



New characterizations of strategy-proofness under single-peakedness

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Abstract

We provide novel representations of strategy-proof voting rules applicable when voters have uni-dimensional single-peaked preferences. In particular, we introduce a ‘grading curve’ representation which is particularly useful when introducing variable electorates. Our analysis recovers, links and unifies existing results in the literature, and provides new characterizations when strategy-proofness is combined with other desirable properties such as ordinality, participation, consistency, and proportionality. Finally, the new representations are used to compute the strategy-proof methods that maximize the ex-ante social welfare for the L_2 -norm and a uniform prior. The resulting strategy-proof welfare maximizer is the linear median (or ‘uniform median’), that we also characterize as the unique proportional strategy-proof voting rule.

Keywords Strategy-proofness · Single-peaked preferences · Voting · Consistency · Participation · Proportionality · Linear/uniform median

A.B. Jennings and E.M. Varloot have contributed more and equally to this work. The authors wish to add the following remarks. A.B. Jennings: In memory of Michel Balinski. His mentorship made my Ph.D. degree possible and without his friendship, I would lack much love of mathematics, many interesting conversations, and a fascination with the theory of voting. R. Laraki: I am deeply indebted to Michel Balinski for introducing me to the world of voting systems by inviting me to participate in the organisation of the first field experiment on electoral voting [1]. This marked the inception of a nearly two-decade-long collaboration focused on devising a voting system that is strategically resilient while encompassing other desirable attributes, a theme that this paper delves into. To me, he was not only a loyal and attentive friend, but also a trusted confidant, an invaluable advisor, and even a paternal figure. The void left by his absence will be profoundly poignant. C. Puppe: I dedicate my part of this work to Michel Balinski. I got to know him personally only late in his life. Nevertheless, he left a lasting and inspiring impression on me for which I am very grateful. E.M. Varloot: I have personally never met Dr. Balinski as he passed away too soon, however I can safely say that without his work my PhD thesis would not be what it is today. We can only hope he will continue to inspire many in years to come.

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1 Introduction

In mechanism design, strategy-proofness (SP) is a desirable property. It implies that whatever agents' beliefs are about others' behavior or information, their best strategy is to sincerely submit their privately-known types, even when their beliefs are wrong or mutually inconsistent. Consequently, strategy-proofness guarantees to the designer that she has implemented the intended choice function, i.e. that the final decision is indeed linked in the intended way to the agents' true types. Of course, depending on the context, there are other desirable properties that one would like to satisfy, such as unanimity, voter sovereignty, efficiency, anonymity, neutrality, proportionality, and with variable electorate, consistency and participation.

When side payments are possible and utilities are quasi-linear, anonymous and efficient strategy-proof mechanisms can be designed (the well-known Vickrey–Clarke–Groves mechanisms). By contrast, in contexts of 'pure' social choice ('voting'), the Gibbard–Satterthwaite [2, 3] Theorem shows that only dictatorial rules can be sovereign ('onto') and strategy-proof on an unrestricted domain of preferences. In particular, no onto voting rule can be anonymous and strategy-proof without restrictions on individual preferences.

To overcome the impossibility, several domain restrictions have been investigated. One of the most popular is one-dimensional single-peakedness. Under this restriction, the path-breaking paper by Moulin [4] showed that there is a large class of onto, anonymous and strategy-proof rules. All of them can be derived by simply adding some fixed ballots (called 'phantom' votes) to the agents' ballots and electing the median alternative of the total. Moulin's paper inspired a large literature that obtained related characterizations for other particular domains or proved impossibility results (see Jordan [5] and Barberà, Gul and Stacchetti [6], Nehring and Puppe [7] or Freeman et al. [8]).

Our contribution

In contrast to Moulin's elegant and simple phantom voter characterization in the anonymous case, the more general characterizations in terms of winning coalitions (called 'generalized median voter schemes' in [6], and 'voting by issues' in [7]), as well as Moulin's own 'inf-sup' characterization in the appendix of his classic paper are complex.¹ A basic objective of our paper is to provide simpler alternative representations that apply also to the non-anonymous case.

First, one of our representations is a natural extension of Moulin's idea: the selected outcome is the median of voters' peaks after the addition of new peaks computed from suitable *phantom functions*. Another representation has a compact functional form which we refer to as a *grading curve* in the anonymous case. This is important as it allows a family of voting rules with variable-sized electorate to be described using

¹ Curiously, Moulin provided completely different proofs for his two characterizations.

one single function. This function may be interpreted as the density of the phantoms in Moulin's median formula.

Secondly, the new representations have additional merits. They can be used to connect the phantom characterization with the 'voting by issues' one, and to provide new characterizations of and insights into special cases. For instance, we show that the case of uniformly distributed phantom voters [9] corresponds to a linear grading curve which in turn is fully characterized by one particular additional axiom: proportionality. We also show that this uniform (= linear) median is the social utility maximizing strategy-proof voting rule when the utilitarian welfare is measured using the L_2 -distance to the voters' peaks.

Thirdly and finally, in the variable electorate environment, the curve representation allows us to tightly characterize all anonymous strategy-proof voting rules that are consistent in the sense of Smith [10] and Young [11]: these are exactly those whose grading curve is independent of the size of the electorate.

Further related literature

A link between Moulin's [4] inf-sup characterization and his phantom median voter characterization in the anonymous case has been provided by Weymark [12] who showed how to derive the phantom median voter representation from the inf-sup one. Our paper shows that both results can be derived from our new characterization in terms of phantom functions, which also implies the one in terms of 'winning coalitions' in [6] and [7].

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 introduces our central phantom functions characterization that is used to reprove all the known representations of strategy-proof voting rules as well as several new representations. Section 4 considers additional properties, such as voter sovereignty and efficiency, strict responsiveness and ordinality, and anonymity. Variable electorate axioms such as consistency and participation (e.g. absence of the no-show paradox) are considered in Sect. 5 where a complete characterization of consistent and/or participant methods is established in the anonymous and the non-anonymous cases.² Section 6 computes the welfare-maximizing voting rule under the strategy-proofness constraint. Section 7 concludes. The Appendix contains missing proofs and additional results.

2 Strategy-proofness and its consequences

The voting problem we are considering can be described by the following elements. First, there is an ordered set of alternatives Λ (for example, political candidates on a left-right spectrum, a set of grades such as "Great, Good, Average, Poor, Terrible" or a set of locations on the line). Second, there is a finite set of voters $N = \{1, \dots, n\}$,

² To our knowledge, consistency has not yet been studied in the context of uni-dimensional strategy-proofness.

with a typical element \vec{r} of Λ^N being called a voting profile. A voting rule φ maps each profile in Λ^N to an element in Λ .

The interpretation is: each voter $i \in N$ has a single peaked preference over the linearly ordered set Λ (see Definition 1 below). He submits his peak (or a strategically chosen ballot) $r_i \in \Lambda$ to the designer who then computes $\varphi(r_1, \dots, r_n) = \varphi(\vec{r})$ and implements (or elects) the computed alternative.

Without loss, we will assume $\Lambda \subseteq \mathbf{R}$ and use the notations $\mu^- := \inf \Lambda$, $\mu^+ := \sup \Lambda$ and $\overline{\Lambda} = \Lambda \cup \{\mu^-, \mu^+\}$. In Moulin's [4] paper, $\Lambda = \mathbf{R}$, $\mu^- = -\infty$ and $\mu^+ = +\infty$. In Barberà, Gul and Stacchetti's [6] paper, Λ is finite.

Definition 1 The (complete) preference order of voter i over the alternatives in Λ is single peaked if there is a unique alternative $x \in \Lambda$ such that for any $y, z \in \Lambda$, if y is between x and z , then voter i prefers x to y and y to z . The alternative x is called the peak of the preference order. It is voter i 's favorite alternative.

This means that the utility function of each voter is strictly increasing from μ^- to his peak and then strictly decreasing from the peak to μ^+ . We wish the voting rule to satisfy some desirable axioms. The main focus is strategy-proofness (SP). Sections 4 and 5 will explore combinations with other axioms.

Axiom 1 (Strategy-Proofness: SP) *A voting rule φ is strategy-proof if for every voting profile \vec{r} and voter $i \in N$, if \vec{s} differs from \vec{r} only in dimension i , then:*

$$\varphi(\vec{s}) \geq \varphi(\vec{r}) \geq r_i \text{ or } \varphi(\vec{s}) \leq \varphi(\vec{r}) \leq r_i.$$

Remark 1 The formulation of SP in Axiom 1 is usually called uncompromisingness [5, 13]. It needs to be explained why it is analogous to the usual definition. The argument is as follows. If $r_i < \varphi(\vec{r})$ and $\varphi(\vec{s}) < \varphi(\vec{r})$ (where \vec{s} differs from \vec{r} only in dimension i) then it is possible to create a single-peaked preference P^{r_i} at r_i such that $\varphi(\vec{s})$ is strictly preferred to $\varphi(\vec{r})$ by P^{r_i} . Hence, a voter with this preference, by reporting the peak s_i instead of r_i , improves his utility, contradicting strategy-proofness. A similar conclusion is obtained for the other cases.

Some useful consequences of strategy-proofness follow.

Definition 2 A voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is weakly responsive if for all voters i , and for all \vec{r} and \vec{s} that only differ in dimension i , if $r_i < s_i$ then $\varphi(\vec{r}) \leq \varphi(\vec{s})$.

Weak responsiveness is sometimes called weak monotonicity.

Lemma 1 *If a voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is strategy-proof, then it is weakly responsive.*

Proof Suppose \vec{r} and \vec{s} only differ in i with $r_i < s_i$. If $r_i < \varphi(\vec{r})$ or $s_i > \varphi(\vec{s})$, then by strategy-proofness $\varphi(\vec{r}) \leq \varphi(\vec{s})$. Otherwise, $\varphi(\vec{r}) \leq r_i < s_i \leq \varphi(\vec{s})$. \square

Lemma 2 *If a voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is strategy-proof, then it is uniformly continuous.*

Proof Suppose that there is \vec{r} and \vec{s} that differ only in dimension i such that $r_i < s_i$. First we show that $\varphi(\vec{s}) - \varphi(\vec{r}) \leq s_i - r_i$. If $s_i < \varphi(\vec{s})$ or $r_i > \varphi(\vec{r})$, then strategy-proofness gives $\varphi(\vec{s}) = \varphi(\vec{r})$. Otherwise, $r_i \leq \varphi(\vec{r}) \leq \varphi(\vec{s}) \leq s_i$. In either case,

$\varphi(\vec{s}) - \varphi(\vec{r}) \leq s_i - r_i$. Now let us use this property to show that φ is uniformly continuous. Let $\epsilon > 0$ be given. For any \vec{r} and \vec{s} with $|r_i - s_i| \leq \frac{\epsilon}{n}$ for all i , we have:

$$\begin{aligned} |\varphi(\vec{r}) - \varphi(\vec{s})| &\leq \sum_i |\varphi(r_1, \dots, r_i, s_{i+1}, \dots, s_n) - \varphi(r_1, \dots, r_{i-1}, s_i, \dots, s_n)| \\ &\leq \sum_i |r_i - s_i| \leq \epsilon. \end{aligned}$$

□

Lemma 3 (Continuous Extension) *If a voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is strategy-proof, then it has a unique continuous extension in $\overline{\Lambda}^N \rightarrow \overline{\Lambda}$. (Proof: See appendix A.)*

It is therefore natural to ask what are the SP voting rules in $\overline{\Lambda}^N \rightarrow \overline{\Lambda}$ that are not continuous extensions of voting rules in $\Lambda^N \rightarrow \Lambda$.

Lemma 4 *A SP voting rule in $\overline{\Lambda}^N \rightarrow \overline{\Lambda}$ is not a continuous extension of a voting rule in $\Lambda^N \rightarrow \Lambda$ iff it is constant valued with a value not in Λ .*

Proof \Rightarrow : Suppose that $\varphi : \overline{\Lambda}^N \rightarrow \overline{\Lambda}$ is not an extension of a function from $\Lambda^N \rightarrow \Lambda$. Therefore, there is a voting profile $\vec{r} \in \Lambda^N$ such that $\varphi(\vec{r}) \notin \Lambda$. Let $\varphi(\vec{r}) = \mu^-$ (resp. μ^+). By strategy-proofness, for all $\vec{s} \in \overline{\Lambda}^N$, $\varphi(\vec{s}) = \mu^-$ (resp. μ^+). Therefore, φ is a constant (equal to μ^- or to μ^+) and its value is not in Λ . \Leftarrow : Immediate. □

From Lemma 3, if we discard the SP voting rules that are constant values not in Λ , we obtain all the SP methods from Λ to itself. Consequently, from now on, we will consider w.l.o.g. that $\overline{\Lambda} = \Lambda$ (and so the voters are allowed to submit to the designer the extreme alternatives μ^- and μ^+).

3 Characterizations of SP voting rules

In this section we start by establishing two new mathematically convenient characterizations of SP voting rules. In the subsequent sections, the second is used to derive all the known characterizations as well as several new ones.

We denote by $\Gamma := \{\mu^-, \mu^+\}^N$ the set of voting profiles where all voters have extreme positions (they submit an extreme alternative). We define $\vec{\mu}^- := (\mu^-, \dots, \mu^-)$, and $\vec{\mu}^+ := (\mu^+, \dots, \mu^+)$. For $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ in Γ we say that $X \leq Y$ if for every voter $i \in N$, $X_i \leq Y_i$.

3.1 Phantom function characterizations

In this subsection we will introduce the concept of phantom functions and two new characterizations of SP voting rules, the second being a direct consequence of the first. We show in the next sections that the second characterization implies not only all the known characterizations (the two by Moulin and the one by Barberà, Gul Stacchetti) but also several new representations.

Definition 3 (*Phantom function*) A function $\alpha : \Gamma \rightarrow \Lambda$ is called a phantom function if α is weakly increasing ($X \leq Y \implies \alpha(X) \leq \alpha(Y)$). We will use the shorthand $\alpha_X := \alpha(X)$.

It is immediate that each SP voting rule φ is associated with a unique phantom function α_φ defined as:

$$\forall X \in \Gamma, \alpha_\varphi(X) := \varphi(X). \quad (1)$$

That is, α_φ provides the outcome of φ when all voters vote at the extremes. Observe that α_φ is necessarily weakly increasing because φ is SP and so is weakly responsive (by Lemma 1). Conversely, the next theorem proves that each phantom function α is associated with a unique SP voting rule φ_α . This is because strategy-proofness implies that we can always let voters vote at the extremes without changing the outcome. To state precisely our result, we denote by θ a function which transforms voters' votes to the extreme alternatives.

Definition 4 (*The θ function*)³

$$\begin{aligned} \theta : \Lambda^N \times \mathbf{R} &\rightarrow \Gamma \\ \theta : \vec{r}, x &\rightarrow X = \theta(\vec{r}, x) \end{aligned}$$

Such that $\forall i; X_i = \mu^- \Leftrightarrow r_i < x$ and $\forall i; X_i = \mu^+ \Leftrightarrow r_i \geq x$.

Hence, θ at x transforms each voter i whose submitted input r_i is strictly below (resp. weakly above) x to the extreme value μ^- (resp. μ^+). The next theorem shows that there is a one to one correspondence between SP rules and phantom functions, and provides a formula that relates them, thanks to the θ function. We note that θ is weakly decreasing with respect to x .

Theorem 1 (Phantom function characterization 1) *The voting function φ is strategy-proof iff there exists a phantom function $\alpha : \Gamma \rightarrow \Lambda$ such that:*

$$\forall \vec{r} \in \Lambda^n; \varphi(\vec{r}) := \begin{cases} \alpha_{\vec{\mu}^-} & \text{if } \forall j, r_j \leq \alpha_{\vec{\mu}^-} \\ \alpha_{\theta(\vec{r}, r_i)} & \text{if } (i \in N) \text{ and } r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, r_i)}\} \\ r_i & \text{if } (i \in N) \text{ and } \forall \epsilon > 0, \alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq r_i \leq \alpha_{\theta(\vec{r}, r_i)} \end{cases} \quad (2)$$

And that phantom function is necessarily unique.

Proof The proof is quite long and so is delegated to Appendix B. □

The interpretation of Theorem 1 is simple. The first and second cases correspond to the fact that for any voting profile, \vec{r} , where $\varphi(\vec{r})$ is not equal to any of the input votes, $\varphi(\vec{r})$ must equal the output after raising (to μ^+) all votes greater than $\varphi(\vec{r})$ (and lowering the rest to μ^-). The final case comes from the fact that for a voting profile, \vec{r} ,

³ The definition of θ is asymmetric because it sends values strictly below the cut-off to μ^- and greater than or equal to μ^+ . That's why our characterization in Theorem 1 "looks" asymmetric. It is not the case in the characterization of Theorem 2.

where $\varphi(\vec{r})$ does equal one of the input votes, $\varphi(\vec{r})$ must be (weakly) between the output for the two extreme profiles generated by raising the votes that are (respectively) strictly greater and weakly greater than $\varphi(\vec{r})$ (and lowering the rest). As will be seen, Theorem 1 implies—easily—the next characterization which implies all the subsequent ones. The next theorem provides a more elegant and compact characterization without the use of the θ function. For $X = (X_1, \dots, X_n) \in \Gamma$, we denote $\mu^-(X)$ (resp. $\mu^+(X)$) the set of voters $i \in N$ such that $X_i = \mu^-$ (resp. $X_i = \mu^+$).

Theorem 2 (Phantom function characterization 2) *The voting function φ is strategy-proof iff there exists a phantom function $\alpha : \Gamma \rightarrow \Lambda$ (the same as the one in Theorem 1) such that:*

$$\begin{aligned} \forall \vec{r}; \varphi(\vec{r}) \\ := \begin{cases} \alpha_X & \text{if } \exists X \in \Gamma \text{ s.t. } \mu^+(X) \subseteq \{j : \alpha_X \leq r_j\} \wedge \mu^-(X) \subseteq \{j : \alpha_X \geq r_j\} \\ r_i & \text{if } \exists X, Y \in \Gamma \text{ s.t. } \alpha_X \leq r_i \leq \alpha_Y \text{ and} \\ & \mu^+(X) = \{j : r_i < r_j\} \wedge \mu^-(Y) = \{j : r_i > r_j\} \end{cases} \end{aligned} \quad (3)$$

Proof Suppose there is a phantom function α such that φ satisfies Eq. 3. Let μ be defined as in Theorem 1. Let us now compare the two.

- If $\varphi(\vec{r})$ is equal to one of the inputs, r_i : Then for X defined by $\mu^+(X) = \{j : r_i < r_j\}$ and Y defined by $\mu^-(Y) = \{j : r_i > r_j\}$, Eq. 3 gives $\alpha_X \leq r_i \leq \alpha_Y$. We also have $Y = \theta(\vec{r}, r_i)$ and $X = \lim_{\epsilon \rightarrow 0^+} \theta(\vec{r}, r_i + \epsilon)$. Therefore $\mu(\vec{r}) = r_i$.
- If $\varphi(\vec{r})$ is not equal to any inputs: The first case of Eq. 3 holds and we must have $\varphi(\vec{r}) = \alpha_X$ with $\mu^+(X) \subseteq \{j : \alpha_X \leq r_j\}$ and $\mu^-(X) \subseteq \{j : \alpha_X \geq r_j\}$. Since α_X is not equal to any inputs, $\mu^+(X) = \{j : \alpha_X < r_j\}$.
If $\mu^+(X) = \emptyset$ then $X = \vec{\mu}^-$ and $\mu(\vec{r}) = \alpha_{\vec{\mu}^-}$.
Otherwise, there exists r_i such that $r_i = \min\{r_j : r_j \geq \alpha_X\}$. Since α_X is not equal to any inputs, $\{j : r_j \geq r_i\} = \{j : r_j > \alpha_X\} = \mu^+(X)$, so $\theta(\vec{r}, r_i) = X$. It follows that $r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, r_i)}\}$ and $\mu(\vec{r}) = \alpha_X$.

It follows that $\mu = \varphi$. □

The intuition behind the characterization of Theorem 2 is simple. By strategy-proofness, when a voter's ballot is strictly smaller than the societal outcome then, it can be replaced by the minimal ballot μ^- without changing the final outcome. Symmetrically, if it is strictly larger than the final outcome, it can be replaced by μ^+ without changing the outcome. As such, if the outcome is not one of the ballots, then it must be α_X where $\mu^+(X)$ is the set of voters whose ballots are greater than the outcome. On the other hand, if the outcome is one of the ballots then by weak responsiveness it is in between α_X and α_Y where $\mu^+(X)$ is the set of voters whose ballots are strictly higher than the outcome and $\mu^-(Y)$ is the set of voters whose ballots are strictly smaller than the outcome.

Remark 2 It is not possible to have both set inclusions in case 1 holding as equalities if the outcome is one of the peaks, for that would require that anyone whose peak is the outcome would have both μ^- and μ^+ associated with it, which is impossible.

3.2 MaxMin characterizations

In his appendix, Moulin [4] proved the following MaxMin characterization of strategy-proof voting rules.

Theorem 3 (Moulin's MaxMin Characterization) *A voting rule φ is strategy-proof iff for each subset $S \subseteq N$ (including the empty set), there is a value $\beta_S \in \Lambda$ such that:*

$$\forall \vec{r} \in \Lambda^n, \varphi(\vec{r}) = \max_{S \subseteq N} \min \left(\beta_S, \min_{i \in S} \{r_i\} \right).$$

Remark 3 Moulin observed in his proof (the appendix in [4]) that without loss, $S \rightarrow \beta_S$ can be taken to be weakly increasing. With this selection, the β in his theorem coincides with the α in Theorem 2 as proved now.

Theorem 4 (MaxMin Characterization 2) *A function φ is strategy-proof iff there exists a phantom function α (the same as the one in Theorems 1 and 2) that verifies:*

$$\forall \vec{r} \in \Lambda^n, \varphi(\vec{r}) = \max_{X \in \Gamma} \min \left(\alpha_X, \min_{i \in \mu^+(X)} \{r_i\} \right). \quad (4)$$

Proof Let φ be a strategy-proof voting rule defined by a phantom function α as in Theorem 2. Let $\mu : \Lambda^N \rightarrow \Lambda$ be defined as in the theorem statement:

$$\forall \vec{r}, \mu(\vec{r}) = \max_{X \in \Gamma} \min \left(\alpha_X, \min_{i \in \mu^+(X)} \{r_i\} \right).$$

Since φ is the unique SP voting rule defined by α we only need to prove that $\varphi = \mu$. Fix $\vec{r} \in \Lambda^n$. Let $x = \mu(\vec{r})$ and $Z \in \Gamma$ be such that $x = \min(\alpha_Z, \min_{i \in \mu^+(Z)} \{r_i\})$. (Equivalently, $\mu^+(Z) \subseteq \{j : r_j \geq x\}$ and $\{j : r_j < x\} \subseteq \mu^-(Z)$.)

- If $x = r_i$ for some i : Set Y such that $\mu^-(Y) = \{j : r_j < r_i\}$. Then $\mu^-(Y) \subseteq \mu^-(Z)$, so $\alpha_Y \geq \alpha_Z \geq r_i$. Set X such that $\mu^+(X) = \{j : r_j > r_i\}$. The maximum in the definition of $\mu(\vec{r})$ applies to X , so $\min(\alpha_X, \inf_{j \in \mu^+(X)} \{r_j\}) \leq r_i$. But $r_j > r_i$ for all $j \in \mu^+(X)$, which forces $\alpha_X \leq r_i$. This verifies that $\alpha_X \leq r_i \leq \alpha_Y$, so by the definition of φ , $\varphi(\vec{r}) = r_i = x$.
- If $x \neq r_i$ for any i : Set X such that $\mu^+(X) = \{j : r_j \geq x\} = \{j : r_j > x\}$. We have $\mu^+(Z) \subseteq \mu^+(X)$ so $\alpha_X \geq \alpha_Z \geq x$. The maximum in the definition of $\mu(\vec{r})$ applies to X , so $\min(\alpha_X, \inf_{j \in \mu^+(X)} \{r_j\}) \leq x$. But $r_j > x$ for $j \in \mu^+(X)$ which forces $\alpha_X \leq x$. It follows that $\alpha_X = x$. By construction, $X \in \Gamma$ verifies $\mu^+(X) \subseteq \{j : \alpha_X \leq r_j\} \wedge \mu^-(X) \subseteq \{j : \alpha_X \geq r_j\}$. By the definition of φ , we therefore have $\varphi(\vec{r}) = \alpha_X = x$.

Therefore, $\varphi = \mu$. \square

3.3 Median characterizations

The most popular of Moulin's representation (Theorem 5) assumes anonymity.

Axiom 2 (Anonymity) *A voting rule ϕ is anonymous iff for any permutation σ of voters and for all voting profiles \vec{r} :*

$$\phi(r_{\sigma(1)}, \dots, r_{\sigma(n)}) = \phi(r_1, \dots, r_n).$$

Anonymity states that all voters must be treated equally.

Theorem 5 (Moulin's Median-Characterization. Anonymous Case) *A voting rule φ is strategy-proof and anonymous iff there is a set of $n + 1$ values $\alpha_0, \dots, \alpha_n$ in Λ (called phantom voters by Moulin) such that:*

$$\forall \vec{r}; \varphi(\vec{r}) = \text{med}(r_1, \dots, r_n, \alpha_0, \dots, \alpha_n).$$

where med denotes the median operator.⁴

The two characterizations of Moulin look quite different: median in the anonymous case (Theorem 5) and maxmin in the general case (Theorem 3). Their proofs are separated in his article. In order to link the two we need to be able to choose $n + 1$ phantom voters among the 2^N outputs of our phantom functions. The next theorem explains how they can be chosen. For all $k = 0, \dots, n$ and $\vec{r} = (r_1, \dots, r_n)$, let $X_k(\vec{r}) \in \Gamma$ be defined in such a way that $\mu^+(X_k(\vec{r}))$ is equal to the set of voters that provides the k largest peaks.⁵ For $k = 0$, we let $X_0(\vec{r}) = \vec{\mu}^- = (\mu^-, \dots, \mu^-)$.

Theorem 6 (Median Characterization. General Case) *A voting rule φ is strategy-proof iff there exists a phantom function α (the same as in Theorem 2) such that:*

$$\forall \vec{r} \in \Lambda^n; \varphi(\vec{r}) := \text{med}(r_1, \dots, r_n, \alpha_{X_0(\vec{r})}, \alpha_{X_1(\vec{r})}, \dots, \alpha_{X_n(\vec{r})}). \quad (5)$$

Proof This is a direct consequence of Theorem 2. See Appendix C.1. □

Remark 4 This characterization implies easily Moulin's median characterization in Theorem 5 (thanks to Proposition 1 below). Hence, we have unified the proofs of the Moulin's characterizations. Weymark [12] unified the two, using a different approach.

Proposition 1 *A strategy-proof voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is anonymous iff its phantom function α is anonymous.*

Proof \Rightarrow : It is immediate from equation (1) that if φ is anonymous then so is α .

\Leftarrow : For any permutation σ , let $\vec{s} = (s_1, \dots, s_n) = (r_{\sigma(1)}, \dots, r_{\sigma(n)})$ be the permutation σ of the peaks. For all k we have that $X_k(\vec{r})$ and $X_k(\vec{s})$ both have k values μ^+

⁴ It is well-defined as we have an odd number of input values.

⁵ If there are more than k voters with the k largest peaks, break ties arbitrarily. The tie breaking rule does not affect the outcome.

and $n - k$ values μ^- . Therefore since α is anonymous, we have $\alpha(X_k(\vec{r})) = \alpha(X_k(\vec{s}))$. Thus, using the median representation (Theorem 6):

$$\begin{aligned}\varphi(\vec{s}) &= \text{med}(r_{\sigma(1)}, \dots, r_{\sigma(n)}, \alpha_{X_0(\vec{s})}, \alpha_{X_1(\vec{s})}, \dots, \alpha_{X_n(\vec{s})}) \\ &= \text{med}(r_1, \dots, r_n, \alpha_{X_0(\vec{r})}, \alpha_{X_1(\vec{r})}, \dots, \alpha_{X_n(\vec{r})}) \\ &= \varphi(\vec{r}).\end{aligned}$$

□

3.4 Curve characterizations

This section establishes two novel characterizations of SP methods.

Theorem 7 (Curve Characterization. General Case) *A voting rule φ is strategy-proof iff there exists a phantom function α (the same as the one in Theorem 2) such that:*

$$\forall \vec{r} \in \Lambda^n; \varphi(\vec{r}) := \sup \{y \in \Lambda : \alpha_{\theta(\vec{r}, y)} \geq y\}.$$

Proof This is another direct consequence of Theorem 2. Let φ be a strategy-proof voting rule defined by a phantom function α . And let μ be defined as

$$\forall \vec{r}; \mu(\vec{r}) := \sup \{y \in \Lambda : \alpha_{\theta(\vec{r}, y)} \geq y\}.$$

Since φ is the unique strategy-proof voting rule defined by α we only need to prove that $\varphi = \mu$. First notice that $y \rightarrow \alpha_{\theta(\vec{r}, y)}$ is weakly decreasing.

- Case $\varphi(\vec{r}) = \alpha_X$: By the characterization of φ , we have $\mu^+(X) \subseteq \{j : \alpha_X \leq r_j\}$ and $\mu^-(X) \subseteq \{j : \alpha_X \geq r_j\}$. The former implies that $\alpha_X \leq \alpha_{\theta(\vec{r}, \alpha_X)}$. The latter implies that $\{j : \alpha_X < r_j\} \subseteq \mu^+(X)$. For $\epsilon > 0$, we have:

$$\{j : \alpha_X + \epsilon \leq r_j\} \subseteq \{j : \alpha_X < r_j\} \subseteq \mu^+(X)$$

and $\alpha_{\theta(\vec{r}, \alpha_X + \epsilon)} \leq \alpha_X < \alpha_X + \epsilon$. Therefore $\{y \in \Lambda : \alpha_{\theta(\vec{r}, y)} \geq y\}$ must contain α_X and not $\alpha_X + \epsilon$. It follows that $\mu(\vec{r}) = \alpha_X$.

- Case $\varphi(\vec{r}) = r_i$: Let X and Y be such that $\alpha_X \leq r_i \leq \alpha_Y$ and $\mu^+(X) = \{j : r_i < r_j\} \wedge \mu^-(Y) = \{j : r_i > r_j\}$. For any $\epsilon > 0$ we have: $\alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq \alpha_X \leq r_i \leq \alpha_Y = \alpha_{\theta(\vec{r}, r_i)}$. Therefore $\alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq r_i < r_i + \epsilon$ and $r_i \leq \alpha_{\theta(\vec{r}, r_i)}$. Thus $\{y \in \Lambda : \alpha_{\theta(\vec{r}, y)} \geq y\}$ must contain r_i and not $r_i + \epsilon$. As such $\mu(\vec{r}) = r_i$.

Therefore $\mu = \varphi$. □

Remark 5 The curve characterization will be useful in the proof and/or characterization of participation (Sect. 5.1), consistency (Sects. 5.2 and 5.3), proportionality (Sect. 5.4), and social welfare maximization (Sect. 6).

In the anonymous case, the curve characterization is simplified as follows.

Theorem 8 (Grading Curve Representation. Anonymous Case) A strategy-proof voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is anonymous iff there exists a weakly increasing function $g^n : [0, 1] \rightarrow \Lambda$ such that:

$$\forall X \in \Gamma; \alpha(X) = g^n \left(\frac{\#\mu^+(X)}{n} \right).$$

Where n is the cardinality of N .

Proof \Rightarrow : By Proposition 1, α is anonymous, so for each i there is α_i with $\alpha_X = \alpha_i$ for all $X \in \Gamma$ with $\#\mu^+(X) = i$. Therefore we can choose g^n weakly increasing with $g^n\left(\frac{i}{n}\right) = \alpha_i$. \Leftarrow : Suppose $\forall X, \alpha_X = g^n\left(\frac{\#\mu^+(X)}{n}\right)$. It follows that if X' is a permutation of X , then $\#\mu^+(X) = \#\mu^+(X')$ and $\alpha_X = \alpha_{X'}$. \square

Theorem 9 (Grading Curve Characterization. Anonymous Case)⁶ A voting rule $\varphi : \Lambda^n \rightarrow \Lambda$ is strategy-proof and anonymous iff there is a weakly increasing function $g^n : [0, 1] \rightarrow \Lambda$ such that:

$$\forall \vec{r}, \varphi(\vec{r}) := \sup \left\{ y : g^n \left(\frac{\#\{r_i \geq y\}}{n} \right) \geq y \right\}.$$

The g^n function is called the grading curve associated to φ .

Proof A direct consequence of Theorems 7 and 8. \square

The g^n function has the very intuitive interpretation as the density of the phantom voters. In fact, the median representation in Theorem 5 needs the specification of $n + 1$ values, and those values change completely with the size n of the electorate. By contrast, under the grading curve representation, the same function g can describe a family of mechanisms for all n simultaneously. For example, the linear function $g(x) = \frac{x}{R}$ corresponds to the phantoms uniformly distributed across the interval $[0, R]$, for all jury size n (see Sect. 5).

3.5 Voting by issues characterizations

The ‘voting by issues’ representation established by Barberà, Gul and Stacchetti [6] appears less practical than the other formulations, but it has the great power of being extendable to multi-dimensional and generalized single-peaked domains (see Nehring and Puppe [7, 15]). In this section, we state the original result and then refine it using our new tools.

Axiom 3 (Voter Sovereignty) A voting rule ϕ is voter sovereign iff for all $x \in \Lambda$ there is a preference profile \vec{r} such that: $\phi(\vec{r}) = x$.

This means that all alternatives can potentially be selected as the outcome. Section 4.1 characterizes voter sovereignty in terms of phantom functions. A voting by

⁶ The result first appeared in the unpublished Ph.D. thesis [14] of our co-author A. Jennings.

issues consists of a property space (Ω, \mathcal{H}) where Ω is a set of alternatives and \mathcal{H} is a set of subsets of Ω called *properties*. Each property H comes together with its complementary property $H^c := \Omega \setminus H$; the pair (H, H^c) is called an *issue*. Each voter i provides a ballot $r_i \in \Omega$. Each issue (H, H^c) is then resolved separably as a binary election. If $r_i \in H$ then we say that voter i votes for issue H or that his ballot verifies issue H . And if $r_i \notin H$ then we say that he voted against H , i.e. he voted for H^c . If an issue is elected then the result of the vote will be one of the elements of the issue. As such, the result of the election is an intersection of elements of \mathcal{H} . A coalition is a subset of voters. A coalition $W \subseteq N$ is said to be winning for $H \in \mathcal{H}$, if when all voters in W voted for H , H is elected. Let W_H be the set of winning coalitions for H . The following result is due to [6]; our present formulation follows [7].

Theorem 10 (Voting by Issues Characterization) *Let Λ be finite. A voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is strategy-proof and voter-sovereign iff it is voting by issues satisfying, for all $G, H \in N, G \subseteq H \Rightarrow \mathcal{W}_G \subseteq \mathcal{W}_H$.*

This result was proved with Λ finite, assuming voter-sovereignty. In the next characterization we explicitly define the winning coalitions in terms of the phantom functions without assuming voter sovereignty nor that Λ is finite.

Theorem 11 (Explicit Voting by Issues Characterization) *A voting rule φ is strategy-proof iff it is voting by issues on the property space (Λ, \mathcal{H}) where \mathcal{H} consists of all properties of the form $\{x \in \Lambda : x \leq a\}$, $\{x \in \Lambda : x \geq a\}$, and their complements, and such that, for all $a \in \Lambda$:*

- $\mu^+(X)$ is a winning coalition for $H = \{y \in \Lambda : y \geq a\}$ if and only if $\alpha_X \geq a$;
- $\mu^-(X)$ is a winning coalition for $H = \{y \in \Lambda : y \leq a\}$ if and only if $\alpha_X \leq a$.

The phantom function α is the one associated to φ , as in Theorem 2.

Proof Let φ be a strategy-proof voting rule defined by a phantom function α as in Theorem 2. Let μ be the vote by issue given in Theorem 11. Since φ is the unique strategy-proof voting rule defined by α we only need to prove that $\varphi = \mu$.

- Case $\varphi(\vec{r}) = \alpha_X$: For $a = \alpha_X$ we have $\mu^+(X) \subseteq \{j : r_j \geq a\}$ is a winning coalition for $\{y \geq a\}$ and $\mu^-(X) \subseteq \{j : r_j \leq a\}$ is a winning coalition for $\{y \leq a\}$. Therefore $\mu(\vec{r}) = a$.
- Case $\varphi(\vec{r}) = r_i$: Let $X, Y \in \Gamma$ be the same voting profiles as found in Theorem 2. For $a = r_i$ we have $\alpha_Y \geq a$ therefore $\mu^+(Y) \subseteq \{j : r_j \geq a\}$ is a winning coalition for $\{y \geq a\}$ and $\alpha_X \leq a$ therefore $\mu^-(X) \subseteq \{j : r_j \leq a\}$ is a winning coalition for $\{y \leq a\}$. Therefore $\mu(\vec{r}) = a$.

We have shown that $\varphi = \mu$. □

4 Additional properties: fixed electorate

The phantom function and the different representations (in particular the curve representation) will be very helpful in understanding the effects of imposing more axioms on the voting rule. This is the subject of this and the next section.

4.1 Voter sovereignty and efficiency

In some applications, it makes little sense to vote for an alternative that is never a possible output. Therefore, one may wish the voting rule to be voter-sovereign: for all $x \in \Lambda$ there is a profile \vec{r} s.t. $\varphi(\vec{r}) = x$ (see Axiom 3).

Proposition 2 *A strategy-proof voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is voter-sovereign iff its phantom function α satisfies $\alpha_{\vec{\mu}^-} = \mu^-$ and $\alpha_{\vec{\mu}^+} = \mu^+$. In that case, φ is unanimous ($\varphi(a, \dots, a) = a$ for all $a \in \Lambda$).*

Proof \Rightarrow If $\alpha_{\vec{\mu}^-} > \mu^-$ (resp. $\alpha_{\vec{\mu}^+} < \mu^+$), by Theorem 1, $\varphi(\vec{\mu}^-) = \alpha_{\vec{\mu}^-}$. Let x be any value such that $x < \alpha_{\vec{\mu}^-}$. By weak responsiveness (Lemma 1), we have $\forall \vec{r}, \varphi(\vec{r}) \geq \alpha_{\vec{\mu}^-} > x$. As such, φ is not voter-sovereign since x cannot be reached.

\Leftarrow Suppose that $\alpha_{\vec{\mu}^-} = \mu^-$ and $\alpha_{\vec{\mu}^+} = \mu^+$. Then according to Theorem 6, $\varphi(a, \dots, a) = a$ for all $a \in \Lambda$. As such it is voter-sovereign. \square

Axiom 4 (Pareto Optimality) *A voting rule is Pareto optimal if no other alternative leads to an improved satisfaction for some voter without loss for all voters.*

Proposition 3 *If a voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is voter-sovereign and strategy-proof then it is efficient (the selected alternative is Pareto optimal).*

This result was proved in Weymark [12]. Here is an alternative proof.

Proof Suppose that φ is strategy-proof and voter-sovereign. Let us use the curve characterization (Theorem 7): $\varphi(\vec{r}) := \sup \{y : \alpha_{\theta(\vec{r}, y)} \geq y\}$. Then for $y = \min\{r_j\}$, we have $\theta(\vec{r}, y) = \vec{\mu}^+$ and $\alpha_{\theta(\vec{r}, y)} = \mu^+ \geq y$. Therefore $\varphi(\vec{r}) \geq \min\{r_j\}$. Similarly, for $y = \max\{r_j\} + \epsilon$, where $\epsilon > 0$, we have $\theta(\vec{r}, y) = \vec{\mu}^-$ and $\alpha_{\theta(\vec{r}, y)} = \mu^- < y$. Therefore $\varphi(\vec{r}) \leq \max\{r_j\}$.

We have shown that $\min\{r_j\} = r_k \leq \varphi(\vec{r}) \leq r_l = \max\{r_j\}$. As such for $x \in \Lambda$ if $x < \varphi(\vec{r})$ (resp. $x > \varphi(\vec{r})$) then voter l (resp. k) has a worse outcome in x than in \vec{r} according to peak r_l (resp. r_k). It follows that no voting profile can obtain a result that is better for at least one voter without being worse for another. \square

4.2 Strict responsiveness and ordinality

Axiom 5 (Strict Responsiveness) *A voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is strictly responsive if for any \vec{r} and \vec{s} such that for all j , $r_j < s_j$ we have $\varphi(\vec{r}) < \varphi(\vec{s})$.*

Strict responsiveness is sometimes called strict monotonicity.

Axiom 6 (Ordinality) *A voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is ordinal iff for all strictly responsive and surjective functions $\pi : \Lambda \rightarrow \Lambda$ we have:*

$$\varphi(\pi(r_1), \dots, \pi(r_n)) = \pi \circ \varphi(\vec{r}).$$

Remark 6 Ordinality says nothing when Λ is finite. In this case, the identity is the only strictly responsive and surjective function from Λ to itself.

Proposition 4 For a strategy-proof voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ the following are equivalent:

- (1) The phantom function α verifies $\alpha(\Gamma) = \{\mu^-, \mu^+\}$
- (2) φ is strictly responsive.

And when moreover Λ is rich,⁷ (1) and (2) are equivalent to:

- (3) φ is ordinal and not constant. (Proof: See appendix D.1.)

In the anonymous case, using Moulin's median representation in the proof above, we deduce that the ordinal / strictly monotonic strategy-proof voting rules are the *order functions*, where for $k = 1, \dots, n$, the k th-order function is the SP-rule which associates to any input $\vec{r} = (r_1, \dots, r_n) \in \Lambda^n$ the k th-highest value $r_{(k)}$, where $r_{(1)} \geq \dots \geq r_{(k)} \geq \dots \geq r_{(n)}$ (See [16], Chapters 10 and 11).

5 Additional properties: variable electorate

We wish to consider situations where casting a vote or not is a choice. As such we need to make a distinction between the electorate \mathcal{E} (that is potentially infinite) and the set of voters $N \subseteq \mathcal{E}$ (which will be assumed finite). Henceforth, we define a *voter* as someone who chooses to cast a ballot and an *elector* as someone who can cast a ballot. Similarly a *ballot* is cast by a voter while a *vote* is the response of an elector. A vote that is not a ballot is represented by the symbol \emptyset (interpreted as abstention). As such we represent the set of votes as an element of $\Lambda^* = \{\vec{r} \in (\Lambda \cup \emptyset)^\mathcal{E} : \#r_i \neq \emptyset < +\infty\}$. Since we seek for strategy-proof methods, it may also be of interest to ensure that no elector can benefit from not casting a ballot (the absence of the no-show paradox or participation [17]). Another desirable property is consistency. It states that when we obtain the same outcome for the voting profiles of two disjoint group of voters (with fixed ballots) then that outcome is also the outcome of the union of their voting profiles. First, we need to extend our definitions and concepts to the variable electorate context.

Definition 5 (*Voting function*) A voting function $\varphi^* : \Lambda^* \rightarrow \Lambda$ is a function such that for any finite set of electors N there is a voting rule $\varphi_N : \Lambda^N \rightarrow \Lambda$ such that φ_N is the restriction of φ^* to the set of voters N .

Intuitively once our set of voters is fixed then we are considering a voting rule and we can use our previous results. Furthermore in the general case the voting rules are independent even if they use very similar (but not identical) set of voters. As such we start by determining what our set of voters is and then we use the corresponding voting rule. This is coherent with the non-anonymous setting where the result of the election strongly depends on who casts a ballot.

Definition 6 (*Redefining concepts for variable electorate*) A voting function φ^* verifies one of the previously mentioned properties (strategy-proofness, continuity, voter sovereignty, strict responsiveness, anonymity) if for all sets of voters N the restriction of φ^* to N verifies the property.

⁷ Λ is rich if for any $\alpha < \beta$ in Λ there exists a $\gamma \in \Lambda$ such that $\alpha < \gamma < \beta$.

As with voting rules, we would like to define a phantom function that completely characterizes the strategy-proof voting functions. We will then be able to characterize participation (e.g. no no-show paradox) and consistency by using the phantom functions.

Let $\Gamma^* := \{\mu^-, \mu^+, \emptyset\}^*$ be the set of voting profiles where voters have extreme positions or can abstain. (Recall that the set of voters is always finite). As before we wish for a phantom $\alpha : \Gamma^* \rightarrow \Lambda$ such that:

$$\forall X \in \Gamma^*, \alpha_X =: \varphi^*(X)$$

The bijection over each set of voters gives us the bijection between a strategy-proof voting function and its associated phantom function. The definition of a phantom function therefore corresponds to all functions that are phantom functions for each voting rule corresponding to a restriction of the voting function to a fixed set of voters.

Definition 7 (*Phantom functions extended to variable electorate*) A function $\alpha : \Gamma^* \rightarrow \Lambda$ is a phantom function if α verifies for any X and Y that differ only in dimension i with $X_i = \mu^-$ and $Y_i = \mu^+$ we have $\alpha_X \leq \alpha_Y$.

Definition 8 (*The θ function extended to variable electorate*)

$$\begin{aligned} \theta : (\mathbf{R} \cup \{\emptyset\})^{\mathcal{E}} \times \mathbf{R} &\rightarrow \{\mu^-, \mu^+, \emptyset\}^{\mathcal{E}} \\ \theta : \vec{r}, x &\rightarrow X = \theta(\vec{r}, x) \end{aligned}$$

Such that $\forall i; X_i = \mu^- \Leftrightarrow r_i < x$ and $\forall i; X_i = \mu^+ \Leftrightarrow r_i \geq x$. It follows that $X_i = \emptyset$ means that the elector i did not vote ($r_i = \emptyset$).

5.1 Participation (or no no-show paradox)

Axiom 7 (*Participation*) A voting function $\varphi^* : \Lambda^* \rightarrow \Lambda$ is said to verify participation if for all \vec{r} where $r_i \neq \emptyset$ and \vec{s} that only differs from \vec{r} in dimension i with $s_i = \emptyset$ we have:

$$\varphi^*(\vec{s}) \geq \varphi^*(\vec{r}) \geq r_i \text{ or } \varphi^*(\vec{s}) \leq \varphi^*(\vec{r}) \leq r_i.$$

Participation is a natural extension of strategy-proofness. “Strategy-proofness + participation” is equivalent to no matter what the elector does, no strategy gives a strictly better outcome than a honest ballot.

Theorem 12 (*Participation and Phantom Functions*) A strategy-proof voting rule $\varphi^* : \Lambda^* \rightarrow \Lambda$ verifies participation iff with the order $\mu^- < \emptyset < \mu^+$, its associated phantom function α is weakly increasing. (Proof: See appendix D.2.)

Strategy-proofness requires phantom functions to be weakly increasing. Hence, there is no surprise that “strategy-proofness + participation” implies that the phantom functions are weakly increasing for the order $\mu^- < \emptyset < \mu^+$. Also, it is of no surprise

that in the voting by issues approach, participation means monotonicity with respect to the addition of a new member.⁸

Proposition 5 (Participation and Winning Coalitions) *In a vote by issue, a strategy-proof voting function verifies participation iff when an elector i decides to become a voter with ballot x then for any property H containing x , if W_H was a winning coalition of H for the initial set of voters then $W_H \cup \{i\}$ is a winning coalition for the new set of voters. (Proof: See appendix D.3.)*

5.2 Consistency

To define consistency, we introduce the function $\sqcup : \Lambda^* \times \Lambda^* \rightarrow \Lambda^*$ that takes the ballots of two disjoint sets of voters and returns the union of the ballots.

Axiom 8 (Consistency) *A voting function φ^* is said to be consistent if for any two disjoint sets of voters R and S , when \vec{r} represents the ballots of R and \vec{s} represents the ballots of S , we have:*

$$\varphi^*(\vec{r}) = \varphi^*(\vec{s}) \Rightarrow \varphi^*(\vec{r}) = \varphi^*(\vec{r} \sqcup \vec{s}).$$

Theorem 13 *A strategy-proof voting function $\varphi^* : \Lambda^* \rightarrow \Lambda$ verifies consistency iff for all $X, Y \in \Gamma^*$ with disjoint sets of voters and $\alpha_X \leq \alpha_Y$ we have:*

$$\alpha_X \leq \alpha_{X \sqcup Y} \leq \alpha_Y.$$

Proof \Rightarrow : By *reductio ad absurdum*. Let us suppose $\alpha_{X \sqcup Y} < \alpha_X \leq \alpha_Y$. Let us define \vec{s} as: if $Y_j = \mu^-$ then $s_j = \mu^-$ and if $Y_j = \mu^+$ then $s_j = \alpha_X$. Therefore $\varphi^*(X) = \alpha_X$, $\varphi^*(\vec{s}) = \alpha_X$ and $\varphi^*(X \sqcup \vec{s}) = \alpha_{X \sqcup Y}$. This contradicts consistency. A similar proof shows that we cannot have $\alpha_X \leq \alpha_Y < \alpha_{X \sqcup Y}$.

\Leftarrow : Suppose that for all $X, Y \in \Gamma^*$ that correspond to two disjoint sets of voters N_1 and N_2 such that $\alpha_X \leq \alpha_Y$ we have $\alpha_X \leq \alpha_{X \sqcup Y} \leq \alpha_Y$.

If $\varphi^*(\vec{r}) = \varphi^*(\vec{s}) = a$ then:

- If $a = r_i = s_k$: then for all $\epsilon > 0$ we have $\alpha_{\theta(\vec{r} \sqcup \vec{s}, r_i + \epsilon)} \leq \max(\alpha_{\theta(\vec{r}, r_i + \epsilon)}, \alpha_{\theta(\vec{s}, r_i + \epsilon)}) \leq r_i$ and $r_i \leq \min(\alpha_{\theta(\vec{r}, r_i)}, \alpha_{\theta(\vec{s}, r_i)}) \leq \alpha_{\theta(\vec{r} \sqcup \vec{s}, r_i)}$. For ϵ small enough $\mu^+(\theta(\vec{r} \sqcup \vec{s}, r_i + \epsilon)) = \{j \in N_1 : r_j > r_i\} \cup \{j \in N_2 : s_j > r_i\}$. Therefore $\varphi(\vec{r} \sqcup \vec{s}) = a$. Consistency is verified.
- If $a = r_i = \alpha_X$ where $X = \vec{s}$, then for all $\epsilon > 0$: we have $\alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq \alpha_{\theta(\vec{r} \sqcup X, r_i + \epsilon)} \leq \alpha_X = r_i$ and $r_i = \alpha_X \leq \alpha_{\theta(\vec{r} \sqcup X, r_i)} \leq \alpha_{\theta(\vec{r}, r_i)}$. For ϵ small enough $\mu^+(\theta(\vec{r} \sqcup X, r_i + \epsilon)) = \{j \in N_1 : r_j > r_i\} \cup \{j \in N_2 : s_j > r_i\}$. Therefore $\varphi(\vec{r} \sqcup \vec{s}) = a$. Consistency is verified.
- If $a = \alpha_X = \alpha_Y$ where the voters of $X = \vec{r}$ and the voters of $Y = \vec{s}$, we have $\alpha_{X \sqcup Y} = \alpha_X = \alpha_Y$. Therefore $\varphi(\vec{r} \sqcup \vec{s}) = a$. Consistency is verified. \square

⁸ While strategy-proofness is thus easily compatible with the absence of the no-show paradox in our context of costless voting, Müller and Puppe [18] have shown that, under quite general conditions, strategy-proofness implies minimal participation in equilibrium once voting is costly (even with infinitesimal cost).

Notice that the set of winning coalitions in a voting by issue context is defined for each fixed set of voters, but there is no consistency imposed a priori when the set of voters changes. When consistency is imposed, in addition to SP, some monotonicity over the set of winning coalitions is obtained.

Proposition 6 (Consistent Voting Coalitions) *In a vote by issues, a strategy-proof voting function verifies consistency iff for any $a \in \Lambda$ if X and $Y \in \Gamma^*$ have disjoint sets of voters:*

- When $\mu^+(X)$ and $\mu^+(Y)$ are winning coalitions for $H = \{y \geq a\}$ then $\mu^+(X) \cup \mu^+(Y)$ is a winning coalition for H .
- When $\mu^-(X)$ and $\mu^-(Y)$ are winning coalitions for $H = \{y \leq a\}$ then $\mu^-(X) \cup \mu^-(Y)$ is a winning coalition for H .

Proof \Rightarrow : Suppose that our strategy-proof voting verifies $\alpha_X \leq \alpha_{X \sqcup Y} \leq \alpha_Y$ for all X and Y that correspond to disjoint sets of voters. A simple inequality consideration for any a gives the result. \Leftarrow : Suppose $\alpha_X \leq \alpha_Y$. $\mu^+(X)$ and $\mu^+(Y)$ are winning coalitions for $\{y \geq \alpha_X\}$ therefore $\alpha_{X \sqcup Y} \geq \alpha_X$. Conversely $\mu^-(X)$ and $\mu^-(Y)$ are winning coalitions for $\{y \leq \alpha_X\}$ therefore $\alpha_{X \sqcup Y} \leq \alpha_Y$. Therefore we verify participation. \square

Proposition 7 *A strategy-proof voting function $\varphi^* : \Lambda^* \rightarrow \Lambda$ that verifies voter sovereignty and consistency also verifies participation.*

Proof Let $X \in \Gamma^*$ and $Y \in \Gamma^*$ be the voting profile where only i is a voter and where he votes respectively μ^- and μ^+ . By voter sovereignty, $\alpha_X = \mu^-$ and $\alpha_Y = \mu^+$. By consistency, for any $Z \in \Gamma^*$ where i is not a voter:

$$\alpha_X \leq \alpha_{X \sqcup Z} \leq \alpha_Z \leq \alpha_{Y \sqcup Z} \leq \alpha_Y.$$

\square

5.3 Combining consistency and anonymity

In this section we show that consistency in the anonymous case is equivalent to removing the dependency of the grading curves on the number of voters n .

Axiom 9 (Homogeneity) *A voting profile is homogeneous iff for any two profiles \vec{r} and \vec{s} such that there exists $k \geq 1$ that verifies:*

$$\forall x \in \Lambda, \#\{j : x = r_j \in \vec{r}\} = k \#\{j : x = r_j \in \vec{s}\}$$

we have

$$\varphi(\vec{s}) = \varphi(\vec{r}).$$

Hence, a voting function is homogeneous if for any $k \geq 1$ when each ballot is replaced by k copies of that ballot the result of the function does not change.

Proposition 8 *A SP function is consistent and anonymous iff it is homogeneous.*

Proof \Rightarrow : Immediate due to the fact that α_X only depends on the fraction $\frac{\#\mu^+(X)}{\#\mu^-(X) + \#\mu^+(X)}$. \Leftarrow : Suppose that we have a homogenous strategy-proof voting function. By definition this implies anonymity. For any X and Y , we can duplicate in order to have X' , Y' and $(X \sqcup Y)'$ with the same number of voters in each and $\alpha_Z = \alpha_{Z'}$ for $Z \in \{X, Y, X \sqcup Y\}$. By barycentric considerations:

$$\mu^+(X') \leq \mu^+((X \sqcup Y)') \leq \mu^+(Y')$$

Therefore by definition of a phantom function $\alpha_X \leq \alpha_{X \sqcup Y} \leq \alpha_Y$. \square

Theorem 14 *A SP voting function $\varphi^* = (\varphi^n) : \Lambda^* \rightarrow \Lambda$ is anonymous and consistent iff there is a weakly increasing function $g : [0, 1] \rightarrow \Lambda$ (electorate size independent) and a constant $x \in \Lambda$ such that the associated phantom function α on Γ^* is defined as:*

$$\alpha_X := \begin{cases} g\left(\frac{\#\mu^+(X)}{\#N}\right) & \text{if } \#N \neq 0 \\ x & \text{if } \#N = 0 \end{cases}$$

Furthermore φ^* verifies participation iff $x \in g([0, 1])$. (Proof: See appendix D.4.)

This is an elegant new result. It says that consistency and anonymity are equivalent to the grading curve being independent of the electorate size.

Axiom 10 (Continuity with Respect to New Members) [10, 11]. *A voting function φ^* is said to be continuous with respect to new members if:*

$$\forall \vec{r}, \vec{s}, \lim_{n \rightarrow +\infty} \varphi^*(\overbrace{\vec{r} \sqcup \cdots \sqcup \vec{r}}^n \sqcup \vec{s}) = \varphi(\vec{r})$$

Not surprisingly, but elegantly, continuity with respect to new members is equivalent to continuity of the grading curve:

Theorem 15 (Continuous Grading Curves) *A strategy-proof, homogeneous (= consistent and anonymous) voting function $\varphi^* : \Lambda^* \rightarrow \Lambda$ is continuous with respect to new members iff its grading curve g is continuous. (Proof: See appendix D.5.)*

5.4 Proportionality

In this subsection we assume that $\Lambda = [0, 1]$.

Definition 9 (Linear=Uniform Median) The strategy-proof voting function $\varphi^* : [0, 1]^* \rightarrow [0, 1]$ defined for any $n = \#N$ and $X \in \{0, 1\}^N$, by $\alpha(X) = \frac{\sum_i X_i}{n}$ is called the linear median. It corresponds to the grading curve $g(x) = x$.

The linear median was first proposed and studied in the unpublished Ph.D. dissertation [14] of our co-author A. Jennings. It was rediscovered independently by Caragiannis et al. [9] for its nice statistical properties under the name ‘uniform median.’

These authors use a different representation based on the Moulin phantom characterization. Namely, if there are n voters and $\Lambda = [0, 1]$ then the linear median can be computed via the Moulin median formula:

$$\varphi(\vec{r}) = \text{med}(r_1, \dots, r_n, \alpha_0, \dots, \alpha_n),$$

where $\alpha_k = \frac{k}{n}$, $\forall k = 0, \dots, n$, that is, the Moulin $n + 1$ phantom voters are uniformly distributed on the interval $[0, 1]$. It is not evident from the median representation why this ‘uniform median’ function satisfies participation, consistency, or continuity. But these properties are immediate consequences of the linear grading curve $g(x) = x$ representation since it is continuous and independent of the size of the electorate. We thus obtain:

Proposition 9 *The linear median satisfies anonymity, continuity, consistency, sovereignty, participation and continuity with respect to new members. On the other hand, it is neither ordinal nor strictly responsive.*

The proof is trivial from previous subsections as g is independent on the electorate size, is continuous, is voter sovereign, etc.

Axiom 11 (Proportionality) *A voting function $\varphi^* : [0, 1]^* \rightarrow [0, 1]$ is proportional if*

$$\forall X \in \{0, 1\}^N, \varphi^*(X) = \frac{\sum_i X_i}{\#N}.$$

Theorem 16 *A voting function $\varphi^* : [0, 1]^* \rightarrow [0, 1]$ is strategy-proof and proportional iff it is the linear median.*

This a direct consequence of the fact that a SP function is completely determined by its phantom function as proved in Theorem 2.

6 Maximizing social welfare

Social welfare is often taken to be the sum of individual utilities. This section deals with the maximization of social welfare under the strategy-proofness constraint. We will measure the individual utilities by the L_q -distance to the peaks and our objective is to compute the SP socially optimal mechanism for $q \in \{1, 2\}$. (In this section we assume that $\Lambda = [m, M]$.)

Definition 10 (*Ex-Post Social Welfare*) The ex-post social welfare for a given voting rule $\varphi : [m, M]^n \rightarrow [m, M]$ and a given norm L_q is defined to be:

$$SW(\varphi, \vec{r}) := - \sum_i \|\varphi(\vec{r}) - r_i\|_q.$$

It is well-known that, if the number of voters is odd, the unique voting rule which maximizes ex-post social welfare for the L_1 norm is the median. If the number of

voters is even, the voting rule must be the median of the votes along with one fixed phantom vote, α . (See for instance [16], Section 12.4.) On the other hand, it is trivial to establish that the unique voting rule that always maximizes the L_2 ex-post social welfare is the arithmetic mean: $\phi(r_1, \dots, r_n) = \frac{1}{n} \sum_i^n r_i$ —which is clearly not a SP voting rule. Since no fixed SP voting rule is ex-post optimal for the L_2 -norm for every realization \vec{r} , we can optimize ex-ante.

Theorem 17 *The linear median, corresponding to $g(x) = m + x(M - m)$ is the unique voting rule that minimizes*

$$E(f) = \int_m^M \cdots \int_m^M \sum_{i=1}^n (x_i - f(\vec{x}))^2 dx_n \dots dx_1$$

over the set of all strategy-proof voting rules f . (Proof: See appendix E.)

7 Conclusion

We introduced the notions of phantom functions and grading curves and demonstrated their usefulness in (i) unifying a number of existing characterizations of strategy-proof voting rules on the domain of single-peaked preferences, and (ii) obtaining insightful new characterizations.

As an important example, we have characterized the linear median as the unique strategy-proof voting rule satisfying proportionality or maximizing the ex-ante social welfare under the L_2 -norm and a uniform ex-ante prior. It has been shown to possess further salient properties such as consistency and participation (because its grading curve, the identity, is size electorate independent). On the other hand, the linear median presupposes a cardinal scale. However, adding the natural condition of ordinality characterizes, in the anonymous case, the class of order (statistics) functions which play an important role in the majority judgment—ordinal—method of voting (Balinski-Laraki [16] Chapters 10–13). A particularly appealing order function is the middlemost (Chapter 13 in [16]).

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Appendix A: Proof of Lemma 3

If a voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ is strategy-proof, then it has a unique continuous extension in $\overline{\Lambda}^N \rightarrow \overline{\Lambda}$.

Proof We define a clamp function $\vec{c} : \overline{\Lambda}^N \times \Lambda \times \Lambda \rightarrow \Lambda^N$ (component-wise) by:

$$\vec{c}(\vec{r}, x^-, x^+)_i := \min\{x^+, \max\{x^-, r_i\}\}.$$

\vec{c} is weakly-increasing with respect to x^+ and x^- .

Fix $\vec{r} \in \overline{\Lambda}$ and a sequence \vec{s}_i in Λ converging to \vec{r} . Define the sequence \vec{t}_i to match \vec{s}_i in every component j where $r_j \in \{\mu^-, \mu^+\}$, and to match \vec{r} in every component where $r_j \notin \{\mu^-, \mu^+\}$. $\lim_{i \rightarrow \infty} |\varphi(\vec{s}_i) - \varphi(\vec{t}_i)| = 0$ by uniform continuity of φ .

- Case $\exists a, b \in \Lambda, a < \varphi(\vec{c}(\vec{r}, a, b)) < b$:

By strategy proofness and monotonicity, $\forall x, y \in \Lambda$ with $x \leq a < b \leq y$, $\varphi(\vec{c}(\vec{r}, x, y)) = \varphi(\vec{c}(\vec{r}, a, b))$. \vec{t}_i is eventually confined to this region. Thus:

$$\lim_{i \rightarrow \infty} \varphi(\vec{s}_i) = \lim_{i \rightarrow \infty} \varphi(\vec{t}_i) = \varphi(\vec{c}(\vec{r}, a, b)).$$

- Case $\nexists a, b \in \Lambda, a < \varphi(\vec{c}(\vec{r}, a, b)) < b$:

Choose arbitrary $a, b \in \Lambda$ with $a < b$. We must have either $\varphi(\vec{c}(\vec{r}, a, b)) \leq a$ or $b \leq \varphi(\vec{c}(\vec{r}, a, b))$.

- Subcase $\varphi(\vec{c}(\vec{r}, a, b)) \leq a$:

Choose arbitrary $c \in \Lambda$ with $c \leq a$. Again, since $c < b$ we must have either $\varphi(\vec{c}(\vec{r}, c, b)) \leq c$ or $b \leq \varphi(\vec{c}(\vec{r}, c, b))$, but in this case the latter would lead to a contradiction (since $\varphi(\vec{c}(\vec{r}, c, b)) \leq \varphi(\vec{c}(\vec{r}, a, b)) \leq a < b$). Thus, $\varphi(\vec{c}(\vec{r}, c, b)) \leq c < b$.

As above, strategy-proofness and monotonicity imply

$$\forall y \in \Lambda, \varphi(\vec{c}(\vec{r}, c, y)) \leq \varphi(\vec{c}(\vec{r}, c, b)) \leq c.$$

Since c was chosen arbitrarily from Λ with no lower bound and \vec{t}_i is eventually confined to this region, it follows that

$$\lim_{i \rightarrow \infty} \varphi(\vec{s}_i) = \lim_{i \rightarrow \infty} \varphi(\vec{t}_i) = \inf \Lambda = \mu^-.$$

- Subcase $b \leq \varphi(\vec{c}(\vec{r}, a, b))$:

A completely analogous argument to the subcase above gives

$$\forall c \geq b, \forall x \in \Lambda, \varphi(\vec{c}(\vec{r}, x, c)) \geq \varphi(\vec{c}(\vec{r}, a, c)) \geq c$$

and

$$\lim_{i \rightarrow \infty} \varphi(\vec{s}_i) = \lim_{i \rightarrow \infty} \varphi(\vec{t}_i) = \sup \Lambda = \mu^+.$$

If both subcases were to exist (for different $a, b \in \Lambda$) then we would have $\varphi(\vec{c}(\vec{r}, a_1, b_1)) \leq a_1 < b_1$ and $a_2 < b_2 \leq \varphi(\vec{c}(\vec{r}, a_2, b_2))$. This leads to a contradiction at $a_0 = \min\{a_1, a_2\}$ and $b_0 = \max\{b_1, b_2\}$:

$$\varphi(\vec{c}(\vec{r}, a_0, b_0)) \leq a_0 < b_0 \leq \varphi(\vec{c}(\vec{r}, a_0, b_0)).$$

Thus, whichever subcase holds for an arbitrarily chosen $a < b$ in Λ must hold for all $a < b$ in Λ .

Since $\vec{s}_i \rightarrow \vec{r}$ was arbitrary and $\lim_{i \rightarrow \infty} \varphi(\vec{s}_i)$ is well-defined, we conclude that this limit is the unique continuous extension of φ to $\overline{\Lambda}^N \rightarrow \Lambda$. \square

Appendix B: Proof of Theorem 1

This section aims to prove the following: the voting function φ is strategy-proof iff there exists a phantom function $\alpha : \Gamma \rightarrow \Lambda$ such that:

$$\forall \vec{r}; \varphi(\vec{r}) := \begin{cases} \alpha_{\vec{\mu}^-} & \text{if } \forall j, r_j \leq \alpha_{\vec{\mu}^-} \\ \alpha_{\theta(\vec{r}, r_i)} & \text{if } (i \in N) \text{ and } r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, r_i)}\} \\ r_i & \text{if } (i \in N) \text{ and } \forall \epsilon > 0, \alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq r_i \leq \alpha_{\theta(\vec{r}, r_i)} \end{cases} \quad (6)$$

Throughout the proofs, x_i often denotes the i th smallest element of the voting profile \vec{r} .

Definition 11 For a given phantom function α , and a given voting profile \vec{r} we define $\alpha_{\vec{r}, i} := \alpha_{\theta(\vec{r}, x_i)}$ where x_i is the i th smallest element of $(r_k)_{k \in N}$ and we define $\alpha_{\vec{r}, n+1} := \alpha_{\vec{\mu}^-} = \alpha(\mu^-, \dots, \mu^-)$.

Lemma 5 *Let us fix a phantom function α . For all \vec{r} , there exists a voter i such that one of the following holds*

- $r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, r_i)}\}$.
- $\forall \epsilon > 0, \alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq r_i \leq \alpha_{\theta(\vec{r}, r_i)}$.
- $\forall j \in N, r_j < \alpha_{\vec{\mu}^-}$.

Proof Fix voting profile \vec{r} . Consider the following cases:

- (i) Case $x_n < \alpha_{\vec{r}, n+1}$: Then $\forall j \in N, r_j \leq x_n < \alpha_{\vec{r}, n+1} = \alpha_{\vec{\mu}^-}$.
- (ii) Case $x_1 \geq \alpha_{\vec{r}, 1}$: Then $x_1 = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, x_1)}\}$.
- (iii) Case $\exists k > 1$ s.t. $x_{k-1} < \alpha_{\vec{r}, k} \leq x_k$: Then $x_k = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, x_k)}\}$.
- (iv) None of the above cases are true:

Let $i = \min\{j : x_j \geq \alpha_{\vec{r}, j+1}\}$. (The set is nonempty, otherwise case (i) would hold.) By this definition, $x_i \geq \alpha_{\vec{r}, i+1}$ and either $i = 1$ or $x_{i-1} < \alpha_{\vec{r}, i}$. In either case, we must have $x_i < \alpha_{\vec{r}, i}$. (If $\alpha_{\vec{r}, i} \leq x_i$ then case (ii) or case (iii) would be true, respectively.)

Now we will show that $\forall \epsilon > 0, \alpha_{\theta(\vec{r}, x_i + \epsilon)} \leq \alpha_{\vec{r}, i+1}$. If $i = n$, this is because $\alpha_{\theta(\vec{r}, x_n + \epsilon)} = \alpha_{\vec{\mu}^-} = \alpha_{\vec{r}, n+1}$. For $i < n$, there is no r_j strictly between x_i and x_{i+1} , so $\{j : r_j \geq x_i + \epsilon\} \subseteq \{j : r_j \geq x_{i+1}\}$. It follows that $\theta(\vec{r}, x_i + \epsilon) \leq \theta(\vec{r}, x_{i+1})$ and $\alpha_{\theta(\vec{r}, x_i + \epsilon)} \leq \alpha_{\theta(\vec{r}, x_{i+1})} = \alpha_{\vec{r}, i+1}$.

We conclude that $\forall \epsilon > 0, \alpha_{\theta(\vec{r}, x_i + \epsilon)} \leq \alpha_{\vec{r}, i+1} \leq x_i \leq \alpha_{\theta(\vec{r}, x_i)}$. \square

Lemma 6 *The function φ , defined from a fixed α as in equation 6 is properly defined for all \vec{r} .*

Proof Given Lemma 5 there is always a voter i that verifies one of the properties that define φ . We must show that if different i satisfy the property this does not lead to an ambiguous definition.

- If the first and second property are verified then $\alpha_{\vec{r},i} \leq r_i \leq \alpha_{\vec{\mu}^-}$. By monotonicity of α , $\alpha_{\vec{\mu}^-} = \alpha_{\vec{r},n+1} \leq \alpha_{\vec{r},i}$, therefore $\alpha_{\vec{r},i} = \alpha_{\vec{\mu}^-}$. There is no contradiction.
- If the first and third property are verified then $\forall \epsilon > 0, \alpha_{\theta(\vec{r},r_i+\epsilon)} \leq r_i \leq \alpha_{\vec{\mu}^-}$. Therefore by monotonicity of α , $\alpha_{\vec{\mu}^-} = r_i$. There is no contradiction.
- If the second is verified for a voter i and the third for a voter k with $r_i \leq r_k$, then we have $r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_i)}\}$ and $\forall \epsilon > 0, \alpha_{\theta(\vec{r},r_k+\epsilon)} \leq r_k \leq \alpha_{\theta(\vec{r},r_k)}$. Since $r_i \leq r_k$, we have $\alpha_{\theta(\vec{r},r_k)} \leq \alpha_{\theta(\vec{r},r_i)}$. Hence $r_i \leq r_k \leq \alpha_{\theta(\vec{r},r_k)} \leq \alpha_{\theta(\vec{r},r_i)} \leq r_i$. This proves that $\alpha_{\theta(\vec{r},r_i)} = r_k$. This case does not lead to an ambiguous definition.
- If the second is verified for a voter i and the third for a voter k with $r_k < r_i$, then we have $r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_i)}\}$ and $\forall \epsilon > 0, \alpha_{\theta(\vec{r},r_k+\epsilon)} \leq r_k \leq \alpha_{\theta(\vec{r},r_k)}$. Therefore by monotonicity of α , for $\epsilon > 0$ such that $r_i > r_k + \epsilon$:

$$\theta(\vec{r}, r_i) \leq \theta(\vec{r}, r_k + \epsilon) \text{ as such } \alpha_{\theta(\vec{r},r_i)} \leq \alpha_{\theta(\vec{r},r_k+\epsilon)} \leq r_k < r_i.$$

This is absurd since $r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_i)}\}$. Therefore this case never happens.

- Suppose the second is true for two different voters i and k . Therefore $r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_i)}\}$ and $r_k = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_k)}\}$. Without loss of generality we can suppose that $r_i \leq r_k$. Therefore by monotonicity of α , $\alpha_{\theta(\vec{r},r_k)} \leq \alpha_{\theta(\vec{r},r_i)} \leq r_i$. Since r_k is the min of $\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_k)}\}$ we have that $r_k \leq r_i$. As such $r_k = r_i$. Therefore $\alpha_{\theta(\vec{r},r_k)} = \alpha_{\theta(\vec{r},r_i)}$. This case does not raise an ambiguous definition.
- Suppose the third is true for two different voters i and k . Therefore for all $\epsilon > 0$, $\alpha_{\theta(\vec{r},r_i+\epsilon)} \leq r_i \leq \alpha_{\theta(\vec{r},r_i)}$ and $\alpha_{\theta(\vec{r},r_k+\epsilon)} \leq r_k \leq \alpha_{\theta(\vec{r},r_k)}$. Let us suppose that $r_i \neq r_k$. Without loss of generality, we will use $r_i < r_k$. Choose $\epsilon > 0$ so that $r_k > r_i + \epsilon$. Then by monotonicity of α we have $\alpha_{\theta(\vec{r},r_i+\epsilon)} \leq r_i < r_k \leq \alpha_{\theta(\vec{r},r_k)} \leq \alpha_{\theta(\vec{r},r_i+\epsilon)}$. Which is absurd. As such $r_i = r_k$. This case does not cause an ambiguous definition of φ .

We have considered all the cases. The definition is not ambiguous. \square

Lemma 7 *If φ verifies strategy-proofness then there is a phantom function α such that equation 6 holds.*

Proof Let us use the phantom function α such that $\alpha(X) := \varphi(X)$. We will show for each case used in the equation 6 that we obtain the desired value of $\varphi(\vec{r})$. Lemma 5 provides that we have studied all the cases and that therefore the proof is complete.

1. Case $\forall j, r_j \leq \alpha_{\vec{\mu}^-}$: Since $\alpha_{\vec{\mu}^-} = \varphi(\mu^-, \dots, \mu^-) \in \Lambda$, $\alpha_{\vec{\mu}^-} \geq \mu^-$. If $\alpha_{\vec{\mu}^-} = \mu^-$, then $\forall j, r_j = \mu^-$ so $\varphi(\vec{r}) = \varphi(\vec{\mu}^-) = \alpha_{\vec{\mu}^-}$. If $\mu^- < \alpha_{\vec{\mu}^-} = \varphi(\mu^-, \dots, \mu^-)$, then we can use strategy-proofness on each dimension to raise each component of the input of φ without changing the output. Thus $\varphi(\vec{r}) = \varphi(\vec{\mu}^-) = \alpha_{\vec{\mu}^-}$.
2. Case $\exists i, r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r},r_i)}\}$: Let $X = \theta(\vec{r}, r_i)$. If $\varphi(\vec{r}) < \alpha_X$, then $\varphi(\vec{r}) < \alpha_X \leq r_i$. By strategy-proofness we could take every component of \vec{r} which is greater than or equal to r_i and raise it to μ^+ , and also lower all other components

to μ^- , without increasing the output of φ . Thus $\varphi(\theta(\vec{r}, r_i)) \leq \varphi(\vec{r}) < \varphi(\theta(\vec{r}, r_i))$, which is a contradiction.

If $\varphi(\vec{r}) > \alpha_X$, we will also find a contradiction. If there is no r_j strictly smaller than r_i , then $\theta(\vec{r}, r_i) = \vec{\mu}^+$ and $\alpha_X = \alpha_{\vec{\mu}^+} = \varphi(\vec{\mu}^+) \geq \varphi(\vec{r})$, a contradiction. Otherwise, let r_j be the largest component of \vec{r} strictly smaller than r_i . Then $r_j < \alpha_{\theta(\vec{r}, r_i)} = \alpha_X < \varphi(\vec{r})$. By strategy-proofness, we can lower all components less than or equal to r_j to μ^- and raise the others to μ^+ , weakly increasing the output of φ . This gives $\varphi(\theta(\vec{r}, r_i)) \geq \varphi(\vec{r}) > \alpha_X = \varphi(\theta(\vec{r}, r_i))$, a contradiction.

Therefore if $\exists i, r_i = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, r_i)}\}$ then $\varphi(\vec{r}) = \alpha_X$.

3. Case $\forall \epsilon > 0, \alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq r_i \leq \alpha_{\theta(\vec{r}, r_i)}$:

Suppose that $\varphi(\vec{r}) < r_i$. By strategy-proofness, we can raise all components greater than or equal to r_i to μ^+ and lower the others to μ^- , weakly decreasing the output of φ . This gives $\varphi(\theta(\vec{r}, r_i)) \leq \varphi(\vec{r}) < r_i \leq \alpha_{\theta(\vec{r}, r_i)} = \varphi(\theta(\vec{r}, r_i))$, a contradiction. Suppose that $\varphi(\vec{r}) > r_i$. Choose $\epsilon > 0$ such that $r_i + \epsilon$ is strictly between r_i and the smallest component of \vec{r} strictly larger than r_i . (If there is no larger component, then any arbitrary $\epsilon > 0$ will do.) By strategy-proofness, we can lower all components less than or equal to r_i to μ^- and raise the others to μ^+ , weakly increasing the output of φ . This gives $\varphi(\theta(\vec{r}, r_i + \epsilon)) \geq \varphi(\vec{r}) > r_i \geq \alpha_{\theta(\vec{r}, r_i + \epsilon)} = \varphi(\theta(\vec{r}, r_i + \epsilon))$, a contradiction.

Therefore if $\forall \epsilon > 0, \alpha_{\theta(\vec{r}, r_i + \epsilon)} \leq r_i \leq \alpha_{\theta(\vec{r}, r_i)}$, then $r_i = \varphi(\vec{r})$.

□

Lemma 8 For a function φ , if there is a phantom function α such that Eq. 6 holds then φ is strategy-proof.

Proof First we prove weak responsiveness then strategy-proofness.

- Weak responsiveness: This proof is by induction. Let \vec{s} differ from \vec{r} only in dimension i and $r_i < s_i$. Let k be such that $\varphi(\vec{r}) \in \{\alpha_{\theta(\vec{r}, r_k)}, r_k\}$ when $\varphi(\vec{r}) \neq \alpha_{\vec{\mu}^-}$.

1. Case $\varphi(\vec{r}) = \alpha_{\vec{\mu}^-}$:

$\alpha_{\vec{\mu}^-}$ is the minimum of φ therefore $\varphi(\vec{r}) \leq \varphi(\vec{s})$.

2. Case $s_i \leq \varphi(\vec{r})$:

We have $\varphi(\vec{r}) \leq r_k$, therefore $s_i \leq r_k$ and $i \neq k$. If $s_i = r_k$ then $r_k \leq \alpha_{\theta(\vec{r}, r_k)}$.

We have $\forall \epsilon > 0, \theta(\vec{s}, s_k + \epsilon) = \theta(\vec{r}, r_k + \epsilon)$ and $\theta(\vec{s}, s_k) \geq \theta(\vec{r}, r_k)$. Therefore,

$\forall \epsilon > 0, \alpha_{\theta(\vec{s}, s_k + \epsilon)} \leq s_k \leq \alpha_{\theta(\vec{s}, s_k)}$. Otherwise $s_i < r_k$ implies $\theta(\vec{s}, s_k) = \theta(\vec{r}, r_k)$ and $\theta(\vec{s}, s_k + \epsilon) = \theta(\vec{r}, r_k + \epsilon)$. Therefore, $\forall \epsilon > 0, \alpha_{\theta(\vec{s}, s_k + \epsilon)} \leq s_k \leq \alpha_{\theta(\vec{s}, s_k)}$. In both cases, we can conclude that $\varphi(\vec{r}) = \varphi(\vec{s})$.

3. Case $\varphi(\vec{r}) < r_i$:

If $r_k = r_i$, then $\varphi(\vec{r}) = \alpha_{\theta(\vec{r}, r_k)}$. Let $s_l = \min\{s_j : s_j \geq \varphi(\vec{r})\}$. Then $\theta(\vec{s}, s_l) = \theta(\vec{r}, r_k)$ and $s_l = \min\{s_j : s_j \geq \alpha_{\theta(\vec{r}, s_l)}\}$. Therefore $\varphi(\vec{r}) = \varphi(\vec{s})$. Otherwise if $k \neq i$, $\theta(\vec{s}, s_k) = \theta(\vec{r}, r_k)$ and for any ϵ such that $0 < \epsilon < r_i - r_k$, we have $\theta(\vec{s}, s_k + \epsilon) = \theta(\vec{r}, r_k + \epsilon)$. We can conclude that $\varphi(\vec{r}) = \varphi(\vec{s})$.

4. Case $\varphi(\vec{r}) = r_i$:

Let X be defined by $X_j = \mu^+$ iff $j = i$ or $r_j > r_i$. We have $\theta(\vec{r}, r_i + \epsilon) < X \leq \theta(\vec{r}, r_i)$. Let $x = \min\{r_j : r_j > r_i\}$.

- (a) Subcase $\alpha_X < r_i$:

If there were no $j \neq i$ such that $r_j = r_i$ then $i = k$ and we would have

$X = \theta(\vec{r}, r_i)$. As such, $\alpha_{\theta(\vec{r}, r_i)} < r_i = \varphi(\vec{r})$. This is absurd. Therefore such a j exists. As such, without loss of generality, we can consider that $i \neq k$. It follows that $\theta(\vec{s}, s_k) = \theta(\vec{r}, r_i)$ and $\theta(\vec{s}, s_k + \epsilon) = \theta(\vec{r}, r_k + \epsilon)$. To conclude $\varphi(\vec{s}) = s_k = r_k = \varphi(\vec{r})$.

(b) Subcase $r_i \leq \alpha_X$ and $s_i \leq x$:

We have $X = \theta(\vec{s}, s_i)$ and $\theta(\vec{s}, s_i + \epsilon) \leq \theta(\vec{r}, r_i + \epsilon)$. As such $\varphi(\vec{s}) = \min(\alpha_X, s_i) \geq r_i = \varphi(\vec{r})$.

(c) Subcase $r_i \leq \alpha_X$ and $s_i > x$:

Let t differ from r only in dimension r_i with $t_i = x$. We can show by induction that $\varphi(r) \leq \varphi(t)$ by using (b). We can show by induction that $\varphi(t) \leq \varphi(s)$ by using (3) or (4). Transitivity of inequalities therefore provides $\varphi(r) \leq \varphi(s)$.

5. Case $r_i < \varphi(\vec{r}) < s_i$:

Let t differ from r only in dimension r_i with $t_i = \varphi(\vec{r})$. We can show by induction that $\varphi(r) \leq \varphi(t)$ by using (2). We can show by induction that $\varphi(t) \leq \varphi(s)$ by using (4). Transitivity of inequalities therefore provides $\varphi(r) \leq \varphi(s)$.

There are a finite number of values r_k and α_X between r_i and s_i . As such by inductive steps we have shown that φ is weakly responsive.

• Strategy-proofness: Let \vec{s} differ from \vec{r} only in dimension i . Suppose that $r_i < \varphi(\vec{r})$:

- If $r_i < s_i$, then by weak responsiveness $\varphi(\vec{r}) \leq \varphi(\vec{s})$.
- If $\varphi(\vec{r}) = \alpha_{\vec{\mu}^-}$ and $s_i < r_i$ then $\varphi(\vec{r}) = \varphi(\vec{s}) = \alpha_{\vec{\mu}^-}$.
- If $s_i < r_i < \varphi(\vec{r}) = \alpha_{\theta(\vec{r}, r_k)}$ then $s_k = \min\{s_j : s_j \geq \alpha_{\theta(\vec{r}, s_k)}\}$ therefore $\varphi(\vec{r}) = \varphi(\vec{s})$.
- If $s_i < r_i < \varphi(\vec{r}) = r_k$ then $\theta(\vec{r}, r_k) = \theta(\vec{s}, s_k)$ and $\theta(\vec{r}, r_k + \epsilon) = \theta(\vec{s}, s_k + \epsilon)$. Therefore $\varphi(\vec{r}) = \varphi(\vec{s})$.

The proof for $r_i > \varphi(\vec{r})$ is symmetrical in every case except:

- If $\alpha_{\theta(\vec{r}, r_k)} < r_i < s_i$ then $r_k = \min\{r_j : r_j \geq \alpha_{\theta(\vec{r}, r_k)}\}$. Let l be such that $s_l = \min\{s_j : s_j \geq \alpha_{\theta(\vec{r}, r_k)}\}$. We have $s_l = \min\{s_j : s_j \geq \alpha_{\theta(\vec{r}, s_l)}\}$. It follows that $\varphi(\vec{r}) = \varphi(\vec{s})$.

□

Theorem 18 (Phantom function characterization) *The voting function φ is strategy-proof iff there exists a phantom function $\alpha : \Gamma \rightarrow \Lambda$ such that Eq. 6 is verified.*

Proof The proof is immediate by using Lemmas 6, 7 and 8. □

Appendix C: Missing proofs for the new characterizations

C.1 Proof of Theorem 6

A voting rule φ is strategy-proof iff there exists a phantom function α (the same as in Theorem 2) such that:

$$\forall \vec{r}; \varphi(\vec{r}) := \text{med}(r_1, \dots, r_n, \alpha_{X_0(\vec{r})}, \alpha_{X_1(\vec{r})}, \dots, \alpha_{X_n(\vec{r})}). \quad (7)$$

Proof Let φ be a strategy-proof voting rule defined by a phantom function α as in Theorem 2. Let $\mu : \Lambda^N \rightarrow \Lambda$ be defined as:

$$\forall \vec{r}, \mu(\vec{r}) := \text{med}(r_1, \dots, r_n, \alpha_{X_0(\vec{r})}, \alpha_{X_1(\vec{r})}, \dots, \alpha_{X_n(\vec{r})}).$$

Since φ is the unique strategy-proof voting rule defined by α we only need to prove that $\varphi = \mu$. We will use the shorthand $X_k := X_k(\vec{r})$ since the context is clear (e.g. \vec{r} will be fixed). Considering the two cases in the formula of Theorem 2:

- Case $\varphi(\vec{r}) = \alpha_X$.

Let us first show that there is k such that $\alpha_X = r_k$ or $\alpha_X = \alpha_{X_k}$.

Suppose that $\forall j, \alpha_X \neq r_j$ then $\mu^+(X) = \{j : r_j > \alpha_X\}$. Therefore there is only one choice for X_k where $k = \#\mu^+(X)$, that is $X_k = X$. Therefore $\alpha_{X_k} = \alpha_X$. Given our assumption we have therefore shown that there is k such that $\alpha_X = r_k$ or $\alpha_X = \alpha_{X_k}$. In other words α_X is one of the arguments given to the median function. It now remains to show that α_X is selected by the median (e.g. is the output of μ).

If $\mu^+(X) = \{j : r_j \geq \alpha_X\}$ then for all $j \in \mu^+(X)$ we have $r_j \geq \alpha_X$. Also, for any l such that $l \geq k$, we have that $\alpha_{X_l} \geq \alpha_X$. Therefore we have $n + 1$ values in the median formula of μ that are greater of equal to α_X . A symmetrical proof gives us $n + 1$ values that are less or equal to α_X .

If $\mu^+(X) \subset \{j : r_j \geq \alpha_X\}$ then there is $r_k = \alpha_X$. For all $j \in \mu^+(X)$ we have $r_k \leq r_j$. For any l such that $l \geq \#\{j : r_j \geq \alpha_X\}$, we have that $\alpha_{X_l} \geq r_k$. Therefore we have $n + 1$ values in the median formula of μ that are greater of equal to $\alpha_X = r_k$. A symmetrical proof gives us $n + 1$ values that are less or equal to α_X . Consequently $\mu(\vec{r}) = \alpha_X$.

- Case $\varphi(\vec{r}) = r_i$.

Let X and Y be such that $\alpha_X \leq r_i \leq \alpha_Y$ and $\mu^+(X) = \{j : r_i < r_j\} \wedge \mu^-(Y) = \{j : r_i > r_j\}$. For any $l \geq \#\mu^+(X)$ we have $\alpha_{X_l} \geq r_i$. Therefore we have (r_i included) $n + 1$ elements that are greater or equal to r_i . A symmetrical proof gives us that there are $n + 1$ that are lesser or equal to r_i . Consequently, r_i is the median of our set of values, that is: $\mu(\vec{r}) = r_i$.

Therefore $\varphi = \mu$. □

Appendix D: Missing proofs for additional properties

D.1 Proof of Proposition 4

For a strategy-proof voting rule $\varphi : \Lambda^N \rightarrow \Lambda$ the following are equivalent:

(1) The phantom function α verifies $\alpha(\Gamma) = \{\mu^-, \mu^+\}$

(2) φ is strictly responsive.

And when moreover Λ is a rich,⁹ (1) and (2) are equivalent to:

(3) φ is ordinal and not constant.

⁹ Λ is rich if for any $\alpha < \beta$ in Λ there exists a $\gamma \in \Lambda$ such that $\alpha < \gamma < \beta$.

Proof (2) \rightarrow (1) : Suppose that φ is strategy-proof and strictly responsive. If there exists $X \in \Gamma$ such that $\alpha_X \notin \{\mu^-, \mu^+\}$, then set \vec{r} and \vec{s} such that if $j \in \mu^+(X)$ then $r_j = \alpha_X, s_j = \mu^+$ and if $j \in \mu^-(X)$ then $r_j = \mu^-, s_j = \alpha_X$. We have $\varphi(\vec{s}) = \varphi(\vec{r})$ which contradicts strict responsiveness. Therefore $\alpha(\Gamma) \subseteq \{\mu^-, \mu^+\}$. If $\#\alpha(\Gamma) = 1$ then the function is a constant and is therefore not strictly responsive. Consequently, $\alpha(\Gamma) = \{\mu^-, \mu^+\}$.

(1) \rightarrow (2) : Suppose that φ is strategy-proof and $\alpha(\Gamma) = \{\mu^-, \mu^+\}$. For any $\varphi(\vec{r})$ and $\varphi(\vec{s})$ such that for each k , $r_k < s_k$. Since all the phantoms are extreme we have: i and j such that $r_i = \varphi(\vec{r})$ and $s_j = \varphi(\vec{s})$. Let \vec{t} be defined as for all k if $r_k < r_i$ then $t_k = r_k$, otherwise if $s_k = r_i$ then $t_k = \frac{r_k + s_k}{2}$, otherwise $t_k = s_k$. We have $\varphi(\vec{t}) \in \{t_k\}$ therefore by weak responsiveness, since $r_i \notin \{t_k\}$, we have $r_i < \varphi(\vec{t}) \leq s_j$.

(3) \rightarrow (1) : Suppose that φ is strategy-proof and ordinal. If there exists $X \in \Gamma$ such that $\alpha_X \notin \{\mu^-, \mu^+\}$, let \vec{r} be such that there are two alternatives $a < b < \alpha_X$ such that if $i \in \mu^+(X)$ then $r_i = b$ otherwise $r_i = a$. Let π be bijective and $\pi(a) < \alpha_X < \pi(b)$. Then:

$$\begin{aligned}\alpha_X &= \varphi(\pi(r_1), \dots, \pi(r_n)) \\ &= \pi \circ \varphi(\vec{r}) \\ &= \pi(b) \\ &> \alpha_X\end{aligned}$$

We have reached a contradiction, therefore $\alpha(\Gamma) = \{\mu^-, \mu^+\}$.

(1) \rightarrow (3) : Here we use the median representation in Theorem 6. Suppose that φ is strategy-proof and $\alpha(\Gamma) = \{\mu^-, \mu^+\}$. For any strictly responsive and bijective π and for any voting profile \vec{r} :

$$\begin{aligned}\varphi(\pi(r_1), \dots, \pi(r_n)) &= med(\pi(r_1), \dots, \pi(r_n), \alpha_{X_0(\pi(\vec{r}))}, \alpha_{X_1(\pi(\vec{r}))}, \dots, \alpha_{X_n(\pi(\vec{r}))}) \\ &= med(\pi(r_1), \dots, \pi(r_n), \alpha_{X_0(\vec{r})}, \alpha_{X_1(\vec{r})}, \dots, \alpha_{X_n(\vec{r})}) \\ &= med(\pi(r_1), \dots, \pi(r_n), \pi(\alpha_{X_0(\vec{r})}), \pi(\alpha_{X_1(\vec{r})}), \dots, \pi(\alpha_{X_n(\vec{r})}))) \\ &= \pi \circ med(r_1, \dots, r_n, \alpha_{X_0(\vec{r})}, \alpha_{X_1(\vec{r})}, \dots, \alpha_{X_n(\vec{r})}) \\ &= \pi \circ \varphi(\vec{r})\end{aligned}$$

Therefore φ is ordinal. □

D.2 Proof of Theorem 12

A strategy-proof voting rule $\varphi^* : \Lambda^* \rightarrow \Lambda$ verifies participation iff with the order $\mu^- < \emptyset < \mu^+$, its associated phantom function α is weakly increasing.

Proof \Rightarrow : Suppose $\mu^- < \emptyset < \mu^+$. We will prove by *reductio ad absurdum*. Assume that α is not weakly increasing. Therefore there is an elector i , such that for X and Y that only differ in i we have $X_i < Y_i$ and $\alpha(X) > \alpha(Y)$. By definition of a phantom function either $X_i = \emptyset$ or $Y_i = \emptyset$. Suppose $X_i = \emptyset$ (therefore $Y_i = \mu^+$), then for

$\vec{r} = Y$ we have that by removing his vote, voter i contradicts the participation property (e.g. participation). A similar proof works for $Y_i = \emptyset$.

\Leftarrow : Suppose that α is weakly increasing. Let $\varphi^*(\vec{r}) = a$ where elector i did not cast a ballot. Let \vec{s} be the voting profile that is identical to \vec{r} except that s_i is a ballot. We will use now the curve characterisation (Theorem 7): $\forall \vec{r}; \varphi^*(\vec{r}) := \sup \{y \in \Lambda : \alpha_{\theta(\vec{r}, y)} \geq y\}$. If $s_i \leq a$ then $\alpha_{\theta(\vec{r}, s_i)} \geq \alpha_{\theta(\vec{r}, a)} \geq s_i$ therefore $\varphi(\vec{s}) \geq s_i$. We also have that $\alpha_{\theta(\vec{r}, a)} \geq \alpha_{\theta(\vec{s}, a)}$ therefore $\varphi^*(\vec{r}) \geq \varphi^*(\vec{s})$. Otherwise if $s_i \geq a$ then $\alpha_{\theta(\vec{r}, s_i)} \leq \alpha_{\theta(\vec{r}, a)} \leq s_i$ therefore $\varphi(\vec{s}) \geq s_i$. We also have that $\alpha_{\theta(\vec{r}, a)} \leq \alpha_{\theta(\vec{s}, a)}$ therefore $\varphi^*(\vec{r}) \leq \varphi^*(\vec{s})$. It follows that not submitting ones true value s_i cannot be beneficial. \square

D.3 Proof of Proposition 5

In a vote by issue, a strategy-proof voting function verifies participation iff when a elector i decides to become a voter with ballot x then for any property H containing x , if W_H was a winning coalition of H for the initial set of voters then $W_H \cup \{i\}$ is a winning coalition for the new set of voters.

Proof \Rightarrow : Suppose that α is weakly increasing for the order $\mu^- < \emptyset < \mu^+$. Let V be a fixed set of voters such that $i \notin V$. For $H = \{y \geq a\}$, $W \subseteq V$ is a winning coalition iff $X \in \Gamma^*$ such $\mu^+(X) = W$ verifies $\alpha_X \geq a$. Let Y that differs from X only in dimension i with $Y_i = \mu^+$, then $\alpha_Y \geq a$. Therefore $\mu^+(Y)$ is a winning coalition. A similar proof works for $H = \{y \leq a\}$. \Leftarrow : Suppose that when a electorate i decides to become a voter with ballot x then for any property H containing x , if W_H was a winning coalition of H for the initial set of voters then $W_H \cup \{i\}$ is a winning coalition for the new set of voters. Let us take $x = \alpha_X$ where i is not a voter for X . Then $\mu^+(X) \cup \{i\}$ (resp. $\mu^-(x)$) is a winning coalition for $\{y \geq x\}$ (resp. $\{y \leq x\}$) therefore $\alpha_Y \geq \alpha_X$ (resp. $\alpha_Y \leq \alpha_X$) where Y differs from X only in dimension i and $Y_i = \mu^+$ (resp. $Y_i = \mu^-$). \square

D.4 Proof of Theorem 14

A SP voting function $\varphi^* = (\varphi^n) : \Lambda^* \rightarrow \Lambda$ is anonymous and consistent iff there is a weakly increasing function $g : [0, 1] \rightarrow \Lambda$ (electorate size independent) and a constant $x \in \Lambda$ such that the phantom function $\alpha : \Gamma^*$ is defined as:

$$\alpha_X := \begin{cases} g\left(\frac{\#\mu^+(X)}{\#N}\right) & \text{if } \#N \neq 0 \\ x & \text{if } \#N = 0 \end{cases}$$

Furthermore the voting function verifies participation iff $x \in g([0, 1])$.

Proof \Rightarrow : Let us first show the existence of x and $g : [0, 1] \rightarrow \Lambda$ such that for all $X \in \Gamma^*$ the equation holds. $x = \alpha_{\emptyset}$ therefore x exists. For any other X if $q = \frac{\#\mu^+(X)}{\#\mu^-(X) + \#\mu^+(X)}$ we define $g(q) = \alpha_X$. Observe that this is well defined as by consistency and anonymity, we can duplicate and merge the electorate and so we

must have $\alpha_X = \alpha_Y = g(q)$ whenever $\frac{\#\mu^+(Y)}{\#\mu^-(Y) + \#\mu^+(Y)} = q$. Now let us show that g is weakly increasing. For any X and Y such that $\frac{\#\mu^+(X)}{\#\mu^-(X) + \#\mu^+(X)} \leq \frac{\#\mu^+(Y)}{\#\mu^-(Y) + \#\mu^+(Y)}$ by consistency we can duplicate X and Y into X' and Y' that have the same number of voters. As such $\alpha_X \leq \alpha_Y$. It follows that g is weakly increasing over the set of rationals. Since the values over the irrationals do not matter we can complete the definition with a weakly increasing g without loss. \Leftarrow : For any $X \in \Gamma^*$ we have $\alpha_X = \alpha_{X \sqcup \emptyset}$. By using the barycentric weights we have that for any X and Y we have:

$$\frac{\#\mu^+(X)}{\#\mu^-(X) + \#\mu^+(X)} \leq \frac{\#\mu^+(X \sqcup Y)}{\#\mu^-(X \sqcup Y) + \#\mu^+(X \sqcup Y)} \leq \frac{\#\mu^+(Y)}{\#\mu^-(Y) + \#\mu^+(Y)}$$

Therefore since g is weakly increasing. $\alpha_X \leq \alpha_{X \sqcup Y} \leq \alpha_Y$. It follows that we verify consistency. Finally we wish to show that if $x \in g([0, 1])$ then our voting function verifies participation. For any $n > 0$ and $0 \leq k < n$ we have:

$$\frac{k}{n+1} < \frac{k}{n} < \frac{k+1}{n+1}$$

Therefore since g is weakly increasing α is weakly increasing except maybe in α_{\emptyset} . Therefore the function verifies participation iff $x \in g([0, 1])$. \square

D.5 Proof of Theorem 15

A strategy-proof, homogeneous (= consistent and anonymous) voting function $\varphi^* : \Lambda^* \rightarrow \Lambda$ is continuous with respect to new members iff its grading curve g is continuous.

Proof \Rightarrow : We have $\forall \vec{r}, \vec{s}, \lim_{n \rightarrow +\infty} \varphi(\overbrace{\vec{r} \sqcup \cdots \sqcup \vec{r}}^n \sqcup \vec{s}) = \varphi(\vec{r})$. Therefore for any $X, Y \in \Gamma^*$ with at least one voter each:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} g\left(\frac{n\#\mu^+(X) + \#\mu^+(Y)}{n(\#\mu^-(X) + \#\mu^+(X)) + \#\mu^-(Y) + \#\mu^+(Y)}\right) \\ &= g\left(\frac{\#\mu^+(X)}{\#\mu^-(X) + \#\mu^+(X)}\right) \end{aligned}$$

Therefore, since g is weakly increasing, g is continuous in all rational numbers. Therefore by monotonicity and density of the rationals within the real numbers we have that g is continuous.

\Leftarrow : Let g be continuous. Then for all X, Y :

$$\begin{aligned} & \lim_{n \rightarrow +\infty} g\left(\frac{n\#\mu^+(X) + \#\mu^+(Y)}{n(\#\mu^-(X) + \#\mu^+(X)) + \#\mu^-(Y) + \#\mu^+(Y)}\right) \\ &= g\left(\frac{\#\mu^+(X)}{\#\mu^-(X) + \#\mu^+(X)}\right) \end{aligned}$$

Therefore by continuity of φ :

$$\forall \vec{r}, \vec{s}, \lim_{n \rightarrow +\infty} \varphi(\overbrace{\vec{r} \sqcup \cdots \sqcup \vec{r}}^n \sqcup \vec{s}) = \varphi(\vec{r}).$$

□

Appendix E: Proof of Theorem 17

Fix natural n . The linear median, corresponding to $\alpha(X) = m + \frac{\sum_i X_i}{n}(M - m)$ is the unique SP voting function that minimizes the L_2 -norm between inputs and output, integrated with the L_2 -norm over the uniform distribution on $[m, M]^n$.

Proof We desire to find the SP voting function f that minimizes

$$E(f) = \int_m^M \cdots \int_m^M \sum_{i=1}^n (x_i - f(\vec{x}))^2 dx_n \dots dx_1$$

Since f is a SP voting function, it is fully characterized by its α_X values as in lemma 1. Thus we can minimize $E(f)$ by optimizing each α_X value independently. Fix X .

$$\frac{\partial E}{\partial \alpha_X} = \sum_{i=1}^n \int_m^M \cdots \int_m^M 2(x_i - f(\vec{x})) \frac{\partial}{\partial \alpha_X} (x_i - f(\vec{x})) dx_n \dots dx_1$$

Let k_1, k_2, \dots, k_a be the indices of X with $X_{k_i} = 1$ and $\hat{k}_1, \hat{k}_2, \dots, \hat{k}_b$ be the indices of X with $X_{\hat{k}_i} = 0$. $\frac{\partial}{\partial \alpha_X} f$ will be 1 where $f(\vec{x}) = \alpha_X$ (and 0 elsewhere) which is exactly in the region where $x_{\hat{k}_1}, x_{\hat{k}_2}, \dots, x_{\hat{k}_b} < \alpha_X < x_{k_1}, x_{k_2}, \dots, x_{k_a}$. Thus,

$$\begin{aligned} \frac{\partial E}{\partial \alpha_X} &= - \left(\sum_{i=1}^b \underbrace{\int_{\alpha_X}^M \cdots \int_{\alpha_X}^M}_{a} \underbrace{\int_m^{\alpha_X} \cdots \int_m^{\alpha_X}}_b 2(x_{\hat{k}_i} - \alpha_X) dx_{\hat{k}_1} \dots dx_{\hat{k}_b} dx_{k_1} \dots dx_{k_a} \right. \\ &\quad \left. + \sum_{i=1}^a \underbrace{\int_{\alpha_X}^M \cdots \int_{\alpha_X}^M}_{a} \underbrace{\int_m^{\alpha_X} \cdots \int_m^{\alpha_X}}_b 2(x_{k_i} - \alpha_X) dx_{\hat{k}_1} \dots dx_{\hat{k}_b} dx_{k_1} \dots dx_{k_a} \right) \\ &= \left(\sum_{i=1}^b (M - \alpha_X)^a (\alpha_X - m)^{b-1} \int_m^{\alpha_X} 2(\alpha_X - x_{\hat{k}_i}) dx_{\hat{k}_i} \right. \\ &\quad \left. - \sum_{i=1}^a (M - \alpha_X)^{a-1} (\alpha_X - m)^b \int_{\alpha_X}^M 2(x_{k_i} - \alpha_X) dx_{k_i} \right) \end{aligned}$$

$$= (M - \alpha_X)^a (\alpha_X - m)^b (a + b) \left(\alpha_X - \left(m + \frac{a}{a + b} (M - m) \right) \right)$$

Since $\frac{a}{a+b} = \frac{\sum_i X_i}{n}$, a minimum for $E(f)$ is

$$\alpha_X = m + \frac{\sum_i X_i}{n} (M - m).$$

Since this is true for all X , this gives the voting function that minimizes the L_2 -norm for votes coming from the uniform distribution on $[m, M]^n$. \square

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