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RESEARCH ARTICLE

# Nonlocal elasticity of Klein-Gordon type with internal length and time scales: Constitutive modelling and dispersion relations

**Eleni Agiasofitou** | **Markus Lazar**

Institute of Engineering Mechanics, Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany

**Correspondence**
Eleni Agiasofitou, Institute of Engineering Mechanics, Karlsruhe Institute of Technology (KIT), D-76131 Karlsruhe, Germany.
Email: eleni.agiasofitou@kit.edu

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**Abstract**
In this work, a nonlocal elasticity theory with nonlocality in space and time is presented by considering nonlocal constitutive equations with a dynamical scalar nonlocal kernel. Based on the proposed theory, we consider the isotropic nonlocal elasticity of Klein-Gordon type including a characteristic internal time scale parameter in addition to the characteristic internal length scale parameter. The dispersion relations for homogeneous isotropic media in the framework of nonlocal elasticity of Klein-Gordon type are analytically determined. The obtained results reveal the ability of the presented nonlocal elasticity model to predict, for the first time in the framework of nonlocal elasticity, in addition to the acoustic modes (low-frequency modes), optic modes (high-frequency modes) as well as frequency band-gaps. The phase and group velocities for all four modes (acoustic and optic branches of longitudinal and transverse waves) are determined showing that all four modes exhibit normal dispersion with positive group velocity. The presented model allows for physically realistic dispersive wave propagation.

## 1 | INTRODUCTION

Nonlocal elasticity belongs to generalized continuum field theories, it considers long-range interatomic interaction, it has a close link to the underlying microstructure [1] and is valid down to the Ångström-scale [2]. Therefore, it can be considered as a generalized continuum theory of Ångström-mechanics [2].

In the literature of nonlocal elasticity, only temporal nonlocality for the modelling of memory effects or only spatial nonlocality has been considered until now (see, e.g., [1, 3]). A systematic development of nonlocal elasticity with nonlocality in space and time is given here deriving from a general anisotropic constitutive relation with a tensorial kernel function for a three-dimensional body an isotropic constitutive relation with one characteristic internal length scale parameter and one characteristic internal time scale parameter, leading in this way to the theory of isotropic nonlocal elasticity of Klein-Gordon type [4]. The determination of the dispersion relations shows that the consideration of the nonlocality in time is of great significance, since we can obtain optic modes and frequency band-gaps in addition to acoustic modes. The characteristic time scale gives rise to the propagation of high-frequency waves. Nonlocal elasticity of Klein-Gordon type is a generalized continuum field theory with only the displacement vector as degree of freedom (three degrees of freedom) and it possesses only four constitutive parameters (two elastic constants, one length scale and one time scale), which are the smallest numbers of degrees of freedom and constitutive parameters appearing in the existing theories of generalized continua in elastodynamics.

## 2 | CONSTITUTIVE MODELLING WITH NONLOCALITY IN SPACE AND TIME

Let us consider a three-dimensional infinite elastic medium occupying the region $\mathcal{V}$. In nonlocal linear elasticity theory, the *equation of motion in presence of body forces* reads

$$\rho\,\partial_t^2 u_i - \partial_j t_{ij} = f_i, \tag{1}$$

where $\rho$ is the *mass density*, $u_i$ is the *displacement vector*, $t_{ij}$ is the *stress tensor of nonlocal elasticity* and $f_i$ is the *body force density*. $\partial_j = \partial/\partial x_j$ denotes partial differentiation with respect to the spatial coordinates $x_i, i = 1,2,3,$ and $\partial_t = \partial/\partial t$ stands for the partial differentiation with respect to time $t$. Inspired from electrodynamics in materials where nonlocality in space and time is used, the corresponding constitutive relation in nonlocal elasticity of elastic bodies with nonlocality in space and time (space-time nonlocality) can be given by

$$t_{ij}(\boldsymbol{x},t) = \int_{-\infty}^{t}\int_{\mathcal{V}} C_{ijkl}(\boldsymbol{x},t;\boldsymbol{x}',t')\, e_{kl}(\boldsymbol{x}',t')\,\mathrm{d}V'\,\mathrm{d}t', \tag{2}$$

where the (compatible) *elastic strain tensor* $e_{kl}$ is defined by

$$e_{kl} = \frac{1}{2}\,(\partial_k u_l + \partial_l u_k) \tag{3}$$

and $C_{ijkl}(\boldsymbol{x},t;\boldsymbol{x}',t')$ is the *tensor function of nonlocal elastic moduli*, which is a tensorial kernel function and $\boldsymbol{x},\boldsymbol{x}' \in \mathcal{V} \subseteq \mathbb{R}^3$. Causality restricts the time integration to $(-\infty, t]$ (see, e.g., [5]). Expressing causality as a property of $C_{ijkl}(\boldsymbol{x},t;\boldsymbol{x}',t')$, one imposes the condition: $C_{ijkl}(\boldsymbol{x},t;\boldsymbol{x}',t') = 0 \quad \text{for} \quad t' > t$.

The integral constitutive relation (2) is the most general linear constitutive relation between stress and elastic strain and states that the stress $t_{ij}(\boldsymbol{x},t)$ (response) at a reference point $\boldsymbol{x}$ and at time $t$ depends on the *elastic strain*$e_{kl}(\boldsymbol{x}',t')$ (disturbance) at all other points $\boldsymbol{x}'$ of the body occupying the region $\mathcal{V}$, and for all past times $-\infty < t' \leq t$. The tensorial kernel $C_{ijkl}(\boldsymbol{x},t;\boldsymbol{x}',t')$ contains the information how the disturbance at $(\boldsymbol{x}',t')$ contributes to the response at $(\boldsymbol{x},t)$. Thus, Equation (2) represents a *nonlocal Hooke law with nonlocality in space and time*. Since the constitutive relation (2) involves space and time integrals, *a medium modelled by nonlocal elasticity with nonlocality in space and time is spatially and temporally dispersive* [6].

Substituting the nonlocal constitutive relation (2) into the equation of motion (1) and using the elastic strain tensor (3), we obtain the following *inhomogeneous integro-partial differential equation*[1] *for the displacement vector* $u_i$:

$$\rho\,\partial_t^2 u_i(\boldsymbol{x},t) - \partial_j \int_{-\infty}^{t}\int_{\mathcal{V}} C_{ijkl}(\boldsymbol{x},t;\boldsymbol{x}',t')\partial'_l u_k(\boldsymbol{x}',t')\,\mathrm{d}V'\,\mathrm{d}t' = f_i(\boldsymbol{x},t), \tag{4}$$

which represents the *equation of motion in terms of the displacement vector in strong nonlocal elasticity*.

In order to separate the two effects of anisotropy, that is, the anisotropy of the elastic moduli of the bulk and the anisotropy of the nonlocality, the multiplicative decomposition of the tensor function of nonlocal anisotropic elastic moduli $C_{ijkl}(\boldsymbol{x}-\boldsymbol{x}')$ is used (see, e.g., [2, 3]). The tensor function of nonlocal anisotropic elastic moduli $C_{ijkl}(\boldsymbol{x}-\boldsymbol{x}', t-t')$ can be decomposed into a tensorial "local" part, $C_{ijkl}$, which is the *tensor of the "local" anisotropic elastic moduli* and a scalar function nonlocal part, $\alpha(\boldsymbol{x}-\boldsymbol{x}', t-t')$, such that

$$C_{ijkl}(\boldsymbol{x}-\boldsymbol{x}', t-t') = C_{ijkl}\,\alpha(\boldsymbol{x}-\boldsymbol{x}', t-t'). \tag{5}$$

[1] Where $\partial'_l = \partial/\partial x'_l$ denotes the partial derivative with respect to $x'_l$.

The *scalar nonlocal kernel function* $\alpha(\boldsymbol{x}-\boldsymbol{x}', t-t')$, being nonlocal in space and time, describes characteristic length scale and time scale effects and is a *dynamical scalar nonlocal kernel*.

Substituting Equation (5) into Equation (2), the stress tensor of nonlocal anisotropic elasticity $t_{ij}$ for homogeneous materials reads

$$t_{ij}(\boldsymbol{x},t) = \int_{-\infty}^{t}\int_{\mathcal{V}} \alpha(\boldsymbol{x}-\boldsymbol{x}', t-t')\,\sigma_{ij}(\boldsymbol{x}',t')\,\mathrm{d}V'\,\mathrm{d}t' = \alpha(\boldsymbol{x},t) * \sigma_{ij}(\boldsymbol{x},t), \tag{6}$$

where $*$ denotes the convolution in space and time (see also [4]) and

$$\sigma_{ij} = C_{ijkl}e_{kl} \tag{7}$$

is nothing but the *classical (local) Hooke law*. Thus, the tensor $\sigma_{ij}$ may be identified with the *Cauchy stress tensor of classical elasticity*.

For an isotropic medium, the stress tensor of nonlocal elasticity $t_{ij}$ for homogeneous materials reads

$$t_{ij}(\boldsymbol{x},t) = \int_{-\infty}^{t}\int_{\mathcal{V}} \alpha(|\boldsymbol{x}-\boldsymbol{x}'|, t-t')\,\sigma_{ij}(\boldsymbol{x}',t')\,\mathrm{d}V'\,\mathrm{d}t', \tag{8}$$

where $\sigma_{ij} = \lambda\,\delta_{ij}e_{kk} + 2\mu\,e_{ij}$ with $\lambda$ and $\mu$ being the *classical Lamé constants*.

By matching the dispersion curves of plane waves with those of lattice dynamics, an important property of the scalar nonlocal kernel with nonlocality in space was proposed by Eringen [1, 7], namely that the scalar nonlocal kernel is the *Green function of a linear differential operator L*. Generalizing this property toward dynamics, it holds for the dynamical nonlocal kernel $\alpha(\boldsymbol{x}-\boldsymbol{x}', t-t')$ in an infinite domain:

$$L\alpha(\boldsymbol{x}-\boldsymbol{x}', t-t') = \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \tag{9}$$

where $\delta(.)$ is the Dirac delta distribution. Applying the differential operator $L$ to the nonlocal constitutive relation (6) and using Equation (9), we obtain a partial differential equation for $t_{ij}$:

$$Lt_{ij} = \sigma_{ij}, \tag{10}$$

where the inhomogeneous part is given by the Cauchy stress tensor $\sigma_{ij}$ of classical elasticity. Thus, *if the scalar nonlocal kernel function is a Green function, then the integral constitutive relation (6) can be reduced to a partial differential equation* (see Equation (10)) and in this way *strong nonlocal elasticity is reduced to weak nonlocal elasticity*.

Moreover, if $L$ is a differential operator with constant coefficients and using Equation (10), then Equation (1) becomes

$$L(\rho\,\partial_t^2 u_i) - \partial_j\sigma_{ij} = Lf_i, \tag{11}$$

which further, using Equations (7) and (3), provides the following *inhomogeneous partial differential equation for the displacement vector $u_i$* for homogeneous media:

$$L(\rho\,\partial_t^2 u_i) - C_{ijkl}\partial_j\partial_l u_k = Lf_i, \tag{12}$$

which represents the *equation of motion (generalized Navier equation) in weak nonlocal elasticity for homogeneous media*.

## 3 | NONLOCAL ELASTICITY MODEL OF KLEIN-GORDON TYPE

We choose a *linear hyperbolic differential operator of second-order*, which is a dynamical extension of the Helmholtz operator, namely the operator

$$L = 1 - \ell^2\Delta + \tau^2\partial_t^2, \tag{13}$$

where $\Delta$ denotes the Laplace operator, $\ell \in \mathbb{R}^+$ is the *characteristic internal length scale parameter* due to nonlocality in space (spatial nonlocality) and $\tau \in \mathbb{R}^+$ is the *characteristic time scale parameter* due to nonlocality in time (temporal nonlocality). Defining a *characteristic velocity of nonlocality*, $c \in \mathbb{R}^+$, in terms of the length scale $\ell$ and time scale $\tau$:

$$c = \frac{\ell}{\tau}, \tag{14}$$

the hyperbolic differential operator $L$ takes the following form

$$L = 1 + \ell^2 \left[ \frac{1}{c^2}\, \partial_t^2 - \Delta \right], \tag{15}$$

which is nothing but the so-called *Klein-Gordon operator*. A realistic value for the characteristic length scale parameter is $\ell \simeq 0.39\, a$, where $a$ is the lattice constant, obtained by Eringen [7]. The characteristic length $\ell$ is given in units of Ångström as it is shown in [2].

With the application of the operator (13) to Equation (11), we obtain the *equation of motion in nonlocal elasticity of Klein-Gordon type in presence of body forces*

$$\big(1 - \ell^2\Delta + \tau^2\partial_t^2\big)(\rho\, \partial_t^2 u_i) - \partial_j \sigma_{ij} = \big(1 - \ell^2\Delta + \tau^2\partial_t^2\big) f_i, \tag{16}$$

or equivalently in terms of the displacement vector

$$\big(1 - \ell^2\Delta + \tau^2\partial_t^2\big)(\rho\, \partial_t^2 u_i) - \mu\, \Delta u_i - (\lambda + \mu)\partial_i\partial_j u_j = \big(1 - \ell^2\Delta + \tau^2\partial_t^2\big) f_i, \tag{17}$$

which is an inhomogeneous partial differential equation of fourth-order with respect to the time derivative and to the mixed space-time derivative. For vanishing body forces, Equation (17) reduces to the following homogeneous partial differential equation

$$\rho\, \partial_t^2 u_i - \mu\, \Delta u_i - (\lambda + \mu)\partial_i\partial_j u_j + \big(\tau^2\partial_t^2 - \ell^2\Delta\big)\rho\, \partial_t^2 u_i = 0, \tag{18}$$

which is a *generalized dynamical Navier equation* due to the last two terms. In particular, Equation (18) compared to the classical Navier equation of elastodynamics contains additionally fourth-order temporal as well as fourth-order mixed spatial-temporal derivatives of the displacement vector.

## 4 | DISPERSION RELATIONS

Consider a three-dimensional elastic body of infinite extent with its equation of motion described by Equation (18). Equation (18) can be decomposed into homogeneous scalar and vector wave equations by using the Helmholtz decomposition of the displacement vector $u_i$ into the Lamé potentials (scalar potential $\phi$, vector potential $\psi_i$) according to (see, e.g., [3])

$$u_i = \partial_i \phi + \varepsilon_{ikl}\partial_k \psi_l \quad \text{with} \quad \partial_i \psi_i = 0, \quad i = 1, 2, 3. \tag{19}$$

Then, Equation (18) is satisfied if

$$c_{\mathrm{L}}^2 \Delta\phi - \big(1 - \ell^2\Delta + \tau^2\partial_t^2\big)\partial_t^2\phi = 0, \qquad c_{\mathrm{T}}^2 \Delta\psi_i - \big(1 - \ell^2\Delta + \tau^2\partial_t^2\big)\partial_t^2\psi_i = 0, \tag{20}$$

where $\phi$ and $\psi_i$ describe *longitudinal* and *transverse waves*, respectively with $c_{\mathrm{L}}$ and $c_{\mathrm{T}}$ denoting the *classical longitudinal* and *transverse velocities*

$$c_{\mathrm{L}}^2 = \frac{\lambda + 2\mu}{\rho}, \qquad c_{\mathrm{T}}^2 = \frac{\mu}{\rho}. \tag{21}$$

We consider linear plane harmonic waves propagating in $x$-direction given by the following complex representation

$$\phi = B\, \mathrm{e}^{\mathrm{i}(kx-\omega t)}, \qquad \psi_i = A_i\, \mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{22}$$

where $k$ is the *wavenumber*, $\omega$ is the *angular* or *circular frequency*, $B$ and $A_i$ are the *amplitudes* of $\phi$ and $\psi_i$, respectively, and $\mathrm{i} = \sqrt{-1}$. Inserting Equation (22) into Equations (20), we obtain the following *dispersion relations for longitudinal and transverse waves*

$$\tau^2\omega_j^4 - (1+\ell^2k^2)\omega_j^2 + c_j^2k^2 = 0, \qquad j = \mathrm{L}, \mathrm{T}, \tag{23}$$

where the subscripts L and T stand for longitudinal and transverse waves, respectively. For travelling waves, the angular frequency should be a real positive quantity [8]. The requirement for all angular frequencies $\omega_j$ to be real in the complete range of wavenumbers imposes the following condition or constraint between the characteristic velocity of nonlocality $c$ and the longitudinal and transverse velocities, $c_{\mathrm{L}}$ and $c_{\mathrm{T}}$ (see[4]):

$$c > c_j, \quad j = \mathrm{L}, \mathrm{T}. \tag{24}$$

Since the velocity of the longitudinal waves is, in principle, greater than the velocity of the transverse waves, the constraint on $c$ for real $\omega$ finally reads: $c > c_{\mathrm{L}} > c_{\mathrm{T}} > 0$.

## 4.1 | Acoustic and optic waves and frequency band-gaps

Solving the dispersion relations (23), we obtain two branches:

- the *acoustic branch*, which is the low-frequency branch, defined by the minus sign

$$\omega_{j\mathrm{A}}(k) = \frac{c}{\ell}\sqrt{\frac{1}{2}\left(1+\ell^2k^2 - \sqrt{(1+\ell^2k^2)^2 - 4\frac{c_j^2}{c^2}\,\ell^2k^2}\right)}, \quad j = \mathrm{L}, \mathrm{T}, \quad k \in \mathbb{R} \tag{25}$$

- the *optic branch*, which is the high-frequency branch, defined by the plus sign

$$\omega_{j\mathrm{O}}(k) = \frac{c}{\ell}\sqrt{\frac{1}{2}\left(1+\ell^2k^2 + \sqrt{(1+\ell^2k^2)^2 - 4\frac{c_j^2}{c^2}\,\ell^2k^2}\right)}, \quad j = \mathrm{L}, \mathrm{T}, \quad k \in \mathbb{R}. \tag{26}$$

Therefore, we have longitudinal acoustic (LA) and longitudinal optic (LO) waves and transverse acoustic (TA) and transverse optic (TO) waves and it yields: $\omega_{j\mathrm{A}} < \omega_{j\mathrm{O}}$. The dispersion curves given by Equations (25) and (26) are plotted for longitudinal waves in Figure 1A for $c_{\mathrm{L}} = 0.8c$, whereas for transverse waves are plotted in Figure 1B for $c_{\mathrm{T}} = 0.4c$.

The longitudinal and transverse acoustic waves attain a horizontal asymptote:

$$\lim_{k\to\infty} \omega_{j\mathrm{A}} = \sup_{k\in\mathbb{R}} \omega_{j\mathrm{A}} = \frac{c_j}{\ell}, \qquad j = \mathrm{L}, \mathrm{T}, \tag{27}$$

which is the supremum (smallest upper bound) of the frequencies of the acoustic modes for $k \in \mathbb{R}$

$$\sup_{k\in\mathbb{R}} \omega_{j\mathrm{A}} > \omega_{j\mathrm{A}}, \qquad j = \mathrm{L}, \mathrm{T}. \tag{28}$$

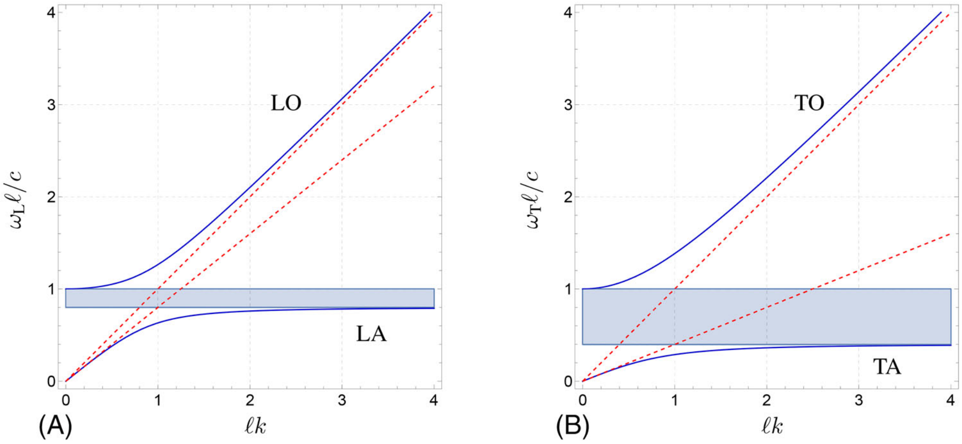


**FIGURE 1** (A) Longitudinal and (B) transverse dispersion curves of nonlocal elasticity of Klein-Gordon type (blue solid lines) and asymptotes (red dashed lines) against the dimensionless wavenumber $\ell k$ ($c_L = 0.8c$, $c_T = 0.4c$).

It holds, $\sup_{k\in\mathbb{R}} \omega_{LA} = c_L/\ell > c_A/\ell = \sup_{k\in\mathbb{R}} \omega_{TA}$. On the other hand, the longitudinal and transverse optic dispersion branches possess a *minimum frequency* or *cut-off frequency* at $k = 0$ denoted by

$$\omega_c := \min_{k\in\mathbb{R}} \omega_{jO} = \omega_{jO}(0) = \frac{c}{\ell} = \frac{1}{\tau}, \qquad j = \mathrm{L}, \mathrm{T}, \tag{29}$$

which is the minimum of $\omega_{jO}(k)$ for $k \in \mathbb{R}$. Hence, optic waves cannot propagate below the cut-off frequency.

Therefore, there exists a *frequency band-gap* between the smallest upper bound $\sup_{k\in\mathbb{R}} \omega_{jA}$ of the acoustic wave and the cut-off frequency $\omega_c$ of the optic wave. The frequency band-gap for which no real solution exists and hence no wave propagation can occur, reads (see Figure 1A and B):

(i) $[\sup_{k\in\mathbb{R}} \omega_{LA},\ \omega_c) = [c_L/\ell,\ c/\ell)$, for longitudinal waves,
(ii) $[\sup_{k\in\mathbb{R}} \omega_{TA},\ \omega_c) = [c_T/\ell,\ c/\ell)$, for transverse waves.

It holds $[c_L/\ell, c/\ell) < [c_T/\ell, c/\ell)$, since $c_L > c_T$. Therefore, the size of the frequency band-gap of longitudinal waves is smaller than that of the transverse waves as one can also see in Figure 1A and B. This means that the transverse acoustic waves stop earlier than the longitudinal acoustic waves. The longitudinal and transverse optic branches have the same cut-off frequency.

Consequently, three regions are distinguished (see also Figure 1A and B):

(i) LA, TA: $0 \le \omega < \sup_{k\in\mathbb{R}} \omega_{jA}$: Acoustic wave propagation.
(ii) $\sup_{k\in\mathbb{R}} \omega_{jA} \le \omega_j < \omega_c$: Frequency band-gap. No wave propagation.
(iii) LO, TO: $\omega_c \le \omega$: Optic wave propagation.

## 4.2 | Phase and group velocities

The *phase velocities of the longitudinal and transverse acoustic waves*, $v^P_{LA}$ and $v^P_{TA}$, are given by

$$v^P_{jA} = \frac{\omega_{jA}}{k}, \quad j = \mathrm{L}, \mathrm{T}, \quad k \in \mathbb{R}_*, \tag{30}$$

where $\omega_{jA}$ is given by Equation (25) and the *phase velocities of longitudinal and transverse optic waves*, $v^P_{LO}$ and $v^P_{TO}$, are given by

$$v^P_{jO} = \frac{\omega_{jO}}{k}, \quad j = \mathrm{L}, \mathrm{T}, \quad k \in \mathbb{R}_*, \tag{31}$$

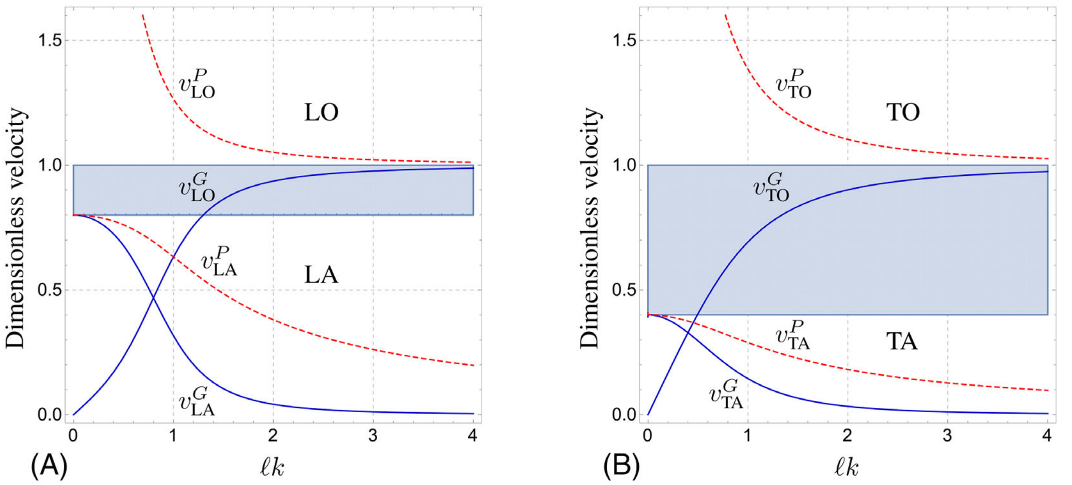


**FIGURE 2** Phase velocities (red dashed lines) and group velocities (blue solid lines) of (A) longitudinal acoustic and optic waves, and (B) transverse acoustic and optic waves against the dimensionless wavenumber ($c_{\mathrm{L}} = 0.8c$, $c_{\mathrm{T}} = 0.4c$).

where $\omega_{j\mathrm{O}}$ is given by Equation (26). We see that the phase velocities of the acoustic branches for both longitudinal and transverse waves are bounded for all wavenumbers $k > 0$, namely from $c_j$. Note that $c_j$ is the phase velocity of classical elasticity which serves here as an upper bound (the smallest one) for the phase velocities of the acoustic modes of nonlocal elasticity of Klein-Gordon type. We see that the frequency band-gap is transferred to a forbidden phase-velocity zone, which corresponds to a so-called *stop-band* for the wave; the wave cannot propagate in this zone. Therefore, there exists a *phase-velocity stop-band*.

Observing the phase velocities for both, acoustic and optic modes of longitudinal and transverse waves (see Figure 2A and B), we see that the larger wavenumbers propagate slower than the smaller wavenumbers which is an important property known from dispersion curves in lattice dynamics. Such a dispersion is characterized as *physically realistic* (see also [9]). Therefore, the nonlocal elasticity model of Klein-Gordon type is able to predict physically realistic dispersion curves.

The *group velocities of the longitudinal and transverse acoustic waves*, $v^G_{\mathrm{LA}}$ and $v^G_{\mathrm{TA}}$, are given by

$$v^G_{j\mathrm{A}} = \frac{\mathrm{d}\omega_{j\mathrm{A}}}{\mathrm{d}k} = \frac{ck\ell}{\sqrt{2}} \frac{1 - \frac{(1+\ell^2k^2)-2c_j^2/c^2}{\sqrt{(1+\ell^2k^2)^2-4(c_j^2/c^2)\,\ell^2k^2}}}{\sqrt{1+\ell^2k^2-\sqrt{(1+\ell^2k^2)^2-4(c_j^2/c^2)\,\ell^2k^2}}}, \quad j = \mathrm{L},\mathrm{T}, \quad k \in \mathbb{R} \tag{32}$$

and the *group velocities of the longitudinal and transverse optic waves*, $v^G_{\mathrm{LO}}$ and $v^G_{\mathrm{TO}}$, read

$$v^G_{j\mathrm{O}} = \frac{\mathrm{d}\omega_{j\mathrm{O}}}{\mathrm{d}k} = \frac{ck\ell}{\sqrt{2}} \frac{1 + \frac{(1+\ell^2k^2)-2c_j^2/c^2}{\sqrt{(1+\ell^2k^2)^2-4(c_j^2/c^2)\,\ell^2k^2}}}{\sqrt{1+\ell^2k^2+\sqrt{(1+\ell^2k^2)^2-4(c_j^2/c^2)\,\ell^2k^2}}}, \quad j = \mathrm{L},\mathrm{T}, \quad k \in \mathbb{R}. \tag{33}$$

The condition $\mathrm{d}^2\omega/\mathrm{d}k^2 \neq 0$ or $\mathrm{d}v^G/\mathrm{d}k \neq 0$ ensures that the group velocity is not constant. This condition together with the original requirement of real angular frequencies for all wavenumbers ensure the existence of dispersive waves [10]. It is easy to see that the group velocities for all four modes, $v^G_{j\mathrm{A}}$ and $v^G_{j\mathrm{O}}$, $j = \mathrm{L},\mathrm{T}$ fulfill the above condition and are positive. Moreover, the group velocity is smaller than the phase velocity for all four modes, therefore normal dispersion occurs (see [4]).

## 5 | CONCLUSION

In this work, the nonlocal elasticity model of Klein-Gordon type has been presented. It describes spatial and temporal nonlocal effects at small scales due to characteristic internal length and time scale parameters. It provides the description of optic modes (high-frequency modes) and frequency band-gaps in addition to the acoustic modes (low-frequency modes). It offers analytical expressions for the dispersion relations, which are derived only for the displacement vector (3 degrees of freedom) without the need of additional degrees of freedom. It possesses a low number of constitutive parameters (only 4 material parameters). All four modes exhibit normal dispersion with positive group velocity. It gives physically realistic dispersive wave propagation.

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### ORCID

*Eleni Agiasofitou* https://orcid.org/0000-0002-2467-4486

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