Multitarget–Multidetection Tracking Using the Kernel SME Filter

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***Abstract*—With the growing availability of high-resolution sensors, processing more than one detection per target becomes increasingly critical when tracking multiple extended objects. However, contemporary sensors often generate spurious detections that need to be considered. Naïvely employing standard multitarget trackers may result in poor tracking performance for multitarget–multidetection tracking in cluttered environments, and the relevant extensions are nontrivial. This paper introduces a version of the kernel symmetric measurement equation (SME) filter that considers both multidetections and clutter. For a simulated scenario, our novel filter achieved a higher accuracy than the global nearest neighbor (GNN) and a fast variant of the joint probabilistic data association filter (JPDAF).**

I. Introduction

Multitarget tracking is a well-researched problem with a long history [1]. While each individual target can often be tracked well with standard trackers, such as the Kalman filter or its extensions, a lack of knowledge as to which target gives rise to which measurement leads to great challenges. Multitarget tracking is further complicated by clutter, i.e., measurements that do not originate from an actual target. With increasing sensor resolution, one may also obtain multidetections, i.e., multiple measurements of a single target, which stands in contrast to the common assumption of point-shaped objects. However, we do not explicitly model the shape of the object, as is done in extended object tracking [2].

Straight-forward approaches perform a one-to-one association decision between targets and measurements. An example is the GNN, which maximizes the likelihood of association when based on the squared Mahalanobis distances [3, Sec. 10.3.1.3]. Due to the Mahalanobis distances, clutter is only incorrectly considered to be true measurements when they are more likely (based on the prior) than the actual measurement. Gating strategies are employed to discard measurements likely not originating from targets early on, which significantly reduces the computational complexity [4] Missed detections are addressed by assigning each track a score that describes the certainty of the existence of this track [5, Sec 6.1]. For multidetections, only one measurement is utilized while the remaining ones may be assigned to other targets, potentially resulting in incorrect associations when targets are closely spaced.

The Probabilistic Data Association Filter (PDAF) is a popular approach for tracking single targets in cluttered environments [6, Ch. 4]. It operates under the assumption that only one (or none in the case of all being clutter) measurement originates from the actual object. The Joint Probabilistic Data Association Filter (JPDAF) [6, Ch.6] extends this concept to multiple targets, accounting for the multitude of possibilities concerning which measurement arises from which track. However, this introduces a significant combinatorial complexity that is commonly addressed using approximations, for instance in [7]. A noteworthy limitation of the JPDAF is that when it integrates all compatible measurements nearby, closely spaced targets can result in overlapping tracks, a phenomenon known as track coalescence [8].

In the past decades, techniques that do not explicitly associate tracks with targets have been proposed, including the probability hypothesis density (PHD) filter [9] or the SME filter [10], [11]. The underlying principle of SMEs is the generation of a measurement equation that is permutation invariant with respect to the measurements. An extension for multidetections based on the original SME approach that relies on partitioning has been proposed in [12].

As we detail later in this paper, the Kernel SME (KSME) filter [13] extends the original SME approach by the consideration of a kernel transformation. The transformation results in a permutation-invariant representation of the measurements as a Gaussian mixture, which is subsequently evaluated at distinct test points.

In this paper, we propose an extension to the KSME filter designed specifically for multitarget–multidetection tracking. This extension facilitates the state estimation for multiple targets, with each target gives potentially rise to multiple measurements. For the novel approach, we present the derivation of the analytic moments of the pseudo-measurements. An extension to the KSME filter that considers clutter and missed detections was previously introduced in [14]. Building upon that foundation, we further expand the KSME filter in this paper to inherently address clutter and multidetections in all its filter equations. We use a Poisson distribution to model the number of detections, thereby implicitly taking into account instances of missed (i.e., zero) detections, without the need for an explicit detection probability [15]. The key result of this paper is the demonstration that the KSME for multidetections can operate without the need for an explicit enumeration of association hypotheses or measurement partitions, given the number of measurements from an object follows a Poisson distribution.

Finally, we present an evaluation of the proposed method through a simulation study. In this evaluation, we focus on scenarios involving closely spaced targets with crossing trajectories in order to analyze potential track coalescence or track repulsion.

The paper is structured as follows. We begin by formally introducing the problem formulation in the next section. Following this, in Sec. III, we elucidate the formulae for the Kernel SME (KSME). Subsequently, Sec. IV presents a simulation-based evaluation of our method, complemented by a thorough discussion of the results. The final section concludes the work.

II. Problem Formulation

We consider a set of N targets $X_k := \{\underline{\boldsymbol{x}}_k^l\}_{i=1}^N$, where each $\underline{\boldsymbol{x}}_k^l \in \mathbb{R}^n$. The state of each target evolves according to the system equation

$$\underline{\boldsymbol{x}}_{k+1}^l = \mathbf{A}_k^l \underline{\boldsymbol{x}}_k^l + \underline{\boldsymbol{w}}_k^l \ ,$$

with i.i.d. Gaussian noise $\underline{\boldsymbol{w}}_k^l \sim \mathcal{N}(\underline{0}, \mathbf{\Sigma}_k^{w_l})$. At every time step, each target can potentially give rise to multiple measurements described by

$$Y_k^l := \{\underline{\boldsymbol{y}}_k^{l,1}, \dots, \underline{\boldsymbol{y}}_k^{l,M_k^l}\} \ ,$$

where the numbers of measurement are Poisson distributed, i.e., $\boldsymbol{M}_k^l \sim \text{Pois}(\lambda^l)$. Missed detections are considered by the case $\boldsymbol{M}_k^l = 0$ with $Y_k^l = \emptyset$. For each target, the probability of a missed detection is $P(\boldsymbol{M}_k^l = 0) = \frac{(\lambda^l)^0}{0!} e^{-\lambda^l} = e^{-\lambda^l}$. Each measurement $\underline{\boldsymbol{y}}_k^{l,i}$ is defined by the measurement equation

$$\underline{\boldsymbol{y}}_k^{l,i} = \mathbf{H}_k^l \underline{\boldsymbol{x}}_k^l + \underline{\boldsymbol{v}}_k^{l,i} \ ,$$

where the measurement noise is i.i.d. Gaussian with $\underline{\boldsymbol{v}}_k^{l,i} \sim \mathcal{N}(\underline{0}, \mathbf{\Sigma}_k^{v_l})$. In addition to the measurements generated by the targets, we consider clutter in each time step, which is described by

$$C_k = \{\underline{\boldsymbol{c}}_k^1, \dots, \underline{\boldsymbol{c}}_k^{M_k^c}\} \ .$$

The number of false measurements is Poisson distributed with $\boldsymbol{M}_k^c \sim \text{Pois}(\lambda^c)$ and all false measurements are independent and uniformly distributed over the surveillance area.

The set of available measurements $Y_k = \{\underline{\boldsymbol{y}}_k^1, \dots, \underline{\boldsymbol{y}}_k^{M_k}\}$ for each time step is described by

$$Y_k = \bigcup_{l=1}^N Y_k^l \cup C_k \ ,$$

where the number of available measurements is $\boldsymbol{M}_k = \sum_{l=1}^N \boldsymbol{M}_k^l + \boldsymbol{M}_k^c$. The available measurements do not obey a specific order and are indistinguishable (except for their position). The objective of multitarget–multidetection tracking is to estimate the targets' true states based on the available measurements.

III. Kernel SME Filter for Multitarget–Multidetection Tracking

For the Kernel SME filter, we consider a kernel transformation that modifies the measurement equation into a permutation-invariant symmetric measurement equation. The original kernel transformation [13]

$$S_{Y_k}(\underline{z}) := \sum_{l=1}^N \mathcal{N}(\underline{z}; \underline{y}_k^l, \Gamma) \ ,$$

with kernel width $\Gamma \in \mathbb{R}_+$, was introduced under the assumption of a fixed number of targets, where each target gives rise to exactly one measurement, without the consideration of clutter or missed detections. Note that the kernel width may be chosen without direct consideration of the involved uncertainties, though our research has shown that taking the uncertainties into account improves the outcomes [16]. The resulting values of $S_{Y_k}(\underline{z})$ are referred to as pseudo-measurements. Thus, the kernel transformation encapsulates the information about the measurements as a Gaussian mixture, where each measurement contributes to the pseudo-measurement. For the consideration of clutter and missed detections, an extension has been proposed [14] which uses a modified kernel transformation

$$S_{Y_k}(\underline{z}) := \sum_{l=1}^N \boldsymbol{d}_k^l \cdot \mathcal{N}(\underline{z}; \underline{y}_k^l, \Gamma) + S_{C_k}(\underline{z}) \ ,$$

where $\boldsymbol{d}_k^l$ is a binary random variable with $\boldsymbol{d}_k^l \sim \text{Bin}(p_d)$, which indicates the detection of the measurement $\underline{y}_k^l$ and

$$S_{C_k}(\underline{z}) := \sum_{\underline{c} \in C_k} \mathcal{N}(\underline{z}; \underline{c}, \Gamma) \ ,$$

represents the contribution of the clutter to the pseudo-measurement. In our approach, we extend the kernel transformation further by considering multiple measurements for each target. The resulting kernel transformation is described by

$$S_{Y_k}(\underline{z}) := \sum_{\underline{y} \in Y_k} \mathcal{N}(\underline{z}; \underline{y}, \Gamma) = \sum_{l=1}^N S_{Y_k^l}(\underline{z}) + S_{C_k}(\underline{z}) \ , \quad (1)$$

where

$$S_{Y_k^l}(\underline{z}) := \sum_{\underline{y} \in Y_k^l} \mathcal{N}(\underline{z}; \underline{y}, \Gamma)$$

represents the proportion contributed by measurements of the target $\underline{\boldsymbol{x}}_k^l$ to the pseudo-measurements. This representation is a generalization of the previous approaches, where scenarios with missed detections are described implicitly by events where $\boldsymbol{M}_k^l = 0$ with probability $P(\boldsymbol{M}_k^l = 0) = \frac{(\lambda^l)^0}{0!} e^{-\lambda^l} = e^{-\lambda^l}$.

It is important to note that the representation of the pseudo-measurements by their components above make the concept of the kernel transformation more comprehensible, the generated pseudo-measurements are not treated differently for the individual measurements. However, this representation will later be useful to derive the analytic moments for the pseudo-measurements.

The kernel transformation represents the available measurements as a Gaussian mixture. Thus, this representation cannot be employed directly for the measurement update and needs to be discretized first. To this end, we express the kernel transformation at $M_k(2n+1)$ discrete locations $\underline{a}_k^1, \ldots, \underline{a}_k^{N_a}$ with

$$\underline{a}_k^{l+i-1} := \underline{\boldsymbol{y}}_k^l + \left(\sqrt{n\Gamma}\right) , \qquad \underline{a}_k^{l+2(i-1)} := \underline{\boldsymbol{y}}_k^l - \left(\sqrt{n\Gamma}\right) \ ,$$

for $i = 1, \ldots, n$ and $l = 1, \ldots, M_k$. The pseudo-measurements $\underline{\boldsymbol{s}}_k = [\underline{\boldsymbol{s}}_k^1, \ldots, \underline{\boldsymbol{s}}_k^{N_a}]$ are obtained by $\underline{\boldsymbol{s}}_k^i = S_{Y_k}(\underline{a}_k^i)$. We consider a Gaussian approximation of the posterior density based on the pseudo-measurements, i.e., $p(\underline{x}_k | \underline{\boldsymbol{s}}_k) \approx \mathcal{N}(\underline{x}_k; \underline{\mu}_k^x, \mathbf{\Sigma}_k^x)$. The measurement update is obtained by a form of the Kalman filter formulae

$$\begin{aligned} \underline{\mu}_{k|k}^x &= \underline{\mu}_{k|k-1}^x + \mathbf{\Sigma}_k^{xs} (\mathbf{\Sigma}_k^s)^{-1} \left(\underline{\boldsymbol{s}}_k - \underline{\mu}_k^s \right) \ , \\ \mathbf{\Sigma}_k^x &= \mathbf{\Sigma}_{k|k-1}^x - \mathbf{\Sigma}_k^{xs} (\mathbf{\Sigma}_k^s)^{-1} \mathbf{\Sigma}_k^{sx} \ . \end{aligned}$$

The measurement update necessitates the analytic moments of the pseudo-measurement, which we describe next.

The mean of the pseudo-measurements is given by

$$\underline{\mu}_k^{s,i} = \sum_{l=1}^{N} \lambda^l P_l^\Gamma(\underline{a}_k^i) + \lambda^c \mathcal{N}(\underline{a}_k^i; \underline{\mu}^c, \mathbf{\Sigma}_k^c + \Gamma) \ ,$$

where

$$P_l^\Gamma(\underline{z}) := \mathcal{N}(\underline{a}_k^i; \mathbf{H}_k^l \underline{\mu}_{k|k-1}^{x_l}, \mathbf{H}_k^l \mathbf{\Sigma}_{k|k-1}^{x_l} (\mathbf{H}_k^l)^\top + \mathbf{\Sigma}_k^{v_l} + \Gamma) \ ,$$

$\underline{\mu}^c$ and $\mathbf{\Sigma}_k^c$ are the mean and covariance of the clutter in the surveillance area. The covariance matrix of the pseudo-measurements is described by

$$\begin{aligned} \mathbf{\Sigma}_k^{s_i, s_j} &= \sum_{l=1}^{N} \lambda^l P_l^\Gamma(\underline{a}_k^i) \sum_{\substack{m=1 \\ m \neq l}}^{N} \lambda^m P_m^\Gamma(\underline{a}_k^j) \\ &\quad + \mathcal{N}(\underline{a}_k^i; \underline{a}_k^j, 2\Gamma) \sum_{l=1}^{N} (\lambda^l)^2 P_l^{\frac{1}{2}\Gamma} \left(\frac{1}{2} (\underline{a}_k^i + \underline{a}_k^j) \right) \\ &\quad + \sum_{l=1}^{N} \lambda^l P_l^\Gamma(\underline{a}_k^i) \cdot \lambda^c \cdot \mathcal{N}(\underline{a}_k^j; \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma) \\ &\quad + \sum_{m=1}^{N} \lambda^m P_m^\Gamma(\underline{a}_k^j) \cdot \lambda^c \cdot \mathcal{N}(\underline{a}_k^i; \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma) \\ &\quad + (\lambda^c)^2 \cdot \mathcal{N}(\underline{a}_k^i; \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma) \cdot \mathcal{N}(\underline{a}_k^j, \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma) \\ &\quad + \lambda^c \cdot \mathcal{N}(\underline{a}_k^i; \underline{a}_k^j, 2\Gamma) \cdot \mathcal{N}\left(\frac{1}{2} (\underline{a}_k^i + \underline{a}_k^j); \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma \right) \\ &\quad - \underline{\mu}_k^{s,i} \underline{\mu}_k^{s,j} \ . \end{aligned}$$

The cross-covariance matrix is given by

$$\begin{aligned} \mathbf{\Sigma}_k^{x, s_i} &= \sum_{l=1}^{N} \lambda^l \cdot P_l^\Gamma(\underline{a}_k^i) \left(\underline{\mu}_{k|k-1}^x + \mathbf{K}_k^l \left(\underline{a}_k^i - \mathbf{H}_k^l \underline{\mu}_{k|k-1}^x \right) \right) \\ &\quad + \underline{\mu}_{k|k-1}^x \cdot \lambda^c \cdot \mathcal{N}(\underline{a}_k^i, \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma) \\ &\quad - \underline{\mu}_{k|k-1}^x \cdot \underline{\mu}_k^{s,i} \ , \end{aligned}$$

where

$$\begin{aligned} \mathbf{K}_k^l =& [\mathbf{\Sigma}_{k|k-1}^{x_1, x_l}, \ldots, \mathbf{\Sigma}_{k|k-1}^{x_N, x_l}] \\ & \cdot \mathbf{H}_k^l \left(\mathbf{H}_k^l \mathbf{\Sigma}_{k|k-1}^x (\mathbf{H}_k^l)^\top + \mathbf{\Sigma}_k^v + \Gamma \right)^{-1} \ . \end{aligned}$$

A rigorous derivation of the analytic moments of the pseudo-measurement is presented in the appendix.

IV. Evaluation

For the simulation experiment, we consider $N = 3$ moving targets traveling along a trajectory, where the target motions are predefined for each time step. The initial true location of the targets are

$$\underline{\tilde{x}}_0^1 = \begin{bmatrix} -5.0 \\ 0.8 \end{bmatrix}, \ \underline{\tilde{x}}_0^2 = \begin{bmatrix} -5.0 \\ 0.0 \end{bmatrix}, \ \underline{\tilde{x}}_0^3 = \begin{bmatrix} -5.0 \\ -0.8 \end{bmatrix}.$$

The motion of the target state vectors are represented by the increments $\Delta \underline{x}_k^l$, which can be described by the model

$$\underline{\tilde{x}}_{k+1}^l = \underline{\tilde{x}}_k^l + \Delta \underline{x}_k^l \ .$$

The target motion increments $\Delta \underline{x}_k^l$ do not include information about the velocity. This model enables the simulation of target motions along specific trajectories. For our simulation, we consider a scenario with closely spaced targets moving along crossing trajectories. The multi-target probabilistic system model is described by

$$\underline{\boldsymbol{x}}_{k+1}^l = \underline{\boldsymbol{x}}_k^l + \Delta \underline{x}_k^l + \underline{\boldsymbol{w}}_k^l \ ,$$

where $\underline{\boldsymbol{w}}_k^l$ is zero-mean Gaussian noise with covariance $(\sigma_w^l)^2 \cdot \mathbf{I}_2$ (with $\mathbf{I}_2$ being the 2×2 identity matrix) and $\sigma_w^l = 0.03$ for $l = 1, 2, 3$. For the measurement model, we employ the identity model, meaning that the measurements directly map from the state space with some added noise. This can be represented as

$$\underline{\boldsymbol{y}}_k^{l,i} = \underline{\tilde{x}}_k^l + \underline{\boldsymbol{v}}_k^{l,i} \ ,$$

where $\underline{\boldsymbol{v}}_k^{l,i}$ are independent zero-mean Gaussian noise terms with covariance

$$\mathbf{\Sigma}_k^{v_l} = \sigma^v \cdot \begin{bmatrix} 1.0 & -0.1 \\ -0.1 & 1.0 \end{bmatrix} \ ,$$

and the noise level is set to $\sigma^v = 0.12$. The Poisson parameters for the number of measurements are set to $\lambda^l = 5$ for $l = 1, 2, 3$, and the false measurement rate is set to $\lambda^c = 5$. In each time step, the measurement set Y_k is permuted in each time step to ensure that their order does not reveal any information about the measurement origin.

In our evaluation, we compared our novel filter, which is part of the Multitarget Tracking Toolbox[1], with a GNN and the

[1] https://github.com/KIT-ISAS/Multitarget-Tracking-Toolbox

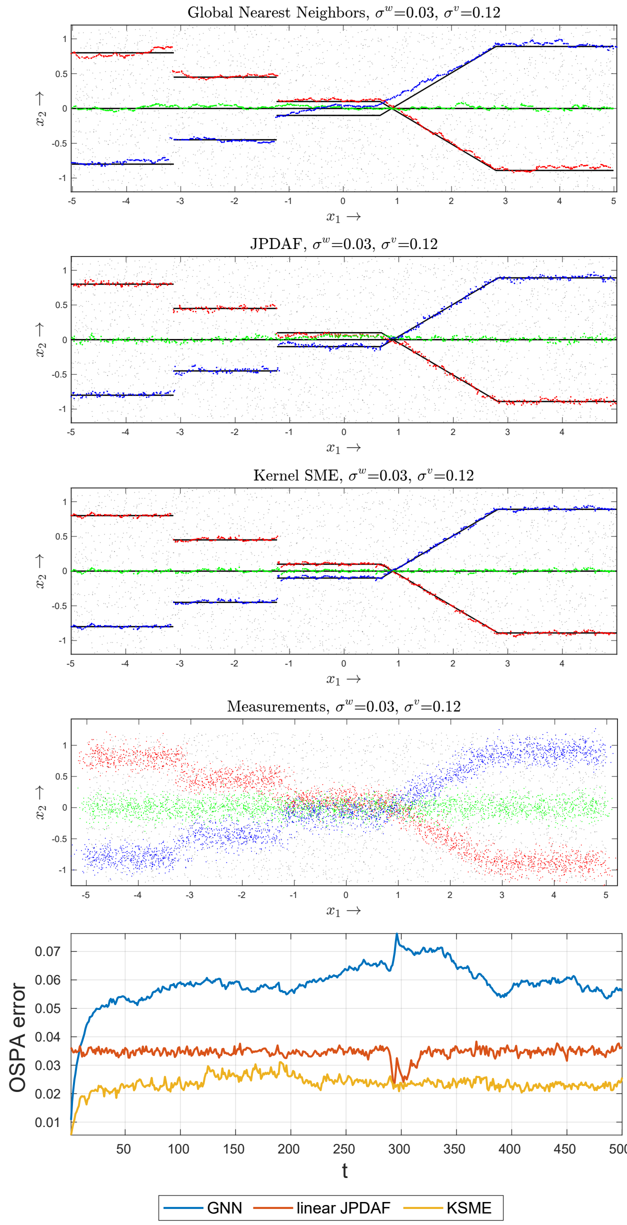

Fig. 1. Top three plots: Trajectories of three target tracks for three different methods. The fourth plot illustrates the available measurements, where each color indicates the origin of the measurements, and gray dots represent false alarms. Bottom plot: The OSPA error for each time step, averaged over 100 simulations.

linear-time JPDAF [7], where the gating probability was set to $p_G = 0.999$. The kernel width for our proposed method is set to $\Gamma = \sigma^v$. We evaluated all methods based on the Optimal Sub-Pattern Assignment (OSPA) metric [17], with order $p = 2$ and cut-off distance $c = 1$. Fig. 1 presents examples of the tracks for each method, along with the average OSPA error over 100 simulations. These plots illustrate the performance of the methods in different phases. The GNN-based method exhibits weaker performance, particularly during phases where the targets are close. Specifically, incorrect associations can lead to deteriorating tracking accuracy when the targets are in

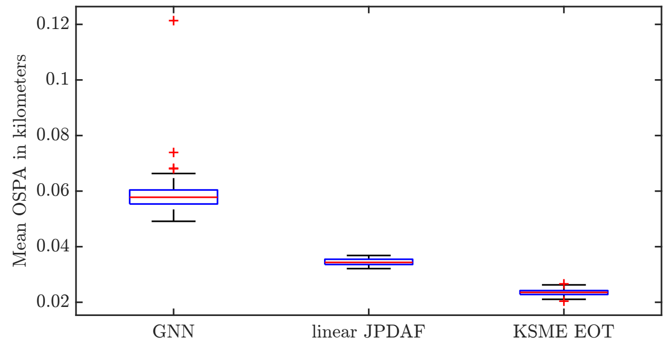

Fig. 2. Boxplot of the overall OSPA error of the considered methods for 100 simulations (a lower value indicates better performance).

close proximity. This is evident in Fig. 1, where the OSPA error for the GNN-based method spikes during phases of crossing target trajectories, indicating track repulsion. The linear-time JPDAF suffers from track coalescence during the phases where the targets are closely spaced. In contrast, the proposed method remains unaffected during these phases. A boxplot detailing the overall OSPA error across 100 simulations is shown in Fig. 2. Notably, our proposed method exhibits significantly better accuracy compared with the other methods.

V. Conclusion

In this paper, we have expanded upon the existing Kernel SME filter to accommodate scenarios involving multiple detections per target. The advantage of Kernel SME methods lies in their ability to estimate target states without the need for explicit enumeration of association hypotheses. In our evaluation, we demonstrated the superior performance of our proposed methods compared with baseline approaches. Unlike other existing strategies, our method avoids common pitfalls such as track coalescence. Therefore, our enhancement of the Kernel SME Filter presents a promising and robust alternative to current methodologies in multitarget–multidetection tracking.

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Appendix

A. Pseudo-Measurement Analytic Moments

In the following part, we derive the analytic moments of the pseudo-measurements $\underline{\boldsymbol{s}}_k^i = S_{Y_k}(\underline{a}_k^i)$, as presented in Sec. III. The key idea is to exploit the representation in equation (1) and determine the moments part by part.

1) Pseudo-Measurement Analytic Mean $\underline{\mu}_k^{s,i}$: The kernel transformation used in our approach is defined by

$$S_{Y_k}(\underline{z}) := \sum_{\underline{y} \in Y_k} \mathcal{N}(\underline{z}; \underline{y}, \Gamma) = \sum_{l=1}^{N} S_{Y_k^l}(\underline{z}) + S_{C_k}(\underline{z}) \ \ .$$

We now proceed to derive the analytic moments of these measurements at the test points $\underline{a}_k^i$ for both parts. Initially, we focus on the clutter term

$$S_{C_k}(\underline{z}) := \sum_{\underline{c} \in C_k} \mathcal{N}(\underline{z}; \underline{c}, \Gamma) \ \ ,$$

with analytic mean

$$\begin{aligned}
&\mathbb{E}\left[S_{C_k}(\underline{z})\right] = \mathbb{E}\left[\sum_{\underline{\boldsymbol{c}} \in C_k} \mathcal{N}(\underline{z}; \underline{\boldsymbol{c}}, \Gamma)\right] \\
&\overset{(*)}{=} \mathbb{E}\left[\boldsymbol{M}_k^c \cdot \mathcal{N}(\underline{z}; \underline{\boldsymbol{c}}, \Gamma)\right] \overset{(**)}{=} \mathbb{E}[\boldsymbol{M}_k^c] \cdot \mathbb{E}\left[\mathcal{N}(\underline{z}; \underline{\boldsymbol{c}}, \Gamma)\right] \\
&= \lambda^c \mathcal{N}\left(\underline{a}_k^i; \underline{\mu}^c, \boldsymbol{\Sigma}_k^c + \Gamma\right) \ \ ,
\end{aligned}$$

where in equation $(*)$ we replaced the sum with the number of clutter measurements $\boldsymbol{M}_k^c$ since the clutter locations are distributed identically and hence can be treated equivalently in the expectation. Furthermore, we leveraged the independence of the number of clutter measurements $\boldsymbol{M}_k^c$ and the clutter location $\underline{\boldsymbol{c}}$ in equation $(**)$. For the former part

$$\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i) := \sum_{l=1}^{N} \sum_{\underline{y} \in Y_k^l} \mathcal{N}(\underline{a}_k^i; \underline{y}, \Gamma) \ \ ,$$

the mean can be determined by

$$\begin{aligned}
&\mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{z})\right] = \mathbb{E}\left[\sum_{l=1}^{N} \sum_{\underline{\boldsymbol{y}}_k^l \in Y_k^l} \mathcal{N}(\underline{a}_k^i; \underline{\boldsymbol{y}}_k^l, \Gamma)\right] \\
&= \sum_{l=1}^{N} \mathbb{E}\left[\sum_{\underline{\boldsymbol{y}}_k^l \in Y_k^l} \mathcal{N}(\underline{a}_k^i; \underline{\boldsymbol{y}}_k^l, \Gamma)\right] \overset{(1)}{=} \sum_{l=1}^{N} \lambda^l \cdot \mathbb{E}\left[\mathcal{N}(\underline{a}_k^i; \underline{\boldsymbol{y}}_k^l, \Gamma)\right] \\
&= \sum_{l=1}^{N} \lambda^l \int \mathcal{N}(\underline{a}_k^i; \underbrace{\mathbf{H}_k^l \underline{\boldsymbol{x}}_k^l + \underline{\boldsymbol{v}}_k}_{\underline{\boldsymbol{y}}_k^l =}, \Gamma) \cdot \mathcal{N}(\underline{a}_k^i; \underline{\mu}_{k|k-1}^x, \boldsymbol{\Sigma}_{k|k-1}^x) \\
&\cdot \mathcal{N}(\underline{\boldsymbol{v}}_k^l; , \underline{\mu}^c, \boldsymbol{\Sigma}_k^v) d\underline{\boldsymbol{x}}_k^l \, d\underline{\boldsymbol{v}}_k \\
&= \sum_{l=1}^{N} \lambda^l P_l^{\Gamma}(\underline{a}_k^i) \ \ ,
\end{aligned}$$

where in (1), we use the same argumentation for the number of generated measurements and measurement locations as before for the clutter and, furthermore, that the expected number of measurements generated by each target is λ^l. Combining both parts results in the analytic mean

$$\underline{\mu}_k^{s,i} = \mathbb{E}\left[S_{Y_k}(\underline{a}_k^i)\right] = \sum_{l=1}^{N} \lambda^l P_l^{\Gamma}(\underline{a}_k^i) + \lambda^c \mathcal{N}(\underline{a}_k^i; \underline{\mu}^c, \boldsymbol{\Sigma}_k^c + \Gamma) \ \ .$$

2) Pseudo-Measurement Analytic Covariance $\boldsymbol{\Sigma}_k^{s_i,s_j}$: The covariance of the pseudo-measurements is defined by

$$\boldsymbol{\Sigma}^{s_i s_j} = \underbrace{\mathbb{E}\left[S_{Y_k}(\underline{a}_k^i) \cdot S_{Y_k}(\underline{a}_k^j)\right]}_{(*)} - \underline{\mu}_k^{s,i} \cdot \underline{\mu}_k^{s,j},$$

where we need to determine the value of the expectation $(*)$. Therefore, we use the following decomposition

$$\begin{aligned}
&\mathbb{E}\left[S_{Y_k}(\underline{a}_k^i) \cdot S_{Y_k}(\underline{a}_k^j)\right] \\
&= \mathbb{E}\left[\left(\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i) + S_{C_k}(\underline{a}_k^i)\right)\left(\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^j) + S_{C_k}(\underline{a}_k^j)\right)\right] \\
&= \underbrace{\mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i) \sum_{m=1}^{N} S_{Y_k^m}(\underline{a}_k^j)\right]}_{(1)} + \underbrace{\mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i) S_{C_k}(\underline{a}_k^j)\right]}_{(2)} \\
&+ \mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^j) S_{C_k}(\underline{a}_k^i)\right] + \underbrace{\mathbb{E}\left[S_{C_k}(\underline{a}_k^i) S_{C_k}(\underline{a}_k^j)\right]}_{(3)} \ \ .
\end{aligned}$$

We determine the analytic covariance of the pseudo-measurements by each term. For the term (1), we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i)\sum_{m=1}^{N} S_{Y_k^m}(\underline{a}_k^j)\right]\\
&=\sum_{l=1}^{N}\sum_{m=1}^{N}\mathbb{E}\left[S_{Y_k^l}(\underline{a}_k^i)S_{Y_k^m}(\underline{a}_k^j)\right]\\
&=\sum_{l=1}^{N}\sum_{m=1}^{N}\lambda^l\lambda^m\mathbb{E}\left[\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{y}}_k^l,\Gamma)\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{y}}_k^m,\Gamma)\right]\\
&=\sum_{l=1}^{N}\sum_{\substack{m=1\\m\neq l}}^{N}\lambda^l\lambda^m P_l^{\Gamma}(\underline{a}_k^i)P_m^{\Gamma}(\underline{a}_k^j)\\
&+\mathcal{N}(\underline{a}_k^i;\underline{a}_k^j,2\Gamma)\cdot\sum_{l=1}^{N}(\lambda^l)^2P_l^{\frac{1}{2}\Gamma}\left(\frac{1}{2}(\underline{a}_k^i+\underline{a}_k^j)\right),
\end{aligned}$$

where we used the identity

$$\mathcal{N}\left(\underline{a}_k^i;\mathbf{H}_k^l\underline{\boldsymbol{x}}_k^l+\underline{\boldsymbol{v}}_k^l,\Gamma\right)\cdot\mathcal{N}\left(\underline{a}_k^j;\mathbf{H}_k^l\underline{\boldsymbol{x}}_k^l+\underline{\boldsymbol{v}}_k^l,\Gamma\right)=$$
$$\mathcal{N}\left(\underline{a}_k^i;\underline{a}_k^j,2\Gamma\right)\cdot\mathcal{N}\left(\frac{1}{2}(\underline{a}_k^i+\underline{a}_k^j);\mathbf{H}_k^l\underline{\boldsymbol{x}}_k^l+\underline{\boldsymbol{v}}_k^l,\frac{1}{2}\Gamma\right)$$

in the last equation.

For the mixed term in (2), we note that the clutter are pairwise independent and identically distributed. Thus, the expected value is given by

$$\begin{aligned}
&\mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i)S_{C_k}(\underline{a}_k^j)\right]\\
&=\mathbb{E}\left[\sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i)\right]\cdot\mathbb{E}\left[S_{C_k}(\underline{a}_k^j)\right]\\
&=\sum_{l=1}^{N}\lambda^l P_l^{\Gamma}(\underline{a}_k^i)\cdot\lambda^c\mathcal{N}(\underline{a}_k^j;\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\ .
\end{aligned}$$

For the term (3) we have to exploit the following decomposition

$$\begin{aligned}
&\mathbb{E}\left[S_{C_k}(\underline{a}_k^i)S_{C_k}(\underline{a}_k^j)\right]\\
&=\mathbb{E}\left[\sum_{\underline{c}^i\in C_k}\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^i,\Gamma)\sum_{\underline{\boldsymbol{c}}_k^j\in C_k}\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)\right]\\
&=\mathbb{E}\Bigg[\sum_{\underline{\boldsymbol{c}}_k^i\in C_k}\sum_{\substack{\underline{\boldsymbol{c}}_k^j\in C_k\\ \underline{\boldsymbol{c}}_k^j\neq\underline{\boldsymbol{c}}_k^i}}\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^i,\Gamma)\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)\\
&+\sum_{\underline{\boldsymbol{c}}_k^j\in C_k}\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^j,\Gamma)\cdot\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)\Bigg]\\
&=\mathbb{E}\Big[\boldsymbol{M}_k^c(\boldsymbol{M}_k^c-1)\cdot\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^i,\Gamma)\cdot\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)\\
&+\boldsymbol{M}_k^c\cdot\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^j,\Gamma)\cdot\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)\Big]\\
&=\mathbb{E}[\boldsymbol{M}_k^c(\boldsymbol{M}_k^c-1)]\cdot\mathbb{E}[\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^i,\Gamma)]\cdot\mathbb{E}[\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)]\\
&+\mathbb{E}[\boldsymbol{M}_k^c]\cdot\mathbb{E}\big[\mathcal{N}(\underline{a}_k^i;\underline{\boldsymbol{c}}_k^j,\Gamma)\cdot\mathcal{N}(\underline{a}_k^j;\underline{\boldsymbol{c}}_k^j,\Gamma)\big]\\
&=(\lambda^c)^2\cdot\mathcal{N}(\underline{a}_k^i;\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\cdot\mathcal{N}(\underline{a}_k^j;\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\\
&+\lambda^c\cdot\mathcal{N}(\underline{a}_k^i;\underline{a}_k^j,2\Gamma)\cdot\mathcal{N}\left(\frac{1}{2}(\underline{a}_k^i+\underline{a}_k^j);\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma\right),
\end{aligned}$$

where we used that the clutter locations $\underline{\boldsymbol{c}}_k$ are identical distributed and the independence of the number of clutter $\boldsymbol{M}_k^c$ and the clutter location $\underline{\boldsymbol{c}}_k$. Furthermore, we used that the expectation of $\mathbb{E}[\boldsymbol{M}_k^c(\boldsymbol{M}_k^c-1)]$ for a Poisson-distribute random variable $\boldsymbol{M}_k^c\sim\text{Pois}(\lambda^c)$ is given by $(\lambda^c)^2$. Combining all results in the analytic covariance for the pseudo measurements

$$\begin{aligned}
\mathbf{\Sigma}_k^{s_i,s_j}&=\sum_{l=1}^{N}\lambda^l P_l^{\Gamma}(\underline{a}_k^i)\sum_{\substack{m=1\\m\neq l}}^{N}\lambda^m P_m^{\Gamma}(\underline{a}_k^j)\\
&+\mathcal{N}(\underline{a}_k^i;\underline{a}_k^j,2\Gamma)\sum_{l=1}^{N}(\lambda^l)^2P_l^{\frac{1}{2}\Gamma}\left(\frac{1}{2}(\underline{a}_k^i+\underline{a}_k^j)\right)\\
&+\sum_{l=1}^{N}\lambda^l P_l^{\Gamma}(\underline{a}_k^i)\cdot\lambda^c\cdot\mathcal{N}(\underline{a}_k^j;\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\\
&+\sum_{m=1}^{N}\lambda^m P_m^{\Gamma}(\underline{a}_k^j)\cdot\lambda^c\cdot\mathcal{N}(\underline{a}_k^i;\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\\
&+(\lambda^c)^2\cdot\mathcal{N}(\underline{a}_k^i;\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\cdot\mathcal{N}(\underline{a}_k^j,\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma)\\
&+\lambda^c\cdot\mathcal{N}(\underline{a}_k^i;\underline{a}_k^j,2\Gamma)\cdot\mathcal{N}\left(\frac{1}{2}(\underline{a}_k^i+\underline{a}_k^j);\underline{\mu}^c,\mathbf{\Sigma}^c+\Gamma\right)\\
&-\underline{\mu}_k^{s,i}\underline{\mu}_k^{s,j}\ .
\end{aligned}$$

3) Pseudo-Measurement Analytic Cross-Covariance $\mathbf{\Sigma}_k^{xs_i}$: For the cross-covariance of the pseudo-measurements and the target states, we have

$$\mathbf{\Sigma}_k^{xs_i}=\underbrace{\mathbb{E}\left[\underline{\boldsymbol{x}}_k S_{Y_k}(\underline{a}_k^i)\right]}_{(**)}-\underline{\mu}_{k|k-1}^x\cdot\underline{\mu}_k^{s,i},$$

where we need to determine the value of the expectations in $\mathbb{E}\left[\underline{\boldsymbol{x}}_k S_{Y_k}(\underline{a}_k^i)\right]$. Therefore, we use the following decomposition

$$\mathbb{E}\left[\underline{\boldsymbol{x}}_k S_{Y_k}(\underline{a}_k^i)\right] = \mathbb{E}\left[\underline{\boldsymbol{x}}_k \sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i)\right] + \mathbb{E}\left[\underline{\boldsymbol{x}}_k S_{C_k}(\underline{a}_k^i)\right] \quad .$$

The value of the first term can be determined by

$$\begin{aligned}
&\mathbb{E}\left[\underline{\boldsymbol{x}}_k \sum_{l=1}^{N} S_{Y_k^l}(\underline{a}_k^i)\right] = \sum_{l=1}^{N} \lambda^l \mathbb{E}\left[\underline{\boldsymbol{x}}_k \cdot \mathcal{N}(\underline{a}_k^i; \underline{\boldsymbol{y}}_k^l, \Gamma)\right] \\
&= \sum_{l=1}^{N} \lambda^l \int \underline{\boldsymbol{x}}_k \cdot \mathcal{N}(\underline{a}_k^i; \underbrace{\mathbf{H}_k^l \underline{\boldsymbol{x}}_k^l + \underline{\boldsymbol{v}}_k}_{\underline{\boldsymbol{y}}_k^l =}, \Gamma) \\
&\cdot \mathcal{N}(\underline{a}_k^i; \underline{\mu}_{k|k-1}^x, \mathbf{\Sigma}_{k|k-1}^x) \cdot \mathcal{N}(\underline{\boldsymbol{v}}_k^l; \underline{\mu}^c, \mathbf{\Sigma}_k^v) d\underline{\boldsymbol{x}}_k^l \, d\underline{\boldsymbol{v}}_k \\
&= \sum_{l=1}^{N} \lambda^l \cdot P_l^\Gamma(\underline{a}_k^i) \left(\underline{\mu}_{k|k-1}^x + \mathbf{K}_k^l \left(\underline{a}_k^i - \mathbf{H}_k^l \underline{\mu}_{k|k-1}^x\right)\right)
\end{aligned}$$

For the second term, we can use the independence of the clutter and the target state, which results in

$$\begin{aligned}
\mathbb{E}\left[\underline{\boldsymbol{x}}_k S_{C_k}(\underline{a}_k^i)\right] &= \mathbb{E}\left[\underline{\boldsymbol{x}}_k\right] \cdot \mathbb{E}\left[S_{C_k}(\underline{a}_k^i)\right] \\
&= \underline{\mu}_{k|k-1}^x \cdot \lambda^c \cdot \mathcal{N}\left(\underline{a}_k^i; \underline{\mu}^c, \mathbf{\Sigma}_k^c + \Gamma\right) .
\end{aligned}$$

Thus, the analytic cross-covariance is given by

$$\begin{aligned}
\mathbf{\Sigma}_k^{x,s_i} = &\sum_{l=1}^{N} \lambda^l \cdot P_l^\Gamma(\underline{a}_k^i) \left(\underline{\mu}_{k|k-1}^x + \mathbf{K}_k^l \left(\underline{a}_k^i - \mathbf{H}_k^l \underline{\mu}_{k|k-1}^x\right)\right) \\
&+ \underline{\mu}_{k|k-1}^x \cdot \lambda^c \cdot \mathcal{N}(\underline{a}_k^i, \underline{\mu}^c, \mathbf{\Sigma}^c + \Gamma) \\
&- \underline{\mu}_{k|k-1}^x \cdot \underline{\mu}_k^{s,i} \quad .
\end{aligned}$$