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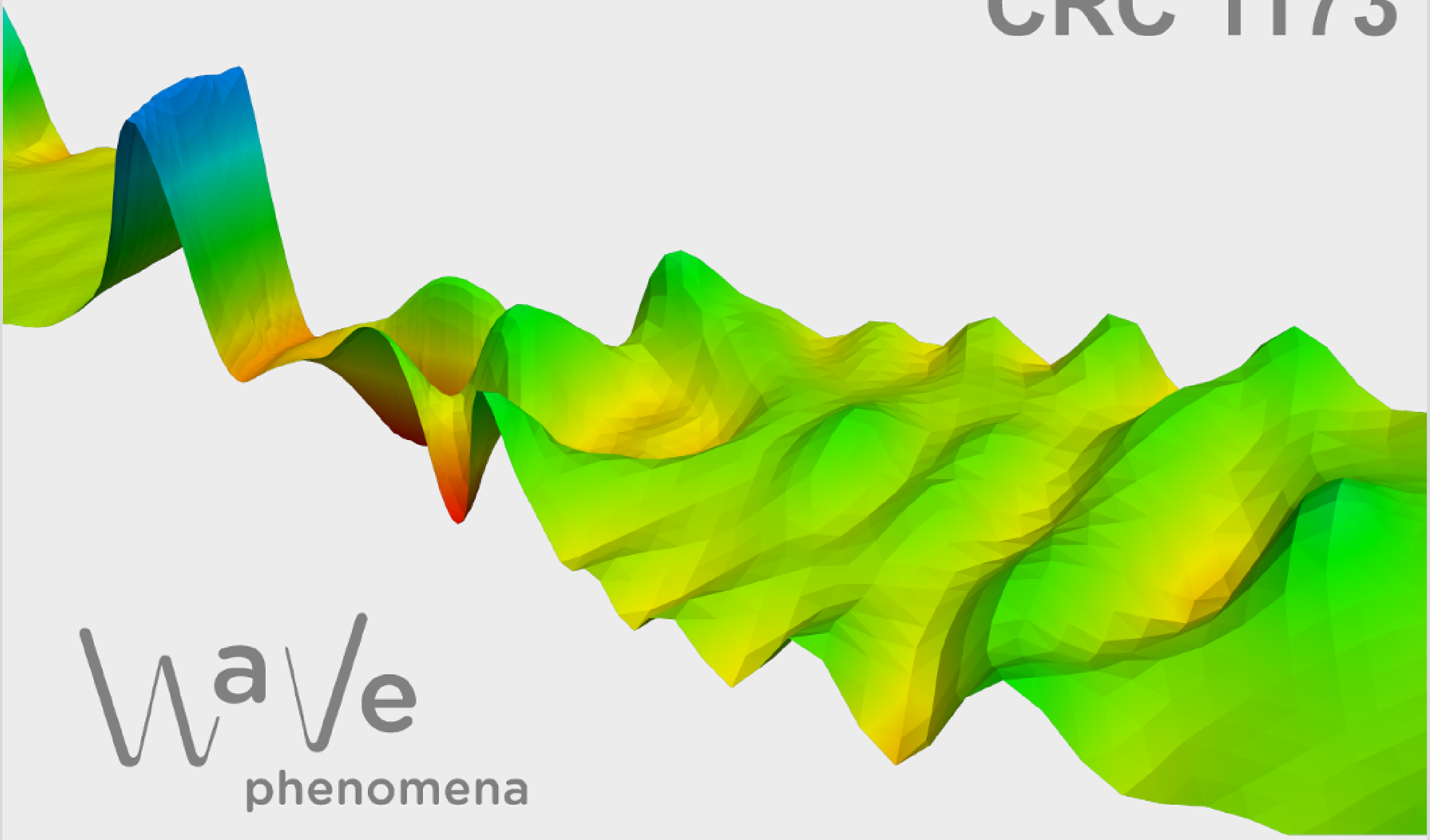

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ROBUST FULLY DISCRETE ERROR BOUNDS FOR THE KUZNETSOV EQUATION IN THE INVISCID LIMIT

BENJAMIN DÖRICH[†] AND VANJA NIKOLIĆ[‡]

Abstract. The Kuznetsov equation is a classical wave model of acoustics that incorporates quadratic gradient nonlinearities. When its strong damping vanishes, it undergoes a singular behavior change, switching from a parabolic-like to a hyperbolic quasilinear evolution. In this work, we establish for the first time the optimal error bounds for its finite element approximation as well as a semi-implicit fully discrete approximation that are robust with respect to the vanishing damping parameter. The core of the new arguments lies in devising energy estimates directly for the error equation where one can more easily exploit the polynomial structure of the nonlinearities and compensate inverse estimates with smallness conditions on the error. Numerical experiments are included to illustrate the theoretical results.

1. Introduction

We consider quasilinear wave equations of the following form:

$$(1+\kappa\partial_t u)\partial_t^2 u - c^2\Delta u - \beta\Delta\partial_t u + \ell\nabla u\cdot\nabla\partial_t u = f. \tag{1.1}$$

This model arises in nonlinear acoustics under the name Kuznetsov equation [24]. It describes propagation of sound waves through fluids and can be understood as an approximation to the Navier–Stokes–Fourier system of governing equations of sound motion that is more accurate than Westervelt's equation [37]. In the context of nonlinear acoustics, $u = u(x,t)$ in (1.1) is the acoustic velocity potential, $c > 0$ denotes the speed of sound in the medium, and $\kappa \in \mathbb{R}$ and $\ell \in \mathbb{R}$ are the nonlinearity coefficients. The quadratic gradient nonlinearity (that is, $(\frac{1}{2}\ell|\nabla u|^2)_t$) captures local (non-cumulative) nonlinear effects in sound propagation, which may be prominent, for example, close to the sound source; see the discussion in [14, Ch. 3] for more details on modeling.

Equation (1.1) is strongly damped when the parameter β, known in acoustics as the sound diffusivity, is positive and it exhibits parabolic-like behavior leading to an exponential decay of the energy of the solutions as time grows; see, e.g., [18, 28] for its well-posedness analysis in this parameter regime. In the case $\beta = 0$, however, smooth solutions are only expected to exist locally in time after which a gradient blow-up is expected; see the analysis in [9], [19], and numerical experiments conducted in [36]. In practice, sound diffusivity is small and it may become negligible in certain (inviscid) propagation media. Investigation of the singular inviscid limit of a Dirichlet boundary-value problem for (1.1) has been conducted in [19]. However, the questions of stability and asymptotic behavior of approximate solutions of (1.1) as $\beta \to 0^+$ are open in the field.

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The first aim of the present work is to establish β-robust error bounds for the finite element and full discretizations of (1.1). As it turns out, the robust discretization is possible when using (at least) quadratic finite elements. Secondly, we determine the behavior of the (semi-)discrete solutions as $\beta \to 0^+$, and the conditions under which it asymptotically preserves the order of convergence of the exact solution established in [19]. In addition, we determine how one has to couple the spatial discretization parameter, time step size, and the damping parameter to allow also linear finite elements in space.

To the best of our knowledge, this is the first work dealing with the robust numerical analysis of the semi-discrete Kuznetsov equation, and the first work analyzing a fully discrete scheme for it. For the strongly damped Kuznetsov equation (with $\beta > 0$ fixed), where one can exploit the parabolic-like evolution, *a priori* analysis of a mixed-approximation has been conducted in [27] and an *a priori* analysis of a discontinuous Galerkin coupling for a nonlinear elasto-acoustic problem based on this model has been performed in [29].

In contrast, quasilinear equations of Westervelt type given by

$$(1 + \kappa_1 u + \kappa_2 \partial_t u)\partial_t^2 u - c^2 \Delta u - \beta \Delta \partial_t u + \kappa_1 (\partial_t u)^2 = f, \quad \kappa_1, \kappa_2 \in \mathbb{R} \tag{1.2}$$

are by now much better understood from the point of view of the numerical analysis as they do not involve quadratic gradient nonlinearities. Again here, the cases $\beta = 0$ and $\beta > 0$ are qualitatively different. Concerning the spatial discretization, the results in [15] yield optimal order of convergence of space discrete solutions in the energy norm for $\beta = \kappa_2 = 0$. A β-uniform analysis of a mixed approximation of (1.2) with $\kappa_1 = 0$ has been conducted in [27]. Fully discrete schemes for (1.2) with $\beta = \kappa_2 = 0$ have been analyzed in [10, 25]. We also point out the works [8, 20, 22, 35], where existence of solutions to undamped quasilinear and nonlinear evolution equations of this type is established, and one can find approximation rates of the implicit and semi-implicit Euler methods. Within an (extended) Kato framework, optimal order for these methods has been determined in [16] and rigorous error bounds for the time discretization by higher-order Runge-Kutta methods are derived in [17, 23].

For the strongly damped Westervelt equation with $\beta > 0$ fixed and $\kappa_2 = 0$, optimal order of convergence of continuous Galerkin methods has been established in [31]. Recently also Westervelt's equation with time-fractional dissipation instead of $-\beta\Delta\partial_t u$ has received attention. A time-stepping method for such a model has been analyzed in [2], and a β-robust finite element analysis for both time-fractional and strongly damped Westervelt's equation has been performed in [30], together with establishing the vanishing β convergence rates of the approximate solution.

We mention also that other quasilinear wave models have been rigorously investigated in the literature. In [13], trigonometric integrators have been analyzed for nonlinear wave equations in the form of

$$\partial_t^2 u = \partial_x^2 u - u + \kappa a(u)\partial_x^2 u + \kappa g(u, \partial_x u)$$

in one space dimension under periodic boundary conditions. Analysis of different time stepping schemes for nonlinear hyperbolic problems can also be found in [3, 4, 11, 33], and two-step methods are considered in [5]. For a class of linearly implicit single-step schemes as well as a linearly and a fully implicit two-step scheme, optimal error bounds are derived in [26].

Compared to the available works, the main challenge here comes from treating the nonlinear term $\ell \nabla u \cdot \nabla \partial_t u$ after discretization, in combination with having to guarantee that the discrete version preserves the non-degeneracy condition:

$$1 + \kappa \partial_t u \geq \tilde{\gamma} > 0$$

and that the bounds are uniform with respect to β. To guarantee β uniformity, we have to work also with the time-differentiated version of the (semi-)discrete problems, which introduces (a discrete version of) the term $\ell \nabla u \cdot \nabla \partial_t^2 u$. A fixed-point argument along the lines of existing results on the damped Kuznetsov equation in [27, 29] would then not allow us to match the order of convergence in the fixed-point iterates. We will instead first show the existence of a unique approximate solution on a discretization-dependent time interval and then derive uniform bounds to extend it beyond it, in the spirit of [15]. To tackle the quadratic gradient nonlinearity, the main idea here is to devise energy estimates directly for the error equation where one can more easily exploit the polynomial structure of the nonlinearities so as to mimic the following identity from the continuous setting:

$$(\nabla u \cdot \nabla \partial_t^2 u, \partial_t^2 u)_{L^2(\Omega)} = -\frac{1}{2}(\Delta u\, \partial_t^2 u, \partial_t^2 u)_{L^2(\Omega)}$$

and then compensate inverse estimates with smallness conditions on the error. We refer to Proposition 3.4 for details.

Organization of the exposition. The rest of the manuscript is organized as follows. We first state our main results in Section 2, both for the spatially semi-discrete and the fully discrete problem, and present numerical experiments which corroborate the theory. In Section 3, we conduct the finite element analysis and establish the β-robust optimal error bounds in the energy norm when using quadratic or higher-order elements as well as the β-limiting behavior of the finite element solution. Section 4 is dedicated to the stability and error analysis of a fully discrete problem based on a semi-implicit Euler method for the time discretization. An extension to linear finite elements is given in Section 5 with non-robust estimates with respect to β.

Notation. Below we use $x \lesssim y$ to denote $x \leq Cy$, where $C > 0$ is a generic constant that does not depend on the discretization parameters nor on the damping coefficient β, but may depend on the exact solution and the final time T. We use $(\cdot,\cdot)_{L^2(\Omega)}$ to denote the scalar product in $L^2(\Omega)$. We omit the temporal domain when writing norms; for example, $\|\cdot\|_{L^p(L^q(\Omega))}$ denotes the norm on $L^p(0,T;L^q(\Omega))$. We use $\|\cdot\|_{L^p_t(L^q(\Omega))}$ to denote the norm on $L^p(0,t;L^q(\Omega))$ for some $t \in (0,T)$.

2. Statements of the main results

In this section, we present the main results of this work. To this end, we first discuss the assumptions on the exact solution. As we are interested in the vanishing β dynamics, we may assume that $\beta \in [0,\bar\beta]$ for some fixed $\bar\beta > 0$.

2.1. Assumptions on the exact solution.

Throughout, we assume that the initial data and source term are sufficiently smooth and small and the final time $T > 0$ short so that the initial boundary-value problem

$$\begin{cases} (1+\kappa\partial_t u)\partial_t^2 u - c^2\Delta u - \beta\Delta\partial_t u + \ell\nabla u\cdot\nabla\partial_t u = f \quad \text{in } \Omega\times(0,T), \\ u|_{\partial\Omega} = 0, \\ (u,u_t)|_{t=0} = (u_0,v_0), \end{cases} \tag{2.1}$$

has a unique solution in

$$\mathcal{U} = L^\infty(0,T;W^{2,\infty}(\Omega)) \cap H^3(0,T;H^{k+1}(\Omega)\cap W^{1,\infty}(\Omega)\cap H^1_0(\Omega)) \cap W^{3,\infty}(0,T;H^2(\Omega)) \cap H^4(0,T;L^2(\Omega)),$$

for $k \geq 2$ (or $k \geq 1$ in Section 5), with β-uniform bounds:

$$\|u\|_{\mathcal{U}} \leq C, \qquad 1+\kappa\partial_t u \geq \tilde\gamma > 0 \quad \text{for all } (x,t) \in \bar\Omega\times[0,T]. \tag{2.2}$$

Note that the $\tilde\gamma$-bound in (2.2) guarantees that the leading term in the Kuznetsov equation does not degenerate. The β-uniform well-posedness analysis of (2.1) with $f = 0$ can be found in [19]. Compared to the results of [19], we require more smoothness from the solution. More precisely, assuming the domain Ω is sufficiently smooth, the results of [19, Thm. 6.2] provide uniform well-posedness in the following space:

$$H^3(0,T;H_0^1(\Omega)) \cap W^{2,\infty}(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \cap W^{1,\infty}(0,T;H^3_\diamondsuit(\Omega)) \cap L^\infty(0,T;H^4_\diamondsuit(\Omega)),$$

where we have denoted

$$H^3_\diamondsuit(\Omega) = \left\{u \in H^3(\Omega) \,:\, u|_{\partial\Omega} = 0,\ \Delta u|_{\partial\Omega} = 0\right\}, \quad H^4_\diamondsuit(\Omega) = H^4(\Omega) \cap H^3_\diamondsuit(\Omega).$$

However, we expect that the techniques in [19] can be extended in a relatively straightforward manner to rigorously prove higher-order uniform well-posedness in $\mathcal{U}$ for sufficiently smooth and small data, as assumed in the present numerical analysis. We also note that the higher regularity for the strongly damped Kuznetsov equation (i.e., with $\beta > 0$) follows by the results of [21].

Our main contributions concern robust error bounds for a finite element discretization of (2.1) and a fully discrete scheme, as well as establishing asymptotic-preserving behavior of respective solutions as β vanishes; we illustrate them in Figure 1.

$$\begin{tikzcd}
u^n_{h,\beta} \arrow[d, "\begin{array}{c}\beta \to 0\\ \text{Theorem 2.4}\end{array}"'] \arrow[rr, bend left, "{\tau, h \to 0,\ \text{Theorem 2.3}}"] & u_h^\beta \arrow[r, "{h \to 0\ ,\ \text{Theorem 2.1}}"'] \arrow[d, "\begin{array}{c}\beta \to 0\\ \text{Theorem 2.2}\end{array}"'] & u^\beta \arrow[d, "\begin{array}{c}\beta \to 0\\ {[19]}\end{array}"] \\
u^n_{h,\beta=0} \arrow[rr, bend right, "{\tau, h \to 0,\ \text{Theorem 2.3}}"'] & u_h^{\beta=0} \arrow[r, "{h \to 0\ ,\ \text{Theorem 2.1}}"] & u^{\beta=0}
\end{tikzcd}$$

FIGURE 1. Diagram representing the main contributions of this work

2.2. **Main results for the finite element discretization.** In the present work, we employ Lagrange finite elements and consider a quasi-uniform triangulation $\mathcal{T}_h$ and the space

$$V_h \coloneqq \{\varphi_h \in C(\Omega) \mid \varphi_h|_K \in \mathcal{P}_k(K) \text{ for all } K \in \mathcal{T}_h\}$$

of piecewise polynomials of degree k. To conduct the error analysis bellow in a β-uniform manner, we assume that $k \geq 2$. The case $k = 1$ with non-uniform bounds is treated separately in Section 5. We introduce the Ritz projection defined for $\varphi \in H^1(\Omega)$ via

$$(\nabla \varphi, \nabla \varphi_h)_{L^2(\Omega)} = (\nabla \mathrm{R}_h \varphi, \nabla \varphi_h)_{L^2(\Omega)}$$

for all $\varphi_h \in V_h$. Further, we make use of the nodal interpolation operator $I_h \colon C(\Omega) \to V_h$, and define the discrete Laplacian operator $\Delta_h \colon V_h \to V_h$ for $\psi_h, \varphi_h \in V_h$ via the relation

$$(\Delta_h \psi_h, \varphi_h)_{L^2(\Omega)} = -(\nabla \psi_h, \nabla \varphi_h)_{L^2(\Omega)}.$$

With these preparations, we consider the spatially discrete Kuznetsov equation

$$\begin{aligned}((1+\kappa\partial_t u_h)\partial_t^2 u_h, \varphi_h)_{L^2(\Omega)} - (c^2\Delta_h u_h, \varphi_h)_{L^2(\Omega)} - (\beta\Delta_h\partial_t u_h, \varphi_h)_{L^2(\Omega)}\\ + \ell(\nabla u_h\cdot\nabla\partial_t u_h, \varphi_h)_{L^2(\Omega)} = (f_h, \varphi_h)_{L^2(\Omega)},\end{aligned} \tag{2.3a}$$

for all $\varphi_h \in V_h$, supplemented by approximate initial data

$$(u_h, \partial_t u_h)|_{t=0} = (u_{0h}, u_{1h}). \tag{2.3b}$$

Our first main results establishes *a priori* error bounds for u_h in the energy norm that are uniform with respect to the damping parameter β.

Theorem 2.1 (Robust finite element estimates)**.** *Let $k \geq 2$ and $\beta \in [0, \bar\beta]$ for some $\bar\beta > 0$. Furthermore, assume that f, $f_h \in H^1(0,T; L^2(\Omega))$ are such that*

$$\|f - f_h\|_{H^1(L^2(\Omega))} \lesssim h^k \tag{2.4}$$

and that the approximate initial data are chosen as

$$u_h(0) = \mathrm{R}_h u_0,\ \ \partial_t u_h(0) = \mathrm{R}_h v_0, \tag{2.5}$$

where $u \in \mathcal{U}$ is the solution of (1.1) satisfying (2.2), and $\partial_t^2 u_h(0)$ is given by

$$\begin{aligned}&((1+\kappa\partial_t u_h(0))\partial_t^2 u_h(0), \varphi_h)_{L^2(\Omega)} - (c^2\Delta_h u_h(0), \varphi_h)_{L^2(\Omega)}\\ &\quad - (\beta\Delta_h\partial_t u_h(0), \varphi_h)_{L^2(\Omega)} + \ell(\nabla u_h(0)\cdot\nabla\partial_t u_h(0), \varphi_h)_{L^2(\Omega)}\\ &\qquad\qquad = (f_h(0), \varphi_h)_{L^2(\Omega)}\end{aligned} \tag{2.6}$$

for all $\varphi_h \in V_h$. Then, there exists $h_0 > 0$ and a constant $C > 0$, independent of h and β, such that for all $h \leq h_0$, the following error bound holds:

$$\begin{aligned}\|\partial_t^2 u(t) - \partial_t^2 u_h(t)\|^2_{L^2(\Omega)} + \|\nabla\partial_t u(t) - \nabla\partial_t u_h(t)\|^2_{L^2(\Omega)}\\ + \int_0^t \|\nabla u(s) - \nabla u_h(s)\|^2_{L^6(\Omega)}\,\mathrm{d}s \leq Ch^{2k}\end{aligned} \tag{2.7}$$

for all $t \in [0,T]$.

A non-robust variant of this result for $k = 1$ is presented later in Theorem 5.1. The second main result confirms that, in the setting of Theorem 2.1, the finite element solution preserves the asymptotic behavior as $\beta \to 0$ of the exact solution established in [19].

Theorem 2.2 (Asymptotic-preserving behavior in the inviscid limit)**.** *Under the assumptions of Theorem 2.1, for $h \in (0, h_0]$, the family $\{u_h^\beta\}_{\beta\in(0,\bar\beta]}$ of finite element solutions of (2.3) converges in the energy norm to the finite element solution $u_h^{\beta=0}$ of the inviscid semi-discrete problem (i.e., with $\beta = 0$) at a linear rate as $\beta \to 0$. In other words,*

$$\|\partial_t u_h^\beta - \partial_t u_h^{\beta=0}\|_{L^\infty(L^2(\Omega))} + \|\nabla(u_h^\beta - u_h^{\beta=0})\|_{L^\infty(L^2(\Omega))} \leq C\beta,$$

where the constant $C > 0$ is independent of β and h.

2.3. **Main results for a fully discrete semi-implicit Euler method.** Our next results concern a full discretization of (2.1) based on a semi-implicit Euler method. To present it, we introduce the discrete derivative ∂_τ as follows:

$$\partial_\tau a^n = \frac{1}{\tau}(a^n - a^{n-1}),\ \ n \geq 1, \qquad\qquad \partial_\tau^{k+1} a^n = \partial_\tau\partial_\tau^k a^n,\ \ k \geq 0\,, \tag{2.8a}$$

and the notational conventions

$$\partial_\tau a^0 = a^0, \qquad\qquad \partial_\tau^{n+j} a^n = \partial_\tau^n a^n,\ \ j \geq 0\,. \tag{2.8b}$$

Then for $1 \leq n \leq N$ with $(N+1)\tau \leq T$, we consider

$$\begin{aligned}((1+\kappa\partial_\tau u_h^n)\partial_\tau^2 u_h^{n+1}, \varphi_h)_{L^2(\Omega)} &- (c^2\Delta_h u_h^{n+1}, \varphi_h)_{L^2(\Omega)} - (\beta\Delta_h\partial_\tau u_h^{n+1}, \varphi_h)_{L^2(\Omega)} \\ &+ (\ell\nabla u_h^n \cdot \nabla\partial_\tau u_h^{n+1}, \varphi_h)_{L^2(\Omega)} = (f_h^{n+1}, \varphi_h)_{L^2(\Omega)},\end{aligned} \tag{2.9}$$

for all $\varphi_h \in V_h$. We set the initial conditions as

$$u_h^0 \coloneqq \mathrm{R}_h u_0, \qquad u_h^1 \coloneqq \mathrm{R}_h\big(u_0 + \tau v_0 + \frac{\tau^2}{2} w_0\big), \tag{2.10a}$$

using the approximation $w_0 \approx \partial_t^2 u(t_0)$ defined as

$$w_0 = \mathrm{R}_h\Big((1+\kappa v_0)^{-1}\big(c^2\Delta u_0 + \beta\Delta v_0 - \ell\nabla u_0 \cdot \nabla v_0 + f\big)\Big), \tag{2.10b}$$

such that u_h^1 resembles a projected Taylor approximation to $u(t_1)$. The quadratic gradient nonlinearity forces us to assume the following CFL condition:

$$\tau \leq Ch^{1+d/6+2\varepsilon} \tag{2.11}$$

for some arbitrarily chosen $\varepsilon > 0$. Without loss of generality, we assume $\varepsilon \in (0, \frac{1}{2} - \frac{d}{12})$, such that $1 + d/6 + 2\varepsilon < 2$.

Theorem 2.3 (Robust fully discrete error bounds). *Let $k \geq 2$ and $\beta \in [0, \bar\beta]$ for some $\bar\beta > 0$, and let the CFL condition (2.11) hold. Furthermore, assume that $f, f_h \in C^1(0,T; L^2(\Omega))$ are such that*

$$\|f - f_h\|_{C^1(L^2(\Omega))} \lesssim h^k, \tag{2.12}$$

and the initial values are chosen as in (2.10). If u_h^n is the solution of (2.9), and the solution $u \in \mathcal{U}$ of (1.1) satisfies (2.2), then for for $h \leq h_0$ and $\tau \leq \tau_0$, it holds

$$\begin{aligned}\|\partial_t^2 u(t_n) - \partial_\tau^2 u_h^n\|_{L^2(\Omega)}^2 &+ \|\nabla\partial_t u(t_n) - \nabla\partial_\tau u_h^n\|_{L^2(\Omega)}^2 \\ &+ \tau\sum_{j=1}^n \|\nabla u(t_j) - \nabla u_h^j\|_{L^6(\Omega)}^2 \leq C\big(\tau + h^k\big)^2,\end{aligned}$$

for all $n = 2, \ldots, N+1$, where the constant C is independent of h, τ, and β.

A non-robust variant of this result for $k = 1$ is presented later in Theorem 5.3. As in the semi-discrete case, our fourth main result confirms that, in the setting of Theorem 2.3, also the fully discrete solution preserves the asymptotic behavior as $\beta \to 0$ of the exact solution established in [19].

Theorem 2.4 (Asymptotic-preserving behavior in the inviscid limit). *Under the assumptions of Theorem 2.3, for $h \in (0, h_0]$ and $n \in \{2, \ldots, N+1\}$ fixed, the family $\{u_{h,\beta}^n\}_{\beta\in(0,\bar\beta]}$ of finite element solutions of (2.9) converges in the discrete energy norm to the finite element solution $u_{h,\beta=0}^n$ of the inviscid fully discrete problem (i.e., with $\beta = 0$) at a linear rate as $\beta \to 0$. In other words,*

$$\|\partial_\tau u_{h,\beta}^n - \partial_\tau u_{h,\beta=0}^n\|_{L^2(\Omega)} + \|\nabla(u_{h,\beta}^n - u_{h,\beta=0}^n)\|_{L^2(\Omega)} \leq C\beta,$$

for all $n = 1, \ldots, N+1$, where the constant $C > 0$ is independent of β, h, and τ.

2.4. **Numerical results.** In this section, we illustrate our theoretical findings with three numerical experiments. We first study the case of a smooth (*a priori* known) exact solution u to show the optimality of the derived convergence rates. In the second experiment, we verify the convergence with respect to the vanishing damping parameter β for given data without a known solution. Thirdly, we study a more realistic scenario of a traveling Gaussian pulse. In all experiments, we observe the optimality of our main results.

Discretization. For the discretization in space with Lagrangian finite elements, we use the open-source Python tool FEniCSx, (https://fenicsproject.org/); see [6] and [1]. For a stable implementation, we introduce the auxiliary quantity for the discrete derivative

$$v_h^{n+1} = \partial_\tau u_h^{n+1}, \qquad n \geq 0.$$

We reformulate this as an update step

$$u_h^{n+1} = u_h^n + \tau v_h^{n+1} \tag{2.13}$$

once we have computed v_h^{n+1}. Note that by (2.10) it holds

$$v_h^1 = \mathrm{R}_h\big(v_0 + \frac{\tau}{2} w_0\big).$$

With (2.13), we eliminate u_h^{n+1} in (2.9) and obtain the following relation for $n \geq 1$:

$$\begin{aligned}&((1+\kappa v_h^n)\partial_\tau v_h^{n+1}, \varphi_h)_{L^2(\Omega)} + \tau c^2(\nabla v_h^{n+1}, \nabla\varphi_h)_{L^2(\Omega)} + \beta(\nabla v_h^{n+1}, \nabla\varphi_h)_{L^2(\Omega)} + \ell(\nabla u_h^n \cdot \nabla v_h^{n+1}, \varphi_h)_{L^2(\Omega)}\\ &= -\, c^2(\nabla u_h^n, \nabla\varphi_h)_{L^2(\Omega)} + (f_h^{n+1}, \varphi_h)_{L^2(\Omega)},\end{aligned}$$

which in turn yields the system

$$\begin{aligned}((1+\kappa v_h^n)v_h^{n+1}, \varphi_h)_{L^2(\Omega)} + c^2\tau^2(\nabla v_h^{n+1}, \nabla\varphi_h)_{L^2(\Omega)} + \tau\beta(\nabla v_h^{n+1}, \nabla\varphi_h)_{L^2(\Omega)}&\\ + \tau\ell(\nabla u_h^n \cdot \nabla v_h^{n+1}, \varphi_h)_{L^2(\Omega)}&\\ = ((1+\kappa v_h^n)v_h^n, \varphi_h)_{L^2(\Omega)} - \tau c^2(\nabla u_h^n, \nabla\varphi_h)_{L^2(\Omega)} + (f_h^{n+1}, \varphi_h)_{L^2(\Omega)}.&\end{aligned} \tag{2.14}$$

Since the mass and stiffness matrix change in each time step, the routines in FEniCSx assemble the mass and stiffness matrix and use the PETSc linear algebra backend to solve the linear system (2.14). The codes to reproduce the results are available at

https://doi.org/10.35097/1871.

2.4.1. *Smooth solution.* In the first example, we consider the domain $\Omega = [0,1]\times[0,1]$ and choose initial data as

$$u_0(x) = c_{\mathrm{sp}} \sin(\pi x_1)\sin(\pi x_2), \quad v_0(x) = c_{\mathrm{sp}}c_{\mathrm{time}} \sin(\pi x_1)\sin(\pi x_2), \quad c_{\mathrm{sp}}, c_{\mathrm{time}} > 0 \tag{2.15}$$

with the parameters

$$\kappa = 0.7, \qquad c^2 = 1, \qquad \ell = 2,$$

and vary $\beta \geq 0$. The forcing term f is chosen such that the exact solution is given by

$$u(x,t) = c_{\mathrm{sp}} e^{c_{\mathrm{time}} t} \sin(\pi x_1)\sin(\pi x_2). \tag{2.16}$$

In Figure 2, we present the computed error

$$\mathbf{E}(t) = \|\nabla\partial_t u(t) - \nabla\partial_t u_h(t)\|_{L^2(\Omega)}$$

for the semi-discrete method (2.3) at time $t = 0.8$, using a small time-step size $\tau = 1.5\cdot 10^{-3}$. We perform experiments for the space and time discretization with elements of order $k = 1,2,3$.

We observe convergence of order k until a plateau caused by the temporal discretization is reached. For smaller time-step sizes the plots look qualitatively similar with a lower plateau. Further, we observe that the plots for $\beta = 0, 10^{-3}, 10^{-2}$ have no visible difference, which is in alignment with Theorem 2.1. Note that the case $k = 1$ is not covered by Theorem 2.1, but appears to work well in practice even for $\beta = 0$.

In Figure 3, we present the computed error

$$\mathbf{E}(t_n) = \|\nabla\partial_t u(t_n) - \nabla\partial_\tau u_h^n\|_{L^2(\Omega)}$$

for the fully discrete method (2.9) at $n = N+1$ with elements of order $k = 2$ and $h \approx 1.1\cdot 10^{-2}$. As predicted by Theorem 2.3, we observe convergence or order $\mathcal{O}(\tau)$ independent of the damping parameter β.

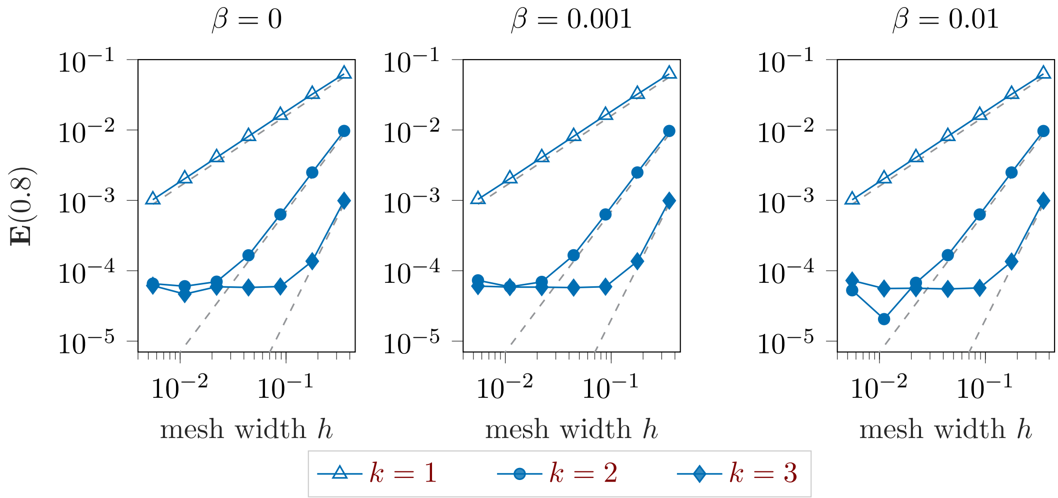

FIGURE 2. Convergence of (2.3) with $\|\nabla\partial_t u(t) - \nabla\partial_t u_h(t)\|_{L^2(\Omega)}$ at $t = 0.8$ with the parameters $c_{\text{sp}} = 0.1$, $c_{\text{time}} = 0.5$ in (2.16) for elements of order $k = 1, 2, 3$ and $\tau \approx 1.5 \cdot 10^{-3}$ and damping parameters $\beta = 0, 10^{-3}, 10^{-2}$ (from left to right). The dashed lines indicate order $\mathcal{O}(h^k)$ for $k = 1, 2, 3$.

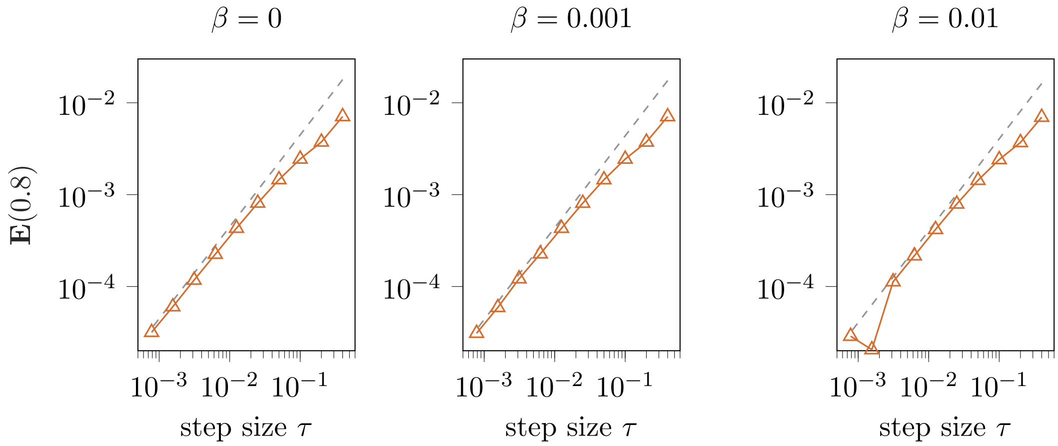

FIGURE 3. Convergence of (2.9) with $\|\nabla\partial_t u(t_n) - \nabla\partial_\tau u_h^n\|_{L^2(\Omega)}$ for $n = N + 1$ with the parameters $c_{\text{sp}} = 0.1$, $c_{\text{time}} = 0.5$ in (2.16) with $k = 2$ and $h \approx 1.1 \cdot 10^{-2}$ and damping parameters $\beta = 0, 10^{-3}, 10^{-2}$ (from left to right). The dashed lines indicate order $\mathcal{O}(\tau)$.

2.4.2. *Convergence in the inviscid limit.* In the next experiment, we verify the sharpness of the results in Theorems 2.2 and 2.4. Here, we use the same domain Ω and initial data as in (2.15), but parameters and source term are chosen as

$$\kappa = 0.3, \qquad c^2 = 1, \qquad \ell = 2, \qquad f = 0\,,$$

with $c_{\text{sp}} = 0.01$ and $c_{\text{time}} = 1$.

Since the estimates only compare the numerical solution, we do not need an exact or a reference solution. We use $k = 2$ and compute the difference between $(u^n_{h,\beta=0}, \partial_\tau u^n_{h,\beta=0})$ and $(u^n_{h,\beta}, \partial_\tau u^n_{h,\beta})$ in the $H^1(\Omega) \times L^2(\Omega)$-norm, i.e.,

$$\overline{\mathbf{E}}(t_n) = \|\nabla u^n_{h,\beta=0} - \nabla u^n_{h,\beta}\|_{L^2(\Omega)} + \|\partial_\tau u^n_{h,\beta=0} - \partial_\tau u^n_{h,\beta}\|_{L^2(\Omega)},$$

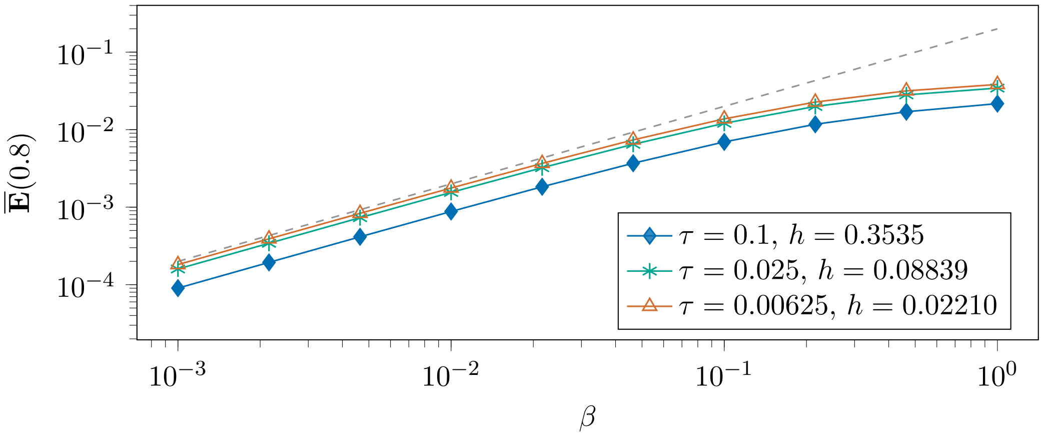

FIGURE 4. Convergence of $(u^n_{h,\beta=0}, \partial_\tau u^n_{h,\beta=0})$ and $(u^n_{h,\beta}, \partial_\tau u^n_{h,\beta})$ in the $H^1(\Omega) \times L^2(\Omega)$-norm at the end time $t = 0.8$ for different values of h and τ. The dashed line indicates order $\mathcal{O}(\beta)$.

for different values of β at the end time $n = N + 1$.

We observe in Figure 4 that for varying values of h and τ, the convergence in β is uniform of order $\mathcal{O}(\beta)$. Different values of h and τ lead to qualitatively similar pictures, with a clustering at the dashed line for finer resolutions which confirms the assertions in Theorems 2.2 and 2.4.

2.4.3. *Gaussian pulse.* In the last experiment, we simulate the propagation of a Gaussian pulse on the larger domain $\Omega = [-4, 4] \times [-4, 4]$. We use the initial states

$$u_0(x) = -e^{-|x|^2}, \qquad v_0(x) = 0,$$

where although u_0 is not zero on the boundary of Ω, by the size of the domain it is still within machine precision. Further, we take the following parameters and source term:

$$\kappa = -0.29, \qquad c^2 = 1, \qquad \ell = 2, \qquad f = 0\,,$$

and vary $\beta \geq 0$. Here we have to choose $\kappa > -0.3$ in order to prevent $1 + \kappa \partial_\tau u^n_h < 0$ after a short time. Since we do not have an exact solution, we first compute a reference solution with finer spatial and temporal resolution, i.e., $h_{\text{ref}} \approx 4 \cdot 10^{-2}$ and $\tau_{\text{ref}} = 4 \cdot 10^{-4}$. Due to the larger domain, we have to increase the number of elements by a factor 16, and hence compute the errors only for a coarser resolution.
In Figure 5, we observe that also in this example we have convergence of optimal order uniformly in the damping parameter β.

3. Uniform finite element analysis

In this section, we conduct a β-uniform analysis of the semi-discrete problem (2.3a) with approximate data (2.3b). We begin by discussing the general strategy. Due to the type of quasilinearity present in the problem and the need to conduct estimates uniformly in β, one would have to resort to higher-order Sobolev spaces to mimic the approach of the β-uniform well-posedness analysis of the Kuznetsov equation in [19]. As we cannot exploit such global spatial smoothness arguments for the approximate solution, we rely instead on inverse finite element estimates in careful combination with working with a

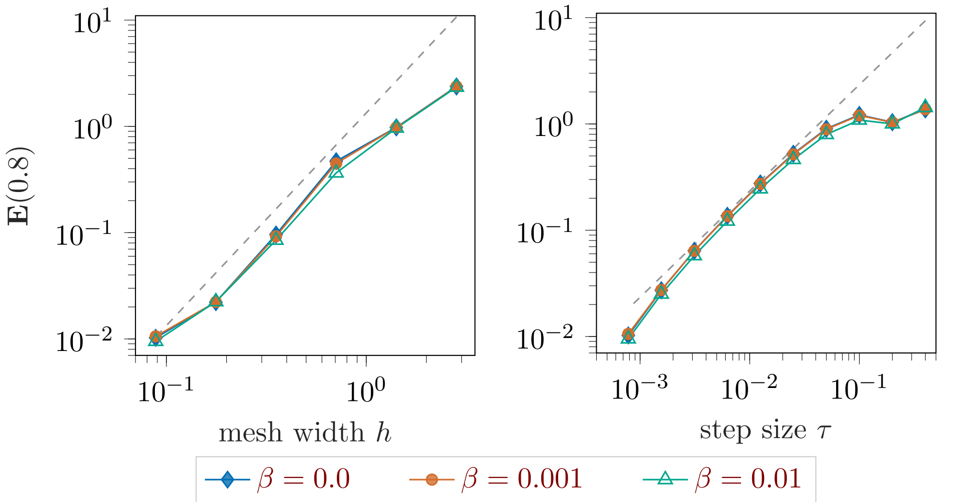

FIGURE 5. **Left**: Convergence of (2.3) with $\|\nabla\partial_t u(t) - \nabla\partial_t u_h(t)\|_{L^2(\Omega)}$ at $t = 0.8$ for elements of order $k = 2$ and $\tau \approx 7.8 \cdot 10^{-4}$ and damping parameters $\beta = 0, 10^{-3}, 10^{-2}$. The dashed line indicates order $\mathcal{O}(h^2)$. **Right**: Convergence of (2.9) with $\|\nabla\partial_t u(t_n) - \nabla\partial_\tau u_h^n\|_{L^2(\Omega)}$ for $n = N+1$ with $k = 2$ and $h \approx 9 \cdot 10^{-2}$ and damping parameters $\beta = 0, 10^{-3}, 10^{-2}$. The dashed line indicates order $\mathcal{O}(\tau)$.

time-differentiated problem given by

$$\begin{aligned}
&((1+\kappa\partial_t u_h)\partial_t^3 u_h, \varphi_h)_{L^2(\Omega)} + \kappa((\partial_t^2 u_h)^2, \varphi_h)_{L^2(\Omega)} - (c^2\Delta_h \partial_t u_h, \varphi_h)_{L^2(\Omega)} \\
&- (\beta\Delta_h\partial_t^2 u_h, \varphi_h)_{L^2(\Omega)} + \ell(\nabla\partial_t u_h \cdot \nabla\partial_t u_h, \varphi_h)_{L^2(\Omega)} + \ell(\nabla u_h \cdot \nabla\partial_t^2 u_h, \varphi_h)_{L^2(\Omega)} \\
&\qquad = (\partial_t f_h, \varphi_h)_{L^2(\Omega)}.
\end{aligned} \tag{3.1}$$

The "problematic" nonlinear term in (3.1) is the one involving $\ell\nabla u_h \cdot \nabla\partial_t^2 u_h$, and it has to be treated as a right-hand side perturbation. In the literature, error bounds for nonlinear (wave-type) problems are often established via some variant of a fixed-point argument for the numerical solution u_h which combines existence and error analysis; see, e.g., [26, 31–33] and the references provided therein. However, such strategies do not transfer easily to our setting as applying an inverse bound to estimate $\nabla\partial_t^2 u_h$ would prevent the fixed-point iterates to match in the order of h-convergence.

Our finite element analysis instead builds upon that of [15] to first show that an accurate approximate solution exists u_h on a discretization-dependent time interval $[0, t_h^*]$. We then derive uniform estimates for

$$e_h = \mathrm{R}_h u - u_h,$$

which in turn allow extending the existence interval and optimal error bounds to the whole time interval $[0, T]$. Crucially, with this approach we can exploit the polynomial structure of the nonlinearity and the fact that

$$\partial_t^2 e_h \nabla\partial_t^2 e_h = \frac{1}{2}\nabla(\partial_t^2 e_h)^2$$

to compensate inverse estimates with smallness conditions on the error e_h; see Proposition 3.4 for details.

3.1. **Auxiliary results.** Before we turn to the proofs of the main results in this section, we recall the relevant known estimates from the literature that we employ frequently within our analysis. We rely on the approximation properties of the Ritz projection for $0 \leq \ell \leq k$:

$$\|\varphi - \mathrm{R}_h\varphi\|_{L^p(\Omega)} + h\|\varphi - \mathrm{R}_h\varphi\|_{W^{1,p}(\Omega)} \leq Ch^{\ell+1}\|\varphi\|_{W^{\ell+1,p}(\Omega)}, \quad \varphi \in W^{\ell+1,p}(\Omega), \tag{3.2}$$

for all $2 \leq p \leq \infty$; see, for example, [7, Thm. 8.5.3]. In addition, we have the following bounds for the interpolant:

$$\|\varphi - I_h\varphi\|_{L^p(\Omega)} + h\|\varphi - I_h\varphi\|_{W^{1,p}(\Omega)} \leq Ch^{\ell+1}\|\varphi\|_{W^{\ell+1,p}(\Omega)}, \quad \varphi \in W^{\ell+1,p}(\Omega),$$

for $2 \leq p \leq \infty$ and $1 \leq \ell \leq k$.

For $\varphi_h \in V_h$ also the discrete Sobolev embedding

$$\|\varphi_h\|_{L^\infty(\Omega)} + \|\varphi_h\|_{W^{1,6}(\Omega)} \leq C\|\Delta_h\varphi_h\|_{L^2(\Omega)} \tag{3.3}$$

with a constant C independent of h is heavily used, see for example [10, 12, 34]. Furthermore, we rely on the following inverse estimates:

$$\|\nabla\varphi_h\|_{L^2(\Omega)} \leq Ch^{-1}\|\varphi_h\|_{L^2(\Omega)}, \tag{3.4a}$$

$$\|\Delta_h\varphi_h\|_{L^2(\Omega)} \leq Ch^{-1}\|\nabla\varphi_h\|_{L^2(\Omega)}, \tag{3.4b}$$

$$\|\varphi_h\|_{L^\infty(\Omega)} \leq Ch^{-d/p}\|\varphi_h\|_{L^p(\Omega)}, \tag{3.4c}$$

for $\varphi_h \in V_h$ and $p \in [1, \infty]$, with constants independent of h.

3.2. **Finite element analysis.** We begin the analysis by defining the (possibly h-dependent) time t_h^* as follows:

$$\begin{aligned} t_h^* := \sup\Big\{t \in (0,T] \mid \text{ a unique solution } u_h \in H^3(0,t;V_h) \text{ of (2.3) exists, and} \\ h^{-1-d/6}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)} \leq C_0, \\ h^{-1-d/6}\|\nabla\partial_t e_h(s)\|_{L^2(\Omega)} \leq C_0 \text{ for all } s \in [0,t]\Big\} \end{aligned}$$

for some $C_0 > 0$. Our first task is to establish that this set is non-empty. To this end, we estimate $\|\partial_t^2 e_h\|_{L^2(\Omega)}$ and $\|\nabla\partial_t e_h\|_{L^2(\Omega)}$ at initial time.

Lemma 3.1. *Under the assumptions of Theorem 2.1, with approximate initial values chosen to be the Ritz projections of the exact ones as in* (2.5) *and $\partial_t^2 u_h(0)$ determined by* (2.6)*, the following estimate holds:*

$$\|\partial_t^2 e_h(0)\|_{L^2(\Omega)} + \|\nabla\partial_t e_h(0)\|_{L^2(\Omega)} \leq Ch^k$$

with a constant $C > 0$ independent of h and β.

Proof. With our choice of the approximate initial data, $e_h(0) = \partial_t e_h(0) = 0$ and thus trivially

$$\|\nabla\partial_t e_h(0)\|_{L^2(\Omega)} \leq Ch^k.$$

It remains to estimate $\partial_t^2 e_h(0)$. We note that the Ritz projection of u satisfies the following problem at $t = 0$:

$$\begin{aligned} &((1+\kappa\partial_t u_h(0))\partial_t^2\mathrm{R}_h u(0), \varphi_h)_{L^2(\Omega)} - (c^2\Delta_h\mathrm{R}_h u(0), \varphi_h)_{L^2(\Omega)} - (\beta\Delta_h\partial_t\mathrm{R}_h u(0), \varphi_h)_{L^2(\Omega)} \\ &\quad + \ell(\nabla u_h(0)\cdot\nabla\partial_t\mathrm{R}_h u(0), \varphi_h)_{L^2(\Omega)} = (f_h(0), \varphi_h)_{L^2(\Omega)} + (\delta_h(0), \varphi_h)_{L^2(\Omega)} \end{aligned} \tag{3.5}$$

for all $\varphi_h \in V_h$, with the defect at zero satisfying

$$\begin{aligned} (\delta_h(0), \varphi_h)_{L^2(\Omega)} = &((1+\kappa\partial_t u_h(0))\partial_t^2\mathrm{R}_h u(0) - (1+\kappa\partial_t u(0))\partial_t^2 u(0), \varphi_h)_{L^2(\Omega)} \\ &+ \ell(\nabla u_h(0)\cdot\nabla\partial_t\mathrm{R}_h u(0) - \ell\nabla u(0)\cdot\nabla\partial_t u(0), \varphi_h)_{L^2(\Omega)} \\ &+ (f(0) - f_h(0), \varphi_h)_{L^2(\Omega)}. \end{aligned}$$

Since $\partial_t^2 u_h(0)$ is determined by (2.6), by subtracting (2.6) from (3.5) and using the fact that $e_h(0) = \partial_t e_h(0) = 0$, we see that $\partial_t^2 e_h(0)$ solves

$$((1 + \kappa\partial_t u_h(0))\partial_t^2 e_h(0), \varphi_h)_{L^2(\Omega)} = (\delta_h(0), \varphi_h)_{L^2(\Omega)} \tag{3.6}$$

for all $\varphi_h \in V_h$. By the inverse estimates (3.4) and the approximation properties of the Ritz projection stated in (3.2), we have

$$\|\partial_t u_h(0)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} + \|v_0 - I_h v_0\|_{L^\infty(\Omega)} + \|I_h v_0 - \mathrm{R}_h v_0\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} + Ch^{1-d/4}\|v_0\|_{W^{1,4}(\Omega)}.$$

Thus, for sufficiently small $h \leq h_0$ (relative to v_0), we can guarantee that

$$|\kappa|\|\partial_t u_h(0)\|_{L^\infty(\Omega)} < 1.$$

This further implies that there exists $\gamma > 0$, independent of h and β, such that

$$1 + \kappa\partial_t u_h(0) \geq \gamma > 0. \tag{3.7}$$

By the approximation properties of the Ritz projection, and the accuracy of f_h assumed in (2.4), we have

$$\|\delta_h(0)\|_{L^2(\Omega)} \lesssim h^k. \tag{3.8}$$

Therefore, by using $\varphi_h = \partial_t^2 e_h(0)$ in (3.6) and relying on (3.7) and (3.8), we immediately obtain

$$\|\partial_t^2 e_h(0)\|_{L^2(\Omega)} \lesssim h^k,$$

which concludes the proof. □

We next aim to prove that $t_h^* > 0$ by applying a local version of the Picard–Lindelöf theorem to the time-differentiated semi-discrete problem.

Lemma 3.2. *Under the assumptions of Theorem 2.1, we have $t_h^* > 0$.*

Proof. For the purposes of stating the time-differentiated problem in a compact manner, we introduce the discrete multiplication operator $\lambda_h = \lambda_h(\partial_t u_h)\colon V_h \to V_h$ defined by

$$(\lambda_h \varphi_h, \psi_h)_{L^2(\Omega)} = ((1 + \kappa\partial_t u_h)\varphi_h, \psi_h)_{L^2(\Omega)}$$

for $\varphi_h, \psi_h \in V_h$, which is invertible at $t = 0$ by (3.7). The time-differentiated semi-discrete problem can then be written as

$$\partial_t^3 u_h = \Big(\lambda_h^{-1}\big(c^2\Delta_h u_h + \beta\Delta_h\partial_t u_h - \ell\nabla u_h \cdot \partial_t\nabla\partial_t u_h - f_h\big)\Big)_t \tag{3.9}$$

and further rewritten as a first-order problem for $(u_h, \partial_t u_h, \partial_t^2 u_h)^T$. Unique solvability then follows by a similar reasoning to that of [15, Lemma 4.2] using a local version of the Picard–Lindelöf theorem on the open set

(3.10)
$$\begin{aligned} U_h = \{(u_h, \partial_t u_h, \partial_t^2 u_h) \in (C([0,t]; V_h))^3 : \; & |\kappa|\|\partial_t u_h(s)\|_{L^\infty(\Omega)} < 1, \\ & h^{-1-d/6}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)} < C_0, \\ & h^{-1-d/6}\|\nabla\partial_t e_h(s)\|_{L^2(\Omega)} < C_0, \; s \in [0,t]\}. \end{aligned}$$

We check first that $(u_h(0), \partial_t u_h(0), \partial_t^2 u_h(0)) \in U_h$. As concluded in the proof of Lemma 3.1, for $h \leq h_0$ small enough, we have

$$|\kappa|\|\partial_t u_h(0)\|_{L^\infty(\Omega)} < 1.$$

By Lemma 3.1, we also have

$$\|\partial_t^2 e_h(0)\|_{L^2(\Omega)} \lesssim h^k.$$

Therefore, since $\partial_t e_h(0) = 0$, for $h \leq h_0$ and $k \geq 2$, we conclude that

$$h^{-1-d/6}\max\{\|\partial_t^2 e_h(0)\|_{L^2(\Omega)}, \|\nabla\partial_t e_h(0)\|_{L^2(\Omega)}\} = h^{-1-d/6}\|\partial_t^2 e_h(0)\|_{L^2(\Omega)} < C_0$$

and thus $(u_h(0), \partial_t u_h(0), \partial_t^2 u_h(0)) \in U_h$.

Equation (3.9) rewritten as a first-order system in time is driven by a locally Lipschitz continuous right-hand side. Indeed, Lipschitz continuity of the right-hand side follows analogously to the arguments of [15, Lemma 4.2] by the fact that V_h is a finite-dimensional space and that we can use inverse estimates (3.4) for functions in V_h.

Thus by the local version of the Picard–Lindelöf theorem, a unique solution $u_h \in H^3(0, t_h^*; V_h) \hookrightarrow C^2([0,T]; V_h)$ of (3.9) supplemented with initial data exists on $[0, \tilde{t}]$ for some $\tilde{t} > 0$. Time integrating (3.9) and using (2.6) shows that u_h solves (2.3a), (2.5). We therefore conclude that $t_h^* > 0$. $\square$

We have shown that a unique approximate solution exists on $[0, t_h^*]$. The next result establishes additional uniform bounds on this time interval.

Lemma 3.3. *Let the assumptions of Theorem 2.1 hold. On the interval $[0, t_h^*]$, the following bound holds for sufficiently small h:*

$$\|u_h\|_{L_t^\infty(W^{1,\infty}(\Omega))} + \|\nabla \partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))} + \|\partial_t^2 u_h\|_{L_t^\infty(L^\infty(\Omega))} \lesssim 1. \tag{3.11}$$

In addition,

$$1 + \kappa \partial_t u_h \geq \gamma > 0, \quad (x,t) \in \Omega \times [0, t_h^*], \tag{3.12}$$

where $\gamma > 0$ does not depend on h, β, or t_h^*.

Proof. Using the stability properties of the Ritz projection in (3.2) and the definition of t_h^*, we obtain the following bound:

$$\begin{aligned} \|u_h\|_{W^{1,\infty}(\Omega)} &\lesssim \|\mathrm{R}_h u\|_{L^\infty(\Omega)} + \|\nabla \mathrm{R}_h u\|_{L^\infty(\Omega)} + \|e_h\|_{W^{1,\infty}(\Omega)} \\ &\lesssim \|u\|_{W^{1,\infty}(\Omega)} + h^{-d/2} \|\nabla e_h\|_{L^2(\Omega)} \leq C, \end{aligned}$$

since $d/2 \geq 1 + d/6$, as well as

$$\|\partial_t^2 u_h\|_{L^\infty(\Omega)} \leq \|\partial_t^2 u\|_{W^{1,\infty}(\Omega)} + C h^{-d/2} \|\partial_t^2 e_h\|_{L^2(\Omega)} \leq C$$

on $[0, t_h^*]$. Furthermore,

$$\|\nabla \partial_t u_h\|_{L^\infty(\Omega)} \lesssim \|\partial_t \mathrm{R}_h u\|_{W^{1,\infty}(\Omega)} + \|\nabla \partial_t e_h\|_{L^\infty(\Omega)} \lesssim \|\partial_t u\|_{W^{1,\infty}(\Omega)} + h^{-d/2} \|\nabla \partial_t e_h\|_{L^2(\Omega)} \leq C$$

for all $t \in [0, t_h^*]$. The bound (3.12) follows by the solvability of the (differentiated) semi-discrete problem in U_h; cf. (3.10). $\square$

Our main task in the remaining of this section is to prove that

$$\|\partial_t^2 e_h(t_h^*)\|_{L^2(\Omega)} + \|\nabla \partial_t e_h(t_h^*)\|_{L^2(\Omega)} \lesssim h^k, \tag{3.13}$$

where the error $e_h = \mathrm{R}_h u - u_h$ satisfies

$$\begin{aligned} ((1 + \kappa \partial_t u_h) \partial_t^2 e_h, \varphi_h)_{L^2(\Omega)} + \kappa (\partial_t e_h \partial_t^2 \mathrm{R}_h u, \varphi_h)_{L^2(\Omega)} - (c^2 \Delta_h e_h, \varphi_h)_{L^2(\Omega)} \\ - (\beta \Delta_h \partial_t e_h, \varphi_h)_{L^2(\Omega)} \\ = -\ell (\nabla u_h \cdot \nabla \partial_t e_h + \nabla e_h \cdot \nabla \partial_t \mathrm{R}_h u, \varphi_h)_{L^2(\Omega)} + (\delta_h, \varphi_h)_{L^2(\Omega)} \end{aligned} \tag{3.14}$$

and the defect is given by

$$\begin{aligned} (\delta_h, \varphi_h)_{L^2(\Omega)} = ((1 + \kappa \partial_t u_h) \partial_t^2 \mathrm{R}_h u - (1 + \kappa \partial_t u) \partial_t^2 u, \varphi_h)_{L^2(\Omega)} \\ + \ell (\nabla u_h \cdot \nabla \partial_t \mathrm{R}_h u - \ell \nabla u \cdot \nabla \partial_t u, \varphi_h)_{L^2(\Omega)} + (f - f_h, \varphi_h)_{L^2(\Omega)} \end{aligned} \tag{3.15}$$

for all $\varphi_h \in V_h$. The bound (3.13) will allow us to extend the existence interval beyond $[0, t_h^*]$.

To prove (3.13), we use a two-step testing procedure. In the first step, we test the time-differentiated error equation with $\partial_t^2 e_h$.

Proposition 3.4. *Let the assumptions of Theorem 2.1 hold. For $t \in [0, t_h^*]$, it holds*

$$
\begin{aligned}
&\|\partial_t^2 e_h(t)\|^2_{L^2(\Omega)} + \|\nabla \partial_t e_h(t)\|^2_{L^2(\Omega)} + \beta \int_0^t \|\nabla \partial_t^2 e_h\|^2_{L^2(\Omega)} \\
&\lesssim \|\partial_t^2 e_h(0)\|^2_{L^2(\Omega)} + \int_0^t \left(\|\nabla \partial_t e_h(s)\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h(s)\|^2_{L^2(\Omega)} + \|\partial_t \delta_h(s)\|^2_{L^2(\Omega)} \right) \mathrm{d}s \\
&\qquad + \alpha \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s.
\end{aligned}
\tag{3.16}
$$

for all $t \in [0, t_h^*]$ and $\alpha > 0$, with the hidden constant independent of h, t_h^*, and β.

Proof. As announced, we test the time-differentiated error equation with $\varphi_h = \partial_t^2 e_h(t)$:

$$
\begin{aligned}
&((1+\kappa \partial_t u_h)\partial_t^3 e_h, \partial_t^2 e_h)_{L^2(\Omega)} + \kappa(\partial_t^2 u_h \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} + \kappa(\partial_t^2 e_h \partial_t^2 \mathrm{R}_h u, \partial_t^2 e_h)_{L^2(\Omega)} \\
&\quad + \kappa(\partial_t e_h \partial_t^3 \mathrm{R}_h u, \partial_t^2 e_h)_{L^2(\Omega)} - (c^2 \Delta_h \partial_t e_h, \partial_t^2 e_h)_{L^2(\Omega)} - (\beta \Delta_h \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} \\
&= -\ell(\nabla \partial_t u_h \cdot \nabla \partial_t e_h + \nabla \partial_t e_h \cdot \nabla \partial_t \mathrm{R}_h u, \partial_t^2 e_h)_{L^2(\Omega)} - \ell(\nabla u_h \cdot \nabla \partial_t^2 e_h + \nabla e_h \cdot \nabla \partial_t^2 \mathrm{R}_h u, \partial_t^2 e_h)_{L^2(\Omega)} \\
&\qquad + (\partial_t \delta_h, \partial_t^2 e_h)_{L^2(\Omega)}
\end{aligned}
$$

for all $t \in [0, t_h^*]$. Using integration by parts in time and Young's inequality yields the following estimate:

$$
\begin{aligned}
&\partial_t \|(1+\kappa \partial_t u_h)^{1/2} \partial_t^2 e_h\|^2_{L^2(\Omega)} + c^2 \partial_t \|\nabla \partial_t e_h\|^2_{L^2(\Omega)} + \beta \|\nabla \partial_t^2 e_h\|^2_{L^2(\Omega)} \\
&\lesssim \|\partial_t^2 u_h \partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h \partial_t^2 \mathrm{R}_h u\|^2_{L^2(\Omega)} + \|\partial_t e_h \partial_t^3 \mathrm{R}_h u\|^2_{L^2(\Omega)} + \|\partial_t e_h\|^2_{L^2(\Omega)} \\
&\quad + \ell^2 \|\nabla \partial_t u_h \cdot \nabla \partial_t e_h\|^2_{L^2(\Omega)} + \ell^2 \|\nabla \partial_t e_h \cdot \nabla \partial_t \mathrm{R}_h u\|^2_{L^2(\Omega)} \\
&\quad + \ell(\nabla u_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} + \ell^2 \|\nabla e_h \cdot \nabla \partial_t^2 \mathrm{R}_h u\|^2_{L^2(\Omega)} + \|\partial_t \delta_h\|^2_{L^2(\Omega)}
\end{aligned}
\tag{3.17}
$$

for $0 \le t \le t_h^*$. We can rely on the bounds on u_h obtained in Lemma 3.3 to further estimate the right-hand side terms. First, using also the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$, $p \in [1,6]$, we estimate the first three terms on the right-hand side of (3.17) as follows:

$$
\begin{aligned}
&\|\partial_t^2 u_h \partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h \partial_t^2 \mathrm{R}_h u\|^2_{L^2(\Omega)} + \|\partial_t e_h \partial_t^3 \mathrm{R}_h u\|^2_{L^2(\Omega)} \\
&\lesssim \|\partial_t^2 u_h\|^2_{L^\infty(\Omega)} \|\partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h\|^2_{L^2(\Omega)} \|\partial_t^2 \mathrm{R}_h u\|^2_{L^\infty(\Omega)} + \|\partial_t e_h\|^2_{L^6(\Omega)} \|\partial_t^3 \mathrm{R}_h u\|^2_{L^3(\Omega)} \\
&\lesssim \|\partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\partial_t e_h\|^2_{H^1(\Omega)},
\end{aligned}
$$

where we have employed $\|\partial_t^2 u_h(t)\|_{L^\infty(\Omega)} \lesssim 1$ on $[0, t_h^*]$. Next, we can bound the ℓ^2 terms in the following manner using (3.11):

$$
\ell^2 \|\nabla \partial_t u_h \cdot \nabla \partial_t e_h\|^2_{L^2(\Omega)} \lesssim \|\nabla \partial_t u_h\|^2_{L^\infty(\Omega)} \|\nabla \partial_t e_h\|^2_{L^2(\Omega)} \lesssim \|\nabla \partial_t e_h\|^2_{L^2(\Omega)}. \tag{3.18}
$$

Similarly, using (3.2),

$$
\ell^2 \|\nabla \partial_t e_h \cdot \nabla \partial_t \mathrm{R}_h u\|^2_{L^2(\Omega)} \lesssim \|\nabla \partial_t e_h\|^2_{L^2(\Omega)} \|\nabla \partial_t \mathrm{R}_h u\|^2_{L^\infty(\Omega)} \lesssim \|\partial_t u\|^2_{W^{1,\infty}(\Omega)} \|\nabla \partial_t e_h\|^2_{L^2(\Omega)},
$$

and

$$
\ell^2 \|\nabla e_h \cdot \nabla \partial_t^2 \mathrm{R}_h u\|^2_{L^2(\Omega)} \lesssim \|\nabla e_h\|^2_{L^2(\Omega)} \|\nabla \partial_t^2 \mathrm{R}_h u\|^2_{L^\infty(\Omega)} \lesssim \|\partial_t^2 u\|^2_{W^{1,\infty}(\Omega)} \|\nabla e_h\|^2_{L^2(\Omega)}. \tag{3.19}
$$

The most salient point of the proof lies in estimating the term $\ell(\nabla u_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}$ in (3.17). To bound this term, we split the scalar product into three components by involving the exact solution:

$$
\begin{aligned}
&(\nabla u_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} \\
&= (\nabla u \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} - (\nabla (u - \mathrm{R}_h u) \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} - (\nabla e_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}.
\end{aligned}
\tag{3.20}
$$

We can then use the fact that

$$\partial_t^2 e_h \nabla \partial_t^2 e_h = \frac{1}{2}\nabla(\partial_t^2 e_h)^2$$

and integration by parts to rewrite the first term on the right-hand side of (3.20) as

$$(\nabla u \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} = -\frac{1}{2}(\Delta u\, \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}.$$

Employing Hölder's and Young's inequalities in (3.20) then yields

$$\begin{aligned}
&(\nabla u_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} \\
&= -\frac{1}{2}(\Delta u\, \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} - (\nabla(u - \mathrm{R}_h u)\cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} - (\nabla e_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} \\
&\lesssim \|\Delta u\|_{L^\infty(\Omega)}\|\partial_t^2 e_h\|^2_{L^2(\Omega)} + \big(\|u - \mathrm{R}_h u\|_{W^{1,\infty}(\Omega)}\big)\|\nabla \partial_t^2 e_h\|_{L^2(\Omega)}\|\partial_t^2 e_h\|_{L^2(\Omega)} \\
&\qquad - (\nabla e_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)} \\
&\lesssim \big(\|\Delta u\|_{L^\infty(\Omega)} + h^{-1}\|u - \mathrm{R}_h u\|_{W^{1,\infty}(\Omega)}\big)\|\partial_t^2 e_h\|^2_{L^2(\Omega)} - (\nabla e_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)},
\end{aligned} \tag{3.21}$$

where we have also used the inverse estimate (3.4) on $\|\nabla \partial_t^2 e_h\|_{L^2(\Omega)}$ in the last line. After integrating in time, we can estimate the last term on the right-hand side of (3.21) as follows:

$$\begin{aligned}
&\int_0^t \|e_h(s)\|_{W^{1,\infty}(\Omega)}\|\nabla \partial_t^2 e_h(s)\|_{L^2(\Omega)}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\lesssim h^{-1-d/6}\int_0^t \|e_h(s)\|_{W^{1,6}(\Omega)}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\lesssim \big(\max_{s\in[0,t_h^*]} h^{-1-d/6}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)}\big)\int_0^t \|\Delta_h e_h(s)\|_{L^2(\Omega)}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\leq \alpha \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + C_\alpha \int_0^t \|\partial_t^2 e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned}$$

for any $\alpha > 0$, where we used the definition of t_h^*. From (3.21), relying also on the estimate

$$h^{-1}\|u(t) - \mathrm{R}_h u(t)\|_{W^{1,\infty}(\Omega)} \lesssim h^{-1}h\|u(t)\|_{W^{2,\infty}(\Omega)} \lesssim \|u(t)\|_{W^{2,\infty}(\Omega)}, \quad t \in [0,T],$$

(which holds by (3.2)), we then have

$$\begin{aligned}
&\ell \int_0^t (\nabla u_h \cdot \nabla \partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}\,\mathrm{d}s \\
&\lesssim \int_0^t \|u(s)\|_{W^{2,\infty}(\Omega)}\|\partial_t^2 e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \alpha \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + C_\alpha \int_0^t \|\partial_t^2 e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned}$$

for any $\alpha > 0$. Integrating over $(0,t)$ in (3.17) and using this estimate together with (3.18)–(3.19) yields (3.16). □

Note that we cannot yet control the $\Delta_h e_h$ term on the right-hand side of (3.16). Therefore, in the second step, we additionally test the error equation (3.14) with $-\Delta_h e_h$.

Proposition 3.5. *Let the assumptions of Theorem 2.1 hold. For $t \in [0, t_h^*]$, it holds*

$$\begin{aligned}
&\|\partial_t^2 e_h(t)\|^2_{L^2(\Omega)} + \|\nabla \partial_t e_h(t)\|^2_{L^2(\Omega)} + \beta\|\Delta_h e_h(t)\|^2_{L^2(\Omega)} + \frac{c^2}{4}\int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\leq C\|\partial_t^2 e_h(0)\|^2_{L^2(\Omega)} + C\int_0^t \big(\|\nabla \partial_t e_h(s)\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h(s)\|^2_{L^2(\Omega)} + \|\partial_t \delta_h(s)\|^2_{L^2(\Omega)} \\
&\qquad + \|\delta_h(s)\|^2_{L^2(\Omega)} + \|\nabla \partial_t e_h(s)\|^2_{L^2(\Omega)} + \|\nabla e_h(s)\|^2_{L^2(\Omega)}\big)\,\mathrm{d}s
\end{aligned} \tag{3.22}$$

for all $t \in [0, t_h^*]$*, with a constant* $C > 0$ *independent of* h*,* t_h^**, and* β*.*

Proof. Testing the error equation (3.14) with $\varphi_h = -\Delta_h e_h$, integrating over $(0,t)$, and using $e_h(0) = 0$ yields

$$
\begin{aligned}
&c^2 \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \beta \|\Delta_h e_h(t)\|^2_{L^2(\Omega)} \\
&= \int_0^t ((1+\kappa\partial_t u_h)\partial_t^2 e_h + \kappa \partial_t e_h \partial_t^2 \mathrm{R}_h u + \ell \nabla u_h \cdot \nabla \partial_t e_h + \nabla e_h \cdot \nabla \partial_t \mathrm{R}_h u - \delta_h, \Delta_h e_h)_{L^2(\Omega)} \,\mathrm{d}s.
\end{aligned} \tag{3.23}
$$

We use Young's inequality to bound the right-hand side:

$$
\begin{aligned}
&\int_0^t ((1+\kappa\partial_t u_h)\partial_t^2 e_h + \kappa \partial_t e_h \partial_t^2 \mathrm{R}_h u + \ell \nabla u_h \cdot \nabla \partial_t e_h + \nabla e_h \cdot \nabla \partial_t \mathrm{R}_h u - \delta_h, \Delta_h e_h)_{L^2(\Omega)} \,\mathrm{d}s \\
&\leq \frac{1}{2c^2} \|(1+\kappa\partial_t u_h)\partial_t^2 e_h + \kappa \partial_t e_h \partial_t^2 \mathrm{R}_h u + \ell \nabla u_h \cdot \nabla \partial_t e_h + \nabla e_h \cdot \nabla \partial_t \mathrm{R}_h u - \delta_h\|^2_{L^2_t(L^2(\Omega))} \\
&\qquad + \frac{c^2}{2} \|\Delta_h e_h\|^2_{L^2_t(L^2(\Omega))}.
\end{aligned} \tag{3.24}
$$

We can conclude similarly to before by using (3.18)–(3.19) that

$$
\begin{aligned}
&\|(1+\kappa\partial_t u_h)\partial_t^2 e_h + \kappa \partial_t e_h \partial_t^2 \mathrm{R}_h u + \ell \nabla u_h \cdot \nabla \partial_t e_h + \nabla e_h \cdot \nabla \partial_t \mathrm{R}_h u - \delta_h\|_{L^2_t(L^2(\Omega))} \\
&\lesssim \|\partial_t^2 e_h\|_{L^2_t(L^2(\Omega))} + \|\nabla \partial_t e_h\|_{L^2_t(L^2(\Omega))} + \|\nabla e_h\|_{L^2_t(L^2(\Omega))} + \|\delta_h\|_{L^2_t(L^2(\Omega))}
\end{aligned}
$$

for $t \in [0, t_h^*]$. Using absorption via the c^2 term in (3.23), we arrive at

$$
\begin{aligned}
&\frac{c^2}{2} \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \beta \|\Delta_h e_h(t)\|^2_{L^2(\Omega)} \\
&\lesssim \int_0^t \Big(\|\partial_t^2 e_h(s)\|^2_{L^2(\Omega)} + \|\delta_h(s)\|^2_{L^2(\Omega)} + \|\nabla \partial_t e_h(s)\|^2_{L^2(\Omega)} + \|\nabla e_h(s)\|^2_{L^2(\Omega)} \Big) \,\mathrm{d}s.
\end{aligned} \tag{3.25}
$$

Then adding estimates (3.16) and (3.25) and choosing $\alpha > 0$ small enough (independently of h, t_h^*, and β) so that the corresponding term can be absorbed by the left-hand side leads to (3.22). □

To show (3.13), it remains to estimate the defect terms on the right-hand side of (3.22).

Lemma 3.6. *Let the assumptions of Theorem 2.1 hold. On* $[0, t_h^*]$*, the defect satisfies the following bounds:*

$$
\begin{aligned}
\|\delta_h\|_{L^2_t(L^2(\Omega))} &\leq C(u) \big(h^k + \|\partial_t e_h\|_{L^2_t(L^2(\Omega))} + \|\nabla e_h\|_{L^2_t(L^2(\Omega))} \big), \\
\|\partial_t \delta_h\|_{L^2_t(L^2(\Omega))} &\leq C(u) \big(h^k + \|\partial_t e_h\|_{L^\infty_t(L^2(\Omega))} + \|\partial_t^2 e_h\|_{L^2_t(L^2(\Omega))} + \|\nabla e_h\|_{L^\infty_t(L^2(\Omega))} \\
&\qquad + \|\nabla \partial_t e_h\|_{L^2_t(L^2(\Omega))} \big),
\end{aligned} \tag{3.26}
$$

where $C(u) = C(1 + \|u\|_{H^3(W^{1,\infty}(\Omega))}) \|u\|_{H^3(H^{k+1}(\Omega))}$ *does not depend on* h *or* β*.*

Proof. We can rewrite the equation for the defect in (3.15) as follows:

$$
\begin{aligned}
(\delta_h, \varphi_h)_{L^2(\Omega)} = &((1+\kappa\partial_t \mathrm{R}_h u)(\partial_t^2 \mathrm{R}_h u - \partial_t^2 u) + \kappa(\partial_t \mathrm{R}_h u - \partial_t u)\partial_t^2 u, \varphi_h)_{L^2(\Omega)} \\
&- \kappa(\partial_t e_h \, \partial_t^2 \mathrm{R}_h u, \varphi_h)_{L^2(\Omega)} + (\ell \nabla \mathrm{R}_h u \cdot \nabla \partial_t \mathrm{R}_h u - \ell \nabla u \cdot \nabla \partial_t u, \varphi_h)_{L^2(\Omega)} \\
&- \ell(\nabla e_h \cdot \nabla \partial_t \mathrm{R}_h u, \varphi_h)_{L^2(\Omega)} + (f - f_h, \varphi_h)_{L^2(\Omega)}.
\end{aligned}
$$

Using estimate (3.2) for the Ritz projection several times and the assumption on $f - f_h$, we arrive at the first bound in (3.26).

The time-differentiated defect solves

$$\begin{aligned}(\partial_t \delta_h, \varphi_h)_{L^2(\Omega)} = {} & ((1+\kappa\partial_t \mathrm{R}_h u)(\partial_t^3 \mathrm{R}_h u - \partial_t^3 u) + \kappa \partial_t^2 \mathrm{R}_h u(\partial_t^2 \mathrm{R}_h u - \partial_t^2 u) \\ & + \kappa(\partial_t^2 \mathrm{R}_h u - \partial_t^2 u)\partial_t^2 u + \kappa(\partial_t \mathrm{R}_h u - \partial_t u)\partial_t^3 u - \kappa \partial_t^2 e_h \, \partial_t^2 \mathrm{R}_h u - \kappa \partial_t e_h \, \partial_t^3 \mathrm{R}_h u \\ & + \ell \nabla \partial_t \mathrm{R}_h u \cdot \nabla \partial_t \mathrm{R}_h u - \ell \nabla \partial_t u \cdot \nabla \partial_t u + \ell \nabla \mathrm{R}_h u \cdot \nabla \partial_t^2 \mathrm{R}_h u - \ell \nabla u \cdot \nabla \partial_t^2 u \\ & - \ell \nabla \partial_t e_h \cdot \nabla \partial_t \mathrm{R}_h u - \ell \nabla e_h \cdot \nabla \partial_t^2 \mathrm{R}_h u + \partial_t (f - f_h), \varphi_h)_{L^2(\Omega)}\end{aligned}$$

for all $\varphi_h \in V_h$. We can similarly bound $\partial_t \delta_h$ in $L^2(0, t_h^*; L^2(\Omega))$ using the stability and approximation properties (3.2) of the Ritz projection to arrive at the second estimate in (3.26). □

We now have all the ingredients to prove our first main result on the robust finite element bounds stated in Theorem 2.1.

Proof of Theorem 2.1. Using the bounds derived in Lemma 3.6 on the defect in (3.22), together with

$$\|\partial_t e_h\|_{L^\infty_t(L^2(\Omega))} \leq \sqrt{T} \|\partial_t^2 e_h\|_{L^2_t(L^2(\Omega))}, \quad \|\nabla e_h\|_{L^\infty_t(L^2(\Omega))} \leq \sqrt{T} \|\nabla \partial_t e_h\|_{L^2_t(L^2(\Omega))}$$

(since $e_h(0) = \partial_t e_h(0) = 0$) and Lemma 3.1, by employing Grönwall's inequality, we arrive at

$$\|\partial_t^2 e_h(t)\|^2_{L^2(\Omega)} + \|\nabla \partial_t e_h(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s \leq C h^{2k} \tag{3.27}$$

for all $t \in [0, t_h^*]$. Therefore, estimate (3.13) holds. In turn, we conclude that for $h \leq h_0$ and $k \geq 2$, the following estimate holds:

$$h^{-1-d/6} \max\{\|\partial_t^2 e_h(t_h^*)\|_{L^2(\Omega)}, \|\nabla \partial_t e_h(t_h^*)\|_{L^2(\Omega)}\} < C_0.$$

Along the previous lines of reasoning, we also have $|\kappa| \|\partial_t u_h(t_h^*)\|_{L^\infty(\Omega)} < 1$. Altogether, we have shown that

$$(u_h(t_h^*), \partial_t u_h(t_h^*), \partial_t^2 u_h(t_h^*)) \in U_h$$

which means that the solution can be extended beyond t_h^*. Hence, we can conclude that $t_h^* = T$. Since the constant in (3.27) does not depend on t_h^*, the bound on e_h is valid on $[0, T]$.

Then writing the overall error as

$$u - u_h = (u - \mathrm{R}_h u) + e_h,$$

and using the approximation property of the Ritz projection (3.2) yields the bound in (2.7). □

3.3. **The inviscid limit of the finite element solutions.** By using the established β-uniform finite element error bound in Theorem 2.1, we can also determine the asymptotic properties of the semi-discrete solution as $\beta \to 0$. The difference $\bar{u}_h = u_h^{\beta=0} - u_h^\beta$ of the semi-discrete solutions $u_h^{\beta=0}$ of the undamped

$$((1+\kappa\partial_t u_h^{\beta=0})\partial_t^2 u_h^{\beta=0} - c^2 \Delta_h u_h^{\beta=0} + \ell \nabla u_h^{\beta=0} \cdot \nabla \partial_t u_h^{\beta=0}, \varphi_h)_{L^2(\Omega)} = (f_h, \varphi_h)_{L^2(\Omega)},$$

and u_h^β of the damped problem

$$((1+\kappa\partial_t u_h^\beta)\partial_t^2 u_h^\beta - c^2 \Delta_h u_h^\beta - \beta \Delta_h \partial_t u_h^\beta + \ell \nabla u_h^\beta \cdot \nabla \partial_t u_h^\beta, \varphi_h)_{L^2(\Omega)} = (f_h, \varphi_h)_{L^2(\Omega)}$$

solves

(3.28)
$$
\begin{aligned}
((1+\kappa\partial_t u_h^{\beta=0})\partial_t^2\bar{u}_h+\kappa\partial_t\bar{u}_h\partial_t^2u_h^{\beta}-c^2\Delta_h\bar{u}_h+\ell\nabla\bar{u}_h\cdot\nabla\partial_t u_h^{\beta}+\ell\nabla u_h^{\beta=0}\cdot\nabla\partial_t\bar{u}_h,\varphi_h)_{L^2(\Omega)}\\
=-\beta(\Delta_h\partial_t u_h^{\beta},\varphi_h)_{L^2(\Omega)}
\end{aligned}
$$

for all $\varphi_h\in V_h$ on $[0,T]$. We next prove the statement of Theorem 2.2 on the inviscid limit of the finite element solutions.

Proof of Theorem 2.2. The proof follows by testing the difference equation (3.28) with $\partial_t\bar{u}_h$. We can rely on the identity

$$
\begin{aligned}
&\int_0^t((1+\kappa\partial_t u_h^{\beta=0})\partial_t^2\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}\,\mathrm{d}s\\
&=\frac{1}{2}((1+\kappa\partial_t u_h^{\beta=0})\partial_t\bar{u}_h(t),\partial_t\bar{u}_h(t))_{L^2(\Omega)}-\frac{1}{2}\int_0^t(\kappa\partial_t^2u_h^{\beta=0}\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}\,\mathrm{d}s
\end{aligned}
$$

and the estimate

$$
\int_0^t(\kappa\partial_t u_h^{\beta=0}\partial_t^2u_h^{\beta},\partial_t\bar{u}_h)_{L^2(\Omega)}\,\mathrm{d}s\lesssim\|\partial_t^2u_h^{\beta=0}\|_{L^\infty(L^\infty(\Omega))}\|\partial_t\bar{u}_h\|^2_{L^2(L^2(\Omega))}.
$$

We note that thanks to the previous analysis and the assumptions on the exact solution the following uniform bound holds :

$$
\|\partial_t^2u_h^{\beta=0}\|_{L^\infty(L^\infty(\Omega))}\lesssim\|\partial_t^2u^{\beta=0}\|_{L^\infty(L^\infty(\Omega))}+h^{-d/2}\|\partial_t^2e_h^{\beta=0}\|_{L^\infty(L^2(\Omega))}\leq C.
$$

To estimate the ℓ terms, we rely on a rewriting with the help of the exact solution and integration by parts in space:

$$
\begin{aligned}
&\int_0^t(\ell\nabla\bar{u}_h\cdot\nabla\partial_t u_h^{\beta}+\ell\nabla u_h^{\beta=0}\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}\,\mathrm{d}s\\
&=\int_0^t\Big\{(\ell\nabla\bar{u}_h\cdot\nabla\partial_t u_h^{\beta},\partial_t\bar{u}_h)_{L^2(\Omega)}+\frac{1}{2}(\ell\Delta u^{\beta=0}\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}-(\ell\nabla(u^{\beta=0}-u_h^{\beta=0})\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}\Big\}\,\mathrm{d}s.
\end{aligned}
$$

We can further transform the last term on the right by decomposing it via the Ritz projection:

$$
\begin{aligned}
&\int_0^t(\ell\nabla(u^{\beta=0}-u_h^{\beta=0})\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}\,\mathrm{d}s\\
&=\int_0^t(\ell\nabla(u^{\beta=0}-\mathrm{R}_hu^{\beta=0})\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}+(\ell\nabla e_h^{\beta=0}\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}\,\mathrm{d}s\\
&\leq Ch\|u^{\beta=0}\|_{L^\infty(W^{2,\infty}(\Omega))}h^{-1}\|\partial_t\bar{u}_h\|^2_{L^2(L^2(\Omega))}+\int_0^t|(\ell\nabla e_h^{\beta=0}\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}|\,\mathrm{d}s.
\end{aligned}
$$

Furthermore, in the last term, we have for any $\alpha_1>0$

$$
\begin{aligned}
&\int_0^t|(\nabla e_h^{\beta=0}\cdot\nabla\partial_t\bar{u}_h,\partial_t\bar{u}_h)_{L^2(\Omega)}|\,\mathrm{d}s\\
&\lesssim\int_0^t h^{-1-d/6}\|\Delta_he_h^{\beta=0}\|_{L^2(\Omega)}\|\partial_t\bar{u}_h\|^2_{L^2(\Omega)}\,\mathrm{d}s\\
&\lesssim\max_{s\in[0,t]}\|\partial_t\bar{u}_h(s)\|_{L^2(\Omega)}h^{-1-d/6}\Big(\int_0^t\|\Delta_he_h^{\beta=0}\|^2_{L^2(\Omega)}\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t\|\partial_t\bar{u}_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\Big)^{1/2}\\
&\leq\alpha_1\max_{s\in[0,t]}\|\partial_t\bar{u}_h(s)\|^2_{L^2(\Omega)}+C_\alpha h^{k-1-d/6}\|\partial_t\bar{u}_h(s)\|^2_{L^2(L^2(\Omega))},
\end{aligned}
$$

where we have used the uniform bound on $\|\Delta_h e_h^{\beta=0}\|_{L^2(L^2(\Omega))}$ by Theorem 2.1. Thanks also to the (assumed) uniform bound on $\|u^{\beta=0}\|_{L^\infty(W^{2,\infty}(\Omega))}$, we arrive at an estimate of the form

$$\begin{aligned}
&\|\partial_t \bar{u}_h(t)\|^2_{L^2(\Omega)} + \|\nabla \bar{u}_h(t)\|^2_{L^2(\Omega)} \\
&\lesssim \beta \left|\int_0^t (\nabla \partial_t u_h^\beta, \nabla \partial_t \bar{u}_h)_{L^2(\Omega)}\, \mathrm{d}s\right| + \alpha_1 \max_{s\in[0,t]} \|\partial_t \bar{u}_h(s)\|^2_{L^2(\Omega)} + \|\partial_t \bar{u}_h\|^2_{L^2_t(L^2(\Omega))} + \|\nabla \bar{u}_h\|^2_{L^2_t(L^2(\Omega))}.
\end{aligned}$$

Observe that we cannot absorb $\nabla \partial_t \bar{u}_h$ by the left-hand side. In the β term above we thus integrate by parts in time:

$$\beta\left|\int_0^t (\nabla \partial_t u_h^\beta, \nabla \partial_t \bar{u}_h)_{L^2(\Omega)}\, \mathrm{d}s\right| = \beta\left|(\nabla \partial_t u_h^\beta(t), \nabla \bar{u}_h(t))_{L^2(\Omega)} - \int_0^t (\nabla \partial_t^2 u_h^\beta, \nabla \bar{u}_h)_{L^2(\Omega)}\, \mathrm{d}s\right|.$$

We can then rely on the uniform bounds

$$\begin{aligned}
\|\nabla \partial_t u_h^\beta\|_{L^\infty(L^2(\Omega))} &\lesssim \|\nabla \partial_t u^\beta\|_{L^\infty(L^\infty(\Omega))} + h^{-1}\|\partial_t e_h\|_{L^\infty(L^2(\Omega))} \leq C, \\
\|\nabla \partial_t^2 u_h^\beta\|_{L^2(L^2(\Omega))} &\lesssim \|\nabla \partial_t^2 u^\beta\|_{L^2(L^\infty(\Omega))} + h^{-1}\|\partial_t^2 e_h\|_{L^2(L^2(\Omega))} \leq C,
\end{aligned} \tag{3.29}$$

and Young's inequality to obtain

$$\begin{aligned}
&\|\partial_t \bar{u}_h(t)\|^2_{L^2(\Omega)} + \|\nabla \bar{u}_h(t)\|^2_{L^2(\Omega)} \\
&\lesssim \beta^2 + \alpha_1 \max_{s\in[0,t]} \|\partial_t \bar{u}_h(s)\|^2_{L^2(\Omega)} + \alpha_2 \max_{s\in[0,t]} \|\nabla \bar{u}_h(s)\|^2_{L^2(\Omega)} + \|\partial_t \bar{u}_h\|^2_{L^2_t(L^2(\Omega))} + \|\nabla \bar{u}_h\|^2_{L^2_t(L^2(\Omega))}
\end{aligned} \tag{3.30}$$

for any α_1, $\alpha_2 > 0$. From (3.30) by taking the maximum over $t \in (0, \tilde{t})$ for some $\tilde{t} < T$, choosing $\alpha_{1,2} > 0$ small enough (independently of h and β) so that the corresponding terms can be absorbed, and then applying Grönwall's inequality, we arrive at

$$\|\partial_t \bar{u}_h\|^2_{L^\infty(L^2(\Omega))} + \|\nabla \bar{u}_h\|^2_{L^\infty(L^2(\Omega))} \lesssim \beta^2,$$

as claimed. □

We note that this result matches the convergence order of the exact solutions of the damped Kuznetsov equation as $\beta \to 0$; see [19, Thm. 7.1].

4. Robust semi-implicit time discretization

We now turn our attention to the analysis of a fully discrete scheme given by (2.9) with the aim of proving Theorems 2.3 and 2.4. We first collect several useful results when using the discrete derivatives ∂_τ defined in (2.8). We state relations which mimic the product rule:

$$\begin{aligned}
\partial_\tau\big(a^{n+1}b^{n+1}\big) &= \big(\partial_\tau a^{n+1}\big)b^{n+1} + a^n\big(\partial_\tau b^{n+1}\big) \\
&= \big(\partial_\tau a^{n+1}\big)b^{n} + a^{n+1}\big(\partial_\tau b^{n+1}\big),
\end{aligned} \tag{4.1}$$

the integration by parts formula:

$$\tau \sum_{n=1}^{N} a^{n+1}\, \partial_\tau b^{n+1} = a^{N+1}b^{N+1} - a^1 b^1 - \tau \sum_{n=1}^{N} \partial_\tau a^{n+1} b^{n+1}, \tag{4.2}$$

as well as the fundamental theorem of calculus:

$$\|a^N\|^2_{L^2(\Omega)} - \|a^0\|^2_{L^2(\Omega)} \leq 2\tau \sum_{j=1}^{N} (a^j, \partial_\tau a^j)_{L^2(\Omega)}; \tag{4.3}$$

see, e.g., [16]. However due to the nonlinear structure in the highest order term, we need the following extension of (4.3).

Lemma 4.1. *Let* $\max_{j=1,\dots,N} \|\omega_j\|_{L^\infty(\Omega)} + \|\partial_\tau \omega_j\|_{L^\infty(\Omega)} \leq C_\omega$ *be uniformly bounded from above and from below with* $\min_{j=1,\dots,N} \omega_j \geq \alpha > 0$. *Then, it holds*

$$\|a^N\|_{L^2(\Omega)}^2 \lesssim \|a^0\|_{L^2(\Omega)}^2 + \tau \sum_{j=1}^N (\omega_j a^j, \partial_\tau a^j)_{L^2(\Omega)} + \|a^j\|_{L^2(\Omega)}^2,$$

with constants that only depend on C_ω *and* α.

Proof. Simply setting $\widetilde{a}_j = \sqrt{\omega_j} a_j$ in (4.3) and using (4.1) leads to the claim. □

In addition, we need the following error bounds in the defects. We state the result here in a general form, but postpone the proof to the Appendix A.

Lemma 4.2. *Let* $m \geq 0$ *and* $u \in H^{m+1}(0,T;H^k(\Omega))$. *Then, for* $0 \leq \ell \leq m$, *it holds*

$$\|\partial_\tau^\ell u(t_n)\|_{H^k(\Omega)} \leq C \|u\|_{H^{\ell+1}(H^k(\Omega))}$$

with a constant C *independent of* τ, *and for* $0 \leq \ell_1, \ell_2 \leq m$ *it holds*

$$\|\partial_\tau^{m-\ell_1} \partial_t^{\ell_1} u(t_n) - \partial_\tau^{m-\ell_2} \partial_t^{\ell_2} u(t_n)\|_{H^k(\Omega)}^2 \leq C_m \, \tau \int_{t_{n-m}}^{t_n} \|\partial_t^{m+1} u(s)\|_{H^k(\Omega)}^2 \, \mathrm{d}s$$

with a constant C_m *that only depends on* m. *If* $u \in H^{m+2}(0,T;H^k(\Omega))$, *then we further have*

$$\|\partial_\tau^{m-\ell_1} \partial_t^{\ell_1} u(t_n) - \partial_\tau^{m-\ell_2} \partial_t^{\ell_2} u(t_n)\|_{H^k(\Omega)} \leq C\tau \|u\|_{H^{m+2}(H^k(\Omega))}.$$

These identities will be used throughout the proofs in the following section. Analogously to the approach in the finite element analysis, we define the fully discrete error by

$$e_h^n := \mathrm{R}_h \widehat{u}^n - u_h^n \tag{4.4}$$

with $\widehat{u}^n = u(t_n)$, and proceed to investigate it.

Proposition 4.3. *Under the assumptions of Theorem 2.3, for all* $n = 2, \dots N+1$, *the approximation* u_h^n *defined in* (2.9) *exists and the error defined in* (4.4) *satisfies*

$$\|\partial_\tau^2 e_h^n\|_{L^2(\Omega)}^2 + \|\nabla \partial_\tau e_h^n\|_{L^2(\Omega)}^2 + \tau \sum_{j=1}^n \|\Delta_h e_h^j\|_{L^2(\Omega)}^2 \leq C\big(\tau + h^k\big)^2, \tag{4.5}$$

as well as

$$\begin{aligned} &h^{-1-d/6-\varepsilon} \|\partial_\tau^2 e_h^n\|_{L^2(\Omega)} \leq C_0, \\ &h^{-1-d/6-\varepsilon} \|\nabla \partial_\tau e_h^n\|_{L^2(\Omega)} \leq C_0, \\ &h^{-1-d/6-\varepsilon} \big(\tau \sum_{j=1}^n \|\Delta_h e_h^j\|_{L^2(\Omega)}^2\big)^{1/2} \leq C_0, \end{aligned} \tag{4.6}$$

with some constants C, $C_0 > 0$ *that are independent of* h, τ, n, *and* β.

The rest of Section 4 is devoted to the proof of Proposition 4.3 which we conduct via induction over n. In Section 4.1, we first show that the statement holds in the case $n = 2$ as induction basis. In the following Section 4.2, we perform the step from n to $n+1$ to conclude that the statement of Proposition 4.3 holds. Then Theorem 2.3 will follow in a straightforward manner.

4.1. **Induction basis.** This part is dedicated to the induction base $n = 2$. We first study the error induced by the initial values for e_h^1, and then proceed to bound e_h^2 in a series of lemmas. To keep the presentation short, we formulate several of them such that they also apply to the induction step, assuming that the bounds in Proposition 4.3 already hold up to some $n \geq 2$.

Lemma 4.4. *Let the assumptions of Theorem 2.3 hold, and let the initial values be defined by (2.10). Then, $e_h^0 = 0$ and*

$$\|\partial_\tau^2 e_h^1\|_{L^2(\Omega)} + \tau^{-1}\|\partial_\tau \nabla e_h^1\|_{L^2(\Omega)} + \tau^{-2}\|\Delta_h e_h^1\|_{L^2(\Omega)} \lesssim \tau + h^k,$$

where the constant is independent of h, τ and β.

Proof. Recalling that by (2.8) we have $\partial_\tau^2 e_h^1 = \partial_\tau e_h^1$, the estimate directly follows from the definitions of u_h^0, u_h^1, and w_0 in (2.10). ☐

Along the lines of Lemma 3.3 in Section 3, we next derive some useful bounds on the numerical solution.

Lemma 4.5. *Let the assumptions of Theorem 2.3 hold. If the assertions of Proposition 4.3 hold for up to n, then the following bound holds for $j = 1, \dots, n$:*

$$\|u_h^j\|_{W^{1,\infty}(\Omega)} + \|\nabla \partial_\tau u_h^j\|_{L^\infty(\Omega)} + \|\partial_\tau^2 u_h^j\|_{L^\infty(\Omega)} \lesssim 1,$$

and, in addition,

$$1 + \kappa \partial_\tau u_h^j \geq \gamma > 0, \quad j = 1, \dots n, \tag{4.7}$$

where the constant γ does not depend on h, τ, n, or β.

In the next steps, we derive the equation solved by e_h^{n+1}. To this end, we insert the projected exact solution $\mathrm{R}_h \widehat{u}^n$ into (2.9) to obtain for $n \geq 1$

$$\begin{aligned}((1 + \kappa \partial_\tau u_h^n)\partial_\tau^2 \mathrm{R}_h \widehat{u}^{n+1} - c^2 \Delta_h \mathrm{R}_h \widehat{u}^{n+1} - \beta \Delta_h \partial_\tau \mathrm{R}_h \widehat{u}^{n+1} + \ell \nabla u_h^n \cdot \nabla \partial_\tau \mathrm{R}_h \widehat{u}^{n+1}, \varphi_h)_{L^2(\Omega)} \\ = (f_h^{n+1} + \delta_h^{n+1}, \varphi_h)_{L^2(\Omega)},\end{aligned}$$

with defect δ_h^{n+1} given below in (4.12). This leads us to the error equation:

$$\begin{aligned}((1 + \kappa \partial_\tau u_h^n)\partial_\tau^2 e_h^{n+1} - c^2 \Delta_h e_h^{n+1} - \beta \Delta_h \partial_\tau e_h^{n+1} + \ell \nabla u_h^n \cdot \nabla \partial_\tau e_h^{n+1}, \varphi_h)_{L^2(\Omega)} \\ = (\delta_h^{n+1}, \varphi_h)_{L^2(\Omega)}\end{aligned} \tag{4.8}$$

for $n \geq 1$. In the fully discrete case, we cannot use a Picard–Lindelöf theorem, and hence we explicitly have to show the existence of the approximation u_h^{n+1}. By (4.4), it is sufficient to show the unique solvability of (4.8) or, in other words, existence of a unique e_h^{n+1}. By multiplying (4.8) with τ^2 and solving for e_h^{n+1}, we rewrite the problem as a linear system of the form

$$(\mathcal{R}^n e_h^{n+1}, \varphi_h)_{L^2(\Omega)} = (\widetilde{f}^n, \varphi_h)_{L^2(\Omega)}, \tag{4.9a}$$

where

$$\mathcal{R}^n = (1 + \kappa \partial_\tau u_h^n)\mathrm{I} - c^2 \tau^2 \Delta_h - \tau \beta \Delta_h + \tau \ell \nabla u_h^n \cdot \nabla, \tag{4.9b}$$

$$\widetilde{f}^n = 2(1 + \kappa \partial_\tau u_h^n) e_h^n + (1 + \kappa \partial_\tau u_h^n) e_h^{n-1} - \tau \beta \Delta_h \partial_\tau e_h^n + \tau \ell \nabla u_h^n \cdot \nabla \partial_\tau e_h^n + \tau^2 \delta_h^{n+1}, \tag{4.9c}$$

and I is the identity operator. This rewriting enables us to prove the following existence result.

Lemma 4.6. *Let the assumptions of Theorem 2.3 hold.*

(a) There exists a unique solution u_h^2 of (2.9) for $n=2$.

(b) If the assertions of Proposition 4.3 hold for up to n, then there exists a unique solution u_h^{n+1} of (2.9).

Proof. Since we consider a finite dimensional solution space, it is sufficient to show injectivity of $\mathcal{R}^n$. We only present the proof of part (b) as part (a) can be proven along the same lines. For $\varphi_h \in V_h$, by using the lower bound in (4.7), we compute

$$\begin{aligned}
&(\mathcal{R}^n\varphi_h, \varphi_h)_{L^2(\Omega)} \\
&= ((1+\kappa\partial_\tau u_h^n)\varphi_h, \varphi_h)_{L^2(\Omega)} + \big(\tau^2c^2+\tau\beta\big)\|\nabla\varphi_h\|_{L^2(\Omega)}^2 + \tau\ell(\nabla u_h^n\cdot\nabla\varphi_h, \varphi_h)_{L^2(\Omega)} \\
&\geq \gamma\|\varphi_h\|_{L^2(\Omega)}^2 + \big(\tau^2c^2+\tau\beta\big)\|\nabla\varphi_h\|_{L^2(\Omega)}^2 - \tau\ell|(\nabla u_h^n\cdot\nabla\varphi_h, \varphi_h)_{L^2(\Omega)}|\,,
\end{aligned}$$

and thus it holds

$$\gamma\|\varphi_h\|_{L^2(\Omega)}^2 + \big(\tau^2c^2+\tau\beta\big)\|\nabla\varphi_h\|_{L^2(\Omega)}^2 \leq (\mathcal{R}^n\varphi_h, \varphi_h)_{L^2(\Omega)} + \tau\ell|(\nabla u_h^n\cdot\nabla\varphi_h, \varphi_h)_{L^2(\Omega)}|.$$

We expand the last term and rely on inverse estimates (3.4) and the discrete embedding (3.3) to obtain

$$\begin{aligned}
&\tau|(\nabla u_h^n\cdot\nabla\varphi_h, \varphi_h)_{L^2(\Omega)}| \\
&= \tau|-(\Delta\widehat{u}^n\varphi_h, \varphi_h)_{L^2(\Omega)} + (\nabla\big(\mathrm{R}_h\widehat{u}^n - \widehat{u}^n\big)\nabla\varphi_h, \varphi_h)_{L^2(\Omega)} - (\nabla e_h^n\nabla\varphi_h, \varphi_h)_{L^2(\Omega)}| \\
&\leq C\tau\big(1+h^{k-1}\big)\|\varphi_h\|_{L^2(\Omega)}^2 + \tau h^{-1-d/6}\|\Delta_h e_h^n\|_{L^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.10}$$

We absorb the first term for τ sufficiently small, and estimate the second term with the C_0 bound in (4.6) and the CFL condition (2.11)

$$\tau h^{-1-d/6}\|\Delta_h e_h^n\|_{L^2(\Omega)} \leq \tau^{1/2}h^{-1-d/6}\big(\tau\sum_{j=1}^n\|\Delta_h e_h^j\|_{L^2(\Omega)}^2\big)^{1/2} \leq C_0 h^{\varepsilon}\tau^{1/2}.$$

Hence, this term can also be absorbed, such that we obtain for some $\alpha>0$

$$\|\varphi_h\|_{L^2(\Omega)}^2 + \big(\tau^2c^2+\tau\beta\big)\|\nabla\varphi_h\|_{L^2(\Omega)}^2 \leq \alpha(\mathcal{R}^n\varphi_h, \varphi_h)_{L^2(\Omega)}, \tag{4.11}$$

where α is independent of h, τ, n, and β. □

The following lemma provides an estimate of the defect. Again here, we state it in its full generality to be used not only for proving the induction basis $n=2$ but also later for completing the induction step.

Lemma 4.7. *Let the assumptions of Theorem 2.3 hold. If the assertions of Proposition 4.3 hold for up to n, then*

$$\tau\sum_{j=1}^n\|\delta_h^{j+1}\|_{L^2(\Omega)}^2 \leq C(u,f)\big(h^k+\tau\big)^2 + \tau\sum_{j=1}^n\big(\|\partial_\tau e_h^j\|_{L^2(\Omega)}^2 + \|\nabla e_h^j\|_{L^2(\Omega)}^2\big),$$

and

$$\begin{aligned}
\tau\sum_{j=1}^n\|\partial_\tau\delta_h^{j+1}\|_{L^2(\Omega)}^2 &\leq C(u,f)\big(h^k+\tau\big)^2 \\
&\quad + \tau\sum_{j=1}^n\big(\|\partial_\tau e_h^j\|_{L^2(\Omega)}^2 + \|\partial_\tau^2 e_h^j\|_{L^2(\Omega)}^2 + \|\nabla e_h^j\|_{L^2(\Omega)}^2 + \|\partial_\tau\nabla e_h^j\|_{L^2(\Omega)}^2\big)
\end{aligned}$$

with constants independent of h, τ, n, and β.

Proof. It is straightforward to check that the defect in (4.8) is given by

$$\begin{aligned}\delta_h^{n+1} &= (1+\kappa\partial_\tau \mathrm{R}_h\widehat{u}^n)\partial_\tau^2\mathrm{R}_h\widehat{u}^{n+1} - (1+\kappa\partial_t\widehat{u}^{n+1})\partial_t^2\widehat{u}^{n+1} - \kappa\partial_\tau e_h^n\partial_\tau^2\mathrm{R}_h\widehat{u}^{n+1} \\ &\quad + \beta\big(\Delta\partial_t\widehat{u}^{n+1} - \Delta\partial_\tau\widehat{u}^{n+1}\big) + f(t_{n+1}) - f_h^{n+1} \\ &\quad + \ell\nabla\mathrm{R}_h\widehat{u}^n\cdot\nabla\partial_\tau\mathrm{R}_h\widehat{u}^{n+1} - \ell\nabla\widehat{u}^{n+1}\cdot\nabla\partial_t\widehat{u}^{n+1} - \ell\nabla e_h^n\cdot\nabla\partial_\tau\mathrm{R}_h\widehat{u}^{n+1}.\end{aligned} \tag{4.12}$$

Most of the terms were already estimated in Lemma 3.6. Using also the estimates provided in Lemma 4.2 and the approximation properties of f_h^n assumed in (2.12), we obtain the first bound. Using the product rule (4.1) several times, by the same strategy, we arrive at the second bound for $\partial_\tau\delta_h^{n+1}$. □

With this preparation, we can show that estimates (4.5) and (4.6) hold for $n=2$ and thus complete the induction basis.

Lemma 4.8. *Let the assumptions of Theorem 2.3 hold, and let the initial values be defined by (2.10). Then, the error e_h^2 satisfies the following bounds*

$$\|\partial_\tau^2 e_h^2\|_{L^2(\Omega)} + \|\partial_\tau\nabla e_h^2\|_{L^2(\Omega)} + \|\Delta_h e_h^2\|_{L^2(\Omega)} \lesssim \tau + h^k\,,$$

with a constant independent of h, τ, and β.

Proof. We employ the estimate (4.11) in Lemma 4.6 and use $n=1$ in (4.9) to obtain

$$\begin{aligned}\|e_h^2\|_{L^2(\Omega)} + \tau\|\nabla e_h^2\|_{L^2(\Omega)} &\lesssim \|\mathcal{R}^1 e_h^2\|_{L^2(\Omega)} \\ &\lesssim \|e_h^1\|_{L^2(\Omega)} + \|e_h^0\|_{L^2(\Omega)} + \tau\beta\|\Delta_h\partial_\tau e_h^1\|_{L^2(\Omega)} \\ &\qquad + \tau\|\nabla\partial_\tau e_h^1\|_{L^2(\Omega)} + \tau^2\|\delta_h^2\|_{L^2(\Omega)} \\ &\lesssim \tau^2\big(\tau + h^k\big),\end{aligned}$$

where we have used $e_h^0 = 0$, Lemma 4.4, and for estimating the defect, Lemma 4.2 with $0 \le m \le 2$. The first two terms can then be bounded using

$$\begin{aligned}\|\partial_\tau\nabla e_h^2\|_{L^2(\Omega)} &\le \tau^{-1}\big(\|\nabla e_h^2\|_{L^2(\Omega)} + \|\nabla e_h^1\|_{L^2(\Omega)}\big), \\ \|\partial_\tau^2 e_h^2\|_{L^2(\Omega)} &\le \tau^{-2}\big(\|e_h^2\|_{L^2(\Omega)} + 2\|e_h^1\|_{L^2(\Omega)} + \|e_h^0\|_{L^2(\Omega)}\big).\end{aligned}$$

Using the inverse estimate (3.4) and the CFL condition (2.11), we additionally have

$$\|\Delta_h e_h^2\|_{L^2(\Omega)} \lesssim h^{-2}\|e_h^2\|_{L^2(\Omega)} \lesssim \tau^{-2}\|e_h^2\|_{L^2(\Omega)},$$

and conclude the desired bound. □

We thus conclude that estimate (4.5) holds for $n=2$. Estimates (4.6) for $n=2$ then directly follow by exploiting the CFL condition in (2.11). Altogether, we conclude that the statement of Proposition 4.3 holds for $n=2$.

4.2. **Completing the induction step.** We next perform the induction step needed to prove estimates (4.5) and (4.6). Note that by Lemma 4.6, we have already shown the existence of u_h^{n+1}. To prove (4.5), we proceed similarly to Propositions 3.4 and 3.5 in two testing steps. First, we test the equation for e_h^{j+1} with $-\Delta_h e_h^{j+1}$ in Proposition 4.9, and then test the discretely differentiated version with $\partial_\tau^2 e_h^{j+1}$ in Proposition 4.10.

Proposition 4.9. *Let the assumptions of Theorem 2.3 hold. If the assertions of Proposition 4.3 hold up to n, then*

$$\begin{aligned}
&\tau \sum_{j=1}^{n} \|\Delta_h e_h^{j+1}\|^2_{L^2(\Omega)} + \beta \|\Delta_h e_h^{n+1}\|^2_{L^2(\Omega)} \\
&\lesssim \beta \|\Delta_h e_h^1\|^2_{L^2(\Omega)} + \tau \sum_{j=1}^{n} \big(\|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)} + \|\delta_h^{j+1}\|^2_{L^2(\Omega)} \big),
\end{aligned} \tag{4.13}$$

with constants independent of h, τ, n and β.

Proof. As announced, we test the error equation for e_h^{j+1} with $\varphi_h = -\Delta_h e_h^{j+1}$ to obtain

$$\begin{aligned}
&c^2 \|\Delta_h e_h^{j+1}\|^2_{L^2(\Omega)} + \beta (\Delta_h \partial_\tau e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} \\
&= ((1+\kappa \partial_\tau u_h^j) \partial_\tau^2 e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} + (\ell \nabla u_h^j \cdot \nabla \partial_\tau e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} + (\delta_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)}.
\end{aligned}$$

Note that since the assertions of Proposition 4.3 hold up to n, we can rely on the uniform bounds stated in Lemma 4.5. Thus, summing from 1 to n, using Lemma 4.1 as well as Young's inequality and the uniform bounds in Lemma 4.5, leads to estimate (4.13). □

We next need a discretely differentiated version of the error equation (4.8), analogously to (3.1). We use the discrete product rule (4.1) to obtain

$$\begin{aligned}
&((1+\kappa \partial_\tau u_h^n) \partial_\tau^3 e_h^{n+1} + \kappa \partial_\tau^2 u_h^n \partial_\tau^2 e_h^n - c^2 \Delta_h \partial_\tau e_h^{n+1} - \beta \Delta_h \partial_\tau^2 e_h^{n+1} \\
&\quad + \ell \nabla u_h^n \cdot \nabla \partial_\tau^2 e_h^{n+1} + \ell \nabla \partial_\tau u_h^n \cdot \nabla \partial_\tau e_h^n, \varphi_h)_{L^2(\Omega)} = (\partial_\tau \delta_h^{n+1}, \varphi_h)_{L^2(\Omega)},
\end{aligned} \tag{4.14}$$

for $n \geq 2$. Further, we introduce the notation

$$\|a_h^j\|_{\ell^\infty(1,n,L^2(\Omega))} \coloneqq \max_{j=1,\dots,n} \|a_h^j\|_{L^2(\Omega)},$$

which allows us to formulate the next proposition.

Proposition 4.10. *Let the assumptions of Theorem 2.3 hold. If the assertions of Proposition 4.3 hold for up to n, then for any $\alpha > 0$ it holds*

$$\begin{aligned}
&\|\partial_\tau^2 e_h^{n+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \nabla e_h^{n+1}\|^2_{L^2(\Omega)} + \tau\beta \sum_{j=2}^{n} \|\nabla \partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} \\
&\lesssim \alpha \|\partial_\tau^2 e_h^j\|^2_{\ell^\infty(2,n+1,L^2(\Omega))} + \|\partial_\tau^2 e_h^2\|^2_{L^2(\Omega)} + \|\partial_\tau \nabla e_h^2\|^2_{L^2(\Omega)} \\
&\quad + \tau \sum_{j=2}^{n} \big(\|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \delta_h^{j+1}\|^2_{L^2(\Omega)} \big),
\end{aligned} \tag{4.15}$$

with constants independent of h, τ, n and β.

Proof. We test the discretely differentiated error for e_h^{j+1} with $\varphi_h = \partial_\tau^2 e_h^{j+1}$ to obtain

$$\begin{aligned}
&((1+\kappa \partial_\tau u_h^j) \partial_\tau^3 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)} + c^2 (\nabla \partial_\tau e_h^{j+1}, \nabla \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)} + \beta \|\nabla \partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} \\
&\leq \kappa |(\partial_\tau^2 u_h^j \partial_\tau^2 e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| + \ell |(\nabla u_h^j \cdot \nabla \partial_\tau^2 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| \\
&\quad + \ell |(\nabla \partial_\tau u_h^j \cdot \nabla \partial_\tau e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| + |(\partial_\tau \delta_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| \\
&\lesssim \|\partial_\tau^2 e_h^j\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \delta_h^{j+1}\|^2_{L^2(\Omega)} + |\ell (\nabla u_h^j \cdot \nabla \partial_\tau^2 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}|,
\end{aligned}$$

where we have also used the uniform bounds on u_h^n stated in Lemma 4.5 in the last line. We sum these inequalities from $j = 2, \dots, n$ and use Lemma 4.1 to conclude that

$$
\begin{aligned}
&\|\partial_\tau^2 e_h^{n+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \nabla e_h^{n+1}\|^2_{L^2(\Omega)} + \tau \sum_{j=2}^{n} \beta \|\nabla \partial_\tau^2 e_h^{n+1}\|^2_{L^2(\Omega)} \\
&\lesssim \tau \sum_{j=2}^{n} \big(\|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \delta_h^{j+1}\|^2_{L^2(\Omega)}\big) + \|\partial_\tau^2 e_h^2\|^2_{L^2(\Omega)} \\
&\qquad + \|\partial_\tau \nabla e_h^2\|^2_{L^2(\Omega)} + \tau \sum_{j=2}^{n} |(\nabla u_h^j \cdot \nabla \partial_\tau^2 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}|.
\end{aligned} \tag{4.16}
$$

It remains to bound the last term. To this end, we use the expansion from (4.10) to obtain

$$
\begin{aligned}
&\tau \sum_{j=2}^{n} |(\nabla u_h^j \cdot \nabla \partial_\tau^2 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| \\
&\lesssim \tau \sum_{j=2}^{n} \big(\|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)}\big) + h^{-1-d/6} \tau \sum_{j=2}^{n} \|\Delta_h e_h^j\|_{L^2(\Omega)} \|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Since the assertions of Proposition 4.3 hold up to n, by the C_0 bounds in (4.6) and Young's inequality, we have

$$
\begin{aligned}
&h^{-1-d/6} \tau \sum_{j=2}^{n} \|\Delta_h e_h^j\|_{L^2(\Omega)} \|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} \\
&\lesssim \|\partial_\tau^2 e_h^j\|_{\ell^\infty(2,\, n+1,\, L^2(\Omega))} h^{-1-d/6} \Big(\tau \sum_{j=2}^{n} \|\Delta_h e_h^j\|^2_{L^2(\Omega)}\Big)^{1/2} \Big(\tau \sum_{j=2}^{n} \|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)}\Big)^{1/2} \\
&\lesssim \alpha \|\partial_\tau^2 e_h^j\|^2_{\ell^\infty(2,\, n+1,\, L^2(\Omega))} + \tau \sum_{j=2}^{n} \|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $\alpha > 0$ can be chosen arbitrarily. Employing this bound in (4.16) leads to (4.15). $\square$

We now combine all previous results in this section to arrive at the statement of Proposition 4.3.

Proof of Proposition 4.3. Recall that the statement of Proposition 4.3 holds for $n = 2$ by the results of Section 4.1. We complete the induction step by showing the existence and proving the estimates (4.5) and (4.6). Since the assertions in Proposition 4.3 are assumed to hold up to n, by Lemma 4.6 we have existence of the solution u_h^{n+1} of (2.9). In addition, by Proposition 4.9 and Proposition 4.10, we have

$$
\begin{aligned}
&\|\partial_\tau^2 e_h^{n+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \nabla e_h^{n+1}\|^2_{L^2(\Omega)} + \tau \sum_{j=1}^{n} \|\Delta_h e_h^{j+1}\|^2_{L^2(\Omega)} + \beta\tau \sum_{j=2}^{n} \|\nabla \partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} \\
&\qquad\qquad + \beta \|\Delta_h e_h^{n+1}\|^2_{L^2(\Omega)} \\
&\lesssim \alpha \|\partial_\tau^2 e_h^j\|^2_{\ell^\infty(2,\, n+1,\, L^2(\Omega))} + \tau \sum_{j=1}^{n} \big(\|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)}\big) \\
&\quad + \|\partial_\tau^2 e_h^2\|^2_{L^2(\Omega)} + \|\partial_\tau \nabla e_h^2\|^2_{L^2(\Omega)} + \beta \|\Delta_h e_h^1\|^2_{L^2(\Omega)} + \tau \sum_{j=1}^{n} \big(\|\delta_h^{j+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \delta_h^{j+1}\|^2_{L^2(\Omega)}\big)
\end{aligned}
$$

with the hidden constant independent of h, τ, n, and β. From here using Lemmas 4.4, 4.7, and 4.8, together with

$$\|e_h^j\|_{L^2(\Omega)} \lesssim \|\nabla e_h^j\|_{L^2(\Omega)}, \qquad \|\nabla e_h^j\|^2_{L^2(\Omega)} \leq T\tau \sum_{k=1}^{j} \|\nabla \partial_\tau e_h^j\|^2_{L^2(\Omega)},$$

due to $e_h^0 = 0$, we infer for $m = n+1$

$$\begin{aligned} &\|\partial_\tau^2 e_h^m\|^2_{L^2(\Omega)} + \|\partial_\tau \nabla e_h^m\|^2_{L^2(\Omega)} + \tau \sum_{j=1}^{m-1} \|\Delta_h e_h^{j+1}\|^2_{L^2(\Omega)} + \beta\tau \sum_{j=2}^{m-1} \|\nabla \partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} \\ &\qquad\qquad + \beta \|\Delta_h e_h^m\|^2_{L^2(\Omega)} \\ &\lesssim \alpha \|\partial_\tau^2 e_h^j\|^2_{\ell^\infty(2,m,L^2(\Omega))} + \big(\tau + h^k\big)^2 + \tau \sum_{j=1}^{m-1} \big(\|\partial_\tau^2 e_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau e_h^{j+1}\|^2_{L^2(\Omega)} \big) \end{aligned}$$

for any $\alpha > 0$. It is straightforward to prove that analogous estimates hold for $m \leq n$. Therefore, taking the maximum of this inequality over $m = 2, \dots, n+1$ and choosing α sufficiently small, together with a Grönwall argument yields the error estimate stated in (4.5) with a constant independent of n.

We then use the bound in (4.5), which is uniform in n, and the CFL condition in (2.11) to obtain estimates in (4.6). This step closes the induction argument. □

The statement of Theorem 2.3 now follows immediately.

Proof of Theorem 2.3. Using the embedding in (3.3) and the best approximation properties of the Ritz projection in (3.2), we obtain the claimed estimate. □

4.3. **The inviscid limit of the fully discrete solution.** We next study the limiting behavior of the fully discrete problem as $\beta \to 0$ and prove Theorem 2.4. Similarly to Section 3.3, we emphasize the β dependence of the fully discrete solution by using the notation $u^n_{h,\beta}$ when $\beta \in (0, \bar\beta]$ and $u^n_{h,\beta=0}$ in the inviscid case $\beta = 0$.

We define the quantity

$$\bar u_h^n = u^n_{h,\beta=0} - u^n_{h,\beta},$$

and estimate it to arrive at Theorem 2.4.

Proof of Theorem 2.4. By subtracting the equation for $u^{j+1}_{h,\beta}$ from the equation for $u^{j+1}_{h,\beta=0}$, we conclude that $\bar u_h^{j+1}$ satisfies

$$\begin{aligned} &((1 + \kappa \partial_\tau u^j_{h,\beta=0}) \partial_\tau^2 \bar u_h^{j+1} - c^2 \Delta_h \bar u_h^{j+1}, \varphi_h)_{L^2(\Omega)} \\ &= (-\kappa \partial_\tau \bar u_h^j \partial_\tau^2 u^{j+1}_{h,\beta} - \ell \nabla u^j_{h,\beta=0} \cdot \nabla \partial_\tau \bar u_h^{j+1} - \ell \nabla \bar u_h^j \cdot \nabla \partial_\tau u^{j+1}_{h,\beta} - \beta \Delta_h \partial_\tau u^{j+1}_{h,\beta}, \varphi_h)_{L^2(\Omega)} \end{aligned}$$

for $j = 1, \dots, n$. We test this problem with $\varphi_h = \partial_\tau \bar u_h^{j+1}$, sum from $j = 1, \dots, n$, and use Lemma 4.1 to obtain

$$\begin{aligned} &\|\partial_\tau \bar u_h^{n+1}\|^2_{L^2(\Omega)} + \|\nabla \bar u_h^{n+1}\|^2_{L^2(\Omega)} \\ &\lesssim \tau \sum_{j=1}^{n} \big(\|\partial_\tau \bar u_h^{j+1}\|^2_{L^2(\Omega)} + \|\nabla \bar u_h^{j+1}\|^2_{L^2(\Omega)} \big) + |\ell (\nabla u^n_{h,\beta=0} \cdot \nabla \partial_\tau \bar u_h^{j+1}, \partial_\tau \bar u_h^{j+1})_{L^2(\Omega)}| \\ &\qquad + |\beta\tau \sum_{j=1}^{n} (\nabla \partial_\tau u^{j+1}_{h,\beta}, \nabla \partial_\tau \bar u_h^{j+1})_{L^2(\Omega)}|, \end{aligned} \tag{4.17}$$

where we have also used the uniform bounds on $u_{h,\beta}^{n+1}$ guaranteed by Lemma 4.5. We proceed to estimate the right-hand side terms. Using the expansion in (4.10), we estimate

$$
\begin{aligned}
&\tau \sum_{j=1}^{n} |\ell(\nabla u_{h,\beta=0}^{n} \cdot \nabla \partial_\tau \bar{u}_h^{j+1}, \partial_\tau \bar{u}_h^{j+1})_{L^2(\Omega)}| \\
&\lesssim \tau \sum_{j=1}^{n} \|\partial_\tau \bar{u}_h^{j+1}\|^2_{L^2(\Omega)} + \|\partial_\tau \bar{u}_h^{j}\|_{\ell^\infty(2,\,n+1,\,L^2(\Omega))} h^{-1-d/6} \tau \sum_{j=1}^{n} \|\Delta_h e_{h,\beta=0}^{j}\|_{L^2(\Omega)} \|\partial_\tau \bar{u}_h^{j+1}\|_{L^2(\Omega)} \\
&\lesssim \tau \sum_{j=1}^{n} \|\partial_\tau \bar{u}_h^{j+1}\|^2_{L^2(\Omega)} + \alpha_1 \|\partial_\tau \bar{u}_h^{j}\|_{\ell^\infty(2,\,n+1,\,L^2(\Omega))}
\end{aligned}
$$

for any $\alpha_1 > 0$. It remains to bound the term involving β in (4.17) to set up a Grönwall argument. To this end, we employ the summation by parts formula (4.2) to obtain

$$
\begin{aligned}
&|\beta\tau \sum_{j=1}^{n} (\nabla \partial_\tau u_{h,\beta}^{j+1}, \nabla \partial_\tau \bar{u}_h^{j+1})_{L^2(\Omega)}| \\
&= |\beta\big((\nabla \partial_\tau u_{h,\beta}^{n+1}, \nabla \bar{u}_h^{n+1})_{L^2(\Omega)} - (\nabla \partial_\tau u_{h,\beta}^{1}, \nabla \bar{u}_h^{1})_{L^2(\Omega)}\big) - \beta\tau \sum_{j=1}^{n} (\partial_\tau^2 \nabla u_{h,\beta}^{j+1}, \nabla \bar{u}_h^{j+1})_{L^2(\Omega)}| \\
&\lesssim \beta^2 \big(\|\nabla \partial_\tau u_{h,\beta}^{n+1}\|^2_{L^2(\Omega)} + \|\nabla \partial_\tau u_{h,\beta}^{1}\|^2_{L^2(\Omega)} + \tau \sum_{j=1}^{n} \|\partial_\tau^2 \nabla u_{h,\beta}^{j+1}\|^2_{L^2(\Omega)}\big) + \|\nabla \bar{u}_h^{1}\|^2_{L^2(\Omega)} \\
&\qquad + \alpha_2 \|\nabla \bar{u}_h^{n+1}\|^2_{L^2(\Omega)} + \tau \sum_{j=1}^{n} \|\nabla \bar{u}_h^{j+1}\|^2_{L^2(\Omega)}
\end{aligned}
$$

for any $\alpha_2 > 0$, where we have also relied on the C_0 bounds given in (4.6). Similarly to the reasoning in Section 3, by Lemma 4.5 we have uniform bounds for the first two terms multiplied with β^2 on the right-hand side, and we can proceed similarly to (3.29) to bound the third term. Further, by (2.10), it holds

$$
\|\nabla \bar{u}_h^1\|_{L^2(\Omega)} = \frac{\beta\tau^2}{2} \|\mathrm{R}_h (1+\kappa v_0)^{-1} \Delta v_0\|_{L^2(\Omega)} \lesssim C(\|u\|_{\mathcal{U}}) \cdot \tau^2 \beta
$$

and we can conclude by reducing α_2 that

$$
\begin{aligned}
&\|\partial_\tau \bar{u}_h^{n+1}\|^2_{L^2(\Omega)} + \|\nabla \bar{u}_h^{n+1}\|^2_{L^2(\Omega)} \\
&\leq C\beta^2 + C\alpha_1 \|\partial_\tau \bar{u}_h^{j}\|^2_{\ell^\infty(2,\,n+1,\,L^2(\Omega))} + C\tau \sum_{j=1}^{n} \big(\|\partial_\tau \bar{u}_h^{j}\|^2_{L^2(\Omega)} + \|\nabla \bar{u}_h^{j}\|^2_{L^2(\Omega)}\big).
\end{aligned}
$$

We now take the maximum of this inequality over $n = 2, \ldots, N+1$, and for small enough α_1 apply a Grönwall argument to obtain

$$
\max_{n=1,\ldots,N+1} \|\partial_\tau u_{h,\beta}^{n} - \partial_\tau u_{h,\beta=0}^{n}\|_{L^2(\Omega)} + \max_{n=1,\ldots,N+1} \|\nabla(u_{h,\beta}^{n} - u_{h,\beta=0}^{n})\|_{L^2(\Omega)} \leq C\beta,
$$

as claimed. □

We see that, under the assumptions of Theorem 2.3, also the fully discrete problem preserves the asymptotic behavior of the exact and semi-discrete solutions as $\beta \to 0$.

5. Non-robust estimates for linear finite elements

In this final section, we extend the results presented in Section 2 to the case of linear finite elements, i.e., $k = 1$. We can prove the qualitatively same error bounds with constants that do not depend on the damping parameter $\beta > 0$, as long as we couple the discretization parameters with the damping parameter correctly.

5.1. **Semi-discretization.** We first consider the error bound for the semi discretization in space, and state a variant of Theorem 2.1 that takes $\beta > 0$ into account. We first state our theorem, and devote the rest of this section to its proof. Since several arguments are unchanged compared to Section 3, we only present the key estimates here.

Theorem 5.1 (Non-robust finite element estimates)**.** *Let the assumptions of Theorem 2.1 hold, but replace the assumptions on k and β with $k \geq 1$ and $\beta > 0$, satisfying the relation*

$$h^{k-d/6-2\varepsilon} \leq C_1\sqrt{\beta}, \tag{5.1}$$

for some C_1, $\varepsilon > 0$ which are independent of h and β. Then there exists $h_0 > 0$ and a constant $C > 0$, independent of h and β, such that for all $h \leq h_0$, the following error bound holds:

$$\|\partial_t^2 u(t) - \partial_t^2 u_h(t)\|^2_{L^2(\Omega)} + \|\nabla\partial_t u(t) - \nabla\partial_t u_h(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s) - \nabla u_h(s)\|^2_{L^6(\Omega)}\,\mathrm{d}s \leq Ch^{2k} \tag{5.2}$$

for all $t \in [0, T]$.

The key idea of the proof remains the same as before, and hence analogously to Section 3, we work on the time interval $[0, t^*_{h,\beta}]$ with

$$\begin{aligned} t^*_{h,\beta} \coloneqq \sup \Big\{ t \in (0, T] \mid \text{ a unique solution } u_h \in H^3(0, t; V_h) \text{ of (2.3) exists, and}\\ \beta^{-1/2}h^{-d/6-\varepsilon}\|\partial_t^2 e_h(s)\|_{L^2(\Omega)} \leq C_0,\\ \beta^{-1/2}h^{-d/6-\varepsilon}\|\nabla\partial_t e_h(s)\|_{L^2(\Omega)} \leq C_0,\\ h^{-d/6-\varepsilon}\|\Delta_h e_h(s)\|_{L^2(\Omega)} \leq C_0 \text{ for all } s \in [0, t] \Big\}, \end{aligned} \tag{5.3}$$

for some fixed $C_0 > 0$ and ε as in (5.1). We then conduct the error analysis on this interval, with the aim of later extending $t^*_{h,\beta}$ to T, analogously to before. By arguing as in Lemma 3.1, we can prove that $t^*_{h,\beta} > 0$, as well as obtain the correct estimates for $e_h(0)$, $\partial_t e_h(0)$, and $\partial_t^2 e_h(0)$. We omit those details here.

Lemma 5.2. *Let the assumptions of Theorem 5.1 hold. Then, we have*

$$1 + \kappa\partial_t u_h \geq \gamma > 0, \quad (x, t) \in \Omega \times [0, t^*_h], \tag{5.4}$$

where γ does not depend on h, β, or $t^*_{h,\beta}$.

Proof. Using the stability properties of the Ritz projection stated in (3.2), we obtain

$$\begin{aligned} \|\partial_t u_h(t)\|_{L^\infty(\Omega)} &\lesssim \|\partial_t u(t)\|_{L^\infty(\Omega)} + \|(\mathrm{I} - \mathrm{R}_h)\partial_t u(t)\|_{L^\infty(\Omega)} + h^{-d/6}\|\nabla\partial_t e_h(t)\|_{L^2(\Omega)}\\ &\leq \|\partial_t u(t)\|_{L^\infty(\Omega)} + Ch^k + C\beta^{1/2}h^\varepsilon \end{aligned}$$

for all $t \in [0, t^*_h]$. Hence we have the uniform lower bound in (5.4) that guarantees non-degeneracy as well as uniform boundedness of $\|\partial_t u_h\|_{L^\infty(L^\infty(\Omega))}$. □

We are now ready to derive the relevant estimates and prove the error bound in (5.2). Before, we briefly comment on the changes compared to Section 3.

By the bounds in Propositions 3.4 and 3.5, one obtains

$$\beta \int_0^t \|\nabla\partial_t^2 e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \beta\|\Delta_h e_h(t)\|^2_{L^2(\Omega)} \leq Ch^{2k}. \tag{5.5}$$

However, in order to stay uniform in β, these bounds were not exploited in the analysis of Section 3. If we consider now fixed $\beta > 0$, this enables us to employ (5.5) while paying with inverse powers of β via Young's inequality. The coupling condition in (5.3) allows us to close the proof even for $k = 1$. More details can be found in the following proof.

Proof of Theorem 5.1. Below whenever the temporal argument is skipped, we assume that the given (in)equality holds for all $t \in [0, t^*_{h,\beta}]$. We proceed in two steps.

(a) Starting from the estimate in (3.17) and using the uniform lower bound in (5.4), we obtain

$$\begin{aligned}
&\partial_t \|(1+\kappa\partial_t u_h)^{1/2}\partial_t^2 e_h\|^2_{L^2(\Omega)} + c^2\partial_t\|\nabla\partial_t e_h\|^2_{L^2(\Omega)} + \beta\|\nabla\partial_t^2 e_h\|^2_{L^2(\Omega)}\\
&\lesssim \|\partial_t^2 u_h \partial_t^2 e_h\|^2_{L^2(\Omega)} + \ell|(\nabla\partial_t u_h\cdot\nabla\partial_t e_h, \partial_t^2 e_h)_{L^2(\Omega)}| + \ell|(\nabla u_h\cdot\nabla\partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}|\\
&\qquad + \|\partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\nabla\partial_t e_h\|^2_{L^2(\Omega)} + \|\partial_t\delta_h\|^2_{L^2(\Omega)}
\end{aligned}$$

and have to treat the terms involving u_h separately. Now exploiting $\beta > 0$, we estimate

$$\begin{aligned}
\|\partial_t^2 u_h\partial_t^2 e_h\|^2_{L^2(\Omega)} &\lesssim \|\partial_t^2 e_h\partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\partial_t^2 \mathrm{R}_h u\,\partial_t^2 e_h\|^2_{L^2(\Omega)}\\
&\lesssim \|\partial_t^2 e_h\|^2_{L^3(\Omega)}\|\partial_t^2 e_h\|^2_{L^6(\Omega)} + \|\partial_t^2 e_h\|^2_{L^2(\Omega)}\\
&\lesssim \big(\beta^{-1}h^{-d/3}\|\partial_t^2 e_h\|^2_{L^2(\Omega)}\big)\,\beta\|\partial_t^2\nabla e_h\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h\|^2_{L^2(\Omega)}\\
&\le \frac{\beta}{4}\|\partial_t^2\nabla e_h\|^2_{L^2(\Omega)} + C\|\partial_t^2 e_h\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have used the C_0 bounds in (5.3) in the last step. Next, we estimate

$$\begin{aligned}
&|(\nabla\partial_t u_h\cdot\nabla\partial_t e_h, \partial_t^2 e_h)_{L^2(\Omega)}|\\
&\lesssim \|\nabla\partial_t e_h\|^2_{L^2(\Omega)}\|\partial_t^2 e_h\|_{L^\infty(\Omega)} + \|\nabla\partial_t \mathrm{R}_h\|_{L^\infty(\Omega)}\|\nabla\partial_t e_h\|_{L^2(\Omega)}\|\partial_t^2 e_h\|_{L^2(\Omega)}\\
&\lesssim \beta^{-1/2}\|\nabla\partial_t e_h\|^2_{L^2(\Omega)}\beta^{1/2}h^{-d/6}\|\nabla\partial_t^2 e_h\|_{L^2(\Omega)} + \|\nabla\partial_t e_h\|_{L^2(\Omega)}\|\partial_t^2 e_h\|_{L^2(\Omega)}.
\end{aligned}$$

We further estimate the first term in the last line above by using the C_0 bounds in (5.3) for $h \le h_0$:

$$\begin{aligned}
&\beta^{-1/2}h^{-d/6}\|\nabla\partial_t e_h\|^2_{L^2(\Omega)}\beta^{1/2}\|\nabla\partial_t^2 e_h\|_{L^2(\Omega)}\\
&= \big(\beta^{-1/2}h^{-d/6}\|\nabla\partial_t e_h\|_{L^2(\Omega)}\big)\beta^{1/2}\|\nabla\partial_t^2 e_h\|_{L^2(\Omega)}\|\nabla\partial_t e_h\|_{L^2(\Omega)}\\
&\le \frac{\beta}{4}\|\nabla\partial_t^2 e_h\|^2_{L^2(\Omega)} + C\|\nabla\partial_t e_h\|^2_{L^2(\Omega)}.
\end{aligned}$$

Next, proceeding as in (3.21) results in

$$|(\nabla u_h\cdot\nabla\partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}| \lesssim \|\partial_t^2 e_h\|^2_{L^2(\Omega)} + (\nabla e_h\cdot\nabla\partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)},$$

and further with the discrete embedding (3.3) we have

$$\begin{aligned}
|(\nabla e_h\cdot\nabla\partial_t^2 e_h, \partial_t^2 e_h)_{L^2(\Omega)}| &\le \|\nabla e_h\|_{L^\infty(\Omega)}\|\nabla\partial_t^2 e_h\|_{L^2(\Omega)}\|\partial_t^2 e_h\|_{L^2(\Omega)}\\
&\lesssim \|\nabla e_h\|_{L^6(\Omega)}\beta^{1/2}\|\nabla\partial_t^2 e_h\|_{L^2(\Omega)}\big(\beta^{-1/2}h^{-d/6}\|\partial_t^2 e_h\|_{L^2(\Omega)}\big)\\
&\le CC_0^2h^{2\varepsilon}\|\Delta_h e_h\|^2_{L^2(\Omega)} + \frac{\beta}{4}\|\nabla\partial_t^2 e_h\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have used the C_0 bounds in (5.3). For $h \leq h_0$, we have thus again derived (3.16) with constants independent of h and β.

(b) Testing the error equation (3.14) with $\varphi_h = -\Delta e_h$ yields with Young's inequality

$$c^2\|\Delta_h e_h\|^2_{L^2(\Omega)} + \beta\partial_t\|\Delta_h e_h\|^2_{L^2(\Omega)}$$
$$\leq \frac{c^2}{4}\|\Delta_h e_h\|^2_{L^2(\Omega)} + C\left(\|\partial_t e_h\|^2_{L^2(\Omega)} + \|\partial_t^2 e_h\|^2_{L^2(\Omega)} + \|\delta_h\|^2_{L^2(\Omega)} + (\ell\nabla u_h\cdot\nabla\partial_t e_h, \Delta_h e_h)_{L^2(\Omega)}\right);$$

cf. (3.24). The last term is here estimated via

$$\begin{aligned}
&|(\ell\nabla u_h\cdot\nabla\partial_t e_h, \Delta_h e_h)_{L^2(\Omega)}| \\
&\leq |(\ell\nabla e_h\cdot\nabla\partial_t e_h, \Delta_h e_h)_{L^2(\Omega)}| + |(\ell\nabla \mathrm{R}_h u\cdot\nabla\partial_t e_h, \Delta_h e_h)_{L^2(\Omega)}| \\
&\lesssim \left(h^{-d/6}\|\Delta_h e_h\|_{L^2(\Omega)}\right)\|\nabla\partial_t e_h\|_{L^2(\Omega)}\|\Delta_h e_h\|_{L^2(\Omega)} + \|\nabla\partial_t e_h\|^2_{L^2(\Omega)} + \frac{c^2}{4}\|\Delta_h e_h\|^2_{L^2(\Omega)} \\
&\lesssim \|\nabla\partial_t e_h\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\Delta_h e_h\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have relied in the last step on the last C_0 bound in (5.3). We absorb the $\|\Delta_h e_h\|^2_{L^2(\Omega)}$ terms and conclude as before by Grönwall's inequality that

$$\begin{aligned}
\|\partial_t^2 e_h(t)\|^2_{L^2(\Omega)} + \|\nabla\partial_t e_h(t)\|^2_{L^2(\Omega)} + \beta\|\Delta_h e_h(t)\|^2_{L^2(\Omega)} + \beta\int_0^t \|\nabla\partial_t^2 e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
+ \int_0^t \|\Delta_h e_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \leq Ch^{2k}
\end{aligned}$$

on $[0, t^*_{h,\beta}]$. Thanks to this uniform bound, we can reason as in Section 3 to close again the arguments with (5.1) and obtain $t^*_{h,\beta} = T$. □

5.2. **Non-robust estimates for a full discretization.** Our last main result for the full discretization is a variant of Theorem 2.3 in the case of fixed $\beta > 0$. The strategy of the proof is similar to the one on Section 4. In order to compensate the inverse powers of β, we have to assume the following coupling:

$$\tau \leq C_1\beta^{1/2}h^{d/6+2\varepsilon}, \qquad h^{k-d/6-2\varepsilon} \leq C_1\beta^{1/2}, \tag{5.6}$$

for constants C_1, $\varepsilon > 0$ which are independent of h, τ, and β.

Theorem 5.3 (Non-robust fully discrete error bounds)**.** *Let the assumptions of Theorem 2.3 hold, but replace the conditions on k and β with $k \geq 1$ and $0 < \beta \leq \bar{\beta}$. Under the coupling conditions (5.6), for $h \leq h_0$ and $\tau \leq \tau_0$, it holds*

$$\begin{aligned}
\|\partial_t^2 u(t_n) - \partial_\tau^2 u_h^n\|^2_{L^2(\Omega)} + \|\nabla\partial_t u(t_n) - \nabla\partial_\tau u_h^n\|^2_{L^2(\Omega)} \\
+ \tau\sum_{j=1}^n \|\nabla u(t_n) - \nabla u_h^n\|^2_{L^6(\Omega)} \leq C\big(\tau + h^k\big)^2,
\end{aligned}$$

where the constant $C > 0$ is independent of h, τ, and β.

In order to prove the result, we set up an induction argument as before, and show that for $n = 2, \ldots, N+1$ the solution u_h^n exists, and similarly to (4.5), it holds

$$\|\partial_\tau^2 e_h^n\|^2_{L^2(\Omega)} + \|\nabla\partial_\tau e_h^n\|^2_{L^2(\Omega)} + \beta\|\Delta_h e_h^{n+1}\|^2_{L^2(\Omega)} + \tau\sum_{j=1}^n \|\Delta_h e_h^j\|^2_{L^6(\Omega)} \leq C\big(\tau + h^k\big)^2, \tag{5.7}$$

as well as, analogously to (4.6),

$$\begin{aligned}
&\beta^{-1/2}h^{-d/6-\varepsilon}\|\partial_\tau^2 e_h^n\|_{L^2(\Omega)} \leq C_0,\\
&\beta^{-1/2}h^{-d/6-\varepsilon}\|\nabla\partial_\tau e_h^n\|_{L^2(\Omega)} \leq C_0,\\
&h^{-d/6-\varepsilon}\|\Delta_h e_h^n\|_{L^2(\Omega)} \leq C_0,\\
&\beta^{-1/2}h^{-d/6-\varepsilon}\big(\tau\sum_{j=1}^n \|\Delta_h e_h^j\|_{L^2(\Omega)}^2\big)^{1/2} \leq C_0,
\end{aligned} \tag{5.8}$$

with some constants C, $C_0>0$ that are independent of h, τ, n, and β and ε chosen as in (5.6).

Lemma 5.4. *Under the assumptions of Theorem 5.3, the assertions of Lemma 4.4, Lemma 4.6, Lemma 4.7, and Lemma 4.8 hold true, and in particular* (5.7) *and* (5.8) *hold for $n=2$.*

Proof. The bounds in Lemmas 4.4 and 4.8 directly follow from the conditions in (5.6). For the existence statement in Lemma 4.6, we estimate the term in (4.10) now via

$$\begin{aligned}
\tau|(\nabla u_h^n\cdot\nabla\varphi_h,\varphi_h)_{L^2(\Omega)}| &\lesssim \tau h^{-d/6}\|\Delta_h e_h^n\|_{L^2(\Omega)}\|\nabla\varphi_h\|_{L^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)} + \tau\|\varphi_h\|_{L^2(\Omega)}^2\\
&\leq \alpha\big(\|\varphi_h\|_{L^2(\Omega)}^2 + \tau^2\|\nabla\varphi_h\|_{L^2(\Omega)}^2\big)
\end{aligned}$$

for any $\alpha>0$, where we have used (5.8) in the last step. □

Further, we have the crucial result in the leading nonlinear term which prevents degeneracy of the problem also for $k=1$.

Lemma 5.5. *Let the assumptions of Theorem 5.1 hold. If the estimates* (5.7) *and* (5.8) *hold up to $n\geq 2$, then we have*

$$1+\kappa\partial_\tau u_h^j \geq \gamma > 0, \quad j=2,\dots,n, \tag{5.9}$$

where γ does not depend on h, τ β, or n.

Proof. Along the lines of Lemma 5.2, we have

$$\begin{aligned}
\|\partial_\tau u_h^j\|_{L^\infty(\Omega)} &\lesssim \|\partial_\tau \widehat{u}^j\|_{L^\infty(\Omega)} + \|(\mathrm{I}-\mathrm{R}_h)\partial_\tau\widehat{u}^j\|_{L^\infty(\Omega)} + h^{-d/6}\|\partial_t e_h\|_{L^6(\Omega)}\\
&\leq \|\partial_t u\|_{L^\infty(L^\infty(\Omega))} + Ch^k + C\beta^{1/2}h^\varepsilon
\end{aligned}$$

and hence the lower bound in (5.9) follows as well as boundedness of $\|\partial_\tau u_h\|_{L^\infty(\Omega)}$. □

With this result, we can prove the principal result on a non-robust fully discrete bound. As already explained in Section 5.1, the appearance of the inverse powers of β comes in by exploiting the following bounds from Proposition 4.9 and 4.10:

$$\beta\|\Delta_h e_h^{n+1}\|_{L^2(\Omega)}^2 + \tau\beta\sum_{j=2}^n\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 \leq C\big(\tau+h^k\big)^2,$$

and applying the relations in (5.6).

Proof of Theorem 5.3. We conduct the proof in two testing steps.

(a) We proceed as in Proposition 4.10 and test the differentiated error equation (4.14) with $\varphi_h = \partial_\tau^2 e_h^{j+1}$ to obtain

$$\begin{aligned}
&((1+\kappa\partial_\tau u_h^j)\partial_\tau^3 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)} + c^2(\nabla\partial_\tau e_h^{j+1}, \nabla\partial_\tau^2 e_h^{j+1})_{L^2(\Omega)} + \beta\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 \\
&\le \kappa|(\partial_\tau^2 u_h^j \partial_\tau^2 e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| + \ell|(\nabla u_h^j\cdot\nabla\partial_\tau^2 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| \\
&\qquad + \ell|(\nabla\partial_\tau u_h^j\cdot\nabla\partial_\tau e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| + \|\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 + \|\partial_\tau\delta_h^{j+1}\|_{L^2(\Omega)}^2,
\end{aligned}$$

and estimate the three terms separately. We use the C_0 bounds in (5.8) to conclude

$$\begin{aligned}
&\kappa(\partial_\tau^2 u_h^j \partial_\tau^2 e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)} \\
&\lesssim \big(\beta^{-1/2}h^{-d/6}\|\partial_\tau^2 e_h^j\|_{L^2(\Omega)}\big)\|\partial_\tau^2 e_h^j\|_{L^2(\Omega)}\beta^{1/2}\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)} + \|\partial_\tau^2 e_h^j\|_{L^2(\Omega)}^2 + \|\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 \\
&\le \frac{\beta}{4}\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 + C\|\partial_\tau^2 e_h^j\|_{L^2(\Omega)}^2 + C\|\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2,
\end{aligned}$$

and absorb the β term by the left-hand side β term. We sum from $j = 2,\dots,n$ and obtain by the expansion in (4.10)

$$\begin{aligned}
&\tau\sum_{j=2}^n |\ell(\nabla u_h^j\cdot\nabla\partial_\tau^2 e_h^{j+1}, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| \\
&\lesssim h^{-d/6}\beta^{-1/2}\Big(\tau\sum_{j=2}^n\|\Delta_h e_h^j\|_{L^2(\Omega)}^2\Big)^{1/2}\Big(\tau\beta\sum_{j=2}^n\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2\Big)^{1/2}\|\partial_\tau^2 e_h^j\|_{\ell^\infty(2,n+1,L^2(\Omega))} \\
&\qquad\qquad + \tau\sum_{j=2}^n\|\partial_\tau^2 e_h^j\|_{L^2(\Omega)}^2 \\
&\le \alpha_1\|\partial_\tau^2 e_h^j\|_{\ell^\infty(2,n+1,L^2(\Omega))} + \frac{\beta}{4}\tau\sum_{j=2}^n\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 + C\tau\sum_{j=2}^n\|\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2
\end{aligned}$$

for any $\alpha_1 > 0$ by the bounds in (5.8) for $h \le h_0$ and $\tau \le \tau_0$. Finally, we estimate

$$\begin{aligned}
|\ell(\nabla\partial_\tau u_h^j\cdot\nabla\partial_\tau e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| &\le |\ell(\nabla\partial_\tau e_h^j\cdot\nabla\partial_\tau e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| \\
&\quad + \|\nabla\partial_\tau e_h^{j+1}\|_{L^2(\Omega)}^2 + \|\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2
\end{aligned}$$

and with this, by the C_0 bounds in (5.8),

$$\begin{aligned}
|\ell(\nabla\partial_\tau e_h^j\cdot\nabla\partial_\tau e_h^j, \partial_\tau^2 e_h^{j+1})_{L^2(\Omega)}| &\lesssim \|\nabla\partial_\tau e_h^j\|_{L^2(\Omega)}\|\nabla\partial_\tau e_h^j\|_{L^2(\Omega)}h^{-d/6}\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)} \\
&\le \frac{\beta}{4}\|\nabla\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 + C\|\nabla\partial_\tau e_h^j\|_{L^2(\Omega)}^2.
\end{aligned}$$

Lemma 4.1 then yields (4.15).

(b) We then test (4.8) with $\varphi_h = -\Delta_h e_h^{j+1}$ to obtain

$$\begin{aligned}
&c^2\|\Delta_h e_h^{j+1}\|_{L^2(\Omega)}^2 + \beta(\Delta_h\partial_\tau e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} \\
&= ((1+\kappa\partial_\tau u_h^j)\partial_\tau^2 e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} + (\ell\nabla u_h^j\cdot\nabla\partial_\tau e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} + (\delta_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)} \\
&\le \frac{c^2}{4}\|\Delta_h e_h^{j+1}\|_{L^2(\Omega)}^2 + C\Big(\|\nabla\partial_\tau e_h^{j+1}\|_{L^2(\Omega)}^2 + \|\partial_\tau^2 e_h^{j+1}\|_{L^2(\Omega)}^2 + \|\delta_h^{j+1}\|_{L^2(\Omega)}^2 + |(\ell\nabla e_h^j\cdot\nabla\partial_\tau e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)}|
\end{aligned}$$

For the last term, we use the C_0 bound in (5.8) on $\|\Delta_h e_h^j\|_{L^2(\Omega)}$ to conclude that

$$\begin{aligned}
|(\ell\nabla e_h^j\cdot\nabla\partial_\tau e_h^{j+1}, \Delta_h e_h^{j+1})_{L^2(\Omega)}| &\lesssim h^{-d/6}\|\Delta_h e_h^j\|_{L^2(\Omega)}\|\nabla\partial_\tau e_h^j\|_{L^2(\Omega)}\|\Delta_h e_h^{j+1}\|_{L^2(\Omega)} \\
&\le C\|\nabla\partial_\tau e_h^j\|_{L^2(\Omega)}^2 + \frac{c^2}{4}\|\Delta_h e_h^{j+1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

We can absorb the $\|\Delta_h e_h^{j+1}\|^2_{L^2(\Omega)}$ terms by the left-hand side and reason as in the proof of Proposition 4.3 to arrive at (5.7). The coupling in (5.6) then implies (5.8), and the claim follows by induction. $\square$

Appendix A. Estimates for discrete derivatives

In order to keep the presentation self-contained, we include the proof of Lemma 4.2 here in the appendix.

Proof of Lemma 4.2. For the sake of readability, we just consider a generic norm a and assume without loss of generality $\ell_1 \leq \ell_2$ to obtain

$$\|\partial_\tau^{m-\ell_1}\partial_t^{\ell_1}\widehat{u}^n - \partial_\tau^{m-\ell_2}\partial_t^{\ell_2}\widehat{u}^n\| = \|\partial_\tau^{m-\ell_2}\big(\partial_\tau^{\ell_2-\ell_1}\partial_t^{\ell_1}\widehat{u}^n - \partial_t^{\ell_2}\widehat{u}^n\big)\|.$$

Applying the fundamental theorem of calculus ℓ-times gives

$$\partial_\tau^{\ell}\widehat{u}^n = \frac{1}{\tau^\ell}\int_0^\tau \dots \int_0^\tau \partial_t^\ell u(t_{n-\ell} + \sigma_1 + \ldots + \sigma_\ell)\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\ell\,,$$

and similarly

$$\partial_\tau\widehat{u}^k - \partial_t\widehat{u}^k = -\tau\int_0^1\int_s^1 \partial_t^2 u(t_{k-1} + \tau\eta)\,\mathrm{d}\eta\,\mathrm{d}s.$$

With the estimate

$$\Big|\int_0^1 \dots \int_0^1 u(\sigma_1 + \ldots + \sigma_\ell)\,\mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_\ell\Big| \leq \int_0^\ell |u(\sigma_1)|\,\mathrm{d}\sigma_1,$$

we may write

$$\partial_\tau^{m-\ell_2}\big(\partial_\tau^{\ell_2-\ell_1}\partial_t^{\ell_1}\widehat{u}^n - \partial_t^{\ell_2}\widehat{u}^n\big) = \sum_{j=\ell_1}^{\ell_2-1}(\partial_\tau - \partial_t)\big(\partial_\tau^{m-1-j}\partial_t^{j}\widehat{u}^n\big)$$

$$= -\tau\sum_{j=\ell_1}^{\ell_2-1}\int_0^1 \dots \int_0^1\int_0^1\int_s^1 \partial_t^{m+1}u(t_{n-(m-j)} + \tau(\eta + \sigma_1 + \ldots + \sigma_{m-1-j}))\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}\sigma_1\ldots\mathrm{d}\sigma_{m-1-j}$$

and hence

$$\begin{aligned}
&\|\partial_\tau^{m-\ell_2}\big(\partial_\tau^{\ell_2-\ell_1}\partial_t^{\ell_1}\widehat{u}^n - \partial_t^{\ell_2}\widehat{u}^n\big)\| \\
&\leq \tau\sum_{j=\ell_1}^{\ell_2-1}\int_0^1 \dots \int_0^1 \|\partial_t^{m+1}u(t_{n-(m-j)} + \tau(\eta + \sigma_1 + \ldots + \sigma_{m-1-j}))\|\,\mathrm{d}\eta\,\mathrm{d}\sigma_1\ldots\mathrm{d}\sigma_{m-1-j} \\
&\leq \tau\sum_{j=\ell_1}^{\ell_2-1}\int_0^{m-j} \|\partial_t^{m+1}u(t_{n-(m-j)} + \tau\eta)\|\,\mathrm{d}\eta \\
&\leq \tau m\int_0^m \|\partial_t^{m+1}u(t_{n-m} + \tau\eta)\|\,\mathrm{d}\eta
\end{aligned}$$

and with this

$$\|\partial_\tau^{m-\ell_2}\big(\partial_\tau^{\ell_2-\ell_1}\partial_t^{\ell_1}\widehat{u}^n - \partial_t^{\ell_2}\widehat{u}^n\big)\|^2 \leq \tau m^3\int_{t_{n-m}}^{t_n} \|\partial_t^{m+1}u(s)\|^2\,\mathrm{d}s,$$

which concludes the proof. $\square$

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