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The *u*-*p* approximation versus the exact dynamic equations for anisotropic fluid-saturated solids. II. Harmonic waves

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Abstract

The paper presents a comparative analysis of three systems of dynamic equations for fluid-saturated solids: the exact equations and two simplified versions known as the *u*-*p* approximations obtained by neglecting certain acceleration terms in the exact equations. The constitutive relations for the solid skeleton are written in the general anisotropic incrementally linear form without considering any specific constitutive model or a particular type of anisotropy. The dynamic equations are compared in relation to the existence of solutions in the form of plane harmonic waves. Emphasis is placed on finding conditions for the non-existence or existence of growing waves whose amplitude increases in time or space as the wave propagates. The conditions are formulated in terms of the acoustic tensor of the skeleton and the compressibility of the pore fluid. In particular, it is shown that for a hyperelastic skeleton, the exact equations and one of the *u*-*p* approximations do not have growing wave solutions, whereas the other *u*-*p* approximation may have such solutions even if the skeleton is hyperelastic.

KEYWORDS

acoustic tensor, fluid-saturated solid, harmonic waves, *u*-*p* approximation

1 | INTRODUCTION

The *u*-*p* approximation is a simplified version of the dynamic equations for porous fluid-saturated solids. It is obtained by neglecting certain acceleration terms in the governing equations in order to eliminate the pore fluid velocity from the set of unknown variables. The *u*-*p* approximation was proposed in the early 1980s[1,2] as a means of reducing computational costs in the numerical solution of earthquake engineering problems for which the use of the simplified equations was shown to be justified. The *u*-*p* approximation is still widely used for the numerical modelling of the dynamic deformation of saturated solids, especially in geomechanics.[3–18] The validity of the *u*-*p* approximation is usually assessed in terms of accuracy understood as the deviation of the solutions obtained with the *u*-*p* equations from the solutions obtained with the exact (full) equations in which all acceleration terms are retained. The main factors that influence the accuracy of the *u*-*p* approximation are the frequency content of the motion and the permeability of the medium. Depending on the number of acceleration terms neglected in the equations, there exist two versions of the u-p approximation, one of them being less accurate than the other.

In Part I of the present study,[19] the two u-p approximations are compared with the exact formulation from the viewpoint of well-posedness of boundary value problems, addressing the question of hyperbolicity of the governing equations. The present paper continues the qualitative analysis of the u-p approximations and the comparison with the exact formulation. Here we study solutions in the form of plane harmonic waves assuming linearly elastic skeleton with arbitrary anisotropy. We consider two types of waves that have either complex frequencies (first type) or complex wave numbers (second type). Emphasis is placed on the existence of *growing* waves whose amplitude increases exponentially in time (for the first type) or in space (for the second type) as the wave propagates. The objective of this study is to find conditions which guarantee the non-existence or existence of growing waves for the exact formulation and the two u-p approximations. The conditions for the non-existence or existence of growing waves are presented in the form of propositions. The conditions involve the acoustic tensor of the skeleton, the pore fluid compressibility, the density of the solid and fluid phases and the porosity.

Hyperbolicity studied in Part I[19] is a characteristic of mathematical acceptability of the model. The study of growing waves concerns physical acceptability: waves with growing amplitudes are not observed in real media and should be regarded as physically unacceptable. For a saturated solid with a hyperelastic skeleton, the non-existence of growing wave solutions may be considered as a necessary condition for physical acceptability of the dynamic equations. As follows from the propositions proved in the paper, the exact equations and one of the u-p approximations meet this requirement, but this is not always the case for the other u-p approximation.

The structure of the paper is as follows. The governing equations of the exact and u-p formulations are given in Section 2. Growing wave solutions are defined in Section 3. Results on the existence of growing waves are presented and discussed in Sections 4 and 5. Proofs of the propositions formulated in Section 4 are provided separately in Sections 6–10.

2 | GOVERNING EQUATIONS

The governing equations are written in Cartesian coordinates x_1, x_2, x_3 in component form with the summation convention for repeated indices. Assuming small strains, the material time derivatives are replaced with the partial derivatives neglecting the convective terms. The skeleton is assumed to be linearly elastic. The constitutive equations for a dry porous solid or a saturated solid under fully drained conditions (no changes in the pore pressure) are written in the rate form

$$\frac{\partial \sigma_{ji}}{\partial t} = C_{jikl} \frac{\partial v_{sk}}{\partial x_l}, \tag{1}$$

where σ_{ji} are the stress components, C_{jikl} are the components of the stiffness tensor, v_{sk} are the velocity components of the skeleton (the first subscripts 's' stands for 'solid', the second subscript indicates the component), and t is the time variable. We do not assume any specific type of anisotropy, so that the stiffness tensor in Equation (1) is arbitrary except that it must possess both minor symmetries (the left minor symmetry is due to the symmetry of the stress tensor, while the right minor symmetry follows from the fact that the stress rate is independent of the skew-symmetric part of the velocity gradient).

For a fluid-saturated solid, σ_{ji} are the components of the effective stresses defined as[20–22]

$$\sigma_{ji} = \sigma_{ji}^{total} + \left(\delta_{ji} - \frac{C_{jikk}}{3K_s} \right) p_f, \tag{2}$$

where σ_{ji}^{total} are the total stress components, p_f is the pore fluid pressure (positive for compression), K_s is the bulk modulus of the solid phase (the material of the skeleton) and δ_{ji} is the Kronecker delta.

The evolution equation for the pore pressure is[21,22]

$$\frac{\partial p_f}{\partial t} = -Q^* \left(\delta_{ji} - n\delta_{ji} - \frac{C_{kkji}}{3K_s} \right) \frac{\partial v_{sj}}{\partial x_i} - Q^* n \frac{\partial v_{fl}}{\partial x_l}, \tag{3}$$

where

$$\frac{1}{Q^*} = \frac{n}{K_f} + \frac{1}{K_s} \left(1 - n - \frac{C_{iijj}}{9K_s} \right), \tag{4}$$

v_{fi} are the velocity components of the pore fluid (the first subscript 'f' stands for 'fluid', the second subscript indicates the component), K_f is the pore fluid bulk modulus and n is the porosity. The porosity gradient is neglected.

If the stiffness tensor of the skeleton is such that

$$C_{jikk} = C_{kkji} = 3K\delta_{ji}, \tag{5}$$

where K is a scalar, then the effective stresses (2) can be written as

$$\sigma_{ji} = \sigma_{ji}^{total} + \alpha p_f \delta_{ji}, \tag{6}$$

where

$$\alpha = 1 - \frac{K}{K_s}. \tag{7}$$

Equation (3) for the pore pressure becomes

$$\frac{\partial p_f}{\partial t} = -Q(\alpha - n)\frac{\partial v_{sk}}{\partial x_k} - Qn\frac{\partial v_{fk}}{\partial x_k}, \tag{8}$$

where

$$\frac{1}{Q} = \frac{n}{K_f} + \frac{\alpha - n}{K_s}. \tag{9}$$

If condition (5) is satisfied, the equations of motion for the solid and fluid phases are[2,3]

$$\frac{\partial \sigma_{ji}}{\partial x_j} - (\alpha - n)\frac{\partial p_f}{\partial x_i} + \frac{n^2}{k}(v_{fi} - v_{si}) = (1 - n)\varrho_s \frac{\partial v_{si}}{\partial t}, \tag{10}$$

$$-n\frac{\partial p_f}{\partial x_i} - \frac{n^2}{k}(v_{fi} - v_{si}) = n\varrho_f \frac{\partial v_{fi}}{\partial t}, \tag{11}$$

where ϱ_s, ϱ_f are the densities of the solid and fluid phases, and k is the permeability with the dimension [length^3×time/mass] connected with the hydraulic conductivity k' [length/time] by the relation $k = k'/(\varrho_f g)$, where g is the acceleration due to gravity.

Condition (5) is satisfied, in particular, for an isotropic skeleton with the bulk modulus K, but is weaker than the condition of isotropy. If condition (5) is not satisfied, Equations (6)–(10) should be viewed as approximate relations in which the scalar α characterizes the bulk compressibility of the skeleton. If $K_s \gg |C_{jikl}|$, the solid phase may be considered incompressible compared with the skeleton, and Equations (6)–(10) with $\alpha = 1$ are correct independently of whether condition (5) is satisfied or not. We will use Equations (8)–(10) with the scalar α and assume in addition that α satisfies the inequality $\alpha > n$ justified for elastic porous solids.[23–25] This inequality will be used in the proofs of propositions. In particular, it guarantees that $Q > 0$.

System (1), (8), (10), (11) with the unknown functions $v_{si}, v_{fi}, \sigma_{ji}, p_f$ will be referred to as the exact formulation.

The first u-p approximation is obtained by neglecting the relative fluid–solid acceleration and writing $\partial v_{si}/\partial t$ in Equation (11) in place of $\partial v_{fi}/\partial t$. This allows the fluid velocity components v_{fi} to be eliminated from the equations (see Refs. 2, 3, 19 for detail). The resulting system for v_{si}, σ_{ji}, p_f consists of the constitutive equations (1) for the effective stresses, the equations of motion for the whole continuum,

$$\frac{\partial \sigma_{ji}}{\partial x_j} - \alpha\frac{\partial p_f}{\partial x_i} = \varrho\frac{\partial v_{si}}{\partial t} \tag{12}$$

with

$$\varrho = (1 - n)\varrho_s + n\varrho_f, \tag{13}$$

TABLE 1 Unknown functions and governing equations. $\mathbf{v}_s, \mathbf{v}_f, \boldsymbol{\sigma}, p_f$ are the solid velocity, the fluid velocity, the effective stress and the pore pressure.

	Unknown functions	Governing equations
Exact formulation	$\mathbf{v}_s, \mathbf{v}_f, \boldsymbol{\sigma}, p_f$	(1), (8), (10), (11)
u-p approximation UP1	$\mathbf{v}_s, \boldsymbol{\sigma}, p_f$	(1), (12), (14)
u-p approximation UP2	$\mathbf{v}_s, \boldsymbol{\sigma}, p_f$	(1), (12), (15)

and the evolution equation for the pore pressure,

$$\frac{\partial p_f}{\partial t} = -Q\alpha \frac{\partial v_{si}}{\partial x_i} + Qk \frac{\partial}{\partial x_i}\left(\varrho_f \frac{\partial v_{si}}{\partial t} + \frac{\partial p_f}{\partial x_i}\right). \tag{14}$$

System (1), (12), (14) will be called the UP1 approximation. This approximation is used in Refs. 4–10

The second *u-p* approximation, which will be called UP2, is a further simplification obtained by neglecting the acceleration term $\partial v_{si}/\partial t$ in Equation (14). The UP2 system includes the same Equations (1), (12) and, instead of Equation (14), its simplified version

$$\frac{\partial p_f}{\partial t} = -Q\alpha \frac{\partial v_{si}}{\partial x_i} + Qk \frac{\partial^2 p_f}{\partial x_i \partial x_i}. \tag{15}$$

Equation (15) can be obtained directly from Equation (8) using the quasi-static Darcy law, that is, Equation (11) without the acceleration term. The UP2 formulation neglects not only the relative fluid acceleration in the equations of motion for the whole continuum, but also the absolute fluid acceleration for the derivation of Equation (15). The UP2 approximation is used in Refs. 11–17

The unknown functions and the equations for the exact and *u-p* approximations are summarized in Table 1.

3 | TIME-HARMONIC WAVES

Consider plane harmonic waves (so-called normal modes) of the form

$$U(x_1, x_2, x_3, t) = U^0 \exp[i(\varkappa x_j n_j - \omega t)], \tag{16}$$

where $U = (U_1, \dots, U_N)^T$ is the column vector of dependent variables, $U^0 = (U_1^0, \dots, U_N^0)^T$ is the column vector of complex constants (complex amplitudes), $\varkappa$ is the wave number, ω is the angular frequency, n_j are the components of a unit vector $\mathbf{n}$ which determines the wave propagation direction, and i is the imaginary unit. For the exact formulation, the vector of dependent variables is $U = (v_{si}, v_{fi}, \sigma_{ji}, p_f)^T$. For the UP1 and UP2 approximations, $U = (v_{si}, \sigma_{ji}, p_f)^T$.

Normal mode solutions can be found by substituting the ansatz (16) into the governing equations and obtaining a linear system for the components of U^0, with $\varkappa$ and ω being parameters of the system. Equating the determinant of the system to zero gives an equation from which possible pairs $\varkappa, \omega$ can be found – for instance, if one of the two quantities is given and the other one is sought. This may be a laborious task if the solid is anisotropic. This study is not restricted to a specific constitutive model or any special forms of anisotropy. The aim is to establish general facts concerning normal mode solutions for the exact and approximate formulations.

We will study two types of waves (16). The first type includes waves with real wave numbers $\varkappa$ and complex frequencies ω. Such solutions describe travelling waves whose scalar amplitudes $|U_i|$ are spatially homogeneous and change exponentially with time. The amplitudes grow if $\operatorname{Im}\omega > 0$ and decrease if $\operatorname{Im}\omega < 0$.

Definition 1. Normal modes (16) with $\varkappa \in \mathbb{R}$ and $\operatorname{Im}\omega > 0$ will be referred to as *growing waves of the first type*.

Waves (16) with real frequencies ω and complex wave numbers $\varkappa$ belong to another type. Such solutions describe travelling waves whose scalar amplitudes $|U_i|$ are constant at each spatial point and change exponentially in the direction of propagation $\mathbf{n}$. For $\omega > 0$, the amplitudes grow as the wave propagates if $\varkappa$ lies in the 2nd or 4th quadrant of the complex plane. For $\omega < 0$, the amplitudes grow if $\varkappa$ lies in the 1st or 3rd quadrant. All these cases are covered by the inequality

$$\omega \operatorname{Im} \varkappa^2 < 0. \tag{17}$$

Definition 2. Normal modes (16) with $\omega \in \mathbb{R}$ and $\omega \operatorname{Im} \varkappa^2 < 0$ will be referred to as *growing waves of the second type*.

Growing waves of either type are not observed in real media. Based on this empirical argument, we may postulate that growing waves cannot exist in a fluid-saturated solid with a linearly hyperelastic skeleton characterized by a positive strain energy density function

$$W = \frac{1}{2} C_{jikl} \varepsilon_{ji} \varepsilon_{kl} > 0 \quad \text{for all } \boldsymbol{\varepsilon} \neq \mathbf{0}, \tag{18}$$

where C_{jikl} are constants with the minor and major symmetries, and ε_{ji} are the components of the strain tensor $\boldsymbol{\varepsilon}$. For a hyperelastic solid with the strain energy function (18), the constitutive relations (1) in rate form contain the same coefficients C_{jikl} as in Equation (18). The requirement that growing waves do not exist in the hyperelastic solid may be viewed as a necessary condition for acceptability of the dynamic equations. We will refer to this condition as the *acceptability criterion*.

There is, in general, no straightforward connection between the existence of growing waves and well-posedness of initial or initial boundary value problems. An elementary discussion on the relation between well-posedness, stability of solutions and growing waves of the first type can be found in Ref. 26, Section 3.5. As is shown there for initial value problems, ill-posedness is not a consequence of the existence of growing waves alone but follows from the unboundedness of $\operatorname{Im} \omega(\varkappa)$ for $\varkappa \in \mathbb{R}$. We will not address the question of connection between growing waves and well-posedness, restricting ourselves to the acceptability criterion formulated above.

4 | EXISTENCE OF GROWING WAVES

In this section, we present and discuss conditions for the existence or non-existence of growing wave solutions for the exact equations and the u-p approximations. The conditions are expressed in terms of the acoustic tensor of the skeleton, $\mathbf{A}$, with the components $A_{ik} = C_{jikl} n_j n_l$, where n_j are the components of the unit vector $\mathbf{n}$ which determines the wave propagation direction in Equation (16). We begin with the exact equations and formulate a proposition which provides sufficient conditions for the absence of both types of growing waves.

Proposition 1. *If for a given wave propagation direction, the acoustic tensor of the skeleton is symmetric and positive definite, then Equations (1), (8), (10) and (11) of the exact formulation have no growing wave solutions for this propagation direction.*

Proposition 1 is proved in Section 6.

For a hyperelastic skeleton with the strain energy function (18), the minor and major symmetries of the stiffness tensor make the acoustic tensor symmetric for all directions $\mathbf{n}$. Owing to the minor symmetries of C_{jikl}, the function W defined by Equation (18) is positive not only for symmetric tensors $\boldsymbol{\varepsilon}$ but also for non-symmetric tensors. This fact ensures that the acoustic tensor is positive definite for all directions $\mathbf{n}$. Then, as follows from Proposition 1, the exact equations have no growing wave solutions for all directions and therefore meet the acceptability criterion formulated above (the non-existence of growing waves for the hyperelastic skeleton). At the same time, the conditions of the symmetry and positive definiteness of the acoustic tensor for all directions are sufficient for the system of the exact equations to be hyperbolic.[27] Thus, the exact equations for the hyperelastic porous solid are hyperbolic and have no growing wave solutions.

The situation is different for the UP1 approximation.

Proposition 2. *If $\varrho < \alpha \varrho_f$, then for each wave propagation direction, the UP1 equations (1), (12) and (14) have growing wave solutions of the first type for any non-zero wave number and growing wave solutions of the second type for any non-zero frequency.*

Proposition 2 is proved in Section 7. The proposition guarantees the existence of growing waves if $\varrho < \alpha \varrho_f$, irrespective of the stiffness tensor of the skeleton, the pore fluid compressibility and the wave propagation direction. The acceptability criterion is violated in this case. Usually, the inequality $\varrho < \alpha\varrho_f$ is not satisfied in applications. For an incompressible solid phase, $\alpha = 1$ and the inequality $\varrho < \alpha\varrho_f$ reduces to $\varrho_s < \varrho_f$. As shown in Ref. 19, the same inequality, $\varrho < \alpha\varrho_f$, is crucial for the hyperbolicity conditions for the UP1 equations. For instance, in the particular case of an isotropic skeleton, the characteristic speed of longitudinal waves becomes imaginary if $\varrho < \alpha\varrho_f$, leading to ill-posed problems.

An important role in the UP1 approximation is played by a tensor **B** with the components

$$B_{ik} = A_{ik} + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} n_i n_j A_{jk}. \tag{19}$$

In tensorial notations,

$$\mathbf{B} = \mathbf{A} + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} \mathbf{n} \otimes \mathbf{n} \cdot \mathbf{A}. \tag{20}$$

The eigenvalues of the tensor **B** will be denoted by $\zeta_1, \zeta_2, \zeta_3$. If they are real and positive, we define

$$Q_{min} = \frac{\varrho_f}{\alpha\varrho} \min\{\zeta_1, \zeta_2, \zeta_3\}, \quad Q_{max} = \frac{\varrho_f}{\alpha\varrho} \max\{\zeta_1, \zeta_2, \zeta_3\}. \tag{21}$$

We now consider the UP1 equations with $\varrho > \alpha\varrho_f$. In the following propositions, Q is given by Equation (9).

Proposition 3. *Let $\varrho > \alpha\varrho_f$, and assume that for a given wave propagation direction, the eigenvalues of the acoustic tensor and the tensor* **B** *are real and positive, and the acoustic tensor has a complete set of eigenvectors. If, in addition, $Q < Q_{min}$, then for this propagation direction, the UP1 equations (1), (12) and (14) have growing wave solutions of the first type for any non-zero wave number and growing wave solutions of the second type for any non-zero frequency.*

Proposition 3 is proved in Section 8.

The requirement that the eigenvalues of the tensor **B** be real and positive for all directions **n** is the hyperbolicity condition introduced in Ref. 19 for the UP1 equations. The requirement that the eigenvalues of the acoustic tensor be real and positive with a complete set of eigenvectors for all directions **n** is necessary and sufficient for hyperbolicity of the equations for the dry solid.[28] If both systems – for the dry solid and for the UP1 formulation – satisfy their hyperbolicity conditions, then the assumptions of Proposition 3 are fulfilled. In particular, they are fulfilled for a hyperelastic skeleton with the strain energy function (18). Proposition 3 states that growing wave solutions are not only possible in this case but even guaranteed if the fluid bulk modulus K_f is small enough so that $Q < Q_{min}$. This violates the acceptability criterion.

Propositions 2 and 3 establish sufficient conditions for the existence of growing waves. A natural question is whether there are sufficient conditions that exclude growing waves for the UP1 formulation. This question is answered by the next proposition.

Proposition 4. *Let $\varrho > \alpha\varrho_f$, and assume that for a given wave propagation direction, the acoustic tensor of the skeleton is symmetric and positive definite. If, in addition, $Q > Q_{max}$, then the UP1 equations (1), (12) and (14) have no growing wave solutions for this propagation direction.*

Proposition 4 is proved in Section 9. The quantity Q_{max} defined by Equation (21) and involved in the proposition implies that the eigenvalues of the tensor **B** are real. Although Proposition 4 does not impose this condition on **B**, there is no inconsistency. It can be shown that, if $\varrho > \alpha\varrho_f$ and the acoustic tensor is symmetric and positive definite, then the eigenvalues of **B** are real and positive (see Ref. 19, the proof of Proposition 4 therein). Proposition 4 is similar to Proposition 1 for the exact formulation in that both assume that the acoustic tensor of the skeleton is symmetric and positive definite. This assumption is fulfilled for a hyperelastic skeleton with the strain energy function (18). In this case, the non-existence of growing waves for the UP1 approximation is guaranteed if $\varrho > \alpha\varrho_f$ and if the fluid bulk modulus K_f is large enough so that $Q > Q_{max}$.

We now turn to the UP2 approximation.

Proposition 5. *If for a given wave propagation direction, the acoustic tensor of the skeleton is symmetric and positive definite, then the UP2 equations (1), (12) and (15) have no growing wave solutions for this propagation direction.*

TABLE 2 Existence of growing wave solutions for a fluid-saturated solid with a hyperelastic skeleton with the strain energy function (18).

	Existence of growing waves of the first type	**Existence of growing waves of the second type**
Exact formulation	No	No
UP1, $\varrho < \alpha\varrho_f$	Yes, for all $\mathbf{n}$ and all $\varkappa \neq 0$	Yes, for all $\mathbf{n}$ and all $\omega \neq 0$
UP1, $\varrho > \alpha\varrho_f$, $Q < Q_{min}$	Yes, for all $\mathbf{n}$ and all $\varkappa \neq 0$	Yes, for all $\mathbf{n}$ and all $\omega \neq 0$
UP1, $\varrho > \alpha\varrho_f$, $Q > Q_{max}$	No	No
UP2	No	No

Proposition 5 is proved in Section 10. The UP2 equations may have growing wave solutions if the acoustic tensor has real positive eigenvalues but is not symmetric. The formulation of Proposition 5 is the same as that of Proposition 1 for the exact equations. Proposition 5 imposes the same conditions on the acoustic tensor as Proposition 4 for the UP1 formulation but without the additional requirements $\varrho > \alpha\varrho_f$ and $Q > Q_{max}$. The UP2 approximation, like the exact formulation, always satisfies the acceptability criterion and, in this sense, might seem to be a better choice than the UP1 approximation, but it is less accurate than the latter.

The results can be summarized as follows (see also Table 2).

- The exact formulation and the UP2 approximation satisfy the acceptability criterion. In the case of a hyperelastic skeleton with the strain energy function (18), these equations have no growing wave solutions (Propositions 1 and 5).
- The UP1 approximation does not, in general, satisfy the acceptability criterion. If $\varrho < \alpha\varrho_f$, the UP1 equations always have growing wave solutions of both types for all wave propagation directions, regardless of whether the skeleton is hyperelastic or not (Proposition 2).
- If $\varrho > \alpha\varrho_f$ and the skeleton is hyperelastic, the UP1 equations have growing wave solutions of both types for all wave propagation directions if, in addition, $Q < Q_{min}$, and have no growing wave solutions if $Q > Q_{max}$ (Propositions 3 and 4). Apart from the question of accuracy, the applicability of the UP1 approximation is, therefore, restricted by the conditions $\varrho > \alpha\varrho_f$, $Q > Q_{max}$.

5 | ISOTROPIC SOLID

In the particular case of an isotropic elastic skeleton with the Lamé constants λ and μ, the components of the acoustic tensor and the tensor $\mathbf{B}$ are

$$A_{ik} = (\lambda + \mu) n_i n_k + \mu \delta_{ik}, \tag{22}$$

$$B_{ik} = (\lambda + \mu) n_i n_k + \mu \delta_{ik} + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f}(\lambda + 2\mu) n_i n_k. \tag{23}$$

The eigenvalues of the tensor $\mathbf{B}$ can easily be found putting $n_1 = 1, n_2 = n_3 = 0$:

$$\zeta_1 = \frac{\varrho}{\varrho - \alpha\varrho_f}(\lambda + 2\mu), \quad \zeta_2 = \zeta_3 = \mu. \tag{24}$$

Assuming $\varrho > \alpha\varrho_f$, we have

$$Q_{min} = \frac{\varrho_f}{\alpha\varrho}\mu, \quad Q_{max} = \frac{\varrho_f}{\alpha(\varrho - \alpha\varrho_f)}(\lambda + 2\mu). \tag{25}$$

The transverse and longitudinal velocity components of the waves in an isotropic solid are uncoupled. For the UP1 equations, the transverse waves propagate without attenuation, whereas the growth or decay of the longitudinal waves depends on Q. For the UP1 formulation with $\varrho > \alpha\varrho_f$, as follows from Propositions 3 and 4, the longitudinal waves of both types (with either real $\varkappa$ or real ω) decay if $Q > Q_{max}$. If $Q < Q_{max}$, there exist growing longitudinal waves of both types. The equality $Q = Q_{max}$, where Q_{max} is given by Equation (25), coincides with the so-called dynamic compatibility condition (see e.g., Ref. 27, Section 6). Since the longitudinal waves propagate independently of the transverse waves, Q_{min} defined by Equation (25) plays no role for the longitudinal waves. The condition $Q < Q_{min}$ of Proposition 3 remains true but is stronger than the condition $Q < Q_{max}$. The pore fluid bulk modulus that corresponds to the equality $Q = Q_{max}$ can be found from Equations (9) and (25):

$$K_f = \frac{n\varrho_f(\lambda + 2\mu)K_s}{\alpha(\varrho - \alpha\varrho_f)K_s - (\alpha - n)\varrho_f(\lambda + 2\mu)}. \tag{26}$$

For an incompressible solid phase with $(\lambda + 2\mu)/K_s = 0$ and $\alpha = 1$, Equation (26) reduces to

$$K_f = \frac{n\varrho_f(\lambda + 2\mu)}{(1 - n)(\varrho_s - \varrho_f)}. \tag{27}$$

The modulus K_f estimated with Equation (26) or (27) for soil is smaller than the bulk modulus of pure water (2.2 GPa), so the UP1 equations for fully saturated soil have no growing wave solutions. Incomplete saturation with a small amount of free gas in the pore water (less than 1% of volume) drastically reduces the bulk modulus of the water–gas mixture compared to pure water. (The presence of free gas also makes the modulus of the mixture strongly dependent of the pore pressure.[29]) The bulk modulus of the mixture can be smaller than the modulus given by Equation (26) or (27). In this case, besides the fact that the UP1 equations do not satisfy the acceptability criterion, the modelling of incomplete saturation with a reduced modulus K_f may be problematic because of numerical instabilities caused by the existence of growing wave solutions.

6 | PROOF OF PROPOSITION 1

Substituting the normal mode solutions (16) into Equations (1), (8), (10) and (11) results in a system for the complex amplitudes $v^0_{si}, v^0_{fi}, \sigma^0_{ji}, p^0_f$:

$$\omega k(1 - n)\varrho_s v^0_{si} + \kappa k n_j \sigma^0_{ji} - \kappa k(\alpha - n)n_i p^0_f - in^2\left(v^0_{fi} - v^0_{si}\right) = 0, \tag{28}$$

$$\omega k n\varrho_f v^0_{fi} - \kappa k n n_i p^0_f + in^2\left(v^0_{fi} - v^0_{si}\right) = 0, \tag{29}$$

$$\omega\sigma^0_{ji} + \kappa n_l C_{jikl} v^0_{sk} = 0, \tag{30}$$

$$\omega p^0_f - \kappa Q(\alpha - n)n_k v^0_{sk} - \kappa Q n n_k v^0_{fk} = 0. \tag{31}$$

Since solutions with $\omega = 0$ are not growing waves, we assume that $\omega \neq 0$ and substitute σ^0_{ji} and p^0_f from Equations (30) and (31) into Equations (28) and (29) to obtain a system for v^0_{si}, v^0_{fi}:

$$\omega^2 k(1 - n)\varrho_s v^0_{si} - \kappa^2 k A_{ik} v^0_{sk} - \kappa^2 k Q(\alpha - n)n_i n_k\left[(\alpha - n)v^0_{sk} + n v^0_{fk}\right]$$

$$-i\omega n^2\left(v^0_{fi} - v^0_{si}\right) = 0, \tag{32}$$

$$\omega^2 k n\varrho_f v^0_{fi} - \kappa^2 k Q n n_i n_k\left[(\alpha - n)v^0_{sk} + n v^0_{fk}\right] + i\omega n^2\left(v^0_{fi} - v^0_{si}\right) = 0, \tag{33}$$

where $A_{ik} = C_{jikl}n_j n_l$. In the following, a bar over a symbol will denote the complex conjugate. Multiplying Equation (32) by $\bar{v}^0_{si}$ and Equation (33) by $\bar{v}^0_{fi}$ with summation and then adding the two equations, we obtain

$$\omega^2 a_1 + i\omega a_2 - \kappa^2 a_3 = 0, \tag{34}$$

where the real quantities a_1, a_2 and a_3 are

$$a_1 = k\left[(1 - n)\varrho_s v^0_{si}\bar{v}^0_{si} + n\varrho_f v^0_{fi}\bar{v}^0_{fi}\right] > 0, \tag{35}$$

$$a_2 = n^2\left(v^0_{fi} - v^0_{si}\right)\left(\bar{v}^0_{fi} - \bar{v}^0_{si}\right) \geq 0, \tag{36}$$

$$a_3 = kA_{ik}v^0_{sk}\bar{v}^0_{si} + kQ\Big[(\alpha - n)n_k v^0_{sk} + nn_k v^0_{fk}\Big]\Big[(\alpha - n)n_i\bar{v}^0_{si} + nn_i\bar{v}^0_{fi}\Big] > 0. \tag{37}$$

The quantity $A_{ik}v^0_{sk}\bar{v}^0_{si}$ is real and non-negative due to the assumed symmetry and positive definiteness of the acoustic tensor. The other terms in Equations (35)–(37) are real and non-negative because they are the products of complex numbers and their conjugates. It is important that a_1 and a_3 are positive. To show this, suppose a_1 is zero. Then v^0_{si} and v^0_{fi} must be zero, and it follows from Equations (30) and (31) that σ^0_{ji}, p^0_f must be zero as well, giving the trivial solution. Now suppose a_3 is zero. Then $v^0_{si} = 0$, $n_k v^0_{fk} = 0$ from Equation (37), $\sigma^0_{ji} = 0$, $p^0_f = 0$ from Equations (30) and (31), and finally $v^0_{fi} = 0$ from Equation (28), leading again to the trivial solution.

If $\varkappa \in \mathbb{R}$, we multiply Equation (34) by $\bar{\omega}$ to obtain

$$\omega|\omega|^2 a_1 + i|\omega|^2 a_2 - \bar{\omega}\varkappa^2 a_3 = 0. \tag{38}$$

Separating the real and imaginary parts of this equation shows that

$$\operatorname{Im}\omega = -\frac{|\omega|^2 a_2}{|\omega|^2 a_1 + \varkappa^2 a_3} \le 0. \tag{39}$$

This proves the non-existence of growing wave solutions of the first type.

If $\omega \in \mathbb{R}$, separating the real and imaginary parts in Equation (34) gives

$$\omega \operatorname{Im}\varkappa^2 = \frac{\omega^2 a_2}{a_3} \ge 0. \tag{40}$$

This proves the non-existence of growing wave solutions of the second type.

7 | PROOF OF PROPOSITION 2

7.1 | Real $\varkappa$, complex ω

Substituting $\partial v_{si}/\partial t$ from Equation (12) into Equation (14), the latter becomes

$$\frac{\partial p_f}{\partial t} = -Q\alpha\frac{\partial v_{si}}{\partial x_i} + \frac{Qk}{\varrho}\left[\varrho_f\frac{\partial^2\sigma_{ji}}{\partial x_i\partial x_j} + (\varrho - \alpha\varrho_f)\frac{\partial^2 p_f}{\partial x_i\partial x_i}\right]. \tag{41}$$

Substituting the normal mode solutions (16) into Equations (1), (12) and (41) gives a system for the complex amplitudes $v^0_{si}, \sigma^0_{ji}, p^0_f$:

$$-\varkappa n_j\sigma^0_{ji} + \varkappa\alpha n_i p^0_f = \omega\varrho v^0_{si}, \tag{42}$$

$$-\varkappa C_{jikl}n_l v^0_{sk} = \omega\sigma^0_{ji}, \tag{43}$$

$$\varkappa Q\alpha\varrho n_i v^0_{si} - i\varkappa^2 Q\varrho_f k n_i n_j\sigma^0_{ji} - i\varkappa^2 Q(\varrho - \alpha\varrho_f)k p^0_f = \omega\varrho p^0_f. \tag{44}$$

System (42)–(44) is an eigenvalue problem which can be written as

$$MU = \omega\varrho U, \tag{45}$$

where $\omega\varrho$ is the eigenvalue, U is the eigenvector with the components $v^0_{si}, \sigma^0_{ji}, p^0_f$ and M is the matrix of the eigenvalue problem. It is readily seen from Equations (42)–(44) that the diagonal elements of the matrix M are zero except for one element which is

$$-i\varkappa^2 Q(\varrho - \alpha\varrho_f)k. \tag{46}$$

The sum of all eigenvalues of a matrix is equal to its trace. The trace of the matrix M is equal to the diagonal element (46). If $\varrho < \alpha\varrho_f$ and κ is real and non-zero, then at least one eigenvalue $\omega\varrho$ must have a positive imaginary part. This proves the existence of growing wave solutions of the first type.

7.2 | Real ω, complex κ

There are no growing wave solutions of the second type for $\kappa = 0$ or $\omega = 0$, so we assume that both are non-zero. Substituting $n_j\sigma^0_{ji}$ from Equation (42) into Equation (44) and σ^0_{ji} from Equation (43) into Equation (42), we eliminate σ^0_{ji} from the equations and obtain the system

$$-\gamma^2\varrho v^0_{si} + A_{ik}v^0_{sk} + \alpha n_i\gamma p^0_f = 0, \tag{47}$$

$$\gamma^2 Q\big(\alpha + i\omega\varrho_f k\big)n_i v^0_{si} - \big(i\omega Qk + \gamma^2\big)\gamma p^0_f = 0, \tag{48}$$

where $\gamma = \omega/\kappa$. System (47), (48) is the generalized eigenvalue problem

$$GU = \gamma^2 DU \tag{49}$$

in which γ^2 is the eigenvalue, $U = (v^0_{s1}, v^0_{s2}, v^0_{s3}, \gamma p^0_f)^T$ is the eigenvector and the matrices are

$$G = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \alpha n_1 \\ A_{21} & A_{22} & A_{23} & \alpha n_2 \\ A_{31} & A_{32} & A_{33} & \alpha n_3 \\ 0 & 0 & 0 & -i\omega Qk \end{pmatrix}, \quad D = \begin{pmatrix} \varrho & 0 & 0 & 0 \\ 0 & \varrho & 0 & 0 \\ 0 & 0 & \varrho & 0 \\ l_1 & l_2 & l_3 & 1 \end{pmatrix}, \tag{50}$$

where

$$l_i = -Q\big(\alpha + i\omega\varrho_f k\big)n_i, \quad i = 1, 2, 3. \tag{51}$$

The matrix D can be inverted, allowing the generalized eigenvalue problem (49) to be converted into the ordinary eigenvalue problem

$$D^{-1}GU = \gamma^2 U, \tag{52}$$

where

$$D^{-1} = \frac{1}{\varrho}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l_1 & -l_2 & -l_3 & \varrho \end{pmatrix}. \tag{53}$$

Condition (17) for growing wave solutions of the second type can equivalently be written for $\gamma = \omega/\kappa$ as

$$\omega\,\mathrm{Im}\,\gamma^2 > 0. \tag{54}$$

To verify whether condition (54) is satisfied, we again make use of the fact that the sum of the eigenvalues of a matrix is equal to its trace. For the eigenvalue problem (52),

$$\mathrm{tr}\big(D^{-1}G\big) = \frac{1}{\varrho}(A_{11} + A_{22} + A_{33}) + \frac{1}{\varrho}Q\big[\alpha^2 + i\omega k(\alpha\varrho_f - \varrho)\big]. \tag{55}$$

Let $\varrho < \alpha\varrho_f$. If $\omega > 0$, then at least one eigenvalue γ^2 must have a positive imaginary part, so condition (54) is satisfied. If $\omega < 0$, then at least one eigenvalue must have a negative imaginary part, and condition (54) is again satisfied. This proves the existence of growing wave solutions of the second type.

8 | PROOF OF PROPOSITION 3

8.1 | Real $\varkappa$, complex ω

We begin with the analysis of normal mode solutions of the first type with real wave numbers $\varkappa \neq 0$ and complex frequencies $\omega \neq 0$. Substituting σ^0_{ji} from Equation (43) into Equations (42) and (44) eliminates σ^0_{ji} from the equations and leads to a system for v^0_{si}, p^0_f consisting of Equation (47) and the equation

$$\gamma Q\alpha\varrho n_i v^0_{si} + i\varkappa Q\varrho_f k n_i A_{ik} v^0_{sk} - i\varkappa Q(\varrho - \alpha\varrho_f)k\gamma p^0_f - \varrho\gamma^2 p^0_f = 0, \tag{56}$$

where $\gamma = \omega/\varkappa$. In the following, we will use calligraphic letters with subscripts to denote polynomials, with the subscript indicating the degree of the polynomial. Equating the determinant of the matrix of system (47), (56) to zero yields an equation of the form

$$\mathcal{P}_7(\gamma, Q) = 0, \tag{57}$$

where $\mathcal{P}_7(\gamma, Q)$ is a seventh-degree polynomial in γ. For our purposes, we have included Q in the function $\mathcal{P}_7(\gamma, Q)$ as an argument. This function is

$$\mathcal{P}_7(\gamma, Q) = \det\begin{pmatrix} -\gamma^2\varrho I + A & \alpha N \\ \gamma Q\alpha\varrho N^T + i\varkappa Q\varrho_f k N^T A & -i\varkappa Q(\varrho - \alpha\varrho_f)k - \gamma\varrho \end{pmatrix}, \tag{58}$$

where A is the matrix of the components of the acoustic tensor, I is the 3×3 identity matrix, N is the column vector with the components of the vector $\mathbf{n}$ and N^T denotes the transpose of N. The function (58) can be written as

$$\mathcal{P}_7(\gamma, Q) = i\varkappa Qk\mathcal{H}_3(\gamma^2) + \gamma\varrho\mathcal{G}_3(\gamma^2, Q), \tag{59}$$

where $\mathcal{H}_3(\gamma^2)$ and $\mathcal{G}_3(\gamma^2, Q)$ are third-degree polynomials in γ^2:

$$\mathcal{H}_3(\gamma^2) = \det\begin{pmatrix} -\gamma^2\varrho I + A & \alpha N \\ \varrho_f N^T A & -(\varrho - \alpha\varrho_f) \end{pmatrix}, \tag{60}$$

$$\mathcal{G}_3(\gamma^2, Q) = \det\begin{pmatrix} -\gamma^2\varrho I + A & \alpha N \\ Q\alpha N^T & -1 \end{pmatrix}. \tag{61}$$

Another form of the function $\mathcal{H}_3(\gamma^2)$ can be obtained by multiplying the first three rows of the matrix in Equation (60) by $-\varrho_f n_1, -\varrho_f n_2, -\varrho_f n_3$, respectively, and adding them to the fourth row. This eliminates the components of A in the fourth row and gives

$$\mathcal{H}_3(\gamma^2) = \varrho\det\begin{pmatrix} -\gamma^2\varrho I + A & \alpha N \\ \gamma^2\varrho_f N^T & -1 \end{pmatrix}. \tag{62}$$

For the evaluation of the determinants we will use two formulae from linear algebra. Let W be a non-singular square matrix, X and Y be column vectors of conformable size, and z be a non-zero scalar. Then (Ref. 30, p. 475)

$$\det\begin{pmatrix} W & Y \\ X^T & z \end{pmatrix} = z\det\left(W - z^{-1}YX^T\right), \tag{63}$$

$$\det\left(W + YX^T\right) = \det(W)\left(1 + X^T W^{-1} Y\right). \tag{64}$$

Formula (63) enables us to reduce the size of the determinants (60)–(62):

$$\mathcal{H}_3(\gamma^2) = -(\varrho - \alpha\varrho_f)\det\left(-\gamma^2\varrho I + A + \frac{\alpha\varrho_f}{\varrho - \alpha\varrho_f} N N^T A\right), \tag{65}$$

$$\mathcal{H}_3(\gamma^2) = -\varrho \det\big(-\gamma^2\varrho I + A + \gamma^2\alpha\varrho_f NN^T\big), \tag{66}$$

$$\mathcal{G}_3(\gamma^2, Q) = -\det\big(-\gamma^2\varrho I + A + Q\alpha^2 NN^T\big). \tag{67}$$

The assumption that the acoustic tensor has a complete set of eigenvectors means that the matrix A can be diagonalized by a real matrix T, so that $\tilde{A} = T^{-1}AT$ is a diagonal matrix. Multiplying the matrices of the determinants (66), (67) by T^{-1} on the left and by T on the right gives

$$\mathcal{H}_3(\gamma^2) = -\varrho \det\big(-\gamma^2\varrho I + \tilde{A} + \gamma^2\alpha\varrho_f RS^T\big). \tag{68}$$

$$\mathcal{G}_3(\gamma^2, Q) = -\det\big(-\gamma^2\varrho I + \tilde{A} + Q\alpha^2 RS^T\big), \tag{69}$$

where $R = T^{-1}N$ and $S = T^T N$ are column vectors. Let η_i, $i = 1, 2, 3$, denote the eigenvalues of the acoustic tensor, that is, the diagonal entries of $\tilde{A}$. Introduce the notation

$$\beta_i = -\gamma^2\varrho + \eta_i, \quad i = 1, 2, 3. \tag{70}$$

Applying formula (64) for the expansion of the determinants (68), (69), we obtain

$$\mathcal{H}_3(\gamma^2) = -\varrho\beta_1\beta_2\beta_3 - \gamma^2\alpha\varrho_f\varrho(s_1r_1\beta_2\beta_3 + s_2r_2\beta_1\beta_3 + s_3r_3\beta_1\beta_2), \tag{71}$$

$$\mathcal{G}_3(\gamma^2, Q) = -\beta_1\beta_2\beta_3 - Q\alpha^2(s_1r_1\beta_2\beta_3 + s_2r_2\beta_1\beta_3 + s_3r_3\beta_1\beta_2), \tag{72}$$

where s_i, r_i, $i = 1, 2, 3$, are the components of the column vectors S, R.

Below we will make use of the following relation obtained by combining Equations (71) and (72):

$$Q\alpha\mathcal{H}_3 - \gamma^2\varrho_f\varrho\mathcal{G}_3 = \varrho\big(\gamma^2\varrho_f - Q\alpha\big)\beta_1\beta_2\beta_3. \tag{73}$$

Let $Q = 0$. Then the six non-zero roots γ of Equation (57) are the roots of the equation $\mathcal{G}_3(\gamma^2, 0) = 0$, which are $\pm\sqrt{\eta_i/\varrho}$, $i = 1, 2, 3$. Since the eigenvalues of the acoustic tensor are assumed to be real and positive, these roots are real and correspond to stationary travelling waves with constant scalar amplitudes $|U_i|$. The aim is to see how the roots change as Q increases (Q is positive in the original differential equations). There are three cases to be considered separately.

Case 1: the eigenvalues of the acoustic tensor are different, and the components s_i, r_i are all non-zero.

We first analyse the roots γ at small values of Q. The analysis is based on the following observation. Let $\mathcal{P}(\gamma, Q)$ be a polynomial in γ whose coefficients depend on the real parameter Q, and let γ be a simple root of the equation

$$\mathcal{P}(\gamma, Q) = 0 \tag{74}$$

for some Q_0. In the vicinity of Q_0, the root γ is a function of Q. As follows from the relation

$$d\mathcal{P} = \frac{\partial\mathcal{P}}{\partial\gamma}\,d\gamma + \frac{\partial\mathcal{P}}{\partial Q}\,dQ = 0, \tag{75}$$

the derivative of the function $\gamma(Q)$ is

$$\frac{d\gamma}{dQ} = -\frac{\partial\mathcal{P}}{\partial Q}\left(\frac{\partial\mathcal{P}}{\partial\gamma}\right)^{-1}. \tag{76}$$

The location of the roots of equation (57) in the complex plane at small values of Q can be revealed by calculating the partial derivatives of $\mathcal{P}_7(\gamma, Q)$ and the resulting derivatives $d\gamma/dQ$ for the roots at $Q = 0$. Since the eigenvalues of the acoustic tensor in Case 1 are different, the roots $\pm\sqrt{\eta_i/\varrho}$ for $Q = 0$ are simple, so formula (76) can be applied.

The differentiation of the function (59) with the use of Equations (71) and (72) yields

$$\left.\frac{\partial\mathcal{P}_7}{\partial\gamma}\right|_{Q=0} = -\varrho\beta_1\beta_2\beta_3 + 2\gamma^2\varrho^2(\beta_2\beta_3 + \beta_1\beta_3 + \beta_1\beta_2) \tag{77}$$

and, for real values of γ,

$$\operatorname{Im}\frac{\partial \mathcal{P}_7}{\partial Q} = -\varkappa k\varrho\left[\beta_1\beta_2\beta_3 + \gamma^2\alpha\varrho_f(s_1r_1\beta_2\beta_3 + s_2r_2\beta_1\beta_3 + s_3r_3\beta_1\beta_2)\right]. \tag{78}$$

Substituting two roots $\gamma = \pm\sqrt{\eta_1/\varrho}$ into Equations (77) and (78) and using formula (76) with $\gamma = \omega/\varkappa$ (where $\varkappa$ is fixed), we obtain

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\varkappa^2 k\alpha\varrho_f}{2\varrho}\, s_1r_1. \tag{79}$$

Taking $\gamma = \pm\sqrt{\eta_2/\varrho}$ and then $\gamma = \pm\sqrt{\eta_3/\varrho}$ and proceeding along the same lines, we find, respectively,

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\varkappa^2 k\alpha\varrho_f}{2\varrho}\, s_2r_2, \tag{80}$$

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\varkappa^2 k\alpha\varrho_f}{2\varrho}\, s_3r_3. \tag{81}$$

Since

$$s_1r_1 + s_2r_2 + s_3r_3 = S^TR = N^TTT^{-1}N = 1, \tag{82}$$

at least one of the derivatives (79), (80), (81) must be positive and therefore at least one pair of the roots must be such that the ω enters the half-plane $\operatorname{Im}\omega > 0$ as Q becomes non-zero. This proves the existence of growing wave solutions of the first type for small non-zero values of Q.

As Q increases, the growing wave solutions will exist as long as there are ω's in the half-plane $\operatorname{Im}\omega > 0$. In order for the ω's to leave this half-plane, they must cross the real axis, so we will look for the real roots γ of Equation (57) for $Q > 0$. These roots must be the real roots of the equation

$$\mathcal{H}_3(\gamma^2) = 0 \tag{83}$$

obtained by equating the imaginary part of $\mathcal{P}_7$ to zero, see Equation (59). As follows from Equation (65), the roots γ^2 of Equation (83) are ζ_i/ϱ, $i = 1, 2, 3$, where ζ_i are the eigenvalues of the tensor $\mathbf{B}$ defined by Equations (19) and (20). In Proposition 3, the eigenvalues of $\mathbf{B}$ are assumed to be real and positive. The three values of Q at which the roots γ^2 become equal to ζ_i/ϱ will be denoted by Q_i. They are determined by the equations

$$\mathcal{G}_3(\zeta_i/\varrho, Q_i) = 0, \quad i = 1, 2, 3, \tag{84}$$

obtained by equating the real part of $\mathcal{P}_7$ to zero, see Equation (59).

To solve Equation (84) for Q_i, we need to show that the product $\beta_1\beta_2\beta_3$ at $\gamma^2 = \zeta_i/\varrho$ is non-zero. Suppose the product is zero. Then $\zeta_i = \eta_i$ for some i, say $\zeta_1 = \eta_1$, and it follows from Equations (71) and (83) that at least one of the equalities $s_1r_1 = 0$, $\eta_2 = \eta_1$, $\eta_3 = \eta_1$ must hold, but all of them are excluded in Case 1. Equation (73) with $\mathcal{H}_3 = 0$, $\mathcal{G}_3 = 0$, $\gamma^2 = \zeta_i/\varrho$ and $\beta_1\beta_2\beta_3 \neq 0$ yields the relation

$$Q_i = \frac{\varrho_f}{\alpha\varrho}\,\zeta_i, \quad i = 1, 2, 3. \tag{85}$$

The existence of growing wave solutions of the first type is guaranteed if Q is positive and less than the smallest of the three values defined by Equation (85).

Case 1 encompasses situations where the eigenvalues of the acoustic tensor are different and the components s_i, r_i are non-zero. All other possibilities where these conditions are not satisfied can be divided into two groups called Case 2 and Case 3.

Case 2: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ determined by Equations (71) and (72) can be represented as

$$\mathcal{H}_3(\gamma^2) = \beta_3\mathcal{H}_2(\gamma^2), \tag{86}$$

$$\mathcal{G}_3(\gamma^2, Q) = \beta_3 \mathcal{G}_2(\gamma^2, Q), \tag{87}$$

where $\mathcal{H}_2, \mathcal{G}_2$ are quadratic polynomials in γ^2:

$$\mathcal{H}_2(\gamma^2) = -\varrho\beta_1\beta_2 - \gamma^2\alpha\varrho_f\varrho(m_1\beta_2 + m_2\beta_1), \tag{88}$$

$$\mathcal{G}_2(\gamma^2, Q) = -\beta_1\beta_2 - Q\alpha^2(m_1\beta_2 + m_2\beta_1), \tag{89}$$

m_1 and m_2 are non-zero constants, and the eigenvalues η_1, η_2 (contained in β_1, β_2) are different. Accordingly, the function $\mathcal{P}_7(\gamma, Q)$ can be written as

$$\mathcal{P}_7(\gamma, Q) = \beta_3 \mathcal{P}_5(\gamma, Q), \tag{90}$$

where $\mathcal{P}_5(\gamma, Q)$ is a fifth-degree polynomial in γ:

$$\mathcal{P}_5(\gamma, Q) = i\kappa Q k \mathcal{H}_2(\gamma^2) + \gamma\varrho\mathcal{G}_2(\gamma^2, Q). \tag{91}$$

The functions $\mathcal{H}_3, \mathcal{G}_3$ are of the form (86), (87) if two eigenvalues of the acoustic tensor coincide, or if one of the terms $s_1 r_1, s_2 r_2, s_3 r_3$ in Equations (71) and (72) vanishes. For instance, if $\eta_3 = \eta_2 \neq \eta_1$ and $s_1 r_1, s_2 r_2$ are non-zero, then $\mathcal{H}_2, \mathcal{G}_2$ are obtained with $m_1 = s_1 r_1$, $m_2 = s_2 r_2 + s_3 r_3$. If η_1, η_2, η_3 are different, $s_1 r_1, s_2 r_2$ are non-zero and $s_3 r_3 = 0$, then $m_1 = s_1 r_1$, $m_2 = s_2 r_2$.

One eigenvalue of the tensor **B** in Case 2 coincides with the eigenvalue η_3 of the acoustic tensor. Equation (57) has two real roots $\gamma = \pm\sqrt{\eta_3/\varrho}$ for any Q. We are interested in the other roots, which are the roots of the polynomial $\mathcal{P}_5$. The analysis proceeds in the same way as in Case 1. For $Q = 0$, the non-zero roots of $\mathcal{P}_5$ are $\pm\sqrt{\eta_i/\varrho}$, $i = 1, 2$. Since the roots are simple, formula (76) can be applied, producing for the roots $\pm\sqrt{\eta_1/\varrho}$ and $\pm\sqrt{\eta_2/\varrho}$, respectively,

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\kappa^2 k\alpha\varrho_f}{2\varrho} m_1, \tag{92}$$

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\kappa^2 k\alpha\varrho_f}{2\varrho} m_2. \tag{93}$$

Close inspection of the transition from $\mathcal{P}_7$ to $\mathcal{P}_5$ reveals that the equality

$$m_1 + m_2 = s_1 r_1 + s_2 r_2 + s_3 r_3 \tag{94}$$

always holds, so that $m_1 + m_2 = 1$ due to Equation (82). Therefore, at least one of the derivatives (92), (93) must be positive, and at least one pair of the ω's must enter the half-plane $\operatorname{Im}\omega > 0$ as Q becomes non-zero.

The real roots γ of the polynomial $\mathcal{P}_5$ for $Q > 0$ are the real roots of the equation

$$\mathcal{H}_2(\gamma^2) = 0. \tag{95}$$

The roots γ^2 of Equation (95) are ζ_i/ϱ, $i = 1, 2$. The two values of Q corresponding to these roots are determined by two equations

$$\mathcal{G}_2(\zeta_i/\varrho, Q_i) = 0, \quad i = 1, 2. \tag{96}$$

In order to find Q_i, we combine Equations (88) and (89) to obtain

$$Q\alpha\mathcal{H}_2 - \gamma^2\varrho_f\varrho\mathcal{G}_2 = \varrho\big(\gamma^2\varrho_f - Q\alpha\big)\beta_1\beta_2. \tag{97}$$

To be able to use this relation, we need to show that the product $\beta_1\beta_2$ at $\gamma^2 = \zeta_i/\varrho$, $i = 1, 2$, is non-zero. Suppose that it is zero because of $\zeta_1 = \eta_1$. Then it follows from Equations (88) and (95) that $m_1(\eta_2 - \eta_1) = 0$, which is impossible in Case 2. Equation (97) with $\mathcal{H}_2 = 0$, $\mathcal{G}_2 = 0$ and $\beta_1\beta_2 \neq 0$ leads to relation (85) for $i = 1, 2$, and hence to the conclusion that there are ω's in the half-plane $\operatorname{Im}\omega > 0$ if Q is positive and less than the smallest of the two values Q_1, Q_2. The statement remains true with the stronger condition that Q be positive and less than the smallest of the three values Q_1, Q_2, Q_3.

Case 3: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ determined by Equations (71) and (72) can be represented as

$$\mathcal{H}_3(\gamma^2) = \beta_2\beta_3\mathcal{H}_1(\gamma^2), \tag{98}$$

$$\mathcal{G}_3(\gamma^2, Q) = \beta_2\beta_3\mathcal{G}_1(\gamma^2, Q), \tag{99}$$

where $\mathcal{H}_1$ and $\mathcal{G}_1$ are linear functions in γ^2:

$$\mathcal{H}_1(\gamma^2) = -\varrho\beta_1 - \gamma^2\alpha\varrho_f\varrho, \tag{100}$$

$$\mathcal{G}_1(\gamma^2, Q) = -\beta_1 - Q\alpha^2. \tag{101}$$

Accordingly, the function $\mathcal{P}_7(\gamma, Q)$ can be written as

$$\mathcal{P}_7(\gamma, Q) = \beta_2\beta_3\mathcal{P}_3(\gamma, Q), \tag{102}$$

where $\mathcal{P}_3(\gamma, Q)$ is a third-degree polynomial in γ:

$$\mathcal{P}_3(\gamma, Q) = i\kappa Qk\mathcal{H}_1(\gamma^2) + \gamma\varrho\mathcal{G}_1(\gamma^2, Q). \tag{103}$$

The functions $\mathcal{H}_3$ and $\mathcal{G}_3$ are of the form (98), (99) if, for instance, $\eta_1 = \eta_2$ and $s_3r_3 = 0$, or if $s_2r_2 = s_3r_3 = 0$. The latter is the case for isotropic elasticity.

The eigenvalues η_2, η_3 of the acoustic tensor in Case 3 are also the eigenvalues of the tensor **B**. Equation (57) has four real roots $\pm\sqrt{\eta_2/\varrho}$, $\pm\sqrt{\eta_3/\varrho}$ for any Q. We are interested in the other roots which are the roots of the polynomial $\mathcal{P}_3$. For $Q = 0$, two non-zero roots of $\mathcal{P}_3$ are $\pm\sqrt{\eta_1/\varrho}$. Formula (76) gives

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\kappa^2 k\alpha\varrho_f}{2\varrho} > 0, \tag{104}$$

so the two ω's enter the half-plane $\operatorname{Im}\omega > 0$ as Q becomes non-zero.

The real roots γ of the polynomial $\mathcal{P}_3$ for $Q > 0$ are the real roots of the equation

$$\mathcal{H}_1(\gamma^2) = 0. \tag{105}$$

The root γ^2 of Equation (105) is ζ_1/ϱ. The corresponding value of Q, denoted by Q_1, is determined by the equation

$$\mathcal{G}_1(\zeta_1/\varrho, Q_1) = 0. \tag{106}$$

Observing that β_1 cannot be zero at $\gamma^2 = \zeta_1/\varrho$, we can eliminate it from Equations (100) and (101) using Equations (105) and (106), and obtain Equation (85) with $i = 1$. Thus, there are ω's in the half-plane $\operatorname{Im}\omega > 0$ if Q is positive and less than Q_1. This completes the proof for normal mode solutions of the first type.

8.2 | Real ω, complex κ

For the analysis of normal mode solutions of the second type, we multiply Equation (59) by ω/κ to obtain Equation (57) in the form

$$\mathcal{S}_4(\gamma^2, Q) = 0, \tag{107}$$

where $\mathcal{S}_4(\gamma^2, Q)$ is a fourth-degree polynomial in γ^2:

$$\mathcal{S}_4(\gamma^2, Q) = i\omega Qk\mathcal{H}_3(\gamma^2) + \gamma^2\varrho\mathcal{G}_3(\gamma^2, Q). \tag{108}$$

The roots of Equation (107) will be analysed in the same way as the roots of Equation (57), with the only difference that, in view of the inequality (54), we will treat γ^2 as an unknown quantity rather than γ. The same three cases as in Section 8.1 are to be considered separately.

Case 1: the eigenvalues of the acoustic tensor are different, and the components s_i, r_i are all non-zero.

For $Q = 0$, the differentiation of the function (108) with the use of Equations (71) and (72) yields

$$\left.\frac{\partial \mathcal{S}_4}{\partial(\gamma^2)}\right|_{Q=0} = -\varrho\beta_1\beta_2\beta_3 + \gamma^2\varrho^2(\beta_2\beta_3 + \beta_1\beta_3 + \beta_1\beta_2) \tag{109}$$

and, for real values of γ^2,

$$\operatorname{Im}\frac{\partial \mathcal{S}_4}{\partial Q} = -\omega k\varrho\left[\beta_1\beta_2\beta_3 + \gamma^2\alpha\varrho_f(s_1r_1\beta_2\beta_3 + s_2r_2\beta_1\beta_3 + s_3r_3\beta_1\beta_2)\right]. \tag{110}$$

The non-zero roots of Equation (107) for $Q = 0$ are $\gamma^2 = \eta_i/\varrho$, $i = 1,2,3$. Calculating the ratio of the derivatives (110) and (109) for these roots gives, respectively,

$$\omega \operatorname{Im}\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, s_1r_1, \tag{111}$$

$$\omega \operatorname{Im}\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, s_2r_2, \tag{112}$$

$$\omega \operatorname{Im}\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, s_3r_3. \tag{113}$$

Owing to the equality (82), at least one of the quantities (111)–(113) must be positive and hence there must be at least one root γ^2 that satisfies the inequality (54) for small non-zero values of Q.

For $Q > 0$, the real roots γ^2 of Equation (107) are ζ_i/ϱ, $i = 1,2,3$. The corresponding quantities Q_i obtained from Equation (84) are given by relations (85).

Case 2: the functions $\mathcal{H}_3$, $\mathcal{G}_3$ can be written in the form (86), (87). The function $\mathcal{S}_4$ becomes

$$\mathcal{S}_4(\gamma^2, Q) = \beta_3\mathcal{S}_3(\gamma^2, Q), \tag{114}$$

where $\mathcal{S}_3(\gamma^2, Q)$ is a third-degree polynomial in γ^2:

$$\mathcal{S}_3(\gamma^2, Q) = i\omega Qk\mathcal{H}_2(\gamma^2) + \gamma^2\varrho\mathcal{G}_2(\gamma^2, Q). \tag{115}$$

Equation (107) has the real root $\gamma^2 = \eta_3/\varrho$ for any Q. For $Q = 0$, two non-zero roots γ^2 of $\mathcal{S}_3$ are η_1/ϱ, η_2/ϱ. For these roots we obtain, respectively,

$$\omega \operatorname{Im}\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, m_1, \tag{116}$$

$$\omega \operatorname{Im}\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, m_2. \tag{117}$$

Since $m_1 + m_2 = 1$, there must be at least one root γ^2 that satisfies the inequality (54) for small non-zero values of Q. For the real roots ζ_1/ϱ, ζ_2/ϱ for $Q > 0$, Equation (97) with $\mathcal{H}_2 = 0$, $\mathcal{G}_2 = 0$ and $\beta_1\beta_2 \neq 0$ leads to relation (85) for Q_1 and Q_2.

Case 3: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ can be written in the form (98), (99). The function $\mathcal{S}_4$ becomes

$$\mathcal{S}_4(\gamma^2, Q) = \beta_2\beta_3\mathcal{S}_2(\gamma^2, Q), \tag{118}$$

where $\mathcal{S}_2(\gamma^2, Q)$ is a quadratic polynomial in γ^2:

$$\mathcal{S}_2(\gamma^2, Q) = i\omega Qk\mathcal{H}_1(\gamma^2) + \gamma^2\varrho\mathcal{G}_1(\gamma^2, Q). \tag{119}$$

Equation (107) has two real roots η_2/ϱ, η_3/ϱ for any Q. For $Q = 0$, the non-zero root γ^2 of $\mathcal{S}_2$ is η_1/ϱ. For this root, we obtain

$$\omega \operatorname{Im}\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho} > 0, \tag{120}$$

so there is a root γ^2 that satisfies the inequality (54) for small non-zero values of Q. Relation (85) for Q_1 is obtained from Equations (100), (101), (105) and (106). This completes the proof of Proposition 3.

9 | PROOF OF PROPOSITION 4

9.1 | Real $\varkappa$, complex ω

The proof of Proposition 4 has much in common with the proof of Proposition 3 presented in Section 8. Some details already elaborated in Section 8 will, therefore, be omitted. The roots of Equations (57) and (107) will first be analysed for small values of Q through the derivatives $d\gamma/dQ$ and $d\gamma^2/dQ$ calculated at $Q = 0$. In addition, for the present proof, we will need the derivatives calculated at real roots γ for $Q > 0$. The assumption of Proposition 4 that the acoustic tensor is symmetric means that the matrix A can be diagonalized by an orthogonal matrix, so that in Equations (71) and (72), we have $s_i = r_i = n_i$, where n_i are the components of the vector $\mathbf{n}$ in a rotated coordinate system in which the matrix A is diagonal. A coordinate system in which the matrix A is diagonal will be referred to as *the rotated system*.

In this subsection, Proposition 4 will be proved for normal mode solutions of the first type with real wave numbers $\varkappa$ and complex frequencies ω. The same three cases as in Section 8.1 are to be considered.

Case 1: the eigenvalues of the acoustic tensor are different, and all components of the vector $\mathbf{n}$ in the rotated system are non-zero.

As is known from the proof of Proposition 3, Equation (57) with $Q = 0$ has one zero root and six real roots $\gamma = \pm\sqrt{\eta_i/\varrho}$, $i = 1, 2, 3$, where η_i are the eigenvalues of the acoustic tensor. Equations (76)–(78) with $\gamma = 0$ give

$$\operatorname{Im}\frac{d\omega}{dQ} = -\varkappa^2 k < 0. \tag{121}$$

Hence, the root $\gamma = 0$ is such that the ω enters the half-plane $\operatorname{Im}\omega < 0$ as Q becomes non-zero. This root does not produce growing wave solutions for small values of Q. For the roots $\gamma = \pm\sqrt{\eta_i/\varrho}$, $i = 1, 2, 3$, Equations (79)–(81) become

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\varkappa^2 k\alpha\varrho_f}{2\varrho}\, n_1^2 > 0, \tag{122}$$

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\varkappa^2 k\alpha\varrho_f}{2\varrho}\, n_2^2 > 0, \tag{123}$$

$$\operatorname{Im}\frac{d\omega}{dQ} = \frac{\varkappa^2 k\alpha\varrho_f}{2\varrho}\, n_3^2 > 0, \tag{124}$$

where n_1, n_2 and n_3 are the components of the vector $\mathbf{n}$ in the rotated system. Inequalities (122)–(124) show that all six roots $\gamma = \pm\sqrt{\eta_i/\varrho}$ are such that the ω's enter the half-plane $\operatorname{Im}\omega > 0$ as Q becomes non-zero, so that for sufficiently small non-zero values of Q, there are six ω's which correspond to growing wave solutions.

In order for the ω's to leave the half-plane $\operatorname{Im}\omega > 0$ as Q increases, they must cross the real axis. Equation (57) with $Q > 0$ has six real roots $\gamma = \pm\sqrt{\zeta_i/\varrho}$, $i = 1, 2, 3$, determined by Equation (83), where ζ_i are the eigenvalues of the tensor $\mathbf{B}$. The condition $\varrho > \alpha\varrho_f$ and the assumed symmetry and positive definiteness of the acoustic tensor guarantee that the eigenvalues ζ_i are real and positive.[19] Three values of Q at which Equation (57) has real roots $\gamma = \pm\sqrt{\zeta_i/\varrho}$ are determined by Equations (84) and (85).

The next step is to find the sign of $\operatorname{Im}(d\omega/dQ)$ for the roots $\gamma = \pm\sqrt{\zeta_i/\varrho}$. Differentiating the function (59) and substituting $\mathcal{H}_3 = 0$ and $\mathcal{G}_3 = 0$ gives

$$\frac{\partial\mathcal{P}_7}{\partial Q} = \gamma\varrho\,\frac{\partial\mathcal{G}_3}{\partial Q}, \quad \frac{\partial\mathcal{P}_7}{\partial\gamma} = 2i\varkappa Qk\gamma\,\frac{d\mathcal{H}_3}{d(\gamma^2)} + 2\gamma^2\varrho\,\frac{\partial\mathcal{G}_3}{\partial(\gamma^2)}. \tag{125}$$

Using Equation (76) with $\gamma = \omega/\varkappa$ (where $\varkappa$ is fixed), we see that for real values of γ,

$$\operatorname{sign}\left(\operatorname{Im}\frac{d\omega}{dQ}\right) = \operatorname{sign}\left(\frac{\partial\mathcal{G}_3}{\partial Q}\right)\operatorname{sign}\left(\frac{d\mathcal{H}_3}{d(\gamma^2)}\right). \tag{126}$$

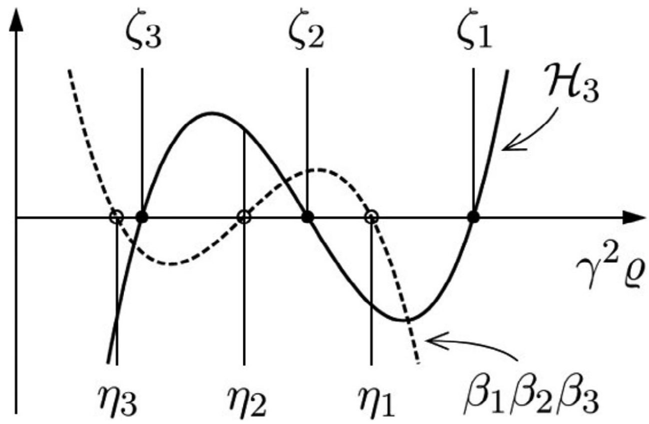

FIGURE 1 Functions $\beta_1\beta_2\beta_3$ and $\mathcal{H}_3$ for a symmetric acoustic tensor in Case 1.

Differentiating Equation (73) with respect to Q shows that for real γ and $\mathcal{H}_3 = 0$, the sign of the derivative $\partial\mathcal{G}_3/\partial Q$ is the same as the sign of the product $\beta_1\beta_2\beta_3$. This product is a real third-degree polynomial in γ^2, negative for $\gamma^2 \to +\infty$, with the roots η_i/ϱ, $i = 1, 2, 3$. The product $\beta_1\beta_2\beta_3$ as a function of γ^2 is shown in Figure 1 with the eigenvalues η_1, η_2, η_3 numbered in decreasing order.

The function $\mathcal{H}_3(\gamma^2)$ is a real third-degree polynomial in γ^2 with the roots ζ_i/ϱ, $i = 1, 2, 3$. Equation (71) with $s_i = r_i = n_i$ and η_1, η_2, η_3 numbered in decreasing order yields

$$\mathcal{H}_3(\eta_1/\varrho) = -\eta_1\alpha\varrho_f(\eta_2 - \eta_1)(\eta_3 - \eta_1)n_1^2 < 0, \tag{127}$$

$$\mathcal{H}_3(\eta_2/\varrho) = -\eta_2\alpha\varrho_f(\eta_1 - \eta_2)(\eta_3 - \eta_2)n_2^2 > 0, \tag{128}$$

$$\mathcal{H}_3(\eta_3/\varrho) = -\eta_3\alpha\varrho_f(\eta_1 - \eta_3)(\eta_2 - \eta_3)n_3^2 < 0. \tag{129}$$

It is seen from Equation (65) that if $\varrho > \alpha\varrho_f$, then $\mathcal{H}_3$ is positive for $\gamma^2 \to +\infty$. This fact together with Equations (127)–(129) means that the eigenvalues $\zeta_1, \zeta_2, \zeta_3$ numbered in decreasing order satisfy the inequalities

$$\eta_3 < \zeta_3 < \eta_2 < \zeta_2 < \eta_1 < \zeta_1. \tag{130}$$

The function $\mathcal{H}_3(\gamma^2)$ is shown in Figure 1. The two curves in Figure 1 show that for $\gamma^2 = \zeta_i/\varrho$, $i = 1, 2, 3$,

$$\beta_1\beta_2\beta_3\frac{d\mathcal{H}_3}{d(\gamma^2)} < 0, \tag{131}$$

and hence $\mathrm{Im}(d\omega/dQ) < 0$ for all six roots $\gamma = \pm\sqrt{\zeta_i/\varrho}$.

Thus, there are six real frequencies ω at $Q = 0$ which enter the half-plane $\mathrm{Im}\,\omega > 0$ as Q becomes non-zero, and six transitions from $\mathrm{Im}\,\omega > 0$ to $\mathrm{Im}\,\omega < 0$ on the real axis at Q_i, $i = 1, 2, 3$. There are no other transitions for $Q > 0$. This means that there are no growing wave solutions of the first type if $Q > Q_{max}$.

Case 2: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ are of the form (86), (87).

The derivation of inequality (121) for the root $\gamma = 0$ at $Q = 0$ does not depend on whether Case 1 or Case 2 is considered. The root $\gamma = 0$ does not produce growing wave solutions for small values of Q. Equations (88)–(91) are valid with the restriction that the non-zero constants m_1, m_2 are now positive, as they are combinations of the non-negative quantities n_1^2, n_2^2, n_3^2. As a consequence, the derivatives (92) and (93) for $Q = 0$ are positive as well, leading to the conclusion that there are four frequencies in the half-plane $\mathrm{Im}\,\omega > 0$ at small non-zero values of Q.

The real roots of the polynomial $\mathcal{P}_5$ for $Q > 0$ are $\gamma = \pm\sqrt{\zeta_i/\varrho}$, $i = 1, 2$. The two values of Q corresponding to these roots are determined by Equation (85) with $i = 1, 2$. Differentiating the function (91) and substituting $\mathcal{H}_2 = 0$ and $\mathcal{G}_2 = 0$ gives equations similar to Equations (125) and (126):

$$\frac{\partial\mathcal{P}_5}{\partial Q} = \gamma\varrho\frac{\partial\mathcal{G}_2}{\partial Q}, \quad \frac{\partial\mathcal{P}_5}{\partial\gamma} = 2i\kappa Qk\gamma\frac{d\mathcal{H}_2}{d(\gamma^2)} + 2\gamma^2\varrho\frac{\partial\mathcal{G}_2}{\partial(\gamma^2)}, \tag{132}$$

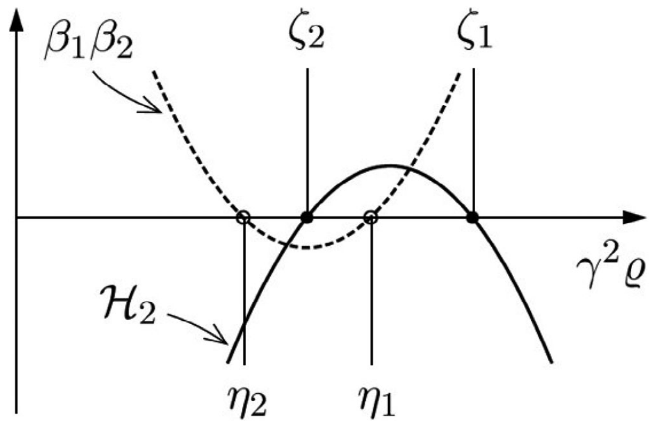

FIGURE 2 Functions $\beta_1\beta_2$ and $\mathcal{H}_2$ for a symmetric acoustic tensor in Case 2.

$$\mathrm{sign}\left(\mathrm{Im}\frac{d\omega}{dQ}\right) = \mathrm{sign}\left(\frac{\partial\mathcal{G}_2}{\partial Q}\right)\mathrm{sign}\left(\frac{d\mathcal{H}_2}{d(\gamma^2)}\right). \tag{133}$$

As follows from Equation (97), the sign of the derivative $\partial\mathcal{G}_2/\partial Q$ for real γ and $\mathcal{H}_2 = 0$ is the same as the sign of the product $\beta_1\beta_2$. This product is a real quadratic polynomial in γ^2, positive for $\gamma^2 \to +\infty$, with the roots η_1/ϱ and η_2/ϱ. The polynomial $\beta_1\beta_2$ is shown in Figure 2, where it is assumed that $\eta_1 > \eta_2$.

The function $\mathcal{H}_2$ is a real quadratic polynomial in γ^2 with the roots ζ_i/ϱ and $i = 1, 2$. Equation (88) with $\eta_1 > \eta_2$ yields

$$\mathcal{H}_2(\eta_1/\varrho) = \eta_1\alpha\varrho_f(\eta_1 - \eta_2)m_1 > 0, \tag{134}$$

$$\mathcal{H}_2(\eta_2/\varrho) = \eta_2\alpha\varrho_f(\eta_2 - \eta_1)m_2 < 0. \tag{135}$$

Taking into account that $m_1 + m_2 = 1$, it can be deduced from Equation (88) that the polynomial $\mathcal{H}_2$ is negative for $\gamma^2 \to +\infty$ if $\varrho > \alpha\varrho_f$. This fact together with Equations (134) and (135) leads to the inequalities

$$\eta_2 < \zeta_2 < \eta_1 < \zeta_1, \tag{136}$$

where it is assumed that $\zeta_1 > \zeta_2$. The function $\mathcal{H}_2(\gamma^2)$ is shown in Figure 2. The two curves in Figure 2 show that for $\gamma^2 = \zeta_i/\varrho$, $i = 1, 2$,

$$\beta_1\beta_2\,\frac{d\mathcal{H}_2}{d(\gamma^2)} < 0, \tag{137}$$

and hence $\mathrm{Im}(d\omega/dQ) < 0$ for all four roots $\gamma = \pm\sqrt{\zeta_i/\varrho}$, $i = 1, 2$. There are four frequencies in the half-plane $\mathrm{Im}\,\omega > 0$ for small non-zero values of Q, and four transitions from $\mathrm{Im}\,\omega > 0$ to $\mathrm{Im}\,\omega < 0$ on the real axis at Q_1 and Q_2. There are no other transitions for $Q > 0$ and therefore no growing wave solutions of the first type for $Q > Q_{max}$.

Case 3: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ are of the form (98) and (99).

Inequality (121) for the root $\gamma = 0$ at $Q = 0$ is obtained in the same way as in the previous cases. Equations (100)–(106) and the corresponding results remain valid. There are two frequencies ω in the half-plane $\mathrm{Im}\,\omega > 0$ for small non-zero values of Q. What we need is to see that the sign of the derivative $\mathrm{Im}(d\omega/dQ)$ for the roots $\gamma = \pm\sqrt{\zeta_1/\varrho}$ of the polynomial $\mathcal{P}_3$ at Q_1 is negative. Differentiating the function (103) and substituting $\mathcal{H}_1 = 0$ and $\mathcal{G}_1 = 0$ gives

$$\frac{\partial\mathcal{P}_3}{\partial Q} = \gamma\varrho\,\frac{\partial\mathcal{G}_1}{\partial Q}, \qquad \frac{\partial\mathcal{P}_3}{\partial\gamma} = 2i\varkappa Qk\gamma\,\frac{d\mathcal{H}_1}{d(\gamma^2)} + 2\gamma^2\varrho\,\frac{\partial\mathcal{G}_1}{\partial(\gamma^2)}, \tag{138}$$

$$\mathrm{sign}\left(\mathrm{Im}\frac{d\omega}{dQ}\right) = \mathrm{sign}\left(\frac{\partial\mathcal{G}_1}{\partial Q}\right)\mathrm{sign}\left(\frac{d\mathcal{H}_1}{d(\gamma^2)}\right). \tag{139}$$

The required derivatives are obtained immediately from Equations (100) and (101):

$$\frac{\partial \mathcal{G}_1}{\partial Q} = -\alpha^2 < 0, \quad \frac{d\mathcal{H}_1}{d(\gamma^2)} = \varrho(\varrho - \alpha\varrho_f) > 0, \tag{140}$$

whence $\mathrm{Im}(d\omega/dQ) < 0$. Two frequencies with $\mathrm{Im}\,\omega > 0$ cross the real axis at Q_1. There are no frequencies with $\mathrm{Im}\,\omega > 0$ for $Q > Q_{max}$. This completes the proof for normal mode solutions of the first type.

9.2 | Real ω, complex κ

For the analysis of normal mode solutions of the second type, we consider Equation (107) in which ω is a fixed real parameter and $\gamma^2 = \omega^2/\kappa^2$ is an unknown complex quantity. The same three cases as in Section 8.2 are to be considered.

Case 1: the eigenvalues of the acoustic tensor are different, and all components of the vector $\mathbf{n}$ in the rotated system are non-zero.

Equation (107) with $Q = 0$ has one zero root $\gamma^2 = 0$ and three real roots $\gamma^2 = \eta_i/\varrho$, $i = 1, 2, 3$. The ratio of the derivatives (109) and (110) for the root $\gamma^2 = 0$ gives

$$\omega\,\mathrm{Im}\,\frac{d\gamma^2}{dQ} = -\omega^2 k < 0. \tag{141}$$

This root does not produce growing wave solutions for small values of Q. For the roots $\gamma^2 = \eta_i/\varrho$, $i = 1, 2, 3$, Equations (111)–(113) become, respectively,

$$\omega\,\mathrm{Im}\,\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, n_1^2 > 0, \tag{142}$$

$$\omega\,\mathrm{Im}\,\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, n_2^2 > 0, \tag{143}$$

$$\omega\,\mathrm{Im}\,\frac{d\gamma^2}{dQ} = \frac{\omega^2 k\alpha\varrho_f}{\varrho}\, n_3^2 > 0, \tag{144}$$

and there are three roots that satisfy the inequality (54) for small non-zero values of Q.

To prove that there are no growing wave solutions for $Q > Q_{max}$, we need to show that the quantity $\omega\,\mathrm{Im}(d\gamma^2/dQ)$ is negative for all three roots $\gamma^2 = \zeta_i/\varrho$ at Q_i, $i = 1, 2, 3$. Differentiating the function (108) and substituting $\mathcal{H}_3 = 0$ and $\mathcal{G}_3 = 0$ gives

$$\frac{\partial \mathcal{S}_4}{\partial Q} = \gamma^2\varrho\,\frac{\partial \mathcal{G}_3}{\partial Q}, \quad \frac{\partial \mathcal{S}_4}{\partial(\gamma^2)} = i\omega Qk\frac{d\mathcal{H}_3}{d(\gamma^2)} + \gamma^2\varrho\,\frac{\partial \mathcal{G}_3}{\partial(\gamma^2)} \tag{145}$$

and, for real values of γ,

$$\mathrm{sign}\left(\omega\,\mathrm{Im}\frac{d\gamma^2}{dQ}\right) = \mathrm{sign}\left(\frac{\partial \mathcal{G}_3}{\partial Q}\right)\mathrm{sign}\left(\frac{d\mathcal{H}_3}{d(\gamma^2)}\right). \tag{146}$$

The right-hand side of Equation (146) is the same as in Equation (126). We have seen in Section 9.1 that the right-hand side of Equation (126) is negative for the roots $\gamma^2 = \zeta_i/\varrho$, $i = 1, 2, 3$.

Case 2: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ are of the form (86) and (87).

The derivation of inequality (141) for the root $\gamma^2 = 0$ at $Q = 0$ does not change. For the roots $\gamma^2 = \eta_i/\varrho$, $i = 1, 2$, at $Q = 0$, the constants m_1 and m_2 in Equations (116) and (117) are positive as combinations of the non-negative quantities n_1^2, n_2^2 and n_3^2. There are two roots that produce growing wave solutions at small non-zero values of Q. We need to show that the quantity $\omega\,\mathrm{Im}(d\gamma^2/dQ)$ is negative for two roots $\gamma^2 = \zeta_i/\varrho$ at Q_i, $i = 1, 2$. Calculating the derivatives gives

$$\frac{\partial \mathcal{S}_3}{\partial Q} = \gamma^2\varrho\,\frac{\partial \mathcal{G}_2}{\partial Q}, \quad \frac{\partial \mathcal{S}_3}{\partial(\gamma^2)} = i\omega Qk\frac{d\mathcal{H}_2}{d(\gamma^2)} + \gamma^2\varrho\,\frac{\partial \mathcal{G}_2}{\partial(\gamma^2)} \tag{147}$$

and, for real values of γ,

$$\operatorname{sign}\left(\omega\,\mathrm{Im}\frac{d\gamma^2}{dQ}\right)=\operatorname{sign}\left(\frac{\partial\mathcal{G}_2}{\partial Q}\right)\operatorname{sign}\left(\frac{d\mathcal{H}_2}{d(\gamma^2)}\right). \tag{148}$$

The right-hand side of Equation (148) is the same as in Equation (133). We have seen in Section 9.1 that the right-hand side of Equation (133) is negative for the roots $\gamma^2=\zeta_i/\varrho$, $i=1,2$.

Case 3: the functions $\mathcal{H}_3$ and $\mathcal{G}_3$ are of the form (98) and (99).

For $Q=0$, inequality (141) for the root $\gamma^2=0$ and inequality (120) for the root $\gamma^2=\eta_1/\varrho$ remain unchanged. There is one root γ^2 that produces growing wave solutions at small non-zero values of Q. For the root $\gamma^2=\zeta_1/\varrho$ at Q_1, we obtain

$$\frac{\partial\mathcal{S}_2}{\partial Q}=\gamma^2\varrho\frac{\partial\mathcal{G}_1}{\partial Q},\quad \frac{\partial\mathcal{S}_2}{\partial(\gamma^2)}=i\omega Qk\frac{d\mathcal{H}_1}{d(\gamma^2)}+\gamma^2\varrho\frac{\partial\mathcal{G}_1}{\partial(\gamma^2)}, \tag{149}$$

$$\operatorname{sign}\left(\omega\,\mathrm{Im}\frac{d\gamma^2}{dQ}\right)=\operatorname{sign}\left(\frac{\partial\mathcal{G}_1}{\partial Q}\right)\operatorname{sign}\left(\frac{d\mathcal{H}_1}{d(\gamma^2)}\right). \tag{150}$$

Inequalities (140) confirm that $\omega\,\mathrm{Im}(d\gamma^2/dQ)<0$ and therefore there are no growing wave solutions for $Q>Q_1$. This completes the proof of Proposition 4.

10 | PROOF OF PROPOSITION 5

Substituting the normal mode solutions (16) into the UP2 system (1), (12), (15) yields Equations (42) and (43) and, instead of Equation (44), the equation

$$\kappa Q\alpha n_k v^0_{sk}-(\omega+i\kappa^2Qk)p^0_f=0. \tag{151}$$

Solutions with $\omega=0$ are not growing waves. Equations (42), (43) and (151) with $\kappa=0,\omega\neq0$ give $v^0_{si}=0,\sigma^0_{ji}=0,p^0_f=0$, that is, only the trivial solution. For the proof of the proposition, we may, therefore, assume that $\omega\neq0$ and $\kappa\neq0$.

Substituting σ^0_{ji} from Equation (43) into Equation (42) gives

$$\varrho\omega^2v^0_{si}-\kappa^2A_{ik}v^0_{sk}-\alpha\kappa\omega n_i p^0_f=0. \tag{152}$$

Let $\bar{v}^0_{si},\bar{p}^0_f$ denote the complex conjugates of v^0_{si},p^0_f. Multiplying the complex conjugate of Equation (151) by $\kappa\omega p^0_f$ and Equation (152) by $\bar{\kappa}Q\bar{v}^0_{si}$ with summation over i, we obtain two equations

$$|\kappa|^2\omega Q\alpha n_k\bar{v}^0_{sk}p^0_f-\kappa\omega(\bar{\omega}-i\bar{\kappa}^2Qk)p^0_f\bar{p}^0_f=0, \tag{153}$$

$$\bar{\kappa}\omega^2\varrho Qv^0_{si}\bar{v}^0_{si}-\kappa|\kappa|^2QA_{ik}v^0_{sk}\bar{v}^0_{si}-|\kappa|^2\omega Q\alpha n_i\bar{v}^0_{si}p^0_f=0. \tag{154}$$

Adding these two equations gives

$$\bar{\kappa}\omega^2a_1-\kappa|\kappa|^2a_2-\kappa\omega(\bar{\omega}-i\bar{\kappa}^2Qk)a_3=0 \tag{155}$$

with the real quantities

$$a_1=\varrho Qv^0_{si}\bar{v}^0_{si}>0,\quad a_2=QA_{ik}v^0_{sk}\bar{v}^0_{si}>0,\quad a_3=p^0_f\bar{p}^0_f\geq0. \tag{156}$$

As follows from Equation (151), if all $v^0_{si}=0$, then $p^0_f=0$ as well, that is why a_1 is non-zero. For the same reason and due to the fact that the acoustic tensor is assumed to be symmetric and positive definite, a_2 is real and positive.

For normal mode solutions with $\kappa\in\mathbb{R}$, we multiply Equation (155) by $\bar{\omega}$ to obtain

$$\omega|\omega|^2a_1-\bar{\omega}\kappa^2a_2-|\omega|^2(\bar{\omega}-i\kappa^2Qk)a_3=0, \tag{157}$$

from which

$$\operatorname{Im}\omega = -\frac{|\omega|^2\kappa^2 Qka_3}{|\omega|^2 a_1 + \kappa^2 a_2 + |\omega|^2 a_3} \le 0. \tag{158}$$

This proves the non-existence of growing wave solutions of the first type.

For normal mode solutions with $\omega \in \mathbb{R}$, we multiply Equation (155) by κ to obtain

$$|\kappa|^2\omega^2 a_1 - \kappa^2|\kappa|^2 a_2 - \kappa^2\omega^2 a_3 + i|\kappa|^4\omega Qka_3 = 0, \tag{159}$$

from which it follow that

$$\omega \operatorname{Im}\kappa^2 = \frac{|\kappa|^4\omega^2 Qka_3}{|\kappa|^2 a_2 + \omega^2 a_3} \ge 0. \tag{160}$$

This proves the non-existence of growing wave solutions of the second type.

11 CONCLUSION

The u-p equations, as approximations of the exact equations, yield approximate solutions. It is known that the accuracy of the u-p approximations depends mainly on the frequency content of the motion and the permeability of the medium. Higher frequencies and higher permeability lead to larger deviations from the exact solutions. The accuracy determines the range of applicability in which the u-p formulations may be considered acceptable for the solution of a particular class of physical problems. Another issue related to applicability of the u-p formulations is hyperbolicity of the equations as a necessary condition for well-posedness of the problem.[19] The hyperbolicity conditions for the two u-p approximations are different and also differ from the hyperbolicity conditions for the exact formulation. The hyperbolicity conditions determine the range of applicability of the u-p approximations in the sense of well-posedness rather than accuracy. The hyperbolicity conditions involve neither frequency nor permeability and have no relation to accuracy.

Yet another aspect of applicability of the u-p formulations concerns time-harmonic growing wave solutions whose amplitude increases exponentially in time or space as the wave propagates. In this paper, the non-existence of such solutions is postulated for saturated solids with a linearly hyperelastic skeleton and is considered as an acceptability criterion for the dynamic equations. It is shown that the exact and UP2 formulations satisfy the acceptability criterion, whereas the UP1 formulation does not. This fact, however, does not mean that the UP2 approximation is preferable to UP1. The latter is more accurate. Whether the UP1 equations have growing wave solutions depends on the density of the solid phase and the compressibility of the pore fluid. The propositions proved in this paper identify the range of applicability in which the UP1 equations have no growing wave solutions. Similar to the hyperbolicity conditions, the existence of growing wave solutions of the UP1 equations depends neither on the frequency nor on the permeability and has no explicit relation to the accuracy of the UP1 approximation.

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CONFLICT OF INTEREST STATEMENT

The author declares no conflicts of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analysed in this study.

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