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The product structure of squaregraphs

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Abstract

A squaregraph is a plane graph in which each internal face is a 4-cycle and each internal vertex has degree at least 4. This paper proves that every squaregraph is isomorphic to a subgraph of the semistrong product of an outerplanar graph and a path. We generalise this result for infinite squaregraphs, and show that this is best possible in the sense that "outerplanar graph" cannot be replaced by "forest".

KEYWORDS

planar graph, product structure, squaregraphs

1 | INTRODUCTION

A *squaregraph* is a plane graph[1] in which each internal face is a 4-cycle and each internal vertex has degree at least 4. These graphs were introduced in 1973 by Soltan, Zambitskiĭ and Prisakaru [25]. They have many interesting structural and metric properties. For example, Bandelt, Chepoi and Eppstein [3] showed that squaregraphs are median graphs and are thus partial cubes, and that every squaregraph can be isometrically embedded[2] into the Cartesian

[1]A *plane graph* is a graph embedded in the plane with no crossings. The word "face" refers to the subgraph on the boundary of the face. A graph is *outerplanar* if it is isomorphic to a plane graph where every vertex is on the outerface.
[2]A graph H can be *isometrically embedded* into a graph G if there exists an isomorphism ϕ from $V(H)$ to a subgraph of G such that $\text{dist}_H(u, v) = \text{dist}_G(\phi(u), \phi(v))$ for all $u, v \in V(H)$.

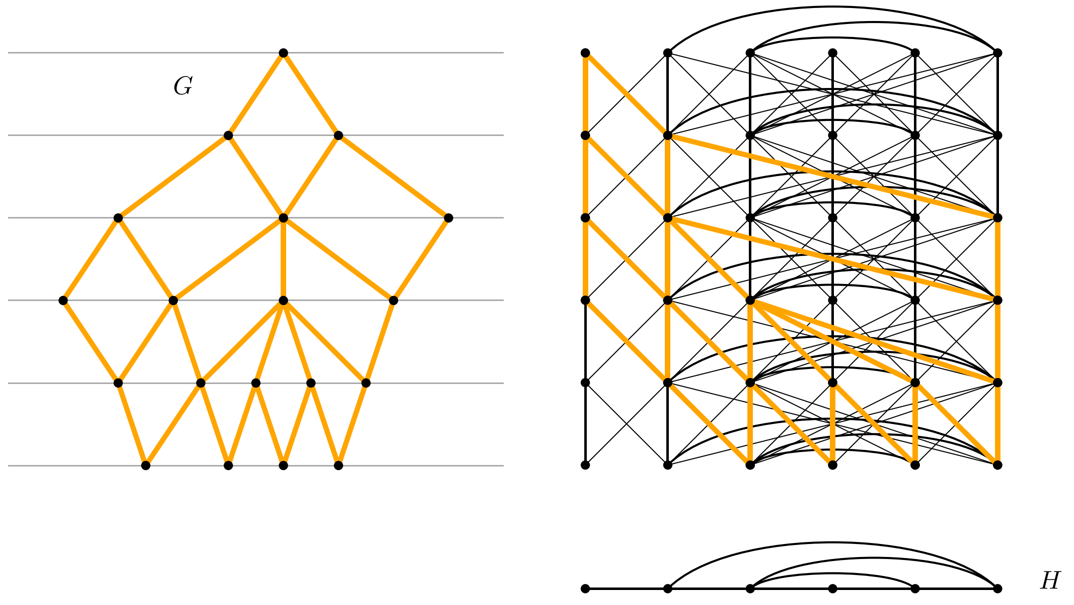

FIGURE 1 A squaregraph G (left) isomorphic to a subgraph of the semistrong product $H \bowtie P$ of an outerplanar graph H and a path P (right). [Color figure can be viewed at wileyonlinelibrary.com]

product[3] of five trees. See the survey by Bandelt and Chepoi [2] for background on metric graph theory.

The primary contribution of this paper is the following product structure theorem for squaregraphs, as illustrated in Figure 1. For graphs G and H, the *semistrong product* $G \bowtie H$ is the graph with vertex-set $V(G) \times V(H)$ with an edge between two vertices (v, w) and (v', w') if $v = v'$ and $ww' \in E(H)$, or $vv' \in E(G)$ and $ww' \in E(H)$; see, for example, [18, 21]. Note that

$$G \times H \subseteq G \bowtie H \subseteq G \boxtimes H.$$

We write $H \lesssim G$ to mean that H is isomorphic to a subgraph of G.

Theorem 1. *For every squaregraph G there is an outerplanar graph H and a path P such that $G \lesssim H \bowtie P$.*

Note that since a path is bipartite, $H \bowtie P$ is also bipartite.

We in fact prove a more general sufficient condition for a plane graph to have such a product structure which implies Theorem 1; see Theorem 5 in Section 2.

The second contribution of this paper is to show that Theorem 1 is best possible in the sense that "outerplanar graph" cannot be replaced by "forest". Moreover, this lower bound holds for strong products. In fact, we prove that for every integer $\ell \in \mathbb{N}$ there is a squaregraph G such

[3]The following are the standard graph products. For graphs G and H, the *Cartesian product* $G \square H$ is the graph with vertex-set $V(G) \times V(H)$ with an edge between two vertices (v, w) and (v', w') if $v = v'$ and $ww' \in E(H)$, or $w = w'$ and $vv' \in E(G)$. The *direct product* $G \times H$ is the graph with vertex-set $V(G) \times V(H)$ with an edge between two vertices (v, w) and (v', w') if $vv' \in E(G)$ and $ww' \in E(H)$. The *strong product* $G \boxtimes H := (G \square H) \cup (G \times H)$.

that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then H contains a cycle (and is therefore not a forest). This result actually follows from a stronger lower bound for bipartite graphs, which has other interesting consequences; see Theorem 11 in Section 3. Also note that Theorem 1 cannot be strengthened by replacing "outerplanar graph" by "graph with bounded pathwidth". Indeed, Bose, Dujmović, Javarsineh, Morin and Wood [8] showed that for every $k \in \mathbb{N}$ there is a tree T (which is a squaregraph) such that for any graph H and path P, if $T \subsetneq H \boxtimes P$ then $\mathrm{pw}(H) \geqslant k$.

In Theorem 1 it is natural to ask whether there is such an outerplanar graph H independent of G. This leads to the study of infinite squaregraphs, previously investigated by Bandelt et al. [3]. Our final contribution is an extension of Theorem 1 in which we show that every (possibly infinite) squaregraph is isomorphic to a subgraph of $O \bowtie \overrightarrow{P}$, where O is the universal outerplanar graph and $\overrightarrow{P}$ is the 1-way infinite path; see Section 4.

Before proving the above results, we provide further motivation by putting Theorem 1 in context. The study of the product structure of graph classes emerged with the following seminal result by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [15], now called the *Planar Graph Product Structure Theorem*. This result describes planar graphs in terms of the strong product of graphs with bounded treewidth[4] and a path. A connected graph has treewidth at most 1 if and only if it is a tree. Treewidth measures how similar a graph is to a tree and is an important parameter in algorithmic and structural graph theory; see [19, 24]. Graphs with bounded treewidth are considered to be a relatively simple class of graphs.

Theorem 2 (Dujmović et al. [15] and Ueckerdt et al. [27]). *For every planar graph G there is a graph H of treewidth at most* 6 *and a path P such that $G \subsetneq H \boxtimes P$.*

The original version of the Planar Graph Product Structure Theorem by Dujmović et al. [15] had "treewidth at most 8" instead of "treewidth at most 6". Ueckerdt et al. [27] proved Theorem 2 with "treewidth at most 6". Since outerplanar graphs have treewidth at most 2, Theorem 1 is stronger than Theorem 2 in the case of squaregraphs. Theorem 1 is also stronger than Theorem 2 in the sense that Theorem 1 uses $\bowtie$ whereas Theorem 2 uses $\boxtimes$. That said, as explained in Section 1.1, it is well known that in the case of bipartite planar graphs G, the proof of Theorem 2 can be adapted to show that $G \subsetneq H \bowtie P$.

Product structure theorems are useful since they reduce problems on a complicated class of graphs (such as planar graphs or squaregraphs) to a simpler class of graphs (bounded treewidth graphs, such as outerplanar graphs). They have been the key tool to resolve several open problems regarding queue layouts [15], nonrepetitive colourings [13], centred colourings [9], clustered colourings [14], adjacency labellings [5, 16, 17], vertex rankings [7], twin-width [6] and infinite graphs [22]. Similar product structure theorems are known for other classes, including graphs with bounded Euler genus [11, 15], apex-minor-free graphs [15], (g, d)-map graphs [12], (g, δ)-string graphs [12], (g, k)-planar graphs [12], powers of planar graphs [12, 20], fan-planar graphs [20] and k-fan-bundle planar graphs [20].

[4]A *tree-decomposition* of a graph G is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the nodes of a tree T, such that (a) for every edge $uv \in E(G)$, some bag B_x contains both u and v, and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a nonempty subtree of T. The *width* of a tree-decomposition is the size of the largest bag minus 1. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G. A *path-decomposition* of a graph G is a tree-decomposition $(B_x \subseteq V(G) : x \in V(T))$ where T is a path. The *pathwidth* of a graph G, denoted by $\mathrm{pw}(G)$, is the minimum width of a path-decomposition of G.

1.1 | Preliminaries

We consider undirected simple graphs G with vertex-set $V(G)$ and edge-set $E(G)$. Unless stated otherwise, graphs are finite. Undefined terms and notation can be found in Diestel's textbook [10].

For $m, n \in \mathbb{Z}$ with $m \leqslant n$, let $[m, n] := \{m, m+1, \ldots, n\}$ and $[n] := [1, n]$.

Let P_n denote a path on n vertices. For graphs G and H, the *complete join* $G + H$ is the graph obtained by the disjoint union of G and H by adding all edges between G and H. For a graph G with $A, B \subseteq V(G)$, let $G[A, B]$ be the subgraph of G with $V(G[A, B]) := A \cup B$ and $E(G[A, B]) := \{uv \in E(G) : u \in A, v \in B\}$.

A *matching* M in a graph G is a set of edges in G such that no two edges in M have a common end vertex. A matching M *saturates* a set $S \subseteq V(G)$ if every vertex in S is incident to some edge in M.

A *model* of H in G is a function μ with domain $V(H)$ such that: $\mu(v)$ is a connected subgraph of G; $\mu(v) \cap \mu(w) = \emptyset$ for all distinct $v, w \in V(H)$; and $\mu(v)$ and $\mu(w)$ are adjacent for every edge $vw \in E(H)$. If, for some $s \in \mathbb{N}_0$, there is a model μ of H in G such that $|V(\mu(v))| \leqslant s$ for each $v \in V(H)$, then H is an *s-small minor* of G.

In a plane graph G, a vertex is *outer* if it is on the outerface of G and is *inner* otherwise. Let I_G denote the set of inner vertices in G.

Let G be a graph. A *partition* of G is a set $\mathcal{P}$ of sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* of $\mathcal{P}$ (with respect to G) is the graph, denoted by $G/\mathcal{P}$, with vertex-set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B. An *H-partition* of G is a partition $\mathcal{P} = (A_x : x \in V(H))$ where $H \cong G/\mathcal{P}$. For an H-partition $(A_x : x \in V(H))$ of G, for each subgraph $J \subseteq G$ the quotient $\tilde{H}$ of the partition $(A_x \cap V(J) : x \in V(H), A_x \cap V(J) \neq \emptyset)$ is called the *subquotient* for J. Note that $\tilde{H}$ is a subgraph of H.

A *layering* of a graph G is an ordered partition $\mathcal{L} := (L_0, L_1, \ldots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$, then $|i - j| \leqslant 1$. $\mathcal{L}$ is a *BFS-layering* (of G) if $L_0 = \{r\}$ for some *root vertex* $r \in V(G)$ and $L_i = \{v \in V(G) : \text{dist}_G(v, r) = i\}$ for all $i \geqslant 1$. A path P is *vertical* (with respect to $\mathcal{L}$) if $|V(P) \cap L_i| \leqslant 1$ for all $i \geqslant 0$.

A *layered partition* $(\mathcal{P}, \mathcal{L})$ of a graph G consists of a partition $\mathcal{P}$ and a layering $\mathcal{L}$ of G. If $\mathcal{P}$ is an H-partition, then $(\mathcal{P}, \mathcal{L})$ is a *layered H-partition*. If $\mathcal{P} = (A_x : x \in V(H))$, then the *width* of $(\mathcal{P}, \mathcal{L})$ is $\max\{|A_x \cap L| : x \in V(H), L \in \mathcal{L}\}$. Layered partitions of width at most 1 are *thin*. Layered partitions were introduced by Dujmović et al. [15] who observed the following connection to strong products (which follows directly from the definitions).

Observation 3 (Dujmović et al. [15]). For all graphs G and H, $G \subsetneq H \boxtimes P \boxtimes K_\ell$ for some path P if and only if G has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most ℓ.

We have the following analogous observation for $\bowtie$ (which also follows directly from the definitions).

Observation 4. For all graphs G and H, $G \subsetneq (H \boxtimes K_\ell) \bowtie P$ for some path P if and only if G has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most ℓ, such that each $L \in \mathcal{L}$ is an independent set in G.

In Observation 4 we may use $G \subsetneq (H \boxtimes K_\ell) \bowtie P$ instead of $G \subsetneq H \boxtimes K_\ell \boxtimes P$ when each $L \in \mathcal{L}$ is an independent set, since no edges in G correspond to edges in $H \boxtimes K_\ell \boxtimes P$ of the form $(v, x, w)(v', y, w)$ where $vv' \in E(H), x, y \in V(K_\ell)$ and $w \in V(P)$.

As mentioned in Section 1, it is well known that in the case of bipartite planar graphs G, the proof of Theorem 2 can be adapted to show that $G \subsetneq H \bowtie P$ for some graph H of treewidth at most 6 and for some path P. To see this, we may assume that G is edge-maximal bipartite planar. Thus G is connected, and each face is a 4-cycle. Let $\mathcal{L} = (L_0, L_1, \ldots)$ be a BFS-layering of G. So each L_i is an independent set. Each face can be written as (a, b, c, d) where $a \in L_i$ and $b, d \in L_{i+1}$ and $c \in L_i \cup L_{i+2}$, for some $i \geqslant 0$. Let G' be the planar triangulation obtained from G by adding the edge bd across each such face. Thus $(L_0, L_1, \ldots)$ is a layering of G'. The proof of Theorem 2 shows that G' has a partition $\mathcal{P}$ such that $\text{tw}(G/\mathcal{P}) \leqslant 6$ and $(\mathcal{P}, \mathcal{L})$ is a thin layered partition. By construction, $(\mathcal{P}, \mathcal{L})$ is a layered partition of G. By Observation 4, $G \subsetneq H \bowtie P$.

A *red-blue colouring* of a bipartite graph G is a proper vertex 2-colouring of G with colours "red" and "blue".

2 | SUFFICIENT CONDITIONS

In this section we prove Theorem 1. We first prove the following, more general sufficient condition for a plane graph to be isomorphic to a subgraph of the strong or semistrong product of an outerplanar graph and a path. Afterwards, we show that this more general result implies Theorem 1.

Theorem 5. *Let G be a plane graph with inner vertices I_G. If G has a layering $\mathcal{L} = (L_0, L_1, \ldots)$ such that $G[L_{i-1}, L_i]$ has a matching saturating $L_{i-1} \cap I_G$ for each $i \geqslant 1$, then $G \subsetneq H \boxtimes P$ for some outerplanar graph H and path P. Moreover, if $V(L_i)$ is an independent set for all $L_i \in \mathcal{L}$, then $G \subsetneq H \bowtie P$.*

Proof. By Observations 3 and 4, it suffices to show that G has a thin layered H-partition $\mathcal{P}$ (with respect to $\mathcal{L}$) for some outerplanar graph H. For each $i \in [n]$, let E_i be a matching in $G[L_{i-1}, L_i]$ that saturates $L_{i-1} \cap I_G$. For vertices $u \in L_{i-1}$ and $v \in L_i$ and an edge $uv \in E_i$, we say that u is the *parent* of v and v is the *child* of u. Observe that each vertex $u \in L_{i-1} \cap I_G$ has exactly one child and each vertex $v \in L_i$ has at most one parent. Let J be the subgraph of G where $V(J) = V(G)$ and $E(J) = \bigcup_{i \in [n]} E_i$.

Let X be a connected component of J. Choose the maximum $j \in [0, n]$ such that there exists some vertex $v \in V(X) \cap L_j$. Vertex v must be outer because each vertex in $L_j \cap I_G$ is adjacent in J to some vertex in L_{j+1}. As illustrated in Figure 2, since each vertex in X has at most one child and at most one parent, X is a vertical path with respect to $\mathcal{L}$.

Let $\mathcal{P}$ be the partition of G determined by the connected components of J. Let $H = G/\mathcal{P}$ be the quotient of $\mathcal{P}$. Since each part in $\mathcal{P}$ is a vertical path with respect to $\mathcal{L}$, it follows that $(\mathcal{P}, \mathcal{L})$ is a thin layered H-partition. It remains to show that H is outerplanar. Since each part in $\mathcal{P}$ is connected, H is a minor of G and is therefore planar. Since each part of $\mathcal{P}$ contains a vertex on the outerface, contracting each part of $\mathcal{P}$ into a single vertex gives a plane embedding of H with each vertex on the outerface; see Figure 2. Therefore H is outerplanar. □

We now work towards showing that squaregraphs satisfy the conditions for Theorem 5.

A plane graph G is *leveled* if the edges are straight line-segments and vertices are placed on a sequence of horizontal lines, $(L_0, L_1, \ldots)$, called *levels*, such that each edge joins two vertices in

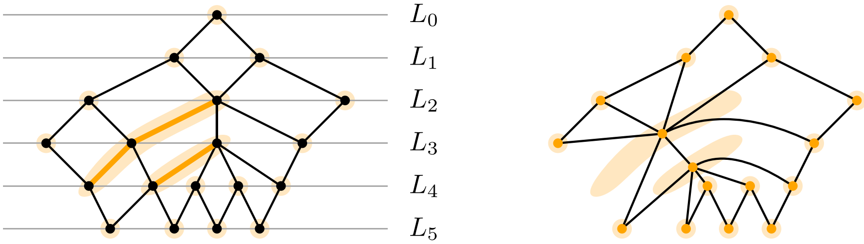

FIGURE 2 (Left) A squaregraph with a BFS-layering and a partition $\mathcal{P}$ into vertical paths (thick orange). The vertical paths are constructed from matchings between consecutive layers, where the leftmost vertex in L_i is chosen for each inner vertex in L_{i-1}. (Right) The lower endpoint of each path is on the outerface, so when each path is contracted we obtain an outerplanar graph. [Color figure can be viewed at wileyonlinelibrary.com]

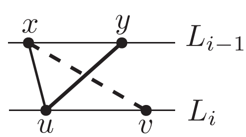

FIGURE 3 Contradiction in the proof of Lemma 6.

consecutive levels. If, in addition, we allow straight-line edges between consecutive vertices on the same level, then G is *weakly leveled*. Observe that the levels in a weakly leveled plane graph G define a layering of G. Leveled plane graphs were first introduced by Sugiyama, Tagawa and Toda [26], and have since been well studied [4].

For a weakly leveled plane graph G with levels $(L_0, L_1, \ldots)$ and a vertex $v \in L_i$, the *up-degree* of v is $|N_G(v) \cap L_{i-1}|$ and the *down-degree* of v is $|N_G(v) \cap L_{i+1}|$. We now give a more natural condition that forces our desired matching between two consecutive levels.

Lemma 6. *Let G be a weakly leveled plane graph with inner vertices I_G. If each vertex in I_G has down-degree at least 2, then $G \subsetneq H \boxtimes P$ for some outerplanar graph H and path P. Moreover, if G is a leveled plane graph, then $G \subsetneq H \bowtie P$.*

Proof. Let $(L_0, L_1, \ldots)$ be the levels of G. Observe that if G is a leveled plane graph, then $V(L_i)$ is an independent set for all $i \geqslant 0$. For each $i \in [n]$, let E_i be the set of edges in $G[L_{i-1}, L_i]$ between each vertex $v \in L_{i-1} \cap I_G$ and its leftmost neighbour in L_i; see Figure 2. For the sake of contradiction, suppose there exists a vertex $u \in L_{i-1} \cup L_i$ that is incident to two edges in E_i. By construction, each vertex in $L_{i-1} \cap I_G$ is incident to at most one edge in E_i so $u \in L_i$. Let x and y be the neighbours of u in L_{i-1}, where x is to the left of y. Since x has down-degree at least 2, x is adjacent to a vertex v that is to the right of u. However, this contradicts G being weakly leveled plane since uy and vx cross; see Figure 3. Therefore, E_i is a matching that saturates $L_{i-1} \cap I_G$. The claim therefore follows by Theorem 5. □

We are ready to prove Theorem 1 which we restate here for convenience.

Theorem 1. *For every squaregraph G there is an outerplanar graph H and a path P such that $G \subsetneq H \bowtie P$.*

Proof. We may assume that G is connected (since if each component of G has the desired product structure, then so does G). Bannister et al. [4] showed that G is isomorphic to a leveled plane graph with levels given by a BFS-layering of G rooted at any vertex r on the outerface. Without loss of generality, assume G is leveled plane with corresponding levels $(L_0, L_1, \ldots)$. Below we show that every inner vertex in G has up-degree at most 2. Since each inner vertex has degree at least 4, each inner vertex has down-degree at least 2. The result thus follows from Lemma 6.

For the sake of contradiction, suppose there exists an inner vertex with up-degree at least 3. Let $i \in [n]$ be minimum such that there is a vertex $v \in L_i \cap I_G$ with up-degree at least 3. Let u_1, u_2, u_3 be neighbours of v in L_{i-1} ordered left to right. Since the levels are defined by a BFS-layering, there is a (u_1, r)-path and a (u_3, r)-path that does not contain u_2; see Figure 4. Hence, u_2 is an inner vertex of G and thus has degree at least 4. However, by planarity, v is the only neighbour of u_2 in L_i. Since u_2 has no neighbours in L_{i-1} (as G is leveled plane), u_2 has three neighbours in L_{i-2}, which contradicts the minimality of i, as required. □

We now give an application of Theorem 1. A colouring ϕ of a graph G is *nonrepetitive* if for every path $v_1, \ldots, v_{2h}$ in G, there exists $i \in [h]$ such that $\phi(v_i) \neq \phi(v_{i+h})$. The *nonrepetitive chromatic number*, $\pi(G)$, is the minimum number of colours in a nonrepetitive colouring of G. Nonrepetitive colourings were introduced by Alon, Grytczuk, Hałuszczak and Riordan [1] and have since been widely studied; see the survey [28].

Kündgen and Pelsmajer [23] showed that $\pi(G) \leqslant 4^{\mathrm{tw}(G)}$ for every graph G. Building upon this result, Dujmović et al. [13] proved the following:

Lemma 7 (Dujmović et al. [13]). *For any graph H and path P, if $G \subsetneq H \boxtimes P$ then $\pi(G) \leqslant 4^{\mathrm{tw}(H)+1}$.*

Using (a variation of) Theorem 2 and Lemma 7, Dujmović et al. [13] resolved a long-standing conjecture of Alon, Grytczuk, Hałuszczak and Riordan [1] by showing that planar graphs G have bounded nonrepetitive chromatic number; in particular, $\pi(G) \leqslant 768$. When G is a squaregraph, Theorem 1 and Lemma 7 imply that $\pi(G) \leqslant 4^3 = 64$.

3 | TIGHTNESS

In this section, we show that Theorem 1 is tight by proving a lower bound for the product structure of bipartite graphs.

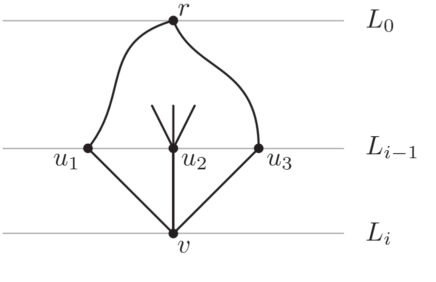

FIGURE 4 Vertex $v \in L_i$ with three neighbours u_1, u_2, u_3 in the preceding layer L_{i-1}. Since u_2 is an inner vertex, it has degree at least 4.

The *row treewidth* of a graph G is the minimum integer k such that $G \subsetneq H \boxtimes P$ for some graph H with treewidth k and path P [8]. Theorem 2 says that every planar graph has row treewidth at most 6. Dujmović et al. [15] showed that the maximum row treewidth of planar graphs is at least 3. They in fact proved the following stronger result.

Theorem 8 (Dujmović et al. [15]). *For all $k, \ell \in \mathbb{N}$ with $k \geqslant 2$ there is a graph G with pathwidth k such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then $K_{k+1} \subsetneq H$ and thus H has treewidth at least k. Moreover, if $k = 2$ then G is outerplanar, and if $k = 3$ then G is planar.*

Theorem 1 says that squaregraphs have row treewidth at most 2. We show that this bound is tight by proving Theorem 11 which is an analogous result to Theorem 8 for bipartite graphs. As an introduction to the key ideas in the proof of Theorem 11, we first establish Proposition 10 which is a slight generalisation of Theorem 8. We need the following lemma for finding long paths in quotient graphs.

Lemma 9. *For every $a, n \in \mathbb{N}$, there exists a sufficiently large $n' \in \mathbb{N}$ such that for every graph G that contains an n'-vertex path and for every H-partition $(A_x : x \in V(H))$ of G where $|A_x| \leqslant a$ for all $x \in V(H)$, for each $w \in V(H)$ the graph $H - w$ contains a path on n vertices.*

Proof. Let m be sufficiently large compared to n and let $n' := (a+1)am + a$. Suppose G has a path on n' vertices. Let $G' = G - A_w$. Since $|V(P) \cap A_w| \leqslant a$, P is split into at most $a + 1$ disjoint subpaths in G'. Thus, there is a path $P_{\max}$ in G' with at least am vertices. Let $\tilde{H}$ be the subquotient of H with respect to $P_{\max}$. Observe that $\tilde{H}$ is connected and that $|V(\tilde{H})| \geqslant am/a = m$. Moreover, $\tilde{H} \subseteq H - w$ since $A_w \cap V(P_{\max}) = \emptyset$. Now $\tilde{H}$ has maximum degree at most $2a$ since every vertex in $P_{\max}$ has degree at most 2. Thus, since m is sufficiently large, $\tilde{H}$ contains a path on at least n vertices, as required. □

The following result generalises Theorem 8 (which is the $n = 2$ case).

Proposition 10. *For all $k, \ell, n \in \mathbb{N}$ there exists a graph G with pathwidth at most $k + 1$ such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then $P_n + K_k \subsetneq H$.*

Proof. We proceed by induction on $k \geqslant 1$. Let n' be sufficiently large compared to n. Let $G^{(1)}$ be the graph obtained from a path on n' vertices plus a dominant vertex v. Observe that $G^{(1)}$ has radius 1 and pathwidth at most 2. Suppose $G^{(1)} \subsetneq H \boxtimes P \boxtimes K_\ell$ for some graph H and path P. By Observation 3, there is a layered H-partition $(A_x : x \in V(H))$ of G of width at most ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Since $G^{(1)}$ has radius 1, every layering of $G^{(1)}$ consists of at most three layers so $|A_x| \leqslant 3\ell$ for all $x \in V(H)$. By Lemma 9 and since n' is sufficiently large, $H - w$ contains a path on n vertices. As v is dominant in $G^{(1)}$, w is also dominant in H. Thus $P_n + K_1 \subsetneq H$.

Now suppose $k > 1$ and let $G^{(k-1)}$ be a graph that satisfies the induction hypothesis for $k - 1$. Let $G^{(k)}$ be obtained by taking 3ℓ disjoint copies of $G^{(k-1)}$ plus a dominant vertex v. Then $G^{(k)}$ has pathwidth at most $k + 1$. As in the base case, let $(A_x : x \in V(H))$ be a layered H-partition of $G^{(k)}$ of width ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Since $G^{(k)}$ has radius 1, it follows that $|A_w - \{v\}| \leqslant 3\ell - 1$. Thus, there is a copy of $G^{(k-1)}$ that contains no vertices from A_w. Now consider the subquotient $\tilde{H}$ of H with respect to this copy of

$G^{(k-1)}$. By induction, $P_n + K_{k-1} \subsetneq \tilde{H}$. Since v is dominant in $G^{(k)}$, w is dominant in H and thus $P_n + K_k \subsetneq H$, as required. □

Note that in Proposition 10, the graph $G^{(1)}$ is outerplanar and the graph $G^{(2)}$ is planar for every $n \in \mathbb{N}$.

We now prove our main lower bound which is a bipartite version of Proposition 10.

Theorem 11. *For all $i, j, k, \ell, n \in \mathbb{N}$ where $i + j = k$, there exists a bipartite graph $G^{(i,j)}$ with pathwidth at most $k + 1$ such that for any graph H and path P, if $G^{(i,j)} \subsetneq H \boxtimes P \boxtimes K_\ell$ then $P_n + K_{i,j}$ is a 2-small minor of H. Moreover, $G^{(1,0)}$ is an outerplanar squaregraph and $G^{(1,1)}$ is a bipartite planar graph.*

Proof. Let $P_n = (a_1, ..., a_n)$ be a path on n vertices. Let $B = \{b_1, ..., b_i\}$ and $C = \{c_1, ..., c_j\}$ be the bipartition of $V(K_{i,j})$. We proceed by induction on k with the following hypothesis: for every $i, j, k, \ell, n \in \mathbb{N}$ where $i + j = k$, there exists a red-blue coloured connected bipartite graph G, such that for any graph H, if $(A_x : x \in V(H))$ is a layered H-partition of G of width at most ℓ, then H contains a model μ of $P_n + K_{i,j}$ such that for each $u \in V(P_n + K_{i,j})$ we have $|V(\mu(u))| \leqslant 2$ and $\bigcup_{a \in V(\mu(u))} A_a$ contains:

1. a red vertex when $u \in B$;
2. a blue vertex when $u \in C$; and
3. a red and a blue vertex when $u \in V(P_n)$.

The claimed theorem follows by Observation 3.

For $k = 1$ we may assume that $i = 1$ and $j = 0$. Let n' be sufficiently large and let $G^{(1,0)}$ be the bipartite graph obtained from a red-blue coloured path $P_G = (u_1, ..., u_{n'})$ on n' vertices plus a red vertex v adjacent to all the blue vertices in $V(P_G)$. Observe that $G^{(1,0)}$ has radius 2 and pathwidth at most 2. Let $(A_x : x \in V(H))$ be a layered H-partition of $G^{(1,0)}$ of width ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Then A_w contains a red vertex. Since $G^{(1,0)}$ has radius 2, every layering of $G^{(1,0)}$ has at most five layers, so $|A_x| \leqslant 5\ell$ for all $x \in V(H)$. By Lemma 9 and since n' is sufficiently large, $H - w$ contains a path $P_H = (a'_1, ..., a'_{2n})$ on $2n$ vertices. Now for every edge $a'_i a'_{i+1} \in E(P_H)$, there exists $j \in [n' - 1]$ such that $u_j, u_{j+1} \in A_{a'_i} \cup A_{a'_{i+1}}$. As such, $A_{a'_i} \cup A_{a'_{i+1}}$ contains a red and a blue vertex. For all $i \in [n]$, let $\mu(a_i) = H[\{a'_{2i-1}, a'_{2i}\}]$ and $\mu(b_1) = \{w\}$. Then μ is a model of $P_n + K_{1,0}$ in H which satisfies the induction hypothesis.

Now suppose $k > 1$ and that there is a red-blue coloured connected bipartite graph $G^{(i-1,j)}$ such that for any graph H, if $(A_x : x \in V(H))$ is a layered H-partition of G of width at most ℓ, then H contains a model $\tilde{\mu}$ of $P_n + K_{i-1,j}$ where $|V(\tilde{\mu}(u))| \leqslant 2$ for all $u \in V(P_n + K_{i-1,j})$ and $\bigcup_{a \in V(\mu(u))} A_a$ contains a red vertex when $u \in B$; a blue vertex when $u \in C$; and a red and a blue vertex when $u \in V(P_n)$. Let $G^{(i,j)}$ be obtained by taking 5ℓ copies of $G^{(i-1,j)}$ plus a red vertex v that is adjacent to all the blue vertices. Then $G^{(i,j)}$ has radius 2 and pathwidth at most $k + 1$. As in the base case, let $(A_x : x \in V(H))$ be a layered H-partition of $G^{(i,j)}$ of width ℓ. Let $w \in V(H)$ be such that $v \in A_w$. Then A_w contains a red vertex. Since $G^{(i,j)}$ has radius 2, $|A_w - \{v\}| \leqslant 5\ell - 1$. Thus, there is a copy of $G^{(i-1,j)}$ that contains no vertices from A_w. Now consider the subquotient $\tilde{H}$ of H with respect to this copy of $G^{(i-1,j)}$. By induction, $\tilde{H}$ contains a model $\tilde{\mu}$ which satisfies the induction hypothesis. Let $\mu(b_i) = \{w\}$ and $\mu(v) = \tilde{\mu}(v)$ for all

$v \in V(P_n + K_{i-1,j})$. Since v is adjacent to all the blue vertices in G, w is adjacent to a vertex in $\bigcup_{a \in V(\mu(u))} A_a$ whenever $u \in V(P_n) \cup C$. Thus μ is a model of $P_n + K_{i,j}$ in H which satisfies the induction hypothesis, as required.

As illustrated in Figure 5, $G^{(1,0)}$ is an outerplanar squaregraph and $G^{(1,1)}$ is a bipartite planar graph. □

We now highlight several consequences of Theorem 11. First, since the graph $G^{(1,0)}$ is an outerplanar squaregraph and $P_2 + K_{1,0}$ is a 3-cycle, we have the following:

Corollary 12. *For every $\ell \in \mathbb{N}$, there exists a squaregraph G such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then H contains a cycle of length at most 6.*

Thus Theorem 1 is best possible in the sense that "outerplanar graph" cannot be replaced by "forest".

Second, since the graph $G^{(1,1)}$ is a bipartite planar graph and $P_2 + K_{1,1} \cong K_4$ which has treewidth 3, we have the following:

Corollary 13. *For every $\ell \in \mathbb{N}$, there exists a bipartite planar graph G such that for any graph H and path P, if $G \subsetneq H \boxtimes P \boxtimes K_\ell$ then H contains a 2-small minor of K_4 and thus $\mathrm{tw}(H) \geqslant 3$.*

Therefore, the maximum row treewidth of bipartite planar graphs is at least 3. We conclude this section with the following open problem: what is the maximum row treewidth of bipartite planar graphs? As in the case of (non-bipartite) planar graphs, the answer is in {3, 4, 5, 6}.

4 | INFINITE SQUAREGRAPHS

In this section by "graph" we mean a graph G with $V(G)$ finite or countably infinite. Huynh et al. [22] showed how Theorem 2 can be used to construct a graph that contains every planar graph as a subgraph and has several interesting properties. Here we adapt their methods to construct an analogous graph that contains every squaregraph as a subgraph.

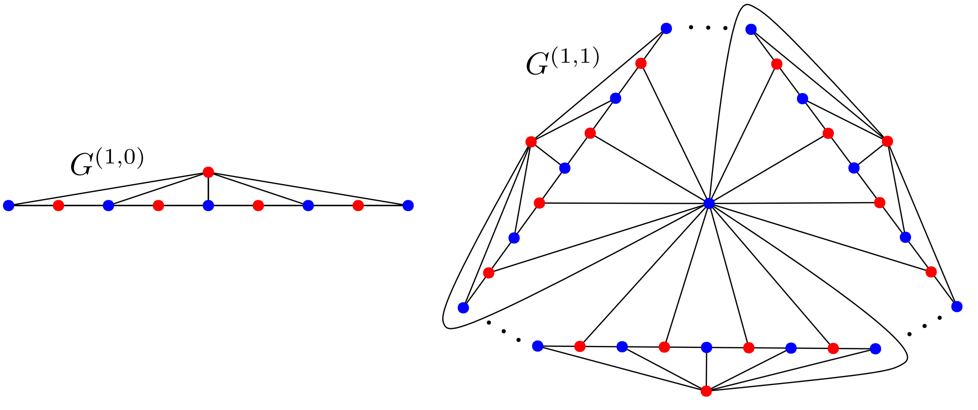

FIGURE 5 The graphs $G^{(1,0)}$ and $G^{(1,1)}$ from Theorem 11. [Color figure can be viewed at wileyonlinelibrary.com]

Bandelt et al. [3] gave several equivalent definitions of an infinite squaregraph. The following definition suits our purposes. Let G be a locally finite[5] graph. For every vertex v of G and every $r \in \mathbb{N}$ the subgraph $G[\{w \in V(G) : \text{dist}_G(v, w) \leqslant r\}]$ is called a *ball*. Since G is locally finite, every ball is finite. An infinite graph G is a *squaregraph* if it is locally finite and every ball in G is a squaregraph. Let $\overrightarrow{P}$ be the 1-way infinite path, which has vertex-set $\mathbb{N}_0$ and edge-set $\{\{i, i+1\} : i \in \mathbb{N}_0\}$. It is well known that there is a *universal* outerplanar graph O. This means that O is outerplanar and every outerplanar graph is isomorphic to a subgraph of O. See Theorem 4.14 in [22] for an explicit definition of O.

Theorem 14. *Every squaregraph is isomorphic to a subgraph of* $O \boxtimes \overrightarrow{P}$.

Theorem 14 follows from Theorem 1 and the next lemma, which is an adaptation of Lemma 5.3 in [22].

Lemma 15. *Let H be a graph. Let G be a locally finite graph such that $B \precsim H \boxtimes \overrightarrow{P}$ for every ball B in G. Then $G \precsim H \boxtimes \overrightarrow{P}$.*

Proof Sketch. Fix $v \in V(G)$. For $n \in \mathbb{N}_0$, let $V_n := \{w \in V(G) : \text{dist}_G(v, w) = n\}$ and $G_n := G[V_0 \cup V_1 \cup \cdots \cup V_n]$. So G_n is a finite ball in G. By assumption, $G_n \precsim H \boxtimes \overrightarrow{P}$. Let X_n be the set of all thin layered H-partitions $(\mathcal{P}, \mathcal{L})$ of G_n, such that L is an independent set in G_n for each $L \in \mathcal{L}$. By Observation 4, $X_n \neq \emptyset$. Since G_n is finite and connected, X_n is finite. For each $n \in \mathbb{N}$ and for each $(\mathcal{P}, \mathcal{L}) \in X_n$, if $\mathcal{P}' := \{Y \setminus V_n : Y \in \mathcal{P}, Y \setminus V_n \neq \emptyset\}$ and $\mathcal{L}' := \{L \setminus V_n : L \in \mathcal{L}, Y \setminus V_n \neq \emptyset\}$ then $(\mathcal{P}', \mathcal{L}') \in X_{n-1}$ (since G_{n-1} is connected). By Kőnig's lemma, there is an infinite sequence $(\mathcal{P}_0, \mathcal{L}_0), (\mathcal{P}_1, \mathcal{L}_1), (\mathcal{P}_2, \mathcal{L}_2), \ldots$ where $\mathcal{P}_{n-1} = \mathcal{P}'_n$ and $\mathcal{L}_{n-1} = \mathcal{L}'_n$ for each $n \in \mathbb{N}$. By construction, $\mathcal{P}_{n-1}$ is a "subpartition" of $\mathcal{P}_n$ and $\mathcal{L}_{n-1}$ is a "subpartition" of $\mathcal{L}_n$. Let $\mathcal{P} := \bigcup_{n \in \mathbb{N}_0} \mathcal{P}_n$ and $\mathcal{L} := \bigcup_{n \in \mathbb{N}_0} \mathcal{L}_n$. Then $(\mathcal{P}, \mathcal{L})$ is a thin layered H-partition of G; see [22] for details. By Observation 4, $G \precsim H \boxtimes \overrightarrow{P}$. □

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[5]A graph G is *locally finite* if every vertex of G has finite degree.

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