Mathematische Zeitschrift (2023) 305:50
https://doi.org/10.1007/s00209-023-03381-9

Mathematische Zeitschrift

On conformal planes of finite area

Alexander Lytchak[1]

Received: 26 November 2022 / Accepted: 25 September 2023 / Published online: 24 October 2023

Abstract
We discuss solutions of several questions concerning the geometry of conformal planes.

Keywords Alexandrov surface · Curvature bounds · Uniformization

1 Introduction

1.1 Applications

Recently, the Liouville equation

$$\Delta u + e^{2u} = 0, \tag{1.1}$$

and its (super-) solutions on $\mathbb{R}^2$ were investigated in a series of work [3, 11, 13], see also [6, 9]. Interesting facts on the geometry of the corresponding conformal planes

$$X^u = (\mathbb{R}^2, e^{2u} \cdot \delta_{Eucl})$$

were proven and the authors formulated several related questions.

Solutions of (1.1) correspond to conformal planes of constant curvature 1 and are closely related to some meromorphic functions on $\mathbb{C}$. Complex analysis can been successfully used to study the solutions and arising geometries [3, 11]. For supersolutions of (1.1), thus for conformal metrics on the plane of curvature ≥ 1, complex analysis does not seem to be such an appropriate tool.

The theory of surfaces with integral curvature bounds in the sense of Alexandrov, see [1, 23, 28] turns out to be more helpful, especially, for questions concerning conformal planes of bounded total area and curvature. This approach implies the following solutions to four questions formulated in [13, 14].

Proposition 1.1 *For a smooth* $u : \mathbb{R}^2 \to \mathbb{R}$ *satisfying*

$$\Delta u + e^{2u} \leq 0, \tag{1.2}$$

Alexander Lytchak was partially supported by the DFG Grants SFB TRR 191 and SPP 2026.

✉ Alexander Lytchak
alexander.lytchak@kit.edu

1 Institute for Algebra and Geometry, KIT Englerstr. 2 , 76131 Karlsruhe, Germany

let the conformal plane X^u have finite area. Then the diameter $\operatorname{diam}(X^u)$ *of the plane X^u can be any number in the interval* $(0, 2\pi)$.

In [13, Theorem 1.4], it was proved that (1.2) implies $\operatorname{diam}(X^u) \leq 2\pi$, and [13, Question 8.2] asks if the inequality $\operatorname{diam}(X^u) \leq \pi$ holds.

Proposition 1.2 *For a smooth $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.2), the area of the conformal plane X^u can be infinite or any positive real number.*

On contrary, for solutions u of (1.1) the conformal planes X^u have area 4π or infinity, [13]. It has been asked in [13, Question 8.3], whether the upper bound of 4π is valid for all conformal planes X^u of finite area corresponding to solutions of (1.2). The above result has been independently observed by Alexandre Eremenko.

As a consequence, we deduce a negative answer to another question formulated in [13, Question 8.7], see Corollary 2.1 below.

1.2 From the sphere to conformal planes

The above results are easy consequences of known theorems on singular metrics on $\mathbb{S}^2$ with bounded integral curvature and of a simple relation between conformal planes and conformal spheres, which we are going to explain now.

By the uniformization theorem, any Riemannian metric on $\mathbb{R}^2$ is either conformally equivalent to the disc or to the plane. While it is easy to construct many (non-complete) Riemannian metrics on $\mathbb{R}^2$ with prescribed curvature properties, (for instance, with constant curvature 1), it seems difficult to verify that such a synthetically constructed metric is a conformal plane. A criterion of conformality is provided by the special case of a classical result of Cheng–Yau [10, Corollary 1]: If a *complete* Riemannian manifold X homeomorphic to $\mathbb{R}^2$ has at most *quadratic area growth* then X is a conformal plane. In particular, all *complete* Riemannian metrics of *finite area* on $\mathbb{R}^2$ are conformal planes.

An easy criterion for *non-complete* planes, sufficient for the Propositions stated above, is the following one.

Proposition 1.3 *Let X be a Riemannian manifold homeomorphic to the plane and of finite area. Assume that the completion $\hat{X}$ of X is homeomorphic to $\mathbb{S}^2$ and that $\hat{X} \setminus X$ has just one point p. If the area of metric balls $B_r(p)$ in $\hat{X}$ around p grows at most quadratically,*

$$\liminf_{r \to 0} \frac{\operatorname{area}(B_r(p))}{r^2} < \infty,$$

then X is conformally equivalent to the plane.

It might be possible to deduce Proposition 1.3 from the theorem by Cheng–Yau mentioned above, applying a conformal change of the metric, which resembles the *inversion* at the point p. Instead, we observe that Proposition 1.3 is a consequence of a very general uniformization theorem in metric geometry [4, 18, 20, 22].

Remark 1.4 Some assumption in Proposition 1.3 on a neighborhood of p in $\hat{X}$ is needed, as the following easy example demonstrates: Consider the unit Euclidean disc with the conformal factor $f(z) = (1 - |z|^2)$. The completion $\hat{X}$ of this conformal disc X has finite area, is homeomorphic to $\mathbb{S}^2$, and $\hat{X} \setminus X$ has just one point.

Thus, in order to construct conformal planes with prescribed properties as in Propositions 1.1, 1.2, it suffices to construct metrics on the sphere with one singularity p and prescribed geometric properties outside the singularity. We construct such a piecewise spherical metric with only 3 vertices, such that the total angle at just one of these vertices (the singularity p) is larger than 2π. Note that all such metics are classified [12, 19]. Smoothing the metric at the singularites with angles smaller than 2π, we obtain the desired examples. These examples have bounded integral curvature in the sense of Alexandrov, [1, 23, 28]; more classical uniformization theorems, [28], imply the conclusion of Proposition 1.3 in this case.

1.3 Completions of conformal planes

A partial converse to Proposition 1.3 is essentially contained in the proof of [13, Theorem 1.4]:

Lemma 1.5 *Let the conformal plane $X = X^u$ have finite area and let $\hat{X}$ denote the completion of X. Then $\hat{X} \backslash X$ has at most one point.*

Thus, either X is complete or $\hat{X} \backslash X$ has exactly one point p. In the latter case, the space $\hat{X}$ can display a rather wild behavior near p. For instance, it may not be locally compact around p, see Example 3.1 below. Even if X has curvature larger than 1 and $\hat{X}$ is compact, thus homeomorphic to $\mathbb{S}^2$, the geometry around p can be rather wild, see Example 3.2 below.

The geometry of the completion $\hat{X}$ at the *singular point* $\hat{X} \backslash X$ turns out to be much tamer if the curvature on X is assumed to be integrable.

Recall first that the Hausdorff (=canonical Riemannian) area $\mathcal{H}^2$ on the conformal plane X^u is the multiple $e^{2u} \cdot \mathcal{L}^2_{\mathbb{R}^2}$ of the Lebesgue area $\mathcal{L}^2$. Thus the *total area* of X^u equals $\mathcal{A}(X^u) = \int_{\mathbb{R}^2} e^{2u}$.

The curvature of the conformal plane X^u equals $K = e^{-2u} \cdot \Delta u$. Thus, the (integral) boundedness of the curvature of X^u, is the analytic assumptions $\Delta u \in L^\infty(\mathbb{R}^2)$ ($\Delta u \in L^1(\mathbb{R}^2)$). If $\Delta(u) \in L^1(\mathbb{R}^2)$ then

$$\mathcal{K}(X^u) := \int_{\mathbb{R}^2} \Delta u \, d\mathcal{L}^2_{\mathbb{R}^2} = \int_{X^u} K \, d\mathcal{H}^2_X$$

is called the total curvature of X^u.

Most parts of the next result are scattered through the literature:

Theorem 1.6 *Let $X = X^u$ be a conformal plane of finite area $\mathcal{A}(X)$ and finite total curvature $\mathcal{K}(X)$. Then $\mathcal{K}(X) \geq 2\pi$. If $\mathcal{K}(X) > 2\pi$ then X^u is not complete.*

If X is not complete then the completion $\hat{X}$ is a sphere which has bounded integral curvature in the sense of Alexandrov.

Upon a conformal identification of $\mathbb{R}^2$ with $\mathbb{S}^2 \backslash \{p\}$, the function u defines a δ-subharmonic function on $\mathbb{S}^2$, in the complete and in the non-complete case.

Recall that a function is called δ-subharmonic if locally around any point it can represented as a difference of two subharmonic functions.

The theory of surfaces with integral curvature bounds implies that in the non-complete case, the area growth is at most quadratic at the point $p = \hat{X} \backslash X$. Moreover, limes inferior arising in Proposition 1.3 is a limit and equals $\frac{\mathcal{K}(X)}{2} - \pi$, see Sect. 4.1.

1.4 Uniformly bounded curvature

A final application answers the question investigated in [14] and relates this question to the theory of manifolds with both-sided curvature bounds, [5]. Slightly weaker results have been obtained in [14] by direct methods.

Proposition 1.7 *Assume that the plane $X = X^u$ has finite area and that the total curvature $\mathcal{K}(X)$ equals 4π. If the curvature K of X is uniformly bounded then the completion $\hat{X}$ of X is a Riemannian manifold conformally equivalent to the round sphere $\mathbb{S}^2$. For the conformal factor $e^{2\hat{u}}$, the function $\hat{u}$ is of class $\mathcal{C}^{1,\alpha}$ on $\mathbb{S}^2$, for every $\alpha < 1$.*

Even if the curvature K is continuous on $\hat{X}$, the function $\hat{u}$ does not need to be $\mathcal{C}^{1,1}$. If K is β-Hoelder on $\mathbb{S}^2$ then $\hat{u}$ is $\mathcal{C}^{2,\beta}$.

2 From the sphere to the plane

2.1 One-point complements in spheres

In the proof of Proposition 1.3 below, we are going to freely use the vocabulary of metric geometry. We refer to [20] for the definitions and properties, in particular for the notion of *weak conformality*.

Proof of Proposition 1.3 By assumption, we have a geodesic metric space $\hat{X}$, homeomorphic to $\mathbb{S}^2$ and a point $p \in \hat{X}$ such that $X = \hat{X} \setminus \{p\}$ has a smooth Riemannian metric. By assumption, the area growth at p is at most quadratic. In particular, $\hat{X}$ has finite 2-dimensional Hausdorff measure.

By [20, Theorem 1.3], there exists a weakly quasiconformal map $h : \mathbb{S}^2 \to \hat{X}$ from the round sphere $\mathbb{S}^2$.

The area growth assumption implies that h is a homeomorphism, [20, Theorem 7.4]. The map h restricts to a weakly quasiconformal map from $\mathbb{S}^2 \setminus h^{-1}(p) \to X$. Since $h^{-1}(p)$ is a singleton, $\mathbb{S}^2 \setminus h^{-1}(p)$ is conformally equivalent to $\mathbb{R}^2$. Therefore, we have a weakly quasiconformal map between smooth Riemannian manifolds $\hat{h} : \mathbb{R}^2 \to X$. If X were a conformal disc, we would obtain a weakly quasiconformal homeomorphism from $\mathbb{R}^2$ to the disc D. Such a homeomorphism cannot exist, see, for instance, [17, p. 2–4]. □

Assuming that $\hat{X}$ has bounded integral curvature in the sense of Alexandrov, [1, 23, 28], a shorter proof of Proposition 1.3 is possible. Indeed, in this case, the uniformization theorem, [28, Section 7] states that the metric on $\hat{X}$ is defined as $e^v \cdot \delta_{\mathbb{S}^2}$, where the function v in the conformal factor is δ-subharmonic on $\mathbb{S}^2$. This directly describes $X = \hat{X} \setminus \{p\}$ as conformally changed $\mathbb{S}^2$ without a point.

2.2 Some examples of conformal planes

We are going to prove Proposition 1.1 and Proposition 1.2. Observe first, that rescaling the metric by a positive constant $\lambda \leq 1$ provides again a metric in the same class (curvature at least 1, finite area). Thus, it suffices to find conformal planes of curvature ≥ 1 and arbitrary large finite area, respectively, finite area and diameter arbitrary close to 2π.

Consider a piecewise spherical metric on $\mathbb{S}^2$ such that the total angle is larger than 2π in at most one singularity p. In the arising metric space Y the curvature is constant 1 outside p

and finitely many further points $p_1, .., p_n$. Around any point p_i the metric is a spherical cone metric over a circle of length less than 2π. The metric around p_i can be smoothened in an arbitrary small neighborhood, such that the arising metric is smooth and has curvature ≥ 1, [16, Lemma 2.4]. Moreover, by construction, the new smooth metric has almost the same diameter and area as the original one.

Performing this operation around every vertex $p_1, \dots, p_n$, we obtain a metric space Y_ε homeomorphic to $\mathbb{S}^2$, such that $X := Y_\varepsilon \setminus p$ is a smooth Riemannian manifold of curvature ≥ 1. This manifold X is a conformal plane by Proposition 1.3; it has finite area and diameter arbitrary close (by the choice of ε) to the area and the diameter of Y.

Thus, in order to prove Propositions 1.1 and 1.2 it suffices to find piecewise spherical metrics Y on $\mathbb{S}^2$ with at most one singularity of total angle larger than 2π and arbitrary large area, respectively, diameter arbitrary close to 2π.

Proof of Proposition 1.2 Consider an interval $I = [a, b]$ of large length N. Let Z be the spherical join of a point p and I. The space Z is topologically a closed disc and it has curvature one in the interior. The boundary of Z is built by two geodesics pa and pb of length $\pi/2$ and by the local geodesic I. The angle at a and b is $\frac{\pi}{2}$, the total angle at p equals N. The area of Z equals N.

Consider the doubling Y of Z along the boundary. Then Y is a piecewise spherical metric on the 2-sphere, with 3 singularities of total angles $\pi, \pi, 2N$ and with total area $2N$. Due to the consideration preceding the proof, this suffices for the conclusion. □

Proof of Proposition 1.1 Fix $\varepsilon < \frac{\pi}{2}$. Consider a triangle $D = pxy$ in the round sphere $\mathbb{S}^2$ with px of length ε, with $\angle pxy = \frac{\pi}{2}$ and with the length of xy equal to $\pi - \varepsilon$. Then $\angle pyx < \frac{\pi}{2} < \angle ypx$.

Consider another isometric copy $D' = pxy'$ of the triangle and glue D and D' along the common side px. The arising space Z is homeomorphic to a closed disc. It has constant curvature 1 in the interior. The boundary is built by 4 geodesics py, py', yx and $y'x$. The angle at x equals π, the angles at y and y' are smaller than π, the angle at p is larger than π. The diameter of Z is at least the distance of y and y' which is larger than $2\pi - 4\varepsilon$.

The doubling Y of Z along the boundary ∂Z is homeomorphic to $\mathbb{S}^2$ and has diameter at least $2\pi - 4\varepsilon$. Moreover, Y has piecewise constant curvature 1 and at exactly one singularity p the total angle is larger than 2π. Due to the consideration preceding the proof, this suffices for the conclusion, since ε can be chosen arbitrary small. □

As a consequence we provide the following negative answer to [13, Question 8.7]. We refer to the discussion in [13, Section 7] for motivation and relation with the Levy–Gromov inequality.

Corollary 2.1 *For any $\epsilon > 0$ there exist a smooth Jordan curve Γ in $\mathbb{R}^2$ bounding a Jordan domain Ω and a smooth $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.2), such that $\int_{R^2} e^{2u} < \infty$ and the following holds true:*

$$\left(\int_\Gamma e^u\right)^2 \leq \varepsilon \cdot \int_\Omega e^{2u} \cdot \int_{\mathbb{R}^2 \setminus \Omega} e^{2u}.$$

Proof The construction in the proof of Proposition 1.2 provides conformal metrics X^u on $\mathbb{R}^2$ with curvature ≥ 1 and arbitrary large but finite area $A = A(u)$. Moreover, by construction, any of this metric spaces X^u contains a metric ball Ω of radius $r = \frac{\pi}{10}$ in the round sphere $\mathbb{S}^2$. Let l_0 and A_0 denote the length of $\partial\Omega$, respectively the area of Ω (both quantities measured in X^u, hence in $\mathbb{S}^2$).

Then the right hand side $(\int_\Gamma e^u)^2$ of the claimed inequality is just l_0^2 while the factors on the right hand side are A_0 and $A - A_0$ respectively. Thus, choosing u such that the area $A = \mathcal{A}(X^u)$ satisfies

$$A \geq A_0 + \frac{l_0}{\sqrt{\varepsilon}},$$

we finish the proof. □

3 Planes of finite area

The next argument is contained in the proof of [13, Theorem 1.4].

Proof of Lemma 1.5 The space X is a length space, hence so is the completion $\hat{X}$, [2, p. 43]. More precisely, for any $x \in \hat{X} \backslash X$ there exists a curve of finite length $\gamma_x : [0, a) \to X$, such that in $\hat{X}$ we have

$$\lim_{t \to a} \gamma_x(t) = x.$$

Assume that we have two different points $x, y \in \hat{X} \backslash X$. Denote by $\varepsilon > 0$ the distance between x and y. Consider curves γ_x, γ_y of finite length in X converging to x and y, as above. By changing the starting points, we may assume that γ_x and γ_y have length smaller than $\frac{\varepsilon}{4}$. In order to obtain a contradiction, it suffices to find points on γ_x and γ_y with distance less than $\frac{\varepsilon}{4}$ from each other.

Our space X is the plane $\mathbb{R}^2$ with the Euclidean metric changed by the conformal factor e^{2u}. Denote by η_r the Euclidean circle around 0 of radius r. We express the finiteness of the area in polar coordinates and obtain by the Hoelder inequality

$$\infty > \mathcal{A}(X) = \int_{\mathbb{R}^2} e^{2u} = \int_0^\infty \left(\int_{\eta_r} e^{2u} \right) dr \geq \int_0^\infty \frac{1}{2\pi r} \left(\int_{\eta_r} e^u \right)^2 dr.$$

The length of η_r in the metric space X is $\int_{\eta_r} e^u$. Therefore, we find a sequence $r_i \to \infty$ such that the length of η_{r_i} is smaller than $\frac{\varepsilon}{4}$.

Since the curves γ_x and γ_y do not have limit points in X, both curves run to infinity in $\mathbb{R}^2$. Hence they both intersect η_r, for all sufficiently large r. Thus, for sufficiently large r_i as above, we find points in the intersection of γ_x and γ_y with η_{r_i}. The distance between these intersection points in X is less than $\frac{\varepsilon}{4}$, in contradiction to our assumption. Hence, $\hat{X} \backslash X$ contains at most one point. □

We are going to explain that $\hat{X}$ does not need to be locally compact at the point $\{p\} = \hat{X} \backslash X$.

Example 3.1 Consider the round sphere $\mathbb{S}^2$ with north pole p. Take a sequence U_j of small metric balls centered on a fixed meridian starting at p. We choose the metric balls pairwise disjoint, not containing p, but converging to p. Change the metric conformally on $\mathbb{S}^2 \backslash \{p\}$ in the following way. The conformal factor is constantly one outside the union of all U_j. The subset U_j has after the conformal change diameter approximately 1 and area approximately $\frac{1}{j^2}$, thus U_j becomes a long and very thin finger sticking out of the sphere. The new metric on $\mathbb{S}^2 \backslash p$ is conformally equivalent to $\mathbb{R}^2$, it has finite area and diameter. Moreover, it has infinitely many points with pairwise distances in the interval [2, 3]. Hence, the completion $\hat{X}$ cannot be locally compact by the theorem of Hopf–Rinow.

The next example shows that even if $\hat{X}$ is compact and X has curvature at least 1, the curvature does not need to be integrable and the area growth at the singularity $p = \hat{X} \backslash X$ can be superquadratic.

Example 3.2 Consider the metric on $\mathbb{R}^2$ with conformal factor $e^{-\frac{2}{|z|}} \cdot |z|^{-4}$ as in [24, Section 5.1], [7, Section 4.1]. The area growth of this metric space Y at $p = 0$ is superquadratic,
[24, p. 19]. Euclidean balls around 0 are metric balls around $p = 0$ in Y and they are convex. Y is smooth outside of 0 and direct computations reveal that the metric has positive curvature outside of p; moreover, the curvature converges to ∞ at p. Consider now a small closed ball B around 0 in Y such that the curvature is larger than 1 outside of $0 = p$ and such that ∂B has length $2\pi s < 2\pi$. Glue to B along ∂B a round hemisphere of radius s. By the gluing theorem (for instance, [21]), the arising sphere has curvature > 1 outside the singularity 0. Smoothing the metric along ∂B, (see for instance, [16]), we obtain a smooth metric $\hat{X}$ on $\mathbb{S}^2$, which has curvature ≥ 1 everywhere outside a single point p and that around p the metric is isometric to Y. By construction (and the uniformization theorem), the metric on $\hat{X} \backslash \{p\}$ is conformally equivalent to $\mathbb{R}^2$.

It seems possible but technically more involved to construct an example of a conformal plane $X = X^u$ of curvature ≥ 1 and finite area, such that the diameter of X is 2π (thus strengthening Proposition 1.1). In such an example the completion $\hat{X}$ has to be non-compact.

4 Planes of finite area and curvature

4.1 Integral bound

If the conformal plane $X = X^u$ has finite total curvature we can control the geometry at infinity much better:

Proof of Theorem 1.6 First assume that $X = X^u$ is complete. Then the curvature estimate $\mathcal{K}(X) \leq 2\pi$ is a classical theorem of Cohn-Vossen, [8, Satz 6], valid also for complete planes of infinite area. Given that the area is finite, the equality $\mathcal{K}(X) = 2\pi$ is proven in [25, Corollary].

Finally, due to [15, Korollar] (or, alternatively, [15, Satz 3]) if X is complete then the function u extends to a δ-subharmonic function on $\mathbb{S}^2$, once $\mathbb{R}^2$ is identified with $\mathbb{S}^2$ without a point by a conformal transformation.

From now on we assume that X is not complete. We consider the completion $\hat{X}$ and let p be the unique point in $\hat{X} \backslash X$, Lemma 1.5.

First, we claim that $\hat{X}$ is compact. Otherwise, we find some $\varepsilon > 0$ and infinitely many points $x_i \in X$ with pairwise distance larger than 2ε. Removing at most one point, we can assume that the distance of any x_i to p is larger than ε. Then the closed balls $\bar{B}_\varepsilon(x_i)$ are pairwise disjoint and compact. Moreover, removing finitely many x_i and using the finiteness of total curvature, we may assume that the total positive curvature of any $\bar{B}_\varepsilon(x_i)$ is at most π. Then, for any i, we can estimate the area of the ball as

$$\mathcal{A}(B_\varepsilon(x_i)) \geq \frac{\pi}{2} \cdot \varepsilon^2,$$

due to [26, Proposition 3.2], [23, Theorem 9.1]. Thus, the finiteness of $\mathcal{A}(X)$ contradicts the disjointness of the balls $\bar{B}_\varepsilon(x_i)$.

Therefore, $\hat{X}$ is compact. Due to the uniqueness of the one-point-compactification, $\hat{X}$ is homeomorphic to $\mathbb{S}^2$.

In order to prove that $\hat{X}$ is a surface with bounded integral curvature we present the metric on $\hat{X}$ as a limit of metrics with a uniform integral bound on curvature, as in [23, Section 8.4].

We claim that there exists a sequence of simple closed curves Γ_j in X, such that for the Jordan domains $p \in O_j$ in $\hat{X}$ of Γ_j the following holds true: The closure $\bar{O}_j = O_j \cup \Gamma_j$ is convex in $\hat{X}$; the diameter of $\bar{O}_j$ and the length of Γ_j are at most $\frac{1}{j}$.

Note, that any such Γ_j would be of bounded turn and the variation from the side of $X \backslash O_j$ (thus the mean curvature) would be non-positive, by convexity of O_j, cf. [23, Theorem 8.1.3]. Moreover, by the Gauss–Bonnet formula and the bound on the total curvature of X, the total curvature of Γ_j would be uniformly bounded.

Once such Γ_j are found, we would cut out O_j and replace it by the round hemisphere $\hat{O}_j$ with boundary of length $\ell(\Gamma_j)$. The arising space $\hat{X}_j$ is a sphere with uniformly bounded integral curvature, [23, Theorems 8.3.1, 8.3.2]. Moreover, identifying $\hat{O}_j$ with O_j by any homeomorphism fixing Γ_j, we obtain a convergence of $\hat{X}_j$ to $\hat{X}$ in the sense of [23, Section 8.4]. Thus, $\hat{X}$ would be of integrally bounded curvature, [23, Theorem 8.4.5].

It remains to find the required curves Γ_j. In order to find them, we fix j and set $\delta = \frac{1}{10j}$. Consider the open ball $U = B_\delta(p)$. We find an index i, such that for all $k \geq i$, the curves $\eta_k := \eta_{r_k}$ constructed in the proof of Lemma 1.5 have length $\ell(\eta_{r_k}) < \delta$. By construction, the Jordan domains $p \in U_k$ of η_k are nested and their intersection consists of the point p only. Choosing i large enough, we may assume in addition, that $U_k \subset U$, for all $k \geq i$.

We fix this η_i. By compactness and local contractibility, there is some $\varepsilon > 0$ such that no closed curve in $\bar{U}_i$ of length at most ε can intersect η_i and be homotopic to η_i within the punctured disc $\bar{U}_j \backslash \{p\}$. We now find some $k > i$ such that $\ell(\eta_k) < \varepsilon$.

In the compact annulus A bounded by η_k and η_i in X we find a shortest non-contractible curve γ. This γ is automatically simple closed. By the choice of k, this curve γ has length at most ϵ, and by the choice of ϵ, the curve γ does not intersect η_i. The Jordan domain $p \in V$ of this curve is contained in U, thus has diameter at most 2δ. If V were not convex, then two points on γ could be connected within A by a shorter curve. But this would contradict the minimal property of γ. This finishes the construction of $\gamma = \gamma_j$ and, therefore, of the statement that $\hat{X}$ has bounded integral curvature.

The final statement that the metric of $\hat{X}$ is conformal to the round metric on the sphere including p is a direct consequence of the uniformization for such surfaces, [23, Section 7], [28]. □

Remark 4.1 We have used some geometric arguments in the non-complete case in the proof above. Possibly, a more analytic proof of the statement using the full strength of [15, Satz 3] can be found.

Some additional comments on the structure of $\hat{X}$ near $p = \hat{X} \backslash X$, in case that X is not complete in Theorem 1.6:

Consider the curvature measure $\hat{\mathcal{K}}$ on the sphere $\hat{X}$ with bounded integral curvature, [1, Chapter 5], [28]. This is a signed measure satisfying $\hat{\mathcal{K}}(\hat{X}) = 4\pi$ by the Gauss–Bonnet theorem [28, p. 20]. On the regular part X, the signed measure $\hat{\mathcal{K}}$ equals $K \cdot \mathcal{H}^2$, where K is the Gaussian curvature. Thus, $\hat{\mathcal{K}}(\{p\}) = 4\pi - \mathcal{K}(X)$. On the other hand, $\hat{\mathcal{K}}(\{p\})$ equals $2\pi - \theta$, where θ is the *total angle* at the point p [23, Lemma 8.1.1]. Moreover, again by [23, Lemma 8.1.1] and the coarea formula (or using [23, Theorem 9.10])

$$\theta = \lim_{r \to 0} \frac{\mathcal{H}^1(\partial B_r(p))}{r} = 2 \lim_{r \to 0} \frac{\mathcal{H}^2(B_r(x))}{r^2}.$$

4.2 Smoothness at infinity

We are going to provide

Proof of Proposition 1.7 We can apply Theorem 1.6. Identifying $\mathbb{R}^2$ conformally with the complement of a point p in $\mathbb{S}^2$, we obtain that the completion $\hat{X}$ is a sphere with curvature bounded in the integral sense of Alexandrov. The *curvature measure* $\hat{\mathcal{K}}$ of $\hat{X}$ coincides on X with the multiple $\hat{\mathcal{K}} = K \cdot \mathcal{H}^2$ of the canonical area measure $\mathcal{H}^2$. By the Gauss-Bonnet theorem, $\hat{\mathcal{K}}(\hat{X}) = 4\pi$,. Thus, by assumption, $\hat{\mathcal{K}}(\{p\}) = 0$. Therefore, the equality $\hat{\mathcal{K}} = K \cdot \mathcal{H}^2$ is valid on all of $\hat{X}$.

Therefore, on all of $\hat{X}$, the metric is defined by a conformal change $e^{2\hat{u}} \cdot \delta_{\mathbb{S}^2}$ of the round metric on $\mathbb{S}^2$, such that the spherical Laplacian of $\hat{u}$ is a bounded function $K + 1$. Elliptic regularity implies that $\hat{u}$ is of class $\mathcal{C}^{1,\alpha}$ for any $\alpha < 1$. Moreover, if the curvature $K : X = \mathbb{R}^2 \to \mathbb{R}$ extends as a β-Hoelder continuous function to $\mathbb{S}^2$ then $\hat{u}$ is $\mathcal{C}^{2,\beta}$-Hoelder.

An example of a conformal metric $e^{2v} \cdot \delta_{\mathbb{R}^2}$ on a disc, which is smooth outside the origin p, not $\mathcal{C}^{1,1}$ at the origin and, such that the Lapalcian Δv is continuous, is presented in [27, p. 693]. This metric (restricted to a subdisc) can clearly be extended to a metric on the sphere, which has continuous curvature but is not $\mathcal{C}^{1,1}$ in conformal coordinates. This example finishes the proof. □

Acknowledgements I thank Qinfeng Li for helpful communication giving rise to this note. I am grateful to Matthew Romney for details about the example appearing in [7, 24] and for the reference [16]. I would like to thank Anton Petrunin for helpful discussions and Dima Panov for sharing his examples of large piecewise spherical spheres, which has simplified the proof of Proposition 1.1. For useful comments I thank Paul Creutz, Qinfeng Li and Matthew Romney.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Aleksandrov, A.D., Zalgaller, V.A.: Two-dimensional manifolds of bounded curvature. (Foundations of the intrinsic geometry of surfaces.). Trudy Mat. Inst. Steklov **63**, 262 (1962)
2. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry. Graduate Studies in Mathematics, vol. 33. American Mathematical Society, Providence (2001)
3. Bergweiler,W., Eremenko, A., Langley, J.: On Conformal Metrics of Constant Positive Curvature in the Plane. arXiv:2208.04672, (2022)
4. Bonk, M., Kleiner, B.: Quasisymmetric parametrizations of two-dimensional metric spheres. Invent. Math. **150**(1), 127–183 (2002)
5. Berestovskij, V.N., Nikolaev, I.G.: Multidimensional generalized Riemannian spaces. In: Geometry, IV, volume of Encyclopaedia Math. Sci., vol. **70**, pp. 165–243, 245–250 (Springer, Berlin, 1993)
6. Chen, W., Li, C.: Classification of solutions of some nonlinear elliptic equations. Duke Math. J. **63**(3), 615–622 (1991)
7. Creutz, P., Romney. M.: The branch set of minimal disks in metric spaces. Int. Math. Res. Not. **2023**(7), 5569–5603 (2023)
8. Cohn-Vossen, S.: Kürzeste Wege und Totalkrümmung auf Flächen. Comp. Math. **2**, 69–133 (1935)

9. Chou, K., Wan, T.: Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc. Pacific J. Math. **163**(2), 269–276 (1994)
10. Cheng, S., Yau, S.: Differential equations on Riemannian manifolds and their geometric applications. Comm. Pure Appl. Math. **28**(3), 333–354 (1975)
11. Eremenko, A., Gui, C., Li, Q., Xu, L.: Rigidity Results on Liouville Equation. arXiv:2207.05587, (2022)
12. Eremenko, A.: Metrics of positive curvature with conic singularities on the sphere. Proc. Am. Math. Soc. **132**(11), 3349–3355 (2004)
13. Gui, C., Li. Q.: Some Geometric Inequalities Related to Liouville Equation. arXiv:2208.03612, (2020)
14. Gui, C., Li, Q.: Completion of $\mathbb{R}^2$ with a conformal metric as a closed surface. Anal. Theory Appl. **37**(1), 59–73 (2021)
15. Huber, A.: Vollständige konforme Metriken und isolierte Singularitäten subharmonischer Funktionen. Comment. Math. Helv. **41**, 105–136 (1966/67)
16. Itoh, J., Rouyer, J., Vîlcu, C.: Moderate smoothness of most Alexandrov surfaces. Int. J. Math **26**(4), 1540004, 14 (2015)
17. Kiernan, P.: Quasiconformal mappings and Schwarz's lemma. Trans. Am. Math. Soc. **148**, 185–197 (1970)
18. Lytchak, A., Wenger, S.: Canonical parameterizations of metric disks. Duke Math. J. **169**(4), 761–797 (2020)
19. Mondello, G., Panov, D.: Spherical metrics with conical singularities on a 2-sphere: angle constraints. Int. Math. Res. Not. IMRN **16**, 4937–4995 (2016)
20. Ntalampekos, D., Romney, M.: Polyhedral approximation of metric surfaces and applications to uniformization. Duke Math. J. **172**(9), 1673–1734 (2023)
21. Petrunin, A.: Applications of quasigeodesics and gradient curves. In Comparison geometry (Berkeley, CA, 1993–94), vol. 30 of Math. Sci. Res. Inst. Publ, pp. 203–219 (Cambridge University Press, Cambridge 1997)
22. Rajala, K.: Uniformization of two-dimensional metric surfaces. Invent. Math. **207**(3), 1301–1375 (2017)
23. Yu. G. Reshetnyak. Two-dimensional manifolds of bounded curvature. In: Geometry, IV, volume 70 of Encyclopaedia Math. Sci., pages 3–163. (Springer, Berlin, 1993)
24. Rajala, K., Rasimus, M., Romney, M.: Uniformization with infinitesimally metric measures. J. Geom. Anal. **31**(11), 11445–11470 (2021)
25. Shiohama, K.: Total curvatures and minimal areas of complete surfaces. Proc. Am. Math. Soc. **94**(2), 310–316 (1985)
26. Shioya, T.: The limit spaces of two-dimensional manifolds with uniformly bounded integral curvature. Trans. Am. Math. Soc. **351**(5), 1765–1801 (1999)
27. Sabitov, I.H., Šefel, S.Z.: Connections between the order of smoothness of a surface and that of its metric. Sibirsk. Mat. Ž. **17**(4), 916–925 (1976)
28. M.. Troyanov. On Alexandrov's Surfaces with Bounded Integral Curvature. Preprint, arXiv:2201.03354. ISBN 978-3-031-24254-0 (2022)