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Generalized Polarity and Weakest Constraint Qualifications in Multiobjective Optimization

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Abstract
In Haeser and Ramos (J Optim Theory Appl, 187:469–487, 2020), a generalization of the normal cone from single objective to multiobjective optimization is introduced, along with a weakest constraint qualification such that any local weak Pareto optimal point is a weak Kuhn–Tucker point. We extend this approach to other generalizations of the normal cone and corresponding weakest constraint qualifications, such that local Pareto optimal points are weak Kuhn–Tucker points, local proper Pareto optimal points are weak and proper Kuhn–Tucker points, respectively, and strict local Pareto optimal points of order one are weak, proper and strong Kuhn–Tucker points, respectively. The constructions are based on an appropriate generalization of polarity to pairs of matrices and vectors.

Keywords Generalized polar cone · Generalized bipolar cone · Multiobjective stationarity condition · Multiobjective Kuhn–Tucker condition · Weakest constraint qualification

Mathematics Subject Classification 90C29 · 90C46

1 Introduction

We consider multiobjective optimization problems (MOPs) of the form

$$\min f(x) \text{ s.t. } g(x) \leqq 0, h(x) = 0 \qquad (MOP)$$

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with a vector-valued objective function $f : \mathbb{R}^n \to \mathbb{R}^m$, a vector-valued inequality constraint function $g : \mathbb{R}^n \to \mathbb{R}^p$ and a vector-valued equality constraint function $h : \mathbb{R}^n \to \mathbb{R}^q$, $m \in \mathbb{N}$, $p, q \in \mathbb{N}_0$. All functions f, g and h are assumed to be continuously differentiable. The feasible set of MOP will be denoted by

$$X = \{x \in \mathbb{R}^n \mid g(x) \leqq 0, h(x) = 0\}. \tag{1}$$

In the present paper, $\leqq$ and $<$ refer to the element-wise non-strict and strict inequalities between vectors, and correspondingly we denote $x \leqq y, x \neq y$ as $x \leq y$ for $x, y \in \mathbb{R}^m$. The set

$$\mathbb{R}^m_{\geqq} = \{d \in \mathbb{R}^m \mid d \geqq 0\}$$

is called the natural ordering cone for $\mathbb{R}^m$. Likewise, we define $\mathbb{R}^m_{\geq}$ and $\mathbb{R}^m_{>}$. We denote the Euclidean norm as $\|\cdot\|$.

In [17], a generalization of the normal cone from single objective to multiobjective optimization is introduced, along with a weakest constraint qualification such that any local weak Pareto optimal point is a weak Kuhn–Tucker point. The aim of the present paper is to extend these results to weakest constraint qualifications for different relevant types of local Pareto optimal points of MOP to fulfill appropriate types of Kuhn–Tucker conditions.

In Sect. 2, we collect preliminaries about multiobjective optimality notions (Sect. 2.1), stationarity notions (Sect. 2.2), and notions of Kuhn–Tucker points (Sect. 2.3). We recall first-order necessary optimality conditions under the Abadie constraint qualification in Sect. 2.4 and explain the Guignard constraint qualification as the weakest constraint qualification from the single-objective case in Sect. 2.5. To the best of the authors' knowledge, some results from Sect. 2 have not been stated in the literature so far.

Section 3 introduces natural generalizations of polarity to pairs of matrices and vectors, which allow us to extend the notion of a multiobjective regular normal cone from [17]. In Sect. 4, we use these constructions to define several types of multiobjective Guignard constraint qualifications, three of which turn out to be central concepts. They are strictly stronger than the standard Guignard constraint qualification and, for $m < n$, strictly weaker than the Abadie constraint qualification. Section 5 shows that they act as weakest constraint qualifications for relevant combinations of stationarity concepts and Kuhn–Tucker notions. Section 6 closes the article with final remarks. For the readers' convenience, some results on relations of local proper Pareto optimality notions are discussed in an appendix.

2 Preliminaries

2.1 Optimality Notions

The single-objective notion of global optimality of a point $\bar{x} \in X$, namely that there is no $x \in X$ with $f(x) < f(\bar{x})$, is transferred to the multiobjective setting by an appropriate generalization of the requirement that no 'better' feasible point exists. In this paper, we focus on the case that the latter is modelled by the ordering relation induced by the natural ordering cone $\mathbb{R}^m_{\geqq}$ (see [8, 9, 20, 29] for more general constructions), resulting in the notion of Pareto optimality (also called Pareto efficiency). Since in the single-objective case global and local optimality of a point $\bar{x} \in X$ may not only be written as the (global or local, resp.) absence of points $x \in X$ with $f(x) < f(\bar{x})$, but equivalently also as the absence of points $x \in X$ with (in vector inequality notation) $f(x) \leqq f(\bar{x})$ and $f(x) \neq f(\bar{x})$, one considers the following notions.

Definition 2.1 A feasible point $\bar{x} \in X$ of MOP is called

(a) *Weak Pareto optimal* ($\bar{x} \in WPO(f, X)$), if there is no $x \in X$ with $f(x) < f(\bar{x})$,
(b) *Local weak Pareto optimal* ($\bar{x} \in LWPO(f, X)$), if there is a neighbourhood U of $\bar{x}$ such that $\bar{x} \in WPO(f, X \cap U)$,
(c) *Pareto optimal* ($\bar{x} \in PO(f, X)$), if there is no $x \in X$ with $f(x) \leq f(\bar{x})$,
(d) *Local Pareto optimal* ($\bar{x} \in LPO(f, X)$), if there is a neighbourhood U of $\bar{x}$ such that $\bar{x} \in PO(f, X \cap U)$.

The inclusions $PO(f, X) \subseteq WPO(f, X)$ and $LPO(f, X) \subseteq LWPO(f, X)$ are easily seen and, as opposed to the single-objective case, they may be strict.

Pareto optimality of a point $\bar{x} \in X$ may be restated as the fact that for any $x \in X$ any improvement in one objective function (i.e. $f_i(x) < f_i(\bar{x})$ for some $i \in \{1, \dots, m\}$) entails a deterioration in at least one other objective function (i.e. $f_j(x) > f_j(\bar{x})$ for some $j \in \{1, \dots, m\}$). In particular, for a decision maker, weak Pareto optimal points which are not Pareto optimal are not desirable since an improvement in one objective is possible without a deterioration in another. In their 1951 seminal paper [25], Kuhn and Tucker argue that the Pareto optimality notion also covers points $\bar{x} \in X$ for which a first-order improvement in one objective function enforces only a second-order deterioration in another objective function. This, however, may be undesirable for decision makers who consider second-order deteriorations an acceptable consequence of first-order improvements. They therefore suggest to also consider *proper* Pareto optimal points in the sense that any first-order improvement of any objective function (in a first order feasible direction) must entail a first-order deterioration of another. Their proposal for an appropriate definition of this fact was subsequently complemented by alternative proper Pareto optimality definitions by Klinger [24], Geoffrion [11], Borwein [5], Benson [3], Henig [18] and Ishizuka-Tuan [19] (see [40] and [15] for an overview). In the present paper, special attention will be paid to proper Pareto optimal points in the sense of Geoffrion and Ishizuka-Tuan. Borwein and Benson proper Pareto optimal points are briefly covered in Appendix A.

Definition 2.2 A feasible point $\bar{x} \in X$ of MOP is called

(a) *Geoffrion proper Pareto optimal* ($\bar{x} \in GPO(f, X)$) if there is a scalar $M > 0$ such that for all $x \in X$ and for all $i \in \{1, \dots, m\}$ with $f_i(x) < f_i(\bar{x})$ there exists some $j \in \{1, \dots, m\}$ with $f_j(x) > f_j(\bar{x})$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M,$$

(b) *Local Geoffrion proper Pareto optimal* ($\bar{x} \in LGPO(f, X)$), if there is a neighbourhood U of $\bar{x}$ with $\bar{x} \in GPO(f, X \cap U)$.

In the original definition from [11] also (local) Pareto optimality of the point $\bar{x} \in X$ is assumed. However, it is easily seen that this requirement is redundant, that is, $GPO(f, X) \subseteq PO(f, X)$ and $LGPO(f, X) \subseteq LPO(f, X)$ hold.

Local Geoffrion proper Pareto optimality may be strengthend further to strict local Pareto optimality of order one, as introduced by Jiménez [21].

Definition 2.3 A feasible point $\bar{x} \in X$ of *MOP* is called *strict local Pareto optimal of order one* ($\bar{x} \in StrLPO(1, f, X)$), if there exist a neighbourhood U of $\bar{x}$ and $\alpha > 0$ such that

$$(f(x) + \mathbb{R}^m_{\geqq}) \cap B(f(\bar{x}), \alpha\|x - \bar{x}\|) = \emptyset \; \forall x \in X \cap U \setminus \{\bar{x}\}$$

holds.

Remark 2.1 In single-objective optimization, the elements of $StrLPO(1, f, X)$ are also called strong local minimal points [37] or strongly unique local minimal points [10, 13], so that it may appear natural to also speak of strong or strongly unique local Pareto optimal points in the multiobjective case. In multiobjective optimization, however, the terminology of strong Pareto optimal points commonly refers to the ones which possess a neighbourhood U with $f(\bar{x}) \leqq f(x)$ for all $x \in X \cap U$ [20].

In any case, without possible confusion strict local Pareto optimal points of order one may alternatively be called strongly unique local Pareto optimal points, which explains our below terminology of strongly stationary points (Definition 2.4) and strong Kuhn–Tucker points (Definition 2.8).

Jiménez and Novo [22] prove that strict local Pareto optimality of order one is stronger than a local proper Pareto optimality notion which is inspired by (global) Borwein proper Pareto optimality. The latter is actually equivalent to local Geoffrion proper Pareto optimality, which we show in Theorem A.1 in the appendix. Altogether this yields the chain of inclusions

$$StrLPO(1, f, X) \subseteq LGPO(f, X) \subseteq LPO(f, X) \subseteq LWPO(f, X). \qquad (2)$$

2.2 Stationarity Notions

With the cones

$$C_<(f, \bar{x}) = \{d \in \mathbb{R}^n \mid \nabla f(\bar{x})^\top d < 0\},$$
$$C_\leq(f, \bar{x}) = \{d \in \mathbb{R}^n \mid \nabla f(\bar{x})^\top d \leq 0\},$$
$$C_\leqq(f, \bar{x}) = \{d \in \mathbb{R}^n \mid \nabla f(\bar{x})^\top d \leqq 0\}$$

of (potential) descent directions for all components of f at $\bar{x} \in X$ and with the tangent cone

$$T(X, \bar{x}) = \{d \in \mathbb{R}^n \mid \exists t^\ell \searrow 0, x^\ell \in X : \lim_\ell (x^\ell - \bar{x})/t^\ell = d\}$$

to X at $\bar{x} \in X$, we define the following stationarity conditions.

Definition 2.4 A feasible point $\bar{x} \in X$ of MOP is called

(a) *Weakly stationary* ($\bar{x} \in WSt(f, X)$) if $C_<(f, \bar{x}) \cap T(X, \bar{x}) = \emptyset$ holds,
(b) *Properly stationary* ($\bar{x} \in PSt(f, X)$) if $C_\leq(f, \bar{x}) \cap T(X, \bar{x}) = \emptyset$ holds,
(c) *Strongly stationary* ($\bar{x} \in SSt(f, X)$) if $C_\leqq(f, \bar{x}) \cap T(X, \bar{x}) = \{0\}$ holds.

The relations $SSt(f, X) \subseteq PSt(f, X) \subseteq WSt(f, X)$ are clear from the definitions. From [4, 36] it is known that any local weak Pareto optimal point is weakly stationary, [19, Prop. 3.2] states that any local Goeffrion proper Pareto optimal point is properly stationary, and [23, Th. 4.1] characterizes the strict local Pareto optimal points of order one by strong stationarity. The following result reformulates these findings in terms of our definitions. Part a is also shown in [17] by a different technique of proof.

Proposition 2.1 *The following relations hold.*

(a) $LWPO(f, X) \subseteq WSt(f, X)$*,*
(b) $LGPO(f, X) \subseteq PSt(f, X)$*,*
(c) $StrLPO(1, f, X) = SSt(f, X)$*.*

The combination of Proposition 2.1 with (2) yields

$$StrLPO(1, f, X) \subseteq LGPO(f, X) \subseteq LPO(f, X) \subseteq LWPO(f, X) \subseteq WSt(f, X) \tag{3}$$

as well as

$$StrLPO(1, f, X) \subseteq LGPO(f, X) \subseteq PSt(f, X). \tag{4}$$

Note that the inclusion $LPO(f, X) \subseteq PSt(f, X)$ does not hold in general. We call local Pareto optima which are properly stationary *local Ishizuka-Tuan proper Pareto optima* and define the corresponding set of points

$$LITPO(f, X) := PSt(f, X) \cap LPO(f, X). \tag{5}$$

There are local Pareto optima which are properly stationary but not local proper in the sense of Geffrion. Therefore, the inclusion $LGPO(f, X) \subseteq LITPO(f, X)$ is in general strict.

Remark 2.2 Points in $LITPO(f, X)$ were introduced by Ishizuka and Tuan [19] under the name Kuhn–Tucker proper Pareto optima. These points are exactly those local Pareto optima in which there is no first-order decrease which is not offset by a first-order increase in another objective. This was also the idea of Kuhn and Tucker for which ‘improper’ points to exclude. In their seminal work [25], however, Kuhn and Tucker use the linearization cone instead of the tangent cone, leading to a different and in general stronger notion as will be discussed at the end of the next subsection. Since the name Kuhn–Tucker proper Pareto optimality is associated in the literature with the original notion of Kuhn–Tucker, we name the notion using the tangent cone after Ishizuka-Tuan who, to our knowledge, first introduced it. Klinger [24] also excludes improper points based on objective decreases of first order, but he considers only directions in a strict subset of the tangent cone. As a consequence, local Klinger proper Pareto optima are in general not properly but only weakly stationary.

In addition to the notion discussed here, Ishizuka and Tuan also introduced an entirely different notion of proper Pareto optimality in the style of Borwein. The latter is equivalent to a local notion by Borwein under the weak assumption of a pointed and convex ordering cone with a compact base as shown in [22]. These solution concepts are based on locality in the image space and are thus stronger than any based on locality in the decision space. Consequently, they will not be discussed further here.

2.3 Notions of Kuhn–Tucker Points

As in the single-objective case (cf., e.g. [2, 16, 31] for details), also in multiobjective optimization, the notion of (Karush–)Kuhn–Tucker points is based on the (weaker) notion of Fritz-John points. For their statements, we use the index set $A(x) = \{i \in \{1, \ldots, p\} \mid g_i(x) = 0\}$ of active inequality constraints at a point $x \in X$ and denote the vector of active inequality constraints at a point by g_A. The Jacobian $\nabla g_A(x)$ is formed by the column vectors $\nabla g_i(x)$, $i \in A(x)$.

Definition 2.5 A point $\bar{x} \in X$ is called a *Fritz-John point* of MOP ($\bar{x} \in FJ(f, g, h)$), if there exist $\lambda \in \mathbb{R}^m_{\geqq}$, $\mu \in \mathbb{R}^{|A(\bar{x})|}_{\geqq}$ and $\nu \in \mathbb{R}^q$ with

$$(\lambda, \mu, \nu) \neq (0, 0, 0) \text{ and } \nabla f(\bar{x})\lambda + \nabla g_A(\bar{x})\mu + \nabla h(\bar{x})\nu = 0.$$

In single-objective optimization, the notion of Fritz-John points is strengthened to the notion of (Karush–)Kuhn–Tucker points by requiring that the (single) multiplier of the objective function is positive. In contrast, in multiobjective optimization, the vector λ of m objective function multipliers arises, so that one needs to distinguish between the case where some of them are positive, i.e. $\lambda \in \mathbb{R}^m_{\geq}$, and the case where all of them are positive, i.e. $\lambda \in \mathbb{R}^m_{>}$. Indeed, just like for the notion of optimality, there is an ambiguity in the generalization from the single-objective to the multiobjective setting. This leads to the following two notions of Kuhn–Tucker points which both generalize the notion of Karush–Kuhn–Tucker points from the single-objective case.

Definition 2.6 A point $\bar{x} \in X$ is called a *weak Kuhn–Tucker point* of MOP ($\bar{x} \in KT_{\geq}(f, g, h)$), if there exist $\lambda \in \mathbb{R}^m_{\geqq}$, $\mu \in \mathbb{R}^{|A(\bar{x})|}_{\geqq}$ and $\nu \in \mathbb{R}^q$ with

$$\lambda \geq 0 \text{ and } \nabla f(\bar{x})\lambda + \nabla g_A(\bar{x})\mu + \nabla h(\bar{x})\nu = 0.$$

Definition 2.7 A point $\bar{x} \in X$ is called a *proper Kuhn–Tucker point* of *MOP* ($\bar{x} \in KT_{>}(f, g, h)$), if there exist $\lambda \in \mathbb{R}^m_{\geqq}$, $\mu \in \mathbb{R}^{|A(\bar{x})|}_{\geqq}$, $\nu \in \mathbb{R}^q$ with

$$\lambda > 0 \text{ and } \nabla f(\bar{x})\lambda + \nabla g_A(\bar{x})\mu + \nabla h(\bar{x})\nu = 0.$$

In addition, we introduce the following strengthened version of a proper Kuhn–Tucker point.

Definition 2.8 A point $\bar{x} \in X$ is called a *strong Kuhn–Tucker point* of *MOP* ($\bar{x} \in KT_{\gg}(f, g, h)$), if $\operatorname{rank}(\nabla f(\bar{x}), \nabla g_A(\bar{x}), \nabla h(\bar{x})) = n$ holds and if there exist $\lambda \in \mathbb{R}^m_{\geqq}$, $\mu \in \mathbb{R}^{|A(\bar{x})|}_{\geqq}$, $\nu \in \mathbb{R}^q$ with

$$\lambda > 0, \;\; \mu > 0 \text{ and } \nabla f(\bar{x})\lambda + \nabla g_A(\bar{x})\mu + \nabla h(\bar{x})\nu = 0.$$

Our terminology *weak*, *proper* and *strong* Kuhn–Tucker points stems from their relation to the above optimality and stationarity notions. The relations $KT_{\gg}(f, g, h) \subseteq KT_{>}(f, g, h) \subseteq KT_{\geq}(f, g, h) \subseteq FJ(f, g, h)$ are clear from the definitions.

The subsequent characterizations of the three types of Kuhn–Tucker points will be useful, where

$$L(g, h, \bar{x}) = \{d \in \mathbb{R}^n \mid \nabla g_A(\bar{x})^\top d \leqq 0, \nabla h(\bar{x})^\top d = 0\}$$

denotes the linearization cone to X at $\bar{x} \in X$. The characterizations are shown by Tucker's theorem of the alternative [27]. We remark that the statement of part c is stronger than the one in [12, Th. 3.4], where the equivalence is shown under the general assumption of the full rank condition. However, the proof from [12] also covers the stronger statement. A proof of the same result from the point of view of linear semi-infinite programming is given in [10, Th. 3.1] and used in, e.g. [7, 13].

Proposition 2.2 *The following assertions hold:*

(a) $\bar{x} \in KT_{\geq}(f, g, h)$ *if and only if* $C_{<}(f, \bar{x}) \cap L(g, h, \bar{x}) = \emptyset$,
(b) $\bar{x} \in KT_{>}(f, g, h)$ *if and only if* $C_{\leq}(f, \bar{x}) \cap L(g, h, \bar{x}) = \emptyset$,
(c) $\bar{x} \in KT_{\gg}(f, g, h)$ *if and only if* $C_{\leqq}(f, \bar{x}) \cap L(g, h, \bar{x}) = \{0\}$.

As already noted in Remark 2.2, Kuhn and Tucker [25] defined local proper Pareto optimality not based on the tangent cone but based on the linearization cone. In fact, the condition $C_{\leq}(f, \bar{x}) \cap L(g, h, \bar{x}) = \emptyset$ from Proposition 2.2b is the one which they require local proper Pareto optima to fulfill, in addition to being locally Pareto optimal. The set of proper Pareto optimal points in the original sense of Kuhn and Tucker is thus $KT_{>}(f, X) \cap LPO(f, X)$.

2.4 First-Order Necessary Optimality Conditions

Since the tangent cone $T(X, \bar{x})$ at $\bar{x} \in X$ may be a proper subset of the linearization cone $L(g, h, \bar{x})$, without some additional assumption, the combination of Proposition 2.1 and Proposition 2.2 does not yield necessary optimality conditions of the form $LWPO(f, X) \subseteq KT_{\geq}(f, g, h)$, etc. Such an additional assumption is the Abadie constraint qualification (ACQ) at $\bar{x} \in X$ which requires the inclusion $L(g, h, \bar{x}) \subseteq T(X, \bar{x})$. In the following result, let $ACQ(X)$ denote the set of points $\bar{x} \in X$ at which ACQ holds.

Corollary 2.1 *The following first-order necessary optimality conditions hold.*

(a) $LWPO(f, X) \cap ACQ(X) \subseteq KT_{\geq}(f, g, h)$,
(b) $LITPO(f, X) \cap ACQ(X) \subseteq KT_{>}(f, g, h)$,
(c) $LGPO(f, X) \cap ACQ(X) \subseteq KT_{>}(f, g, h)$,
(d) $StrLPO(1, f, X) \cap ACQ(X) \subseteq KT_{\gg}(f, g, h)$.

The assertion of Corollary 2.1a was shown by Wang [38]. Earlier, Maruşciac [28] gave a proof implicitly employing the ACQ, but he used the stronger Kuhn–Tucker constraint qualification (KTCQ) in the statement of his theorem. More importantly, his proof holds only for (global) weak Pareto optima. Singh [34] stated and proved the assertion of part a also only for the global case. Interestingly, in the articles of both Maruşciac and Singh, one finds without further discussion a quotation of a result by Lin [26, Th. 5.1] on tangent vectors in the image space which would allow their proofs to be extended to local weak Pareto optima. However, their quotations of Lin's correct result are false. Lin states his result only for images of locally weak minimal elements of $f(X)$ which is in general not true for all local weak Pareto optima, as a counterexample by Wang [38] shows. A correction to Singh's article has been issued [35] which quotes Lin's result only in the global case. Corollary 2.1b was shown by Ishizuka and Tuan [19]. The assertion of Corollary 2.1c was stated by Haeser and Ramos [17], while the result is also a corollary of Proposition 2.1b (originally shown by Ishizuka and Tuan [19]) and part b. Similarly, part d is an immediate corollary of Proposition 2.1c (originally shown by Jimenez and Novo [23]) but, to the best of our knowledge, so far it has not been stated explicitly.

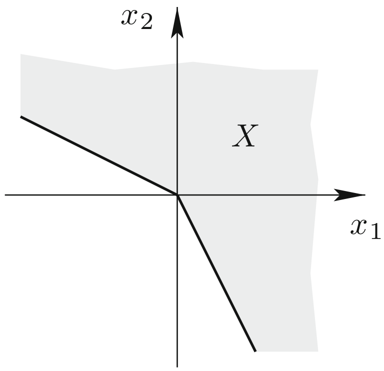
2.5 The Weakest Constraint Qualification in Single-Objective Optimization

In single-objective optimization ($m = 1$), weak and proper stationarity of $\bar{x} \in X$ for f collapse to the condition

$$-\nabla f(\bar{x}) \in T^{\circ}(X, \bar{x}) = \left\{ v \in \mathbb{R}^n \mid v^{\top} d \leqq 0 \, \forall d \in T(X, \bar{x}) \right\} \tag{6}$$

with the polar cone $T^{\circ}(X, \bar{x})$ of the tangent cone $T(X, \bar{x})$. The Guignard constraint qualification (GCQ) at $\bar{x} \in X$ requires $T^{\circ}(X, \bar{x}) \subseteq L^{\circ}(g, h, \bar{x})$, where the Farkas lemma yields some explicit description of $L^{\circ}(g, h, \bar{x})$. With it, one can show that each local minimal point $\bar{x}$ of f on X at which GCQ holds is a Karush–Kuhn–Tucker point.

Fig. 1 Example for the gap between multi- and single-objective optimization



Since both $T^\circ(X, \bar{x})$ and $L^\circ(g, h, \bar{x})$ are closed convex cones, the GCQ at $\bar{x}$ can equivalently be restated as $L(g, h, \bar{x}) \subseteq T^{\circ\circ}(X, \bar{x})$ with the bipolar cone $T^{\circ\circ}(X, \bar{x})$ of $T(X, \bar{x})$. Thus, in view of $T(X, \bar{x}) \subseteq T^{\circ\circ}(X, \bar{x})$, the GCQ is weaker than the ACQ at $\bar{x}$. Indeed, the GCQ is the weakest condition under which a point $\bar{x} \in X$ is a Karush–Kuhn–Tucker point for every at $\bar{x}$ continuously differentiable function f possessing $\bar{x}$ as a local minimizer on X [14].

One might expect that analogously, in Corollary 2.1, one can replace the ACQ by the GCQ. However, examples show that the inclusion $LWPO(f, X) \cap GCQ(X) \subseteq KT_{\geq}(f, g, h)$ does not hold in general, so that the GCQ is too weak to act as a constraint qualification for local weak Pareto optimal points to be weak Kuhn–Tucker points. This effect is known as the 'gap' between multiobjective and single-objective optimization [1, 6, 39]. The following example covers the crucial aspect of the one treated in [1, 6, 39], but with a somewhat simpler geometry and with an emphasis on strict local Pareto optimal points of order one.

Example 2.1 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be some function with $\varphi(x) = 0$ for all $x \leqq 0$ and $\varphi(x) > 0$ else. For example, choose the C^k-function $\varphi(x) = (\max\{0, x\})^{k+1}, k \in \mathbb{N}_0$, or the C^∞-function with $\varphi(x) = 0$, $x \leqq 0$, and $\varphi(x) = \exp(-1/x)$, $x > 0$. Then, $\varphi(a)\varphi(b) \leqq 0$ is equivalent to $\min\{a, b\} \leqq 0$. Therefore, with $g(x) := \varphi(-x_1 - 2x_2)\varphi(-2x_1 - x_2)$, the set $X = \{x \in \mathbb{R}^2 \mid g(x) \leqq 0\}$ possesses the shape shown in Fig. 1.

Due to $\nabla g(\bar{x}) = 0$ at $\bar{x} = 0 \in X$, we have $L(g, \bar{x}) = \mathbb{R}^2$. Moreover, $T(X, \bar{x}) = X$, $T^\circ(X, \bar{x}) = \{0\}$ and $T^{\circ\circ}(X, \bar{x}) = \mathbb{R}^2 = L(g, \bar{x})$ hold, so that X satisfies the GCQ (but not the ACQ) at $\bar{x}$. At the same time, the function $f(x) = x$ possesses a strict local Pareto optimal point of order one at $\bar{x}$ which is not even a weak Kuhn–Tucker point.

Note that in the single-objective setting, the set X from Example 2.1 does not provide a counterexample to the fact that the GCQ is the weakest condition under which a point $\bar{x} \in X$ is a Karush–Kuhn–Tucker point for every at $\bar{x}$ continuously differentiable function f possessing $\bar{x}$ as a local minimizer on X, since no continuously differentiable function f can possess $\bar{x}$ as a local minimizer on X.

3 Generalized Polarity and Stationarity

3.1 Motivation

To prepare the subsequent constructions, recall (e.g. from [32]) that the regular normal cone to a set X at $\bar{x} \in X$ can be defined as

$$\widehat{N}(X, \bar{x}) = \left\{ v \in \mathbb{R}^n \mid \limsup_{x \to \bar{x}, x \in X} \frac{v^\top (x - \bar{x})}{\|x - \bar{x}\|} \leqq 0 \right\},$$

and that it coincides with the polar cone $T^\circ(X, \bar{x})$ (cf. (6)) of the tangent cone $T(X, \bar{x})$. Therefore, the stationarity condition (6) for minimizing a C^1-function $f : X \to \mathbb{R}^1$ may be rewritten as $-\nabla f(\bar{x}) \in \widehat{N}(X, \bar{x})$. We remark that our subsequent arguments will solely rely on generalizations of the regular normal cone $\widehat{N}(X, \bar{x})$, but not on the limiting construction $N(X, \bar{x}) = \limsup_{x \to \bar{x}, x \in X} \widehat{N}(X, \bar{x})$ (cf., e.g. [30, 32] for details on this normal cone).

In [17], the generalization

$$\widehat{N}_{\not>,m}(X, \bar{x}) = \left\{ V \in \mathbb{R}^{n \times m} \mid \limsup_{x \to \bar{x}, x \in X} \min_{k=1,\ldots,m} \frac{(v^k)^\top (x - \bar{x})}{\|x - \bar{x}\|} \leqq 0 \right\}$$

of the regular normal cone in $\mathbb{R}^n$ to a matrix cone in $\mathbb{R}^{n \times m}$ is introduced, where the matrix $V = (v^1, \ldots, v^m)$ with columns $v^k \in \mathbb{R}^n$, $k = 1, \ldots, m$, possesses the dimensions of the Jacobian $\nabla f(\bar{x})$ of the objective function f of MOP at $\bar{x}$. This justifies to call $\widehat{N}_{\not>,m}(X, \bar{x})$ a 'multiobjective regular normal cone' although the vector function f does not appear explicitly in its definition.

In [17, Th. 3.1], the identity

$$\widehat{N}_{\not>,m}(X, \bar{x}) = \left\{ V \in \mathbb{R}^{n \times m} \mid \min_{k=1,\ldots,m} (v^k)^\top d \leqq 0 \, \forall d \in T(X, \bar{x}) \right\}$$

is shown, which is reminiscent of the definition of a polar cone. Our subsequent constructions are based on the 'negative formulation'

$$\widehat{N}_{\not>,m}(X, \bar{x}) = \left\{ V \in \mathbb{R}^{n \times m} \mid V^\top d \not> 0 \, \forall d \in T(X, \bar{x}) \right\}$$

of the latter description, which also explains our notation for this cone. Observe that, while for $m = 1$ the set $\widehat{N}_{\not>,m}(X, \bar{x})$ collapses to the standard regular normal cone

$$\widehat{N}_{\not>,1}(X, \bar{x}) = \left\{ v \in \mathbb{R}^n \mid v^\top d \not> 0 \, \forall d \in T(X, \bar{x}) \right\} = \widehat{N}(X, \bar{x}),$$

the same is also true for the matrix cone

$$\widehat{N}_{\not\geq,m}(X, \bar{x}) = \left\{ V \in \mathbb{R}^{n \times m} \mid V^\top d \not\geq 0 \, \forall d \in T(X, \bar{x}) \right\},$$

since the vector inequalities $>$ and $\geq$ both generalize the scalar inequality $>$. In addition, we define the matrix cone

$$\widehat{N}_{\ngeqq,m}(X,\bar{x}) = \left\{V \in \mathbb{R}^{n\times m} \mid V^\top d \ngeqq 0\,\forall d \in T(X,\bar{x}) \setminus \{0\}\right\}$$

as a generalization of the (possibly empty) cone

$$\operatorname{int} \widehat{N}(X,\bar{x}) = \left\{v \in \mathbb{R}^n \mid v^\top d < 0\,\forall d \in T(X,\bar{x}) \setminus \{0\}\right\} = \widehat{N}_{\ngeqq,1}(X,\bar{x}).$$

These multiobjective regular normal cones satisfy the relations $\widehat{N}_{\ngeqq,m}(X,\bar{x}) \subseteq \widehat{N}_{\ngeq,m}(X,\bar{x}) \subseteq \widehat{N}_{\ngtr,m}(X,\bar{x})$. The crucial observation for the following is that they may be viewed as generalized polar cones of the tangent cone $T(X,\bar{x})$, where different generalizations of polarity come into play.

3.2 Definition of Generalized Polarity and Relations to Stationarity

Recall that the polar cone of a cone $A \subseteq \mathbb{R}^n$ is defined as $A^\circ = \{v \in \mathbb{R}^n \mid v^\top a \leqq 0\,\forall a \in A\}$. The negative formulation $A^\circ = \{v \in \mathbb{R}^n \mid v^\top a \ngtr 0\,\forall a \in A\}$ of the polar cone gives rise to its two natural generalizations

$$\begin{aligned} A^{\ngtr,m} &= \left\{V \in \mathbb{R}^{n\times m} \mid V^\top a \ngtr 0\,\forall a \in A\right\}, \\ A^{\ngeq,m} &= \left\{V \in \mathbb{R}^{n\times m} \mid V^\top a \ngeq 0\,\forall a \in A\right\}, \end{aligned}$$

as well as to the introduction of the (possibly empty) matrix cone

$$A^{\ngeqq,m} = \left\{V \in \mathbb{R}^{n\times m} \mid V^\top a \ngeqq 0\,\forall a \in A \setminus \{0\}\right\}.$$

The identities $\widehat{N}_{\ngtr,m}(X,\bar{x}) = T^{\ngtr,m}(X,\bar{x})$, $\widehat{N}_{\ngeq,m}(X,\bar{x}) = T^{\ngeq,m}(X,\bar{x})$ and $\widehat{N}_{\ngeqq,m}(X,\bar{x}) = T^{\ngeqq,m}(X,\bar{x})$ imply the equivalences

$$\begin{aligned} -\nabla f(\bar{x}) \in \widehat{N}_{\ngtr,m}(X,\bar{x}) &\Leftrightarrow C_<(f,\bar{x}) \cap T(X,\bar{x}) = \emptyset, \\ -\nabla f(\bar{x}) \in \widehat{N}_{\ngeq,m}(X,\bar{x}) &\Leftrightarrow C_\leq(f,\bar{x}) \cap T(X,\bar{x}) = \emptyset, \\ -\nabla f(\bar{x}) \in \widehat{N}_{\ngeqq,m}(X,\bar{x}) &\Leftrightarrow C_\leqq(f,\bar{x}) \cap T(X,\bar{x}) = \{0\}, \end{aligned}$$

and therefore the alternative descriptions of the stationarity sets (Definition 2.4)

$$WSt(f,X) = \{x \in X \mid -\nabla f(x) \in \widehat{N}_{\ngtr,m}(X,x)\}, \tag{7}$$

$$PSt(f,X) = \{x \in X \mid -\nabla f(x) \in \widehat{N}_{\ngeq,m}(X,x)\}, \tag{8}$$

$$SSt(f,X)=\{x\in X\mid -\nabla f(x)\in \widehat{N}_{\ngeqq,m}(X,x)\}. \tag{9}$$

Likewise, the characterizations of the Kuhn–Tucker sets from Proposition 2.2 may be rewritten as

$$KT_{\geq}(f,g,h)=\{x\in X\mid -\nabla f(x)\in L^{\not>,m}(g,h,x)\}, \tag{10}$$

$$KT_{>}(f,g,h)=\{x\in X\mid -\nabla f(x)\in L^{\ngeq,m}(g,h,x)\}, \tag{11}$$

$$KT_{\gg}(f,g,h)=\{x\in X\mid -\nabla f(x)\in L^{\ngeqq,m}(g,h,x)\}. \tag{12}$$

3.3 Basic Properties of Generalized Polar Cones

The crucial properties of the three generalized normal cones stem from their polarity structure. Therefore, in the following, we collect basic facts about the three generalized polar cones of an arbitrary cone $A\subseteq\mathbb{R}^n$.

For any cone A, the inclusions

$$A^{\ngeqq,m}\cup\{0\}\subseteq A^{\ngeq,m}\subseteq A^{\not>,m} \tag{13}$$

are clear, and it is not hard to see that all three sets are cones. However, as opposed to the standard polar cone A°, neither of them is necessarily convex. This is illustrated by the subsequent example which uses that, with any V from one of the three generalized polar cones, also all matrices resulting from V by permutations of its columns lie in the respective cone. Convex combinations of a matrix V and one of its permutations, however, do not need to lie in the respective cone.

Example 3.1 For the cone $X=\{x\in\mathbb{R}^2\mid \varphi(-x_1-2x_2)\varphi(-2x_1-x_2)\leqq 0\}$ from Example 2.1, consider $V^1=(-e_1,-e_2)$ and $V^2=(-e_2,-e_1)$ with the first and second unit vectors e_1 and e_2, respectively. Then, due to $X\setminus\{0\}\subseteq\{x\in\mathbb{R}^2\mid x_1>0 \text{ or } x_2>0\}$, for all $x\in X$, the relations $-e_1^\top x<0$ or $-e_2^\top x<0$ and, therefore, $V^1,V^2\in X^{\ngeqq,2}\subseteq X^{\ngeq,2}\subseteq X^{\not>,2}$ hold. However, with the all-ones vector e and the point $\bar{x}=(-2,1)^\top\in X$, the matrix $V^3=\frac{1}{2}V^1+\frac{1}{2}V^2=-\frac{1}{2}(e,e)$ satisfies $V^3\bar{x}=\frac{1}{2}e>0$ and, thus, $V^3\in(X^{\not>,2})^c\subseteq(X^{\ngeq,2})^c\subseteq(X^{\ngeqq,2})^c$, where the notation A^c refers to the complement of a set A. Consequently, none of the three generalized polar cones of X is convex.

While the standard polar cone A° of a cone $A\subseteq\mathbb{R}^n$ is a closed set, for the generalized polar cones, we list the following properties. For their proofs, it will be convenient that with the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}\subseteq\mathbb{R}^n$, we may alternatively write

$$A^{\not>,m}=\{V\in\mathbb{R}^{n\times m}\mid V^\top a\not>0\;\forall a\in A\cap\mathbb{S}^{n-1}\},$$

$$A^{\ngeq,m}=\{V\in\mathbb{R}^{n\times m}\mid V^\top a\ngeq 0\;\forall a\in A\cap\mathbb{S}^{n-1}\},$$

$$A^{\ngeqq,m}=\{V\in\mathbb{R}^{n\times m}\mid V^\top a\ngeqq 0\;\forall a\in A\cap\mathbb{S}^{n-1}\}.$$

Proposition 3.1 *For any cone $A \subseteq \mathbb{R}^n$, the set $A^{\not>,m}$ is closed, for any closed cone $A \subseteq \mathbb{R}^n$, the set $A^{\not\geq,m}$ is open, and in the case $m \geqq 2$ even for closed cones A, the set $A^{\not\geq,m}$ is in general neither open nor closed.*

Proof The set complement of $A^{\not>,m}$ coincides with the union of open sets $\bigcup_{a\in A}\{V \in \mathbb{R}^{n\times m} \mid V^\top a > 0\}$, which shows the closedness of $A^{\not>,m}$. This also covers the case $A = \emptyset$ with the closed set $A^{\not>,m} = \mathbb{R}^{n\times m}$.

For $A \subseteq \{0\}$, the set $A^{\not\geq,m} = \mathbb{R}^{n\times m}$ is open, and for $A \supsetneq \{0\}$, we may write

$$A^{\not\geq,m} = \left\{V \in \mathbb{R}^{n\times m} \mid \max_{a\in A\cap\mathbb{S}^{n-1}} \min_{k=1,\dots,m} (v^k)^\top a < 0\right\}$$

with the nonempty and compact set $A\cap\mathbb{S}^{n-1}$ and the, for given parameter V, continuous function $\min_{k=1,\dots,m}(v^k)^\top a$. Since the latter function is also continuous in V, so is the optimal value function $\varphi(V) = \max_{a\in A\cap\mathbb{S}^{n-1}} \min_{k=1,\dots,m}(v^k)^\top a$, which proves the openness of $A^{\not\geq,m}$.

Finally, for the closed cone $A = \mathbb{R}_{\geqq} \times \{0\} \subseteq \mathbb{R}^2$, the set

$$\begin{aligned} A^{\not\geq,2} &= \{V \in \mathbb{R}^{2\times 2} \mid V^\top a \not\geq 0\,\forall a \in A\cap\mathbb{S}^1\} = \{V \in \mathbb{R}^{2\times 2} \mid V^\top e_1 \not\geq 0\} \\ &= \{V \in \mathbb{R}^{2\times 2} \mid (v_{11}, v_{12})^\top \not\geq 0\} \end{aligned}$$

is neither open nor closed (where e_1 denotes the first unit vector). $\square$

We remark that the openness of $A^{\not\geq,m}$ for a closed cone A implies

$$0 \in A^{\not\geq,m} \iff A^{\not\geq,m} = \mathbb{R}^{n\times m}. \tag{14}$$

Recall that, in the single-objective case, one may have $\nabla f(\bar{x}) = 0$ at a constrained locally minimal point, namely when the feasible set X may locally be replaced by $\mathbb{R}^n$. In the stationarity condition $-\nabla f(\bar{x}) \in \widehat{N}(X, \bar{x})$, this case is covered by the property $0 \in \widehat{N}(X, \bar{x})$. In the multiobjective setting, the generalized normal cones contain strictly larger sets than $\{0\}$ in the case that locally X can be replaced by $\mathbb{R}^n$, except for the cone $A^{\not\geq,m}$ when $m \leqq n$.

Proposition 3.2 *For any cone $A \subseteq \mathbb{R}^n$ and $m \geqq 2$, the following relations hold.*

(a) $A^{\not>,m} \supseteq (\mathbb{R}^n)^{\not>,m} = \{V \in \mathbb{R}^{n\times m} \mid \exists\lambda \geq 0 : V\lambda = 0\} \supsetneq \{0\}$.
(b) $A^{\not\geq,m} \supseteq (\mathbb{R}^n)^{\not\geq,m} = \{V \in \mathbb{R}^{n\times m} \mid \exists\lambda > 0 : V\lambda = 0\} \supsetneq \{0\}$.
(c) $A^{\not\geq,m} \supseteq (\mathbb{R}^n)^{\not\geq,m} = \{V \in \mathbb{R}^{n\times m} \mid \operatorname{rank}(V) = n$ *and* $\exists\lambda > 0 : V\lambda = 0\} \begin{cases} \supsetneq \{0\}, & m > n, \\ = \emptyset, & m \leqq n. \end{cases}$

Proof The first inclusions in all three assertions are clear from the definitions of the generalized polar cones. To see the equation in part a, observe that any matrix V from

$(\mathbb{R}^n)^{\ngtr,m}$ is characterized by the unsolvability of the system $V^\top y > 0$ with $y \in \mathbb{R}^n$. By Gordan's theorem [27], the latter is equivalent to the existence of some $\lambda \geq 0$ with $V\lambda = 0$, which shows the assertion. The identity in part b is shown analogously, with Stiemke's theorem [27] in place of Gordan's theorem, and the identity in part c is shown with the arguments from the proof of [12, Th. 3.4]. To see the second inclusion in parts a and b, choose $V = (e_1, -e_1, 0, \ldots, 0)$ with the first unit vector $e_1 \in \mathbb{R}^n$. The second inclusion in part c for the case $m > n$ is shown by the choice $V = (e_1, \ldots, e_n, -e, 0, \ldots, 0)$, where $e \in \mathbb{R}^n$ denotes the all-ones vector. In the case $m \leqq n$, the condition $\mathrm{rank}(V) = n$ can only be satisfied for $m = n$, and in this case, the equation $V\lambda = 0$ can only be solved with $\lambda = 0$. This shows the remaining assertion. □

We remark that the combination of Proposition 3.2c with Proposition 2.1c and (9) yields in particular that unconstrained strict local Pareto optimal points of order one exist exactly in the case $m > n$. Moreover, for later use, let us introduce the following definition.

Definition 3.1 We call a matrix $V \in \mathbb{R}^{n\times m}$ *starlike* if $\mathrm{rank}(V) = n$ holds and if there exists some $\lambda > 0$ with $V\lambda = 0$.

In the proof of Proposition 3.2c, we have seen that in $\mathbb{R}^{n\times m}$ starlike matrices exist exactly in the case $m > n$. Moreover, by Stiemke's theorem, the existence of some $\lambda > 0$ with $V\lambda = 0$ is equivalent to the unsolvability of $V^\top y \geq 0$, $y \in \mathbb{R}^n$, that is, to $V \in (\mathbb{R}^n)^{\ngeqq,m}$. Therefore, V is starlike if and only if $\mathrm{rank}(V) = n$ and $V \in (\mathbb{R}^n)^{\ngeqq,m}$ hold.

3.4 Geometrical Characterizations

For the geometrical intuition about generalized polar cones the following constructions are useful. For any $V \in \mathbb{R}^{n\times m}$ the set $\Gamma_{\geqq}(V) = \{y \in \mathbb{R}^n \mid V^\top y \geqq 0\}$ is a polyhedral cone. Subsequently, it will be important that it is described by m linear inequalities, so that we refer to it as an *m-polyhedral cone*. With this notation, we have

$$A^{\ngeq,m} = \{V \in \mathbb{R}^{n\times m} \mid \Gamma_{\geqq}(V) \cap A = \{0\}\}, \tag{15}$$

and $A^{\ngeq,m}$ is nonempty if and only if $\Gamma_{\geqq}(V) \cap A = \{0\}$ holds for some $V \in \mathbb{R}^{n\times m}$.

Example 3.2 Consider the cone $A = \mathbb{R}_{\geqq} \times \mathbb{R}^2 \subseteq \mathbb{R}^3$ and $m = 2$. Since no 2-polyhedral cone in $\mathbb{R}^3$ is pointed, the condition $\Gamma_{\geqq}(V) \cap A = \{0\}$ is violated for any $V \in \mathbb{R}^{3\times 2}$, and $A^{\ngeq,2}$ is empty.

For analogous characterizations of $A^{\ngeqq,m}$ and $A^{\ngtr,m}$, observe that $V \in A^{\ngeqq,m}$ and $V \in A^{\ngtr,m}$ hold if and only if the sets $\{y \in \mathbb{R}^n \mid V^\top y \geq 0\} \cap A$ and $\{y \in \mathbb{R}^n \mid V^\top y > 0\} \cap A$ are empty, respectively. With the (nonclosed) generalized m-polyhedral cones

$\Gamma_{\geq}(V) = \{y \in \mathbb{R}^n \mid V^\top y \geq 0\}$ and $\Gamma_{>}(V) = \{y \in \mathbb{R}^n \mid V^\top y > 0\}$, one may thus write

$$A^{\ngeq,m} = \{V \in \mathbb{R}^{n\times m} \mid \Gamma_{\geq}(V) \cap A = \emptyset\}, \tag{16}$$
$$A^{\ngtr,m} = \{V \in \mathbb{R}^{n\times m} \mid \Gamma_{>}(V) \cap A = \emptyset\}. \tag{17}$$

Observe that for $V \in \mathbb{R}^{n\times m}$ the lineality space of $\Gamma_{\geqq}(V)$ is the set $\Gamma_{=}(V) = \{y \in \mathbb{R}^n \mid V^\top y = 0\}$, so that

$$\Gamma_{\geq}(V) = \Gamma_{\geqq}(V) \setminus \Gamma_{=}(V) \tag{18}$$

holds. Also note that $\Gamma_{>}(V) = \operatorname{int} \Gamma_{\geqq}(V)$ only holds if V does not contain zero vectors as columns.

4 Multiobjective Guignard Constraint Qualifications

4.1 Generalized Polar Cones of Matrix Cones

While for a cone $A \subseteq \mathbb{R}^n$ the generalized polar cones are matrix cones, we will also consider generalized polar cones of such matrix cones. Indeed, for a cone B of matrices in $\mathbb{R}^{n\times m}$ we define the respective generalized polar cones

$$B^{m,\ngtr} = \{a \in \mathbb{R}^n \mid V^\top a \ngtr 0 \,\forall V \in B\},$$
$$B^{m,\ngeq} = \{a \in \mathbb{R}^n \mid V^\top a \ngeq 0 \,\forall V \in B\},$$
$$B^{m,\ngeqq} = \{a \in \mathbb{R}^n \mid V^\top a \ngeqq 0 \,\forall V \in B \setminus \{0\}\}$$

(where we use the notation $B^{m,\ngtr}$ rather than $B^{\ngtr,m}$, etc., to indicate that we 'switch back' from matrix cones to cones in $\mathbb{R}^n$). In analogy to (13), for any matrix cone B, the relations

$$B^{m,\ngeqq} \cup \{0\} \subseteq B^{m,\ngeq} \subseteq B^{m,\ngtr} \tag{19}$$

are clear. It will be useful that with the notation from Sect. 3.4, they may be written as

$$B^{m,\ngtr} = \{a \in \mathbb{R}^n \mid a \notin \Gamma_{>}(V) \,\forall V \in B\}, \tag{20}$$
$$B^{m,\ngeq} = \{a \in \mathbb{R}^n \mid a \notin \Gamma_{\geq}(V) \,\forall V \in B\}, \tag{21}$$
$$B^{m,\ngeqq} = \{a \in \mathbb{R}^n \mid a \notin \Gamma_{\geqq}(V) \,\forall V \in B \setminus \{0\}\}. \tag{22}$$

Lemma 4.1 *The following relations hold for any cones $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^{n\times m}$.*

(a) $A \subseteq B^{m,\ngtr} \iff B \subseteq A^{\ngtr,m}$.
(b) $A \subseteq B^{m,\ngeq} \iff B \subseteq A^{\ngeq,m}$.

(c) $A \setminus \{0\} \subseteq B^{m,\ngeqq} \iff B \setminus \{0\} \subseteq A^{\ngeqq,m}$.

Proof The proof of $\Rightarrow$ in part a is trivial in the case $B = \emptyset$. Otherwise, choose some $V \in B$. Then, for all $a \in B^{m,\not>}$, we have $V^\top a \not> 0$. The condition $A \subseteq B^{m,\not>}$ implies $V^\top a \not> 0$ for all $a \in A$ and thus $V \in A^{\not>,m}$. The proof of $\Leftarrow$ is trivial for $A = \emptyset$. Otherwise, take some $a \in A$. Then, for all $V \in A^{\not>,m}$, we have $V^\top a \not> 0$. Under the condition $B \subseteq A^{\not>,m}$, this yields $V^\top a \not> 0$ for all $V \in B$ and thus $a \in B^{m,\not>}$. While the proofs of part b and c run along the same lines, note that for part c one needs to account for $0 \notin B^{m,\ngeqq}$ and $0 \notin A^{\ngeqq,m}$. □

The following result justifies the introduction of generalized polar cones of matrix cones.

Lemma 4.2 *For $\bar{x} \in X$, let $-\nabla f(\bar{x}) \in B$ hold for some cone $B \subseteq \mathbb{R}^{n\times m}$. Then, the following assertions hold.*

(a) $L(g, h, \bar{x}) \subseteq B^{m,\not>}$ *implies* $\bar{x} \in KT_{\geq}(f, g, h)$.
(b) $L(g, h, \bar{x}) \subseteq B^{m,\ngeq}$ *implies* $\bar{x} \in KT_{>}(f, g, h)$.
(c) $L(g, h, \bar{x})\setminus\{0\} \subseteq B^{m,\ngeqq}$ *and* $\nabla f(\bar{x}) \neq 0$ *imply* $\bar{x} \in KT_{\gg}(f, g, h)$.
(d) $L(g, h, \bar{x}) \setminus \{0\} \subseteq B^{m,\ngeqq}$ *and the openness of* B *imply* $\bar{x} \in KT_{\gg}(f, g, h)$.

Proof For the proof of part a observe that, by Lemma 4.1a, the inclusion $L(g, h, \bar{x}) \subseteq B^{m,\not>}$ is equivalent to $B \subseteq L^{\not>,m}(g, h, \bar{x})$. The assumption $-\nabla f(\bar{x}) \in B$ thus implies $-\nabla f(\bar{x}) \in L^{\not>,m}(g, h, \bar{x})$, and (10) yields the assertion. The proof of part b runs along the same lines, using Lemma 4.1b, $-\nabla f(\bar{x}) \in L^{\ngeq,m}(X, \bar{x})$, and (11).

In the proof of part c, Lemma 4.1c yields $B \setminus \{0\} \subseteq L^{\ngeqq,m}(X, \bar{x})$, so that $-\nabla f(\bar{x}) \in B\setminus\{0\}$ provides $-\nabla f(\bar{x}) \in L^{\ngeqq,m}(X, \bar{x})$, and (12) proves the assertion. The assertion of part d follows from c in the case $\nabla f(\bar{x}) \neq 0$. Otherwise, in analogy to (14), the condition $0 = -\nabla f(\bar{x}) \in B$ implies $B = \mathbb{R}^{n\times m}$. The set $B^{m,\ngeqq} = (\mathbb{R}^{n\times m})^{m,\ngeqq} = \{a \in \mathbb{R}^n \mid V^\top a \ngeqq 0 \, \forall V \in \mathbb{R}^{n\times m}\setminus\{0\}\}$ is empty, since it clearly does not contain $a = 0$ and since for any $a \neq 0$ the matrix $V = ae^\top \neq 0$ satisfies $V^\top a = \|a\|^2 e > 0$. Therefore, under $L(g, h, \bar{x}) \setminus \{0\} \subseteq B^{m,\ngeqq}$ we obtain $L(g, h, \bar{x}) = \{0\}$, and the statement follows in view of (12). □

4.2 Generalized Bipolar Cones

The combinations of the stationarity conditions (7), (8) and (9) with the assertions from Lemma 4.2 allow us to formulate nine conditions under which a stationary point of one of the three considered types is a Kuhn–Tucker point of one of its three considered types. For example, with the choice $B = \widehat{N}_{\not>,m}(X, \bar{x})$, the condition $L(g, h, \bar{x}) \subseteq \widehat{N}^{m,\not>}_{\not>,m}(X, \bar{x})$ guarantees that a weakly stationary point $\bar{x}$ is a weak Kuhn–Tucker point. The condition $L(g, h, \bar{x}) \subseteq \widehat{N}^{m,\not>}_{\not>,m}(X, \bar{x})$ is thus a constraint qualification. In view of $\widehat{N}^{m,\not>}_{\not>,m}(X, \bar{x}) = (T^{\not>,m})^{m,\not>}(X, \bar{x})$, it is actually reminiscent of the Guignard constraint qualification $L(g, h, \bar{x}) \subseteq T^{\circ\circ}(X, \bar{x})$ from the single-objective case, where the standard bipolar cone $T^{\circ\circ}(X,\bar{x})$ of $T(X,\bar{x})$ is replaced by the generalized bipolar cone $(T^{\not>,m})^{m,\not>}(X,\bar{x})$.

The mentioned nine conditions rely on nine corresponding generalized bipolar cones, whose properties will play a crucial role in the sequel. For their investigation, we will use the following explicit descriptions which are based on the concept of generalized m-polyhedral cones from Sect. 3.4.

Lemma 4.3 *For any closed cone $A \subseteq \mathbb{R}^n$, the following assertions hold.*

(a) $(A^{\not>,m})^{m,\not>} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m} : \ \Gamma_>(V) \cap A = \emptyset,\ d \in \Gamma_>(V)\}^c$,
(b) $(A^{\ngeq,m})^{m,\not>} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m} : \ \Gamma_\geq(V) \cap A = \emptyset,\ d \in \Gamma_>(V)\}^c$,
(c) $(A^{\not\geqq,m})^{m,\not>} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m} : \ \Gamma_\geqq(V) \cap A = \{0\},\ d \in \Gamma_>(V)\}^c$,
(d) $(A^{\not>,m})^{m,\ngeq} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m} : \ \Gamma_>(V) \cap A = \emptyset,\ d \in \Gamma_\geq(V)\}^c$,
(e) $(A^{\ngeq,m})^{m,\ngeq} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m} : \ \Gamma_\geq(V) \cap A = \emptyset,\ d \in \Gamma_\geq(V)\}^c$,
(f) $(A^{\not\geqq,m})^{m,\ngeq} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m} : \ \Gamma_\geqq(V) \cap A = \{0\},\ d \in \Gamma_\geq(V)\}^c$,
(g) $(A^{\not>,m})^{m,\not\geqq} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m}\backslash\{0\} : \ \Gamma_>(V) \cap A = \emptyset,\ d \in \Gamma_\geqq(V)\}^c$,
(h) $(A^{\ngeq,m})^{m,\not\geqq} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m}\backslash\{0\} : \ \Gamma_\geq(V) \cap A = \emptyset,\ d \in \Gamma_\geqq(V)\}^c$,
(i) $(A^{\not\geqq,m})^{m,\not\geqq} = \{d \in \mathbb{R}^n \mid \exists\, V \in \mathbb{R}^{n\times m}\backslash\{0\} : \ \Gamma_\geqq(V) \cap A = \{0\},\ d \in \Gamma_\geqq(V)\}^c$.

Proof Choose some $d \notin (A^{\not>,m})^{m,\not>}$. By (20), this is equivalent to the existence of some $V \in A^{\not>,m}$ with $d \in \Gamma_>(V)$. In view of (17), the latter means that there exists some $V \in \mathbb{R}^{n\times m}$ with $\Gamma_>(V) \cap A = \emptyset$ and $d \in \Gamma_>(V)$. This shows the assertion of part a. The other eight assertions are shown analogously, by appropriate combinations of (20), (21) and (22) with (15), (16) and (17), respectively. □

4.3 Definition of Multiobjective Guignard Constraint Qualifications

The constraint qualifications $L(g,h,\bar{x}) \subseteq \widehat{N}^{m,\not>}_{\not>,m}(X,\bar{x})$ and $L(g,h,\bar{x}) \subseteq \widehat{N}^{m,\ngeq}_{\not>,m}(X,\bar{x})$ were introduced and studied by Haeser and Ramos [17]. More generally, Lemma 4.2 allows to formulate seven Guignard-type constraint qualifications for *MOP* as well as two conditions involving the objective function. These are not proper constraint qualifications, but rather regularity conditions. Let $MOGCQ^{m,\ngeq}_{\not>,m}(X)$ denote the set of $\bar{x} \in X$ at which $L(g,h,\bar{x}) \subseteq \widehat{N}^{m,\ngeq}_{\not>,m}(X,\bar{x})$ holds, etc. For the proofs of Theorem 4.1g, h and i with Lemma 4.2c and d, respectively, note that by Proposition 3.1, the cones $\widehat{N}_{\not>,m}(X,\bar{x})$ and $\widehat{N}_{\ngeq,m}(X,\bar{x})$ are in general not open, while $\widehat{N}_{\not\geqq,m}(X,\bar{x})$ is.

Theorem 4.1 *The following assertions hold.*

(a) $WSt(f,X) \cap MOGCQ^{m,\not>}_{\not>,m}(X) \subseteq KT_\geq(f,g,h)$.
(b) $PSt(f,X) \cap MOGCQ^{m,\not>}_{\ngeq,m}(X) \subseteq KT_\geq(f,g,h)$.
(c) $SSt(f,X) \cap MOGCQ^{m,\not>}_{\not\geqq,m}(X) \subseteq KT_\geq(f,g,h)$.

Table 1 Attributions of MOGCQs to stationarity and Kuhn–Tucker notions

	$WSt(f, X)$	$PSt(f, X)$	$SSt(f, X)$
$KT_{\geq}(f, g, h)$	$MOGCQ^{m,\not>}_{\not>,m}$	$MOGCQ^{m,\not>}_{\ngeq,m}$	$MOGCQ^{m,\not>}_{\ngeqq,m}$
$KT_{>}(f, g, h)$		$MOGCQ^{m,\ngeq}_{\ngeq,m}$	$MOGCQ^{m,\ngeq}_{\ngeqq,m}$
$KT_{\gg}(f, g, h)$			$MOGCQ^{m,\ngeqq}_{\ngeqq,m}$

(d) $WSt(f, X) \cap MOGCQ^{m,\ngeq}_{\not>,m}(X) \subseteq KT_{>}(f, g, h)$.

(e) $PSt(f, X) \cap MOGCQ^{m,\ngeq}_{\ngeq,m}(X) \subseteq KT_{>}(f, g, h)$.

(f) $SSt(f, X) \cap MOGCQ^{m,\ngeq}_{\ngeqq,m}(X) \subseteq KT_{>}(f, g, h)$.

(g) $WSt(f, X) \cap MOGCQ^{m,\ngeqq}_{\not>,m}(X) \cap \{\bar{x} \in X \mid \nabla f(\bar{x}) \neq 0\} \subseteq KT_{\gg}(f, g, h)$.

(h) $PSt(f, X) \cap MOGCQ^{m,\ngeqq}_{\ngeq,m}(X) \cap \{\bar{x} \in X \mid \nabla f(\bar{x}) \neq 0\} \subseteq KT_{\gg}(f, g, h)$.

(i) $SSt(f, X) \cap MOGCQ^{m,\ngeqq}_{\ngeqq,m}(X) \subseteq KT_{\gg}(f, g, h)$.

By Proposition 2.1c, the set $SSt(f, X)$ can be replaced by $StrLPO(1, f, X)$ in Theorem 4.1c, f and i. Moreover, the combination of (3) and (4) with the results of Theorem 4.1a, b, d, e, g and h, respectively, shows that the MOGCQs can be applied to stronger optimality notions while yielding the same Kuhn–Tucker properties. For example, Theorem 4.1a and (3) yield $LWPO(f, X) \cap MOGCQ(X)^{m,\not>}_{\not>,m} \subseteq KT_{\geq}(f, g, h)$, which is also shown in [17, Th. 4.1]. Analogously, Theorem 4.1d and (3) imply $LWPO(f, X) \cap MOGCQ(X)^{m,\ngeq}_{\not>,m} \subseteq KT_{>}(f, g, h)$, which Haeser and Ramos show in [17, Th. 4.2]. They point out, however, that for $m > 1$, the underlying constraint qualification $L(g, h, \bar{x}) \subseteq \widehat{N}^{m,\ngeq}_{\not>,m}(X, \bar{x})$ is too strong for practical applications, since it is not even implied by LICQ at $\bar{x} \in X$.

The latter observation is not surprising, since simple examples show that local weak Pareto optimal points which are not local Geoffrion proper Pareto optimal may only be expected to be strong Kuhn–Tucker points under rather strong additional assumptions. Similar reasoning yields that the two regularity conditions from Theorem 4.1g and h are too strong to be relevant for applications. Therefore, we will not consider the three conditions from Theorem 4.1d, g and h any further in this paper. The attributions of the remaining six MOGCQs to stationarity and Kuhn–Tucker notions, as stated in Theorem 4.1, are visualized in Table 1.

The most 'natural' of the six remaining constraint qualifications are the ones from Theorem 4.1a, e and i, which use the same generalized polarity type twice. For easier reference, we denote them as follows.

Table 2 Straightforward inclusions between generalized bipolar cones

$$
\begin{array}{ccccc}
(A^{\ngtr,m})^{m,\ngtr} \subseteq & (A^{\ngeq,m})^{m,\ngtr} & \subseteq & (A^{\ngeqq,m})^{m,\ngtr} & \\
 & \cup\!\!| & & \cup\!\!| & \\
 & (A^{\ngeq,m})^{m,\ngeq} & \subseteq & (A^{\ngeqq,m})^{m,\ngeq} & \\
 & & & \cup\!\!| & \\
 & & & (A^{\ngeqq,m})^{m,\ngeqq} \cup \{0\} &
\end{array}
$$

Definition 4.1 At $\bar{x} \in X$, we call

(a) $L(g, h, \bar{x}) \subseteq \widehat{N}^{m,\ngtr}_{\ngtr,m}(X, \bar{x})$ the weak multiobjective Guignard constraint qualification (WMOGCQ),
(b) $L(g, h, \bar{x}) \subseteq \widehat{N}^{m,\ngeq}_{\ngeq,m}(X, \bar{x})$ the proper multiobjective Guignard constraint qualification (PMOGCQ),
(c) $L(g, h, \bar{x}) \setminus \{0\} \subseteq \widehat{N}^{m,\ngeqq}_{\ngeqq,m}(X, \bar{x})$ the strong multiobjective Guignard constraint qualification (SMOGCQ).

As already mentioned, the WMOGCQ was introduced in [17].

While the remaining three constraint qualifications from Theorem 4.1b, c and f yield weaker results like a sufficient condition for proper stationary local Pareto optimal points to be weak Kuhn–Tucker points, such results are potentially helpful in applications. Rather than naming them, subsequently, we will refer to them as $\text{MOGCQ}^{m,\ngtr}_{\ngeq,m}$, $\text{MOGCQ}^{m,\ngtr}_{\ngeqq,m}$ and $\text{MOGCQ}^{m,\ngeq}_{\ngeqq,m}$, respectively. In fact, the next section will show that they coincide with natural MOGCQs.

4.4 Relations Between the Multiobjective Guignard Constraint Qualifications

For any cone $A \subseteq \mathbb{R}^n$, the inclusions in Table 2 are clear from (13) and (19). Therefore, the inclusions $MOGCQ^{m,\ngtr}_{\ngtr,m}(X) \subseteq MOGCQ^{m,\ngtr}_{\ngeq,m}(X)$, etc., hold.

In the following, we will clarify which of the inclusions from Table 2 are actually identities, and which ones may hold strictly.

Lemma 4.4

(a) For any nonempty closed cone $A \subseteq \mathbb{R}^n$, the identity $(A^{\ngtr,m})^{m,\ngtr} = (A^{\ngeq,m})^{m,\ngtr}$ holds.
(b) For any nonempty closed cone $A \subseteq \mathbb{R}^n$, the identities $(A^{\ngeqq,m})^{m,\ngeqq} \cup \{0\} = (A^{\ngeqq,m})^{m,\ngeq}$ and $(A^{\ngeqq,m})^{m,\ngeq} = (A^{\ngeqq,m})^{m,\ngtr}$ hold.
(c) There exists a closed cone $A \subseteq \mathbb{R}^3$ with $(A^{\ngeq,2})^{2,\ngeq} \subsetneq (A^{\ngeq,2})^{2,\ngtr}$.
(d) There exists a closed cone $A \subseteq \mathbb{R}^3$ with $A^{\ngeqq,2} \neq \emptyset$, $(A^{\ngeq,2})^{2,\ngtr} \subsetneq (A^{\ngeqq,2})^{2,\ngtr}$ and $(A^{\ngeq,2})^{2,\ngeq} \subsetneq (A^{\ngeqq,2})^{2,\ngeq}$.

Proof In view of Table 2, for the proof of part a, it remains to show the inclusion $(A^{\not>,m})^{m,\not>} \supseteq (A^{\not\geq,m})^{m,\not>}$. Let $d \notin (A^{\not>,m})^{m,\not>}$. Then, there is a $V \in \mathbb{R}^{n\times m}$ such that

$$\min_j (v^j)^\top a \leqq 0 \;\; \forall a \in A \tag{23}$$

and

$$V^\top d > 0 \tag{24}$$

hold. From V, we will construct a matrix $\widetilde{V} \in A^{\not\geq,m}$ with $\widetilde{V}^\top d > 0$, which shows $d \notin (A^{\not\geq,m})^{m,\not>}$ and completes the proof of part a.

Indeed, for all $k \in \{1, \ldots, m\}$, we define

$$\widetilde{v}^k := v^k - \varepsilon \sum_{j\neq k} v^j \tag{25}$$

with some $0 < \varepsilon < 1/m$ which is sufficiently small to guarantee, together with (24),

$$(\widetilde{v}^k)^\top d = (v^k)^\top d - \varepsilon \sum_{j\neq k} (v^j)^\top d > 0,\; k = 1, \ldots, m,$$

that is, $\widetilde{V}^\top d > 0$. It remains to show $\widetilde{V} \in A^{\not\geq,m}$.

To this end, for any $a \in A$, we need to show that either $\min_j (\widetilde{v}^j)^\top a < 0$ or $\widetilde{V}^\top a = 0$ hold. Indeed, for given $a \in A$, choose some $k \in \{1, \ldots, m\}$ with $(v^k)^\top a = \min_j (v^j)^\top a$. By (23), we have $(v^k)^\top a \leqq 0$.

Case 1: $(v^k)^\top a < 0$.

The choice of k and $\varepsilon < 1/m$ yield

$$(\widetilde{v}^k)^\top a = (v^k)^\top a - \varepsilon \sum_{j\neq k} (v^j)^\top a \leqq (v^k)^\top a\,(1 - (m-1)\varepsilon) < (v^k)^\top a/m < 0,$$

and therefore $\min_j (\widetilde{v}^j)^\top a < 0$.

Case 2: $(v^k)^\top a = 0$.

Case 2.1: $\exists\, j \neq k : (v^j)^\top a > (v^k)^\top a$.

Under the present assumption, one obtains

$$(\widetilde{v}^k)^\top a = (v^k)^\top a - \varepsilon \sum_{j\neq k} (v^j)^\top a < (v^k)^\top a\,(1 - (m-1)\varepsilon) = 0,$$

and therefore $\min_j (\widetilde{v}^j)^\top a < 0$.

Case 2.2: $\forall\, j \neq k : (v^j)^\top a = (v^k)^\top a$.

In this case, we have $V^\top a = 0$ which, by the definition of $\widetilde{V}$ in (25), implies $\widetilde{V}^\top a = 0$.

Altogether, this shows $\widetilde{V} \in A^{\not\geq,m}$.

Due to Table 2, for the proof of the two assertions in part b, it is sufficient to show $(A^{\not\gneqq,m})^{m,\not\gneqq} \cup \{0\} \supseteq (A^{\not\gneqq,m})^{m,\not>}$. Let $d \notin (A^{\not\gneqq,m})^{m,\not\gneqq}$ and $d \neq 0$. Then, there exists some $V \in A^{\not\gneqq,m}$ with $V^\top d \geqq 0$. Since $A^{\not\gneqq,m}$ is open by Proposition 3.1, for all sufficiently small ε, we have $\widetilde{V} := V + \varepsilon d e^\top \in A^{\not\gneqq,m}$. Moreover, $d \neq 0$ implies $\widetilde{V}^\top d = V^\top d + \varepsilon e \|d\|^2 > 0$ for any $\varepsilon > 0$, which shows $d \notin (A^{\not\gneqq,m})^{m,\not>}$.

For the proof of part c, let $A_1 = \{a \in \mathbb{R}^3 \mid a_1^2 + (a_2 - a_3)^2 \leqq a_3^2, a_1 \leqq 0, a_3 \geqq 0\}$, $A_2 = \{a \in \mathbb{R}^3 \mid (-2, -1, 0)a \leqq 0 \text{ or } (-1, -2, 0)a \leqq 0\}$ (which equals $X \times \mathbb{R}$ with the set X from Example 2.1) and $A = A_1 \cup A_2$. A_1 is a convex closed cone, A_2 is a nonconvex closed cone, and so is A. We will prove the assertion by showing that $d = -e_1$ lies in $(A^{\ngeqq,2})^{2,\not>}$, but not in $(A^{\ngeqq,2})^{2,\ngeqq}$.

Indeed, to show $d \notin (A^{\ngeqq,2})^{2,\ngeqq}$, consider $V = (-e_1, -e_2) \in \mathbb{R}^{3\times 2}$. The condition $\Gamma_{\geq}(V) \cap A_2 = \emptyset$ is clear from the shape of A_2 so that in view of (16), we have $V \in (A_2)^{\ngeqq,2}$. Moreover, since $a_2 \geqq 0$ holds for all $a \in A_1$, $V^\top a = -a \geq 0$ implies $a_2 = 0$ and $a_1 < 0$. However, $a \in A_1$ with $a_2 = 0$ requires $a_1^2 \leqq a_3^2 - (-a_3)^2 = 0$, i.e. $a_1 = 0$, a contradiction. Therefore, we also have $V \in (A_1)^{\ngeqq,2}$ and, altogether, $V \in A^{\ngeqq,2}$. From $V^\top d = e_1 \geq 0$, we thus obtain $d \notin (A^{\ngeqq,2})^{2,\ngeqq}$.

It remains to show $d \in (A^{\ngeqq,2})^{2,\not>}$. Assume that this is not the case. Then, by Lemma 4.3b, there exists some $\widetilde{V} \in \mathbb{R}^{3\times 2}$ with $\Gamma_{\geq}(\widetilde{V}) \cap A = \emptyset$ and $d \in \Gamma_{>}(\widetilde{V})$. From $\Gamma_{\geq}(\widetilde{V}) \cap A_2 = \emptyset$, one obtains firstly $\Gamma_{=}(\widetilde{V}) = \{0_2\} \times \mathbb{R}$ (where 0_2 denotes the zero vector in $\mathbb{R}^2$), implying

$$\widetilde{v}_3^1 = \widetilde{v}_3^2 = 0, \tag{26}$$

and secondly

$$\widetilde{v}_2^1 \leqq 2\widetilde{v}_1^1, \qquad \widetilde{v}_2^2 \geqq \tfrac{1}{2}\widetilde{v}_1^2 \tag{27}$$

(or the same inequalities with switched vectors $\widetilde{v}^1$, $\widetilde{v}^2$; w.l.o.g. we shall consider the first case). Moreover, the condition $0 < \widetilde{V}^T d = (-\widetilde{v}_1^1, -\widetilde{v}_1^2)^\top$ yields

$$\widetilde{v}_1^1, \widetilde{v}_1^2 < 0 \tag{28}$$

and, in combination with (27), also

$$\widetilde{v}_2^1 < 0. \tag{29}$$

The condition $\Gamma_{\geq}(\widetilde{V}) \cap A = \emptyset$ also implies that all $a \in A_1$ satisfy $\widetilde{V}^\top a \ngeqq 0$. This holds in particular for all points $a^\ell = \left(-\frac{1}{\ell}, 1 - \sqrt{1 - \frac{1}{\ell^2}}, 1\right)^\top$, $\ell \in \mathbb{N}$, for which $a^\ell \in A_1$ is easily seen. In view of (26), we thus have

$$0 \nleqq \widetilde{V}^\top a^\ell = \begin{pmatrix} \widetilde{v}_1^1(-\frac{1}{\ell}) + \widetilde{v}_2^1(1 - \sqrt{1 - \frac{1}{\ell^2}}) \\ \widetilde{v}_1^2(-\frac{1}{\ell}) + \widetilde{v}_2^2(1 - \sqrt{1 - \frac{1}{\ell^2}}) \end{pmatrix} \tag{30}$$

for all $\ell \in \mathbb{N}$. For all sufficiently large ℓ, the second entry in (30) satisfies

$$\widetilde{v}_1^2(-\tfrac{1}{\ell}) + \widetilde{v}_2^2(1 - \sqrt{1 - \tfrac{1}{\ell^2}}) \geqq \widetilde{v}_1^2 \left(-\tfrac{1}{\ell} + \tfrac{1}{2}(1 - \sqrt{1 - \tfrac{1}{\ell^2}})\right) > 0$$

where the first inequality is due to (27), and the second inequality follows from (28) and the negativity of the second factor for sufficiently large ℓ (the latter can be derived from $\lim_\ell \left(\ell - \sqrt{\ell^2 - 1}\right) = 0$).

For sufficiently large ℓ (30) can therefore only hold if the first entry is negative. Together with (28) and (29), we obtain

$$0 < \widetilde{v}_1^1/\widetilde{v}_2^1 < \ell(1 - \sqrt{1 - \tfrac{1}{\ell^2}}) = \ell - \sqrt{\ell^2 - 1}.$$

Since $\widetilde{v}_1^1/\widetilde{v}_2^1$ cannot be positive and simultaneously bounded above by terms which converge to zero for $\ell \to \infty$, we have arrived at a contradiction. We have thus shown $d \in (A^{\ngeq,2})^{2,\ngtr}$, which completes the proof of part c.

For the proof of part d, consider the nonempty closed cone $A = (\mathbb{R}^2_{\geqq} \setminus \mathbb{R}^2_{>}) \times \mathbb{R} \subseteq \mathbb{R}^3$. The matrix $\bar{V} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}$ satisfies $\Gamma_{\geqq}(\bar{V}) \cap A = \{0\}$, so that $A^{\ngeqq,2} \neq \emptyset$ holds in view of (15). In the following, we will show that the vector $d = (1,1,0)^\top$ lies in $(A^{\ngeqq,2})^{2,\ngtr}$, but not in $(A^{\ngeq,2})^{2,\ngtr}$, which shows the strictness of the inclusion $(A^{\ngeq,2})^{2,\ngtr} \subseteq (A^{\ngeqq,2})^{2,\ngtr}$. In fact, the matrix $V = \begin{pmatrix} -1 & 2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$ satisfies $\Gamma_{\geq}(V) \cap A = \emptyset$ and $V^\top d > 0$ so that Lemma 4.3b yields $d \notin (A^{\ngeq,2})^{2,\ngtr}$. Assume that also $d \notin (A^{\ngeqq,2})^{2,\ngtr}$ holds. Then, by Lemma 4.3c, there exists some $\widetilde{V} \in \mathbb{R}^{3\times 2}$ with $\Gamma_{\geqq}(\widetilde{V}) \cap A = \{0\}$ and $\widetilde{V}^\top d > 0$. On the other hand, due to the shape of A, any matrix $\widetilde{V}$ with $\widetilde{V}^\top d > 0$ satisfies $\Gamma_{=}(\widetilde{V}) = \{0_2\} \times \mathbb{R} \subseteq A$. This rules out $\Gamma_{\geqq}(\widetilde{V}) \cap A = \{0\}$ and, thus, implies $d \in (A^{\ngeqq,2})^{2,\ngtr}$.

To see the third assertion $(A^{\ngeq,2})^{2,\ngeq} \subsetneq (A^{\ngeqq,2})^{2,\ngeq}$, note that Table 2, the second assertion of part d and part b yield $(A^{\ngeq,2})^{2,\ngeq} \subseteq (A^{\ngeq,2})^{2,\ngtr} \subsetneq (A^{\ngeqq,2})^{2,\ngtr} = (A^{\ngeqq,2})^{2,\ngeq}$. □

Table 3 illustrates how the results from Lemma 4.4 improve the straighforward ones from Table 2. As seen in Lemma 4.4c,d, the three remaining inclusions are in general not identities, not even under the condition $A^{\ngeqq,m} \neq \emptyset$.

The choice $A = T(X, \bar{x})$ in Lemma 4.4 a,b,d and Table 3 yields the following result.

Theorem 4.2 *The following assertions hold.*

(a) At any $\bar{x} \in X$ the WMOGCQ coincides with the $MOGCQ^{m,\ngtr}_{\ngeq,m}$.

Table 3 Inclusions and identities between generalized bipolar cones

$$
\begin{array}{ccccc}
(A^{\ngtr,m})^{m,\ngtr} = & (A^{\ngeq,m})^{m,\ngtr} & \subseteq & (A^{\ngeqq,m})^{m,\ngtr} & \\
& \cup\!\mid & & \parallel & \\
& (A^{\ngeq,m})^{m,\ngeq} & \subseteq & (A^{\ngeqq,m})^{m,\ngeq} & \\
& & & \parallel & \\
& & & (A^{\ngeqq,m})^{m,\ngeqq} \cup \{0\} &
\end{array}
$$

(b) At any $\bar{x} \in X$ the SMOGCQ coincides with the MOGCQ$^{m,\ngeq}_{\ngeqq,m}$ and the MOGCQ$^{m,\ngtr}_{\ngeqq,m}$.

(c) At any $\bar{x} \in X$ the PMOGCQ is stronger than the WMOGCQ.

(d) At any $\bar{x} \in X$ the WMOGCQ is stronger than the SMOGCQ. An example with $n = 3$, $m = 2$ and $\widehat{N}_{\ngeqq,2}(X, \bar{x}) \neq \emptyset$ exists, for which at some $\bar{x} \in X$ both, the WMOGCQ and the PMOGCQ, are strictly stronger than the SMOGCQ.

Proof For the proof of part d, consider the closed cone $X = (\mathbb{R}^2_{\geqq} \setminus \mathbb{R}^2_{>}) \times \mathbb{R} = \{x \in \mathbb{R}^3 \mid g(x) \leqq 0\}$ with $g(x) = (x_1 x_2, -x_1, -x_2)^\top$. At $\bar{x} = 0$, we have $T(X, \bar{x}) = X$, so that from the proof of Lemma 4.4d, we know $\emptyset \neq X^{\ngeqq,2} = \widehat{N}_{\ngeqq,2}(X, \bar{x})$. Furthermore, $\nabla g(\bar{x}) = (0, -e_1, -e_2)$ yields $L(g, h, \bar{x}) = \mathbb{R}^2_{\geqq} \times \mathbb{R}$, so that $L(g, h, \bar{x})$ contains the vector $\bar{d} = (1, 1, 0)^\top$. In the proof of Lemma 4.4d, we have seen $\bar{d} \notin (X^{\ngeq,2})^{2,\ngtr} = \widehat{N}^{2,\ngtr}_{\ngeq,2}(X, \bar{x})$ and, hence, by Theorem 4.2a, the WMOGCQ is violated at $\bar{x}$.

Finally, due to the shape of $T(X, \bar{x})$, for each $d \in L(g, h, \bar{x})$ and $V \in \mathbb{R}^{3\times 2}$, $\Gamma_{\geq}(V) \cap T(X, \bar{x}) = \emptyset$ and $d \in \Gamma_{>}(V)$ imply $\Gamma_{=}(V) = \{0_2\} \times \mathbb{R} \subseteq T(X, \bar{x})$, so that for no $d \in L(g, h, \bar{x})$ there is a $V \in T^{\ngeqq,2}(X, \bar{x})$ with $d \in \Gamma_{>}(V)$. Consequently, the SMOGCQ $L(g, h, \bar{x}) \subseteq (T^{\ngeqq,2})^{2,\ngtr}(X, \bar{x}) = \widehat{N}^{2,\ngtr}_{\ngeqq,2}(X, \bar{x})$ holds at $\bar{x}$. □

From Lemma 4.4c, one may expect that, in analogy to Theorem 4.2d, in part c, an example with $n = 3$, $m = 2$ exists, for which at some $\bar{x} \in X$, the PMOGCQ is strictly stronger than the WMOGCQ. While such an example may exist, unfortunately, it cannot be constructed from the example in Lemma 4.4c, since there the closed cone A is neither contained in a halfspace, nor does $(A^{\ngeq,2})^{2,\ngtr} = \mathbb{R}^3$ hold. On the other hand, $L(g, h, \bar{x})$ is either contained in a halfspace or coincides with $\mathbb{R}^3$, so that the choice $A = T(X, \bar{x})$ requires that either also $A \subseteq L(g, h, \bar{x})$ is contained in a halfspace, or the validity of the WMOGCQ at $\bar{x}$ means $\mathbb{R}^3 = L(g, h, \bar{x}) \subseteq \widehat{N}^{2,\ngtr}_{\ngeq,2}(X, \bar{x}) = (A^{\ngeq,2})^{2,\ngtr}$. We postpone the construction of such an example, or the proof that the WMOGCQ and the PMOGCQ coincide, to future research.

Table 4 illustrates how the results from Theorem 4.2a and b improve Table 1. Theorem 4.2d shows that even in the case $\widehat{N}_{\ngeqq,m}(X, \bar{x}) \neq \emptyset$ both, the PMOGCQ and the WMOGCQ, are strictly stronger than the SMOGCQ.

Table 4 Improved attributions of MOGCQs to stationarity and Kuhn–Tucker notions

	$WSt(f, X)$	$PSt(f, X)$		$SSt(f, X)$
$KT_{\geq}(f, g, h)$	$WMOGCQ$	$WMOGCQ$	$\overset{\Rightarrow}{\not\Leftarrow}$	$SMOGCQ$
		$\Uparrow$		
$KT_{>}(f, g, h)$		$PMOGCQ$	$\overset{\Rightarrow}{\not\Leftarrow}$	$SMOGCQ$
$KT_{\gg}(f, g, h)$				$SMOGCQ$

4.5 Relations to the Standard Guignard Constraint Qualification

This section clarifies how the three MOGCQs are related to the standard GCQ.

Lemma 4.5 *For any nonempty cone $A \subseteq \mathbb{R}^n$, the following assertions hold.*

(a) The inclusion $(A^{\ngeqq,m})^{m,\nless} \subseteq A^{\circ\circ}$ is true, and there exists a closed cone A for which it is satisfied strictly.

(b) For $A^{\ngeqq,m} \neq \emptyset$ also $(A^{\ngeqq,m})^{m,\nless} \subseteq A^{\circ\circ}$ is true, and there exists a closed cone A for which the inclusion is satisfied strictly.

(c) For $A^{\ngeqq,m} = \emptyset$, one may have $(A^{\ngeqq,m})^{m,\nless} \nsubseteq A^{\circ\circ}$, even for a closed convex cone A.

Proof The first assertion in part a is trivially satisfied in the case $A^{\circ\circ} = \mathbb{R}^n$. Otherwise, choose some $d \notin A^{\circ\circ}$. Then, there exists some $v \in \mathbb{R}^n$ with $v^\top d > 0$ and $v^\top a \leqq 0$ for all $a \in A$. The matrix $V = ve^\top$ (with e denoting the all-ones vector) thus satisfies $V^\top d > 0$ and $V^\top a \leqq 0$ for all $a \in A$, which implies $V \in A^{\ngeqq,m}$. Since we have constructed a matrix $V \in A^{\ngeqq,m}$ with $V^\top d > 0$, we have shown $d \notin (A^{\ngeqq,m})^{m,\nless}$ and, therefore, $(A^{\ngeqq,m})^{m,\nless} \subseteq A^{\circ\circ}$.

An example for the second assertion of part a is given by the closed cone $A := X$ from Example 2.1. It satisfies $A^{\circ\circ} = \mathbb{R}^2$. Assume, on the other hand, $-e \in (A^{\ngeqq,2})^{2,\nless}$. Then, all $V \in A^{\ngeqq,2}$ fulfill $V^\top e \nless 0$. However, in Example 3.1 we have seen that the matrix $V = (-e_1, -e_2) \in A^{\ngeqq,2}$ satisfies $V^\top e = -e < 0$. Therefore, $(A^{\ngeqq,m})^{m,\nless} \subsetneq A^{\circ\circ}$ holds.

To see the first assertion of part b (which is trivially true for $A^{\circ\circ} = \mathbb{R}^n$), choose again some $d \notin A^{\circ\circ}$, so that there exists some $v \in \mathbb{R}^n$ with $v^\top d > 0$ and $v^\top a \leqq 0$ for all $a \in A$. Due to $A^{\ngeqq,m} \neq \emptyset$, we may also choose a matrix $W \in A^{\ngeqq,m}$ and define $V_\varepsilon = ve^\top + \varepsilon W$ with some $\varepsilon > 0$. In view of $ev^\top d > 0$, we have $V_\varepsilon^\top d > 0$ for all sufficiently small $\varepsilon > 0$. Due to $W = (w_1, \ldots, w_m) \in A^{\ngeqq,m}$, for all $a \in A \setminus \{0\}$, there exists some $k \in \{1, \ldots, m\}$ with $w_k^\top a < 0$ and, in view of $v^\top a \leqq 0$, this implies $(v + \varepsilon w_k)^\top a < 0$ for any $\varepsilon > 0$. Therefore, all $a \in A \setminus \{0\}$ satisfy $V_\varepsilon^\top a \ngeqq 0$. This means that for some sufficiently small $\varepsilon > 0$, we have constructed a matrix $V_\varepsilon \in A^{\ngeqq,m} \setminus \{0\}$ with $V_\varepsilon^\top d > 0$. This shows $d \notin (A^{\ngeqq,m})^{m,\nless}$, and we have proved the inclusion $(A^{\ngeqq,m})^{m,\nless} \subseteq A^{\circ\circ}$.

The cone $A := X$ from Example 2.1 also provides an example for the second assertion of part b, since the matrix $V = (-e_1, -e_2)$ also lies in $A^{\ngeq,2}$.

As an example for the assertion of part c consider $A = \mathbb{R}_{\geqq} \times \mathbb{R}^2 \subseteq \mathbb{R}^3$ from Example 3.2 which, being a closed convex cone, satisfies $A^{\circ\circ} = A$. On the other hand, in Example 3.2 we have seen $A^{\ngeq,2} = \emptyset$, so that $(A^{\ngeq,2})^{2,\ngeq} = \mathbb{R}^3 \nsubseteq A^{\circ\circ}$ holds. □

From Example 2.1, we know that the standard GCQ is too weak even to guarantee that a strict local Pareto optimal point of order one is a weak Kuhn–Tucker point, while Table 4 provides three sufficiently strong constraint qualifications for six different situations. The following result verifies that these three constraint qualifications are not independent from, but strictly stronger than the standard GCQ.

Theorem 4.3 *The following assertions hold at any $\bar{x} \in X$.*

(a) The WMOGCQ (and therefore also the stronger PMOGCQ) is stronger than the standard GCQ at $\bar{x}$. An example with $n = m = 2$ exists, for which at some $\bar{x} \in X$ the WMOGCQ is strictly stronger than the standard GCQ.

(b) In the case $\widehat{N}_{\ngeq,m}(X, \bar{x}) \neq \emptyset$ also the SMOGCQ is stronger than the standard GCQ at $\bar{x}$, and an example with $n = m = 2$ exists, for which at some $\bar{x} \in X$ the SMOGCQ is strictly stronger than the standard GCQ.

Proof By Table 3 and Lemma 4.5a, the chain of inclusions $(A^{\ngeq,m})^{m,\ngeq} \subseteq (A^{\ngeq,m})^{m,\ngtr} \subseteq A^{\circ\circ}$ holds, so that the choice $A = T(X, \bar{x})$ yields the first assertion of part a. Moreover, in Example 2.1, we have $T^{\circ\circ}(X, \bar{x}) = L(g, \bar{x}) = \mathbb{R}^2$, but $\widehat{N}^{2,\ngtr}_{\ngeq,2}(X, \bar{x}) = (T^{\ngeq,2})^{2,\ngtr}(X, \bar{x}) \subsetneq \mathbb{R}^3$, as seen in the proof of the second assertion of Lemma 4.5a. Therefore, at $\bar{x}$ the standard GCQ holds, whereas the WMOGCQ is violated. Part b follows analogously from Lemma 4.5b. □

4.6 Relations to the Abadie Constraint Qualification

This section clarifies how the three MOGCQs are related to the Abadie constraint qualification. It will turn out that this depends on the relation between m and n.

As mentioned above, we are not interested in analysing the strong constraint qualification and the regularity conditions from Theorem 4.1d, g and h. Part b of the following result supports this point of view in that the corresponding generalized bipolar cones from Lemma 4.3d, g, and h possess an undesirable property, as opposed to the other six ones.

Lemma 4.6 *The following assertions hold.*

(a) Any cone $A \subseteq \mathbb{R}^n$ is a subset of its six generalized bipolar cones from Table 3 (where A is a subset of $(A^{\ngeq,m})^{m,\ngeq} \cup \{0\}$).

(b) Even a closed convex cone A does not need to be contained in its generalized bipolar cone $(A^{\ngtr,m})^{m,\ngeq}$, and $A \setminus \{0\}$ does not need to be contained in $(A^{\ngtr,m})^{m,\ngeq}$ and $(A^{\ngeq,m})^{m,\ngeq}$.

Proof For the proof of part a observe that by Lemma 4.1a, the inclusion $A^{\not>,m} \subseteq A^{\not>,m}$ is equivalent to $A \subseteq (A^{\not>,m})^{m,\not>}$, by Lemma 4.1b the inclusion $A^{\ngeq,m} \subseteq A^{\ngeq,m}$ is equivalent to $A \subseteq (A^{\ngeq,m})^{m,\ngeq}$, and by Lemma 4.1c the inclusion $A^{\not\geqq,m} \setminus \{0\} \subseteq A^{\not\geqq,m}$ is equivalent to $A \setminus \{0\} \subseteq (A^{\not\geqq,m})^{m,\not\geqq}$. The remaining three assertions follow from Table 3.

As a counterexample for the inclusion $A \subseteq (A^{\not>,m})^{m,\ngeq}$ in part b consider the closed convex cone $A = \mathbb{R}^2 \times \mathbb{R}_{\leqq}$ and $m = 2$. For $V = (e_2, e_3)$ we have $\Gamma_>(V) = \mathbb{R} \times \mathbb{R}^2_>$ and $\Gamma_>(V) \cap A = \emptyset$. Furthermore, $\Gamma_{\geq}(V) = \mathbb{R} \times (\mathbb{R}^2_{\geqq} \setminus \{0_2\})$ contains the vector $d = e_2$, so that Lemma 4.3d implies $d \notin (A^{\not>,2})^{2,\ngeq}$. Due to $d \in A$, this shows $A \nsubseteq (A^{\not>,2})^{2,\ngeq}$.

As a counterexample for the inclusion $A\setminus\{0\} \subseteq (A^{\ngeq,m})^{m,\not\geqq}$, consider again $A = \mathbb{R}^2 \times \mathbb{R}_{\leqq}$ and $m = 2$. The choice $V = (e_2 + e_3, -e_2 + e_3)$ yields $\Gamma_{\geqq}(V) \cap A = \{x \in \mathbb{R}^3 \mid |x_2| \leqq x_3 \leqq 0\} = \mathbb{R} \times \{0_2\} = \Gamma_=(V)$ and, therefore, $\Gamma_{\geq}(V) \cap A = \emptyset$. Furthermore, $\Gamma_{\geqq}(V)$ contains the vector $d = e_1$, so that Lemma 4.3h yields $d \notin (A^{\ngeq,2})^{2,\not\geqq}$. Due to $d \in A\setminus\{0\}$, this shows $A \setminus \{0\} \nsubseteq (A^{\ngeq,2})^{2,\not\geqq}$.

Finally, the previous counterexample and the inclusion $(A^{\not>,m})^{m,\not\geqq} \subseteq (A^{\ngeq,m})^{m,\not\geqq}$ show the third assertion of part b. □

The choice $A = T(X, \bar{x})$ in Lemma 4.6 yields the following result.

Theorem 4.4 *The following assertions hold.*

(a) At any $\bar{x} \in X$ the ACQ is stronger than the WMOGCQ, the PMOGCQ, and the SMOGCQ.

(b) Examples exist in which the ACQ holds at $\bar{x} \in X$, but the constraint qualification and regularity conditions from Theorem 4.1d, g and h are violated.

The proof of Theorem 4.4a is based on the inclusions $T(X, \bar{x}) \subseteq \widehat{N}^{m,\not>}_{\not>,m}(X, \bar{x})$, etc. Observe that, if also $T(X, \bar{x}) \subseteq \widehat{N}^{m,\ngeq}_{\not>,m}(X, \bar{x})$ was true, then the ACQ would be stronger than the $\text{MOGCQ}^{m,\ngeq}_{\not>,m}$ at $\bar{x}$ and thus, by Proposition 2.1 and Theorem 4.1d, yield that local weak Pareto optimal points are proper Kuhn–Tucker points, which is in general not true. Haeser and Ramos [17] give examples showing that even the LICQ and the $\text{MOGCQ}^{m,\ngeq}_{\not>,m}$ are unrelated.

Lemma 4.7 *The following assertions hold.*

(a) For $A = \mathbb{R}^n$, all six generalized bipolar cones from Table 3 coincide with $\mathbb{R}^n$ (where $((\mathbb{R}^n)^{\not\geqq,m})^{m,\not\geqq} \cup \{0\}$ coincides with $\mathbb{R}^n$).

(b) Let $A \subsetneq \mathbb{R}^n$ be a nonempty closed cone and $m \geqq n$. Then, all six generalized bipolar cones from Table 3 coincide with A (where $(A^{\not\geqq,m})^{m,\not\geqq} \cup \{0\}$ coincides with A).

(c) In the case $m < n$, there exists a closed cone $A \subseteq \mathbb{R}^n$ which is a proper subset of its six generalized bipolar cones from Table 3 (where A is a proper subset of $(A^{\not\geqq,m})^{m,\not\geqq} \cup \{0\}$).

Proof By Lemma 4.6a, the set A, respectively $A\setminus\{0\}$, is contained in all six generalized bipolar cones. For $A = \mathbb{R}^n$, this yields the assertion of part a and, for the proof of part b, by Table 3 it suffices to show $(A^{\ngeqq,m})^{m,\ngtr} \subseteq A$. Indeed, the set A^c is a nonempty open cone which does not coincide with $\mathbb{R}^n$. Therefore, $0 \notin A^c$ holds, and it suffices to show $A^c \cap \mathbb{S}^{n-1} \subseteq ((A^{\ngeqq,m})^{m,\ngtr})^c \cap \mathbb{S}^{n-1}$.

Choose any $d \in A^c \cap \mathbb{S}^{n-1}$. Then, its orthogonal complement $\{d\}^\perp$ possesses dimension $n-1$. Under the assumption $m \geqq n$, we may choose a starlike matrix S in $\{d\}^\perp$ in the sense of Definition 3.1, that is, a matrix $S \in \mathbb{R}^{n\times m}$ with $S^\top d = 0$, $\text{rank}(S) = n-1$, and $S\lambda = 0$ for some $\lambda > 0$. Then, by Stiemke's theorem, the system $S^\top y \geq 0$, $y \in \mathbb{R}^n$, is not solvable, that is, $S \in (\mathbb{R}^n)^{\ngeqq,m}$ holds.

For any $\varepsilon > 0$ we define $V_\varepsilon = S + \varepsilon d e^\top$. Then, we have

$$V_\varepsilon^\top d = S^\top d + \varepsilon e \,\|d\|^2 = \varepsilon e > 0,$$

that is, $d \in \Gamma_>(V_\varepsilon)$, for all $\varepsilon > 0$. Moreover, all $y \in \Gamma_{\geqq}(V_\varepsilon)$ satisfy

$$0 \leqq V_\varepsilon^\top y = S^\top y + \varepsilon e\, d^\top y$$

and therefore $-d^\top y\, e \leqq S^\top y/\varepsilon$. In view of $S \in (\mathbb{R}^n)^{\ngeqq,m}$, this implies $-d^\top y \leqq 0$ and, thus, $\Gamma_{\geqq}(V_\varepsilon) \subseteq \{y \in \mathbb{R}^n \mid -d^\top y \leqq 0\}$. Because of $\text{dist}(-d, \{y \in \mathbb{R}^n \mid -d^\top y \leqq 0\}) = \|-d\| = 1$, we obtain

$$\text{dist}(-d, \Gamma_{\geqq}(V_\varepsilon)) \geqq 1. \tag{31}$$

Assume that for all $\varepsilon > 0$, some $a^\varepsilon \in (\Gamma_{\geqq}(V_\varepsilon)\setminus\{0\}) \cap A$ exists. Then, without loss of generality, we may assume $a^\varepsilon \in \mathbb{S}^{n-1}$ and that there exists a sequence $(a^\ell) \subseteq \Gamma_{\geqq}(V_{1/\ell}) \cap A \cap \mathbb{S}^{n-1}$ with $\lim_\ell a^\ell = \bar{a} \in A \cap \mathbb{S}^{n-1}$. From $a^\ell \in \Gamma_{\geqq}(V_{1/\ell})$, we obtain

$$0 \leqq V_{1/\ell}^\top a^\ell = S^\top a^\ell + \tfrac{1}{\ell} e\, d^\top a^\ell \to S^\top \bar{a}$$

and hence $S^\top \bar{a} \geqq 0$. In view of $S \in (\mathbb{R}^n)^{\ngeqq,m}$, we have $S^\top \bar{a} = 0$, which implies $\bar{a} \in S^\perp \cap \mathbb{S}^{n-1} = \{td \mid t \in \mathbb{R}\} \cap \mathbb{S}^{n-1} = \{\pm d\}$. The case $\bar{a} = -d$ is ruled out by (31), $(a^\ell) \subseteq \Gamma_{\geqq}(V_{1/\ell})$ and $\lim_\ell a^\ell = \bar{a}$. Therefore, we have $\bar{a} = d \notin A$ and, since A is closed, $a^\ell \notin A$ for all sufficiently large ℓ, in contradiction to the choice $(a^\ell) \subseteq A$. In summary, we obtain $\Gamma_{\geqq}(V_\varepsilon) \cap A = \{0\}$ as well as $d \in \Gamma_>(V_\varepsilon)$ for some sufficiently small $\varepsilon > 0$ and, by Lemma 4.3c, $d \in ((A^{\ngeqq,m})^{m,\ngtr})^c \cap \mathbb{S}^{n-1}$.

For the proof of part c, consider the closed cone $A = \mathbb{R}^n_{\geqq}\setminus\mathbb{R}^n_>$. We will see that each of the sets $(A^{\ngtr,m})^{m,\ngtr}$, $(A^{\ngeq,m})^{m,\ngeq}$ and $(A^{\ngeqq,m})^{m,\ngeqq}$ contains the set $\mathbb{R}^n_>$ which, in view of Table 3, shows the assertion. Indeed, assume that there exists some $d > 0$ with $d \notin (A^{\ngtr,m})^{m,\ngtr}$. By Lemma 4.3a, there is some $V \in \mathbb{R}^{n\times m}$ with $\Gamma_>(V) \cap A = \emptyset$ and $d \in \Gamma_>(V)$. In view of $m < n$, the dimension of the lineality space $\Gamma_=(V) = \{y \in$

$\mathbb{R}^n \mid V^\top y = 0\}$ is at least one, so that we may choose a nonzero element $\bar{y} \in \Gamma_=(V)$. Defining $y(t) = d + t\bar{y}$ with $t \in \mathbb{R}$ yields $y(t) \in \Gamma_>(V)$ for any $t \in \mathbb{R}$.

In the case that $\bar{y}$ possesses positive entries $\bar{y}_i$ choose $\bar{t}$ as the largest of the (negative) fractions $-d_i/\bar{y}_i$ with indices i such that $\bar{y}_i > 0$. For a corresponding index j, we then obtain $y_j(\bar{t}) = 0$ and $y_i(\bar{t}) \geqq 0$ for all $i \neq j$, that is, $y(\bar{t}) \in A$. If $\bar{y}$ does not possess positive entries, due to $\bar{y} \neq 0$, we may define $\bar{t}$ as the smallest of the (positive) fractions $-d_i/\bar{y}_i$ with indices i such that $\bar{y}_i < 0$, and analogously obtain $y(\bar{t}) \in A$. We have thus constructed a vector $y(\bar{t}) \in \Gamma_>(V) \cap A$, in contradiction to the choice of V with $\Gamma_>(V) \cap A = \emptyset$. The same construction shows $\mathbb{R}^n_> \subseteq (A^{\ngeq,m})^{m,\ngeq}$ and $\mathbb{R}^n_> \subseteq (A^{\ngeqq,m})^{m,\ngeqq}$. □

The choice $A = T(X, \bar{x})$ in Lemma 4.7 yields the following result.

Theorem 4.5 *The following assertions hold.*

(a) For $\bar{x} \in X$, let at least one of the conditions $T(X, \bar{x}) = \mathbb{R}^n$ or $m \geqq n$ hold. Then, the ACQ at $\bar{x}$ coincides with the WMOGCQ, the PMOGCQ, and the SMOGCQ at $\bar{x}$.

(b) In the case $m < n$, the ACQ at $\bar{x}$ is in general strictly stronger than the WMOGCQ, the PMOGCQ, and the SMOGCQ at $\bar{x}$.

We remark that Theorem 4.5b complements the claim from [17] that the ACQ is in general strictly stronger than the WMOGCQ, which is there only supported by the reference to the case $m = 1$. Our result shows that also for $m > 1$ such examples exist, but only for $n > m$.

We close this section by a brief look at the special case of convex and closed cones $A \subseteq \mathbb{R}^n$. In this case, [32, Cor. 6.21] yields $A^{\circ\circ} = A$, so that Lemma 4.5 and Lemma 4.6a imply

$$A \subseteq (A^{\not>,m})^{m,\not>}, (A^{\ngeq,m})^{m,\ngeq} \subseteq A^{\circ\circ} = A.$$

Under the additional assumption $A^{\ngeqq,m} \neq \emptyset$, even

$$A \subseteq (A^{\not>,m})^{m,\not>}, (A^{\ngeq,m})^{m,\ngeq}, (A^{\ngeqq,m})^{m,\ngeqq} \cup \{0\} \subseteq A^{\circ\circ} = A$$

holds. Since $A = T(X, \bar{x})$ is a closed cone, we obtain the following result.

Theorem 4.6 *For $\bar{x} \in X$ let $T(X, \bar{x})$ be convex. Then, the ACQ, the standard GCQ, the WMOGCQ and the PMOGCQ coincide at $\bar{x}$. In the case $\widehat{N}_{\ngeqq,m}(X, \bar{x}) \neq \emptyset$, these four constraint qualifications also coincide with the SMOGCQ at $\bar{x}$.*

5 Weakest Constraint Qualifications

Since all MOGCQs introduced in Sect. 4 are of Guignard type, one may expect that they are weakest possible in some appropriate sense. The present section verifies this.

In the following, for $\bar{x} \in \mathbb{R}^n$, let $C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ denote the set of functions $f : \mathbb{R}^n \to \mathbb{R}^m$ which are continuously differentiable at $\bar{x}$.

Lemma 5.1 *For $\bar{x} \in X$, let $B(X, \bar{x}) \subseteq \mathbb{R}^{n\times m}$ be a matrix cone, and let $O_B(f, X)$ be a subset of X with the property*

$$V \in B(X, \bar{x}) \iff \exists\, f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x}) : \ \bar{x} \in O_B(f, X) \text{ and } V = -\nabla f(\bar{x}). \tag{32}$$

Then, the following assertions hold.

(a) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{\geq}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$, is $L(g, h, \bar{x}) \subseteq B^{m,\not>}(X, \bar{x})$.
(b) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{>}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$, is $L(g, h, \bar{x}) \subseteq B^{m,\not\geq}(X, \bar{x})$.
(c) In addition, let $B(X, \bar{x})$ be open. Then, the weakest condition under which a point $\bar{x} \in X$ lies in $KT_{\gg}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$, is $L(g, h, \bar{x}) \setminus \{0\} \subseteq B^{m,\not\geqq}(X, \bar{x})$.

Proof For the proof of part a, we first show that $L(g, h, \bar{x}) \subseteq B^{m,\not>}(X, \bar{x})$ is some condition under which for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$ also $\bar{x} \in KT_{\geq}(f, g, h)$ holds. Indeed, choose any $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$. Then by (32), we have $-\nabla f(\bar{x}) \in B(X, \bar{x})$. Therefore, $L(g, h, \bar{x}) \subseteq B^{m,\not>}(X, \bar{x})$ and Lemma 4.2a yield $\bar{x} \in KT_{\geq}(f, g, h)$.

On the other hand, let $\bar{x} \in KT_{\geq}(f, g, h)$ hold for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$. We will show that then $L(g, h, \bar{x}) \subseteq B^{m,\not>}(X, \bar{x})$ is necessarily satisfied at $\bar{x}$. To this end, choose $d \in L(g, h, \bar{x})$ and $V \in B(X, \bar{x})$. By (32), there exists some $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in O_B(f, X)$ and $V = -\nabla f(\bar{x})$. This implies $\bar{x} \in KT_{\geq}(f, g, h)$ and, by Proposition 2.2a, $C_{<}(f, \bar{x}) \cap L(g, h, \bar{x}) = \emptyset$. The latter rules out $V^\top d > 0$ and, hence, implies $d \in B^{m,\not>}(X, \bar{x})$.

The proofs of parts b and c run along the same lines, employing Lemma 4.2b and d as well as Proposition 2.2b and c, respectively. □

In a next step, we clarify for which sets $O_B(f, X)$ the condition (32) is valid if $B(X, \bar{x})$ is chosen to be one of the cones $\widehat{N}_{\not>,m}(X, \bar{x})$, $\widehat{N}_{\not\geq,m}(X, \bar{x})$ and $\widehat{N}_{\not\geqq,m}(X, \bar{x})$. We begin by strengthening the characterization of $\widehat{N}_{\not>,m}(X, \bar{x})$ from [17, Th. 3.1], which is a multiobjective generalization of the gradient characterization of regular normals from [32, Th. 6.11]. In [17], the result is only shown for $\bar{x} \in LWPO(f, X)$ since there the authors are interested in local weak Pareto optimal points.

Lemma 5.2 *For any $\bar{x} \in X$, we have $V \in \widehat{N}_{\not>,m}(X, \bar{x})$ if and only if there exists some $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in LPO(f, X)$ and $V = -\nabla f(\bar{x})$.*

Proof The 'if' part of the assertion follows from (3) and (7). To see the 'only if' part, let $V = (v^1, \ldots, v^m) \in \widehat{N}_{\not>,m}(X, \bar{x})$. Then, we have analogously to the proof of [17, Th. 3.1] that

$$\eta_0(r) := \sup\{\min_{k=1,\ldots,m} (v^k)^\top (x - \bar{x}) \mid x \in X, \|x - \bar{x}\| \leqq r\} \leqq r \min_{k=1,\ldots,m} \|v^k\|$$

is nondecreasing on $[0, \infty)$ with $0 \leqq \eta_0(0) \leqq \eta_0(r) \leqq o(r)$. By using arguments analogous to the ones in the proofs of [32, Th. 6.11] and [17, Th. 3.1], there exists a continuously differentiable function $\eta : \mathbb{R}_{\geqq} \to \mathbb{R}$ with $\eta_0(0) = \eta(0)$, $\eta_0(r) < \eta(r)$ for $r > 0$ as well as $\eta'(r) \to 0$ and $\frac{\eta(r)}{r} \to 0$ as $r \to 0$. Define $f = (f_k)_{k=1}^p$ with

$$f_k(x) = -(v^k)^\top (x - \bar{x}) + \eta(\|x - \bar{x}\|).$$

Then, f is continuously differentiable at $\bar{x}$ with $\nabla f(\bar{x}) = -V$ and $f(\bar{x}) = 0$.

Assume that $\bar{x} \notin LPO(f, X)$. Then, there are a sequence $(x^\ell) \subseteq X$, $\lim_\ell x^\ell = \bar{x}$ and (after possibly choosing a subsequence) a constant index $i \in \{1, \dots, m\}$ such that for all $\ell \in \mathbb{N}$ $f_i(x^\ell) < f_i(\bar{x}) = 0$ and $f_j(x^\ell) \leqq f_j(\bar{x}) = 0$, $j \in \{1, \dots, m\} \setminus \{i\}$, hold. Since $\eta_0(r) < \eta(r)$ for $r > 0$, and by definition of f_k, we have for all sufficiently large ℓ

$$0 > f_i(x^\ell) = -(v^i)^\top (x^\ell - \bar{x}) + \eta(\|x^\ell - \bar{x}\|) > -(v^i)^\top (x^\ell - \bar{x}) + \eta_0(\|x^\ell - \bar{x}\|)$$

and

$$0 \geqq f_j(x^\ell) = -(v^j)^\top (x^\ell - \bar{x}) + \eta(\|x^\ell - \bar{x}\|) > -(v^j)^\top (x^\ell - \bar{x}) + \eta_0(\|x^\ell - \bar{x}\|)$$

for all $j \in \{1, \dots, m\} \setminus \{i\}$. Hence $\eta_0(\|x^\ell - \bar{x}\|) < \min_{k=1,\dots,m} (v^k)^\top (x^\ell - \bar{x})$ holds, contradicting the definition of η_0. This shows $\bar{x} \in LPO(f, X)$. □

The following result addresses the set $LITPO(f, X)$ of local Ishizuka-Tuan proper Pareto optimal points from (5).

Lemma 5.3 *For any $\bar{x} \in X$, we have $V \in \widehat{N}_{\ngeq,m}(X, \bar{x})$ if and only if there exists some $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in LITPO(f, X)$ and $V = -\nabla f(\bar{x})$.*

Proof The 'if' part of the assertion follows from (5) and (8). For the proof of the 'only if' part, choose $V = (v^1, \dots, v^m) \in \widehat{N}_{\ngeq,m}(X, \bar{x})$. Due to $\widehat{N}_{\ngeq,m}(X, \bar{x}) \subseteq \widehat{N}_{\not\geq,m}(X, \bar{x})$, we may use the same construction as in the proof of Lemma 5.2 and obtain the continuously differentiable function f with components

$$f_k(x) = -(v^k)^\top (x - \bar{x}) + \eta(\|x - \bar{x}\|)$$

such that $\bar{x} \in LPO(f, X)$ and $\nabla f(\bar{x}) = -V$ hold. Since $-\nabla f(\bar{x}) = V \in \widehat{N}_{\ngeq,m}(X, \bar{x})$ is equivalent to $\bar{x} \in PSt(f, X)$, we arrive at $\bar{x} \in LITPO(f, X)$. □

The next result concerns strict local Pareto optimal points of order one. While, as in the proof of Lemma 5.3, we could construct a continuously differentiable function f with the techniques from the proof of Lemma 5.2, actually here it is sufficient to construct a linear function f.

Lemma 5.4 *For any $\bar{x} \in X$, we have $V \in \widehat{N}_{\ngeqq,m}(X, \bar{x})$ if and only if there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ with $\bar{x} \in StrLPO(1, f, X)$ and $V = -\nabla f(\bar{x})$.*

Proof The 'if' part of the assertion follows from Proposition 2.1c and (9). For the proof of the 'only if' part, choose $V = (v^1, \dots, v^m) \in \widehat{N}_{\ngeq,m}(X, \bar{x})$ and define $f(x) = -V^\top(x - \bar{x})$. The resulting condition $-\nabla f(\bar{x}) = V \in \widehat{N}_{\ngeq,m}(X, \bar{x})$ is equivalent to $\bar{x} \in SSt(f, X) = StrLPO(1, f, X)$. □

The combination of Lemma 5.1a and Lemma 5.2 yields the following result. A weaker version of this theorem is shown in [17, Th. 4.1], where it only covers points $\bar{x} \in LWPO(f, X)$.

Theorem 5.1 *The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{\geq}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in LPO(f, X)$, is the WMOGCQ at $\bar{x}$.*

Likewise, the next result follows from the combination of Lemma 5.1a and b, respectively, with Lemma 5.3. Note that in the proof of part a, this combination yields the MOGCQ$^{m,\not>}_{\ngeq,m}$ as the weakest condition, and that in Theorem 4.2a, we have seen the identity of the MOGCQ$^{m,\not>}_{\ngeq,m}$ with the WMOGCQ at $\bar{x}$.

Theorem 5.2 *The following assertions hold.*

(a) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{\geq}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in LITPO(f, X)$, is the WMOGCQ at $\bar{x}$.
(b) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{>}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in LITPO(f, X)$, is the PMOGCQ at $\bar{x}$.

Finally, we state the weakest constraint qualifications resulting from the combination of Lemma 5.1a, b and c, respectively, with Lemma 5.4, where the appearing constraint qualifcations MOGCQ$^{m,\not>}_{\ngeq,m}$ and MOGCQ$^{m,\ngeq}_{\ngeq,m}$ can be replaced by SMOGCQ in view of Theorem 4.2b.

Theorem 5.3 *The following assertions hold.*

(a) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{\geq}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in StrLPO(f, X)$, is the SMOGCQ at $\bar{x}$.
(b) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{>}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in StrLPO(f, X)$, is the SMOGCQ at $\bar{x}$.
(c) The weakest condition under which a point $\bar{x} \in X$ lies in $KT_{\gg}(f, g, h)$, for every $f \in C^1(\mathbb{R}^n, \mathbb{R}^m, \bar{x})$ with $\bar{x} \in StrLPO(1, f, X)$, is the SMOGCQ at $\bar{x}$.

6 Final Remarks

As mentioned in Sect. 4.4, from Lemma 4.4c one may expect that in the assertion of Theorem 4.2c an example with $n = 3$, $m = 2$ exists, for which at some $\bar{x} \in X$ the PMOGCQ is strictly stronger than the WMOGCQ. Since we were not able to provide such an example, we leave the question of whether it exists, or if the WMOGCQ and the PMOGCQ coincide, to future research.

Since $LGPO(f, X) \subseteq LITPO(f, X)$ holds, by Theorem 4.1e and (5), the PMOGCQ at $\bar{x} \in LGPO(f, X)$ yields $\bar{x} \in KT_{>}(f, g, h)$. However, it is not clear whether Lemma 5.3 holds for $\bar{x} \in LGPO(f, X)$ instead of $\bar{x} \in LITPO(f, X)$. Therefore, it is neither clear whether in Theorem 5.2b the set $LITPO(f, X)$ may be replaced by $LGPO(f, X)$, that is, the question remains what the weakest constraint qualification for local Geoffrion proper Pareto optima to guarantee positive objective function multipliers is. Nevertheless, for the local Geoffrion proper Pareto optima, we were able to provide a weaker constraint qualification than the previously known ACQ, in the form of the PMOGCQ.

Finally, we expect that, with appropriate modifications, the results of the present article may be transferred to more general ordering cones and possibly also to the infinite-dimensional setting. We leave these questions for future research.

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A Decision Space Local Borwein Properness

Since the definition of (local) Geoffrion proper Pareto optimality is restricted to the natural ordering cone, Borwein and Benson introduced two cone-based definitions which can easily be generalized to more general ordering cones. In fact, the following two definitions employ the tangent cone

$$T(Y, \bar{y}) = \{v \in \mathbb{R}^m \mid \exists t^\ell \searrow 0, y^\ell \in Y : \lim_\ell (y^\ell - \bar{y})/t^\ell = v\}$$

to Y at $\bar{y} \in Y$ and the projecting cone

$$\text{cone}(Y) = \{\alpha y \mid y \in Y, \alpha \in \mathbb{R}_{\geqq}\}$$

of Y. In the following definition, the global variant is due to Borwein [5]. He also presented a local variant, but in an image space sense. Instead, we cite here a decision space version of local Borwein properness that was given by Jiménez and Novo in [22].

Definition A.1 A feasible point $\bar{x} \in X$ of MOP is called

(a) *Borwein proper Pareto optimal* ($\bar{x} \in BoPO(f, X)$), if

$$T(f(X) + \mathbb{R}^m_{\geqq}, f(\bar{x})) \cap (-\mathbb{R}^m_{\geqq}) = \{0\}$$

holds,

(b) *local Jiménez-Novo-Type-1 proper Pareto optimal* ($\bar{x} \in LJNPO(f, X)$), if there exists a neighbourhood U of $\bar{x}$ with $\bar{x} \in BoPO(f, X \cap U)$.

In the original definition of Borwein proper Pareto optimality from [5], also Pareto optimality of $\bar{x} \in X$ is assumed, but for the natural ordering cone this requirement is redundant [33], and one has $BoPO(f, X) \subseteq PO(f, X)$. In [22], analogously the relation $LJNPO(f, X) \subseteq LPO(f, X)$ is shown for the natural ordering cone.

Also Benson introduced his definition in order to generalize Geoffrion's definition. His definition is global and, to the best of the authors' knowledge, so far a local variant of Benson's proper Pareto optimality has not been considered. We provide it as a link which allows us to show the equivalence of the local definitions of Geoffrion and Jiménez-Novo.

Definition A.2 A feasible point $\bar{x} \in X$ of MOP is called

(a) *Benson proper Pareto optimal* ($\bar{x} \in BePO(f, X)$), if

$$\mathrm{cl}(\mathrm{cone}(f(X) + \mathbb{R}^m_{\geqq} - f(\bar{x}))) \cap (-\mathbb{R}^m_{\geqq}) = \{0\}$$

holds,

(b) *local Benson proper Pareto optimal* ($\bar{x} \in LBePO(f, X)$), if there exists a neighbourhood U of $\bar{x}$ with $\bar{x} \in BePO(f, X \cap U)$.

Benson showed that his (global) definition and that of Geoffrion coincide and that his definition is stronger than that of Borwein [3], that is,

$$GPO(f, X) = BePO(f, X), \tag{33}$$

$$BePO(f, X) \subseteq BoPO(f, X) \tag{34}$$

hold.

Equality holds in (34) under a convexity assumption [3] and, alternatively, under boundedness of $f(X)$ as the following result from [18] states.

Lemma A.1 *Let $f(X)$ be bounded. Then, $BePO(f, X) = BoPO(f, X)$ holds.*

It is not hard to see that in analogy to their global counterparts (33), the local definitions in the sense of Geoffrion and Benson coincide as well, that is,

$$LGPO(f, X) = LBePO(f, X) \tag{35}$$

holds. Moreover, using Lemma A.1, we can establish the equivalence of the local definitions of Geoffrion and of Jiménez-Novo.

Theorem A.1 *The identity*

$$LGPO(f, X) = LJNPO(f, X) \tag{36}$$

holds.

Proof In view of (35), we may as well show the identity $LBePO(f, X) = LJNPO(f, X)$. The inclusion $\subseteq$ is clear from Definitions A.1b and A.2b as well as (34).

To see the reverse inclusion, let $\bar{x} \in LJNPO(f, X)$. Then, a compact neighbourhood U of $\bar{x}$ with $\bar{x} \in BoPO(f, X \cap U)$ exists. Since X is closed by the continuity of g and h, the set $X \cap U$ is compact. The continuity of f therefore implies the boundedness of $f(X \cap U)$ and, by Lemma A.1, $BePO(f, X \cap U) = BoPO(f, X \cap U)$, which yields $\bar{x} \in LBePO(f, X)$. □

By (35) and (36), we have thus seen the identities $LGPO(f, X) = LBePO(f, X) = LJNPO(f, X)$, that is, the concepts of local proper Pareto optimality in the sense of Geoffrion, Benson and Borwein (in the formulation of Jiménez-Novo) coincide. Since in [22] the relation $StrLPO(1, f, X) \subseteq LJNPO(f, X)$ is shown, we obtain in particular

$$StrLPO(1, f, X) \subseteq LGPO(f, X)$$

and, thus, (2).

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