Received: 22 February 2023 | Revised: 8 June 2023 | Accepted: 22 August 2023

DOI: 10.1002/mana.202300094

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

A note on subsets of positive reach

Alexander Lytchak

Institute of Algebra and Geometry, KIT, Karlsruhe, Germany

Correspondence
Alexander Lytchak, Institute of Algebra and Geometry, KIT, Englerstr. 2, 76131 Karlsruhe, Germany.
Email: alexander.lytchak@kit.edu

Funding information
Deutsche Forschungsgemeinschaft, Grant/Award Numbers: 81071066, SFB TRR 191

Abstract

We provide new structural results on sets of positive reach in Euclidean spaces and Riemannian manifolds. In particular, we describe neighborhoods of points, whose tangent cones have maximal dimensions.

KEYWORDS
semiconvex functions, semiconvex sets, regular points

MSC (2020)
53C20, 53C45, 52A30

1 | INTRODUCTION

Sets of positive reach in Euclidean spaces were introduced by Federer [3], as common generalizations of convex subsets and $C^{1,1}$-submanifolds. They have turned out to be relevant in Riemannian, integral, and metric geometry, cf. [6, 9, 10, 19].

A subset C of a Riemannian manifold M has *positive reach* if the closest-point projection is uniquely defined in a neighborhood of C in M. The notion of positive reach is invariant under $C^{1,1}$-diffeomorphisms and it is local. Bangert verified in [1] that being of positive reach in a Riemannian manifold does not depend on the choice of the Riemannian metric, but only on the $C^{1,1}$-atlas.

Geometry and topology of subsets of positive reach has been investigated in many papers including [3, 7, 13, 14, 17, 18]. We refer to Section 2 for a summary of the main properties and recall here only the facts needed to motivate and to state the results of this paper.

At any $x \in C$, there is a well-defined tangent cone T_xC which is a convex cone in the Euclidean space T_xM. The maximal dimension k of the cones T_xC coincides with the Hausdorff dimension of C and is called the *dimension* of C [3, Theorem 4.8, Remark 4.15]. A connected m-dimensional subset C of positive reach is a topological manifold if and only if it is a $C^{1,1}$-submanifold, [14, Proposition 1.4]. Moreover, this happens if and only if all tangent cones T_xC are Euclidean spaces.

Our results describe what happens if these equivalent conditions are not satisfied. The connected set of positive reach C fails to be a $C^{1,1}$-manifold *without boundary* if and only if there is a point with the infinitesimal structure of a manifold *with boundary*:

Theorem 1.1. *Let C be a connected m-dimensional subset of positive reach in a Riemannian manifold M. Either C is a $C^{1,1}$-submanifold without boundary of M or, for some $x \in C$, the tangent cone T_xC is isometric to an m-dimensional Euclidean half-space.*

A slightly stronger statement can be found in Theorem 4.1.

A neighborhood in C of any point $x \in C$ as in Theorem 1.1 is $C^{1,1}$-equivalent to a convex body in $\mathbb{R}^m$, as stated in the following main result of this paper:

Theorem 1.2. *Let $C \subset M$ be an m-dimensional subset of positive reach in an n-dimensional manifold M. Let $x \in C$ be such that the convex cone T_xC has dimension m.*

Then, there exists a closed convex subset $K \subset \mathbb{R}^m$ with non-empty interior in $\mathbb{R}^m$, an open neighborhood O of K in $\mathbb{R}^m$ and a $C^{1,1}$-embedding $\Phi : O \to M$ such that $\Phi(K)$ is a neighborhood of x in C.

The statement is not quite obvious even if $\dim M = m$. In this case, some forms of Theorem 1.2 appear in [4, 6, 16, Theorem 5.6] and as a statement without proof in [12, Appendix B].

If the tangent cone T_xC in Theorem 1.2 is an m-dimensional Euclidean space, then a neighborhood of x in C is a $C^{1,1}$-submanifold without boundary, [17, Theorem 7.5]. This implies:

Corollary 1.3. *Let $C \subset M$ be an m-dimensional subset of positive reach. Then, the set C^+ of all points $x \in C$ with m-dimensional tangent cone T_xC is an open subset of C, homeomorphic to an m-dimensional manifold with boundary. The boundary of the manifold C^+ is the set of points $x \in C^+$ at which T_xC is not a Euclidean space.*

In other words, $x \in C^+$ is a boundary point of the manifold C^+ if and only if the tangent space T_xC has non-empty boundary, analogous to the structure of boundaries of Alexandrov spaces [15].

1.1 | Proofs and further comments

The proof of Theorem 1.1 is presented in Section 4 along the following lines. We consider the maximal non-empty m-dimensional $C^{1,1}$ manifold without boundary U contained in C. We take an arbitrary point $p \in U$ and find a point x in the complement $C \setminus U$, which is a closest point to p with respect to the intrinsic metric of C. The tangent cone T_xC must then contain an m-dimensional half-space and, therefore, coincide with it.

The proof of Theorem 1.2 is more technical. In the full-dimensional case $\dim(C) = \dim(M)$, the result is essentially contained in the paper [16] preceding the investigations of sets of positive reach in [3]. We only need to adapt the vocabulary of [16] to our situation.

The case $\dim(M) < \dim(C)$ is proven by finding a $C^{1,1}$-diffeomorphism of a neighborhood of x which sends (a neighborhood of x in) C into an m-dimensional submanifold. Then, one could apply the previously discussed full-dimensional case. The construction of the diffeomorphism relies on Whitney's extension theorems and a technical result obtained in Section 3. This result, Corollary 3.4, states that the tangent cones of a set of positive reach vary *Lipschitz semi-continuously* in a precise sense. This semi-continuity may be of some independent interest.

We finish the introduction with a few comments and questions.

Remark 1.4. The Riemannian manifold M below and in the formulation of the main results above is always assumed to be smooth. However, all results are literally valid for all Riemannian manifolds with $C^{1,1}$ Riemannian metrics, as the class of sets of positive reach is independent on the metric and only depends on the $C^{1,1}$-atlas of distance coordinates [1, 8].

Remark 1.5. The term *positive reach* used in this paper coincides with the terminology of [1]. This terminology differ from the one used in [3]: Following [3] our *subsets of positive reach* should be called *subsets of locally positive reach*. We have decided to keep Bangert's terminology since in the realm of general Riemannian manifolds this notion seems more suitable to us.

Remark 1.6. Applications of the above results to the theory of submetries will be presented in a separate paper.

The precise classification of (local) structures of subsets of positive reach up to (biLipschitz) homeomorphisms or even up to $C^{1,1}$ diffeomorphisms seems impossible. However, it seems possible to obtain reasonable answers to the following less ambitious questions.

Question 1.7. *Is it true that any connected two-dimensional subset of positive reach is locally bi-Lipschitz equivalent to a subset of positive reach in* $\mathbb{R}^2$*? See [17] for an explicit description of subsets of positive reach in the plane.*

Question 1.8. *Can one obtain an infinitesimal chracterization of topological manifolds with boundary among all subsets of positive reach?*

Question 1.9. *Is it possible to describe up to homeomorphisms all germs of three-dimensional subsets of positive reach?*

2 | PRELIMINARIES

2.1 | Slightly generalized definition and localization

We say that a *locally closed* subset C of a smooth Riemannian manifold M has positive reach in M if there exists an open neighborhood O of C in M such that the closest-point projection P^C onto C is uniquely defined on O.

This definition is usually given for *globally closed* subsets C. However, replacing O by a smaller neighborhood if needed, we may always assume that C is closed in O.

The advantage of this generalized definition is the following locality: for any subset C of positive reach in M, any subset U of C, open in C, is a subset of positive reach in M. On the other hand, a locally closed subset C of M has positive reach if it is covered by relatively open subsets of positive reach in M.

For a closed subset C of a manifold M, the property of being of positive reach does not depend on the Riemannian metric [1, Corollary], moreover, it is invariant under $C^{1,1}$-diffeomorphisms [1, 3, Theorem 4.19]. Due to the locality stated above, the same statements apply to locally closed subsets.

Given a locally closed subset C of positive reach in M and any point $p \in C$, we can find a small chart U around p, such that $C \cap U$ is closed in U. Changing the metric on U to a Euclidean metric, $C \cap U$ becomes a closed subset of positive reach in a Euclidean space.

If a closed subset C is of positive reach in the Euclidean space $\mathbb{R}^n$ then, for any $p \in C$, the intersection of C with any sufficiently small closed ball $\bar{B}_r(p)$ is a compact contractible subset of positive reach [3, Theorem 4.10, Remark 4.15], [17, Lemma 2.3].

For a *compact* subset of positive reach C in a manifold M, there exists a positive number r such that the closest-point projection is uniquely defined on the open r-tubular neighborhood $B_r(C)$ of C. The supremum of such r is usually called the *reach of the subset* C.

Remark 2.1. Note that for *non-compact subsets of positive reach* C, the number *reach of* C defined as above may be 0.

The above consideration allows us to reduce all local statements about arbitrary subsets of positive reach in a Riemannian manifold to compact connected subsets of positive reach in the Euclidean space. We will freely use this observation below.

Finally, we refer [3, 4.18], [1, 14, Theorem 1.3], [8, Proposition 1.3] for many characterizations of the positive reach.

2.2 | Basic properties of subsets of the positive reach

The topological dimension $\dim(C)$ of a subset of the positive reach coincides with its Hausdorff-dimension [3, Remark 4.15]. Moreover, $\dim(C)$ is the maximum of the dimensions of convex cones T_xC.

For $m = \dim(C)$, we denote by C_{reg} the set of all $x \in C$ such that the tangent cone T_xC is isometric to $\mathbb{R}^m$. The subset C_{reg} is non-empty, open in C and it is a $C^{1,1}$-submanifold of M [17, Theorem 7.5].

The complement $C \setminus \bar{C}_{\text{reg}}$ is a locally closed subset of positive reach of dimension at most $m-1$ [17, Theorem 7.5]. As in the introduction, we denote by C^+ the set of points $x \in C$ with $\dim(T_xC) = m$. The previous statement implies $C^+ \subset \bar{C}_{\text{reg}}$.

The tangent cones T_xC depend lower semi-continuously on $x \in C$, [3, Theorem 4.8], [17, Proposition 3.1], or Proposition 3.3 below: if x_i converge to x in C then (in any fixed Euclidean chart around x) any pointwise Hausdorff limit (of a subsequence) of convex cones $T_{x_i}C$ contains the tangent cone T_xC. This immediately implies:

Lemma 2.2. *Let $C \subset M$ be of positive reach in M with $\dim(C) = m$. The set C^+ of all $x \in C$ with $\dim(T_xC) = m$ is open in C.*

For any point $x \in C$, denote by $\hat{T}_xC$ the Euclidean subspace of T_xM generated by the convex cone T_xC. This is a Euclidean space of the same dimension as T_xC. The semi-continuity of the tangent cones T_xC implies the semi-continuity of the Euclidean spaces $\hat{T}_xC$. In the case that the dimensions are constant this implies:

Lemma 2.3. *Let C be a subset of positive reach in M. Let the sequence $x_i \in C$ converge to $x \in C$. Assume that $\dim(T_{x_i}C) = \dim(T_xC)$ for all sufficiently large i. Then, the linear spaces $\hat{T}_{x_i}C$ converge to $\hat{T}_xC$.*

Note that the assumptions on the dimensions are satisfied if $x \in C^+$.

2.3 | Intrinsic metric on subsets of positive reach

Let C be a connected subset of positive reach of a manifold M. The intrinsic metric d_C is defined as usual, [2, Section 2.3], by letting $d_C(x, y)$ be the infimum of lengths of curves in C connecting x and y. Any curve realizing this infimum and parameterized by arclength is called a *C-geodesic* between x and y. If C is compact then any pair of points in C is connected by a C-geodesic.

Let C be of positive reach in M and $p \in C$ be arbitrary. As observed above, there exists a compact neighborhood K of p in C, which is of positive reach in M. We may then change the topology and the metric on M outside a neighborhood U of K in M and embed U isometrically into a *compact* smooth manifold N. By this procedure, the metric in a neighborhood of p in M and the intrinsic metric in a neighborhood of p in C are not changed.

Applying now [14, Remark 6.4, Theorem 1.3] and [13, Theorem 1.1, Theorem 1.2] to the pair $K \subset N$ we deduce:

Proposition 2.4. *Let C be a locally closed subset of positive reach in a manifold M. For arbitrary $p \in C$ and $\delta > 0$ there exist $\kappa > 0$ and $0 < r_0 < \frac{\pi}{\sqrt{\kappa}}$ such that for all $r < r_0$ the following hold true:*

- *$\bar{B}_r(p) \cap C$ is a compact subset of positive reach in M.*
- *The intrinsic distance d_C on $\bar{B}_r(p) \cap C$ differs from the M-distance on $\bar{B}_r(p) \cap C$ at most by the factor $(1 + \delta)$.*
- *With respect to d_C, the subset $\bar{B}_r(p) \cap C$ is convex in C.*
- *With respect to d_C, the subset $\bar{B}_r(p) \cap C$ is a CAT(κ) space.*

The last point above implies that $\bar{B}_r(p) \cap C$ with respect to the intrinsic metric is uniquely geodesic. In particular, it is contractible.

3 | SEMI-CONTINUITY OF TANGENT SPACES

We are going to discuss a Lipschitz-version of the semi-continuity of tangent cones in this section. Recall from [14, Theorem 1.2, Theorem 1.3, Example 3.4]:

Lemma 3.1. *There exists some universal constant $\mu_1 > 0$ with the following property. If C is a compact subset of $\mathbb{R}^n$ of reach ≥ 1 then any C-geodesic $\gamma : [a, b] \to C$ parameterized by arclength is a $C^{1,1}$ curve, and $\gamma' : [a, b] \to \mathbb{R}^n$ is μ_1-Lipschitz continuous.*

The optimal value of μ_1 does not play a role here and it does not follow from [14], but it might be of some independent interest:

Question 3.2. *What is the optimal value of μ_1 in Lemma 3.1 and the optimal value of μ in Proposition 3.3?*

We can now deduce the following semi-continuity statement:

Proposition 3.3. *There exists a universal constant $\mu > 0$ with the following property. Let $C \subset \mathbb{R}^n$ be compact subset of reach ≥ 1. Let $\varepsilon \leq 1$ and $\gamma : [0, \varepsilon] \to C$ be a C-geodesic parameterized by arclength. Then, for $p = \gamma(0)$, $v = \gamma'(0) \in T_pC$ and any $q \in C$, with $||p - q|| \leq \varepsilon^2$, the distance from v to T_qC can be estimated as:*

$$d(v, T_qC) \leq \mu \cdot ||p - q|| \,.$$

Proof. Without loss of generality, we may assume that p is the origin 0. We may further assume that $||p - q|| = ||q|| = \varepsilon^2$, otherwise we just replace γ by a shorter subcurve.

Set $u = \gamma(\varepsilon)$. Then, due to Lemma 3.1,

$$||u - \varepsilon \cdot v|| \leq \frac{\mu_1}{2}\varepsilon^2 \,.$$

Hence,

$$||(u - q) - \varepsilon \cdot v|| \leq \left(1 + \frac{\mu_1}{2}\right)\varepsilon^2 \,.$$

On the other hand, by [3, Theorem 4.18],

$$d(u - q, T_qC) \leq \frac{||u - q||^2}{2} \,.$$

Since T_qC is a cone, the triangle inequality implies

$$d(v, T_qC) \leq \left(1 + \frac{\mu_1}{2}\right)\varepsilon + \frac{||u - q||^2}{2\varepsilon} \,.$$

Since $||u|| \leq \varepsilon \leq 1$, the second summand is at most 2ε. Thus, we deduce the required inequality with $\mu = 3 + \frac{\mu_1}{2}$. □

We extend the above conclusion from a single $v \in T_pC$ to large convex subcones of T_pC, more precisely to the set of all vectors lying at least at some distance from the boundary of T_pC.

For any $\varepsilon > 0$, we consider the set $T_{p,\varepsilon}C$ of all $v \in T_pC$, such that the ball of radius $\varepsilon \cdot ||v||$ around v inside the affine hull $\hat{T}_pC$ is contained in T_pC. If $T_pC = \hat{T}_pC$ then $T_{p,\varepsilon}C = T_pC$, for any ε. In general, $T_{p,\varepsilon}C$ is a convex subcone of T_pC. The subcones $T_{p,\varepsilon}C$ increase with decreasing ε and their union is the set of inner points of T_pC relative to $\hat{T}_pC$.

Now, we can deduce from Proposition 3.3:

Corollary 3.4. *For any compact subset C in $\mathbb{R}^n$ of reach $\geq \delta$ in $\mathbb{R}^n$, for any point $p \in C$ and any $\varepsilon > 0$ the following holds true.*

There exists some $s = s(p, \varepsilon) > 0$ such that for any $q \in C \cap B_s(p)$ and any vector v in the convex subcone $T_{p,\varepsilon}C \subset T_pC$, the distance from v to T_qC is estimated by

$$d(v, T_qC) \leq 2\mu \cdot \delta \cdot ||p - q|| \cdot ||v|| \,,$$

where μ is the constant obtained in Proposition 3.3.

Proof. Rescaling the space we may assume $\delta = 1$. We fix some $\varepsilon_0 \ll \varepsilon$ and adjust it in the course of the proof.

The unit sphere S in the cone T_pC is the closure of the set of starting directions of C-geodesics emanating from p. Hence, we find some $t = t(\varepsilon_0) > 0$ and C-geodesics $\gamma_1, \ldots, \gamma_k : [0, t] \to C$ starting at p in unit directions $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ is ε_0-dense in S.

By Proposition 3.3, for any $q \in C$ with $||q - p|| \leq t^2$ we get

$$d(v, T_qC) \leq \mu \cdot ||p - q|| \,, \tag{3.1}$$

for $v = v_i$, for any $i = 1, \ldots, k$. By convexity of the distance function to the convex cone T_qC, the inequality (3.1) holds true for any v in the convex hull K_{ε_0} of the unit vectors v_i and the origin 0.

For $\varepsilon_0 \to 0$ the convex subsets K_{ε_0} converge to the unit ball in T_pC. Thus, for ε_0 small enough and any unit vector $w \in T_{p,\varepsilon}C$, the convex subset K_{ε_0} contains $\frac{1}{2}w$. Then

$$d\left(\frac{1}{2}w, T_qC\right) \leq \mu \cdot ||p-q|| \leq 2 \cdot \mu \cdot ||p-q|| \cdot ||\frac{1}{2}w|| \,.$$

This implies that the statement of the corollary holds true for $s = t(\varepsilon_0)$, for such sufficiently small ε_0. □

4 EXISTENCE OF BOUNDARY POINTS

The following result is a minor generalization of Theorem 1.1, which is more suitable for localization. Clearly, it implies Theorem 1.1.

Theorem 4.1. *Let C be an m-dimensional set of positive reach in a Riemannian manifold M of dimension n. Let C_{reg} be the maximal m-dimensional $C^{1,1}$-submanifold without boundary contained in C.*

Consider the set of boundary points of C_{reg} in C:

$$\partial C_{reg} := (\bar{C}_{reg} \cap C) \setminus C_{reg}\,.$$

Consider the subset G of all points $x \in \partial C_{reg}$ with T_xC isometric to an m-dimensional Euclidean half-space. Then, G is dense in ∂C_{reg}.

Proof. If ∂C_{reg} is empty, there is nothing to be proven. Thus, assume that ∂C_{reg} is not empty. We fix any $p \in \partial C_{\text{reg}}$ and $\varepsilon > 0$ and are going to find some $x \in B_\varepsilon(p) \cap G$.

We apply Proposition 2.4 with $\delta = 1$ and obtain some $r_0 < \varepsilon$ such that, for all $r \leq r_0$, the intersection $\bar{B}_r(p) \cap C$ is a compact subset of positive reach in M. Moreover, $\bar{B}_r(p) \cap C$ is convex in the intrinsic metric d_C of C.

Choose $r = \frac{r_0}{4}$ and an arbitrary $y \in B_r(p) \cap C_{\text{reg}}$. Consider a closest point $x \in \partial C_{\text{reg}}$ to y with respect to the intrinsic distance d_C. Then, $x \in B_{3r}(p) \subset B_\varepsilon(p)$. It remains to verify $x \in G$.

Consider the C-geodesic $\gamma : [0,a] \to C$ connecting x and y and parameterized by arclength. Since x is a closest point to y on ∂C_{reg}, for any $0 < s \leq a$ the following holds: the open ball W_s of radius s around $\gamma(s)$ with respect to d_C does not intersect ∂C_{reg}. Thus, this ball W_s is completely contained in the $C^{1,1}$-manifold C_{reg}.

In particular, any C-geodesic in W_s extends as a C-geodesic up to points with distance s from $\gamma(s)$, [11, Theorem 1.5]. Moreover, W_s is uniformly bi-Lipschitz to the s-ball in $\mathbb{R}^m$, since W_s is a geodesically convex $C^{1,1}$-manifold which is uniquely geodesic and has curvature uniformly bounded from both sides [8, Proposition 1.7].

Identify the tangent cone T_xC at x with the blow-up of C at x [14, Remark 6.1]. For the starting direction v of γ, which is a unit vector in T_xC, we deduce: the open unit ball around v is m-dimensional. Moreover, no geodesic in T_xC terminates at a point with distance less than 1 to v. Thus, the convex cone T_xC contains the closed unit m-dimensional ball W around v. Since T_xC is a cone, it contains the tangent cone T_0W which is an m-dimensional Euclidean half-space.

The tangent cone T_xC is a convex cone of dimension at most $m = \dim(C)$ containing an m-dimensional half-space T_0W. Moreover, T_xC is not a Euclidean m-dimensional space, since x is not contained in C_{reg}. Therefore, $T_xC = T_0W$. Hence, $x \in G$. □

5 STRUCTURE AROUND THE BOUNDARY POINTS

5.1 Preparation

We are going to prove Theorem 1.2 in this section. Thus, let C be an m-dimensional subset of positive reach in an n-dimensional Riemannian manifold M. Let $x \in C$ be a point with $\dim(T_xC) = m$. If T_xM is a Euclidean m-dimensional space, then x is contained in the manifold C_{reg} [17, Theorem 7.5]. The statement of Theorem 1.2 follows directly in this case.

We assume from now on that T_xC is m-dimensional, but not a Euclidean space. The statement of Theorem 1.2 is local. Arguing as in Section 2.1, we may assume $M = \mathbb{R}^n$ and that C is compact.

We may further assume $x = 0$ and that the m-dimensional Euclidean space $V := \hat{T}_0C$, generated by the cone T_0C, is the coordinate subspace

$$V = \mathbb{R}^m = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n .$$

We find some unit vector $v_0 \in T_0C$ and $\varepsilon > 0$ such that T_0C contains the ball of radius 4ε in $V = \mathbb{R}^m$. After a rotation, we may assume that v_0 is the last coordinate vector $v_0 = e_m$ in $\mathbb{R}^m$.

We fix some $\delta > 0$ smaller than the reach of the compact subset C. Applying Lemma 2.2, Proposition 2.4, and the semi-continuity of tangent cones, we find some $r > 0$ with the following properties, for $C' = \bar{B}_r(0) \cap C$:

- C' has reach at least δ in $\mathbb{R}^n$.
- For any $p \in C'$, we have $\dim(T_pC) = m$.
- T_pC contains some m-dimensional Euclidean ball of radius 2ε around some unit vector $v_p \in T_pC$.
- The subset C' is convex with respect to the intrinsic metric d_C.
- The intrinsic metric d_C and the Euclidean metric differ on C' at most by the factor 2.

5.2 | The full-dimensional case

Assume first that $m = n$, thus that C is full-dimensional. In this case, the result is essentially contained in [16], as we are going to explain now.

Consider the family $\mathcal{O}$ of all closed balls of radius δ which have exactly one point in common with C. This family is closed in the Hausdorff topology and every point in ∂C lies in some ball $O \in \mathcal{O}$, compare [17, Proposition 3.1 (vi)]. In terms of [16], this means that ∂C is an O^*_δ subset of $\mathbb{R}^m$.

The topological boundary ∂C is precisely the set of points $p \in C$ at which T_pC is not $\mathbb{R}^m$, [17, Theorem 9.5]. In particular, $0 \in \partial C$.

By assumption e_m is an interior point of T_0C. By the semi-continuity of tangent cones, T_pC contains e_m as an interior point, for any p in C close to 0. Making r smaller, we may assume that e_m is an interior point of T_pC, for all $p \in C'$.

Then, for any $p \in C' \cap \partial C$, the intersection of all balls in the family $\mathcal{O}$ which contain p is non-empty (more precisely, all these balls contain the point $-t \cdot e_m$, for a sufficiently small $t > 0$). In the notation used in [16], this means that $C' \cap \partial C$ is an O_δ set and the main results of [16] can be applied.

In particular, we find a ball W_0 in the orthogonal complement $\mathbb{R}^{m-1}$ of e_m with the following property: for any $w \in W_0$ the line $\gamma_w(t) = w + te_m$ intersects $C' \cap \partial C$ in at most one point [16, Theorem 1].

On the other hand, for all small $t > 0$, the point $\gamma_0(t)$ lies in the interior of C [17, Lemma 3.5]. Hence, choosing W_0 smaller, if necessary, we may assume that γ_w intersects the interior of C' for any $w \in W_0$.

For all small $t < 0$, the point $\gamma_0(t)$ is not contained in C. Otherwise, $-e_m \in T_0C$ and, since e_m is an inner point of the convex cone T_0C, this would imply that $T_0C = \mathbb{R}^m$, in contradiction to our assumption $0 \notin C_{\mathrm{reg}}$. Hence, making W_0 smaller if needed, we may assume that $\gamma_w(t_0)$ is not in C for some fixed small $t_0 < 0$ and any $w \in W_0$.

Therefore, for any $w \in W_0$ the intersection of γ_w and C' is a compact segment $I_w = [\gamma_w(f_w), \gamma_w(g_w)]$, for some $f_w < g_w$. Moreover, $\gamma_w(f_w) \in \partial C$ and $\gamma_w(g_w)$ lies on the sphere $\partial B_r(0)$ and $g_w > 0$.

By compactness of ∂C, we deduce that the map $w \to f_w$ is continuous. Thus, near the origin 0, the set C is given as the supergraph in $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ of the continuous function $f : W_0 \to \mathbb{R}$, $f(w) := f_w$.

Applying [16, Theorem 5], we deduce that the function f is semi-convex on W_0, thus there exists some $c \in \mathbb{R}$ such that the function

$$\hat{f}(w) := f(w) + c \cdot ||w||^2$$

is convex on W_0.

Consider the diffeomorphism $\Phi : \mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$

$$\Phi(w, t) := (w, t + c \cdot ||w^2||) .$$

Thus, $K = \Phi(C) \cap (W_0 \times \mathbb{R})$ is given in a neighborhood of 0 as the supergraph of the convex function $\hat{f}$. Thus, the smooth diffeomorphism Φ^{-1} sends a neighborhood of 0 in the convex set K onto a neighborhood of 0 in C. This finishes the proof of Theorem 1.2 in case $m = n$.

5.3 | Lipschitz continuity of tangent spaces

We turn to the general case $n \geq m$ and fix $C' \subset C, \varepsilon, \delta, r$ as in Section 5.1. As in Section 2.2, we denote by $\hat{T}_pC$ the m-dimensional Euclidean subspace of T_pM generated by the cone T_pC, for any $p \in C'$. We are going to deduce from Corollary 3.4, the following strengthening of Lemma 2.3:

Proposition 5.1. *The map $p \to \hat{T}_pC$ from C' to the Grassmanian $\mathbf{Gr}_{m,n}$ of m-dimensional subspaces of $\mathbb{R}^n$ is Lipschitz continuous.*

Before we embark on the proof, recall that one (of many biLipschitz equivalent) metric on the Grassmannian $\mathbf{Gr}_{m,n}$ is defined by

$$d(U, W) \leq \sup_{u \in U, ||u||=1} d(u, W) ,$$

for $U, W \in \mathbf{Gr}_{m,n}$. The distance $d(u, W)$ is the norm $||P(u)||$, where $P = P^{W^\perp}$ denotes the projection onto the orthogonal complement $W^\perp$ of W. The linearity of P directly implies the following.

Claim. If $u_0 \in U$ is a unit vector and for any u in the ball $B_\varepsilon(u_0) \cap U$ we have $d(u, W) \leq t$ then

$$d(U, W) \leq \frac{2t}{\varepsilon} .$$

Now, we can proceed with

Proof of Proposition 5.1. On C' the intrinsic metric is 2-biLipschitz to the induced one. Thus, it suffices to prove the Lipschitz property pointwise, hence, to verify the following.

Claim. There is some $\lambda = \lambda(\varepsilon, \delta) > 0$ such that, for any $p \in C'$,

$$\limsup_{q \in C, q \to p} \frac{d(\hat{T}_pC, \hat{T}_qC)}{||p - q||} \leq \lambda .$$

□

By Corollary 3.4, for any v in the ball O in T_pC of radius ε around a unit vector v_p and all q, sufficiently close to p, we have

$$d(v, T_qC) \leq 4\mu \cdot \delta \cdot ||p - q|| ,$$

with some universal constant μ. The observation preceding the proof of the proposition implies

$$d(\hat{T}_pC, \hat{T}_qC) \leq \frac{8\mu}{\varepsilon} \cdot \delta \cdot ||p - q|| ,$$

for all such q. This shows the claim with $\lambda = \rho \cdot 2\mu \cdot \delta$ and finishes the proof of Proposition 5.1.

5.4 | Finding a larger submanifold

In the setting of Section 5.1, we are going to show

Lemma 5.2. *There exists $s > 0$ and a $C^{1,1}$-diffeomorphism Ψ between two neighborhoods of $x = 0$ in $\mathbb{R}^n$, such that $\Psi(C' \cap \bar{B}_s(0)) \subset V = \mathbb{R}^m$.*

Once the lemma is verified, the image $Q := \Psi(C' \cap \bar{B}_s(0))$ is an m-dimensional subset of positive reach in V [3, Theorem 4.19]. Applying the result in the case $m = n$ obtained in Section 5.2, we find a diffeomorhism Φ_0 between two neighborhoods O_1, O_2 of 0 in V, which sends $O_1 \cap Q$ onto a convex subset K. Then, the statement of Theorem 1.2 follows by taking Φ to be the composition Φ_0 and Ψ^{-1}.

Therefore, it remains to provide:

Proof of Lemma 5.2. Rescaling the space we may assume that the reach ρ is at least 1. We fix some $\lambda \geq 1$ such that the map $p \to \hat{T}_p C$ is λ-Lipschitz on C', Proposition 5.1.

For any $0 < s < \frac{1}{4\lambda} \leq \frac{1}{4}$ and for any $p \in C' \cap \bar{B}_s(0)$, we have

$$d(\hat{T}_p C, V) \leq \frac{1}{4}\,.$$

We may replace r by s in the definition of C' and assume that $C' = C' \cap \bar{B}_s(0)$. For any $p \neq q \in C'$, we deduce from the above inequality and [3, Theorem 4.8] that

$$d(\frac{p-q}{||p-q||}, V) \leq d\left(\frac{p-q}{||p-q||}, \hat{T}_q C\right) + d\big(\hat{T}_q C, V\big) \leq \frac{s}{2} + \frac{1}{4} < \frac{1}{2}\,.$$

Consider the orthogonal projection $P : C' \to V$ and denote by $Q \subset V$ the compact image $Q := P(C')$. The above inequality implies that the 1-Lipschitz map $P : C' \to Q$ is injective and has a 2-Lipschitz continuous inverse $\Phi = P^{-1} : Q \to C'$.

We claim that it is sufficient to find an extension of Φ to a $C^{1,1}$-map $\Phi : V \to \mathbb{R}^n$. Once this is done, the differential of Φ at 0 will automatically be the identity. Hence, the map Φ will be a $C^{1,1}$-embedding of a neighborhood O_1 of 0 in V into $\mathbb{R}^n$. Then, making O_1 smaller if necessary, we can extend Φ to a diffeomorphism between two neighborhoods of 0 in $\mathbb{R}^n$. In this case, the inverse map Ψ will be a $C^{1,1}$ diffeomorphism between two neighborhoods of 0 in $\mathbb{R}^n$ and such that $\Psi|_{C'} = P$. This would finish the proof.

It remains to find the extension of $\Phi : Q \to C'$ to a $C^{1,1}$-map $\Phi : V = \mathbb{R}^m \to \mathbb{R}^n$. In order to do so, we will rely on Whitney's extension theorem [20] in the form of [5].

For any $z \in Q$ with $\bar{z} = \Phi(z)$, consider the inverse map $f_z^1 : V \to \hat{T}_{\bar{z}} C \subset \mathbb{R}^n$ of the linear isomorphism $P : \hat{T}_{\bar{z}} C \to V$. Denote by $f_z : V \to \mathbb{R}^n$ the affine map ("Taylor plynomial of degree one")

$$f_z(q) := \Phi(z) + f_z^1(q - z)\,.$$

Whitney's extension theorem in the form of [5, Proposition VII] implies the following. There exists an extension of $\Phi : Q \to \mathbb{R}^n$ to a $C^{1,1}$-map $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ with $D_z \Phi = f_z^1$ for all $z \in Q$ if and only if for some $c > 0$ and all $y, z \in$ the two subsequent conditions are satisfied:

- $||f_z^1 - f_y^1|| \leq c \cdot ||z - y||$.
- $||f_z(z) - f_y(z)|| = ||\Phi(z) - \Phi(y) - f_y^1(z - y)|| \leq c \cdot ||z - y||^2$.

Set $\bar{z} := \Phi(z)$ and $\bar{y} := \Phi(y)$.

In order to verify the first item above, consider an arbitrary unit vector $v \in V$. Then, the vector $f_z^1(v) \in \hat{T}_{\bar{z}} C$ has norm at most 2. By Proposition 5.1, we can find some vector $w \in \hat{T}_{\bar{y}} C$ with

$$||w - f_z^1(v)|| \leq 2 \cdot \lambda \cdot ||\bar{z} - \bar{y}|| \leq 4 \cdot \lambda \cdot ||z - y||\,,$$

where λ is the Lipschitz constant provided in Proposition 5.1. Then

$$||P(w) - v|| = ||P(w - f_z^1(v))|| \leq 4 \cdot \lambda \cdot ||z - y||\,.$$

Hence,

$$||w - f_y^1(v)|| = ||f_y^1(P(w) - v)|| \leq 8 \cdot \lambda \cdot ||z - y||$$

With $c := 12\lambda$ the triangle inequality implies

$$||f_z^1(v) - f_y^1(v)|| \leq c \cdot ||z - y|| \; .$$

Since v was an arbitrary unit vector, this shows the first item.

In order to verify the second item, we apply [3, Theorem 4.18] and find a vector $w \in T_{\bar{y}}C \subset \hat{T}_y C$ with

$$w - (\bar{z} - \bar{y}) \leq \frac{1}{2}||\bar{z} - \bar{y}||^2 \leq 2 \cdot ||z - y||^2$$

Therefore

$$||P(w) - (z - y)|| \leq 2 \cdot ||z - y||^2 \; .$$

Hence

$$||w - f_y^1(z - y)|| \leq 4 \cdot ||z - y||^2 \; .$$

By the triangle inequality, we deduce

$$||\bar{z} - \bar{y} - f_y^1(z - y)|| \leq 6 \cdot ||z - y||^2 \; .$$

Thus, for $c = \max\{12\lambda, 6\}$ this finishes the verification of both conditions, the proof of Lemma 5.2, and of Theorem 1.2. □

ACKNOWLEDGMENTS

I am grateful to Joe Fu, Daniel Hug, and Jan Rataj for helpful comments and discussions. I would like to thank the anonymous referees for careful reading and suggestions. The author was partially supported by the DFG grants SFB TRR 191.

Open access funding enabled and organized by Projekt DEAL.

REFERENCES

[1] V. Bangert, *Sets with positive reach*, Arch. Math. (Basel) **38** (1982), no. 1, 54–57.
[2] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001.
[3] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.
[4] J. Fu, *Tubular neighborhoods in Euclidean spaces*, Duke Math. J. **52** (1985), no. 4, 1025–1046.
[5] G. Glaeser, *Étude de quelques algèbres tayloriennes*, J. Anal. Math. **6** (1958), no. 2, 1–124, erratum.
[6] D. Hug and M. Santilli, *Curvature measures and soap bubbles beyond convexity*, Adv. Math. **411** (2022), no. part A, 108802.
[7] D. Hug, *Generalized curvature measures and singularities of sets with positive reach*, Forum Math. **10** (1998), no. 6, 699–728.
[8] V. Kapovitch and A. Lytchak, *Remarks on manifolds with two-sided curvature bounds*, Anal. Geom. Metr. Spaces **9** (2021), no. 1, 53–64.
[9] V. Kapovitch and A. Lytchak, *The structure of submetries*, Geom. Topol. **26** (2022), no. 6, 2649–2711.
[10] N. Kleinjohann, *Nächste Punkte in der Riemannschen Geometrie*, Math. Z. **176** (1981), no. 3, 327–344.
[11] A. Lytchak and V. Schroeder, *Affine functions on* CAT(κ)*-spaces*, Math. Z. **255** (2007), no. 2, 231–244.
[12] G. Loeper and C. Villani, *Regularity of optimal transport in curved geometry: the nonfocal case*, Duke Math. J. **151** (2010), no. 3, 431–485.
[13] A. Lytchak, *On the geometry of subsets of positive reach*, Manuscr. Math. **115** (2004), no. 2, 199–205.
[14] A. Lytchak, *Almost convex subsets*, Geom. Dedicata **115** (2005), 201–218.
[15] G. Perelman, *A. D. Alexandrov spaces with curvature bounded below II*, 1991, preprint.
[16] Yu. G. Reshetnyak, *On a generalization of convex surfaces*, Mat. Sb. N.S. **40** (1956), no. 82, 381–398.
[17] J. Rataj and L. Zajíček, *On the structure of sets with positive reach*, Math. Nachr. **290** (2017), no. 11–12, 1806–1829.
[18] J. Rataj and M. Zähle, *Curvature measures of singular sets*, Springer Monographs in Mathematics, Springer, Cham, 2019.

[19] Ch. Thaele, *50 years sets with positive reach—a survey*, Surv. Math. Appl. **3** (2008), 123–165.
[20] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89.

How to cite this article: A. Lytchak, *A note on subsets of positive reach*, Math. Nachr. (2023), 1–11.
https://doi.org/10.1002/mana.202300094