutions satisfy the boundary conditions

$$\widetilde{\mathbf{E}} \times \nu = -\mathbf{J}_{\mathbf{E}} \times \nu, \quad \widetilde{\mathbf{H}} \cdot \nu = -\mathbf{J}_{\mathbf{H}} \cdot \nu \qquad \text{on } \partial Q.$$

Depending on the regularity of the surface current j_{surf}, we discuss transmission and regularity properties of the modified solution. To employ a semigroup approach, we formulate (3.3) as an evolution equation. We define the vectors

$$\mathbf{w} = \begin{pmatrix} \widetilde{\mathbf{E}} \\ \widetilde{\mathbf{H}} \end{pmatrix}, \qquad \check{\mathbf{J}} = \begin{pmatrix} \check{\mathbf{J}}^1 \\ \check{\mathbf{J}}^2 \end{pmatrix},$$

and consider the Maxwell operator $\mathbf{M}$ defined in (2.1) on $\mathcal{D}(\mathbf{M})$. This yields an equivalent formulation of (3.3) in $X_0 = L^2(Q)^6$ as

$$\mathbf{w}'(t) = \mathbf{M}\mathbf{w}(t) - \check{\mathbf{J}}(t), \qquad t \geq 0. \tag{3.5}$$

In the following, we discuss wellposedness for (3.5) and consider $\check{\mathbf{J}}$ as a given quantity. From the regularity of j_{surf} and the extension results in Sect. 4, we can then conclude the results in Sect. 2.

3.2. Functional analytic framework for shifted Maxwell system

Recall the spaces X_1, X_2 defined in Sect. 2. We next show that fields in the space X_1 are piecewise H^1-regular.

Lemma 3.1. *The space X_1 embeds into $PH^1(Q)^6$.*

Proof. (1) Let $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in X_1$. In view of Proposition 9.8 in [41], it suffices to analyze the magnetic field component. We next use a well-known technique to deduce the existence of a function $\psi \in PH^2(Q)$ with $\Delta\psi^{(i)} = \operatorname{div} \widetilde{\mathbf{H}}^{(i)}$ and $\nabla\psi \cdot \nu = 0$ on ∂Q from Proposition 8.2 in [41].

Set $V := \{w \in H^1(Q) \mid [w] = 0\}$, where $[\cdot]$ denotes the mean of an integrable function on Q. Note that V is a closed subspace of $H^1(Q)$. Combining the generalized Poincaré inequality with the Lax–Milgram Lemma, there is a unique function $\psi \in V$ with

$$\int_Q \mu(\nabla\psi) \cdot (\nabla\varphi) \mathrm{d}x = -\int_Q \operatorname{div}(\mu\widetilde{\mathbf{H}})\varphi \mathrm{d}x, \qquad \varphi \in V.$$

Using that the mean of $\operatorname{div}(\mu\widetilde{\mathbf{H}})$ is zero on Q, we then infer

$$\int_Q \mu(\nabla\psi) \cdot (\nabla\varphi)\mathrm{d}x = \int_Q \mu(\nabla\psi) \cdot \nabla(\varphi - [\varphi])\mathrm{d}x = -\int_Q \operatorname{div}(\mu\widetilde{\mathbf{H}})(\varphi - [\varphi])\mathrm{d}x = -\int_Q \operatorname{div}(\mu\widetilde{\mathbf{H}})\varphi \mathrm{d}x$$

for all $\varphi \in H^1(Q)$. In particular, the formula

$$\int_Q \mu^2\psi\varphi + \mu(\nabla\psi) \cdot (\nabla\varphi)\mathrm{d}x = \int_Q (\mu^2\psi - \operatorname{div}(\mu\widetilde{\mathbf{H}}))\varphi \mathrm{d}x, \qquad \varphi \in H^1(Q),$$

is valid. We then set $\Psi := \mu\psi \in PH^1(Q)$. Let also $\varphi \in H^1(Q)$, and put $\Phi := \mu\varphi$ as well as $f := \Psi - 1/\mu \operatorname{div}(\mu\widetilde{\mathbf{H}})$. By construction, the identities

$$[\![\tfrac{1}{\mu}\Psi]\!] = 0, \qquad \sum_{i=1}^{2} \int_{Q_i} \Psi^{(i)}\Phi^{(i)} + \tfrac{1}{\mu^{(i)}}(\nabla\Psi^{(i)}) \cdot (\nabla\Phi^{(i)})\mathrm{d}x = \int_Q f\Phi \mathrm{d}x$$

are then valid. Since $\varphi \in H^1(Q)$ is chosen arbitrarily, we infer the last formula for all functions $\Phi \in PH^1(Q)$ with $[\![\frac{1}{\mu}\Phi]\!] = 0$. Now Proposition 8.2 in [41] implies that ψ belongs to $PH^2(Q)$ with $\nabla\psi \cdot \nu = 0$ on ∂Q, and $[\![\mu\partial_1\psi]\!] = 0$ on F_{int}. The construction of ψ additionally implies the desired formula $\operatorname{div}(\mu\nabla\psi) = \operatorname{div}(\mu\widetilde{\mathbf{H}})$ on Q. Together with Proposition 8.2 in [41], we additionally obtain

$$\|\psi\|_{PH^2(Q)} \le C\big(\|\psi\|_{L^2(Q)} + \|\operatorname{div}(\mu\widetilde{\mathbf{H}})\|_{L^2(Q)}\big), \tag{3.6}$$

with a uniform constant $C = C(\mu, Q) > 0$.

(2) We next estimate $\|\psi\|_{L^2(Q)}$. Using the generalized Poincaré inequality as well as an integration by parts, we conclude

$$\|\psi\|^2_{L^2(Q)} \le \tfrac{C_P}{\delta}\|\sqrt{\mu}\nabla\psi\|^2_{L^2(Q)} = -\tfrac{C_P}{\delta}\int_Q \operatorname{div}(\mu\widetilde{\mathbf{H}})\psi \mathrm{d}x \le \tfrac{C_P}{\delta}\|\operatorname{div}(\mu\widetilde{\mathbf{H}})\|_{L^2(Q)}\|\psi\|_{L^2(Q)},$$

where C_P is the Poincaré constant on Q, and $\delta > 0$ is a lower bound for μ. We hence conclude

$$\|\psi\|_{L^2(Q)} \le \tfrac{C_P}{\delta}\|\operatorname{div}(\mu\widetilde{\mathbf{H}})\|_{L^2(Q)}. \tag{3.7}$$

(3) Due to the choice of ψ, the difference $\widetilde{\mathbf{H}} - \nabla\psi$ belongs to the space

$$\{\check{\mathbf{H}} \in H(\operatorname{curl}, Q) \mid \operatorname{div}(\mu\check{\mathbf{H}}) = 0,\ \mu\check{\mathbf{H}} \cdot \nu = 0 \text{ on } \partial Q\}.$$

Proposition 9.7 in [41] consequently yields that $\widetilde{\mathbf{H}} - \nabla\psi$ belongs to $PH^1(Q)^3$ with

$$\|\widetilde{\mathbf{H}} - \nabla\psi\|_{PH^1(Q)} \le C\big(\|\widetilde{\mathbf{H}} - \nabla\psi\|_{L^2(Q)} + \|\operatorname{curl}\widetilde{\mathbf{H}}\|_{L^2(Q)}\big).$$

Combining (3.6) and (3.7), we conclude $\|\widetilde{\mathbf{H}}\|_{PH^1(Q)} \le C\|(0, \widetilde{\mathbf{H}})\|_{X_1}$. □

We next establish piecewise H^2-regularity for fields in X_2. To that end, we start with the first component of the magnetic field. Note that we only sketch the relevant arguments here, as we modify the proof for Lemma 9.15 in [41] in a straightforward way, see also Lemma 3.7 in [21].

Lemma 3.2. *Let* $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in X_2$. *Then* $\widetilde{\mathbf{H}}_1$ *belongs to* $PH^2(Q)$ *with*

$$\|\widetilde{\mathbf{H}}_1\|_{PH^2(Q)} \le C\|(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})\|_{X_2},$$

involving a uniform constant $C = C(\varepsilon, \mu, Q) > 0$.

Proof. (1) We define

$$\begin{aligned} V &:= \{\varphi \in PH^1(Q) \mid [\![\mu\varphi]\!] = 0,\ \varphi = 0 \text{ on } \Gamma_1\}, \\ W &:= \{\varphi \in V \mid \varphi^{(i)} \text{ is smooth on } \overline{Q_i},\ \operatorname{supp}(\varphi) \subseteq (-1,1)\times[0,1]\times[0,1]\}. \end{aligned}$$

Note that W is dense in V in the norm of $PH^1(Q)$. We also use the subcuboids

$$Q_{1,n} := (-1+\tfrac{1}{n}, -\tfrac{1}{n}) \times (\tfrac{1}{n}, 1-\tfrac{1}{n})^2, \qquad Q_{2,n} := (\tfrac{1}{n}, 1-\tfrac{1}{n})^3,$$

for $n \ge 3$. Moreover, we denote by $\Gamma_{j,n}^{(i)}$, for $j \in \{1,2,3\}$ and $i \in \{1,2\}$, the boundary parts of $Q_{i,n}$ with normal vector e_j. We note that $\widetilde{\mathbf{H}}$ is H^2-regular on $Q_{i,n}$. (This is a consequence of $\operatorname{curl}\operatorname{curl}\widetilde{\mathbf{H}}^{(i)} \in L^2(Q_i)$ and standard elliptic regularity theory.)

(2) Let $\varphi \in W$. Integrating by parts, we first obtain

$$\sum_{i=1}^{2}\int_{Q_i} \mu^{(i)}(\nabla\widetilde{\mathbf{H}}_1^{(i)})\cdot(\nabla\varphi^{(i)})\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{2}\left(\int_{Q_{i,n}} -\mu^{(i)}(\Delta\widetilde{\mathbf{H}}_1^{(i)})\varphi^{(i)}\mathrm{d}x + \int_{\partial Q_{i,n}} \mu^{(i)}(\nabla\widetilde{\mathbf{H}}_1^{(i)}\cdot\nu^{(i)})\varphi^{(i)}\mathrm{d}\sigma\right).$$

We next analyze the last summand on the right-hand side. Inserting $\operatorname{curl}\widetilde{\mathbf{H}}$ and $\operatorname{div}(\mu\widetilde{\mathbf{H}})$, we infer

$$\begin{aligned} \sum_{i=1}^{2}\int_{\partial Q_{i,n}} \mu^{(i)}(\nabla\widetilde{\mathbf{H}}_1^{(i)}\cdot\nu^{(i)})\varphi^{(i)}\mathrm{d}\sigma = \sum_{i=1}^{2}\Bigg(&\int_{\Gamma_{1,n}^{(i)}} \nu_1^{(i)}(\operatorname{div}(\mu\widetilde{\mathbf{H}}^{(i)}) - \mu\partial_2\widetilde{\mathbf{H}}_2^{(i)} - \mu\partial_3\widetilde{\mathbf{H}}_3^{(i)})\varphi^{(i)}\mathrm{d}\sigma \\ &- \int_{\Gamma_{2,n}^{(i)}} \mu\nu_2^{(i)}(\operatorname{curl}\widetilde{\mathbf{H}}^{(i)})_3\varphi^{(i)}\mathrm{d}\sigma + \int_{\Gamma_{3,n}^{(i)}} \mu\nu_3^{(i)}(\operatorname{curl}\widetilde{\mathbf{H}}^{(i)})_2\varphi^{(i)}\mathrm{d}\sigma \\ &+ \int_{\Gamma_{2,n}^{(i)}} \mu\nu_2^{(i)}(\partial_1\widetilde{\mathbf{H}}_2^{(i)})\varphi^{(i)}\mathrm{d}\sigma + \int_{\Gamma_{3,n}^{(i)}} \mu\nu_3^{(i)}(\partial_1\widetilde{\mathbf{H}}_3^{(i)})\varphi^{(i)}\mathrm{d}\sigma\Bigg). \end{aligned}$$

Taking the boundary condition $\operatorname{curl}\widetilde{\mathbf{H}} \times \nu = 0$ on ∂Q into account, the second and third integral terms on the right-hand side converge to zero as $n \to \infty$. Combining Green's formula for curl with Lemma 9.14 in [41], one can moreover show that

$$\lim_{n\to\infty}\sum_{i=1}^{2}\left(\int_{\Gamma_{1,n}^{(i)}} -\nu_1^{(i)}(\mu\partial_2\widetilde{\mathbf{H}}_2^{(i)} + \mu\partial_3\widetilde{\mathbf{H}}_3^{(i)})\varphi\mathrm{d}\sigma + \int_{\Gamma_{2,n}^{(i)}} \mu\nu_2^{(i)}(\partial_1\widetilde{\mathbf{H}}_2^{(i)})\varphi\mathrm{d}\sigma + \int_{\Gamma_{3,n}^{(i)}} \mu\nu_3^{(i)}(\partial_1\widetilde{\mathbf{H}}_3^{(i)})\varphi\mathrm{d}\sigma\right) = 0.$$

Altogether, we conclude

$$\sum_{i=1}^{2}\int_{Q_i} \mu^{(i)}(\nabla\widetilde{\mathbf{H}}_1^{(i)})\cdot(\nabla\varphi^{(i)})\mathrm{d}x = \sum_{i=1}^{2}\int_{Q_i} -\mu^{(i)}(\Delta\widetilde{\mathbf{H}}_1^{(i)})\varphi^{(i)}\mathrm{d}x - \int_{F_{\mathrm{int}}} [\![\tfrac{1}{\mu}\operatorname{div}(\mu\widetilde{\mathbf{H}})]\!]\mu\varphi\mathrm{d}\sigma.$$

By density of W in V, this formula is also valid for all $\varphi \in V$.

(3) We next use the Neumann–Laplacian Δ_N on $F_{\text{int}} \cong (0,1)^2$. Note that $I - \Delta_N$ is positive definite and self-adjoint on $L^2(F_{\text{int}})$. As a result, it has well-defined positive definite and self-adjoint fractional powers $(I - \Delta_N)^\gamma$, $\gamma > 0$. The latter generate analytic semigroups $(e^{-t(I-\Delta_N)^\gamma})_{t \geq 0}$.

Let $\chi : [-1,1] \to [0,1]$ be a smooth cutoff function with $\chi = 1$ on $[-1/4, 1/4]$ and $\operatorname{supp} \chi \subset [-1/2, 1/2]$. We put

$$\psi^{(1)}(x) := \chi(x_1) x_1 (e^{-x_1(I-\Delta_N)^{1/2}} g)(x_2, x_3), \qquad \psi^{(2)}(x) := 0,$$

with $g := -[\![\frac{1}{\mu} \operatorname{div}(\mu \widetilde{\mathbf{H}})]\!] \in H^{1/2}(F_{\text{int}}) = \mathcal{D}(I - \Delta_N)^{1/4}$. Similar arguments as in the proof of Lemma 4.1 then imply that $\psi \in PH^2(Q)$, $\psi = 0$ on $\Gamma_1 \cup F_{\text{int}}$, $[\![\nabla \psi \cdot \mathbf{n}_{\text{int}}]\!] = -g$, and

$$\|\psi\|_{PH^2(Q)} \leq C \|g\|_{H^{1/2}(F_{\text{int}})} \leq C \|\operatorname{div}(\mu \widetilde{\mathbf{H}})\|_{PH^1(Q)},$$

with a uniform constant $C > 0$.

(4) Proposition 8.1 in [41] provides a function $\Psi \in PH^2(Q)$ with $\Psi = 0$ on Γ_1, $[\![\mu \Psi]\!] = 0$ and

$$\sum_{i=1}^{2} \int_{Q_i} \mu^{(i)} (\nabla \Psi^{(i)}) \cdot (\nabla \varphi^{(i)}) \mathrm{d}x = \sum_{i=1}^{2} \int_{Q} \left(-\mu^{(i)} \Delta \widetilde{\mathbf{H}}_1^{(i)} + \mu^{(i)} \Delta \psi^{(i)} \right) \varphi^{(i)} \mathrm{d}x, \quad \varphi \in V.$$

By uniqueness, $\widetilde{\mathbf{H}}_1 = \Psi + \psi \in PH^2(Q)$. The asserted energy estimate is a consequence of the estimate in Proposition 8.1 for Ψ, the bound for ψ, and the identity $\operatorname{curl} \operatorname{curl} \widetilde{\mathbf{H}}^{(i)} = -\Delta \widetilde{\mathbf{H}}^{(i)} + \nabla \operatorname{div} \widetilde{\mathbf{H}}^{(i)}$. □

We continue with the remaining magnetic field components.

Lemma 3.3. *Let $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in X_2$. Then $\widetilde{\mathbf{H}}_2$ and $\widetilde{\mathbf{H}}_3$ belong to $PH^2(Q)$ with*

$$\|(\widetilde{\mathbf{H}}_2, \widetilde{\mathbf{H}}_3)\|_{PH^2(Q)} \leq C \|(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})\|_{X_2},$$

involving a uniform constant $C = C(\varepsilon, \mu, Q) > 0$.

Proof. In the presence of Lemmas 3.1–3.2, the proof for Lemma 9.16 in [41] implies also in our setting that $\widetilde{\mathbf{H}}_2$ and $\widetilde{\mathbf{H}}_3$ are elements of $PH^2(Q)$ with

$$\|\widetilde{\mathbf{H}}_j\|_{PH^2(Q)} \leq C \left(\sum_{i=1}^{2} \|\Delta \widetilde{\mathbf{H}}_j^{(i)}\|_{L^2(Q_i)} + \|\operatorname{curl} \widetilde{\mathbf{H}}\|_{PH^1(Q)} + \|\widetilde{\mathbf{H}}_1\|_{PH^2(Q)} \right).$$

(Note here that $(\frac{1}{\varepsilon} \operatorname{curl} \widetilde{\mathbf{H}}, 0) \in X_1$.) Lemmas 3.1–3.2, as well as the identity $\operatorname{curl} \operatorname{curl} \widetilde{\mathbf{H}}^{(i)} = -\Delta \widetilde{\mathbf{H}}^{(i)} + \nabla \operatorname{div} \widetilde{\mathbf{H}}^{(i)}$, then yield the asserted inequality. □

In view of Theorem 9.17 in [41], the electric field component of each vector $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in X_2$ is piecewise H^2-regular, and the PH^2-norm can be estimated in terms of the X_2-norm of $(\widetilde{\mathbf{E}}, 0)$. We have consequently established:

Proposition 3.4. *The space X_2 embeds continuously into $PH^2(Q)^6$.*

3.3. Wellposedness of the shifted Maxwell system

Let $\mathbf{M}_i$ denote the part of $\mathbf{M}$ in $\widehat{X}_i$ for $i \in \{0, 1, 2\}$ defined in Sect. 2, with the notation $\widehat{X}_2 := X_2$. We first show that the domain of $\mathbf{M}_i$ coincides with X_{i+1}. This turns out to be useful for the formulation of the wellposedness statements for (3.5).

Observe first that $\mathbf{M}(\mathcal{D}(\mathbf{M}) \cap \widehat{X}_0) \subset \widehat{X}_0$, and hence $\mathcal{D}(\mathbf{M}_0) = X_1$. We further note that $\mathcal{D}(\mathbf{M}_1) \subset \mathcal{D}(\mathbf{M}^2) \cap \widehat{X}_1$ by definition of $\widehat{X}_1$. Additionally, $\mathbf{M}(\mathcal{D}(\mathbf{M}^2) \cap \widehat{X}_1) \subset \mathcal{D}(\mathbf{M}) \cap \widehat{X}_1 = \widehat{X}_1$, and thus $\mathcal{D}(\mathbf{M}_1) = X_2$. The remaining claim can be verified in the same way.

We next derive wellposedness of the shifted Maxwell equations (3.5) in $\widehat{X}_0$, $\widehat{X}_1$, and $X_2 = \widehat{X}_2$. To that end, we modify arguments in the proofs of Proposition 2.3 from [17] and Proposition 9.22 from [41].

Lemma 3.5. *Let $j \in \{0, 1, 2\}$. The part $\mathbf{M}_j$ of $\mathbf{M}$ in $\widehat{X}_j$ generates a contractive C_0-semigroup on $\widehat{X}_j$. It is the restriction of $(e^{t\mathbf{M}})_{t\geq 0}$ to $\widehat{X}_j$.*

Proof. (1) We first deal with the case $j = 0$. It suffices to show that $\mathbf{M}_0$ is dissipative on $\widehat{X}_0$. By the Lumer–Phillips Theorem, $\mathbf{M}_0$ then generates a contractive strongly continuous semigroup on $\widehat{X}_0$. Note that Proposition 3.5 in [21] shows that $\mathbf{M}$ is skew-adjoint and generates a unitary C_0-semigroup on $X_0 = L^2(Q)^6$. Due to the unique solvability of the shifted Maxwell equations, we then consequently infer also the asserted restriction statement.

As $\mathbf{M}(\mathcal{D}(\mathbf{M}) \cap \widehat{X}_0) \subset \widehat{X}_0$, we conclude that $(\lambda I - \mathbf{M})^{-1}$ leaves $\widehat{X}_0$ invariant for $\lambda \in \mathbb{R} \setminus \{0\}$. Hence, $\lambda I - \mathbf{M}_0 : \mathcal{D}(\mathbf{M}_0) \to \widehat{X}_0$ is bijective for $\lambda \in \mathbb{R} \setminus \{0\}$.

Let $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(\mathbf{M}_0)$, and put $(u, v) := \mathbf{M}_0(\mathbf{E}, \mathbf{H})$. We denote by $(\cdot, \cdot)_{\widehat{X}_0}$ the canonical inner product inducing the norm in (2.2). Note that $\operatorname{div}(\varepsilon u) = \operatorname{div}(\mu v) = 0$ on Q, and $[\![\varepsilon u \cdot \mathbf{n}_{\text{int}}]\!] = 0$. In view of the skew-adjointness of $\mathbf{M}$ on X_0, we then infer that $(\mathbf{M}_0(\mathbf{E}, \mathbf{H}), (\mathbf{E}, \mathbf{H}))_{\widehat{X}_0} = 0$, whence $\mathbf{M}_0$ is dissipative and the asserted generator property is shown.

(2) We next restrict ourselves to the case $j = 1$, as the remaining one $j = 2$ can be obtained by a straightforward modification. In view of subspace theory for semigroups, see Paragraph II.2.3 in [18], it suffices to show that $(e^{t\mathbf{M}})_{t\geq 0}$ leaves $\widehat{X}_1$ invariant and that it is continuous on it. Let $(u, v) \in \widehat{X}_1$, and put $(\widetilde{\mathbf{E}}(t), \widetilde{\mathbf{H}}(t)) := e^{t\mathbf{M}}(u, v)$, $t \geq 0$. Then $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in C([0, \infty), \mathcal{D}(\mathbf{M})) \cap C^1([0, \infty), L^2(Q)^6)$ by standard semigroup theory. The shifted Maxwell system furthermore leads to the formulas

$$\partial_t \operatorname{div}(\varepsilon \widetilde{\mathbf{E}}(t)) = \operatorname{div}(\operatorname{curl} \widetilde{\mathbf{H}}(t)) = 0, \quad \partial_t \operatorname{div}(\mu \widetilde{\mathbf{H}}(t)) = -\operatorname{div}(\operatorname{curl} \widetilde{\mathbf{E}}(t)) = 0$$

in $H^{-1}(Q)$. This shows that

$$\operatorname{div}(\varepsilon \widetilde{\mathbf{E}}^{(i)}(t)) = \operatorname{div}(\varepsilon u^{(i)}) \in H^1_{00}(Q_i), \quad \operatorname{div}(\mu \widetilde{\mathbf{H}}(t)) = \operatorname{div}(\mu v) \in PH^1(Q).$$

Using the continuity of the normal trace operator from $H(\operatorname{div}, Q)$ into $H^{-1/2}(\partial Q)$ and $H^{-1/2}(F_{\text{int}})$, we furthermore conclude in the same way

$$[\![\varepsilon \widetilde{\mathbf{E}} \cdot \mathbf{n}_{\text{int}}]\!] = [\![\varepsilon u \cdot \mathbf{n}_{\text{int}}]\!] \in H^{3/2}_0(F_{\text{int}}), \quad \mu \widetilde{\mathbf{H}}(t) \cdot \nu = 0 \text{ on } \partial Q.$$

As a result, $(\widetilde{\mathbf{E}}(t), \widetilde{\mathbf{H}}(t))$ belongs to $\widehat{X}_1$. The above arguments furthermore imply that $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$ is an element of $C([0, \infty), \widehat{X}_1)$. □

Lemma 3.5 yields the following direct consequences for the inhomogeneous problem (3.3).

First-order regularity result. The first result is the shifted analogue of Theorem 2.3 and yields a classical solution of the system (3.3).

Proposition 3.6. *Let the initial values of the modified fields (3.2) satisfy $(\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0) \in X_1$ and the current from (3.4a) $\check{\mathbf{J}} \in C^1\big([0, T], \widehat{X}_0\big) + C\big([0, T], X_1\big)$. Then, there is a unique solution $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) \in C^1([0, T], \widehat{X}_0) \cap C([0, T], X_1)$ of (3.3). In addition, it holds the energy estimate*

$$\sum_{j=0}^{1} \big\|(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})\big\|_{C^j([0,T],X_{1-j})} \leq C\big(\big\|(\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0)\big\|_{X_1} + (1 + T)\big\|\check{\mathbf{J}}\big\|_{C^1([0,T],\widehat{X}_0)+C([0,T],X_1)}\big)$$

with a uniform constant $C = C(\varepsilon, \mu, Q) > 0$.

Proof. By the standard results, see, e.g., Theorem 4.2.4, Corollaries 4.2.5 and 4.2.6 in [29], and the condition on the initial value and the current $\check{\mathbf{J}}$, we immediately obtain that there is a unique (classical) solution which satisfies $\mathbf{w}(t) \in C^1([0, T], \widehat{X}_0) \cap C([0, T], X_1)$.

For the energy estimates, we consider the variation-of-constants formula

$$\mathbf{w}(t) = \mathrm{e}^{t\mathbf{M}}\,\mathbf{w}_0 - \int_0^t \mathrm{e}^{(t-s)\mathbf{M}}\,\check{\mathbf{J}}(s)\,ds. \tag{3.8}$$

Note that by the integration-by-parts formula, we can exchange spatial for temporal regularity via

$$\mathbf{M}\int_0^t \mathrm{e}^{(t-s)\mathbf{M}}\,\check{\mathbf{J}}(s)\,ds = \mathrm{e}^{t\mathbf{M}}\,\check{\mathbf{J}}(0) - \check{\mathbf{J}}(t) + \int_0^t \mathrm{e}^{(t-s)\mathbf{M}}\,\check{\mathbf{J}}'(s)\,ds$$

in case of $\check{\mathbf{J}} \in C^1([0,T],\widehat{X}_0)$, and the energy estimate is shown. □

Second-order regularity result. Next, we turn to the shifted analogue of Theorem 2.5 which yields derivatives up to order 2.

Proposition 3.7. *Let the initial values of the modified fields (3.2) satisfy* $(\widetilde{\mathbf{E}}_0,\widetilde{\mathbf{H}}_0) \in X_2$ *and the current from (3.4a)* $\check{\mathbf{J}} \in C^1\big([0,T],\widehat{X}_1\big)$*. Then, there is a unique solution* $(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})$ *of (3.3) with*

$$(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in C^2\big([0,T],\widehat{X}_0\big) \cap C^1\big([0,T],\widehat{X}_1\big) \cap C\big([0,T],X_2\big).$$

In addition, it holds the energy estimate

$$\sum_{j=0}^{2}\big\|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\big\|_{C^j([0,T],X_{2-j})} \le C\Big(\big\|(\widetilde{\mathbf{E}}_0,\widetilde{\mathbf{H}}_0)\big\|_{X_2} + (1+T)\big\|\check{\mathbf{J}}\big\|_{C^1([0,T],\widehat{X}_1)}\Big)$$

with a uniform constant $C = C(\varepsilon,\mu,Q) > 0$.

Proof. By the standard results, we immediately obtain that there is a unique (classical) solution which satisfies
$\mathbf{w}(t) \in C^1([0,T],\widehat{X}_1) \cap C([0,T],X_2)$.
The additional regularity in time of the solution follows from the identity

$$\partial_t^2 \int_0^t \mathrm{e}^{(t-s)\mathbf{M}}\,\check{\mathbf{J}}(s)\,ds = \check{\mathbf{J}}'(t) + \mathrm{e}^{t\mathbf{M}}\,\mathbf{M}\check{\mathbf{J}}(0) + \int_0^t \mathrm{e}^{(t-s)\mathbf{M}}\,\mathbf{M}\check{\mathbf{J}}'(s)\,ds. \tag{3.9}$$

In particular, we deduce from (3.8) and (3.9) the desired energy estimates. □

Higher-order regularity result. In the last wellposedness result, we establish solutions with derivatives up to order 3, roughly speaking. This enables us to show Theorem 2.6.

Proposition 3.8. *Let the initial values of the modified fields (3.2) satisfy* $(\widetilde{\mathbf{E}}_0,\widetilde{\mathbf{H}}_0) \in X_3$ *and the current from (3.4a) satisfy* $\check{\mathbf{J}} \in C^1\big([0,T],X_2\big)$*. Then, there is a unique solution* $(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})$ *of (3.3) with*

$$(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in C^2\big([0,T],\widehat{X}_1\big) \cap C^1\big([0,T],X_2\big) \cap C\big([0,T],X_3\big).$$

If, in addition, the current satisfies $\check{\mathbf{J}} \in C^2\big([0,T],X_0\big)$*, then we further obtain* $(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in C^3\big([0,T],X_0\big)$*, and it holds the energy estimate*

$$\sum_{j=0}^{3}\big\|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\big\|_{C^j([0,T],X_{3-j})} \le C\Big(\big\|(\widetilde{\mathbf{E}}_0,\widetilde{\mathbf{H}}_0)\big\|_{X_3} + (1+T)\big\|\check{\mathbf{J}}\big\|_{C^1([0,T],X_2)} + (1+T)\big\|\check{\mathbf{J}}\big\|_{C^2([0,T],X_0)}\Big)$$

with a uniform constant $C = C(\varepsilon,\mu,Q) > 0$.

Proof. We obtain immediately the existence and uniqueness of a solution $(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in C^1\big([0,T],X_2\big) \cap C\big([0,T],X_3\big)$, and we have to establish the additional regularity. For the differentiability in X_1, we employ (3.9), and the fact that $\mathbf{M}$ maps X_2 to $\widehat{X}_1$. The additional regularity of $\check{\mathbf{J}}$, as well as the evolution equation $\partial_t(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) = \mathbf{M}(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) + \check{\mathbf{J}}$, implies the desired regularity $(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in C^3([0,T],X_0)$. □

4. Extension of the surface current

In this section, we establish the connection between the surface current j_{surf} and the volume currents $\check{\mathbf{J}}$, $\mathbf{J_E}$ and $\mathbf{J_H}$, used in the transformation in Sect. 3.1. The results are essential to prove our main findings presented in Sect. 2. To extend j_{surf} from the interface F_{int} to Q, we proceed in two steps. First, we provide two stationary extension results in Sect. 4.1. (The proofs of the statements are postponed to Sect. 4.3.) By means of our stationary findings, we can then establish time-dependent extension results in Sect. 4.2.

4.1. Stationary extension results

Our first extension statement is crucial to construct mappings $\mathbf{J_H}$ and $\mathbf{J_E}$ so that the tangential components of the shifted fields $\widetilde{\mathbf{H}}$ and $\widetilde{\mathbf{E}}$ from (3.2) are continuous across the interface F_{int}, see Sect. 3.1. In view of the regularity results in Propositions 3.6–3.8, it is also important that the mapping $\check{\mathbf{J}}$ from (3.4) fulfills additional boundary, transmission, and regularity conditions. As a result, we furthermore study the transmission relations of our extension operators. For the statements, recall the spaces $\mathcal{X}_j^{\gamma} = \mathcal{D}(-\Delta_j)^{\gamma/2}$ from (2.4). For the sake of presentation, we denote by curl the piecewise defined curl-operator.

Lemma 4.1. *There are bounded linear mappings* $L_1 : \mathcal{X}_3^{-1/2} \times \mathcal{X}_2^{-1/2} \to \{0\} \times L^2(Q)^2$ *and* $L_2 : \mathcal{X}_3^{-3/2} \times \mathcal{X}_2^{-3/2} \to \{0\} \times L^2(Q)^2$ *satisfying the following statements.*

a) Let $v_1 \in \mathcal{X}_3^{1/2}$ *and* $v_2 \in \mathcal{X}_2^{1/2}$. *Then*

$$\begin{aligned}
&[\![L_1(v_1,v_2) \times \mathbf{n}_{int}]\!] = (0, v_1, v_2), && L_1(v_1,v_2)\cdot \nu = 0 \text{ on } \partial Q,\\
&L_2(v_1,v_2) \in H_0(\mathrm{curl}, Q), && [\![\tfrac{1}{\mu}\,\mathrm{curl}\, L_2(v_1,v_2) \times \mathbf{n}_{int}]\!] = -(0, v_1, v_2),\\
&\mathrm{curl}\, L_2(v_1,v_2)\cdot\nu = 0 = \nabla L_2(v_1,v_2)\cdot \nu \text{ on } \partial Q, && L_2(v_1,v_2) = 0 \text{ on } F_{int}.
\end{aligned}$$

b) Let $v_1 \in \mathcal{X}_3^{3/2}$ *and* $v_2 \in \mathcal{X}_2^{3/2}$. *Then* $\frac{1}{\varepsilon}\,\mathrm{curl}\, L_1(v_1,v_2) \in H_0(\mathrm{curl}, Q)$, *and*

$$[\![\tfrac{1}{\varepsilon}\,\mathrm{curl}\,\tfrac{1}{\mu}\,\mathrm{curl}\, L_2(v_1,v_2) \times \mathbf{n}_{int}]\!] = \{\{\tfrac{1}{\varepsilon}\}\} \begin{pmatrix} 0 \\ -(-\Delta_2)^{1/2} v_2 \\ (-\Delta_3)^{1/2} v_1 \end{pmatrix}.$$

c) Let $v_1 \in \mathcal{X}_3^{5/2}$ *and* $v_2 \in \mathcal{X}_2^{5/2}$. *Then*

$$[\![\tfrac{1}{\mu}\,\mathrm{curl}\,\tfrac{1}{\varepsilon}\,\mathrm{curl}\, L_1(v_1,v_2) \times \mathbf{n}_{int}]\!] = \frac{1}{2} \begin{pmatrix} 0 \\ -\{\{\tfrac{1}{\varepsilon\mu}\}\}\partial_2\, \mathrm{div}_{F_{int}}(v_1,v_2) + \{\{\tfrac{\varepsilon}{\mu}\}\}\Delta_3 v_1 \\ -\{\{\tfrac{1}{\varepsilon\mu}\}\}\partial_3\, \mathrm{div}_{F_{int}}(v_1,v_2) + \{\{\tfrac{\varepsilon}{\mu}\}\}\Delta_2 v_2 \end{pmatrix}.$$

Additionally, L_1 *is bounded from* $\mathcal{X}_3^{5/2-j} \times \mathcal{X}_2^{5/2-j}$ *into* $\{0\} \times \bigcap_{l=0}^{3-j} PH^l((-1,1), \mathcal{X}_2^{3-j-l} \times \mathcal{X}_3^{3-j-l})$ *for* $j \in \{0,1,2\}$. L_2 *is bounded from* $\mathcal{X}_3^{3/2-j} \times \mathcal{X}_2^{3/2-j}$ *into* $\{0\} \times \bigcap_{l=0}^{3-j} PH^l((-1,1), \mathcal{X}_3^{3-j-l} \times \mathcal{X}_2^{3-j-l})$ *for* $j \in \{0,1,2\}$.

As indicated above, the mapping $\check{\mathbf{J}}$ from (3.4) has to satisfy several transmission conditions to apply the regularity results in Propositions 3.6–3.8. To account for the higher order transmission conditions and to shift away the contributions from the operators L_1 and L_2 in Lemma 4.1b–c), we still need a second stationary extension result.

Lemma 4.2. *There are bounded linear operators* $\tilde{L}_1 : \mathcal{X}_3^{-3/2} \times \mathcal{X}_2^{-3/2} \to \{0\} \times \bigcap_{l=0}^{1} PH^l((-1,1), \mathcal{X}_2^{1-l} \times \mathcal{X}_3^{1-l})$ *and* $\tilde{L}_2 : \mathcal{X}_2^{-3/2} \times \mathcal{X}_3^{-3/2} \to \{0\} \times \bigcap_{l=0}^{1} PH^l((-1,1), \mathcal{X}_3^{1-l} \times \mathcal{X}_2^{1-l})$ *with*

$$\begin{aligned}
&\tilde{L}_1(v_1,v_2) = 0 = \operatorname{curl} \tilde{L}_1(v_1,v_2) \ \text{on } F_{int}, && \tilde{L}_1(v_1,v_2)\cdot\nu = 0 \ \text{on } \partial Q,\\
&[\![\tfrac{1}{\mu}\operatorname{curl}\tfrac{1}{\varepsilon}\operatorname{curl}\tilde{L}_1(v_1,v_2)\times \mathbf{n}_{int}]\!] = (0,v_1,v_2), && \operatorname{curl}\tilde{L}_1(v_1,v_2)\times\nu = 0 \ \text{on } \partial Q,\\
&\tilde{L}_2(v_3,v_4) = 0 = \operatorname{curl} \tilde{L}_2(v_3,v_4) \ \text{on } F_{int}, && [\![\tfrac{1}{\varepsilon}\operatorname{curl}\tfrac{1}{\mu}\operatorname{curl}\tilde{L}_2(v_3,v_4)\times \mathbf{n}_{int}]\!] = (0,v_3,v_4)\\
&\tilde{L}_2(v_3,v_4) \in H_0(\operatorname{curl},Q), && \operatorname{curl}\tilde{L}_2(v_3,v_4)\cdot\nu = 0 \ \text{on } \partial Q,
\end{aligned}$$

for $v_1, v_4 \in \mathcal{X}_3^{1/2}$ *and* $v_2, v_3 \in \mathcal{X}_2^{1/2}$. *Moreover, both operators are bounded as mappings*

$$\tilde{L}_1 : \mathcal{X}_3^{1/2-j} \times \mathcal{X}_2^{1/2-j} \to \{0\} \times \bigcap_{l=0}^{3-j} PH^l((-1,1), \mathcal{X}_2^{3-j-l} \times \mathcal{X}_3^{3-j-l}),$$
$$\tilde{L}_2 : \mathcal{X}_2^{1/2-j} \times \mathcal{X}_3^{1/2-j} \to \{0\} \times \bigcap_{l=0}^{3-j} PH^l((-1,1), \mathcal{X}_3^{3-j-l} \times \mathcal{X}_2^{3-j-l}),$$

for $j \in \{0,1,2\}$.

For the sake of readability, we move the proofs of Lemmas 4.1–4.2 to Sect. 4.3.

4.2. Time-dependent extension

The next three corollaries provide useful extension results for the surface current j_{surf}. The statements are implications of Lemmas 4.1–4.2. Recall the function spaces $V_{T,j}$, $W_{T,j}$, $Y_{T,j}$, and $Z_{T,j}$ from (2.5), (2.10), and (2.13).

Corollary 4.3. *Let* $T > 0$, $j^2_{surf} \in V_{T,3}$ *and* $j^3_{surf} \in V_{T,2}$. *There is a function*

$$\mathbf{J_H} = \begin{pmatrix} 0 \\ \mathbf{J}_{\mathbf{H},2} \\ \mathbf{J}_{\mathbf{H},3} \end{pmatrix} \in \bigcap_{l=0}^{1} C^{1+l}([0,T], PH^{2-l}(Q)^3),$$

with $[\![\mathbf{J_H}(t) \times \mathbf{n}_{\mathrm{int}}]\!] = j_{\mathrm{surf}}(t)$, and $\mathbf{J_H}(t)\cdot\nu = 0$ on ∂Q for $t \in [0,T]$. Choosing $\mathbf{J}_\Sigma = \mathbf{J_E} = 0$, the function $\check{\mathbf{J}}$ from (3.4a) satisfies the conditions in Proposition 3.6. Moreover, the energy estimate

$$\sum_{l=0}^{1} \|\mathbf{J_H}\|_{C^{1+l}([0,T],PH^{2-l}(Q))} \le C \|(j^2_{\mathrm{surf}}, j^3_{\mathrm{surf}})\|_{V_{T,3}\times V_{T,2}},$$

is valid with a uniform constant $C = C(\varepsilon,\mu,Q)$.

Proof. We choose $\mathbf{J_H} := L_1(j^2_{\mathrm{surf}}, j^3_{\mathrm{surf}})$. The statements then follow from Lemma 4.1 a) and by choosing $j = 2$ in the addendum. □

For the next statement, we denote by Δ_S the two-dimensional Laplacian on the square $S = [0,1]^2$.

Corollary 4.4. *Let* $T > 0$, $j^2_{surf} \in W_{T,3}$ *and* $j^3_{surf} \in W_{T,2}$. *There are two functions*

$$\mathbf{J_H} = \begin{pmatrix} 0 \\ \mathbf{J}_{\mathbf{H},2} \\ \mathbf{J}_{\mathbf{H},3} \end{pmatrix}, \ \mathbf{J_E} = \begin{pmatrix} 0 \\ \mathbf{J}_{\mathbf{E},2} \\ \mathbf{J}_{\mathbf{E},3} \end{pmatrix} \in C^2([0,T], PH^2(Q)^3),$$

with $\operatorname{curl} \mathbf{J}_{\mathbf{H}}^{(i)} \in C^1([0,T], H^1(Q_i)^3)$, $\Delta_S \operatorname{curl} \mathbf{J}_{\mathbf{H}}^{(i)} \in C^1([0,T], L^2(Q_i)^3)$, *and*

$$[\![\mathbf{J}_{\mathbf{H}}(t) \times \mathbf{n}_{int}]\!] = j_{surf}(t), \qquad \mathbf{J}_{\mathbf{H}}(t) \cdot \nu = 0 \text{ on } \partial Q, \qquad \tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{J}_{\mathbf{H}}(t) \in H_0(\operatorname{curl}, Q),$$
$$\mathbf{J}_{\mathbf{E}}(t) \in H_0(\operatorname{curl}, Q), \qquad [\![\tfrac{1}{\mu} \operatorname{curl} \mathbf{J}_{\mathbf{E}}(t) \times \mathbf{n}_{int}]\!] = -\partial_t j_{surf}(t), \qquad \operatorname{curl} \mathbf{J}_{\mathbf{E}}(t) \cdot \nu = 0 \text{ on } \partial Q,$$

for $t \in [0,T]$. *Choosing* $\mathbf{J}_\Sigma = 0$, *the function* $\check{\mathbf{J}}$ *from* (3.4a) *then satisfies the conditions from Proposition 3.7. Moreover, the estimate*

$$\|(\mathbf{J}_{\mathbf{H}}, \mathbf{J}_{\mathbf{E}})\|_{C^2([0,T],PH^2(Q))} + \sum_{i=1}^{2} \big(\|\partial_1 \operatorname{curl} \mathbf{J}_{\mathbf{H}}^{(i)}\|_{C^1([0,T],H^1(Q_i)^3)} + \|\Delta_S \operatorname{curl} \mathbf{J}_{\mathbf{H}}^{(i)}\|_{C^1([0,T],L^2(Q_i))} \big)$$
$$\leq C \|(j^2_{surf}, j^3_{surf})\|_{W_{T,3} \times W_{T,2}},$$

is valid with a uniform constant $C = C(\varepsilon, \mu, Q)$.

Proof. We choose $\mathbf{J}_{\mathbf{H}} = L_1(j^2_{\mathrm{surf}}, j^3_{\mathrm{surf}})$, and $\mathbf{J}_{\mathbf{E}} = L_2(\partial_t j^2_{\mathrm{surf}}, \partial_t j^3_{\mathrm{surf}})$. Then all statements follow from Lemma 4.1. (We choose $j \in \{1,2\}$ in the addendum of Lemma 4.1.) □

We finally provide an extension of j_{surf} that meets the conditions in Proposition 3.8. Recall also Definition (2.15).

Corollary 4.5. *Let* $T > 0$, $j^2_{surf} \in Y_{T,3}$, $j^3_{surf} \in Y_{T,2}$, *and* $\operatorname{div}_{F_{int}} j_{surf} \in C^1([0,T], H_0^{3/2}(F_{int}))$. *Let further*

$$g := \begin{pmatrix} \frac{1}{2}\{\{\frac{1}{\varepsilon\mu}\}\}\partial_2 \operatorname{div}_{F_{int}} j_{surf} - \frac{1}{2}\{\{\frac{\varepsilon}{\mu}\}\}\Delta j^3_{surf} - \partial_t^2 j^2_{surf} \\ \frac{1}{2}\{\{\frac{1}{\varepsilon\mu}\}\}\partial_3 \operatorname{div}_{F_{int}} j_{surf} - \frac{1}{2}\{\{\frac{\varepsilon}{\mu}\}\}\Delta j^2_{surf} - \partial_t^2 j^3_{surf} \end{pmatrix} \in Z_{T,1},$$
$$\tilde{g} := \{\{\tfrac{1}{\varepsilon}\}\} \begin{pmatrix} -(-\Delta_2)^{1/2}\partial_t j^3_{surf} \\ (-\Delta_3)^{1/2}\partial_t j^2_{surf} \end{pmatrix} \in Z_{T,2}.$$

Then there are two functions

$$\mathbf{J}_{\mathbf{H}}, \mathbf{J}_{\mathbf{E}} \in \bigcap_{l=0}^{2} C^{1+l}([0,T], PH^{2-l}(Q)^3) \cap C([0,T], \mathcal{D}(\widetilde{\mathbf{M}}^3)),$$

satisfying the following properties. The statements in Corollary 4.4 are valid,

$$[\![\tfrac{1}{\varepsilon} \operatorname{curl} \tfrac{1}{\mu} \operatorname{curl} \mathbf{J}_{\mathbf{E}} \times \mathbf{n}_{int}]\!] = 0, \qquad [\![\tfrac{1}{\mu} \operatorname{curl} \tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{J}_{\mathbf{H}} \times \mathbf{n}_{int}]\!] = -\partial_t^2 j_{surf},$$

and the mapping $\check{\mathbf{J}}$ *from* (3.4a) *with* $\mathbf{J}_\Sigma = 0$ *satisfies the conditions in Proposition 3.8. The mappings* $\check{\mathbf{J}}, \mathbf{J}_{\mathbf{H}}, \mathbf{J}_{\mathbf{E}}$ *can be estimated by*

$$\|\check{\mathbf{J}}\|_{C^1([0,T],X_2)} + \|\check{\mathbf{J}}\|_{C^2([0,T],X_0)} + \sum_{l=0}^{2} \|(\mathbf{J}_{\mathbf{E}}, \mathbf{J}_{\mathbf{H}})\|_{C^{1+l}([0,T],PH^{2-l}(Q))} + \|(\mathbf{J}_{\mathbf{E}}, \mathbf{J}_{\mathbf{H}})\|_{C([0,T],\mathcal{D}(\widetilde{\mathbf{M}}^3))}$$
$$\leq C \big(\|(j^2_{surf}, j^3_{surf})\|_{Y_{T,3} \times Y_{T,2}} + \|\operatorname{div}_{F_{int}} j_{surf}\|_{C^1([0,T],H_0^{3/2}(F_{int}))} + \|g\|_{Z_{T,1}} + \|\tilde{g}\|_{Z_{T,2}} \big),$$

with a uniform constant $C = C(\varepsilon, \mu, Q) > 0$.

Proof. We choose $\mathbf{J}_{\mathbf{H}} = L_1 j_{\mathrm{surf}} + \tilde{L}_1 g$ and $\mathbf{J}_{\mathbf{E}} = L_2 \partial_t j_{\mathrm{surf}} - \tilde{L}_2 \tilde{g}$. The asserted statements then follow from Lemmas 4.1–4.2, as well as the relation $[\![\operatorname{curl} \mathbf{J}_{\mathbf{H}} \cdot \mathbf{n}_{\mathrm{int}}]\!] = \operatorname{div}_{F_{\mathrm{int}}} j_{\mathrm{surf}}$ (with curl denoting the piecewise defined curl-operator). □

4.3. Proof of the stationary extension results

Proof of Lemma 4.1. In the following, we use ideas and arguments from [16,41]. Recall the Laplacians Δ_2 and Δ_3 on F_{int} from (2.3), as well as the notation $\mathcal{X}_j^\gamma := \mathcal{D}(-\Delta_j)^{\gamma/2}$ from (2.4). Set $\alpha := \min\{1, \varepsilon^{(1)}/\varepsilon^{(2)}\}$, and let additionally $\chi : [-1,1] \to [0,1]$ be a smooth cutoff function with $\operatorname{supp}\chi \subset [-\frac{3}{4}\alpha, \frac{3}{4}\alpha]$ and $\chi = 1$ on $[-\frac{1}{2}\alpha, \frac{1}{2}\alpha]$.

(1) Let $v_{1,1} \in \mathcal{X}_3^{-1/2}$, $v_{1,2} \in \mathcal{X}_2^{-1/2}$, $v_{2,1} \in \mathcal{X}_3^{-3/2}$, and $v_{2,2} \in \mathcal{X}_2^{-3/2}$. We put $L_j(v_{j,1}, v_{j,2}) := (0, \Phi_{j,2}, \Phi_{j,3})$ for $j \in \{1,2\}$ with

$$
\begin{aligned}
\Phi_{1,2}^{(2)}(x) &:= -\frac{1}{2}\chi(x_1)(-\Delta_2)^{1/4}\big(\mathrm{e}^{-\varepsilon^{(2)}x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-1/4}v_{1,2}\big)(x_2,x_3),\\
\Phi_{1,2}^{(1)}(x) &:= -\Phi_{1,2}^{(2)}\big(-\tfrac{\varepsilon^{(1)}}{\varepsilon^{(2)}}x_1, x_2, x_3\big),\\
\Phi_{1,3}^{(2)}(x) &:= \frac{1}{2}\chi(x_1)(-\Delta_3)^{1/4}\big(\mathrm{e}^{-\varepsilon^{(2)}x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-1/4}v_{1,1}\big)(x_2,x_3),\\
\Phi_{1,3}^{(1)}(x) &:= -\Phi_{1,3}^{(2)}\big(-\tfrac{\varepsilon^{(1)}}{\varepsilon^{(2)}}x_1, x_2, x_3\big),\\
\Phi_{2,2}^{(2)}(x) &:= -\frac{\mu^{(2)}}{2}\chi(x_1)x_1(-\Delta_3)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-3/4}v_{2,1}\big)(x_2,x_3),\\
\Phi_{2,2}^{(1)}(x) &:= -\frac{\mu^{(1)}}{\mu^{(2)}}\Phi_{2,2}^{(2)}(-x_1,x_2,x_3),\\
\Phi_{2,3}^{(2)}(x) &:= -\frac{\mu^{(2)}}{2}\chi(x_1)x_1(-\Delta_2)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-3/4}v_{2,2}\big)(x_2,x_3),\\
\Phi_{2,3}^{(1)}(x) &:= -\frac{\mu^{(1)}}{\mu^{(2)}}\Phi_{2,3}^{(2)}(-x_1,x_2,x_3),
\end{aligned}
$$

where $\Phi_{j,2}^{(2)}, \Phi_{j,3}^{(2)}$ are defined on $(0,\infty)\times(0,1)^2$, and $\Phi_{j,2}^{(1)}, \Phi_{j,3}^{(1)}$ are defined on $(-\infty,0)\times(0,1)^2$ for $j \in \{1,2\}$.

By Remark 2 in Section 6.1 of [27], L_1 is then bounded from $\mathcal{X}_3^{-1/2} \times \mathcal{X}_2^{-1/2}$ into $\{0\} \times L^2(Q)^2$. Proposition 6.4 in [27] implies that L_2 is bounded from $\mathcal{X}_3^{-3/2} \times \mathcal{X}_2^{-3/2}$ into $\{0\} \times L^2(Q)^2$.

(2) Let now $v_{1,2}, v_{2,2} \in \mathcal{X}_2^{3/2}$. We calculate

$$
\begin{aligned}
\partial_1\Phi_{1,2}^{(2)} &= \frac{1}{2}\big(-\chi' + \varepsilon^{(2)}\chi(-\Delta_2)^{1/2}\big)\big(\mathrm{e}^{-\varepsilon^{(2)}x_1(-\Delta_2)^{1/2}}\,v_{1,2}\big),\\
\partial_1^2\Phi_{1,2}^{(2)} &= \frac{1}{2}\big(-\chi'' + 2\varepsilon^{(2)}\chi'(-\Delta_2)^{1/2} + (\varepsilon^{(2)})^2\chi\Delta_2\big)\cdot\big(\mathrm{e}^{-\varepsilon^{(2)}x_1(-\Delta_2)^{1/2}}\,v_{1,2}\big),\\
\partial_1^3\Phi_{1,2}^{(2)} &= \frac{1}{2}\big(-\chi''' + 3\varepsilon^{(2)}\chi''(-\Delta_2)^{1/2} + 3(\varepsilon^{(2)})^2\chi'\Delta_2 + (\varepsilon^{(2)})^3\chi(-\Delta_2)^{3/2}\big)\\
&\qquad\cdot\big(\mathrm{e}^{-\varepsilon^{(2)}x_1(-\Delta_2)^{1/2}}\,v_{1,2}\big),\\
\partial_1\Phi_{2,3}^{(2)} &= -\frac{\mu^{(2)}}{2}\big(\chi + x_1\chi' - x_1\chi(-\Delta_2)^{1/2}\big)\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_{2,2}\big),\\
\partial_1^2\Phi_{2,3}^{(2)} &= -\frac{\mu^{(2)}}{2}\big(2\chi' + x_1\chi'' - 2\chi(-\Delta_2)^{1/2} - 2x_1\chi'(-\Delta_2)^{1/2} - x_1\chi\Delta_2\big)\\
&\qquad\cdot\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_{2,2}\big),\\
\partial_1^3\Phi_{2,3}^{(2)} &= -\frac{\mu^{(2)}}{2}\big(3\chi'' + x_1\chi''' - 6\chi'(-\Delta_2)^{1/2} - 3x_1\chi''(-\Delta_2)^{1/2} - 3\chi\Delta_2 - 3x_1\chi'\Delta_2\\
&\qquad - x_1\chi(-\Delta_2)^{3/2}\big)\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_{2,2}\big)
\end{aligned}
\tag{4.1}
$$

on Q_2. Using Proposition 6.4 in [27], we derive the estimates

$$\sum_{l=0}^{3}\int_0^1 \|\partial_1^l \Phi_{2,3}^{(2)}(x_1,\cdot)\|^2_{\mathcal{X}_2^{3-l}}\mathrm{d}x_1$$
$$\le C\int_0^1 \|\Delta_2\,\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_{2,2}\|^2_{L^2((0,1)^2)} + \|x_1(-\Delta_2)^{3/2}\,\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_{2,2}\|^2_{L^2((0,1)^2)}\mathrm{d}x_1$$
$$\le C\|v_{2,2}\|_{\mathcal{X}_2^{3/2}}.$$

Arguing similarly for the remaining mappings in the definition of L_2, we conclude that L_2 is bounded from $\mathcal{X}_3^{3/2}\times\mathcal{X}_2^{3/2}$ to $\{0\}\times\bigcap_{l=0}^3 PH^l((-1,1),\mathcal{X}_3^{3-l}\times\mathcal{X}_2^{3-l})$. The remaining boundedness statements for L_1 and L_2 are obtained with analogous calculations.

(3) It remains to verify the asserted interface and boundary conditions. For $v_{1,1},v_{2,1}\in\mathcal{X}_3^{1/2}$ and $v_{1,2},v_{2,2}\in\mathcal{X}_2^{1/2}$, we calculate

$$[\![L_1(v_{1,1},v_{1,2})\times\mathbf{n}_{\mathrm{int}}]\!] = \begin{pmatrix}0\\ \Phi_{1,3}^{(2)}(0,\cdot)-\Phi_{1,3}^{(1)}(0,\cdot)\\ -(\Phi_{1,2}^{(2)}(0,\cdot)-\Phi_{1,2}^{(1)}(0,\cdot))\end{pmatrix} = \begin{pmatrix}0\\ v_{1,1}\\ v_{1,2}\end{pmatrix},\qquad L_2(v_{2,1},v_{2,2})|_{F_{\mathrm{int}}}=0,$$
$$[\![\tfrac{1}{\mu}\,\mathrm{curl}\,L_2(v_{2,1},v_{2,2})\times\mathbf{n}_{\mathrm{int}}]\!] = \begin{pmatrix}0\\ \frac{1}{\mu^{(2)}}\partial_1\Phi_{2,2}^{(2)}(0,\cdot)-\frac{1}{\mu^{(1)}}\partial_1\Phi_{2,2}^{(1)}(0,\cdot)\\ \frac{1}{\mu^{(2)}}\partial_1\Phi_{2,3}^{(2)}(0,\cdot)-\frac{1}{\mu^{(1)}}\Phi_{2,3}^{(1)}(0,\cdot)\end{pmatrix} = -\begin{pmatrix}0\\ v_{2,1}\\ v_{2,2}\end{pmatrix}.$$

We next combine the analyticity of the semigroups $(\mathrm{e}^{-s(-\Delta_j)^{1/2}})_{s\ge 0}$ for $j\in\{2,3\}$, the choice of χ, and Lemma 2.1 in [17]. In this way, we infer the identities

$$L_1(v_{1,1},v_{1,2})\cdot\nu=0,\quad L_2(v_{2,1},v_{2,2})\times\nu=0,\quad \mathrm{curl}\,L_2(v_{2,1},v_{2,2})\cdot\nu=\begin{pmatrix}\partial_2\Phi_{2,3}-\partial_3\Phi_{2,2}\\ -\partial_1\Phi_{2,3}\\ \partial_1\Phi_{2,2}\end{pmatrix}=0,$$
$$\nabla L_2(v_{2,1},v_{2,2})\cdot\nu=\begin{pmatrix}0\\ \partial_2\Phi_{2,2}\\ \partial_3\Phi_{2,3}\end{pmatrix}\cdot\nu=0$$

on ∂Q. In case $v_{1,1},v_{2,1}\in\mathcal{X}_3^{3/2}$ and $v_{1,2},v_{2,2}\in\mathcal{X}_2^{3/2}$, we furthermore infer

$$[\![\tfrac{1}{\varepsilon}\,\mathrm{curl}\,L_1(v_{1,1},v_{1,2})\times\mathbf{n}_{\mathrm{int}}]\!] = \begin{pmatrix}0\\ \frac{1}{\varepsilon^{(2)}}\partial_1\Phi_{1,2}^{(2)}(0,\cdot)-\frac{1}{\varepsilon^{(1)}}\partial_1\Phi_{1,2}^{(1)}(0,\cdot)\\ -\frac{1}{\varepsilon^{(1)}}\partial_1\Phi_{1,3}^{(1)}(0,\cdot)+\frac{1}{\varepsilon^{(2)}}\partial_1\Phi_{1,3}^{(2)}(0,\cdot)\end{pmatrix}=0,$$
$$[\![\tfrac{1}{\varepsilon}\,\mathrm{curl}\,\tfrac{1}{\mu}\,\mathrm{curl}\,L_2(v_{2,1},v_{2,2})\times\mathbf{n}_{\mathrm{int}}]\!] = \left[\!\!\left[\begin{pmatrix}0\\ -\frac{1}{\varepsilon\mu}\partial_1^2\Phi_{2,3}\\ \frac{1}{\varepsilon\mu}\partial_1^2\Phi_{2,2}\end{pmatrix}\right]\!\!\right] = \{\!\{\tfrac{1}{\varepsilon}\}\!\}\begin{pmatrix}0\\ -(-\Delta_2)^{1/2}v_{2,2}\\ (-\Delta_3)^{1/2}v_{2,1}\end{pmatrix},$$
$$\frac{1}{\varepsilon^{(i)}}\,\mathrm{curl}\,L_1(v_{1,1},v_{1,2})^{(i)}\times\nu=\frac{1}{\varepsilon^{(i)}}\begin{pmatrix}\partial_2\Phi_{1,3}^{(i)}-\partial_3\Phi_{1,2}^{(i)}\\ -\partial_1\Phi_{1,3}^{(i)}\\ \partial_1\Phi_{1,2}^{(i)}\end{pmatrix}\times\nu=0\quad\text{on }\partial Q_i\cap\partial Q,\qquad i\in\{1,2\}.$$

Let finally $v_{1,1} \in \mathcal{X}_3^{5/2}$ and $v_2 \in \mathcal{X}_2^{5/2}$. With Lemma 2.1 in [17] and (4.1), we then obtain

$$[\![\tfrac{1}{\mu} \operatorname{curl} \tfrac{1}{\varepsilon} \operatorname{curl} L_1(v_{1,1}, v_{1,2}) \times \mathbf{n}_{\mathrm{int}}]\!] = \begin{pmatrix} 0 \\ -\frac{1}{2}\{\{\frac{1}{\varepsilon\mu}\}\}\partial_2 \operatorname{div}_{F_{\mathrm{int}}}(v_{1,1}, v_{1,2}) + \frac{1}{2}\{\{\frac{\varepsilon}{\mu}\}\}\Delta_3 v_{1,1} \\ -\frac{1}{2}\{\{\frac{1}{\varepsilon\mu}\}\}\partial_3 \operatorname{div}_{F_{\mathrm{int}}}(v_{1,1}, v_{1,2}) + \frac{1}{2}\{\{\frac{\varepsilon}{\mu}\}\}\Delta_2 v_{1,2} \end{pmatrix}.$$

□

Proof of Lemma 4.2. (1) We argue similarly as in the proof of Lemma 4.1, whence we only sketch our reasoning. In particular, we use the smooth cutoff function χ from the proof for Lemma 4.1. Let $v_1, v_4 \in \mathcal{X}_3^{-3/2}$ and $v_2, v_3 \in \mathcal{X}_2^{-3/2}$. Set $\tilde{L}_1(v_1, v_2) := (0, \Phi_{1,2}, \Phi_{1,3})$ and $\tilde{L}_2(v_3, v_4) := (0, \Phi_{2,2}, \Phi_{2,3})$ with

$$\begin{aligned}
\Phi_{1,2}^{(2)}(x) &:= \tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\chi(x_1)x_1^2(-\Delta_2)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-3/4}v_2\big)(x_2, x_3), && \Phi_{1,2}^{(1)} := 0,\\
\Phi_{1,3}^{(2)}(x) &:= -\tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\chi(x_1)x_1^2(-\Delta_3)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-3/4}v_1\big)(x_2, x_3), && \Phi_{1,3}^{(1)} := 0,\\
\Phi_{2,2}^{(2)}(x) &:= \tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\chi(x_1)x_1^2(-\Delta_3)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-3/4}v_4\big)(x_2, x_3), && \Phi_{2,2}^{(1)} := 0,\\
\Phi_{2,3}^{(2)}(x) &:= -\tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\chi(x_1)x_1^2(-\Delta_2)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-3/4}v_3\big)(x_2, x_3), && \Phi_{2,3}^{(1)} := 0.
\end{aligned}$$

We calculate

$$\begin{aligned}
\partial_1\Phi_{1,2}^{(2)} &= \tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\big(\chi' x_1^2 + 2x_1\chi - \chi x_1^2(-\Delta_2)^{1/2}\big)(-\Delta_2)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-3/4}v_2\big),\\
\partial_1^2\Phi_{1,2}^{(2)} &= \tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\big(\chi'' x_1^2 + 4x_1\chi' + 2\chi - (2\chi' x_1^2 + 4\chi x_1)(-\Delta_2)^{1/2}\\
&\quad + \chi x_1^2(-\Delta_2)\big)(-\Delta_2)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-3/4}v_2\big),\\
\partial_1^3\Phi_{1,2}^{(2)} &= \tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\big(\chi''' x_1^2 + 6x_1\chi'' + 6\chi' - (3\chi'' x_1^2 + 12\chi' x_1 + 6\chi)(-\Delta_2)^{1/2}\\
&\quad + (3\chi' x_1^2 + 6\chi x_1)(-\Delta_2) - \chi x_1^2(-\Delta_2)^{3/2}\big)(-\Delta_2)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-3/4}v_2\big),
\end{aligned}$$

and analogously

$$\begin{aligned}
\partial_1\Phi_{1,3}^{(2)} &= -\tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\big(\chi' x_1^2 + 2x_1\chi - \chi x_1^2(-\Delta_3)^{1/2}\big)(-\Delta_3)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-3/4}v_1\big),\\
\partial_1^2\Phi_{1,3}^{(2)} &= -\tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\big(\chi'' x_1^2 + 4x_1\chi' + 2\chi - (2\chi' x_1^2 + 4\chi x_1)(-\Delta_3)^{1/2}\\
&\quad + \chi x_1^2(-\Delta_3)\big)(-\Delta_3)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-3/4}v_1\big),\\
\partial_1^3\Phi_{1,3}^{(2)} &= -\tfrac{\varepsilon^{(2)}\mu^{(2)}}{2}\big(\chi''' x_1^2 + 6x_1\chi'' + 6\chi' - (3\chi'' x_1^2 + 12\chi' x_1 + 6\chi)(-\Delta_3)^{1/2}\\
&\quad + (3\chi' x_1^2 + 6\chi x_1)(-\Delta_3) - \chi x_1^2(-\Delta_3)^{3/2}\big)(-\Delta_3)^{3/4}\big(\mathrm{e}^{-x_1(-\Delta_3)^{1/2}}(-\Delta_3)^{-3/4}v_1\big).
\end{aligned}$$

Note that the derivatives of $\Phi_{2,2}^{(2)}$ and $\Phi_{2,3}^{(2)}$ have the same structure. By means of Proposition 6.4 in [27], we then conclude the stated boundedness results for $\tilde{L}_1$ and $\tilde{L}_2$.

(2) Let $v_1 \in \mathcal{X}_3^{1/2}$ and $v_2 \in \mathcal{X}_2^{1/2}$. We only analyze the traces of $\tilde{L}_1(v_1, v_2)$ at the interface and the boundary faces. The mapping $\tilde{L}_2(v_3, v_4)$ can be handled in the same way. In the following, we combine the analyticity of $(\mathrm{e}^{-x_1(-\Delta_j)^{1/2}})_{x_1\geq 0}$ with Lemma 2.1 in [17] several times. We then note the identities

$$x_1^2(-\Delta_2)\,\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_2 = 0 = x_1(-\Delta_2)\,\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}(-\Delta_2)^{-1/2}v_2 = x_1(-\Delta_2)^{1/2}\,\mathrm{e}^{-x_1(-\Delta_2)^{1/2}}\,v_2,$$

on F_{int}. Analogous statements are true for the summands in the formulas for $\partial_1^j \Phi_{1,3}^{(2)}$. Altogether, we conclude

$$\tilde{L}_1(v_1, v_2)^{(2)} = 0 = \begin{pmatrix} \partial_2 \Phi_{1,3}^{(2)} - \partial_3 \Phi_{1,2}^{(2)} \\ -\partial_1 \Phi_{1,3}^{(2)} \\ \partial_1 \Phi_{1,2}^{(2)} \end{pmatrix} = \operatorname{curl} \tilde{L}_1(v_1, v_2)^{(2)},$$

$$\frac{1}{\mu^{(2)}\varepsilon^{(2)}} \operatorname{curl}\operatorname{curl} \tilde{L}_1(v_1, v_2)^{(2)} \times \mathbf{n}_{\mathrm{int}} = \frac{1}{\mu^{(2)}\varepsilon^{(2)}} \begin{pmatrix} 0 \\ \partial_2\partial_3 \Phi_{1,2}^{(2)} - \partial_1^2 \Phi_{1,3}^{(2)} - \partial_2^2 \Phi_{1,3}^{(2)} \\ -(\partial_2\partial_3 \Phi_{1,3}^{(2)} - \partial_1^2 \Phi_{1,2}^{(2)} - \partial_3^2 \Phi_{1,2}^{(2)}) \end{pmatrix}$$

$$= \frac{1}{\mu^{(2)}\varepsilon^{(2)}} \begin{pmatrix} 0 \\ -\partial_1^2 \Phi_{1,3}^{(2)} \\ \partial_1^2 \Phi_{1,2}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 \\ v_2 \end{pmatrix},$$

on F_{int}. On the boundary, the relations $\tilde{L}_1(v_1, v_2) \cdot \nu = 0$, and $\operatorname{curl} \tilde{L}_1(v_1, v_2) \times \nu = 0$ are finally also valid. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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Benjamin Dörich
Institut für Angewandte und Numerische Mathematik
Englerstr. 2
76131 Karlsruhe
Germany
e-mail: benjamin.doerich@kit.edu

Konstantin Zerulla
Institut für Analysis
Englerstr. 2
76131 Karlsruhe
Germany
e-mail: konstantin.zerulla@kit.edu

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