Journal of Fourier Analysis and Applications (2023) 29:62
https://doi.org/10.1007/s00041-023-10046-y

Heat Equations and Wavelets on Mumford Curves and Their Finite Quotients

Patrick Erik Bradley[1]

Received: 24 August 2022 / Revised: 13 April 2023 / Accepted: 28 August 2023

Abstract
A class of heat operators over non-archimedean local fields acting on L_2-function spaces on holed discs in the local field are developed and seen as being operators previously introduced by Zúñiga-Galindo, and if the underlying trees are regular, they are associated here with certain finite Kronecker product graphs. L_2-spaces and integral operators invariant under the action of a finite group acting on a holed disc are studied, and then applied to Mumford curves. It is found that the spectral gap in families of Mumford curves can become arbitrarily small.

Keywords p-Adic numbers · Heat equation · Mumford curves · Wavelets

1 Introduction

Ever since the introduction of the Taibleson–Vladimirov operator, a pseudodifferential operator on a non-archimedean local field [1–3], there has been active research on heat equations on such a field, in particular the field of p-adic numbers, which studies this operator or generalisations of it. Whereas in the classical case, heat equations are also extensively studied on manifolds, there is according to [4] no comparable theory of pseudodifferential operators over p-adic manifolds, not to say over manifolds defined over a non-archimedean local field. A construction of a certain pseudodifferential operator on a certain p-adic manifold invariant under the action of certain finite groups was undertaken in [5], however the generalised diffusion obtained cannot be considered a heat equation, because it does not give rise to a stochastic semigroup. A study of how p-adic pseudodifferential operators transform under group actions on the Bruhat–Tits

Communicated by Alexey Karapetyants.

✉ Patrick Erik Bradley
bradley@kit.edu

1 Institute of Photogrammetry and Remote Sensing, Karlsruhe Institute of Technology, Englerstr. 7, 76131 Karlsruhe, Germany

Published online: 02 October 2023

tree was undertaken in [6]. A different approach was pursued in [7], where p-adic integral operators on closed open (clopen) subsets of $\mathbb{Q}_p$ were constructed which are direct analogues of graph Laplacians. In fact, they can be seen as p-adic analogues of graph Laplacians in light of the dictionary developed in [8].

A 'nice' theory of p-adic heat equations is one in which the heat operator is diagonalisable by the p-adic Fourier transform [9]. More general operators can be diagonalised using wavelets on ultrametric spaces [10]. Kozyrev's well-known p-adic wavelets were found to diagonalise the Taibleson–Vladimirov operator [11]. More general wavelet bases in the p-adic and also adelic setting are studied in [12]. Certain pseudodifferential operators have eigenfunctions which are such wavelets [13]. Important applications of p-adic diffusion can be found in p-adic physics, initiated by I. Volovich in [14], and in which there has been extensive research in the last decades [15]. More recent applications include porous media and fluid dynamics [16–18]. However, the graph-based heat operators from the previous paragraph are not entirely diagonalisable by Kozyrev wavelets, one also needs the eigenvectors of the graph Laplacian [7]. The work of W. Zúñiga-Galindo contains many classes of p-adic heat equations, for which the Cauchy problem is solved in the affirmative. This includes those with graph-based operators, like also those in [19].

The non-archimedean counterpart of Riemann surfaces are the *Mumford curves* which are projective algebraic curves defined over non-archimedean fields allowing a Schottky uniformisation [20]. Locally, they are holed discs inside the base field. That means that if the base field is a non-archimedean local field, then the Haar measure allows integration of functions defined on these local pieces. The question is, how to glue together local operators on overlaps in a meaningful way. The stable reduction theorem [21] states that there is a model over a finite extension of the base field such that the special fibre, aka the reduction curve over the residue field, is a singular projective curve whose irreducible components are all rational curves, and the singularities are ordinary double points. Consequently, the intersection graph of the reduction curve of a Mumford curve has first Betti number equal to the genus of the curve. The rigid analytic theory of Mumford curves [22] allows to construct the intersection graph with the help of a covering of the curve by holed discs. This fact, together with the rigid analytic proof of the stable reduction theorem from [23] gives insight into how to obtain an integral operator on Mumford curves. Namely, these are graph-based integral operators which can be viewed as Zúñiga-Galindo's operators on the set of K-rational points of a Mumford curve by taking disjoint covers by holed discs, and where each maximal ball in each patch corresponds to a vertex in a graph associated with the curve. This is one contribution of the present article.

Another contribution is that the action of a finite group of automorphisms of the Mumford curve leads to integral operators on the space of invariant L^2-functions on the K-rational points of the curve via the induced graph automorphism invariance. The invariant L_2-functions on the curve (or, more precisely, its fundamental domain) then decompose into eigenfunctions of our graph-based operators which consist of such wavelets and of functions coming from eigenvectors of a graph Laplacian. It turns out that if the residue field has at least two elements, then the spectrum of these operators consist entirely of eigenvalues coming from wavelets.

Finally, it is proven that the spectral gap of the new operators can be arbitrarily small in families of Mumford curves with isomorphic stable reduction graphs. As an example, the eigenvalues of the heat operator invariant under the involution of a Tate curve are calculated explicitly.

We view the methods here are as a starting point for also investigating non-linear integral equations on Mumford curves, in order to generalise the results obtained on p-adic balls for p-adic non-linear evolution equations, like e.g. the p-adic analogue of the porous medium equation [24, 25].

This article is subdivided into five numbered sections, the present one being the introduction. The following Sect. 2 fixes some notation. This is followed in Sect. 3 by a generalisation of the dictionary developed in [8] to the more general situation of this article and now contains a correspondence between matrix eigenvalues and operator eigenvalues. Section 4 studies the invariant heat equation on a holed disc. Section 5 applies the results of the previous section to the case of a Mumford curve and an action of a group of automorphisms. It concludes with a study of the eigenvalues of the operators invariant under the involution of a Tate curve.

2 Notation

Let K be a non-archimedean local field whose absolute value is denoted as $|\cdot|_K$. The multiplicative group of K is denoted as $K^\times$. The unit ball of K is denoted as O_K. The Haar measure μ on K is chosen such that $\mu(O_K) = 1$. It is known that O_K is a discrete valuation ring whose uniformiser is denoted as π. It has the property

$$|\pi|_K = p^{-\frac{1}{e}}$$

for some $e \geq 1$, and where p is a prime number. In the case that K is a p-adic number field of degree n over $\mathbb{Q}_p$, then there is the well-known formula

$$n = e \cdot f$$

where f is the degree of the residue field k over the finite field $\mathbb{F}_p$ with p elements. Let

$$\tau \colon k \to O_K$$

be a lift of the residue map which takes a residue class modulo πO_K to a representative in O_K. The Bruhat–Tits tree for the projective-linear group $\mathrm{PGL}_2(K)$ will be denoted as $\mathcal{T}_K$. Its vertices are in one-one correspondence with discs of the projective line $\mathbb{P}^1(K)$, where a disc is a subset of the form

$$\{|x-a|_K \leq r\} \quad \text{or} \quad \{|x-a|_K \geq r\} \cup \{\infty\}$$

where $a \in K$ and $r \in \left|K^\times\right|$. When integrating a complex-valued function $f(x)$ on K, the Haar measure will be denoted as dx, so that the integral has the form

$$\int_K f(x)\,dx$$

if it exists. Finally, we will make use of indicator functions which we will write as

$$\Omega(x \in U) = \begin{cases} 1, & x \in U \\ 0, & \text{otherwise} \end{cases}$$

where U is a measurable subset of K.

3 A π-Adic Dictionary

The aim of this section is to generalise the dictionary of [8, Sec. 2] to the setting of this article.

Let $Z \subset K$ be a compact measurable set, and $\mathcal{U}$ a finite covering of Z by disjoint sets of finite measure. The space of continuous complex-valued functions on Z will be denoted as $C(Z, \mathbb{C})$ or simply as $C(Z)$. The space of bounded linear operators on a Banach space F is denoted as $\mathcal{B}(F)$.

Let $n = |\mathcal{U}|$, and let $n \times n$-matrices be indexed by $\mathcal{U}$. We call these $\mathcal{U}$-matrices. If $A = (A_{UV})$ is a $\mathcal{U}$-matrix, then we define

$$\|A\|_{\mathcal{U}} = \sqrt{\sum_{U,V \in \mathcal{U}} |A_{UV}|^2\, \mu(U)\mu(V)}$$

this defines a norm on the algebra of $\mathcal{U}$-matrices, called $\mathcal{U}$-norm.

Lemma 1 (Dictionary) *There is an injective isometric homomorphism between algebras*

$$\begin{aligned} b\colon \mathbb{C}^{n\times n} &\to \mathcal{B}(C(Z, \mathbb{C})) \\ A = (A_{UV}) &\mapsto A(x, y) = \sum_{U,V \in \mathcal{U}} A_{UV}\,\Omega(x \in U)\Omega(y \in V) \end{aligned}$$

where the first space has the $\mathcal{U}$-norm, and the second space the norm

$$\|\mathcal{A}\| := \sqrt{\int_Z \int_Z |A(x, y)|^2\; dy\,dx}$$

where $\mathcal{A} \in \mathcal{B}(C(Z))$ has kernel $A(x, y)$.

Proof First observe that $b(A)$ is indeed a bounded linear operator:

$$\begin{aligned}\|b(A)u\|_2^2 &= \int_Z \left| \int_Z A(x,y)u(y)\,dy \right|^2 dx \\ &\leq \sum_{U,V\in\mathcal{U}} |A_{UV}|^2 \int_Z \Omega(x\in U) \int_V |u(y)|^2\,dy\,dx \\ &\leq \sum_{U,V\in\mathcal{U}} |A_{UV}|^2\,\mu(U)\mu(V)\,\|u\|_2^2 \\ &= \|A\|_{\mathcal{U}}^2\,\|u\|_2^2\end{aligned}$$

Now, linearity and multiplicativity of b are straightforward calculations. In the case of mutliplication, one may consult the corresponding part in the proof of [8, Prop. 2.2].

Isometry follows from the observation that

$$\|A\|_{\mathcal{U}}^2 = \int_Z \int_Z |A(x,y)|^2\,dy\,dx = \|\mathcal{A}\|^2$$

Thus the assertion is proven. □

A $\mathcal{U}$-vector is an n-tupel with entries in $\mathbb{C}$ indexed by $\mathcal{U}$. If $e = (e_V)$ is an $\mathcal{U}$-vector, then there is an associated function $e(x) \in C(Z)$ defined as:

$$e(x) = \sum_{V\in\mathcal{U}} \mu(V)^{-1} e_V\,\Omega(x\in V)$$

There is a map

$$b\colon \mathbb{C}^n \to C(Z),\ e = (e_v) \mapsto e(x)$$

We are convinced that the same letter b as in Lemma 1 will not cause confusion.

Now, there is a product defining an action of $\mathcal{B}(C(Z))$ on $C(Z)$. Namely,

$$u \mapsto \mathcal{A}u(x) = \int_Z A(x,y)u(y)\,dy.$$

We will write this in the usual way as $\mathcal{A}u$.

Lemma 2 *Let $A = (A_{UV})$ be a $\mathcal{U}$-matrix, and $e = (e_V)$ a $\mathcal{U}$-vector. Then*

$$b(A)b(e) = b(Ae).$$

Proof We have

$$Ae = c = (c_U)$$

with

$$c_U = \sum_{V \in \mathcal{U}} A_{UV} e_V.$$

Hence, $b(c)$ is the function

$$c(x) = \sum_{U \in \mathcal{U}} \sum_{V \in \mathcal{U}} A_{UV} e_V \Omega(x \in U).$$

On the other hand, $b(A)b(e)$ is the function

$$\begin{aligned} \int_Z A(x,y)e(y)\,dy &= \sum_{U,V \in \mathcal{U}} A_{UV} \mu(V)^{-1} e_V \int_V \Omega(y \in V)\,dy\,\Omega(x \in U) \\ &= \sum_{U,V \in \mathcal{U}} A_{UV} e_V \Omega(x \in U) = b(Ae). \end{aligned}$$

This proves the assertion. □

Corollary 3 *If $e = (e_V)$ is an eigenvector of $\mathcal{U}$-matrix A with eigenvalue $\lambda \in \mathbb{C}$, then $b(e)$ is an eigenfunction of $b(A)$ for the same eigenvalue.*

Proof We have

$$b(A)b(e) = b(Ae) = b(\lambda e) = \lambda b(e)$$

where the first equality is due to Lemma 2. □

4 Invariant Heat Equations Associated with Holed Discs

In this section, we introduce and study integral operators and heat equations on holed discs, and also study the action of a finite group on holed discs. In the case that the reduction tree of the holed disc is regular, then the operators turn out to be Zùñiga-type operators on certain Kronecker product graphs, also in the invariant case, where invariance becomes such under graph automorphisms.

4.1 Zúñiga Operators on Holed Discs

Let $Z \subset K$ be a holed disc. It is a measurable subset, and the disjoint union of finitely many balls in K. There is a tree T_Z associated with Z, which is a subtree of the Bruhat–Tits tree $\mathcal{T}_K$. Related to the the *projective dendrogram* for a finite set S consisting of one point in K for each hole in Z from [26], it is given as follows: The projective dendrogram $T^*(S)$ for S is the smallest subtree of the Bruhat–Tits tree $\mathcal{T}_K$ having the set $S \cup \{\infty\}$ at its boundary. The finite part $T(S)$ of $T^*(S)$ is the finite subtree of $\mathcal{T}_K$ whose vertices correspond to the joins of the geodesic lines in $\mathcal{T}_K$ between ∞

and $x \in S$. And T_Z is the intermediate finite tree in $T^*(S)$ containing $T(S)$, obtaind by cutting off halflines such that the endpoints correspond to the holes of Z as disks making up $\mathcal{T}_K$, and the root is the disc obtained by "filling" the holes of Z.

Each vertex v of T_Z itself corresponds to a holed disc $U_v \subset K$ obtained by removing from the disc associated with v a maximal strict subdisk for each edge attached to v leading away from the root. We call such a holed disc *thin*, and we can write

$$Z = \coprod_{v \in V(T_Z)} U_v \tag{1}$$

where $V(G)$ denotes the vertex set of a graph G.

Now, let G be a weighted simple finite graph having T_Z as a spanning tree. Its adjacency matrix will be denoted as $A = (A_{vw})_{v,w \in V(G)}$. We assume that the weights A_{vw} are all non-negative, and that $A_{vv} = 0$. This simply states that G has no self-loops, as it is simple. We consider the following operator:

$$\mathcal{D}_A^\alpha f(x) = \int_Z A_Z^\alpha(x,y)(f(x) - f(y))\, dy$$

whose kernel function is

$$A_Z^\alpha(x,y) = \sum_{v,w \in V(G)} A_{uv} \, |x-y|^\alpha \, \Omega(x \in U_v) \Omega(y \in U_w)$$

and $\alpha \in \mathbb{C}$.

Definition 1 The operator $\mathcal{D}_A^\alpha$ is called the *Zúñiga operator* for the weighted graph G (or the matrix A).

Lemma 4 *The operator $\mathcal{D}_A^\alpha$ on $L^2(Z)$ is of the type introduced in [7] for a graph whose vertices are represented by p-adic balls.*

Proof Each thin holed disk U_v in (1) is itself a union of balls:

$$U_v = \coprod_{i=1}^{m_v} B_{i,v}. \tag{2}$$

Let x, y be two points from Z. Then

$$x \in B_{i,v}, \quad y \in B_{j,w}$$

where $v, w \in V(G)$, and

$$i \in \{1, \dots, m_v\}, \quad j \in \{1, \dots, m_w\}.$$

Since $A_{vv} = 0$, we assume that $v \neq w$. In this case,

$$|x - y|_K = d(B_{i,v}, B_{j,w})$$

where d is the π-adic distance between disjoint sets in K. Hence, $\mathcal{D}_Z^\alpha$ is the operator from [7] associated with the matrix $C = (C_{(i,v),(j,w)}) \in \mathbb{R}^{I \times I}$ with

$$C_{(i,v),(j,w)} = A_{vw} \cdot d(B_{i,v}, B_{j,w})^\alpha$$

with index set

$$I = \coprod_{v \in V(G)} \{1, \ldots, m_v\} \times V(G)$$

and the assertion follows. □

The eigenvectors of the matrix C with

$$C_{(i,v),(j,w)} = A_{vw} \cdot d(B_{i,v}, B_{j,w}) \tag{3}$$

as an adjacency matrix of a graph whose n vertices are the balls $B_{i,v}$ taken from the proof of Lemma 4 occur in the eigendecomposition of $L^2(Z)$ for $\mathcal{D}_A^\alpha$:

Corollary 5 *Let $\alpha \in \mathbb{R}$. Then there is an orthogonal decomposition*

$$L^2(Z) = L_0^2(Z) \oplus L_A^2(Z)$$

where the Kozyrev wavelets supported in Z are an orthonormal basis of $L_0^2(Z)$, and the functions

$$e(x) = \sum_{v \in V(G)} \sum_{i=1}^{m_v} \mu(B_{i,v})^{-1} e_{i,v} \Omega(x \in B_{i,v})$$

where $(e_{i,v}) \in \mathbb{R}^{|I|}$ is a normalised Laplacian eigenvector of the matrix C, are an orthonormal basis of the finite-dimensional summand $L_A^2(Z)$.

Proof This is shown in [7, Thm. 10.1] for the operators defined there, and Lemma 4 states that $\mathcal{D}_A^\alpha$ is such an operator, except that the balls are now in K instead of $\mathbb{Q}_p$, but the proof of that theorem carries over to the field K. □

Before we extend Zúñiga's theory to the case of finite group actions in the next subsection, we look at an example.

Example 1 Let $K = \mathbb{Q}_p$, $U = p\mathbb{Z}_p$, $V = 1 + p\mathbb{Z}_p$, and $Z = U \cup V$ with the covering $\mathcal{U} = \{U, V\}$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Even if U, V are not thin holed discs, we have a corresponding Zúñiga operator $\mathcal{D}_A^\alpha$ with kernel function

$$A_Z^\alpha(x,y) = |x-y|_K^\alpha\,\Omega(x\in U)\Omega(y\in V) + |x-y|_K^\alpha\,\Omega(x\in V)\Omega(y\in U).$$

Let $W_n = p^{2n}\mathbb{Z}_p$ with $n \geq 1$. Then the wavelet

$$\psi_n(x) = p^n\chi(p^{-2n-1}x)\Omega(x\in W_n)$$

is an eigenfunction of $\mathcal{D}_A^\alpha$. Namely, we have

$$\begin{aligned}\mathcal{A}_Z^\alpha\psi_n(x) &= \int_{W_n}|x-y|_K^\alpha\,\psi_n(y)\,dy\,\Omega(x\in V)\\ &= C_{UV}^\alpha\int_{W_n}\psi_n(y)\,dy\,\Omega(x\in V) = 0\end{aligned}$$

because the integral vanishes by [27, Thm. 3.29]. And we have

$$\begin{aligned}\int_Z A_Z^\alpha(x,y)\,dy\,\psi_n(x) &= \int_V |x-y|_K^\alpha\;dy\,\psi_n(x)\\ &= \mu(V)\psi_n(x).\end{aligned}$$

Hence, $\psi_{W_n}(x)$ with $n \geq 1$ is indeed an eigenfunction for eigenvalue $-\mu(V)$.

This example shows that, in general, a Zúñiga operator cannot be expected to be a compact operator, not even for $\alpha = 0$.

4.2 Zúñiga Operators Invariant Under Finite Group Actions

Now, let H be a finite group acting as automorphisms on a finite graph G. Since the vertices of G are represented by thin holed discs U_v, the group H acts on Z by permuting the sets U_v. We may and will assume that H acts by affine-linear transformations, i.e. locally as

$$h(x) = a_h + b_h x$$

for $h \in H$. So, we can define

$$L^2(Z)^H = \left\{f\in L^2(Z) \mid \forall h\in H\colon\ f(hx) = f(x)\right\}$$

this is the space of H-invariant L_2-functions on Z. An H-invariant operator is

$$\mathcal{D}_{A,H}^\alpha = \sum_{h\in H}\mathcal{D}_{A,h}^\alpha$$

where $\mathcal{D}^{\alpha}_{A,h}$ is the operator

$$\mathcal{D}^{\alpha}_{A,h}(f)(x) = \mathcal{D}^{\alpha}_{A}(f)(hx)$$

for $h \in H$.

Lemma 6 *Given the matrix $A_h = (A_{hv,w})_{v,w \in V(G)}$ for $h \in H$, we have that*

$$\mathcal{D}^{\alpha}_{A,H} = \frac{1}{|H|} \sum_{h \in H} \mathcal{D}^{\alpha}_{A_h}$$

as an operator on $L^2(Z)$.

Proof We calculate

$$\begin{aligned} A^{\alpha}_{Z,h}(x,y) &= \sum_{v,w \in V(G)} A_{vw} \, |hx - y|^{\alpha}_K \, \Omega(x \in U_v)\Omega(y \in U_w) \\ &= \sum_{v,w \in V(G)} A_{hv,w} \, |x - y|^{\alpha}_K \, \Omega(x \in U_v)\Omega(y \in U_w). \end{aligned}$$

This shows that $A^{\alpha}_{Z,h}(x,y)$ is the kernel function of $\mathcal{D}^{\alpha}_{A_h}$, which implies the assertion. □

Let $\psi(x)$ be a Kozyrev wavelet for K. It is given as

$$\psi(x) = \mu(B)^{\frac{1}{2}} \chi_K(p^{\frac{d-1}{e}} \tau(j)x)\Omega(x \in B)$$

where $B \subset K$ is a ball of radius $p^{-\frac{d}{e}}$, $j \in k$, and $\tau : k \to O_K$ a lift of the residue field k. Now, we define the function

$$\psi^h(x) = \psi(hx)$$

Lemma 7 *It holds true that*

$$\psi^h(x) = \chi(c)\psi^{(h)}(x)$$

where $c \in K$, and $\psi^{(h)}(x)$ is a Kozyrev wavelet for K supported in the ball $h^{-1}(B)$.

Proof This is a simple calculation. □

Define the matrices $C^h = (C^h_{(i,v),(j,w)})$ with

$$C^h_{(i,v),(j,w)} = A_{hv,w} \, d(B_{i,hv}, B_{j,w})^{\alpha}$$

and the quantities

$$\gamma^h_{i,v} = \gamma_{i,hv} = \sum_{(j,w)} C^h_{(i,v),(j,w)} = \sum_{w \in V(G)} \sum_{j=1}^{m_w} A_{hv,w} \, d(B_{i,hv}, B_{j,w})^{\alpha}$$

for $h \in H$. According to [7, Thm. 10.1], the quantity $-\gamma_{i,v}^h$ is an eigenvalue of the Zúñiga operator $\mathcal{D}_{A_h}^\alpha$ corresponding to a Kozyrev wavelet supported in $B_{i,v}$.

Lemma 8 *For $h \in H$, it holds true that $-\gamma_{i,v}^h$ is an eigenvalue of any $\mathcal{D}_{A_g}^\alpha$ with $g \in H$, correspondong to wavelet $\psi^{(h)}(x)$ as an eigenfunction.*

Proof This is clear by definition of $\psi^{(h)}$ and Zúñiga's theorem [7, Thm. 10.1]. □

Theorem 9 *Let $\alpha \in \mathbb{R}$. The H-invariant space $L^2(Z)^H$ has an orthogonal basis of eigenfunctions of $\mathcal{D}_{A,H}^\alpha$ of the following form:*

$$\text{from Kozyrev wavelets:} \quad \frac{1}{|H|}\sum_{h\in H}\psi^{(h)}(x), \quad \text{eigenvalue:} \quad -\frac{1}{|H|}\sum_{h\in H}\gamma_{i,v}^h$$
$$\text{from a Laplacian matrix:} \quad e^H(x), \quad \text{eigenvalue:} \quad \lambda^H$$

where

$$e^H(x) = \sum_{v\in V(G)} e_v^H \Omega(x \in U_v)$$

and $(e^H)_{v\in V(G)}$ is an H-invariant eigenvector for the common eigenvalue λ^H of all the matrices L_h with $h \in H$, where L_h is the graph Laplacian associated with the simple graph having adjacency matrix C^h. There is a decomposition

$$L(Z)^H = L_0^2(Z)^H \oplus L_A^2(Z)^H$$

where $L_0^2(Z)$ is spanned by the averages of H-orbits of Kozyrev wavelets, and $L_A^2(Z)^H$ by the H-invarant functions coming from the graph Laplacians.

Proof First, observe that $\mathcal{D}_{A,H}^\alpha$ is the Zúñiga operator associated (in our sense) with the matrix

$$A_H = \frac{1}{|H|}\sum_{h\in H} A_h.$$

Hence, in Zùñiga's sense, the operator is associated with the matrix

$$C_H = \frac{1}{|H|}\sum_{h\in H} C^h = \frac{1}{|H|}P_H C$$

where

$$P_H = \sum_{h\in H} P_h$$

and

$$P_h = (\delta_{(i,hv),(j,w)})$$

is a row permutation matrix associated with $h \in H$. The associated Laplacian matrix is

$$L_H = D_H - C_H$$

where

$$D_H = \sum_{h \in H} D_h$$

and D_h is the degree matrix of C^h. It follows that

$$L_H = \sum_{h \in H} L_h = \frac{1}{|H|} P_H L$$

where L_h is the Laplacian matrix of C^h, and $L = L_{\mathrm{id}}$.

An H-invariant vector u is an eigenvector of L_H if and only if

$$L_H u = \lambda u \;\Leftrightarrow\; Lu = \lambda\, |H|\, P_H^{-1} u \;\Leftrightarrow\; Lu = \lambda u$$

where the latter equivalence holds true, because

$$|H|\, P_H u = u \;\Leftrightarrow\; u = \frac{1}{|H|} P_H u = \frac{1}{|H|} \sum_{h \in H} P_h u \overset{(*)}{=} \frac{1}{|H|} \,|H|\, u = u$$

where $(*)$ holds true, because u is H-invariant. This means that an H-invariant vector u is an eigenvector of L_H for eigenvalue λ, if and only if u is an eigenvector of the Laplacian $L = L_{\mathrm{id}}$ for the same eigenvalue λ. Actually even iff u is an eigenvector of L_h for eigenvalue λ, because

$$L_H = \Phi_H(L_h)$$

for any $h \in H$, where

$$\Phi_H(M) = \frac{1}{|H|} \sum_{h \in H} M_h$$

is the H-averaging operator. This proves the assertion in the Laplacian matrix case, using Zúñiga's Theorem (Corollary 5) for $\mathcal{D}^{\alpha}_{A,H}$ in order to see that eigenvectors of A_H create by the dictionary (Lemma 2) eigenfunctions of $\mathcal{D}^{\alpha}_{A,H}$ in the first place.

As for the Kozyrev case, we do a calculation for

$$\Psi(x)=\frac{1}{|H|}\sum_{h\in H}\psi^{(h)}(x)$$

where $\psi^{(h)}(x)$ is a Kozyrev wavelet supported in $h^{-1}B_{i,v}$, where $B_{i,v}$ is a ball as in (2), in order to see that with

$$\gamma_{i,v}^{H}=\frac{1}{|H|}\sum_{h\in H}\gamma_{i,v}^{h}$$

the quantity $-\gamma_{i,v}^{H}$ is an eigenvalue of $\mathcal{D}_{A,H}^{\alpha}$ with eigenfunction $\Psi(x)$. Namely,

$$\begin{aligned}\mathcal{D}_{A,H}^{\alpha}\Psi(x)&=\frac{1}{|H|}\sum_{h\in H}\mathcal{D}_{A,H}^{\alpha}\psi^{(h)}(x)=\frac{1}{|H|}\sum_{h\in H}\frac{1}{|H|}\sum_{g\in H}\mathcal{D}_{A,g}^{\alpha}\psi^{(h)}(x)\\&=-\frac{1}{|H|}\sum_{h\in H}\frac{1}{|H|}\sum_{g\in H}\gamma_{i,v}^{h}\psi^{(h)}(x)=-\frac{1}{|H|}\sum_{g\in H}\gamma_{i,v}^{h}\frac{1}{|H|}\psi^{(h)}(x)\\&=-\gamma_{i,v}^{H}\Psi(x)\end{aligned}$$

Since $L^2(Z)^H$ is spanned by the H-averages (i.e. $\frac{1}{|H|}$ times the sum of the H-orbits) of the elements of any orthonormal basis, the Kozyrev part of the assertion now follows.

The asserted decomposition of $L^2(Z)^H$ now follows, using the decomposition of Corollary 5. □

Definition 2 The operator $\mathcal{D}_{A,H}^{\alpha}$ is called the H*-averaging Zúñiga operator* associated with α, matrix A or graph G (and group H).

We learn from Theorem 9 that the spectrum of the H-averaging Zúñiga operator $\mathcal{D}_{A,H}^{\alpha}$ contains the H-averages of the Kozyrev part of the spectrum of the individual $\mathcal{D}_{A,h}^{\alpha}$ for $h\in H$. And the size of the graph Laplacian part of its spectrum depends on the dimension of the part of the common eigenspaces of the matrices A^h, on which the finite group H acts trivially. In other words, it depends on the linear representations of H as permutation subgroups of $\mathrm{GL}_n(\mathbb{R})$. But this is left for future research.

Notice that, if the tree T_Z is m-regular, then the matrix C from (3) is a Kronecker product:

$$C=B^{\alpha}\otimes\mathbb{1}_r$$

with $\mathbb{1}_n$ the $n\times n$-matrix of constant value 1, and $B^{\alpha}=(B_{vw}^{\alpha})$, where

$$B_{vw}^{\alpha}=A_{vw}d(U_v,U_w)^{\alpha}$$

and the sets U_v with $v\in V(G)$ are thin holed disks covering the holed disc Z. If $m>1$, then the following result says that the Laplacian eigenvalues of C are all vertex degree values.

Lemma 10 *Assume that all U_v for $v \in V(G)$ contain at least $m_v > 1$ maximal balls. Then the Laplacian spectrum of C contains all the quantities* $\deg_{B^\alpha}(v)$. *with multiplicity $m_v - 1$ for given $v \in V(G)$, i.e. if a vertex degree repeats itself ℓ times, then the multiplicity of* $\deg_{B^\alpha}(v)$ *is $\ell \cdot (m_v - 1)$.*

Proof The Laplacian of C has the structure of a block matrix with constant rectangular blocks of size $m_v \cdot m_w$ with $v \neq w \in V(G)$ outside the diagonal, and for each vertex $v \in V(G)$ a diagonal block of size m_v^2 which is a diagonal matrix with constant diagonal entry $\deg_{B^\alpha}(v)$. Hence, since all $m_v > 1$, any vector $e = (e_{(i,w)})$ with

$$e_{(i,w)} = 0 \quad \text{for any } w \neq v \text{, and with sum of entries zero} \tag{4}$$

is a non-zero eigenvector associated with eigenvalue $\deg_{B^\alpha}(v)$. The multiplicity of such eigenspaces is clearly $m \cdot (m_v - 1)$, where m is the multiplicity of the vertex degree $\deg_{B^\alpha}(v)$, because condition (4) defines a co-dimension 1 subspace of $\mathbb{R}^{m_v}$. □

Corollary 11 *If $p^f - 1$ is greater than the largest number of holes in some U_v as a thin holed disc for $v \in V(G)$, then the spectrum of $\mathcal{D}^\alpha_{A,H}$ coming from a Laplacian matrix contains the quantities*

$$-\frac{1}{|H|}\sum_{h\in H}\gamma^h_{i,v}$$

i.e. the wavelet eigenvalues of $\mathcal{D}^\alpha_{A,H}$.

Proof According to the proof of Theorem 9, the H-invariant eigenvectors of the Laplacian L of C are the common eigenvectors of all Laplacians L^h having all the same eigenvalue. Hence, by Lemma 10, the assertion follows, since each U_v contains $m_v > 1$ maximal balls. Namely, the degree eigenvalues of

$$L_H = \frac{1}{|H|}\sum_{h\in H} L^h$$

are also the wavelet eigenvalues of $\mathcal{D}^\alpha_{A,H}$. □

We will just look at one kind of further decomposition of the finite-dimensional part of $L^2(Z)^H$. Namely, there is a well-known exact sequence

$$0 \longrightarrow c(G,\mathbb{C}) \longrightarrow A(G,\mathbb{C}) \xrightarrow{d} H(G,\mathbb{C}) \xrightarrow{\phi} \mathbb{C}^{b_0(G)} \longrightarrow 0 \tag{5}$$

where $b_0(G)$ is the 0-th Betti number of G, and $c(G,\mathbb{C})$ is the kernel of the linear map d, cf. e.g. [28, Lemma 6.1], whose proof immediately generalises to the case of disconnected graphs. The map d is defined as follows:

$$d(\beta)\colon v \mapsto \sum_{\substack{e\in E(G)\\ e\dashv v}} \beta(e)$$

where $e \dashv v$ means that edge e is attached to vertex v. The map ϕ is defined as follows:

$$f \mapsto \left(\sum_{v\in C} f(v)\right)_{C\in C(G)}$$

where $C(G)$ is the set of all connected components of G.

Lemma 12 *There is an exact sequence*

$$0 \longrightarrow c(G,\mathbb{C})^H \longrightarrow A(G,\mathbb{C})^H \xrightarrow{d^H} H(G,\mathbb{C})^H \xrightarrow{\phi^H} \left(\mathbb{C}^{b_0(G)}\right)^H \longrightarrow 0$$

where H acts on functions via averaging.

Proof We first show that

$$d\left(A(G,\mathbb{C})^H\right) \subset H(G,\mathbb{C})^H.$$

Namely, let

$$\beta^H : e \mapsto \frac{1}{|H|}\sum_{h\in H}\beta^h(e)$$

with $\beta^h(e) = \beta(he)$. Then

$$d(\beta^H) = \frac{1}{|H|}\sum_{h\in H} d(\beta^h)$$

takes $v \in V(G)$ to

$$\sum_{e\dashv v}\frac{1}{|H|}\sum_{h\in H}\beta(he) = \frac{1}{|H|}\sum_{h\in H}\sum_{e\dashv hv}\beta(e) = \frac{1}{|H|}\sum_{h\in H} d(\beta)^h(v)$$

where again $d(\beta)^h(v)$ means $d(\beta)(hv)$. This shows that there is a well-defined map

$$d^H\colon A(G,\mathbb{C})^H \to H(G,\mathbb{C})^H$$

which restricts the original d. Similary, one shows that ϕ restricts to

$$\varphi^H\colon H(G,\mathbb{C})^H \to \left(\mathbb{C}^{b_0(G)}\right)^H.$$

Also, the exactness of the origina sequence descends to the exactness of this sequence. Namely, $\ker\phi^H$ consists of those elements of $\ker\phi = d(A(G,\mathbb{C}))$ which are H-invariant. So, we have

$$\ker\phi^H = d(A(G,\mathbb{C}))^H = d\left(A(G,\mathbb{C})^H\right)$$

where the last equality is immediate. As

$$c(G,\mathbb{C})^H = c(G,\mathbb{C}) \cap \ker d^H = \ker d^H$$

and $\left(\mathbb{C}^{\beta_0(G)}\right)^H$ is the image of ϕ^H, the exactness of the sequeuence now follows. □

Furthermore, there is a linear map

$$\sigma\colon L^2(Z) \to H(G,\mathbb{C}),\, f \mapsto \left(v \mapsto \int_{U_v} f(x)\,dx\right)$$

which clearly induces a linear map

$$\sigma^H\colon L^2(Z)^H \to H(G,\mathbb{C})^H$$

Lemma 13 *The map σ^H is surjective, and it holds true that*

$$\ker(\phi^H \circ \sigma^H) = L^2_{C(G)}(Z)^H$$

where

$$L^2_{C(G)}(Z)^H = \left\{ f \in L^2(Z)^H : \forall C \in C(G)\colon \int_C f(x)\,dx = 0 \right\}$$

and, further,

$$\ker(\sigma^H) = L^2_0(Z)^H$$

Proof This is immediate. □

Corollary 14 *It holds true that*

$$L^2_{C(G)}(Z)^H / L^2_{\mathcal{U}}(Z)^H \cong d(A(G,\mathbb{C}))^H$$

and

$$L^2(Z)^H / L^2_0(Z)^H \cong H(G,\mathbb{C})^H.$$

In particular, it holds true that

$$\begin{aligned} L^2(Z)^H &\cong L^2_0(Z)^H \oplus d(A(G,\mathbb{C}))^H \oplus \left(\mathbb{C}^{b_0(G)}\right)^H \\ &\cong L^2_0(Z)^H \oplus \frac{A(G,\mathbb{C})^H}{c(G,\mathbb{C})^H} \oplus \left(\mathbb{C}^{b_0(G)}\right)^H. \end{aligned}$$

Proof This follows from the exact sequence of Lemma 12 and from Lemma 13. In particular, there is a decomposition

$$H(G, \mathbb{C}) \cong \frac{A(G, \mathbb{C})^H}{c(G, \mathbb{C})^H} \oplus \left(\mathbb{C}^{b_0(G)}\right)^H$$

from the exact sequence. □

Remark 1 Notice that the invariant direct summand $\left(\mathbb{C}^{b_0(G)}\right)^H$ equals the eigenspace of $\mathcal{D}^{\alpha}_{A,H}$ for eigenvalue zero.

4.3 Heat Equations Invariant Under Finite Group Actions

In [29], there is a formulation of the Yosida-Hille-Ray Theorem which is valid in our case only for the space $C(K)$. We formulate and prove a version which is valid also for closed subspaces of $C(E)$.

Proposition 15 *Let D be a linear operator on $C(Z)$, and $\bar{A}$ its closure, assumed to be single-valued and generating a strongly continuous, positive contraction semigroup $\{T(t)\}$ on $C(K)$. Let E be a closed subspace invariant under A. Then the restriction $A|_E : E \to E$ generates a strongly continuous, positive contraction semigroup $T_B(t)$ on E.*

Proof The operator $A|_E$ inherits from A the following properties:

1. The domain of $A|_E$ is dense in E.
2. $A|_E$ is dissipative.
3. The resolvent set $R(\lambda - A|_E)$ is dense in E for some $\lambda > 0$.

From [29, Thm. 1.2.12], it follows that

$$\overline{A|_E} = \bar{A}|_E$$

generates a strongly continuous semigroup $\{T_E(t)\}$.

We now need to show that $\{T_E(t)\}$ is positive. But since

$$T_E = T|_E$$

and $\{T(t)\}$ is a positve semigroup, it follows that $\{T_E(t)\}$ is positive. □

Corollary 16 *Let $\varepsilon > 0$. Then $\varepsilon\mathcal{D}^{\alpha}_{A,H}$ with $\alpha \in \mathbb{R}$ generates a strongly continuous, positive contraction semigroup $\exp(t\varepsilon\mathcal{D}^{\alpha}_{A,H})$ on $C(Z)^H$.*

Proof According to [7, Lemma 4.1], $\varepsilon\mathcal{D}^{\alpha}_{A,H}$ is a closed linear operator on $C(Z)$ generating a strongly continuous, positive contraction semigroup. It follows from Proposition 15 that the restriction to the invariant closed subspace $C(Z)^H$ also generates a semigroup with the desired properties. □

We are now going to look at the following Cauchy problem which makes sense, beacause $C(Z)^H$ is a closed subspace of $C(Z)$:

Task 1 (Cauchy Problem) Find $h(t,x) \in C^1\left((0,\infty), C(Z)^H\right)$ such that

$$\left(\frac{\partial}{\partial t} - \epsilon \mathcal{D}^{\alpha}_{A,\Gamma}\right) h(t,x) = 0 \tag{6}$$

for $t \geq 0$, $x \in Z$ which satisfies the initial condition

$$h(0,x) = h_0(x)$$

where $h_0 \in C(Z)^H$ is fixed.

As the corresponding semigroup is Feller, it describes a π-adic heat equation on Z. Consequently there is a π-adic diffusion process in Z attached to the differential Eq. (6)

Theorem 17 *There exists an H-invariant probability measure $p_t(x,\cdot)$ with $t \geq 0$, $x \in Z$ on the Borel σ-algebra of Z such that the Cauchy Problem (Task 1) has a unique solution of the form*

$$h(t,x) = \int_Z h_0(y) p_t(x,dy).$$

In addition, $p_t(x,\cdot)$ is the transition function of a Markov process whose paths are right continuous and have no discontinuities other than jumps.

Proof The proof of [7, Thm. 4.2] carries over word for word. □

5 Invariant Heat Equations on Mumford Curves

5.1 Mumford Curves

Mumford curves were first constructed in [20] as a successful attempt to generalise Tate's analytic uniformisation of p-adic elliptic curves, which can be found in [30]. Through this construction, Mumford revealed a one-to-one correspondence between conjugacy classes of Schottky groups in $\mathrm{PGL}_2(K)$ and a certain class of projective algebraic curves defined over K.

In order to present more details of this correspondence, we will give a brief summary of [22, Ch. I, III, IV, V], adapted to the setting of a non-archimedean local field. The slightly over two pages in [23, Ch. 5.4] also contain a brief overview over Mumford curves.

A Möbius transformation $\gamma \in \mathrm{PGL}_2(K)$ is called *hyperbolic*, if

$$\left|\mathrm{Tr}(A_\gamma)\right|_K^2 > 1$$

where $A_\gamma \in \mathrm{SL}_2(K)$ is the representative of γ as a special linear matrix.

A discrete subgroup of $\mathrm{PGL}_2(K)$ which is freely generated by $g \geq 1$ hyperbolic transformations is called a *Schottky group*.

A *limit point* of a subgroup Γ of $\mathrm{PGL}_2(K)$ is a point $x \in \mathbb{P}^1(K)$ such that there exists a point $x_0 \in \mathbb{P}^1$ and a sequence $\gamma_n \in \Gamma$ such that

$$\lim_{n\to\infty} \gamma_n(x_0) = x.$$

The set of limit points is denoted as $\mathscr{L}(\Gamma)$, and $\Omega = \mathbb{P}^1(K) \setminus \mathscr{L}(\Gamma)$ is the set of *regular points* of Γ. A Schottky group Γ has the property

$$\Omega \neq \emptyset.$$

This is an open subset of $\mathbb{P}^1(K)$.

The *Mumford curve* associated with a Schottky group Γ with $g \geq 1$ generators is the quotient space

$$X = \Omega/\Gamma.$$

It is a projective algebraic curve over K of genus g [22, Ch. III].

A *rational affinoid domain* is a holed disc in $\mathbb{P}^1$. A Mumford curve has a finite covering $\mathcal{U}$ by rational affinoid domains [22, Ch. V]. In this covering, two overlapping patches $U \in \mathcal{U}$ are glued with another along a boundary component (which is a circle).

5.2 Finite Group Actions on Mumford Curves

The theory of finitely generated groups of projective-linear transformations shows that there are many finite quotients of Mumford curves:

Theorem 18 (Gerritzen, van der Put (1980)) *Any finitely generated discrete subgroup N of $\mathrm{PGL}_2(K)$ has a normal subgroup Γ of finite index which is a Schottky group.*

Proof [22, Thm. I.3.1]. $\square$

Let H be a finite group acting on a Mumford curve X. According to the theory of Mumford curves [22], the group H is part of the following diagram

$$\begin{tikzcd} \Omega \arrow[r, "\Gamma"] \arrow[dr, "N"'] & X \arrow[d, "H"] \\ & X/H \end{tikzcd} \tag{7}$$

where $\Omega \subset \mathbb{P}^1_K$ is the universal topological covering of X. The finite group H is the quotient of a finitely generated discrete subgroup N of $\mathrm{PGL}_2(K)$, and a Schottky group Γ which is normal and of finite index in N. For any such group N, there exists such a Γ by Theorem 18. Both groups Γ and N act on a subtree $\mathscr{T}_N$ of the Bruhat–Tits tree $\mathscr{T}_K$ of $\mathrm{PGL}_2(K)$. Its construction is given in [22]: the vertices are the intersection vertices of $\mathscr{T}_K$ of the geodesic lines between any three limit points of Γ (or N, which does not matter). The quotient graphs are part of the following diagram:

$$\begin{tikzcd} \mathscr{T}_N \arrow[r, "\Gamma"] \arrow[dr, "N"'] & G \arrow[d, "H"] \\ & G/H \end{tikzcd} \tag{8}$$

Definition 3 The graph G in (8) is called the *reduction graph* of the Mumford curve X.

A fundamental domain of the tree action of Γ is given by lifting a spanning tree T of G/H to $\mathscr{T}_N$. This tree corresponds to a holed disc $\mathscr{F}$ in $\mathbb{P}^1(K)$, We may assume that ∞ is not a regular point of the N- or Γ-action. Thus we may and will assume that $\mathscr{F} \subset K$. The action of N on Ω then induces an action of the finite group H on $\mathscr{F}$ and T. We have

$$\mathscr{F}/H \cong X(K)/H$$

because as K-analytic manifolds the fundamental domain $\mathscr{F}$ is analytically isomorphic to the K-rational points $X(K)$ of the Mumford curve X. This means that we are in the situation of Sect. 4. Consequently, we can define an H-invariant Zùñiga operator $\mathcal{D}^\alpha_{A,H}$ on $\mathscr{F}$, where A is a weighted adjacency matrix of the graph G, and view it as an invariant operator on $X(K)$. The results for the H-invariant semigroup, Cauchy problem for the associated heat equation, and probability measures for a Markov process, i.e. Corollary 16 and Theorem 17 are now interpreted as being on the Mumford curve X, or more precisely, on $X(K)$.

5.3 The Spectral Gap of Heat Operators on Mumford Curves

In order to eliminate the dependence of the spectrum on the volume of the fundamental domain $\mathscr{F}$, we make the following assumption:

Assumption 1 We assume that for a given Mumford curve X, the fundamental domain $\mathscr{F}$ is a holed disc inside O_K containing an element of absolute value 1.

For the Zúñiga operator

$$\mathcal{D}^\alpha_{X(K)} = \mathcal{D}^\alpha_A$$

on $L^2_s(\Omega)^H$, we now assume that A is the combinatorial adjacency matrix of the graph G in diagram (8).

The decomposition of $L^2(\mathscr{F})^H$ as in Theorem 9 allows us to say something about the spectral gap of

$$\mathcal{D}^\alpha_{X(K)} = \mathcal{D}^\alpha_{X(K),1}$$

(i.e. H being the trivial group 1) if X varies over all Mumford curves having the same combinatorial reduction graph.

A graph is called *stable* if every vertex not attached to a loop-edge is attached to at least three edges. By the stable reduction Theorem [21], the stable intersection graph is unique for a given Mumford curve X.

Corollary 19 *Let $\alpha > 0$, and let $M_g(G)$ be the set of isomorphism classes of Mumford curves of genus $g \geq 1$ defined over K having fixed stable reduction graph G. Then, under Assumption 1, for every $\epsilon > 0$, M_G contains a curve $X = \Omega/\Gamma$ such that the spectral gap of the operator $\mathcal{D}^\alpha_{X(K)}$ is smaller than ϵ.*

Proof From Corollary 5, we find that we can look at the quantities

$$\gamma_{i,v} = \sum_{w \in V(G)} \sum_{j=1}^{m_w} A_{vw}\, d(B_{i,v}, B_{j,w})^\alpha$$

where v is a vertex of G, and $B_{i,v}$ as in (1) (where $Z = \mathscr{F}$, the fundamental domain in question) and (2).

For each Mumford curve in $M_g(G)$, choose a fundamental domain $\mathscr{F}_X$ satisfying Assumption 1, such that the reduction trees T_X of $\mathscr{F}_X$ are isomorphic after deleting all vertices of degree 2. These are then spanning trees of the stable graph G. Denote the actual reduction graph of X as G_X. The trees T_X are the trees T_Z considered in the beginning of Sect. 4.1. As they are finite subtrees of the Bruhat–Tits tree, we may now vary X within $M_g(G)$, such that the edges of G_X become longer beyond any bound. Under Assumption 1, this implies that there exists a vertex $v \in V(G_X)$ having neighbours $w \in V(G_X)$ such that

$$D_v(X, \alpha) = \max_{w\colon (v,w) \in E(G_X)} \{d(U_v, U_w)\}$$

becomes smaller than any given bound. Since

$$\gamma_{i,v} \leq (p^f - 1)\deg(v) \cdot D_v(X, \alpha)$$

because each U_w for $w \in V(G_X)$ is a thin holed disc covered by at most $p^f - 1$ maximal subballs, and since $\deg_v$ is bounded (by p^f for fixed K), it follows that for some vertex $v \in G_X$, the quantity $\gamma_{i,v}$ becomes arbitrarily small in this family of Mumford curves. This proves the assertion about the spectral gap, as $-\gamma_{i,v}$ is an eigenvalue of $\mathcal{D}^\alpha_{X(G)}$ by Corollary 5. $\square$

5.4 Degree Eigenvalues of Heat Operators on Tate Curves

It is well-known in graph theory that the spectral gap of the Laplacian of a connected weighted graph, also known as the algebraic connectivity, is bounded from above by the minimal vertex degree. In general, it is not known how the spectrum of a Kronecker product depends on the spectra of its factors. For this reason, we look only at the degree eigenvalues in the case of a Tate curve. From Corollary 19, we already know that the algebraic connectivity of a Mumford curve with fixed stable graph can be arbitrarily small, because there are arbitrarily small degree eigenvalues in such families of curves.

Here, we will exhibit an explicit calculation of the degree eigenvalues

$$\gamma_{i,v}^H = \frac{1}{|H|}\sum_{h\in H}\gamma_{i,v}^h$$

under Assumption 1 for Tate curves. Let $X = K^\times/\langle q\rangle$ with $|q| = p^{-\frac{d}{e}}$ be a Tate curve. A fundamental domain $\mathscr{F} \subset K^\times$ can be chosen of the form

$$\mathscr{F} = \left\{|\pi|_K^d < |x|_K \le 1\right\}$$

so that it satisfies Assumption 1. The finite group H is here taken as the group of order 2 generated by the involution σ on X. This exists, because an elliptic curve has a Legendre equation of the form

$$y^2 = x(x-1)(x-\lambda)$$

with $\lambda \in K \setminus \{1\}$, and the involution σ on X takes a point (x, y) to the point $(x, -y)$.

In fact, with the help of a Möbius transformation fixing the set $\{0, 1, \infty\}$, we may assume that $|\lambda|_K = 1$. It is well-known that this equation gives a Tate elliptic curve, if and only if

$$|\lambda - 1|_K < 1.$$

Let us mention that in the case that K is of positive characteristic, the inequality characterising Tate curves is

$$|\lambda - 1|_K < |2|_K^2$$

according to [31, Example 3.8]. The diagram (7) becomes in our case

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\langle q\rangle"] \arrow[dr, "N"'] & X \arrow[d, "H"] \\ & \mathbb{P}^1 \end{tikzcd}$$

where $\mathbb{G}_m$ is the multiplicative group. Its K-rational points are $\mathbb{G}_m(K) = K^\times$.

We can describe the action of H on $\mathscr{F}$ as follows:

$$\sigma : \mathscr{F} \to \mathscr{F},\ x \mapsto \begin{cases} \frac{x}{q}, & |x|_K = |\pi|_K^k\,,\ k = 1, \ldots, d-1 \\ \frac{1}{x}, & |x|_K = 1 \end{cases}$$

A simple calculation yields that σ is does take $\mathscr{F}$ to itself, and is of order 2.

The annulus $\mathscr{F}$ has a disjoint covering by sets of the form

$$U_k = \left\{ |x|_K = |\pi|_K^k \right\}$$

with $k = 0, \ldots, d-1$, and each circle U_k is covered by $p^f - 1$ balls of radius $|\pi|_K^k$. Let T be the reduction tree of $\mathscr{F}$.

The Zúñiga operator $\mathcal{D}_A^\alpha$ for the combinatorial adjacency matrix of T has the kernel function

$$A(x, y) = \sum_{\substack{k,\ell=0 \\ k \equiv \ell \pm 1 \bmod d}}^{d-1} \sum_{i,j=1}^{p^f-1} d(B_{i,k}, B_{j,\ell})^\alpha\, \Omega(x \in B_{i,k}) \Omega(y \in B_{j,\ell})$$

where

$$B_{i,k} = \left\{ x \in K :\ \left| x - \pi^k \tau(i) \right|_K < |\pi|_K^k \right\}$$

for a given lift $\tau : O_K/\pi O_K \to O_K$.

Lemma 20 *It holds true that*

$$d(B_{i,k}, B_{j,\ell}) = \begin{cases} |\pi|_K^k\,, & k = \ell + 1,\ \ell \in \{0, \ldots, d-2\} \\ |\pi|_K^\ell\,, & k = \ell - 1,\ \ell \in \{1, \ldots, d-1\} \\ 1, & \{k, \ell\} = \{0, d-1\} \end{cases}$$

for $k \equiv \ell \pm 1 \mod d$.

Proof This is a simple calculation. □

Theorem 21 *It holds true that*

$$\gamma_{i,k}^H = \begin{cases} \frac{1}{2}(p^f - 1)\left(|\pi|_K^{k\alpha} + 2\,|\pi|_K^{(d-k)\alpha} + |\pi|_K^{(d-k-1)\alpha} \right), & k \notin \{0, d-1\} \\ 2(p^f - 1), & k = 0 \\ \frac{1}{2}(p^f - 1)\left(|\pi|_K^{\alpha} + |\pi|_K^{(d-2)\alpha} \right), & k = d-1 \end{cases}$$

where $\alpha > 0$.

Proof The involution σ induces the following map

$$\sigma\colon \{0,\dots,d-1\} \to \{0,\dots,d-1\},\ k \mapsto \begin{cases} d-k, & k \in \{1,\dots,d-1\} \\ 0, & k=0 \end{cases}$$

Assume first that $k \notin \{0, d-1\}$. We have

$$\begin{aligned}\gamma_{i,k} &= \sum_{j=1}^{p^f-1} d(B_{i,k}, B_{j,k+1})^\alpha + d(B_{i,k}, B_{j,k-1})^\alpha \\ &= (p^f-1)\left(|\pi|_K^{k\alpha} + |\pi|_K^{(k-1)\alpha}\right)\end{aligned}$$

and, similarly,

$$\gamma_{i,k}^\sigma = \gamma_{i,d-1} = (p^f-1)\left(|\pi|_K^{(d-k)\alpha} + |\pi|_K^{(d-k-1)\alpha}\right).$$

It follows that

$$\gamma_{i,k}^H = \frac{1}{2}(p^f-1)(\gamma_{i,k} + \gamma_{i,k}^\sigma)$$

is as asserted in this case.

Now, assume that $k=0$. Then

$$\gamma_{i,0} = (p^f-1)\cdot 2 = \gamma_{i,0}^\sigma = \gamma_{i,0}^H.$$

And now, assume that $k=d-1$. Then

$$\begin{aligned}\gamma_{i,d-1} &= (p^f-1)\left(1 + |\pi|_K^{(d-2)\alpha}\right) \\ \gamma_{i,d-1}^\sigma &= \gamma_{i,1} = (p^f-1)\left(1 + |\pi|_K^\alpha\right)\end{aligned}$$

and the average $\gamma_{i,d-1}^H$ is asserted. □

Acknowledgements We are indebted to Wilson Zúñiga-Galindo for presenting the article [7] at the p-adics.2019 conference in Covilhã, Portugal, and for further fruitful discussions towards attempts in constructing integral operators on p-adic manifolds, as well as for straightening out the nomenclature. Also David Weisbart is to be credited for inciting discussions about topics in p-adic analysis, where my attention was brought towards the major helpful idea towards constructing Zúñiga operators on Mumford curves, and also for suggestions for this article. Àngel Moràn Ledezma is thanked for inspiring discussions. We thank the anonymous referees for helpful suggestions, in particular one of them for suggesting on how to extend the results and simultaneously simplify the exposition significantly. This research is partially supported by the Deutsche Forschungsgemeinschaft (DFG) under project number 469999674.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Taibleson, M.H.: Fourier Analysis on Local Fields. Princeton University Press, Princeton, NJ (1975)
2. Saloff-Coste, L.: Opérateurs pseudo-différentiels sur un corps local. C. R. Acad. Sci. Paris Sér. **I**(297), 171–174 (1983)
3. Vladimirov, V.S., Volovich, I.V., Zelenov, E.I.: p-adic Analysis and Mathematical Physics. Series on Soviet and East European Mathematics, vol. 1. World Scientific Publishing Co., Inc., River Edge, NJ (1994)
4. Chacón-Cortés, L.F., Zúñiga-Galindo, W.A.: Heat traces and spectral zeta functions for p-adic Laplacians. St. Petersburg Math. J. **29**, 529–544 (2018)
5. Bradley, P.E.: Generalised diffusion on moduli spaces of p-adic Mumford curves. p-Adic Numbers Ultrametric Anal. Appl. **12**(2), 73–89 (2020)
6. Albeverio, S., Kozyrev, S.V.: Pseudodifferential p-adic vector fields and pseudodifferentiation of a composite p-adic function. p-Adic Numbers Ultrametric Anal. Appl. **2**(1), 21–34 (2010)
7. Zúñiga-Galindo, W.A.: Reaction-diffusion equations on complex networks and Turing patterns via p-adic analysis. J. Math. Anal. Appl. **491**(1), 124239 (2020)
8. Bradley, P.E.: p-Adic wave equations on finite graphs and T_0-spaces. In: Zuniga, W., Toni, B. (eds.) Advances in Non-Archimedean Analysis and Applications. STEAM-H, pp. 275–295. Springer, New York (2021)
9. Zúñiga-Galindo, W.A.: The non-Archimedean stochastic heat equation driven by Gaussian noise. J. Fourier Anal. Appl. **21**, 600–627 (2015)
10. Khrennikov, A.Y., Kozyrev, S.V.: Pseudodifferential operators on ultrametric spaces and ultrametric wavelets. Izvestiya **69**(5), 989–1003 (2005)
11. Kozyrev, S.V.: Wavelet theory as p-adic spectral analysis. Izv. Math. **66**(2), 367–376 (2002)
12. Khrennikov, A.Y., Shelkovich, V.M., van der Walt, J.H.: Measure-free viewpoint on p-adic and adelic wavelets. P-Adic Numbers Ultrametric Anal. Appl. **5**, 204–217 (2013)
13. Khrennikov, A.Y., Shelkovich, V.M., van der Walt, J.H.: Adelic multiresolution analysis, construction of wavelet bases and pseudo-differential operators. J. Fourier Anal. Appl. **19**, 1323–1358 (2013)
14. Volovich, I.V.: Number theory as the ultimate physical theory. CERN-TH.4781/87, Geneva. p-Adic Numbers Ultrametric Anal. Appl. **2**(1), 77–87 (2010) (1987)
15. Dragovich, B., Khrennikov, A.Y., Kozyrev, S.V., Volovich, I.V.: On p-adic mathematical physics. p-Adic Numbers Ultrametric Anal. Appl. **1**(1), 1–17 (2009)
16. Khrennikov, A., Oleschko, K., López, M.J.C.: Application of p-adic wavelets to model reaction diffusion dynamics in random porous media. J. Fourier Anal. Appl. **22**, 809–822 (2016)
17. Khrennikov, A., Oleschko, K., López, M.J.C.: Modeling fluid's dynamics with master equations in ultrametric spaces representing the treelike structure of capillary networks. Entropy **18**(7), 249 (2016)
18. Khrennikov, A., Oleschko, K., Correa Lopez, M.J.: p-adic analog of Navier-Stokes equations: dynamics of fluid's flow in percolation networks (from discrete dynamics with hierarchic interactions to continuous universal scaling model). Entropy **19**, 161 (2017)
19. Torresblanca-Badillo, A., Zúñiga-Galindo, W.A.: Ultrametric diffusion, exponential landscapes, and the first passage time problem. Acta Appl. Math. **157**(1), 93–116 (2018)
20. Mumford, D.: An analytic construction of degenerating curves over complete local rings. Compos. Math. **24**(2), 129–174 (1972)
21. Deligne, P., Mumford, D.: The irreducibility of the space of curves of a given genus. Publ. Math. I.H.E.S. **36**, 79–109 (1969)

22. Gerritzen, L., van der Put, M.: Schottky Groups and Mumford Curves. Lecture Notes in Mathematics, vol. 817. Springer, Heidelberg (1980)
23. Fresnel, J., van der Put, M.: Rigid Analytic Geometry and Its Applications. Progress in Mathematics, vol. 218. Birkhäuser, Boston, MA (2004)
24. Khrennikov, A.Y., Kochubei, A.N.: p-Adic analogue of the porous medium equation. J. Fourier Anal. Appl. **24**, 1401–1424 (2018)
25. Antoniouk, A.V., Khrennikov, A.Y., Kochubei, A.N.: Multidimensional nonlinear pseudo-differential evolution equation with p-adic spatial variables. J. Pseudo-Differ. Oper. Appl. **11**, 311–343 (2020)
26. Bradley, P.E.: Degenerating families of dendrograms. J. Classif. **25**, 27–42 (2008)
27. Khrennikov, A., Kozyrev, S., Zúniga-Galindo, W.A.: Ultrametric Pseudodifferential Equations and Its Applications. Encyclopedia of Mathematics and Its Applications, vol. 168. Cambridge University Press, Cambridge (2018)
28. van der Put, M.: Discrete groups, Mumford curves and theta functions. Annales de la faculté des sciences de Toulouse 6 e série **1**(3), 399–438 (1992)
29. Ethier, S.N., Kurtz, T.G.: Markov Processes - Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics. Wiley, New York (1986)
30. Roquette, P.: Analytic Theory of Elliptic Functions over Local Fields. Hamburger Math. Einzelschriften, Neue Folge, Heft 1. Vandenhoeck & Ruprecht, Göttingen (1970)
31. Bradley, P.E.: Cyclic coverings of the p-adic projective line by Mumford curves. Manuscr. Math. **124**, 77–95 (2007)