Journal of
Mathematical Physics

RESEARCH ARTICLE | NOVEMBER 06 2023

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J. Math. Phys. 64, 113502 (2023)
https://doi.org/10.1063/5.0152374

Journal of
Mathematical Physics

ARTICLE

pubs.aip.org/aip/jmp

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Cite as: J. Math. Phys. 64, 113502 (2023); doi: 10.1063/5.0152374
Submitted: 29 March 2023 • Accepted: 12 October 2023 •
Published Online: 6 November 2023

View Online Export Citation CrossMark

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ABSTRACT

For a finite graph, a spectral curve is constructed as the zero set of a two-variate polynomial with integer coefficients coming from p-adic diffusion on the graph. It is shown that certain spectral curves can distinguish non-isomorphic pairs of isospectral graphs, and can even reconstruct the graph. This allows the graph reconstruction from the spectrum of the associated p-adic Laplacian operator. As an application to p-adic geometry, it is shown that the reduction graph of a Mumford curve and the product reduction graph of a p-adic analytic torus can be recovered from the spectrum of such operators.

I. INTRODUCTION

The aim of spectral geometry is to describe the relationship between the geometry of certain objects like surfaces, or more general Riemannian manifolds, and the spectra of differential operators, like Laplacians, defined on them. In other words, as stated by Kac,[1] "Can one hear the shape of a drum?" Ideally, one would like to be able to recover the geometric object, up to isometry, from the spectra of one or several naturally defined operators. Many counter-examples for Riemanian manifolds have appeared showing that isospectral but non-isometric manifolds exist, which gives a negative answer to the question. For example for the drum problem:

$$\begin{cases} \nabla u = -\lambda u \\ u|_{\partial D} = 0 \end{cases}$$

Gordon, Webb, and Wolpert in 1992, showed the existence of a pair of drums with different shapes but which are isospectral.[2] On the other hand, information about the geometry of the object can be extracted from the spectrum. This kind of problems is known as "inverse problems." Many famous results towards this direction have been established, for example the famous Weyl asymptotic law.[3] These problems extend to objects other than Riemannian manifolds, like graphs, where for the adjacency matrix and the Laplacian, non-isomorphic isospectral graphs have been found. This problem has been intensively studied.[4–6] Recovering the structure of a graph from the spectrum of an operator may lead to an invariant to describe the topology of the graph. This could lead to new applications, like recovering the structure of a graph from a diffusion process, which has many potential applications, e.g., for topological access methods for spatial data,[7] to name only one.

Prime numbers play a fundamental role in many mathematical theories and applications to sciences. From the realm of arithmetic as the fundamental blocks or "atoms" of integers, to applications in physics, from quantum physics to the theory of complex disordered systems and geophysics, information processing, biology, and cognitive science[8] and the references therein. One powerful framework for the application of number theory in sciences is the so called p-adic analysis or more general ultrametric analysis.[9,10] An important example is given in the theory of disordered systems (spin glasses) where the p-adic structure is encoded in a Parisi matrix which arises by the intrinsic hierarchical structure inside the spin glasses.[11] This lead in the middle of the 1980s to the idea of using ultrametric spaces to describe the state of complex systems. A central idea in physics of complex systems (like proteins) states that the dynamics on such systems is generated by a random walk (diffusion equation) in the corresponding energy landscape. By using interbasin kinetics methods, an energy landscape is approximated by an ultrametric space and a function on this space describing the distribution of the activation barriers[12] and the references therein. Most of the applications towards this direction recquire well-defined and natural pseuddifferential operators constructed on ultrametric structures such as non-Archimedean fields, where the Taibleson–Vladimirov operator plays a fundamental role for diffusion on the field p-adic numbers.[9] Differential operators and spectral geometry on Riemannian manifolds have been extensively studied, nevertheless there is no comparable theory of non-Archimedean spectral geometry of pseudodifferential operators over p-adic structures. Many other operators have been developed, some of them with the aim of applications, and others as generalisations to more general structures. For the former we have many classes of p-adic operators from the work of Zúñiga, Kozyrev, and Khrennikov, where the relation of graph theory and p-adic integral and pseudodifferential operators is explicitly stated,[10,13,14] and the reference therein.

For the latter one of the authors initiated the study of heat equations and integral operators on the non-Archimedean kind of Riemannian surfaces i.e., Mumford Curves.[15] All those developments in the theory of pseudodifferential equations over non-Archimedean spaces clearly deal (indirectly) with one of the main problems in spectral geometry, that is, direct problems in which a description of the eigenvalues is needed.

In this article we initiate the study of inverse problems of spectral geometry in the non-Archimedean framework. Moreover, a new invariant for an arbitrary combinatorial simple graph is introduced, showing that the spectra of certain p-adic operators defined on the graph lead to a complete characterisation of its isomorphisim class. The question "Can you hear the shape of a graph?" has already been answered in different contexts. The question was posed in the context of quantum graphs, and was answered in the affirmative, that is, they showed that the spectrum of the Schrödinger operator on a finite, metric graph determines uniquely the connectivity matrix and the bond lengths under certain conditions.[16] A new spectral invariant in quantum graphs has been introduced.[17] It was proved that the spectral determinant of the Laplace operator on a finite connected metric graph determines the number of spanning trees under certain conditions.[18] Understanding how the spectra of certain operators in general graphs determine the geometry of a graph is an important task for applications like graph comparison in graph analytics. For example, the Network Laplacian Spectral Descriptor, a graph representation method that allows for straightforward comparisons of large graphs, is proposed.[19] Moreover, our results are applied to p-adic structures like Mumford curves and p-adic analytic tori. Hence these results initiate the study of inverse problems in spectral geometry in the non-Archimedean framework.

Given a graph G and a matrix $\Delta \in \mathbb{N}^{|G|\times|G|}$, we study a generalisation of a graph Laplacian Λ_G^Δ defined in $L^2(G\times K)$, where K is a non-Archimedean local field. The space $L^2(G\times K)$ can be decomposed as a direct sum of finite dimensional spaces of dimension $|G|$, this leads to the following representation of Λ_G^Δ,

$$\Lambda_G^\Delta = \bigoplus_{G\mathcal{K}} L(G_r^\Delta),$$

where the matrices $L(G_r^\Delta)$ are the Laplacian matrix of a weighted version of the graph, and for $r = 1$, we have that $G_1 = G$. Therefore, this operator can be understood as a direct sum of scaled replica of the original graph. The spectrum of each copy belongs to a common plane algebraic curve $V(P_G^\Delta)$ called the spectral curve of the graph. For a suitable choice of Δ we prove that P_G^Δ is an invariant of the graph G. This leads to a reconstruction theorem which enable us to reconstruct the graph through the spectra of the operator Λ_G^Δ (see Theorem 4.7 and Corollary 4.9). Finally using these results we are able to reconstruct the reduction graph of a Mumford curve and the product graph coming from the reduction of a p-adic analytic torus using the spectrum of a p-adic Laplacian.

II. NOTATION AND SOME RESULTS FROM p-ADIC ANALYSIS

In this section we review some results from p-adic analysis, for a complete exposition of the subject and proofs the reader may consult Kochubei's book.[20]

Let K be a non-Archimedean local field. Let $|\cdot|_K$ denote the absolute value of the field K. Denote the local ring of K by $\mathcal{O}_K = \{x \in K : |x|_K \le 1\}$ and its maximal ideal by $\mathfrak{m}_K = \{x \in K : |x|_K < 1\}$. Let χ be a fixed non-constant complex-valued additive character on K. We denote by dx the Haar measure on the additive group of K, normalised such that the measure of $\mathcal{O}_K$ is equal to 1. The Fourier transform of an absolute integrable complex-valued function $f \in L^1(K)$ will be written as

$$\mathcal{F}(f)(\xi) = \int_K \chi(\xi x) f(x)dx,\ \ \xi \in K.$$

If $\mathcal{F}(f) = \hat{f} \in L^1(K)$, we get the inversion formula

$$f(x) = \int_K \chi(-x\xi)\hat{f}(\xi)d\xi.$$

Since the mapping $\mathcal{F} : L^1(K) \cap L^2(K) \to L^2(K)$ is an isometry, this mapping has an extension to an L^2-isometry from $L^2(K)$ into $L^2(K)$, where the inverse Fourier transform will be denoted as $\mathcal{F}^{-1}$.

Now we introduce the Vladimirov–Taibleson operator. Let $\mathcal{D} \subset L^2(K)$ be its domain given by the set of those $f \in L^2(K)$, for which $|\xi|^\alpha \hat{u}(\xi) \in L^2(K)$. The Vladimirov operator $(\Delta^\alpha, \mathcal{D})$, $\alpha > 0$, for $f \in \mathcal{D}$ is defined by

$$\Delta^\alpha f(x) = \mathcal{F}^{-1}_{\xi\mapsto x}(|\xi|_K^\alpha \mathcal{F}_{x\mapsto\xi}(f)(\xi))(x),\ \ x \in K.$$

where $\mathcal{F}_{\xi\mapsto x}$ means that the Fourier transform takes functions f in the variable x to functions $\hat{f}$ in the new variable ξ. The operator Δ^α is an unbonded operator in $L^2(K)$, and since it is unitarily equivalent to the operator of multiplication by $|\xi|_K^\alpha$, it is self-adjoint, its spectrum consists of the eigenvalues $\lambda_r = q^{\alpha r}$, where $r \in \mathbb{Z}$ and q is the cardinality of the residue field $\mathcal{O}_K/\mathfrak{m}_K$. Moreover we have the following result

Theorem II.1 *(Kozyrev). There exist a complete orthonormal system of eigenfunctions of the operator Δ^α of the form $\psi_{r,n}(x) \in L^2(K)$, where $r \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that*

$$\Delta^\alpha \psi_{r,n}(x) = q^{\alpha(1-r)}\psi_{r,n}.$$

Proof. For the case of $\mathbb{Q}_p$, this was proven by Kozyrev.[21] The case of a general non-Archimedean local field K can be found e.g., in Kochubei's book.[20] □

Henceforth this basis of $L^2(K)$ from Theorem II.1 will be denoted by $\mathcal{K}$.

III. SOME CONCEPTS FROM ALGEBRAIC GEOMETRY

In order to appreciate the last part of this article, we review some related concepts from algebraic geometry. First of all, our main focus in this area is on algebraic curves. These appear in the integrable chiral Potts model. Loosely speaking, an *affine algebraic curve* is the solution set of a system of algebraic equations

$$\begin{aligned} F_1(x_1,\ldots,x_n) &= 0 \\ &\vdots \\ F_{n-1}(x_1,\ldots,x_n) &= 0 \end{aligned}$$

with polynomials $F_1,\ldots,F_{n-1}$ in n variables $X_1,\ldots,X_n$, as long as the solution space in the algebraic closure of the field, from which the coefficients are taken from, is one-dimensional. In this article, we will not need all of the highly sophisticated machinery developped in the 19th and 20th century to both describe and to study such a system of equations. In order to define later in this article the concept of *spectral curve*, we simply need the concept of an *affine plane algebraic curve*. This is the case when $n = 2$, i.e., when we have

$$F(x,y) = 0$$

for some polynomial $F(X,Y) \in k[X,Y]$, where $k[X,Y]$ stands for the space of all polynomials in two variables X, Y and coefficients in the field k. Indeed, we allow the field k to be arbitrary, but will use only the case $k = \mathbb{C}$, when defining the term *spectral curve*. In fact, the coefficients will be integers, but we allow the solution space to be inside $\mathbb{C}^2$. This allows to define a plane algebraic curve in this form:

$$V(F)$$

where $V(F) \subset \mathbb{C}^2$ means the *vanishing set* of the polynomial F in the complex 2-plane $\mathbb{C}^2$.

The applications of our results that we have in mind is the case of certain algebraic curves defined over a local field, of which the p-adic numbers are a special case. Such curves have appeared also in p-adic string theory, as they can be seen as a counterpart of Riemann surfaces. These algebraic curves are called *Mumford curves*. They are not affine, but *projective*. This means that they are obtained by patching together affine pieces. There is a well-defined procedure for doing this, but it is beyond the scope of this article to present all the details. Let us just say that this procedure generalises the way in which the Riemann sphere is obtained by patching together two copies of the affine algebraic curve $\mathbb{C}$. The parts of $\mathbb{C}$ that are patched toegether are $\mathbb{C}\setminus\{0\}$, and the identification map

$$z \mapsto z^{-1}$$

is used to glue the two pieces together. The result is the projective line $\mathbb{P}^1(\mathbb{C})$ which extends $\mathbb{C}$ by an extra point ∞. In a similary way, all Riemann surfaces are obtained by patching together finitely many pieces which are each affine algebraic curves. This method reveals the fact that all Riemann surfaces are isomorphic with non-singular projective algebraic curves defined over $\mathbb{C}$, and vice versa. The converse means that projective algebraic curves (locally given by algebraic equations as above) can be obtained analytically by pasting finitely many open domains in the complex plane. Topologically, a Riemann surface is determined by the number of "tunnels" known as its *genus*. So, through the correspondence with projective algebraic curves, these also have the genus as a topological invariant.

The non-Archimedean world is in this respect more complicated. In this world, not all non-singular projective algebraic curves defined over a local field can be obtained by pasting finitely many open domains in the non-Archimedean complex plane. Those curves that do, are called *Mumford curves*. They allow for a coarse combinatorial description, since open domains in non-Archimedean fields correspond to a rooted tree, and the pasting operation produces a finite graph. It is a deep result that the number of independent holes of this graph coincides with the genus of the Mumford curve. And knowing the graph structure of a Mumford curve tells us something about its non-Archimedean analytic structure. For this reason, to be able to "hear" the graph structure from a Laplacian-like operator is of interest.

The second concept related to a projective algebraic curve is a so-called *abelian variety*. A special case of these occur as *Jacobians* of projective Riemann surfaces. These encode their hidden periodic structure, and have themselves become the subject of research. A Jacobian is a manifold of dimension g, where g is the genus of the Riemann surface it is derived from. The name "abelian variety" comes from the fact that they have an analytic group structure which turns them into abelian groups. The Jacobian associated with a Mumford curve has a simple form. Namely, it is given as the g-fold product of the multiplicative group $K^\times$ modulo the action of a multiplicative lattice Λ of rank g. The notation used for it in Sec. VI below uses the multiplicative group $\mathbb{G}_m$ as a group variety, and is actually more correct than using $K^\times$. But again, this would mean to invoke too much sophisticated machinery from algebraic geometry, if this is to be elucidated here. However, the points of $\mathbb{G}_m$ having coordinates in the field K are precisely the multiplicative group $K^\times$. So, it does not really matter here. . .. In any case, this notion of non-Archimedean Jacobian is the multiplicative version of its classical counterpart as the g-fold product of the additve group of complex numbers modulo an additve lattice of rank $2g$. Such a manifold is topologically a g-dimensional torus. That is the reason why non-Archimedean geometers call its non-Archimedean counterpart a *p-adic analytic torus* (if K is the field of p-adic numbers). And hearing its shape also in principle reveals some periodic information about the Mumford curve, if it happens to be its Jacobian.

IV. SPECTRAL CURVES FOR DIFFUSION PAIRS

In this section, we introduce the objects necessary for constructing the spectral curve of a so-called *diffusion pair* which is actually nothing but a weighted graph, where the weights are integer powers of a fixed variable Y. These objects are p-adic matrix-valued Laplacian operators reflecting the adjacency structure of a graph.

A. *p*-Adic Laplacians for graphs

Let $G \subset K/O_K$ be a finite set. Then we have isomorphisms

$$L^2(G\times K) \cong L^2(G)\otimes L^2(K) \cong \bigoplus_{a\in G} L^2(K_a)$$

where K_a is a copy of K for each $a \in G$. We define maps:

$$\bigoplus_{a\in G} L^2(K_a) \xrightarrow{H_G} \bigoplus_{a\in G} L^2(K_a) \tag{1}$$

where we write

$$L^2(K_a) = \bigoplus_{\psi\in\mathcal{K}} \mathbb{C}\psi_a$$

using the set $\mathcal{K}$ of Kozyrev wavelets on K, and

$$\psi_a(x) = \psi(x).$$

The map H_G is given as follows:

$$H_G : (u_g)_{g\in G} \mapsto (f_g)_{g\in G},\ f_g = \sum_{a\in G} C_{g,a}\Delta_{g,a}u_a$$

where $\Delta_{g,a}$ is the Vladimirov operator

$$\Delta_{g,a} : L^2(K_a) \longrightarrow L^2(K_g), \psi_a \mapsto \Delta^{\alpha_{g,a}}\psi_g$$

where

$$\Delta^{\alpha_{g,a}} = \mathcal{F}^{-1}|\cdot|_K^{\alpha_{g,a}}\mathcal{F}$$

behaves like the usual Vladimirov operator, except for being applied to different copies of Kozyrev wavelets indexed by vertices of G. In particular, it simply multiplies the indexed Kozyrev wavelet ψ_g by an integer power of $q^{\alpha_{g,a}}$.

Notice that in the basis of $L^2(G\times K)$ given by

$$G\mathcal{K} = \{\psi_g : g\in G,\ \psi\in\mathcal{K}\}$$

we can represent H_G by the $|G|\times|G|$-matrix

$$(C_{a,g}\Delta_{a,g}).$$

And the matrix $C = (C_{a,g}) \in \mathbb{C}^{|G|\times|G|}$ is uniquely determined by the given bases, and can be viewed as an adjacency matrix of a simple graph with vertex set G.

In order to obtain a graph Laplacian matrix, we consider instead of H_G the operator

$$\Lambda_G^\Delta : L^2(K)^{|G|} \to L^2(K)^{|G|} \tag{2}$$

represented by the matrix

$$(L^{\Delta}_{a,b})_{a,b\in G}$$

with

$$L^{\Delta}_{a,b} = \begin{cases} -C_{a,b}\Delta_{a,b}, & a \neq b \\ \sum_{g\in G} C_{a,g}\Delta_{a,g}, & a = b. \end{cases}$$

Here, $\Delta = (\Delta_{g,a})$ can be viewed as a matrix in $\mathbb{N}^{|G|\times|G|}$ having entry $\alpha_{g,a}$ whenever (g,a) represents an edge of the graph.

Later, we will show that there exist choices of diffusion parameters $\alpha_{a,g} \in \mathbb{N}$ such that the spectrum of the operator Λ^{Δ}_G determines the isomorphism class of the combinatorial simple graph G.

Definition IV.1. The operator Λ^{Δ}_G is called the p-adic Laplacian associated with the diffusion pair (G,Δ).

B. The spectral curve of a diffusion pair

Let (G,Δ) be a diffusion pair. Recall that

$$L = D - A$$

is the (combinatorial) Laplacian of the graph G. Here, D is the diagonal matrix containing the vertex degrees, and A the adjacency matrix of G. The latter has an entry $a_{i,j} = 1$ precisely when (i,j) represents an edge of the graph, and 0 otherwise. We begin with the following observation:

Lemma IV.2. It holds true that

$$\mathrm{Spec}(L) \subset \mathrm{Spec}(\Lambda^{\Delta}_G)$$

as an inclusion of multi-sets.

Proof. Observe first that in the basis $G\mathcal{K}$, the operator Λ^{Δ}_G from (2) is represented by an $\mathbb{N}\times\mathbb{N}$-matrix having a block-diagonal structure with blocks of size $|G|\times|G|$ after a suitable linear ordering of the basis. Now, the non-zero entries of each block away from the diagonal consist of Vladimirov eigenvalues. By Theorem II.1, they are of the form

$$q^{\alpha_{a,g}(1-r)}$$

with $r \in \mathbb{Z}$. For $r = 1$ we identify the Laplacian matrix L as one of the blocks. Hence, the eigenvalues of L are contained in the spectrum of Λ^{Δ}_G. □

Observe further that the block-diagonal structure found in the proof of the above lemma is in fact a replication of Laplacian matrices for the same combinatorial graph structure on G, except that now the edge lengths are powers of p of the form $p^{\alpha_{a,b}(1-r)}$ for fixed $r \in \mathbb{Z}$. This means that there is a family of graphs G^{Δ}_r parametrised by $r \in \mathbb{Z}$ having the same (combinatorial) Laplacian matrix L. And in order to find all eigenvalues of Λ^{Δ}_G, it is necessary and sufficient to find the Laplacian eigenvalues for each graph in the family G^{Δ}_r with $r \in \mathbb{Z}$.

We will now examine the characteristic polynomial of each graph Laplacian L^{Δ}_r associated with graph G^{Δ}_r from the family. Notice that

$$L^{\Delta}_1 = L_1 = L, \quad G^{\Delta}_1 = G_1 = G,$$

are independent of the diffusion parameters symbolised by Δ. We also assume that the parameters $\alpha_{a,b}$ are all pairwise different positive natural numbers. The characteristic polynomial of G^{Δ}_r is

$$P^{\Delta}_r(X) \in \mathbb{Q}[X]$$

and its degree is $|G|$. Again, we have

$$P_1(X) = P^{\Delta}_1(X)$$

is independent of Δ, and coincides with the characteristic polynomial of the graph Laplacian L.

The coefficients of P^{Δ}_r are given by the Leibniz formula for the determinant as polynomials with integer coefficients in another variable Y evaluated in p^{1-r}. Hence, we obtain a polynomial

$$P^{\Delta}_G(X,Y) \in \mathbb{Z}[X,Y]$$

whose zero set in $\overline{\mathbb{Q}}^2$ contains the spectrum of Λ^{Δ}_G as the first coordinate of some of its points. Here, we mean by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$. Observe that

$$P^{\Delta}_G(X,p^{1-r}) = P^{\Delta}_r(X)$$

for $r \in \mathbb{Z}$.

Definition IV.3. The complex plane algebraic curve $V(P_G^\Delta)$ is called the spectral curve of the pair (G, Δ).

Recall that $V(P_G^\Delta)$ is defined by the equation

$$P_G^\Delta(x, y) = 0$$

with $x, y \in \mathbb{C}$, and is a so-called *affine algebraic curve* in the complex 2-plane.

C. Recovering spectral curves

Assume that we are given the $\mathrm{Spec}(\Lambda_G^\Delta)$ as a multi-set, where Λ_G^Δ is the p-adic Laplacian associated with diffusion pair (G, Δ). Assume also that the task is to recover the graph G from that spectrum. One way would be to try to recover the spectral polynomial $P(X, Y) = P_G^\Delta(X, Y)$ and use the Reconstruction theorem (Theorem V.7) proved below.

In this situation, an algorithm which terminates in finite time cannot be expected, because each individual eigenvalue has to be associated with one of the graphs G_r $(r \in \mathbb{Z})$ in the family induced by the spectral pair. But from a purely existential standpoint, we can say that there exists a classification of eigenvalues (including their multiplicities) such that each class is $\mathrm{Spec}(L_r)$ with $r \in \mathbb{Z}$. Once this classification is made, then each coefficient

$$a_i(p^{1-r}), \quad i = 1, \ldots, n$$

of the polynomial

$$P(X, Y) = \sum_{i=1}^{n} a_i(Y) X^i$$

with $r \in \mathbb{Z}$ can be calculated in each class of eigenvalues. All that is then needed, is for each $r \in \mathbb{Z}$ the value of

$$P(X, p^{1-r})$$

in finitely many places x_s. Then interpolation yields the coefficients of $P(X, Y)$.

Definition IV.4. A set of pairs (x_s, p^{1-r}) with $s, r \in R \subset \mathbb{N}$ is called a recovery datum, if R is a finite set and $P(X, Y)$ can be interpolated after evaluating the polynomial in that set of pairs.

Theorem IV.5. *Let (G, Δ) be a diffusion pair. Given $\mathrm{Spec}(L_r)$ as distinguished multi-sets for sufficiently but finitely many $r \in \mathbb{Z}$, it is possible to obtain recovery data for the spectral polynomial P_G^Δ with a terminating algorithm.*

Proof. Since all graphs G_r are simple and have the same underlying combinatorial graph with n vertices and

$$|E| \le \frac{1}{2} n(n-1) =: b_n$$

edges, it follows that the number of places to interpolate

$$P(X, Y) = P_G^\Delta(X, Y)$$

is bounded. As n is given as the size of each multi-set $\mathrm{Spec}(L_r)$, it follows that b_n such spectra are sufficient in order to reproduce the characteristic polynomials

$$P(X, p^{1-r}) = P_r(X)$$

of the graphs G_r. Evaluating the b_n polynomials $P_r(X)$ at b_n places $x_s \in \mathbb{R}$ yields pairs (x_s, p^{1-r}) which form a recovery datum, as now interpolation of $P(X, Y)$ is possible. Together, this is an algorithm terminating after finitely many steps. □

The question is now, whether it is possible to extract distinguished multi-sets $\mathrm{Spec}(L_r)$ somehow by clustering spectral values. If it is allowed to vary the prime number p, then this can be done in the following game:

Game 1. Assume that you are allowed to choose diffusion parameters Δ once, and a prime number p as many times as you wish. Then you will receive for each p the multi-set $\mathrm{Spec}(\Lambda_G^\Delta)$ of an unknown simple finite connected graph. If you manage to recover the spectral curve for the diffusion pair (G, Δ) from these spectra, then you win, otherwise you lose.

The following theorem states that there exist winning strategies for the Game 1:

Theorem IV.6. *For the p-adic Laplacian associated with any diffusion pair, there exists a winning strategy in order to obtain distinguished multi-sets $\mathrm{Spec}(L_r)$ with $r \in \mathbb{Z}$.*

Proof. In the case that the choice of diffusion parameters is to have them all equal to 1 (when belonging to an edge, otherwise, it is zero), a winning strategy for connected graphs is to pick a large prime p. In this case, we have

$$L_r = p^{1-r}L$$

where L is the Laplacian of the graph G.

Since the non-zero part of the spectrum of L lies inside a compact interval not containing 0, as has been proven by Fiedler,[22] it then suffices to ask for higher and higher prime numbers until there are increasing gaps between clusters with relatively small inter-cluster distances between neigbouring points. With increasing p, this phenomenon becomes more and more clearly visible.

If not all parameters are chosen equal to 1, we restrict to $r \le 1$ and again vary the prime p. From Matrix Perturbation Theory,[23] we get that if p is sufficiently large, then in the range $r < 1$, again the inter-cluster distances of neighbouring eigenvalues will be smaller than the intra-cluster distances between neighbouring clusters. Hence, choosing p sufficiently large, again removes overlaps between the clusters. This effect can be explained in the same way as in the Proof of Theorem V.3 below. Namely, Eq. (3) shows how an edge weight of $\epsilon = p^{1-r}$ with integer $r < 1$ and $p \to \infty$ perturbates the eigenvalues in the way described above.

Since in both cases of diffusion parameter choices, the spectrum of L remains fixed for any choice of varying the prime p, the reference cluster for $r = 1$ can be extracted, and then the spectra $\mathrm{Spec}(L_r)$ for more values of $r \in \mathbb{Z}$, or of $r \le 1$ in the second case. After having extracted sufficently many of these finite spectra, one can proceed to the interpolation method and compute a recovery datum, as now the requirements for Theorem IV.5 are met. □

V. SPECTRAL CURVES WHICH ARE SEPARATING

In this section, we first separate pairs of non-isomrophic, but isospectral, graphs via spectral curves for suitable diffusion pairs using analytic matrix perturbation theory. After that, we prove for every finite graph the existence of a diffusion pair such that the graph can be reconstructed from the spectral polynomial. Although this generalises the first result, we believe that the matrix perturbation method is of general interest, nevertheless.

Example V.1. According to Mednykh and Mednykh,[5] the two graphs in Fig. 1 are isospectral. Their first Betti number equals 3. This is the smallest example of an isospectral pair of non-isomorphic simple graphs without bridges. Recall that a bridge is an edge which separates a connected component of a graph into two disconnected parts if removed. The label "2" on an edge indicates a diffusion parameter value of 2, i.e., an edge weight Y^2. Unlabelled edges have diffusion parameter value 1, i.e., edge weight Y. Their respective spectral polynomials P_1 for the left, and P_2 for the right graph of Fig. 1 are:

$$P_1(X,Y) = \det\begin{pmatrix} X-3Y & Y & 0 & 0 & 0 & 0 & Y & Y \\ Y & X-2Y & Y & 0 & 0 & 0 & 0 & 0 \\ 0 & Y & X-2Y & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & Y & X-2Y & Y & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & X-2Y & Y & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & X-3Y & Y & Y \\ Y & 0 & 0 & 0 & 0 & Y & X-(2Y+Y^2) & Y^2 \\ Y & 0 & 0 & 0 & 0 & Y & Y^2 & X-(2Y+Y^2) \end{pmatrix}$$

$$P_2(X,Y) = \det\begin{pmatrix} X-3Y & Y & Y & 0 & 0 & 0 & 0 & Y \\ Y & X-2Y & Y & 0 & 0 & 0 & 0 & 0 \\ Y & Y & X-3Y & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & Y & X-2Y & Y & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & X-2Y & Y & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & X-(2Y+Y^2) & Y & Y^2 \\ 0 & 0 & 0 & 0 & 0 & Y & X-2Y & Y \\ Y & 0 & 0 & 0 & 0 & Y^2 & Y & X-(2Y+Y^2) \end{pmatrix}$$

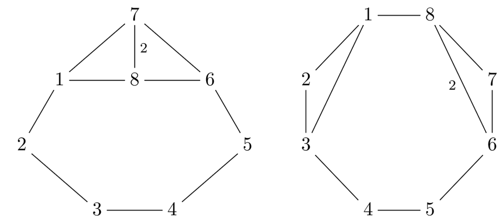

FIG. 1. A pair of non-isomorphic, but isospectral graphs whose first Betti number is 3.

Their tangent cones T_1 of P_1 and T_2 of P_2 are:

$$T_1(X,Y) = \det\begin{pmatrix} X-3Y & Y & 0 & 0 & 0 & 0 & Y & Y \\ Y & X-2Y & Y & 0 & 0 & 0 & 0 & 0 \\ 0 & Y & X-2Y & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & Y & X-2Y & Y & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & X-2Y & Y & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & X-3Y & Y & Y \\ Y & 0 & 0 & 0 & 0 & Y & X-2Y & 0 \\ Y & 0 & 0 & 0 & 0 & Y & 0 & X-2Y \end{pmatrix}$$

$$T_2(X,Y) = \det\begin{pmatrix} X-3Y & Y & Y & 0 & 0 & 0 & 0 & Y \\ Y & X-2Y & Y & 0 & 0 & 0 & 0 & 0 \\ Y & Y & X-3Y & Y & 0 & 0 & 0 & 0 \\ 0 & 0 & Y & X-2Y & Y & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & X-2Y & Y & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & X-2Y & Y & 0 \\ 0 & 0 & 0 & 0 & 0 & Y & X-2Y & Y \\ Y & 0 & 0 & 0 & 0 & 0 & Y & X-2Y \end{pmatrix}$$

These are spectral polynomials of the two bridgeless graphs of genus two, where each edge has the same variable Y, shown in Fig. 2. According to Theorem 3.1 of Mednykh and Mednykh,[5] these graphs are not isospectral, because they are not isomorphic. It follows that the two tangent cones T_1 and T_2 are not equal.

We saw that replacing one certain edge weight in each graph by Y^2 leads to two polynomials $P_1(X,Y)$ and $P_2(X,Y)$ which are not the same. So, in this case, the isospectral pair is separated by these two polynomials. It only happens that

$$P_1(X,1) = P_2(X,1)$$

resulting in two identical characteristic polynomials for the two non-isomorphic graphs. This example motivates the remainder of this article.

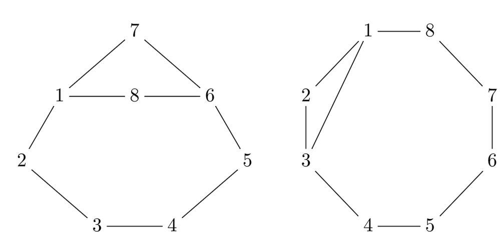

FIG. 2. A pair of non-isomorphic bridgeless graphs whose first Betti number is 2. According to Mednykh and Mednykh,[5] it follows that they are not isospectral.

A. Separating isospectral pairs via matrix perturbation

An introduction to matrix perturbation theory can be found in Matrix Perturbation Theory.[23] We will use this method in order to construct distinct spectral polynomials for non-isomorphic, but isospectral graphs. All our calculations are explicit and most of them reproduced here, even if many can also be found in that bibliographic reference in a more general setting.

Let $k \neq \ell$ be natural numbers in $\{1, \dots, n\}$. We define the matrix

$$U(k, \ell) = (u_{ij})$$

with

$$u_{ij} = \begin{cases} 1, & i = j = k, \text{ or } i = k = \ell \\ -1, & (i, j) = (k, \ell), \text{ or } (i, j) = (\ell, k) \\ 0, & \text{otherwise} \end{cases}$$

This is the Laplacian of the graph on n vertices having precisely one undirected edge (k, ℓ), since $U(k, \ell) = U(\ell, k)$.

Let $v = (v_1, \dots, v_n) \in \mathbb{R}^n$. Then

$$v^\top U(k, \ell) v = (v_k - v_\ell)^2,$$

as can be verified by a simple calculation.

Now, let E be a symmetric set of pairs (i, j) of numbers $1, \dots, n$ with $i \neq j$. Then

$$U(E) := \sum_{(k,\ell) \in E} U(k, \ell),$$

and we have

$$v^\top U(E) v = \sum_{(k,\ell) \in E} (v_k - v_\ell)^2.$$

We can now define

$$\|v\|_E^2 = v^\top U(E) v.$$

The function $\|\cdot\|_E$ is a semi-norm on $\mathbb{R}^n$, and we have

$$\|v\|_E^2 = 0 \quad \Leftrightarrow \quad \forall\ (i, j) \in E : v_i = v_j.$$

Lemma V.2. Let L be the Laplacian of a simple graph G with n vertices, and let E, E' be two disjoint sets of pairs from $\{1, \dots, n\}$. Then there exists an eigenvector v of L such that

$$\|v\|_E^2 \neq \|v\|_{E'}^2.$$

Proof. This follows from the fact that L is symmetric, i.e., from the Spectral theorem for symmetric real-valued matrices. Namely, if for all eigenvectors v of L we had

$$\|v\|_E = \|v\|_{E'},$$

then it would be impossible to generate via linear combination a vector $x \in \mathbb{R}^n$ having $\|x\|_E \neq \|x\|_{E'}$, because both seminorms scale identically when an eigenvector is multiplied with a scalar. □

An equivalence of diffusion pairs (G_1, Δ_1) and (G_2, Δ_2) where G_i are graphs on the same vertex set V, is given by a bijection $E_1 \to E_2$ between the edge sets of G_1 and G_2 which takes a weighted edge to an edge having the same weight. If two graphs G_1, G_2 are isospectral, then it is possible to define diffusion parameters Δ_1, Δ_2 such that there is an equivalence of diffusion pairs $(G_1, \Delta_1) \sim (G, \Delta_2)$. The following theorem shows that this already allows to distinguish non-isomorphic graphs, if suitable choices are taken.

Theorem V.3. *Assume that G_1, G_2 is a pair of non-isomorphic, but isospectral graphs. Then there exist equivalent diffusion pairs (G_1, Δ_1), (G_2, Δ_2) such that*

$$P_{G_1}^{\Delta_1}(X, Y) \neq P_{G_2}^{\Delta_2}(X, Y)$$

as bivariate polynomials.

Proof. Denote the edge set of graph G_i as E_i for $i = 1, 2$, and let $C = E_1 \cap E_2$, and $C_1 = E_1 \setminus E_2$, $C_2 = E_2 \setminus E_1$. Since G_1 and G_2 are isospectral, it follows that E_1 and E_2 have the same number of elements.

We are interested in the spectrum of the matrices

$$L_{\varepsilon,i} = U(C) + \varepsilon U(C_i)$$

for $i = 1, 2$, and with $\varepsilon > 0$. In first order with respect to the parameter ε, we have

$$\lambda^{(1)}_{\varepsilon,i} = \lambda + \varepsilon \|v\|^2_{C_i} + \text{terms of higher order in } \varepsilon, \tag{3}$$

where $v \in \mathbb{R}^n$ is an eigenvector of $U(C)$ associated with eigenvalue λ, and $\lambda^{(1)}_{\varepsilon,i}$ approximates in first order an eigenvalue $\lambda_{\varepsilon,i}$ of $L_{\varepsilon,i}$. According to Lemma V.2, one can find an eigenvector v of $U(C)$ such that

$$\|v\|_{C_1} \neq \|v\|_{C_2}.$$

This implies that the first order approximations of the eigenvalues $\lambda_{\varepsilon,i}$ are different. It follows that the eigenvalues $\lambda_{\varepsilon,1}$ and $\lambda_{\varepsilon,2}$ are different for the range of $\varepsilon > 0$ in which an analytic expansion in ε is possible and $\varepsilon > 0$ is sufficiently small.

Now, we have varied one eigenvalue of $U(C)$ analytically in two different ways. The other eigenvalues of $U(C)$ also vary analytically as in (3), except that possibly it could be that some eigenvalues corresponding to $L_{\varepsilon,1}$ and $L_{\varepsilon,2}$ are equal in this case. In any case, the spectrum of $U(C) = L_{\varepsilon,0}$ is varied continuously in two different manners. Hence, for $\varepsilon > 0$ small, the discrete subsets $\mathrm{Spec}(L_{\varepsilon,1})$ and $\mathrm{Spec}(L_{\varepsilon,2})$ of $\mathbb{R}$ are different. This implies that the corresponding characteristic polynomials

$$P_i(X, \varepsilon), \quad i = 1, 2$$

are different. But these are evaluations of the spectral polynomials

$$P_i(X, Y)$$

evaluated at $Y = \varepsilon$ for all $\varepsilon > 0$ sufficiently small, and where these polynomials are given for edge weight maps:

$$\Delta_i : C \cup E_i \to \mathbb{N}, \quad e \mapsto \begin{cases} 1, & e \in C \\ 0, & e \in E_i. \end{cases}$$

It follows that these two spectral polynomials are different.

In order to obtain positive diffusion constants, now look at the spectrum of the matrix

$$\varepsilon L_{\varepsilon,i} = \varepsilon U(C) + \varepsilon^2 U(C_i),$$

which amounts to taking the diffusion parameters as

$$\Delta_i : C \cup E_i, \quad e \mapsto \begin{cases} 2, & e \in E_i \\ 1, & e \in C. \end{cases}$$

Hence, there exists a bijection $E_1 \to E_2$ between sets of weighted edges, i.e., an equivalence of diffusion pairs $(G_1, \Delta_1) \sim (G_2, \Delta_2)$, such that $P^{\Delta_1}_{G_1}(X, Y) \neq P^{\Delta_2}_{G_2}(X, Y)$ as asserted. □

B. A reconstruction theorem

Let $G = (V(G), E(G))$ be a graph with n vertices. We begin with the following observation:

Theorem V.4 *(Kel'mans, 1967). Let*

$$P(X) = X^n - \sum_{i=1}^{n-1} (-1)^i c_i X^{n-i}$$

be the characteristic polynomial of the graph G. Then

$$c_i = \sum_{\substack{S \subset V \\ |S| = n-i}} T(G_S),$$

where $T(H)$ is the number of spanning trees of graph H, and G_S is the quotient graph obtained by identifying all vertices in S with a single vertex.

Proof. Kel'mans,[24] and Kel'mans and Chelnokov.[25] □

We recover these quantities from the spectral polynomial $P(X,Y)$ as follows:

$$c_i(G) = a_{n-i}(1)$$

for $i = 1,\dots,n-1$.

We define

$$\mathcal{F}(G) = \{F : F \text{ is a forest with } V(F) = V(G) \text{ and } E(F) \subset E(G)\}$$

and call $\mathcal{F}(G)$ the *spanning forest set of* G. The following subsets of $\mathcal{F}(G)$ are of interest:

$$\mathcal{F}^i(G) = \{F \in \mathcal{F}(G) : b_0(F) = i\}.$$

This allows us to formulate a generalisation of Theorem V.4 which is also a known result, but formulated here in the guise of spectral curves:

Theorem V.5 *(Buslov, 2014). Let* (G,Δ) *be a diffusion pair. Then the coefficients of the spectral polynomial*

$$P_G^\Delta(X,Y) = \sum_{i=1}^n a_i(Y)X^i$$

of a diffusion pair (G,Δ) *are given as*

$$a_i(Y) = (-1)^{n-i}\sum_{F\in\mathcal{F}^i(G)}\pi_F,\quad \pi_F = \prod_{e\in E(F)} Y^{\alpha_e}$$

for $i = 1,\dots,n$.

Proof. Theorem 2 of Buslov.[26] □

Let G' be another graph. An isomorphism

$$\mathcal{F}(G) \cong \mathcal{F}(G')$$

between spanning forest sets is given by a bijection $f : \mathcal{F}(G) \to \mathcal{F}(G')$ and an ismorphism $F \cong F'$ with $F \in \mathcal{F}(G)$ and $F' = f(F)$ for all $F \in \mathcal{F}(G)$.

Lemma V.6. Let G, G' *be two graphs on n vertices. Then G and* G' *are isomorphic if and only if the strict forest subsets* $\mathcal{F}(G)$ *and* $\mathcal{F}(G')$ *are isomorphic.*

Proof. If the graphs are isomorphic, then clearly their strict forest sets are isomorphic.

Assume now that $G = T$ and $G' = T'$ are trees. Since trees are their own spanning forests, we clearly must have that $T \cong T'$.

Now, assume that G is not a tree. If G is a forest, then we can apply the result for trees to each individual connected component of G. So, we may assume that $b_1(G) > 0$, and that G is connected. If $\mathcal{F}(G) \cong \mathcal{F}(G')$, then w.l.o.g. we may assume that these two sets are equal, and that the vertex sets of the two graphs coincide. Let T be a spanning tree of G. Then

$$\mathcal{F}(T) \subset \mathcal{F}(G) = \mathcal{F}(G').$$

Hence, by symmetry, each spanning tree of G is a spanning tree of G' and vice versa. This implies that $G = G'$, as otherwise there is an edge of G not in G'. But then a spanning tree of G containing that edge is not a spanning tree of G', a contradiction. Hence, $G \cong G'$. □

Theorem V.7 *(Reconstruction theorem). For any finite graph G, there exists a p-adic Laplacian given by diffusion parameters* Δ *such that the diffusion pair* (G,Δ) *has a spectral polynomial* $P(X,Y)$ *such that for any diffusion pair* (G',Δ) *its spectral polynomial* $P'(X,Y)$ *satisfies:*

$$P(X,Y) = P'(X,Y) \iff (G,\Delta) \cong (G',\Delta).$$

In other words, the isomorphism class of G is the unique family of graphs having spectral polynomial $P(X,Y)$.

Proof. We need only prove that if $P(X,Y) = P'(X,Y)$, then $(G,\Delta) \cong (G',\Delta')$.

Let

$$\Delta : E(G) \longrightarrow \left\{\alpha_1,\dots,\alpha_{|E(G)|}\right\}, e \mapsto \alpha_e$$

be a bijection. W.l.o.g. we may assume that the edges of G are numbered as α_e, and that Δ is the identity map, so that we may write Y^e instead of Y^{α_e} in our polynomials. We further make the following assumption:

We assume that no edge label equals the finite sum of any other edge labels (they are all positive integers). (4)

Let $I \subset E(G)$. Then we define

$$\mathcal{F}^i_I(G) = \{F \in \mathcal{F}^i(G) : E(F) = I\}$$
$$\mathcal{F}^i(I) = \{F \in \mathcal{F}^i(G) : E(F) \subseteq I\}.$$

Then $\mathcal{F}^i(G)$ is the disjoint union of all the sets $\mathcal{F}^i_I(G)$ for $I \subset E(G)$. Also, $\mathcal{F}^i_I(G)$ is either empty or consists of precisely one forest.

Let $E(G) =: E$, and let H be a spanning subgraph of G. Then

$$a_{H,i}(Y) = (-1)^{n-i} \sum_{F \in \mathcal{F}^i(H)} \prod_{e \in E(F)} Y^e,$$

and for any spanning tree T of G, we have

$$a_i(Y) = a_{T,i}(Y) + a_{G\setminus T,i}(Y), \tag{5}$$

where $G\setminus T$ is obtained from G by removing the edges of T.

Now, given a spectral polynomial $P(X,Y)$, the polynomial

$$a_1(Y) = \sum_{k=1}^{M} a_{1k} Y^k$$

with $M \gg 0$ recovers the set of spanning trees as follows: a monomial $a_{1k}Y^k$ means that there are $|a_{1k}| \in \mathbb{N}$ spanning trees whose total sum of edge labels equals k. Because of our assumption (4), we have that

$$|a_{1k}| \leq 1$$

for all $k = 1, 2, \ldots$. Hence, each non-zero coefficient of $a_1(Y)$ encodes precisely one spanning tree of G. Also, the two parts of the decomposition (5) have no monomials in common.

In order to recover the diffusion constants, we look at

$$a_{n-1}(Y).$$

Again, assumption (4) ensures that all coefficients are either 1 or zero. So, the exponents of the non-zero monomials retrieve all the distinct labelled edges of G.

We can go now further to extract for every spanning tree with a given set of edge labels, the set of all spanning forests, thereby knowing their numbers of connected components. Beginning with all pairs of distinct edges, we can also extract the sets of edges in each connected component of any spanning forest. The assumption (4) makes this possible. In this way, a unique spanning tree is constructed. Doing this for all spanning trees, we obtain a unique set of spanning forests $\mathcal{F}(G)$ for some graph G. By Lemma V.6, the isomorphism class of G is now uniquely determined. □

Remark V.8. We remark that the spectral polynomial for any diffusion pair satisfying (4) has coefficients $0, \pm 1$, *as has been seen in the proof of the Reconstruction theorem (Theorem V.7).*

Corollary V.9. Playing Game 1 with choosing diffusion parameters which satisfy (4), allows you to reconstruct the unknown graph G.

Proof. This is an immediate consequence of Theorem V.7, because Game 1 has a winning strategy for any choice of diffusion parameters according to Theorem IV.6. □

Remark V.10. Notice that we are not solving the graph isomorphism problem in polynomial time, as the computation of the coefficients of the spectral polynomial can be expected to be far too time-consuming.

VI. HEARING SHAPES OF p-ADIC GEOMETRIC OBJECTS

Corollary V.9 can also be applied to objects of p-adic geometry which have an underlying graph sructure. The first kind of objects consists of Mumford curves which have reduction graphs whose first Betti number equals the genus of the curve. The second kind are analytic tori which after a base change look like products of Mumford curves of genus 1, so-called *Tate curves*. In order to be able to do this, we will construct embeddings of the sets of K-rational points of these objects into K. The details of this procedure are presented in the following two subsections.

A. Hearing the shape of a Mumford curve

Mumford curves are explained in some detail in Chap. 5 of Fresnel and van der Put's book.[27] However, we will not need to much of their construction. All we need is that they are projective algebraic curves admitting a finite cover by holed disks. Certain types of coverings by holed disks in K are introduced in Ref. 15. These produce certain types of reduction graphs.

Let X be a Mumford curve. In Ref. 15, the concept of covering of the set $X(K)$ of its K-rational points was explained. Let such a covering be given, and let G be the corresponding reduction graph. It is a connected graph whose vertices all have degree at least 2, and its first Betti number equals the genus g of the curve. We require Mumford curves to have positive genus.

The covering of $X(K)$ consists of holed disks which are in one-to-one correspondence with the vertices of G. The edges of G correspond to annuli (as rigid analytic spaces) with minimal positive thickness, i.e., they do not contain K-rational points. These annuli connect two otherwise disjoint holed disks in $X(K)$ without introducing extra K-rational points.

As a p-adic manifold, $X(K)$ is simply a disjoint union of finitely many holed disks. This compact manifold can be embedded into K as a closed-open subset in such a way that each patch of the embedded covering of $X(K)$ contains a distinct point representing a class in K/O_K. Now, we are in the setting of Sec. IV A, and have a p-adic Laplacian operator acting on the space $L^2(K)^{|G|}$.

Corollary VI.1. Playing Game 1 with diffusion parameters satisfying (4) allows to reconstruct a verticial reduction graph of a Mumford curve.

Proof. This is immediate from the construction above and Corrolary V.9. □

B. Hearing the shape of a p-adic analytic torus

The theory of p-adic analytic tori is outlined in Chap. 6 of Fresnel and van der Put's book.[27] We will collect the data in what follows, and then proceed with the application.

Let A be an analytic torus

$$A = \mathbb{G}_m/\Lambda$$

with $\mathbb{G}_m$ the multiplicative group, and a multiplicative lattice Λ generated by a basis

$$\tilde{q}_i = q_1^{\alpha_{i1}} \cdots q_g^{\alpha_{ig}}$$

for $i = 1, \ldots, g$ with

$$q_i = qe_i,$$

where $q = p^f$, e_i is the unit vector of the i-th component in the product space $K^\times \times \cdots \times K^\times$, and $\alpha_{ij} \in \mathbb{Z}$.

The lattice basis yields a decomposition

$$K^g = \bigoplus_{i=1}^{g} K\tilde{q}_i.$$

On each component $K\tilde{q}_i$, we can define an ultrametric norm as follows: the generator $\tilde{q}_i$ is determined by an integer vector

$$\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ig}) \in \mathbb{Z}^g.$$

Its associated *primitive vector* is a vector

$$\lambda_i \in \mathbb{N}^g$$

such that

$$\alpha_i = k \cdot \lambda_i$$

with $k \in \mathbb{Z}$, and $|k|$ is maximal with this property. If we write

$$\tilde{q} = q_1 \cdots q_g$$

and

$$\tilde{q}^\beta = q_1^{\beta_1} \cdots q_g^{\beta_g}$$

for

$$\beta = (\beta_1, \ldots, \beta_g) \in \mathbb{Z}^g,$$

then we have

$$\tilde{q}_i = \tilde{q}^{\alpha_i}.$$

Its associated *primitive generator* of the line $K\tilde{q}_i$ is defined as

$$\tilde{b}_i := \tilde{q}^{\lambda_i},$$

where λ_i is the primitive vector associated with α_i.

Definition VI.2. The ultrametric norm associated with the line $K\tilde{q}_i$ is defined as

$$\|x\|_i = \|\tilde{b}_i\|_{K^g}^{-\log_q|\lambda|_K},$$

where $x = \lambda\tilde{b}_i \in K\tilde{q}_i$ with primitive generator $\tilde{b}_i$, and $\lambda \in K$. The Haar measure μ_i on $K\tilde{q}_i$ is normalised such that the unit ball with respect to $\|\cdot\|_i$ has measure one.

The element $\tilde{q}_i$ defines a Tate curve, whose reduction graph we assume to be simplicial, as follows: as we have

$$\alpha_i = n_i\lambda_i$$

with natural $n_i > 2$, it follows that

$$\|\tilde{q}_i\|_i = \|\tilde{b}_i\|^{n_i} = (q^{-c_i})^{n_i}$$

with integer $c_i > 0$. Hence, the component Tate curve here is

$$T_i = (K^\times\tilde{b}_i)/\langle\tilde{q}_i\rangle \cong K^\times/\langle(q^{c_i})^{n_i}\rangle.$$

However, there is a difference in the reduction graph structures on both sides of the isomorphism: a covering as in Sec. VI A of the curve on the right has $c_i n_i$ vertices, whereas anyone on the left has n_i vertices.

We have a decomposition

$$L^2(K^g) = \bigoplus_{i=1}^{g} L^2(K\tilde{q}_i).$$

On each factor, we repeat the construction of the previous Subsection VI A and obtain an operator

$$\Lambda : \bigoplus_{i=1}^{g} L^2(K\tilde{q}_i)^{|G_i|} \to \bigoplus_{i=1}^{g} L^2(K\tilde{q}_i)^{|G_i|}$$

where G_i is the reduction graph of T_i. This operator generalises the product space operator from Rajkumar and Weisbart[28] and decomposes as

$$\Lambda_G = \Lambda_1 + \cdots + \Lambda_g,$$

where

$$\Lambda_i = 1 \otimes \cdots \otimes 1 \otimes \Lambda_{G_i}^{\Delta_i} \otimes 1 \otimes \cdots \otimes 1$$

for $i = 1, \ldots, g$.

Instead of playing Game 1 for each component graph G_i, we recover the product graph for the torus A:

Corollary VI.3. Playing Game 1 is possible for p-adic analytic tori in a successful way in order to recover the product graph composed of $G_1, \ldots, G_g$.

Proof. Let G be the product graph of $G_1, \ldots, G_g$. Then Λ_G can be viewed in fact as an operator

$$\Lambda_G : L^2(K)^{|G|} \to L^2(K)^{|G|}$$

by taking suitable isomorphisms. Then play Game 1 using assumption (4). This recovers G. □

ACKNOWLEDGMENTS

Evgeny Zelenov is thanked for posing this problem to one of the authors. David Weisbart is thanked for valuable discussions. Also, the anonymous referee is thanked for pointing out errors and making helpful suggestions towards improving the exposition of this article. This research is partially supported by the Deutsche Forschungsgemeinschaft under Project No. 469999674.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Patrick Erik Bradley: Conceptualization (lead); Formal analysis (equal); Funding acquisition (lead); Investigation (lead); Methodology (lead); Supervision (lead); Writing – original draft (lead); Writing – review & editing (equal). **Ángel Morán Ledezma**: Formal analysis (equal); Funding acquisition (supporting); Investigation (supporting); Methodology (supporting); Writing – original draft (supporting); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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