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On nonlinear Landau damping and Gevrey regularity

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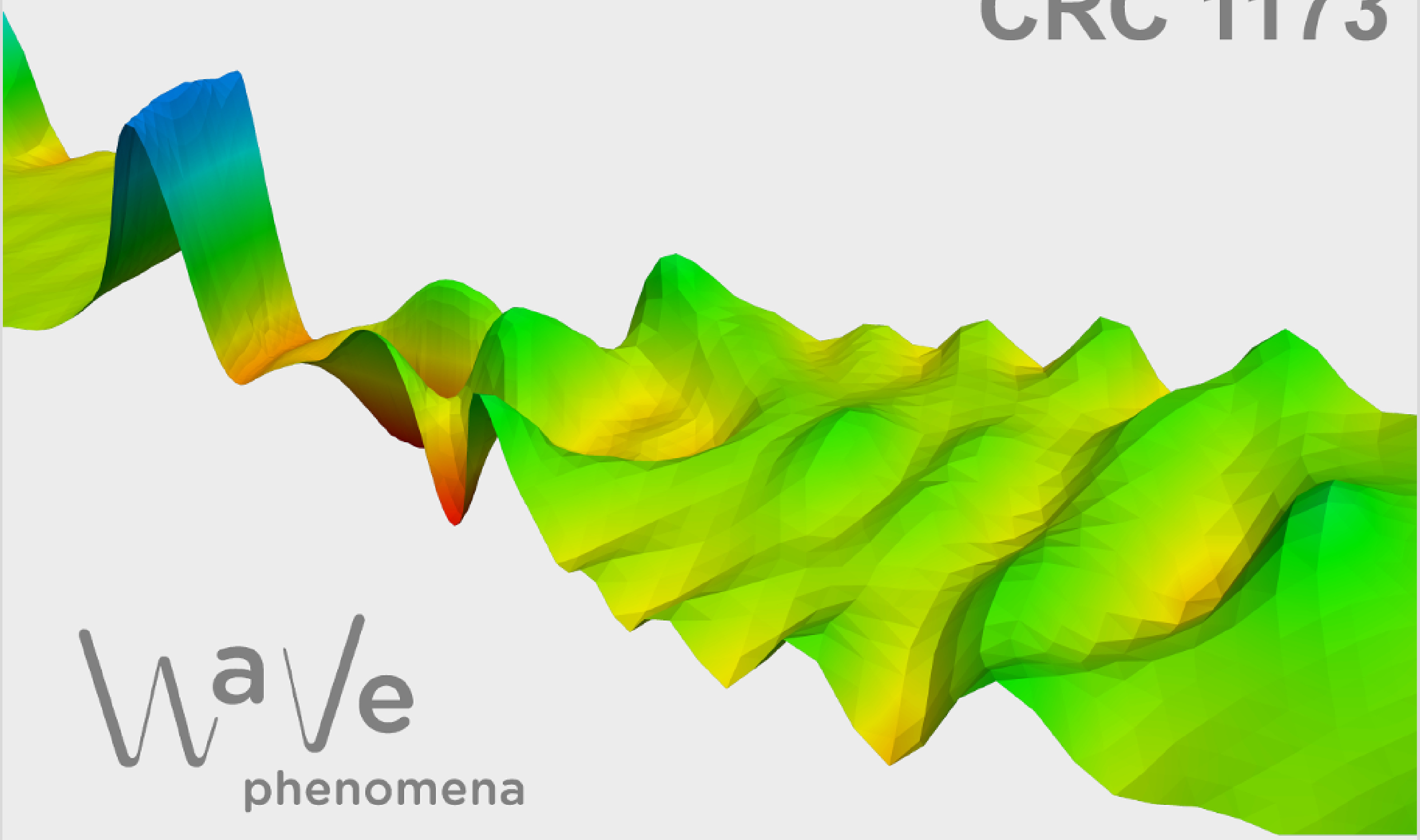

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ON NONLINEAR LANDAU DAMPING AND GEVREY REGULARITY

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Abstract. In this article we study the problem of nonlinear Landau damping for the Vlasov-Poisson equations on the torus. As our main result we show that for perturbations initially of size $\epsilon > 0$ and time intervals $(0, \epsilon^{-N})$ one obtains nonlinear stability in regularity classes larger than Gevrey 3, uniformly in ϵ. As a complementary result we construct families of Sobolev regular initial data which exhibit nonlinear Landau damping. Our proof is based on the methods of Grenier, Nguyen and Rodnianski [GNR20].

Contents

1. Introduction

In this article we consider the nonlinear stability problem for the Vlasov-Poisson equations

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \\ &(t,x,v) \in \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d, \\ &\rho(t,x) = \int f(t,x,v) dv, \\ &F(t,x) = \nabla \Delta^{-1} \rho(t,x), \\ &(t,x,v) \in (0, \epsilon^{-N}) \times \mathbb{T} \times \mathbb{R}, \end{aligned} \tag{1}$$

for *finite*, but very large times and for perturbations initially of size $\epsilon > 0$. For simplicity of presentation we restrict to perturbations around the zero solution $f(t,x,v) = 0$. Here $f(t,x,v) \in \mathbb{R}$ models the phase space-density of a plasma and $F(t,x) \in \mathbb{R}$ corresponds to a mean-field electric force field generated by the spatial density $\rho(t,x) \in \mathbb{R}$.

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The study of the long-time behavior of the Vlasov-Poisson equations and, in particular, the phenomenon of Landau damping (decay of $F(t,x)$ as $t \to \infty$ at very fast rates), is a very active field of research. For an overview we refer to the seminal works of Villani and Mouhot [MV11, MV10, BMM16] who for the first time established nonlinear stability and Landau damping.

Due to nonlinear resonances [MWGO68] these results require extremely strong regularity assumptions. Indeed in [Bed20] (see also [Zil21] and [GNR22]) it is shown that the nonlinear equations exhibit chains of resonances and associated growth by

$$\exp(|\epsilon\eta|^{1/3})$$

for perturbations frequency localized at η (with respect to v). Hence no uniform (in time) stability results can be expected to hold in weaker than Gevrey 3 regularity (that is, L^2 spaces with such an exponential decay in Fourier space). Nevertheless, as shown in [Zil21] in principle the "physical notion" of Landau damping (that is, $F(t,x)$ decays in time) does not require stability of $f(t,x,v)$. In particular, that notion of damping might be more "robust" than suggested by the high regularity requirements and suggests that Landau damping might persist even at lower than Gevrey regularity of perturbations.

As a first step towards such a result in this article we adapt the methods of [GNR20] and ask the following questions:

- If we consider perturbations initially of size $0 < \epsilon \ll 1$, how does the possible (optimal) norm inflation depend on ϵ?
- Can the methods of [GNR20] be modified to reach optimal Gevrey classes?
- If we consider a finite time interval

$$(0, \epsilon^{-N}),$$

  how does this change upper and lower bounds and can we establish stability in better than Gevrey 3 regularity uniformly in ϵ?

This coupling between the size of the perturbation and the time scale is motivated by works on echo chains in the inviscid Boussinesq equations [BBCZD21, Zil22], where one naturally is restricted to a time scale $(0, \epsilon^{-2})$.

Our main results are summarized in the following theorem.

Theorem 1.1. *Let $0 < \epsilon \ll 0.1$ and $T = \epsilon^{-N}$. Then there exists $\gamma = \gamma(N)$ independent of ϵ such that the nonlinear Vlasov-Poisson equations are stable in Gevrey $\frac{1}{\gamma} > 3$.*

More precisely, there exists $\beta \geq \frac{1}{12}$ and $C > 0$ such that if the Fourier transform of initial data satisfies

$$\sum_k \int \langle k,\eta\rangle^8 \exp(C\log(T)\min(\epsilon^\beta \langle k,\eta\rangle^{1/3}, \langle k,\eta\rangle^\gamma)) \tag{2}$$

$$(|\mathcal{F}|f_0(k,\eta)|^2 + |\partial_\eta \mathcal{F} f_0(k,\eta)|^2)d\eta \leq \frac{1}{100}\epsilon, \tag{3}$$

then the bound (2) remains true for $g(t,x,v) = f(t,x-tv,v)$ for all times $t \in (0,T)$ up to a loss in the constants C and $\frac{1}{100}$. Moreover, on that same time interval the force field perturbation satisfies the bound

$$|F(t,x)| \lesssim \epsilon \exp(-\epsilon^\beta t^\gamma).$$

We in particular emphasize the following differences and improvements compared to [GNR20]:

- In [GNR20], whose methods we adapt, a similar result is established with
$$\exp(C\langle k,\eta\rangle^{1/3+\delta}); \delta>0,$$
instead and T is allowed to be infinite.
- The present result reaches $\frac{1}{3}$ ($\delta=0$) at the cost of time-dependent prefactor $\log(T)$. However, for $T=\epsilon^{-N}$ this can be absorbed into a slight loss of β (and γ).
- As our main novelties we highlight the frequency cut-off and the improved Gevrey classes. To the author's knowledge this is the first nonlinear Landau damping result (for finite time) for *generic small data* in sub Gevrey 3 regularity.
- As we discuss in the following Lemma 1.2, it is easy to construct special data at arbitrarily low regularity which exhibits Landau damping as $t\to\infty$. However, this data needs to satisfy rather restrictive Fourier support assumptions and is unstable as $t\to-\infty$. As seen from the norm inflation results of [Bed20, Zil21] for generic data some level of Gevrey regularity is necessary for uniform in time stability.
- As we discuss in Section 2, a model for plasma echoes suggests that optimal bounds should be given by $\beta=\frac{1}{3}$ and $\gamma_N=\frac{1}{3}\frac{3N-2}{3N-1}$. However, since the current method of proof not only requires solutions to remain bounded but to decay with an integrable rate $(1+|t|)^{-\sigma+2}$, this restricts us to slightly smaller values of β and hence larger values of $\gamma=\frac{1}{3}-\frac{\beta}{2N}$.

As an independent result, the following lemma constructs examples of "trivial" solutions, which are only Sobolev regular but nevertheless exhibit Landau damping (see also [GNR22]).

Lemma 1.2. *Let $\psi\in\mathcal{S}(\mathbb{R})$ be a Schwartz function whose Fourier transform is supported in a ball of radius 0.1 around 0. Then for any $s\geq 0$ and any sequence $(c_k)_k\in\ell^2(\mathbb{N})$ the function*

$$f(t,x,v)=\Re\sum_{k\geq 1}c_k(1+k^2)^{\frac{s}{2}}e^{ikx}e^{i(-k-kt)v}\psi(v) \tag{4}$$

is an element of the Sobolev space $H^s(\mathbb{T}\times\mathbb{R})$ for all times $t>0$ and is a solution of the Vlasov-Poisson equations for $t\in(0,\infty)$. Moreover, $f(t,x+tv,v)=f(0,x,v)$ is independent of time and $F(t,x)=0$. Hence this solution trivially exhibits both scattering and Landau damping.

Proof of Lemma 1.2. We note that by the assumption on the Fourier support of ψ, the Fourier transform of the functions

$$(1+k^2)^{\frac{s}{2}}e^{ikx}e^{i(-k-kt)v}\psi(v)$$

is supported in a ball of size 0.1 around the frequency $(k,-k-kt)$. Since $k\geq 1$ and $t>0$, it follows that these supports are disjoint for different values of k and hence for $\psi\neq 0$ the series is convergent in H^s if and only if $(c_k)_k\in\ell^2(\mathbb{N})$. Furthermore, since $k + kt \geq 1 > 0.1$ it follows that the spatial density $\rho(t, x)$ satisfies

$$\begin{aligned}\rho(t,x) &= \int f(t, x + tv, v)dv = \int \Re \sum_{k\geq 1} c_k e^{ikx} e^{i\eta_k v}\psi(v) \\ &= \Re \sum_{k \geq 1} e^{ikx}\mathcal{F}(\psi)(\eta_k) = 0\end{aligned}$$

and hence also $F(t, x) = \nabla\Delta^{-1}\rho = 0$ is trivial. Thus the Vlasov-Poisson equations reduce to the free transport equations, which $f(t, x, v)$ solves by construction. □

Such "traveling wave" solutions also form the core of nonlinear instability results [Zil21, Bed20], where small high-frequency perturbations of these waves exhibit norm inflation. More generally one can consider a factor $e^{i(-\eta_k - kt)v}$ with $\eta_k > 0.1$ positive and increasing in k. However, we emphasize that the behavior changes drastically when this sign condition is allowed to be violated or if we require Landau damping for both $t \to \infty$ and $t \to -\infty$. In these cases one needs to require c_k to decay sufficiently rapidly to satisfy the assumptions of Theorem 1.1.

The remainder of this article is structured as follows:

- In Section 2 we briefly discuss the plasma echo mechanism in terms of a toy model, which exhibits exactly the growth expressed in Theorem 1.1.
- In Section 3 we recall the generator function method of [GNR20]. Here the added ϵ dependence and the cut-off constitute the main new effects and require more precise estimates. Furthermore, we restructure the proof to highlight the role of traveling waves and establish improved bounds above a frequency cut-off.
- The estimates use a bootstrap approach. Here the most important step is given by Subsection 4 establishing improved control of ρ assuming control of f.
- Conversely, Subsection 5 establishes improved control of f given control of ρ. Both conditional results are then combined to prove Theorem 1.1.

2. Plasma echoes and two heuristic models

In this section we briefly discuss the main norm inflation mechanism of the Vlasov-Poisson equations, known as *plasma echoes* and the effects of a time cut-off. These resonances are also experimentally observed [MWGO68]. The interested reader is referred to the seminal works of Villani, Mouhot and Bedrossian [MV10, Bed20, Zil21] for a more detailed discussion.

As a heuristic model, let $\eta \in \mathbb{R}$ and $k \in \mathbb{Z}$ be given and let $\psi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function as in Lemma 1.2. Then by a similar argument as in Lemma 1.2 the function

$$f(t, x, v) = \epsilon \cos(x - tv)\psi(v) + \epsilon \sin(kx + (\eta - kt)v)\psi(v)$$

is a solution of nonlinear Vlasov-Poisson equations (1) for all times t such that $|t| > 0.1$ and $|\eta - kt| > 0.1$. We refer to both summands as (traveling) waves. According to the linearized dynamics around 0 (that is, the free transport dynamics), both waves do not interact and exhibit weak convergence in L^2 as $t \to \infty$. However, when considering the full nonlinear problem with the same initial data at time $t = 0.1$, the nonlinearity $F \cdot \nabla_v f$ introduces a correction around the time $t : |\eta - kt| < 0.1$.

For our model problems we insert different waves for f and $F[\rho]$, which corresponds to considering parts of the first Duhamel iteration. We thus obtain a correction involving a time integral of

$$F[\int \epsilon \sin(kx + (\eta - kt)V)\psi(V)dV] \cdot \nabla_v \epsilon \cos(x - tv)\psi(v) \tag{5}$$

and

$$F[\int \epsilon \cos(x - tV)\psi(V)dV]\nabla_v \epsilon \sin(kx + (\eta - kt)v)\psi(v),$$

respectively. In the naming of [BMM16, MV10] these are model problems for the "reaction" and "transport" terms.

We begin by considering the model associated to (5). Changing to coordinates $(x - tv, v)$, taking a Fourier transform and inserting the choice of $F = \nabla\Delta^{-1}\rho$, one deduces (see [MV10]) that the correction is estimated to be of the size

$$\epsilon^2 \frac{|\eta|}{|k|^3} \|\mathcal{F}\psi\|_{L^1},$$

is localized to the frequency $k \pm 1$ in x and η in v and occurs at around the time $\frac{\eta}{k}$. This time-localized, large correction is the physically observed *echo*.

Furthermore, in principle it could happen that this correction in turn results in another correction at the later time $\frac{\eta}{k-1}$, then at the time $\frac{\eta}{k-2}$ and so on. One thus obtains the upper estimate of the toy model of [MV11] by

$$\prod_{l=1}^{k} \epsilon \frac{|\eta|}{|l|^3},$$

which is maximized for $k \approx \sqrt[3]{\epsilon|\eta|}$ and suggests an upper bound on the norm inflation by

$$\sup_k \prod_{l=1}^{k} \epsilon \frac{|\eta|}{|l|^3} \approx \exp(\sqrt[3]{\epsilon|\eta|}). \tag{6}$$

While this toy model only suggests an upper bound for stability results, in [Bed20] Bedrossian showed that such resonance chain can indeed occur in the nonlinear problem (with certain constraints on ϵ and η). Moreover, in [Zil21] for a related model (linearizing around the *wave solution* $\epsilon \cos(xe_j - tv)\psi(v)$) we showed that infinitely many such resonance chains can lead to norm blow-up as $t \to \infty$ in any regularity class below Gevrey 3.

However, we emphasize that for resonance to provide a large contribution, two competing conditions have to be satisfied (see also [Zil22]):

- One the one hand, the frequency l should satisfy an *upper bound* such that $\epsilon \frac{|\eta|}{|l|^3} \gtrsim 1$ to actually yield a large correction.
- On the other hand, the frequency l needs to satisfy a *lower bound* so that the resonant time $\frac{\eta}{l}$ occurs before the time cut-off T.

Thus, when considering a finite time interval $(0, T)$ the heuristic estimate (6) should be modified to

$$\prod_{\frac{\eta}{T} \leq l \leq \sqrt[3]{\epsilon\eta}} \epsilon \frac{|\eta|}{|l|^3} \lesssim \begin{cases} \exp(\sqrt[3]{\epsilon|\eta|}) & \text{if } \frac{|\eta|}{\sqrt[3]{\epsilon|\eta|}} \leq T, \\ 1 & \text{else.} \end{cases}$$

The transition between these cases occurs at the cut-off frequency $\eta_* = \sqrt{\epsilon T^3}$.

In particular, for this toy model and $T = \epsilon^{-N}$ we thus obtain that for all $|\eta| \leq \eta_* = \epsilon^{-3N+1}$,

$$\epsilon|\eta| \leq |\eta|^{\frac{3N-2}{3N-1}}.$$

The norm inflation in this model can thus be estimated by

$$\exp(|\min(\eta, \eta_*)|^{\gamma_N})$$

with

$$\gamma_N = \frac{1}{3}\frac{3N-2}{3N-1},$$
$$\eta_* = \epsilon^{-\frac{3N-1}{2}}.$$

A major aim of Theorem 1.1 is to establish that such a cut-off effect holds also for the full nonlinear problem. However, in view of technical challenges we here allow for different cut-offs and different powers of ϵ. As we discuss in Section 4 and Section 5 this does not seem to be just a technical issue. More precisely, while for the frequency regimes corresponding to this first model it seems possible to reach exactly these powers, other frequency regimes pose greater challenges, which we illustrate in the following model.

In our second model we consider the contribution by

$$\begin{aligned} &F[\int \epsilon\cos(x-tV)\psi(V)dV]\nabla_v \epsilon\sin(kx+(\eta-kt)v)\psi(v) \\ &\approx \epsilon\tilde{\psi}(t)\cos(x)\epsilon(\eta-kt)\cos(kx+(\eta-kt)v)\psi(v), \end{aligned} \tag{7}$$

where $\tilde{\psi}$ denotes the Fourier transform of ψ with respect to v. Hence for frequencies η much larger than kt, this contribution suggests that $g(t,x,v) := f(t, x-tv, v)$ should behave as a solution of

$$\partial_t g \approx \epsilon\tilde{\psi}(t)\cos(x-tv)\partial_v g.$$

We stress that this is not an effect one would see for $\rho = \int f dv$, since there $\eta \approx kt$ due to the velocity integral.

Even assuming that ψ is very smooth and hence that $\tilde{\psi}(t)$ is decaying rapidly, this transport type equation poses great challenges for estimates since ∂_v is an unbounded operator. In particular, while for analytic regularity this contribution could be easily "hidden" in a loss of constant (that is, consider a weight $\exp(z(t)\langle\eta\rangle)$ with $-\partial_t z \gg \tilde{\psi}(t)$) any weaker Gevrey class will have to account for the fact that z-derivative a priori only gains fractional regularity in v. Hence, in Section 5 we need to exploit that ∂_v is an anti-symmetric operator on L^2 and that corresponding commutators in our L^2 based Gevrey spaces provide a "gain" of one derivative.

The requirements highlighted by both of these models and the fact that the method of proof not only requires that ρ remains bounded, but rather

$$\|\rho(t,x)\| \leq \epsilon(1+|t|)^{-\sigma+1}$$

for a given power $\sigma > 3$ determines our cut-off and ϵ dependences in Theorem 1.1.

In the remainder of the article we to adapt the method of proof of [GNR20] to establish non-linear stability estimates and incorporate these effects.

3. Generator functions and cut-offs

In our proof we follow the method of [GNR20] with (major) modifications to account for the frequency cut-off and the ϵ dependence. We briefly discuss the overall strategy of the proof and state the main estimates as propositions, which we use to establish Theorem 1.1. The proofs of these propositions is given in Sections 4 and 5.

Considering the structure of the Vlasov-Poisson equations (1), we study solutions in coordinates moving with free transport and denote

$$g(t,x,v) = f(t,x+tv,v).$$

Then the Vlasov-Poisson equations can be equivalently expressed as a coupled system for ρ and g:

$$\begin{aligned} \partial_t g &= -F[\rho](t,x+tv)\cdot(\nabla_v + t\nabla_x)g, \\ \rho &= \int g(t,x-tv,v)dv, \\ F &= \nabla W *_x \rho, \\ \partial_t \rho &= -\int F(t,x)(\nabla_v + t\nabla_x)g(t,x-tv,v)dv. \end{aligned} \tag{8}$$

Similarly to usual Cauchy-Kowaleskaya approaches using time-dependent Fourier multipliers [BMM16], one further introduces two parameter dependent energy functionals, where at a later stage the parameter will also be chosen depending on time.

Definition 3.1 (Generator functions (c.f. [GNR20]))**.** *Let $0 < \epsilon \ll 1$, $C > 0$, $\sigma > 3$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $\eta_* \gg 1$ be given constants. We further introduce the short-hand-notation*

$$\ulcorner k,\eta \urcorner = \begin{cases} \epsilon^{\beta}(1+k^2+\eta^2)^{\frac{1}{6}} & \text{if } |k|+|\eta| \leq \eta_*, \\ \epsilon^{\beta'}(1+k^2+\eta^2)^{\frac{\gamma}{2}} & \text{if } |k|+|\eta| \geq 2\eta_*, \end{cases}$$

with a smooth interpolation in the remaining region. We refer to these cases as below the cut-off *and* above the cut-off*, respectively. Here the constant are chosen under the constraints*

$$\begin{aligned} 0 \leq \beta &\leq \frac{1}{3\sigma}, \\ 0 \leq \beta' &\leq \frac{\gamma}{\sigma}, \\ \frac{1-\alpha}{2} \leq \gamma &\leq \frac{1}{3}, \\ \epsilon^{\beta'-\beta} &= \eta_*^{\frac{1}{3}-\gamma}, \\ \eta_* &\geq T^2. \end{aligned} \tag{9}$$

Then for given functions g and ρ and a given parameter $\gamma > 0$, for any $z \geq 0$ we define the (possibly infinite) energies

$$E_1(z) = \| \exp(Cz \ulcorner k,\eta \urcorner^{1/3}) \langle k,\eta\rangle^{\sigma} (\tilde{g}, \partial_\eta \tilde{g})\|^2_{L^2(\mathbb{Z}\times\mathbb{R})} \tag{10}$$

and

$$E_2(z) = \| |k|^{-\alpha} \exp(Cz \ulcorner k,kt \urcorner^{1/3}) \langle k,kt\rangle^{\sigma} \tilde{\rho}\|_{l^\infty(\mathbb{Z})}, \tag{11}$$

where $\tilde{g}(k,\eta)$ and $\tilde{\rho}(k)$ denote the respective Fourier transforms.

For the proof of Theorem 1.1 with $T = \epsilon^{-N}$ our constants are chosen as

$$\begin{aligned}\eta_* &= \epsilon^{-2N},\\ \beta &= \frac{1}{3\sigma},\\ \beta' &= 0,\\ \gamma &= \frac{1}{3} - \frac{\beta}{2N}.\end{aligned}$$

We thus obtain stability in the Gevrey class $\frac{1}{\gamma} > 3$ uniformly in $0 < \epsilon \ll 1$.

We remark that in [GNR20] stability is established with an exponential weight

$$\exp(z\langle k,\eta\rangle^{1/3+\delta})$$

for $\delta > 0$. These new generator functions introduce an improved exponent $1/3$, the gain of a factor ϵ^β and, most importantly, an improved sub $\frac{1}{3}$ growth past a cut-off.

Our main aim in the following is to show that for the above choices of constants one may find $z(t) \geq 1$ such that, if initially

$$\sqrt{E_1} + E_2 \lesssim \epsilon,$$

then this estimate remains valid for all times smaller than T.

More precisely, we claim that E_1 satisfies the following estimate.

Proposition 3.2. *Let g, ρ be a solution of (8) and let η_* and β be as in Theorem 1.1. Let further $E_1(z), E_2(z)$ denote the now time-dependent generator functions as in Definition 3.1. Then there exist universal constants $C_1, C_2 > 0$ such that for all times $0 < t < T$ it holds that*

$$\partial_t E_1(z) \leq C_1 E_2(z) E_1(z) + C_2 \epsilon^{-\beta} C^{-1}(1+t) E_2(z) \partial_z E_1(z).$$

This result is analogous to [GNR20, Proposition 4.1] but considers our modified generator functions and hence include the improved ϵ dependence and frequency cut-off. The proof of this proposition is given in Section 4. Assuming these results for the moment, we note that if we can show that

$$C_2 \epsilon^{-\beta} C^{-1}(1+t) E_2(z) \leq \frac{1}{1+t}, \tag{12}$$

then choosing

$$z(t) = 2\log(T) - \log(t),$$

it follows that

$$\begin{aligned}&\frac{d}{dt} E_1(z(t)) \leq C_1 E_2(z(t)) E_1(z(t))\\ &\rightsquigarrow E_1(z(t)) \leq E_1(z(t))|_{t=0} \exp\Big(\int_0^t E_2 ds\Big) \lesssim E_1(z(t))|_{t=0}.\end{aligned} \tag{13}$$

Thus, the main challenge to the proof of Theorem 1.1 is given by establishing a suitable decay bound on $E_2(z(t))$ with this choice of $z(t)$.

Proposition 3.3. *Let E_1, E_2 be as in Proposition 3.2. Furthermore, choose z to be time-dependent as*

$$z(t) = 2\log(T) - \log(t)$$

and suppose that on a time interval $[0, t_*] \subset [0.T]$ *it holds that*

$$E_1(z(t)) \leq 16\epsilon^2. \tag{14}$$

Then on that same time interval we have the estimate

$$\begin{aligned}(1+t)^{\sigma-1}E_2(z(t)) \leq \epsilon(1+t)^{\sigma-1} \\ + c \sup_{0\leq s\leq t}(1+s)^{\sigma-1}E_2(z(s)),\end{aligned}$$

where $c > 0$ *only depends on the constant* C *in Definition 3.1 and* $c < 1/2$ *if* C *is sufficiently large.*

This estimate is similar to [GNR20, Lemma 4.4] with the following key differences:

- The frequency transition $\ulcorner \cdot, \cdot \urcorner$ and the associated cut-off constitute a main new effect.
- We here reach $1/3$ in the exponent. However, in turn the estimates only remain valid while $z(t)$ remains bounded below, which requires restricting to times smaller than T.
- An analogous result can be established for $T \leq \infty$ with $\partial_t z(t) \approx (1+t)^{-1-\delta}$ and exponents $\ulcorner k, \eta \urcorner^{1/3+\delta}$ (and smaller β) instead.
- In comparison to [GNR20], the additional ϵ dependence of E_1, E_2 implies a loss of powers of ϵ in some estimates. In particular, choosing β, β' maximally under the constraints (9), choosing ϵ small does not yield any further improvements. Hence, we need to carefully estimate all terms and establish a bound by $c < 1$ uniformly in $\epsilon, t, \eta_*(\epsilon, N)$.

Given these result, we can establish Theorem 1.1.

Proof of Theorem 1.1. By assumption it holds that at time 0,

$$\begin{aligned}E_1|_{t=0} &\leq \epsilon^2, \\ E_2|_{t=0} &\leq \epsilon.\end{aligned}$$

In particular, at that time the estimates (14) and (12) are satisfied with improved constants. Thus, by continuity these estimates remain true at least for some positive time. We hence define $0 < t_* \leq T$ as the maximal time such that

$$\begin{aligned}E_1 &\leq 16\epsilon^2, \\ E_2 &\leq 4\epsilon(1+t)^{-\sigma+1},\end{aligned}$$

holds for all times $0 \leq t \leq t_*$.

If $t_* = T$, then Theorem 1.1 immediately follows from the results of Propositions 3.3 and the discussion following Proposition 3.2. Thus suppose for the sake of contradiction, that the maximal time t_* is strictly smaller than T. Then by the estimates of Proposition 3.3 at the time t_* it holds that

$$E_2(t) \leq \frac{1}{1-c}\epsilon(1+t)^{-\sigma+1} \leq 8\epsilon(1+t)^{-\sigma+1},$$

where we used that c is sufficiently small. Thus equality in (12) is not attained.

Similarly, by the results of Proposition 3.2 and (13) it follows that

$$E_1 \leq \epsilon^2(1 + \int_0^t c(1+s)^{-\sigma+2} ds) \leq 8\epsilon^2.$$

Hence, equality is also not attained at time t_* for (14). Thus, by by continuity the estimates (12), (14) remain valid at least for a small additional time past t_*. However, this contradicts the maximality of t_* and thus it is not possible that $t_* < T$, which concludes the proof. □

It thus remains to establish the bounds for E_1 and E_2 as claimed in Propositions 3.2 and 3.3. We emphasize that (except for the cut-off) the estimate for E_1 follows the same abstract and rather rough argument as in [GNR20], which requires $\sigma > 3$. In contrast the proof of the estimates of E_2 could in principle be modified to allow $\sigma \geq 1$ and to match the bounds of the echo chains discussed in Section 2. We expect that using methods closer to the ones of [BMM16] it should be possible to obtain the case $\sigma = 1$ also for E_2, however the method of [GNR20] seems to require that $\sigma > 3$.

4. Control of ρ

In this section we assume that $E_1(z(t))$ remains small as stated in (14):

$$E_1(z(t)) \leq 16\epsilon^2,$$

and estimate the possible norm growth of $\rho(t)$.

We may thus consider $g(t,x,v)$ as given in (8) and study the evolution equation for ρ:

$$\partial_t \rho = -\int F(t,x)(\nabla_v + t\nabla_x) g(t, x-tv, v) dv.$$

Here we note that $F = F[\rho]$ is given by a Fourier multiplier

$$\mathcal{F}F(t,l) = \frac{1}{il}\tilde{\rho}(t,l)$$

and that multiplication turns into a (discrete) convolution under a Fourier transform. We may thus equivalently express our equation in integral form as

$$\tilde{\rho}(t,k) = \tilde{g}(0,k,kt) + \sum_{l \neq 0} \int_0^t \frac{k(t-s)}{l} \tilde{\rho}(s,l)\tilde{g}(s,k-l,kt-ls) ds. \tag{15}$$

As discussed in the echo model of Section 2, here our assumed control of g is rather weak if $k-l$ is small and $kt - ls \approx 0$. Indeed, at a heuristic level we may only expect estimates of the form

$$|\tilde{g}(s,k-l,kt-ls)| \lesssim \epsilon(1+|kt-ls|^2)^{-1},$$

whose time integral is estimated by $\frac{\epsilon}{l}$. Furthermore, inserting the assumption that $kt - ls \approx 0$ and denoting $\eta := kt$, the first factor can be estimated by

$$\frac{k(t-s)}{l} \approx \frac{(l-k)s}{l} \approx (l-k)\frac{\eta}{l^2}.$$

Hence, we again arrive at the estimate of the growth factor by $\epsilon\frac{\eta}{l^3}$ (for $l-k=1$) as in (6). In particular, using that $l \approx k = \frac{\eta}{t}$ this factor is much smaller than 1 if η is very large, which will allow us to introduce a cutoff in η.

For ease of reference, we also note some general techniques to be used throughout the proof:

- The cut-off weight of Definition 3.1
$$\ulcorner k,\eta \urcorner = \begin{cases} \epsilon^{\beta}\langle k,\eta\rangle^{1/3}, \\ \epsilon^{\beta'}\langle k,\eta\rangle^{\gamma} \end{cases}$$
satisfies a triangle inequality
$$\ulcorner k,\eta \urcorner \leq \ulcorner k-l,\eta-\xi \urcorner + \ulcorner l,\xi \urcorner .$$
Moreover, we emphasize that due to the exponents $0<\gamma\leq 1/3$ the right-hand-side in general is much larger than the left-hand-side unless (l,ξ) (or $(k-l,\eta-\xi)$) is small compared to (k,η).
- Let $C_1>0$, then for k,η it holds that
$$\exp(-C_1 \ulcorner k,\eta \urcorner) \leq \begin{cases} C_1^{-3}\epsilon^{-3\beta}\langle k,\eta\rangle^{-1} & \text{below the cut-off,} \\ C_1^{-\frac{1}{\gamma}}\epsilon^{-3\beta'}\langle k,\eta\rangle^{-1} & \text{above the cut-off.} \end{cases} \tag{16}$$
We stress that this limits the choice of β in Definition 3.1 to $\beta\leq\frac{1}{3},\beta'\leq\gamma$. Furthermore, it deteriorates for $C_1>0$ small.
- Similarly, in view of the time decay with $t^{-\sigma+1}$ as required in the estimate (12), for some estimates we require an improved bound of the form
$$\exp(-C_1 \ulcorner k,\eta \urcorner) \leq \begin{cases} C_1^{-3\sigma}\epsilon^{-3\sigma\beta}\langle k,\eta\rangle^{-\sigma} & \text{below the cut-off,} \\ C_1^{-\frac{\sigma}{\gamma}}\epsilon^{-3\sigma\beta'}\langle k,\eta\rangle^{-\sigma} & \text{above the cut-off.} \end{cases} \tag{17}$$
uniformly in t. This imposes the stronger condition $\beta\leq\frac{1}{3\sigma}$.

With these preparations, we turn to our estimate of E_2.

Proof of Proposition 3.3. Recalling the Definition 3.1 of E_2, we consider a weighted version of the integral equation (15):
$$\begin{aligned} &e^{Cz(t)\epsilon^{\beta}\ulcorner k,kt\urcorner}\langle k,kt\rangle^{\sigma}|k|^{-\alpha}\tilde{\rho}(t,k) \\ &= e^{Cz(t)\epsilon^{\beta}\ulcorner k,kt\urcorner}\langle k,kt\rangle^{\sigma}|k|^{-\alpha}\tilde{g}(0,k,\eta+kt) \\ &\quad + \sum_{l\neq 0}\int_0^t e^{Cz(t)\epsilon^{\beta}\ulcorner k,kt\urcorner}\langle k,kt\rangle^{\sigma}|k|^{-\alpha}\frac{k(t-s)}{l}\tilde{\rho}(s,l)\tilde{g}(s,k-l,kt-ls)ds. \end{aligned}$$
The first summand here is rapidly decaying due to the regularity of the initial data, in particular, it holds that
$$\sup_k e^{Cz(t)\epsilon^{\beta}\ulcorner k,kt\urcorner}\langle k,kt\rangle^{\sigma}|k|^{-\alpha}\tilde{g}(0,k,\eta+kt) \leq \epsilon(1+t)^{-\sigma}.$$

For the integral term we insert the weights of Definition 3.1 for both $\tilde{\rho}$ and $\tilde{g}$ thus arrive at
$$\begin{aligned} &\sum_{l\neq 0}\int_0^t e^{C\epsilon^{\beta}(z(t)\ulcorner k,kt\urcorner^{1/3}-z(s)\ulcorner k-l,kt-ls\urcorner^{1/3}-z(s)\ulcorner l,ls\urcorner^{1/3}} \\ &\quad \langle k,kt\rangle^{\sigma}\langle k-l,kt-ls\rangle^{-\sigma}\langle l,ls\rangle^{-\sigma}|k|^{-\alpha}|l|^{\alpha}\frac{k(t-s)}{l} \\ &\quad \left(A(s,l,ls)|l|^{-\alpha}\tilde{\rho}(s,l)\right)\left(A(s,k-l,kt-ls)\tilde{g}(s,k-l,kt-ls)\right)ds, \end{aligned}$$

where we denote the weights as A for brevity. Using the Sobolev embedding $H^1 \subset L^\infty$ and (14) to estimate

$$\| (A(s, k-l, kt-ls)\tilde{g}(s, k-l, kt-ls)) \|_{\ell^\infty} \leq \sqrt{E_1(z(s))} \lesssim \epsilon$$

and that by definition

$$\| \left(A(s,l,ls)|l|^{-\alpha}\tilde{\rho}(s,l)\right) \|_{\ell^\infty} = E_1(z(s)),$$

we thus arrive at an integral inequality of the form

$$E_1(z(t)) \leq \epsilon(1+t)^{-\sigma} + \sup_k \sum_{l\neq 0} \int_0^t \epsilon C_{k,l}(t,s) E_1(z(s)) ds.$$

Here we use a similar notation as [GNR20] and defined

$$\begin{aligned} \epsilon C_{k,l}(t,s) = e^{C(z(t)-z(s))\ulcorner k,kt\urcorner} & e^{Cz(s)(\ulcorner k,kt\urcorner - \ulcorner k-l,kt-ls\urcorner - \ulcorner l,ls\urcorner)} \\ & \langle k,kt\rangle^\sigma \langle k-l,kt-ls\rangle^{-\sigma}\langle l,ls\rangle^{-\sigma}|k|^{-\alpha}|l|^\alpha \\ & \epsilon\frac{k(t-s)}{l}. \end{aligned} \tag{18}$$

In view of the time decay encoded in (14), we define (with the same notation as [GNR20])

$$\begin{aligned} \zeta(t) &= \sup_{0\leq\tau\leq t} E_1(z(\tau))\langle s\rangle^{\sigma-1}, \\ \langle s\rangle &:= \sqrt{1+s^2}. \end{aligned}$$

Inserting this definition, we obtain the following integral inequality:

$$\zeta(t) \leq \epsilon + \sup_k \int_0^t \sum_{l\neq 0} \epsilon C_{k,l}(t,s)\langle s\rangle^{-\sigma+1}\langle t\rangle^{\sigma-1} ds\zeta(t). \tag{19}$$

It thus suffices to show that

$$\sup_k \int_0^t \sum_{l\neq 0} \epsilon C_{k,l}(t,s)\langle s\rangle^{-\sigma+1}\langle t\rangle^{\sigma-1} ds \leq \frac{1}{2}. \tag{20}$$

Compared to [GNR20] we here additionally make use of the power ϵ^1. We further highlight the strong control required for small times $s \ll t$ when $\sigma > 1$. In that region we need to rely on the exponential factors in $C_{k,l}(t,s)$ to provide decay, which we noted as (17).

In the following we distinguish multiple cases for the estimates in (20) and possibly restrict to sub-intervals of $(0,t)$ to study

$$\int_I \epsilon C_{k,l}(t,s)\langle s\rangle^{-\sigma+1}\langle t\rangle^{\sigma-1} ds \tag{21}$$

for fixed k and $l \neq 0$.

As a first instructive case we study the setting $l = k$, where we use different arguments if $|kt - ls| \geq \frac{kt}{2}$ and when $|kt - ls| \leq \frac{kt}{2}$.

The diagonal case $l = k$:
We note that in this special case $l = k$ (recall that $l \neq 0$) our estimate (21) reduces

to

$$\int_I \epsilon e^{C(z(t)-z(s))\ulcorner k,kt\urcorner} e^{Cz(s)(\ulcorner k,kt\urcorner-\ulcorner 0,k(t-s)\urcorner-\ulcorner k,ks\urcorner)}$$
$$\langle k,kt\rangle^{\sigma}\langle 0,k(t-s)\rangle^{-\sigma}\langle k,ks\rangle^{-\sigma}(t-s)\langle t\rangle^{\sigma-1}\langle s\rangle^{-\sigma+1}ds.$$

We split this integral into the regions

$$|kt-ks|\geq \frac{kt}{2}\Leftrightarrow s\leq \frac{t}{2},$$
$$|kt-ks|\leq \frac{kt}{2}\Leftrightarrow s\geq \frac{t}{2}.$$

Both regimes exhibit similar behavior as the model problems of Section 2 and hence require different arguments.

We begin our discussion with the region $s\geq \frac{t}{2}$, where it holds that

$$\langle k,kt\rangle^{\sigma}\langle 0,k(t-s)\rangle^{-\sigma}\langle k,ks\rangle^{-\sigma}(t-s)\langle t\rangle^{\sigma-1}\langle s\rangle^{-\sigma+1}\leq 2^{2\sigma-1}\langle t-s\rangle^{-\sigma+1}k^{-\sigma}.$$

If $\sigma>2$ this integral is bounded uniformly in t and smaller than a constant times $\frac{\epsilon}{k^{\sigma}}$. We emphasize that in this region we did not require any bounds involving $\ulcorner k,kt\urcorner$.

We next turn to the case $s\leq \frac{t}{2}$, where we emphasize that

$$\begin{aligned}&\epsilon\langle k,kt\rangle^{\sigma}\langle 0,k(t-s)\rangle^{-\sigma}\langle k,ks\rangle^{-\sigma}(t-s)\langle t\rangle^{\sigma-1}\langle s\rangle^{-\sigma+1}\\ &\leq \epsilon\langle t\rangle^{\sigma}\langle k,ks\rangle^{-\sigma}\langle s\rangle^{-\sigma+1}\end{aligned}$$

is integrable but the value of the integral might be of size

$$\epsilon\langle t\rangle^{\sigma}\langle k\rangle^{-\sigma}. \tag{22}$$

Since $t\leq T$, this bound becomes small if $k:|k|\gg T$ is sufficiently large, which hence allows for a cutoff at $\eta_*\geq T^2$. However, for smaller values of k this integral might be very large and we thus need to rely on our exponential factor to improve our estimate.

For this purpose we note that for $s\leq \frac{t}{2}$, by the intermediate value theorem and monotonicity of $\partial_t z$ it holds that

$$|z(t)-z(s)|=|\partial_t z(\bar{t})|(t-s)\geq \frac{t-s}{t}\geq \frac{1}{2}.$$

We may thus use the exponential estimate (17) to conclude that

$$e^{C(z(t)-z(s))\ulcorner k,kt\urcorner}\lesssim \begin{cases}C^{-3\sigma}\epsilon^{-3\sigma\beta}\langle k,kt\rangle^{-\sigma}, & \text{below the cutoff,}\\ C^{-\sigma/\gamma}\epsilon^{-\sigma\beta'/\gamma}\langle k,kt\rangle^{-\sigma}, & \text{above the cutoff.}\end{cases}$$

Combining this with (22) and the requirements (9), we hence obtain a bound uniformly in t and ϵ, which further decays in k.

In the following we discuss the various cases $0\neq l\neq k$, where in view of Section 2 we expect to encounter contributions due to resonances. Furthermore, as seen already in the simple example of the diagonal case, the weight $\langle t\rangle^{\sigma-1}\langle s\rangle^{-\sigma+1}$ in (20) poses challenges to deriving good estimates in terms of powers of ϵ.

The reaction case: $\left\{s : |kt - ls| \leq \frac{kt}{2}\right\}$.

We note that in this case by the triangle inequality it holds that $|ls| \geq \frac{kt}{2}$ and thus (21) reduces to

$$\int_I e^{C(z(t)-z(s))\ulcorner k,kt \urcorner} l \langle k-l, kt-ls\rangle^{-\sigma} |k|^{-\alpha}|l|^{\alpha} \epsilon \frac{k(t-s)}{l^2} \langle t\rangle^{\sigma-1}\langle s\rangle^{-\sigma+1} ds.$$

We first consider the sub-case $s \leq \frac{t}{2}$. Here we note that

$$\begin{aligned} |z(t)-z(s)| &\geq \frac{1}{2}, \\ t-s &\leq t, \\ |l| &\geq k. \end{aligned}$$

Therefore, a rough estimate is given by

$$e^{-\frac{C}{2}\ulcorner k,kt \urcorner} \epsilon \langle t\rangle^{\sigma} \int_I l\langle k-l, kt-ls\rangle^{-\sigma}\langle s\rangle^{-\sigma+1} ds.$$

The integral here is bounded by $\langle k-l\rangle^{-\sigma+1}$, which is summable in l provided $\sigma > 2$. For the prefactor, we use (17) and (9) to control

$$e^{-\frac{C}{2}\ulcorner k,kt \urcorner} \epsilon \langle t\rangle^{\sigma} \lesssim C^{-3\sigma}.$$

Therefore, we indeed obtain a bound by a small constant provided C is sufficiently large.

We next consider the sub-case $\frac{t}{2} \leq s \leq t$. Here we note that

$$\begin{aligned} |z(t)-z(s)| &\geq \frac{t-s}{t}, \\ \langle t\rangle^{\sigma-1}\langle s\rangle^{-\sigma+1} &\leq 2^{\sigma-1}, \\ \frac{k}{2} &\leq l \leq 2k. \end{aligned}$$

Therefore, may equivalently consider

$$\int_I e^{-C\frac{t-s}{t}\ulcorner k,kt \urcorner} \langle k-l, kt-ls\rangle^{-\sigma} \epsilon(t-s)|k|^{1-\alpha}|l|^{\alpha-1}.$$

If $|kt-ls| \geq \frac{t}{2} \geq \frac{t-s}{2}$, we may simply estimate $|k|^{1-\alpha}|l|^{\alpha-1} \leq 2^{\alpha-1}$ and

$$\langle k-l, kt-ls\rangle^{-\sigma}\epsilon(t-s) \leq 2\epsilon\langle k-l, kt-ls\rangle^{-\sigma+1},$$

which is integrable. Furthermore, the value of the integral is bounded by $\epsilon\langle k-l\rangle^{-\sigma+2}$ and hence summable and small provided $\sigma > 3$. It thus remains to study the sub-case, where $|kt-ls| \leq \frac{t}{2}$, $s \geq \frac{t}{2}$, which implies that

$$|t-s| \approx \frac{|k-l|}{k}$$

We thus need to control

$$e^{-C\frac{|k-l|}{k}\ulcorner k,kt \urcorner}(k-l)\epsilon\frac{kt}{k^3} l\langle k-l, kt-ls\rangle^{-\sigma+1}. \tag{23}$$

Applying the estimate (16) *below the cut-off*, as in [GNR20, Lemma 4.4] we hence obtain a bound by

$$\epsilon^{1-3\beta}|k-l|^{-2}k^3\frac{k-l}{k^3}\langle k-l, kt-ls\rangle^{-\sigma+1}.$$

The powers of k exactly cancel and we obtain the desired bound.

We emphasize that *above the cut-off* this argument breaks, since we would obtain a bound by $k^{\frac{1}{\gamma}-3}$ which grows unbounded as $k \to \infty$. However, this problem does not occur in the case of a finite time interval, since one then *cannot* independently let kt and k^3 tend to infinity. More precisely, since $0 \leq t \leq T$ we may very roughly control (23) by

$$\epsilon \frac{kT}{k^3} \langle k-l, kt-ls \rangle^{-\sigma+2},$$

irrespective of the definition of $\ulcorner k, kt \urcorner$! In particular, if k is sufficiently large such that

$$\epsilon \frac{kT}{k^3} \ll 1$$

no further argument is required. Since our cut-off is defined in terms of

$$\langle k, kt \rangle \geq \eta_*,$$

this can for instance be achieved by choosing $\eta_* \geq T^2$, so that $k \geq T$.

As discussed in Section 2, the choice of η_* can surely be further optimized, but for the purposes of this article a rough bound is sufficient.

The transport case: $\left\{ s : \frac{kt}{2} \leq |kt - ls| \right\}$.
Similarly to the second model of Section 2, in this case ls and l might be very small and hence cannot compensate for powers of k and t. For this reason we crucially rely on the decay of the exponential factor. In particular, in contrast to the $l = k$ case, we here need to control positive powers of k even if kt is much larger than the cut-off.

Inserting our assumptions we need to estimate

$$\begin{aligned}
&\int_I \epsilon e^{-C\epsilon^\beta (z(t)-z(s)) \ulcorner k, kt \urcorner^{1/3}} \\
&\qquad 2^\sigma l \langle l, ls \rangle^{-\sigma} |k|^{-\alpha} |l|^\alpha \\
&\qquad \frac{k(t-s)}{l^2} \langle t \rangle^{\sigma-1} \langle s \rangle^{-\sigma+1} ds.
\end{aligned}$$

We argue as in [GNR20] (see Lemma 4.4, case 1 and, in particular, (4.24) there), but more discussion is required for the cut-off.

We first discuss the case $s \leq \frac{t}{2}$. Following a similar argument as in the resonant case we bound $|z(t) - z(s)| \geq \frac{1}{2}$ below and apply (17) to arrive at bound by

$$\begin{aligned}
&k^{1-\alpha} \langle t \rangle^\sigma |l|^{\alpha-1} \langle l, ls \rangle^{-\sigma} \langle s \rangle^{-\sigma+1} \langle k, kt \rangle^{-\sigma} C^{-3\sigma} \\
&\leq C^{-3\sigma} k^{1-\alpha-\sigma} |l|^{\alpha-1} \langle l, ls \rangle^{-\sigma} \langle s \rangle^{-\sigma+1}.
\end{aligned} \tag{24}$$

Here we estimated $C^{-\frac{\sigma}{\gamma}} \leq C^{-3\sigma}$ for simplicity of notation.

For $s \geq \frac{t}{2}$ also further discussion is needed. If $l \geq \frac{k}{2}$, we simply bound by

$$\begin{aligned}
&\int_I \epsilon \langle l, lt \rangle^{-\sigma} |k|^{-\alpha} |l|^\alpha \frac{kt}{l^2} ds \\
&\leq \langle t \rangle^{-\sigma+2} l^{-\sigma+\alpha-2} |k|^{-\alpha+1},
\end{aligned}$$

which is summable in $l \geq \frac{k}{2}$ provided $\sigma > 2$.

For $|l| \leq \frac{k}{2}$ we instead can only use that $|l|^{\alpha-2}$ is summable and need to show that

$$\int_I \epsilon e^{-C\epsilon^\beta (z(t)-z(s))\ulcorner k,kt\urcorner^{1/3}} |k|^{1-\alpha}(t-s)\langle t\rangle^{-\sigma} ds$$

is uniformly bounded. Following the argument of [GNR20] we employ a variant of (17) with the exponent $3(1-\alpha)$ when *below the cut-off* to obtain

$$|t-s|^{1-3(1-\alpha)} t^{3(1-\alpha)} k^{1-\alpha} \langle k, kt\rangle^{-(1-\alpha)} \langle t\rangle^{-\sigma}.$$

At this point we require that

$$|t-s|^{1-3(1-\alpha)}$$

is locally integrable and hence that

$$1-3(1-\alpha) > -1 \Leftrightarrow \alpha > \frac{1}{3}.$$

Integrating and bounding

$$\int_I |t-s|^{1-3(1-\alpha)} ds \leq \langle t\rangle^{2-3(1-\alpha)},$$

We hence arrive at a bound by

$$\langle t\rangle^{2-\sigma} \leq 1.$$

If we are *above the cut-off* we argue similarly and apply (17) with the exponent $\frac{1-\alpha}{\gamma}$, which imposes the stronger constraint

$$1 - \frac{1-\alpha}{\gamma} > -1 \Leftrightarrow \alpha > 1-2\gamma. \tag{25}$$

Thus the transport case should be understood to impose constraints on α given γ. □

This concludes our estimate of $\rho(t)$ incorporating a frequency cut-off and improved ϵ dependence. The estimate of $g(t,x,v) = f(t,x-tv,v)$ in comparison uses a much more abstract and lossy argument. In particular, we can follow the strategy of [GNR20] more closely and only need to discuss the cut-off in some detail.

5. Control of $f(t, x-tv, v)$

In this section we establish growth bounds on E_1 with $\rho(t,x)$ considered given and, in particular, prove Proposition 3.2. Here we argue similarly as in Proposition 4.1 of [GNR20], but need to account for the following changes:

- Our cut-off $\ulcorner \cdot, \cdot \urcorner$ include a factor ϵ^β in the exponent, hence compared to [GNR20] our derivative ∂_z is rescaled by $\epsilon^{-\beta}$.
- Above the cut-off we use a different exponent γ, which we need to account for in our estimates.

Proof of Proposition 3.2. We follow the same strategy as in [GNR20, Proposition 4.1] and denote our weight as

$$A_{k,\eta} := e^{Cz\ulcorner k,\eta\urcorner^{1/3}} \langle k,\eta\rangle^\sigma.$$

We remark that these cut-offs preserve triangle inequalities and that $A_{k,\eta} \leq C A_{k-l,\eta-\xi} A_{l,\xi}$ satisfies an algebra property (with constant independent of η_*).

Since we consider the case of perturbations around 0, after a Fourier transform the Vlasov-Poisson equations read

$$\partial_t \tilde{g}(k,\eta) = -i \sum_l (\eta - kt)\tilde{F}(t,l)\tilde{g}(t,k-l,\eta-lt).$$

Testing with $A_{k,\eta}^2 \tilde{g}(k,\eta)$, we note that by anti-symmetry

$$\int \sum_{k,l} -i(\eta-kt)\tilde{F}(t,l)A_{k-l,\eta-lt}\tilde{g}(t,k-l,\eta-lt)A_{k,\eta}\tilde{g}(t,k,\eta)d\eta = 0$$

and hence our estimate reduces to controlling

$$\int -i \sum_{k,l} (\eta-kt)\frac{A_{k,\eta}-A_{k-l,\eta-lt}}{A_{l,lt}A_{k-l,\eta-lt}} A\tilde{F}(t,l)A\tilde{g}(t,k-l,\eta-lt)A\tilde{g}(t,k,\eta)d\eta. \tag{26}$$

in terms of $\partial_z E_1$.

Here we argue as in [GNR20] and first estimate

$$(\eta-kt)\frac{A_{k,\eta}-A_{k-l,\eta-lt}}{A_{l,lt}A_{k-l,\eta-lt}}$$

beginning with the case where

$$\langle l, lt\rangle \geq \frac{1}{2}\langle k,\eta\rangle.$$

Then it follows that

$$\begin{aligned}
&\langle t\rangle\langle k,\eta\rangle \frac{A_{k,\eta}-A_{k-l,\eta-lt}}{A_{l,lt}A_{k-l,\eta-lt}} \\
&\leq \langle t\rangle\langle k,\eta\rangle \frac{\langle k,\eta\rangle^\sigma + \langle k-l,\eta-lt\rangle^\sigma}{\langle l,lt\rangle^\sigma \langle k-l,\eta-lt\rangle^\sigma} \\
&\leq \langle t\rangle(\langle k-l\rangle^{-\sigma+1} + \langle l\rangle^{-\sigma+1}).
\end{aligned} \tag{27}$$

If instead $\langle l, lt\rangle \leq \frac{1}{2}\langle k,\eta\rangle$ is possibly much smaller, we need to exploit cancellation in

$$\langle k,\eta\rangle(A_{k,\eta}-A_{k-l,\eta-lt}).$$

More precisely, we recall that $A_{k,\eta}$ is of the form

$$\langle x\rangle^\sigma e^{\ulcorner x \urcorner}.$$

We thus split

$$\begin{aligned}
&\langle x\rangle^\sigma e^{\ulcorner x \urcorner} - \langle x+y\rangle^\sigma e^{\ulcorner x+y \urcorner} \\
&= (\langle x\rangle^\sigma - \langle x+y\rangle^\sigma)e^{\ulcorner x \urcorner} \\
&+\langle x+y\rangle^\sigma (e^{\ulcorner x \urcorner} - e^{\ulcorner x+y \urcorner}),
\end{aligned}$$

By the chain rule and the intermediate value theorem

$$|\langle x\rangle^\sigma - \langle x+y\rangle^\sigma| \leq c_\sigma \frac{\langle y\rangle}{\langle x\rangle}\langle x\rangle^\sigma,$$

where we used that $\frac{|x|}{2} \leq |x+y| \leq 2|x|$. Hence, for that part we easily obtain a bound by

$$\langle l, lt\rangle(A_{k,\eta}-A_{k-l,\eta-lt}).$$

It thus remains to discuss the difference

$$e^{\ulcorner x \urcorner} - e^{\ulcorner x+y \urcorner}.$$

Since x and $x+y$ are of comparable magnitude and we consider a smooth cut-off (that is, matching powers β, β'), it suffices to consider the case when either both x and $x+y$ are above or below the cut-off. In either case this hence reduces to considering

$$\begin{aligned}
e^{C_1\langle x\rangle^\theta} - e^{C_1\langle x+y\rangle^\theta} &= \int_0^1 \frac{d}{d\tau} e^{C_1\langle x+\tau y\rangle^\theta} d\tau \\
&\leq \int_0^1 C_1\theta\langle y\rangle\langle x+\tau y\rangle^{\theta-1} e^{C_1\langle x+\tau y\rangle^\sigma} d\tau \\
&\leq 2C_1\langle y\rangle\langle x\rangle^{\theta-1}(e^{C_1\langle x\rangle^\theta} + e^{C_1\langle x+y\rangle^\theta}) \\
&\leq 2\frac{\langle y\rangle}{\langle x\rangle}(C_1\langle x\rangle^\theta e^{C_1\langle x\rangle^\theta} + C_1\langle x+y\rangle^\theta e^{C_1\langle x+y\rangle^\theta}).
\end{aligned}$$

We remark that here by our notational conventions

$$C_1\langle x\rangle^\theta e^{C_1\langle x\rangle^\theta} = C^{-1}\epsilon^{-\beta}\partial_z e^{Cz\ulcorner x\urcorner}.$$

We hence obtain the desired control and may conclude as in [GNR20]. In the interest of a self-contained presentation, we recall the proof below: Inserting the previous estimates (26) may be controlled by

$$\epsilon^{-\beta}\sum_{k,l}\langle t\rangle(\langle k-l\rangle^{-\sigma+1} + \langle l\rangle^{-\sigma+1})$$

$$\int (AF)_{l,lt}\sqrt{1+\ulcorner k,\eta\urcorner}(Ag)_{k,\eta}\sqrt{1+\ulcorner k-l,\eta-lt\urcorner}(Ag)_{k-l,\eta-lt}.$$

Here we now make use of the power $|l|^{-\alpha}$ in the definition of E_2 and the fact that $F_l = \frac{1}{il}\rho_l$ and bound

$$|(AF)_{l,lt}| \leq \langle l\rangle^{\alpha-1}E_2.$$

where

$$\langle l\rangle^{\alpha-1} \in \ell^2(\mathbb{Z}) \Leftrightarrow \alpha < \frac{1}{2}.$$

Thus, by Young's and Hölder's inequality we over all obtain a control by

$$c\epsilon^{-\beta}E_2(1+\partial_z)E_1.$$

The estimate for $\partial_\eta g_{k\eta}$ follows analogously and is hence omitted (see also [GNR20, Proposition 4.1]). $\square$

To the author's knowledge this is the first nonlinear Landau damping result (for finite times ϵ^{-N}) in Gevrey classes larger than 3 (see also Lemma 1.2 and [GNR22]). As mentioned throughout the proofs, we expect that optimal classes are determined by plasma echoes as captured in the first model problem of Section 2. However, the method of [GNR20] which we adapt trades further decay for simplicity of proof. That is, the additional requirement that

$$E_2 \leq \epsilon(1+t)^{-\sigma+1}$$

with $\sigma > 3$ further restricts our choice of parameters and seems to be optimal for this method of proof.

While we for simplicity here have considered the case of Landau damping around the special case $f = 0$, we expect that these results can be extended to Penrose stable equilibria by similar arguments as in [GNR20]. Moreover, in future work we plan to establish nonlinear stability of the wave-type solutions underlying the echo chain construction of [Bed20] and [Zil21] in suitable Gevrey classes.

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