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On the Constitutive Modelling of Piezoelectric Quasicrystals

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Abstract: Quasicrystals endowed with piezoelectric properties belong nowadays to novel piezoelectric materials. In this work, the basic framework of generalized piezoelectricity theory of quasicrystals is investigated by providing an improvement of the existing constitutive modelling. It is shown, for the first time, that the tensor of phason piezoelectric moduli is fully asymmetric without any major or minor symmetry, which has important consequences on the constitutive relations as well as on its classification with respect to the crystal systems and Laue classes. The exploration of the tensor of phason piezoelectric moduli has a significant impact on the understanding of the piezoelectric properties of quasicrystals. Using the group representation theory, the classification of the tensor of phason piezoelectric moduli with respect to the crystal systems and Laue classes is given for one-dimensional quasicrystals. The number of independent components of the phason piezoelectric moduli is determined for all 31 point groups of one-dimensional quasicrystals. It is proven that the 10 centrosymmetric crystallographic point groups have no piezoelectric effects and that the remaining 21 non-centrosymmetric crystallographic point groups exhibit piezoelectric effects due to both phonon and phason fields. Moreover, the constitutive relations for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 9 with point group 6 and Laue class 10 with point group $6mm$ are explicitly derived, showing that the constitutive relations for piezoelectric quasicrystals depend on the considered Laue class as well as on the point group. Comparisons with existing results in the literature and discussion are also given.

Keywords: quasicrystals; piezoelectricity; novel piezoelectric materials; phason; symmetries; constitutive tensors; constitutive relations; classification

1. Introduction

It is well-known that an important class of materials with numerous applications in technology are the piezoelectric crystals (e.g., [1–3]). For a piezoelectric material, the elastic and electric response is coupled and anisotropic. Piezoelectricity exists only in crystals with non-centrosymmetric point groups [2]. Nowadays, industrial, manufacturing, and the automotive industry are the largest application market for piezoelectric devices.

Nowadays, an interesting class of novel materials are the quasicrystals. Quasicrystals were discovered by Shechtman in 1982 and later reported by Shechtman et al. [4] in 1984. In 2011, Shechtman received the Nobel Prize in Chemistry for the discovery of quasicrystals. Quasicrystals belong to aperiodic crystals and possess long-range orientational order but no translational symmetry in the aperiodic directions. The basis of the continuum theory of solid quasicrystals is set up by two elementary excitations: the phonons and the phasons [5,6]. Quasicrystals can have crystallographic and non-crystallographic point groups. There exist one-, two- and three-dimensional quasicrystals. In particular, for one-dimensional quasicrystals, there exist 31 point groups, which are crystallographic point groups consisting of triclinic, monoclinic, orthorhombic, tetragonal, trigonal and hexagonal systems and 10 Laue classes [7]. The two-dimensional quasicrystals have 57 point groups, consisting of 31 crystallographic point groups and 26 non-crystallographic

point groups (pentagonal, decagonal, octagonal, dodecagonal systems) [8]. The three-dimensional quasicrystals have 60 point groups, consisting of 32 crystallographic point groups (i.e., cubic system) and 28 non-crystallographic point groups (i.e., icosahedral system) [9]. To study the elastic properties of quasicrystals, the framework of linear elasticity theory of quasicrystals has been investigated. The elastic energy density was derived for two-dimensional pentagonal and three-dimensional icosahedral quasicrystals by Levine et al. [6] and for octagonal and dodecagonal quasicrystals by Socolar [10]. The generalized linear elasticity theory of quasicrystals was first proposed by Ding et al. [11]. Yang et al. [12] investigated the linear elasticity theory of cubic quasicrystals. The generalized linear elasticity theory of one-dimensional quasicrystals was developed by Wang et al. [7].

The desirable properties of the quasicrystalline materials belong to their piezoelectric properties, as pointed out by Yang et al. [13,14], Hu et al. [15,16], and Gong et al. [17]. In the theory of piezoelectricity of quasicrystals, the phonon fields as well as the phason fields are coupled with the electric field. Due to the piezoelectric effects, quasicrystals may have applications in electronic systems such as sensor and actuator devices, by which an electric voltage can induce an elastic deformation and vice versa. Wang and Pan [18] derived analytical solutions for the fields involved in the problem of uniformly moving screw dislocations in a one-dimensional hexagonal piezoelectric quasicrystal. By using the generalized Stroh formalism, analytical solutions of the electric and elastic fields induced by a straight dislocation in a one-dimensional piezoelectric quasicrystal have been derived by Yang et al. [19]. Quasicrystals with piezoelectric properties represent novel piezoelectric materials. The search for a clean, reusable source of energy is driving the interest in the exploration of piezoelectricity of quasicrystals. Quasicrystals may be used to develop new piezoelectric materials with promising applications in technology that could include energy harvesting practices in the future [20].

Using group representation theory, Hu et al. [15] explored the piezoelectric effects in quasicrystals, including the contribution from both phonon and phason fields, and investigated piezoelectric effects in all two-dimensional quasicrystals with crystallographic and non-crystallographic symmetries and three-dimensional icosahedral and cubic quasicrystals. Hu et al. [15] found that the piezoelectric behavior of quasicrystals is more complicated than that of ordinary crystals due to the presence of the phason field and gave the non-vanishing phonon and phason piezoelectric constants for two-dimensional quasicrystals and three-dimensional icosahedral and cubic quasicrystals. In particular, Hu et al. [15] showed that two-dimensional quasicrystals with non-centrosymmetric crystallographic point groups have piezoelectric effects induced by both phonon and phason fields. Two-dimensional quasicrystals with non-crystallographic point groups can be divided into four classes. The first class consists of quasicrystals with central symmetry ($\bar{5}, \bar{5}m, N/m, N/mmm$ for $N = 10, 8, 12$) having no piezoelectric effects. The second class consists of quasicrystals with five-fold symmetry ($5, 52, 5m$) having piezoelectric effects induced by both phonon and phason fields. The third class consists of quasicrystals with even-fold rotoinversion symmetry ($\bar{N}, \bar{N}m2$ for $N = 10, 8, 12$) having a piezoelectric effect induced only by the phason field. The fourth class consists of quasicrystals with even-fold proper rotation symmetry ($N, N22, Nmm$ for $N = 10, 8, 12$) having a piezoelectric effect induced only by the phonon field. On the other hand, three-dimensional quasicrystals can be divided into three classes. The first class ($m\bar{3}, m\bar{3}m, 235, m3\bar{5}$) has no piezoelectric effects. The second class ($23, \bar{4}3m$) has piezoelectric effects induced by both phonon and phason fields. The third class (432) has a piezoelectric effect induced only by the phason field.

Concerning one-dimensional quasicrystals, Wang et al. [7] derived the independent components of the elastic constitutive tensors for the 31 point groups of one-dimensional quasicrystals based on the fact that "all the point groups belonging to the same Laue class possess the same elastic properties" given by Yang et al. [21]. However, this does not hold for the piezoelectric quasicrystals, since the piezoelectric properties are not the same for point groups belonging to the same Laue class. This happens because the tensor of phonon

piezoelectric moduli (the corresponding "usual" tensor of piezoelectric moduli for crystals) depends on the point group (see Nye [2]). An important aspect that is highlighted in this work is that, for the derivation of the explicit form of the constitutive relations for piezo-electric quasicrystals, Laue classes as well as the corresponding point groups have to be taken into account. For the classification of the elastic constitutive tensors C_{ijkl}, $E_{\alpha j \gamma l}$, $D_{ij\gamma l}$, the Laue class is enough. On the other hand, in general, for the classification of the phonon and phason piezoelectric constitutive tensors e_{jkl} and $f_{j\gamma l}$, the Laue class is not enough and the classification has to be given with respect to the point groups. Only for one-dimensional quasicrystals, the classification of the tensor of phason piezoelectric moduli f_{j3l} is enough with respect to the Laue class as we will show in this work. The above aspects are often ignored in the literature of elasticity (see, e.g., [9,22,23]) and piezoelectricity (see, e.g., [24–27]) of one-dimensional hexagonal quasicrystals. The significance of the classification of the constitutive tensors and its influence on the constitutive relations is revealed in Section 3.

As far as one-dimensional piezoelectric quasicrystals are concerned, the independent components of the piezoelectric constitutive tensors have been given by Li and Liu [28] in matrix form for all 31 point groups of one-dimensional quasicrystals. However, the tensor of phason piezoelectric moduli has been erroneously given in [28] as a 3×6 matrix instead of a 3×3 matrix, as it will be shown in this work, leading to an incorrect classification of the tensor of phason piezoelectric moduli and consequently to an incorrect interpretation of the piezoelectric properties of one-dimensional quasicrystals.

On the other hand, concerning the state of the art of constitutive modelling in the literature of piezoelectric quasicrystals, Altay and Dökmeci [29] developed the three-dimensional basic equations in differential and variational invariant forms for piezoelectric quasicrystals. However, they provided the constitutive relations for general quasicrystals with the constitutive tensor of phason piezoelectric moduli possessing an invalid minor symmetry with respect to the last two indices, and all these basic equations have been adopted by many researchers in the field of piezoelectric quasicrystals.

In this work, developing the constitutive modelling of piezoelectric quasicrystals, it is shown first of all that the tensor of phason piezoelectric moduli is an asymmetric, two-point tensor of rank 3 without possessing any minor symmetry in contrast to the statement given by Altay and Dökmeci [29]. The fully asymmetric character of the tensor of phason piezoelectric moduli has a significant impact on its classification and this in turn has physical consequences for the piezoelectric properties of quasicrystals and mathematical consequences for the number of independent components of the phason piezoelectric moduli and consequently for the explicit expressions of the corresponding constitutive relations.

This paper is organized as follows. In Section 2, the basic framework of generalized piezoelectricity theory of quasicrystals is given by improving the constitutive modelling concerning the symmetries of the tensor of phason piezoelectric moduli. In Section 3, the piezoelectric behavior of one-dimensional quasicrystals is investigated. By using group representation theory, the piezoelectric constitutive tensors as well as the tensors with "phasonic character" are studied with a focus on the classification of the tensor of phason piezoelectric moduli for one-dimensional piezoelectric quasicrystals, giving rise to the examination of the piezoelectric behavior of one-dimensional quasicrystals. Based on this classification, the constitutive relations for one-dimensional hexagonal piezoelectric quasicrystals are explicitly derived for Laue class 9 with point group 6 and Laue class 10 with point group 6*mm*. Conclusions are given in Section 4. The classification of the constitutive tensors with "phasonic character" for one-dimensional piezoelectric quasicrystals is given in the Appendices A–C.

2. Generalized Piezoelectricity Theory of Quasicrystals

In this section, we give the basic framework of generalized linear piezoelectricity theory of quasicrystals, which is the generalization of linear (compatible) elasticity of quasicrystals toward the piezoelectricity of quasicrystals. The focus of this section lies on

the constitutive modelling and its improvement and the matrix representation form of the phason piezoelectric moduli tensor.

2.1. Basic Framework

An $(n-3)$-dimensional quasicrystal can be generated by the projection of an n-dimensional periodic structure to the three-dimensional physical space ($n = 4, 5, 6$). The n-dimensional hyperspace E^n can be decomposed into the direct sum of two orthogonal subspaces,

$$E^n = E_\parallel^3 \oplus E_\perp^{(n-3)}, \tag{1}$$

where $E_\parallel^3$ is the three-dimensional physical or parallel space of the phonon fields and $E_\perp^{(n-3)}$ is the $(n-3)$-dimensional perpendicular space of the phason fields. For $n = 4, 5, 6$, we speak of one-dimensional, two-dimensional, and three-dimensional quasicrystals and the dimension of the hyperspace is 4D, 5D, and 6D, respectively. It is considerably helpful to use two different coordinate systems with different indices [30,31]. Indices in the parallel space are denoted by small Latin letters i, j, k with $i, j, k = 1, 2, 3$. Indices in the perpendicular space are denoted by small Greek letters α, γ with $\alpha, \gamma = 3$ for one-dimensional quasicrystals (with quasiperiodicity in z-direction), $\alpha, \gamma = 1, 2$ for two-dimensional quasicrystals, and $\alpha, \gamma = 1, 2, 3$ for three-dimensional quasicrystals. The reason for this distinction is based on the fact that the phonon and phason fields transform, in general, under different representations, as it has been shown by Yang et al. [14,30] using group representation theory. In particular, the phonon fields having indices i, j, k transform under Γ_A, where Γ_A is a three-dimensional representation of the symmetry group in the parallel space, and the phason fields having indices α, γ transform under Γ_B, where Γ_B is a representation in the perpendicular space (three-dimensional representation for three-dimensional quasicrystals, two-dimensional representation for two-dimensional quasicrystals, one-dimensional representation for one-dimensional quasicrystals). Irreducible representations of phonon tensor fields and of phason tensor fields "live" in the parallel tensor space and in the perpendicular tensor space, respectively. It is important to highlight here that indices with small Latin letters cannot be interchanged with indices with small Greek letters, since they transform under different representations and "live" in different spaces. Throughout the article, phonon fields will be denoted by $(\cdot)^\parallel$ and phason fields by $(\cdot)^\perp$. All field quantities (phonon and phason fields) depend on the so-called material space coordinates (or spatial coordinates) $x \in \mathbb{R}^3$. Notice that, in the linear theory of quasicrystals, the material space coincides with the parallel space.

The (elastic) *phonon and phason distortion tensors*, $\beta_{kl}^\parallel$ and $\beta_{\gamma l}^\perp$, are defined as the gradient of the *phonon displacement vector* $u_k^\parallel$ and of the *phason displacement vector* $u_\gamma^\perp$, respectively,

$$\beta_{kl}^\parallel := u_{k,l}^\parallel, \qquad \beta_{\gamma l}^\perp := u_{\gamma,l}^\perp, \tag{2}$$

and they fulfill the following *compatibility conditions*:

$$\epsilon_{ijl} \beta_{kl,j}^\parallel = 0, \qquad \epsilon_{ijl} \beta_{\gamma l,j}^\perp = 0, \tag{3}$$

where ϵ_{ijl} denotes the three-dimensional Levi-Civita tensor. The subscript comma denotes the partial differentiation $\partial_j = \partial/\partial x_j$ with respect to the spatial coordinates x_j. In the geometrically linearized theory of elasticity, the symmetric part of $\beta_{kl}^\parallel = u_{k,l}^\parallel$ is the *phonon strain tensor*

$$e_{kl}^\parallel = \frac{1}{2} (u_{k,l}^\parallel + u_{l,k}^\parallel). \tag{4}$$

With the use of the two different coordinate systems and indices, it becomes evident that the phason distortion tensor $\beta_{\gamma l}^{\perp}$ is an asymmetric two-point tensor, that is,

$$\beta_{\gamma l}^{\perp} \neq \beta_{l\gamma}^{\perp} \tag{5}$$

and consequently a phason strain tensor, a quantity that is often misleadingly used in the literature, cannot be defined. A simple example is the case of one-dimensional quasicrystals, where $\gamma = 3$ and $l = 1, 2, 3$ and an interchange of the indices is not possible (e.g., $\beta_{31}^{\perp}$ exists but $\beta_{13}^{\perp}$ does not exist). The lack of distinction between phonon and phason indices has caused a great confusion in the literature of quasicrystals.

The *electric field strength vector* E_l is defined as the negative gradient of the *electrostatic potential* φ as follows:

$$E_l := -\varphi_{,l} \tag{6}$$

and satisfies the *Bianchi identity*

$$\epsilon_{ijl} E_{l,j} = 0 \,, \tag{7}$$

which is the compatibility condition for the electric field. Physically, Equation (7) is the *electrostatic version of the Faraday law* [32].

2.2. Constitutive Modelling

We consider a variational problem with the Lagrangian density $\mathcal{L}$ depending on the variables $\boldsymbol{u}^{\parallel}$, $\boldsymbol{u}^{\perp}$ and φ for given body forces and body charges

$$\mathcal{L} = \mathcal{L}(u_i^{\parallel}, u_{i,j}^{\parallel}, u_{\alpha}^{\perp}, u_{\alpha,j}^{\perp}, \varphi, \varphi_{,j}) = -\left[\Psi(e_{ij}^{\parallel}, \beta_{\alpha j}^{\perp}, E_j) + V(u_i^{\parallel}, u_{\alpha}^{\perp}, \varphi) \right], \tag{8}$$

where Ψ is the so-called *electric enthalpy density for piezoelectric quasicrystals* and V is the *electro-elastic potential* of (phonon and phason) body forces and body charges. For vanishing phason fields, the above electric enthalpy density and the electro-elastic potential reduce to the corresponding forms of classical piezoelectricity [33–35].

For a linear piezoelectric quasicrystal, the electric enthalpy density Ψ is given by the following quadratic form:

$$\Psi = \frac{1}{2} C_{ijkl} e_{ij}^{\parallel} e_{kl}^{\parallel} + D_{ij\gamma l} e_{ij}^{\parallel} \beta_{\gamma l}^{\perp} + \frac{1}{2} E_{\alpha j\gamma l} \beta_{\alpha j}^{\perp} \beta_{\gamma l}^{\perp} - e_{jkl} E_j e_{kl}^{\parallel} - f_{j\gamma l} E_j \beta_{\gamma l}^{\perp} - \frac{1}{2} \varepsilon_{jl} E_j E_l \,, \tag{9}$$

where the *constitutive tensors*, which are the *physical property tensors*, are defined by

$$C_{ijkl} = \frac{\partial^2 \Psi}{\partial e_{ij}^{\parallel} \partial e_{kl}^{\parallel}} \,, \qquad E_{\alpha j\gamma l} = \frac{\partial^2 \Psi}{\partial \beta_{\alpha j}^{\perp} \partial \beta_{\gamma l}^{\perp}} \,, \qquad D_{ij\gamma l} = \frac{\partial^2 \Psi}{\partial e_{ij}^{\parallel} \partial \beta_{\gamma l}^{\perp}} \,, \tag{10}$$

$$e_{jkl} = -\frac{\partial^2 \Psi}{\partial E_j \partial e_{kl}^{\parallel}} \,, \qquad f_{j\gamma l} = -\frac{\partial^2 \Psi}{\partial E_j \partial \beta_{\gamma l}^{\perp}} \,, \qquad \varepsilon_{jl} = -\frac{\partial^2 \Psi}{\partial E_j \partial E_l} \,. \tag{11}$$

C_{ijkl} is the *tensor of the elastic moduli of phonons*, $E_{\alpha j\gamma l}$ is the *tensor of the elastic moduli of phasons*, $D_{ij\gamma l}$ is the *tensor of the elastic moduli of the phonon–phason coupling*, e_{jkl} is the *tensor of the phonon piezoelectric moduli*, $f_{j\gamma l}$ is the *tensor of the phason piezoelectric moduli*, and ε_{jl} is the *tensor of the dielectric moduli*. For these constitutive tensors, it holds:

- The tensor of the elastic moduli of phonons possesses the *major symmetry* and the *minor symmetries* (see, e.g., [36])

$$C_{ijkl} = C_{klij} \,, \qquad C_{ijkl} = C_{ijlk} = C_{jikl} \,, \tag{12}$$

since the electric enthalpy density Ψ is quadratic in terms of the phonon strain tensor $e_{ij}^{\parallel}$ and the tensor $e_{ij}^{\parallel}$ is symmetric, respectively. The tensor C_{ijkl} has 21 independent components.

- The tensor of the elastic moduli of phasons possesses only a *major symmetry* (see also [11])

$$E_{\alpha j\gamma l} = E_{\gamma l\alpha j}, \tag{13}$$

since the electric enthalpy density is quadratic in terms of the phason distortion tensor $\beta_{\alpha j}^{\perp}$, but $\beta_{\alpha j}^{\perp}$ is an asymmetric tensor.

- The tensor of the elastic moduli of the phonon–phason coupling possesses only a *minor symmetry* with respect to the first two indices (see also [11])

$$D_{ij\gamma l} = D_{ji\gamma l}, \tag{14}$$

due to the symmetric character of $e_{ij}^{\parallel}$.

- The tensor of the phonon piezoelectric moduli possesses a *minor symmetry* with respect to the last two indices (see, e.g., [2])

$$e_{kij} = e_{kji}, \tag{15}$$

since $e_{ij}^{\parallel}$ is a symmetric tensor. The tensor e_{kji} has $3 \times 6 = 18$ independent components.

- The tensor of the phason piezoelectric moduli $f_{j\gamma l}$ is an asymmetric tensor without a minor symmetry with respect to the last two indices

$$f_{j\gamma l} \neq f_{jl\gamma}, \tag{16}$$

since the phason distortion tensor $\beta_{\gamma l}^{\perp}$ is an asymmetric two-point tensor (see Equation (5) and also Remark 2).

- The tensor of the dielectric moduli exhibits the *major symmetry* (see, e.g., [2])

$$\varepsilon_{ij} = \varepsilon_{ji}, \tag{17}$$

since the electric enthalpy density is quadratic with respect to the electric field strength E_j and it has six independent components.

Evidently, the tensors $D_{ij\gamma l}, e_{kij}, f_{j\gamma l}$ do not possess a major symmetry, since they are coupling tensors (see also Remark 3).

To sum up, it holds for the *symmetries of the constitutive or physical property tensors of piezoelectric quasicrystals*:

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jikl}, \quad E_{\alpha j\gamma l} = E_{\gamma l\alpha j}, \quad D_{ij\gamma l} = D_{ji\gamma l}, \tag{18}$$

$$e_{kij} = e_{kji}, \quad f_{j\gamma l} \neq f_{jl\gamma}, \quad \varepsilon_{ij} = \varepsilon_{ji}. \tag{19}$$

Remark 1. *The number of components of the constitutive tensors with phasonic character depends, on the first step, on the dimension of the considered quasicrystal, that is if it is one-dimensional, two-dimensional, or three-dimensional. Accordingly, on the second step, the number of the independent components depends on the considered symmetry and Laue class (see, e.g., the classification of the constitutive tensors D_{ij3l} and f_{j3l} for one-dimensional piezoelectric quasicrystals in Appendix B and Appendix C, respectively).*

Remark 2. *As it has been shown above, the tensor of phason piezoelectric moduli $f_{j\gamma l}$ is not symmetric with respect to the last two indices, as erroneously postulated (see Equation (2.30)) by Altay and Dökmeci [29] and adopted by other researchers (see, e.g., [25,37]), leading to erroneous*

constitutive relations. The impact of the asymmetric character of phason piezoelectric moduli will become clear in Section 3.

The three response quantities are defined through the following *linear constitutive relations:*

$$\sigma_{ij}^{\parallel} = \frac{\partial \Psi}{\partial e_{ij}^{\parallel}} = C_{ijkl}e_{kl}^{\parallel} + D_{ij\gamma l}\beta_{\gamma l}^{\perp} - e_{lij}E_l \,, \tag{20}$$

$$\sigma_{\alpha j}^{\perp} = \frac{\partial \Psi}{\partial \beta_{\alpha j}^{\perp}} = D_{kl\alpha j}e_{kl}^{\parallel} + E_{\alpha j\gamma l}\beta_{\gamma l}^{\perp} - f_{l\alpha j}E_l \,, \tag{21}$$

$$D_j = -\frac{\partial \Psi}{\partial E_j} = e_{jkl}e_{kl}^{\parallel} + f_{j\gamma l}\beta_{\gamma l}^{\perp} + \varepsilon_{jl}E_l \,, \tag{22}$$

where $\sigma_{ij}^{\parallel}$ and $\sigma_{\alpha j}^{\perp}$ are the *phonon and phason stress tensors*, respectively, and D_j is the *electric displacement vector* or *electric excitation*. From Equations (20) and (21), it is obvious that the phonon stress tensor is symmetric, $\sigma_{ij}^{\parallel} = \sigma_{ji}^{\parallel}$, whereas the phason stress tensor $\sigma_{\alpha j}^{\perp}$ is an asymmetric two-point tensor, $\sigma_{\alpha j}^{\perp} \neq \sigma_{j\alpha}^{\perp}$ (see also [31,38]). Moreover, one can see in Equations (20)–(22) that the electric field strength gives a contribution to the phonon and phason stress fields and the elastic phonon strain tensor and phason distortion tensor contribute to the electric displacement vector, as it should be for a piezoelectric quasicrystal by its definition.

Using the constitutive relations (20)–(22), the electric enthalpy density (9) can be rewritten in the following "compact" form:

$$\Psi(e_{ij}^{\parallel}, \beta_{\alpha j}^{\perp}, E_j) = \frac{1}{2}\sigma_{ij}^{\parallel}e_{ij}^{\parallel} + \frac{1}{2}\sigma_{\alpha j}^{\perp}\beta_{\alpha j}^{\perp} - \frac{1}{2}D_j E_j \,. \tag{23}$$

The *electro-elastic potential V* is defined by

$$V(u_i^{\parallel}, u_{\alpha}^{\perp}, \varphi) = -f_i^{\parallel}u_i^{\parallel} - f_{\alpha}^{\perp}u_{\alpha}^{\perp} + q\varphi \,, \tag{24}$$

where $f_i^{\parallel}$ is the conventional *phonon body force density*, $f_{\alpha}^{\perp}$ is the *phason body force density* (e.g., [38,39]), and q denotes the *body charge density*. The phonon and phason body force densities and the body charge density are given as the excitations due to the phonon displacement, phason displacement, and electrostatic potential, respectively, as follows:

$$f_i^{\parallel} = -\frac{\partial V}{\partial u_i^{\parallel}} \,, \qquad f_{\alpha}^{\perp} = -\frac{\partial V}{\partial u_{\alpha}^{\perp}} \,, \qquad q = \frac{\partial V}{\partial \varphi} \,. \tag{25}$$

The associated Euler–Lagrange equations are as follows:

$$E_i^{u^{\parallel}}(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial u_i^{\parallel}} - \partial_j\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}^{\parallel}}\right) = 0 \,, \tag{26}$$

$$E_{\alpha}^{u^{\perp}}(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial u_{\alpha}^{\perp}} - \partial_j\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j}^{\perp}}\right) = 0 \,, \tag{27}$$

$$E^{\varphi}(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_j\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,j}}\right) = 0 \,. \tag{28}$$

Using Equations (8), (9), (20)–(22), (24), and (25) as well as the relations (2), (4), and (6), Equations (26)–(28) give the (*force*) *equilibrium conditions*

$$\sigma_{ij,j}^{\parallel} + f_i^{\parallel} = 0, \tag{29}$$

$$\sigma_{\alpha j,j}^{\perp} + f_\alpha^{\perp} = 0, \tag{30}$$

and the *Gauss law of electrostatics* (e.g., [32])

$$D_{j,j} = q. \tag{31}$$

Substituting Equations (20)–(22) into Equations (29)–(31) and using Equations (2), (4), and (6), we obtain the *coupled equations of equilibrium for the phonon and phason displacement fields and the electrostatic potential*:

$$C_{ijkl}u_{k,lj}^{\parallel} + D_{ij\gamma l}u_{\gamma,lj}^{\perp} + e_{lij}\varphi_{,lj} = -f_i^{\parallel}, \tag{32}$$

$$D_{kl\alpha j}u_{k,lj}^{\parallel} + E_{\alpha j\gamma l}u_{\gamma,lj}^{\perp} + f_{l\alpha j}\varphi_{,lj} = -f_\alpha^{\perp}, \tag{33}$$

$$e_{jkl}u_{k,lj}^{\parallel} + f_{j\gamma l}u_{\gamma,lj}^{\perp} - \varepsilon_{jl}\varphi_{,lj} = q, \tag{34}$$

which are non-homogeneous partial differential equations with sources being the body forces and body charges.

In absence of body forces and body charges, the corresponding coupled equations of equilibrium are given by the following homogeneous partial differential equations:

$$C_{ijkl}u_{k,lj}^{\parallel} + D_{ij\gamma l}u_{\gamma,lj}^{\perp} + e_{lij}\varphi_{,lj} = 0, \tag{35}$$

$$D_{kl\alpha j}u_{k,lj}^{\parallel} + E_{\alpha j\gamma l}u_{\gamma,lj}^{\perp} + f_{l\alpha j}\varphi_{,lj} = 0, \tag{36}$$

$$e_{jkl}u_{k,lj}^{\parallel} + f_{j\gamma l}u_{\gamma,lj}^{\perp} - \varepsilon_{jl}\varphi_{,lj} = 0. \tag{37}$$

Remark 3. *It should be noticed that the electric enthalpy density Ψ given by Equation (9) is in accordance with the corresponding one given by Altay and Dökmeci [29] (see Equation (2.26) in [29]). However, attention should be given to the last terms of the constitutive Equations (20) and (21) in comparison with the corresponding terms in Equations (2.27) and (2.28) in [29], since there is a difference in the order of indices. In particular, in Equations (2.27) and (2.28) in [29], a major symmetry has been used for two coupling tensors, the tensors of phonon and phason piezoelectric moduli. However, these major symmetries do not exist.*

2.3. Matrix Representation Form of the Tensor of Phason Piezoelectric Moduli $f_{i\alpha j}$

Special attention is given here to the tensor of phason piezoelectric moduli $f_{i\alpha j}$, since its asymmetric character has important physical consequences. The tensor of phason piezoelectric moduli $f_{i\alpha j}$ is a physical property tensor with the indices i and j to transform under Γ_A and the index α to transform under Γ_B. Hence, $f_{i\alpha j}$ transforms under the representation $\Gamma_A \times \Gamma_B \times \Gamma_A$. Let us write the tensor $f_{i\alpha j}$ in a matrix representation form.

- For three-dimensional triclinic piezoelectric quasicrystals, the tensor of phason piezo-electric moduli $f_{i\alpha j}$ ($i = 1, 2, 3$, $\alpha = 1, 2, 3$, $j = 1, 2, 3$) has $3 \times 3 \times 3 = 27$ independent components. Following [16] for the arrangement of the last two indices in the order

$$(\alpha j) = 11, \quad 22, \quad 33, \quad 23, \quad 31, \quad 12, \quad 32, \quad 13, \quad 21,$$

the tensor $f_{i\alpha j}$ can be written in a 3×9 matrix representation as follows:

$$f_{i\alpha j} = \begin{pmatrix} f_{111} & f_{122} & f_{133} & f_{123} & f_{131} & f_{112} & f_{132} & f_{113} & f_{121} \\ f_{211} & f_{222} & f_{233} & f_{223} & f_{231} & f_{212} & f_{232} & f_{213} & f_{221} \\ f_{311} & f_{322} & f_{333} & f_{323} & f_{331} & f_{312} & f_{332} & f_{313} & f_{321} \end{pmatrix}_{(27)}. \tag{38}$$

The subscript $(\cdot)$ in the matrices stands for the number of independent components. The classification of the tensor of phason piezoelectric moduli $f_{i\alpha j}$ has been given by Hu et al. [15] for three-dimensional icosahedral and cubic piezoelectric quasicrystals.

- For two-dimensional triclinic piezoelectric quasicrystals, the tensor $f_{i\alpha j}$ ($i = 1, 2, 3$, $\alpha = 1, 2$, $j = 1, 2, 3$) has $3 \times 2 \times 3 = 18$ independent components. Following [16] for the arrangement of the last two indices in the order

$$(\alpha j) = 11, \quad 22, \quad 23, \quad 12, \quad 13, \quad 21,$$

the tensor $f_{i\alpha j}$ can be written in a 3×6 matrix representation as follows:

$$f_{i\alpha j} = \begin{pmatrix} f_{111} & f_{122} & f_{123} & f_{112} & f_{113} & f_{121} \\ f_{211} & f_{222} & f_{223} & f_{212} & f_{213} & f_{221} \\ f_{311} & f_{322} & f_{323} & f_{312} & f_{313} & f_{321} \end{pmatrix}_{(18)}, \tag{39}$$

which is in full agreement with the corresponding representation ($d_{ijk}^{(2)}$) given by Hu et al. [15]. The classification of the tensor of phason piezoelectric moduli $f_{i\alpha j}$ has been given by Hu et al. [15] for all two-dimensional piezoelectric quasicrystals with crystallographic and non-crystallographic symmetries.

- For one-dimensional triclinic piezoelectric quasicrystals, the tensor $f_{i\alpha j}$ ($i = 1, 2, 3$, $\alpha = 3$, $j = 1, 2, 3$) has $3 \times 1 \times 3 = 9$ independent components. Since $\alpha = 3$ is a fixed index, the tensor f_{i3j} is a tensor of rank 2 and can be written as usual in a 3×3 matrix representation

$$f_{i3j} = \begin{pmatrix} f_{131} & f_{132} & f_{133} \\ f_{231} & f_{232} & f_{233} \\ f_{331} & f_{332} & f_{333} \end{pmatrix}_{(9)}. \tag{40}$$

Therefore, it is clear that the phason piezoelectric moduli for one-dimensional piezoelectric quasicrystals is represented by 3×3 matrices and not by 3×6 matrices, as it has been erroneously given by Li and Liu [28]. The classification of the tensor of phason piezoelectric moduli f_{i3j} for one-dimensional piezoelectric quasicrystals is given in Section 3, which is devoted exclusively to one-dimensional piezoelectric quasicrystals.

3. Piezoelectricity of One-Dimensional Quasicrystals

Based on the improved constitutive modelling that has been developed in the previous section, we give here the basic equations governing the one-dimensional piezoelectric quasicrystals and explore their piezoelectric behavior through the classification of the tensors with "phasonic character". This gives rise to show, considering one-dimensional hexagonal piezoelectric quasicrystals as a representative example, that the constitutive relations depend on the considered Laue class as well as on the point group.

3.1. Basic Equations

In the class of one-dimensional quasicrystals, the atom arrangement is quasiperiodic in one direction, for example, in the x_3-direction, and periodic in the plane perpendicular to this direction ($x_1 x_2$-plane). One-dimensional quasicrystals are quasicrystals with crystallographic symmetries. For one-dimensional quasicrystals, the phason displacement field is a scalar field, $u_\alpha^\perp \equiv u_3^\perp$, the phason distortion tensor becomes a vector $\beta_{3l}^\perp = u_{3,l}^\perp$, and the indices α, $\gamma = 3$.

From the viewpoint of group representation theory, $u_i^{\parallel}, i = 1, 2, 3$ transforms under a three-dimensional representation Γ_A, whereas $u_3^{\perp}$ transforms under a one-dimensional representation Γ_B. The latter may be an identity representation or sometimes a one-dimensional non-identity representation with:

(i)　A character 1 for the first-type operations, which are proper rotations n ($n = 1, 2, 3, 4, 6$) about the vertical axis and reflections m across vertical mirror planes;

(ii)　A character -1 for the second-type operations, which are compound operations being a product of inversion $\bar{1}$ and one of the first-type operations [7].

For one-dimensional piezoelectric quasicrystals, the set of the *equilibrium Equations* (29)–(31) becomes

$$\sigma_{ij,j}^{\parallel} + f_i^{\parallel} = 0\,, \tag{41}$$

$$\sigma_{3j,j}^{\perp} + f_3^{\perp} = 0\,, \tag{42}$$

$$D_{j,j} = q \tag{43}$$

and the corresponding *constitutive relations* (20)–(22) take the form

$$\sigma_{ij}^{\parallel} = C_{ijkl}e_{kl}^{\parallel} + D_{ij3l}\beta_{3l}^{\perp} - e_{lij}E_l\,, \tag{44}$$

$$\sigma_{3j}^{\perp} = D_{kl3j}e_{kl}^{\parallel} + E_{3j3l}\beta_{3l}^{\perp} - f_{l3j}E_l\,, \tag{45}$$

$$D_j = e_{jkl}e_{kl}^{\parallel} + f_{j3l}\beta_{3l}^{\perp} + \varepsilon_{jl}E_l\,. \tag{46}$$

Accordingly, the *field Equations* (32)–(34) read

$$C_{ijkl}u_{k,lj}^{\parallel} + D_{ij3l}u_{3,lj}^{\perp} + e_{lij}\varphi_{,lj} = -f_i^{\parallel}\,, \tag{47}$$

$$D_{kl3j}u_{k,lj}^{\parallel} + E_{3j3l}u_{3,lj}^{\perp} + f_{l3j}\varphi_{,lj} = -f_3^{\perp}\,, \tag{48}$$

$$e_{jkl}u_{k,lj}^{\parallel} + f_{j3l}u_{3,lj}^{\perp} - \varepsilon_{jl}\varphi_{,lj} = q\,. \tag{49}$$

In the constitutive relations (44)–(46), the *constitutive tensors* C_{ijkl}, D_{ij3l}, E_{3j3l}, e_{lij}, f_{l3j}, and ε_{jl} appear. It is remarkable to observe that all these tensors transform as tensors in the three-dimensional physical or parallel space, since only the indices in the parallel space are varying. The classification of the tensors of the elastic moduli C_{ijkl}, D_{ij3l}, and E_{3j3l} for one-dimensional quasicrystals has been given by Wang et al. [7]. On the other hand, the tensors of the phonon piezoelectric moduli e_{lij} and the dielectric moduli ε_{jl} are actually the tensors of classical piezoelectricity in the absence of phason fields and follow the classification of the corresponding tensors of classical piezoelectricity of crystals, as it can be found in Nye [2]. Therefore, what remains to be given is the classification of the tensor of phason piezoelectric moduli f_{l3j}, which follows in the next section.

3.2. Piezoelectric Behavior of One-Dimensional Quasicrystals

The tensors of elastic moduli of phonons C_{ijkl}, phonon piezoelectric moduli e_{ijk}, and dielectric moduli ε_{ij} are actually the tensors of classical piezoelectricity in absence of phason fields and follow the classification of the corresponding tensors of classical piezoelectricity of crystals, as it is given by Nye [2]. Our primary interest here for the examination of the piezoelectric behavior of one-dimensional quasicrystals is the classification of the tensor of phason piezoelectric moduli. In order to illustrate the method clearly, we give the classification of all three constitutive tensors with, let us say, "phasonic character", that is, D_{ij3l}, E_{3j3l}, and f_{l3j}, following Nye [2].

Let A_{ij} be a coordinate transformation matrix in the parallel space, $B = 1$ for first-type operations in the perpendicular space and $B = -1$ for second-type operations in the perpendicular space of one-dimensional quasicrystals.

- The tensor D_{ij3l} has 18 independent components and transforms under the representation $\{\Gamma_A \times \Gamma_A\} \times \Gamma_B \times \Gamma_A$, where the symbol $\{\cdot\}$ denotes the symmetric part. Therefore, D_{ij3l} transforms as a tensor of rank 3 in the parallel space times a scalar in the perpendicular space according to the following law:

$$D'_{ij3l} = A_{ik}A_{jm}BA_{ln}D_{km3n}\,. \tag{50}$$

- The tensor E_{3j3l} has six independent components and transforms under the representation $\{(\Gamma_B \times \Gamma_A) \times (\Gamma_B \times \Gamma_A)\}$. Accordingly, E_{3j3l} transforms as tensor of rank 2 in the parallel space times a scalar in the perpendicular space two times according to the following law:

$$E'_{3j3l} = BA_{jm}BA_{ln}E_{3m3n}\,. \tag{51}$$

- The tensor of phason piezoelectric moduli f_{i3j} has nine independent components and transforms under the representation $\Gamma_A \times \Gamma_B \times \Gamma_A$. Therefore, f_{i3j} transforms as tensor of rank 2 in the parallel space times a scalar in the perpendicular space according to the following law:

$$f'_{i3j} = A_{ik}BA_{jl}f_{k3l}\,. \tag{52}$$

Let us now consider a one-dimensional quasicrystal possessing a center of symmetry (inversion). The corresponding transformation reads

$$A_{ik} = -\delta_{ik}\,, \quad B = -1\,, \quad A_{jl} = -\delta_{jl}\,, \tag{53}$$

where δ_{ik} is the Kronecker delta. We examine the behavior of the tensors D_{ij3l}, E_{3j3l}, and f_{i3j} under this transformation. Using the transformation (53), the transformed tensors (50) and (51) read

$$D'_{ij3l} = D_{ij3l}\,, \tag{54}$$

$$E'_{3j3l} = E_{3j3l}\,. \tag{55}$$

It can be seen that both tensors D_{ij3l} and E_{3j3l} remain unchanged under the action of the symmetry operation "inversion". Therefore, "elastic properties possess an intrinsic (or inherent) centrosymmetry, and hence all point groups belonging to the same Laue class possess the same elastic properties", as mentioned by Hu et al. [8]. Therefore, D_{ij3l} and E_{3j3l} are non-zero for one-dimensional quasicrystals with centrosymmetric point groups. The independent elastic constants and invariants for each Laue class of one-dimensional quasicrystals have been given by Wang et al. [7]. The corresponding classification of the tensors D_{ij3l} and E_{3j3l} in matrix form is given in Appendix B and Appendix A, respectively.

Next, we explore how the phonon and phason fields influence the piezoelectric behavior of one-dimensional quasicrystals through the tensors of phonon and phason piezoelectric moduli, e_{lij} and f_{i3j}. Using the transformation (53), the transformed tensor of phason piezoelectric moduli (52) reads

$$f'_{i3j} = -f_{i3j}\,. \tag{56}$$

Since the considered one-dimensional quasicrystal possesses a center of symmetry, the transformation does not change the components: f_{i3j} must transform to itself, that is,

$$f'_{i3j} = f_{i3j}\,. \tag{57}$$

Therefore, from Equations (56) and (57), we obtain that

$$f_{i3j} = 0 \qquad (58)$$

for one-dimensional quasicrystals with a center of symmetry. This means that one-dimensional quasicrystals with centrosymmetric point groups cannot exhibit piezoelectric effects induced by the phason fields. The same argument holds for the phonon piezoelectric moduli which obeys the transformation law

$$e'_{lij} = A_{lk}A_{im}A_{jn}e_{kmn} , \qquad (59)$$

which, under the transformation of inversion (53), yields $e'_{lij} = -e_{lij}$. Therefore, $e_{lij} = 0$ for one-dimensional quasicrystals with a center of symmetry. Hence, under the transformation of inversion, both piezoelectric moduli become zero and this prevents the appearance of piezoelectricity in one-dimensional quasicrystals with centrosymmetric crystallographic point groups. The outcome of the aforementioned analysis is: *for one-dimensional quasicrystals with centrosymmetric point groups both phonon and phason piezoelectric moduli are zero and consequently they do not exhibit piezoelectricity.*

One-dimensional quasicrystals have 31 point groups; 21 of them are non-centrosymmetric. The classification of the tensor of phonon piezoelectric moduli e_{lij} is the same as in classical piezoelectricity and can be found in Nye [2]. Note that the classification of the phonon piezoelectric moduli depends not only on the Laue class but also on the point group. Using group representation theory [2,40], we give the classification of the tensor of phason piezoelectric moduli f_{i3j} with respect to the crystal system and Laue classes of one-dimensional quasicrystals with non-centrosymmetric crystallographic point groups in Appendix C. The results of this classification are collected in Table 1.

Table 1. Matrix form of the tensor of phason piezoelectric moduli f_{i3j} for one-dimensional piezoelectric quasicrystals with respect to the crystal system and Laue classes.

System	Laue Class	Matrix Form of f_{i3j}	Orientation of Axes
Triclinic	1	$\begin{pmatrix} f_{131} & f_{132} & f_{133} \\ f_{231} & f_{232} & f_{233} \\ f_{331} & f_{332} & f_{333} \end{pmatrix}_{(9)}$	arbitrary
Monoclinic	2	$\begin{pmatrix} f_{131} & f_{132} & 0 \\ f_{231} & f_{232} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(5)}$	$2 \parallel x_3$
Monoclinic	3	$\begin{pmatrix} f_{131} & 0 & f_{133} \\ 0 & f_{232} & 0 \\ f_{331} & 0 & f_{333} \end{pmatrix}_{(5)}$	$2_h \parallel x_2$
Orthorhombic	4	$\begin{pmatrix} f_{131} & 0 & 0 \\ 0 & f_{232} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(3)}$	$2 \parallel x_i$
Tetragonal Trigonal Hexagonal	5 7 9	$\begin{pmatrix} f_{131} & f_{132} & 0 \\ -f_{132} & f_{131} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(3)}$	$4 \parallel x_3$ $3 \parallel x_3$ $6 \parallel x_3$
Tetragonal Trigonal Hexagonal	6 8 10	$\begin{pmatrix} f_{131} & 0 & 0 \\ 0 & f_{131} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(2)}$	$4 \parallel x_3$ $3 \parallel x_3$ $6 \parallel x_3$

In addition, all 31 point groups of one-dimensional quasicrystals and the corresponding numbers of the independent phonon and phason piezoelectric moduli show exactly

which point groups exhibit piezoelectric effects and which not, which is detailed in Table 2. It can be seen in Table 2 that *the 21 non-centrosymmetric point groups of one-dimensional quasicrystals produce piezoelectricity due to both phonon and phason fields.* Note that for the usual crystals, only 20 from 21 non-centrosymmetric point groups possess piezoelectricity (see, e.g., Nye [2]).

Table 2. The 31 point groups of one-dimensional quasicrystals and piezoelectricity. (n_e: number of independent phonon piezoelectric moduli e_{ijk}. n_f: number of independent phason piezoelectric moduli f_{i3j}).

System	Laue Class	Point Group	Inversion	n_e	n_f
Triclinic	1	1	–	18	9
		$\bar{1}$	$\bar{1}$	0	0
Monoclinic	2	$2\ (2 \parallel x_3)$	–	8	5
		m_h	–	10	5
		$2/m_h$	$\bar{1}$	0	0
	3	$2_h\ (2 \parallel x_2)$	–	8	5
		m	–	10	5
		$2_h/m$	$\bar{1}$	0	0
Orthorhombic	4	$2_h 2_h 2$	–	3	3
		$mm2\ (2 \parallel x_3)$	–	5	3
		$2_h m m_h\ (2 \parallel x_1)$	–	5	3
		mmm_h	$\bar{1}$	0	0
Tetragonal	5	4	–	4	3
		$\bar{4}$	–	4	3
		$4/m_h$	$\bar{1}$	0	0
	6	$42_h 2_h$	–	1	2
		$4mm$	–	3	2
		$\bar{4}2_h m\ (2 \parallel x_1)$	–	2	2
		$4/m_h mm$	$\bar{1}$	0	0
Trigonal	7	3	–	6	3
		$\bar{3}$	$\bar{1}$	0	0
	8	$32_h\ (2 \parallel x_1)$	–	2	2
		$3m$	–	4	2
		$\bar{3}m$	$\bar{1}$	0	0
Hexagonal	9	6	–	4	3
		$\bar{6}$	–	2	3
		$6/m_h$	$\bar{1}$	0	0
	10	$62_h 2_h$	–	1	2
		$6mm$	–	3	2
		$\bar{6}m2_h$	–	1	2
		$6/m_h mm$	$\bar{1}$	0	0

Remark 4. *The latter result is in contradiction with the result of Li and Liu [28], where there exist two point groups, namely hexagonal Laue class 9 with point group $\bar{6}$ and hexagonal Laue class 10 with point group $\bar{6}m2_h$, possessing piezoelectric effects only due to phonon fields, since the phonon piezoelectric moduli is non-zero whereas the phason piezoelectric moduli is zero for these two point groups; a case that cannot be true as it has been shown above.*

Remark 5. *Comparing Table 2 with the corresponding Table 1 in [28], one can see that there are differences in the number of the independent components of phason piezoelectric moduli. Notice that the tensor of phason piezoelectric moduli, as a tensor of rank 2, has the same properties, that*

is, the same number of independent components for point groups belonging to the same Laue class. However, this is not the case in [28] (see also Remark 6).

To sum up: *from the 31 crystallographic point groups of one-dimensional quasicrystals, the 10 centrosymmetric crystallographic point groups have no piezoelectric effects and the remaining 21 non-centrosymmetric crystallographic point groups exhibit piezoelectric effects due to both phonon and phason fields.*

Furthermore, it is worth mentioning that the 21 non-centrosymmetric point groups which produce piezoelectricity due to phonon and phason fields in one-dimensional quasicrystals (see Table 2) are the same as the 21 non-centrosymmetric point groups which produce piezoelectricity due to phonon and phason fields in two-dimensional quasicrystals with crystallographic symmetries, as shown by Hu et al. [15].

Remark 6. *Notice that the phason piezoelectric moduli for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 9 with point group $\bar{6}$ and Laue class 10 with point group $\bar{6}m2_h$ are zero according to Li and Liu [28], producing no piezoelectric effect due to the phasons. However, as it can be seen in Table 1, there exist three and two independent components of phason piezoelectric moduli for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 9 (independent of the point group) and Laue class 10 (independent of the point group), respectively, and therefore there is a piezoelectric contribution due to phasons. The misleading result of Li and Liu [28] can also be explained by the fact that point groups belonging to the same Laue class, such as hexagonal Laue class 9 and Laue class 10, have different number of independent components of phason piezoelectric moduli. However, the independent components of the tensor of phason piezoelectric moduli, as an asymmetric tensor of rank 2, are the same for the point groups belonging to the same Laue class in the case of one-dimensional quasicrystals (see also Table 2).*

3.3. One-Dimensional Hexagonal Piezoelectric Quasicrystals

In order to gain more insight into the significance of the piezoelectric constitutive tensors, we proceed to explicitly give the constitutive relations for one-dimensional hexagonal piezoelectric quasicrystals of Laue classes 9 and 10. The forthcoming comparisons between the two sets of the constitutive relations reveal important differences between the two Laue classes, which are often ignored in the literature, not only in the framework of piezoelectricity (e.g., [24–27]), but also in the framework of elasticity (see e.g., [9,22,23]) of one-dimensional hexagonal quasicrystals.

For one-dimensional piezoelectric quasicrystals, Laue classes are not enough for the explicit form of the constitutive relations, since the tensor of phonon piezoelectric moduli e_{lij} depends on the considered point group (see [2]). In what follows, we examine two Laue classes of one-dimensional piezoelectric quasicrystals with a representative point group.

- For Laue class 9 with point group 6, using Equations (A5), (A11), and (A19) and the expressions for the tensors of the elastic moduli and phonon piezoelectric moduli given by Nye [2], the constitutive relations (44)–(46) take the following explicit form:

$$\sigma_{11}^{\parallel} = C_{1111}e_{11}^{\parallel} + C_{1122}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp} - e_{311}E_3 \,, \tag{60}$$

$$\sigma_{22}^{\parallel} = C_{1122}e_{11}^{\parallel} + C_{1111}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp} - e_{311}E_3 \,, \tag{61}$$

$$\sigma_{33}^{\parallel} = C_{3311}(e_{11}^{\parallel} + e_{22}^{\parallel}) + C_{3333}e_{33}^{\parallel} + D_{3333}\beta_{33}^{\perp} - e_{333}E_3 \,, \tag{62}$$

$$\sigma_{23}^{\parallel} = 2C_{2323}e_{23}^{\parallel} + D_{2331}\beta_{31}^{\perp} + D_{2332}\beta_{32}^{\perp} - e_{123}E_1 - e_{113}E_2 \,, \tag{63}$$

$$\sigma_{13}^{\parallel} = 2C_{2323}e_{13}^{\parallel} + D_{2332}\beta_{31}^{\perp} - D_{2331}\beta_{32}^{\perp} - e_{113}E_1 + e_{123}E_2 \,, \tag{64}$$

$$\sigma_{12}^{\parallel} = 2C_{1212}e_{12}^{\parallel}, \tag{65}$$

$$\sigma_{31}^{\perp} = 2D_{2331}e_{23}^{\parallel} + 2D_{2332}e_{13}^{\parallel} + E_{3131}\beta_{31}^{\perp} - f_{131}E_1 + f_{132}E_2, \tag{66}$$

$$\sigma_{32}^{\perp} = 2D_{2332}e_{23}^{\parallel} - 2D_{2331}e_{13}^{\parallel} + E_{3131}\beta_{32}^{\perp} - f_{132}E_1 - f_{131}E_2, \tag{67}$$

$$\sigma_{33}^{\perp} = D_{1133}\left(e_{11}^{\parallel} + e_{22}^{\parallel}\right) + D_{3333}e_{33}^{\parallel} + E_{3333}\beta_{33}^{\perp} - f_{333}E_3, \tag{68}$$

$$D_1 = 2e_{123}e_{23}^{\parallel} + 2e_{113}e_{13}^{\parallel} + f_{131}\beta_{31}^{\perp} + f_{132}\beta_{32}^{\perp} + \varepsilon_{11}E_1, \tag{69}$$

$$D_2 = 2e_{113}e_{23}^{\parallel} - 2e_{123}e_{13}^{\parallel} - f_{132}\beta_{31}^{\perp} + f_{131}\beta_{32}^{\perp} + \varepsilon_{11}E_2, \tag{70}$$

$$D_3 = e_{311}\left(e_{11}^{\parallel} + e_{22}^{\parallel}\right) + e_{333}e_{33}^{\parallel} + f_{333}\beta_{33}^{\perp} + \varepsilon_{33}E_3 \tag{71}$$

with five elastic moduli of phonons, C_{1111}, C_{1122}, C_{1133}, C_{3333}, C_{2323}, C_{1212} = $(C_{1111} - C_{1122})/2$; two elastic moduli of phasons, E_{3131} and E_{3333}; four elastic moduli of phonon–phason coupling, D_{2331}, D_{2332}, D_{1133}, D_{3333}; four phonon piezoelectric moduli, e_{311}, e_{333}, e_{113}, e_{123}; three phason piezoelectric moduli, f_{131}, f_{132}, f_{333}; and two dielectric moduli, ε_{11} and ε_{33}.

Thus, a *one-dimensional hexagonal piezoelectric quasicrystal of Laue class 9 with point group 6 has 20 material moduli.*

- For Laue class 10 with point group 6*mm*, using Equations (A5), (A12), and (A20) and the expressions for the tensors of the elastic moduli and phonon piezoelectric moduli given by Nye [2], the constitutive relations (44)–(46) take the following explicit form:

$$\sigma_{11}^{\parallel} = C_{1111}e_{11}^{\parallel} + C_{1122}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp} - e_{311}E_3, \tag{72}$$

$$\sigma_{22}^{\parallel} = C_{1122}e_{11}^{\parallel} + C_{1111}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp} - e_{311}E_3, \tag{73}$$

$$\sigma_{33}^{\parallel} = C_{3311}\left(e_{11}^{\parallel} + e_{22}^{\parallel}\right) + C_{3333}e_{33}^{\parallel} + D_{3333}\beta_{33}^{\perp} - e_{333}E_3, \tag{74}$$

$$\sigma_{23}^{\parallel} = 2C_{2323}e_{23}^{\parallel} + D_{2332}\beta_{32}^{\perp} - e_{113}E_2, \tag{75}$$

$$\sigma_{13}^{\parallel} = 2C_{2323}e_{13}^{\parallel} + D_{2332}\beta_{31}^{\perp} - e_{113}E_1, \tag{76}$$

$$\sigma_{12}^{\parallel} = 2C_{1212}e_{12}^{\parallel}, \tag{77}$$

$$\sigma_{31}^{\perp} = 2D_{2332}e_{13}^{\parallel} + E_{3131}\beta_{31}^{\perp} - f_{131}E_1, \tag{78}$$

$$\sigma_{32}^{\perp} = 2D_{2332}e_{23}^{\parallel} + E_{3131}\beta_{32}^{\perp} - f_{131}E_2, \tag{79}$$

$$\sigma_{33}^{\perp} = D_{1133}\left(e_{11}^{\parallel} + e_{22}^{\parallel}\right) + D_{3333}e_{33}^{\parallel} + E_{3333}\beta_{33}^{\perp} - f_{333}E_3, \tag{80}$$

$$D_1 = 2e_{113}e_{13}^{\parallel} + f_{131}\beta_{31}^{\perp} + \varepsilon_{11}E_1, \tag{81}$$

$$D_2 = 2e_{113}e_{23}^{\parallel} + f_{131}\beta_{32}^{\perp} + \varepsilon_{11}E_2, \tag{82}$$

$$D_3 = e_{311}\left(e_{11}^{\parallel} + e_{22}^{\parallel}\right) + e_{333}e_{33}^{\parallel} + f_{333}\beta_{33}^{\perp} + \varepsilon_{33}E_3 \tag{83}$$

with five elastic moduli of phonons, C_{1111}, C_{1122}, C_{1133}, C_{3333}, C_{2323}, C_{1212} = $(C_{1111} - C_{1122})/2$; two elastic moduli of phasons, E_{3131} and E_{3333}; three elastic moduli of phonon–phason coupling, D_{2332}, D_{1133}, D_{3333}; three phonon piezoelectric moduli, e_{311}, e_{333}, e_{113}; two phason piezoelectric moduli f_{131} and f_{333}; and two dielectric moduli, ε_{11} and ε_{33}.

Thus, *a one-dimensional hexagonal piezoelectric quasicrystal of Laue class 10 with point group 6mm has 17 material moduli.*

In the framework of piezoelectricity, the constitutive relations for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 10 with point group 6*mm*, Equations (72)–(83), are in accordance with the corresponding ones given by Wang and Pan [18], where $D_{1133} = R_1$, $D_{3333} = R_2$, $D_{2332} = R_3$, $E_{3131} = K_2$, $E_{3333} = K_1$, $e_{311} = e_{31}^{(1)}$, $e_{333} = e_{33}^{(1)}$, $e_{113} = e_{15}^{(1)}$,

$f_{131} = e_{15}^{(2)}$, $f_{333} = e_{33}^{(2)}$. The constitutive relations for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 9 with point group 6, Equations (60)–(71), are given here for the first time in the literature.

It can be seen that the constitutive relations (60)–(71) for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 9 with point group 6 are different than the constitutive relations (72)–(83) for the hexagonal piezoelectric quasicrystals of Laue class 10 with point group 6*mm*. In particular, comparing the above two sets of the constitutive relations, we observe that two additional terms appear in each of the Equations (63), (64), (66), (67), (69), and (70) due to the different matrix representation of three tensors, namely the tensor of elastic moduli of phonon–phason coupling D_{ij3l} (see Equations (A11) and (A12)), the tensor of phason piezoelectric moduli f_{i3j} (see Equations (A19) and (A20)), and finally the tensor of phonon piezoelectric moduli e_{lij}. This difference in the constitutive relations is often ignored and neglected in the literature of one-dimensional piezoelectric quasicrystals (e.g., [24–27]). In general, for one-dimensional hexagonal piezoelectric quasicrystals, there are five non-centrosymmetric point groups (see Table 2), leading to different constitutive relations. For the explicit form of the constitutive relations of piezoelectric quasicrystals, the corresponding point group is relevant and must be taken into account.

Furthermore, as it has been mentioned above, a reason for the difference in the two sets of the constitutive relations lies on the different matrix representation of the tensor of the elastic moduli of phonon–phason coupling D_{ij3l}. Therefore, a difference in the constitutive relations of one-dimensional hexagonal quasicrystals between the two Laue classes 9 and 10 is expected also in the framework of elasticity. This is a fact that is often ignored in the literature of elasticity theory of one-dimensional hexagonal quasicrystals, since no attention is given to the distinction of the two Laue classes (see, e.g., [9,22,23]). The constitutive relations of one-dimensional hexagonal quasicrystals for the two Laue classes 9 and 10 are derived from Equations (60)–(71) and Equations (72)–(83), respectively, for vanishing electric fields, and they are given explicitly below.

- The constitutive relations for one-dimensional hexagonal quasicrystals of Laue class 9 are as follows:

$$\sigma_{11}^{\parallel} = C_{1111}e_{11}^{\parallel} + C_{1122}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp}, \tag{84}$$

$$\sigma_{22}^{\parallel} = C_{1122}e_{11}^{\parallel} + C_{1111}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp}, \tag{85}$$

$$\sigma_{33}^{\parallel} = C_{3311}(e_{11}^{\parallel} + e_{22}^{\parallel}) + C_{3333}e_{33}^{\parallel} + D_{3333}\beta_{33}^{\perp}, \tag{86}$$

$$\sigma_{23}^{\parallel} = 2C_{2323}e_{23}^{\parallel} + D_{2331}\beta_{31}^{\perp} + D_{2332}\beta_{32}^{\perp}, \tag{87}$$

$$\sigma_{13}^{\parallel} = 2C_{2323}e_{13}^{\parallel} + D_{2332}\beta_{31}^{\perp} - D_{2331}\beta_{32}^{\perp}, \tag{88}$$

$$\sigma_{12}^{\parallel} = 2C_{1212}e_{12}^{\parallel}, \tag{89}$$

$$\sigma_{31}^{\perp} = 2D_{2331}e_{23}^{\parallel} + 2D_{2332}e_{13}^{\parallel} + E_{3131}\beta_{31}^{\perp}, \tag{90}$$

$$\sigma_{32}^{\perp} = 2D_{2332}e_{23}^{\parallel} - 2D_{2331}e_{13}^{\parallel} + E_{3131}\beta_{32}^{\perp}, \tag{91}$$

$$\sigma_{33}^{\perp} = D_{1133}(e_{11}^{\parallel} + e_{22}^{\parallel}) + D_{3333}e_{33}^{\parallel} + E_{3333}\beta_{33}^{\perp} \tag{92}$$

  with five elastic moduli of phonons, C_{1111}, C_{1122}, C_{1133}, C_{3333}, C_{2323}, $C_{1212} = (C_{1111} - C_{1122})/2$; two elastic moduli of phasons, E_{3131} and E_{3333}; and four elastic moduli of phonon–phason coupling, D_{2331}, D_{2332}, D_{1133}, D_{3333}.
  Thus, a *one-dimensional hexagonal quasicrystal of Laue class 9 has 11 material moduli*.
- The constitutive relations of one-dimensional hexagonal quasicrystals of Laue class 10 read:

$$\sigma_{11}^{\parallel} = C_{1111}e_{11}^{\parallel} + C_{1122}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp}, \tag{93}$$

$$\sigma_{22}^{\parallel} = C_{1122}e_{11}^{\parallel} + C_{1111}e_{22}^{\parallel} + C_{1133}e_{33}^{\parallel} + D_{1133}\beta_{33}^{\perp}, \tag{94}$$

$$\sigma_{33}^{\parallel} = C_{3311}(e_{11}^{\parallel} + e_{22}^{\parallel}) + C_{3333}e_{33}^{\parallel} + D_{3333}\beta_{33}^{\perp}, \tag{95}$$

$$\sigma_{23}^{\parallel} = 2C_{2323}e_{23}^{\parallel} + D_{2332}\beta_{32}^{\perp}, \tag{96}$$

$$\sigma_{13}^{\parallel} = 2C_{2323}e_{13}^{\parallel} + D_{2332}\beta_{31}^{\perp}, \tag{97}$$

$$\sigma_{12}^{\parallel} = 2C_{1212}e_{12}^{\parallel}, \tag{98}$$

$$\sigma_{31}^{\perp} = 2D_{2332}e_{13}^{\parallel} + E_{3131}\beta_{31}^{\perp}, \tag{99}$$

$$\sigma_{32}^{\perp} = 2D_{2332}e_{23}^{\parallel} + E_{3131}\beta_{32}^{\perp}, \tag{100}$$

$$\sigma_{33}^{\perp} = D_{1133}(e_{11}^{\parallel} + e_{22}^{\parallel}) + D_{3333}e_{33}^{\parallel} + E_{3333}\beta_{33}^{\perp} \tag{101}$$

with five elastic moduli of phonons, C_{1111}, C_{1122}, C_{1133}, C_{3333}, C_{2323}, $C_{1212} = (C_{1111} - C_{1122})/2$; two elastic moduli of phasons, E_{3131}, and E_{3333}; and three elastic moduli of phonon–phason coupling, D_{2332}, D_{1133}, D_{3333}.

Thus, a *one-dimensional hexagonal quasicrystal of Laue class 10 has 10 material moduli.*

It has to be mentioned that: (i) the non-zero elastic constants given above for both Laue classes 9 and 10 are in accordance with the corresponding non-zero elastic constants given by Wang et al. [7]; (ii) the constitutive relations (93)–(101) are in accordance with the constitutive relations of one-dimensional hexagonal quasicrystals of Laue class 10 given by Wang et al. [7].

4. Conclusions

The main favorable conclusions of this work are summarized as follows:

- It is proven that the tensor of phason piezoelectric moduli $f_{i\alpha j}$ is fully asymmetric without any major or minor symmetry in contrast to the work of Altay and Dökmeci [29], where it is claimed that the tensor $f_{i\alpha j}$ is symmetric with respect to the last two indices. Therefore, the tensor of phason piezoelectric moduli $f_{i\alpha j}$ is a two-point tensor of rank 3 in the case of three-dimensional piezoelectric quasicrystals and is represented by a 3×9 matrix with 27 independent components. It is not represented by a 3×6 matrix with 18 independent components, which would be the case if possessing a minor symmetry with respect to the last two indices, as claimed in [29], which is however not the case.

- The classification of the tensor of phason piezoelectric moduli f_{i3j} for one-dimensional piezoelectric quasicrystals has been derived in a 3×3 matrix form for all crystal systems and Laue classes. Hence, f_{i3j} cannot be represented in a 3×6 matrix form as claimed in [28].

- It has been shown that all non-centrosymmetric crystallographic point groups of one-dimensional quasicrystals exhibit piezoelectric effects induced by both phonon and phason fields in contrast to the result of Li and Liu [28], where there are two cases of one-dimensional hexagonal piezoelectric quasicrystals, namely of Laue class 9 with point group $\bar{6}$ and Laue class 10 with point group $\bar{6}m2_h$, in which there is a piezoelectric effect induced only by the phonon and not by the phason fields.

- It has been shown that one-dimensional quasicrystals with centrosymmetric crystallographic point groups have no piezoelectric effects.

- The piezoelectric behavior of one-dimensional quasicrystals has been explored here. It is shown that one-dimensional quasicrystals can be divided into two classes: the first class, consisting of one-dimensional quasicrystals with centrosymmetry, has no piezoelectric effects and the second class, consisting of one-dimensional quasicrystals without centrosymmetry, has piezoelectric effects induced by both phonon and phason fields.

- For the derivation of the constitutive relations of elasticity of quasicrystals, the Laue class has to be taken into account. As a representative example, the explicit constitutive relations for Laue class 9 and Laue class 10 have been derived. A one-dimensional hexagonal quasicrystal of Laue class 9 has 11 material moduli, whereas that of Laue class 10 has 10 material moduli.
- For the derivation of the constitutive relations of piezoelectricity of quasicrystals, the point group has to be taken into account in addition to the Laue class. As a representative example, the explicit constitutive relations for Laue class 9 with point group 6 and Laue class 10 with point group $6mm$ have been derived. A one-dimensional hexagonal piezoelectric quasicrystal of Laue class 9 with point group 6 has 20 material moduli, whereas that of Laue class 10 with point group $6mm$ has 17 material moduli.

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Appendix A. Classification of the Tensor E_{3j3l} for One-Dimensional Quasicrystals

In this Appendix, following Wang et al. [7], we give the classification of the symmetric tensor of rank 2, $E_{3j3l} = E_{3l3j}$, in a 3×3 matrix representation for different Laue classes.

Appendix A.1. Triclinic

For one-dimensional triclinic quasicrystals of Laue class 1, there are six independent components:

$$E_{3j3l} = \begin{pmatrix} E_{3131} & E_{3132} & E_{3133} \\ E_{3132} & E_{3232} & E_{3233} \\ E_{3133} & E_{3233} & E_{3333} \end{pmatrix}_{(6)}. \tag{A1}$$

Appendix A.2. Monoclinic

(i) For one-dimensional monoclinic quasicrystals of Laue class 2, there are four independent components:

$$E_{3j3l} = \begin{pmatrix} E_{3131} & E_{3132} & 0 \\ E_{3132} & E_{3232} & 0 \\ 0 & 0 & E_{3333} \end{pmatrix}_{(4)}. \tag{A2}$$

The unique axis of the point groups 2, m_h and $2/m_h$ is the x_3-axis.

(ii) For one-dimensional monoclinic quasicrystals of Laue class 3, there are four independent components:

$$E_{3j3l} = \begin{pmatrix} E_{3131} & 0 & E_{3133} \\ 0 & E_{3232} & 0 \\ E_{3133} & 0 & E_{3333} \end{pmatrix}_{(4)}. \tag{A3}$$

The unique axis lies in the horizontal plane, e.g., along the x_2-axis.

Appendix A.3. Orthorhombic

For one-dimensional orthorhombic quasicrystals of Laue class 4, there are three independent components:

$$E_{3j3l} = \begin{pmatrix} E_{3131} & 0 & 0 \\ 0 & E_{3232} & 0 \\ 0 & 0 & E_{3333} \end{pmatrix}_{(3)}. \tag{A4}$$

Appendix A.4. Tetragonal, Trigonal, Hexagonal

For one-dimensional tetragonal quasicrystals of Laue classes 5 and 6; for one-dimensional trigonal quasicrystals of Laue classes 7 and 8; and for one-dimensional hexagonal quasicrystals of Laue classes 9 and 10, there are two independent components:

$$E_{3j3l} = \begin{pmatrix} E_{3131} & 0 & 0 \\ 0 & E_{3131} & 0 \\ 0 & 0 & E_{3333} \end{pmatrix}_{(2)}. \tag{A5}$$

Appendix B. Classification of the Tensor D_{ij3l} for One-Dimensional Quasicrystals

In this Appendix, following Wang et al. [7], we give the classification of the tensor D_{ij3l}, which is a tensor of rank 3 symmetric with respect to the first two indices $D_{ij3l} = D_{ji3l}$, as a 6×3 matrix for different Laue classes. To obtain a 6×3 matrix representation for the tensor D_{ij3l}, the following arrangement is used for the first two indices

$$(ij) = 11, \quad 22, \quad 33, \quad 23, 32, \quad 31, 13, \quad 12, 21. \tag{A6}$$

Appendix B.1. Triclinic

For one-dimensional triclinic quasicrystals of Laue class 1, there are 18 independent components:

$$D_{ij3l} = \begin{pmatrix} D_{1131} & D_{1132} & D_{1133} \\ D_{2231} & D_{2232} & D_{2233} \\ D_{3331} & D_{3332} & D_{3333} \\ D_{2331} & D_{2332} & D_{2333} \\ D_{1331} & D_{1332} & D_{1333} \\ D_{1231} & D_{1232} & D_{1233} \end{pmatrix}_{(18)}. \tag{A7}$$

Appendix B.2. Monoclinic

(i)   For one-dimensional monoclinic quasicrystals of Laue class 2, the eight independent tensor components are $D_{1133}, D_{2233}, D_{3333}, D_{1233}, D_{2331}, D_{2332}, D_{1331}, D_{1332}$.

$$D_{ij3l} = \begin{pmatrix} 0 & 0 & D_{1133} \\ 0 & 0 & D_{2233} \\ 0 & 0 & D_{3333} \\ D_{2331} & D_{2332} & 0 \\ D_{1331} & D_{1332} & 0 \\ 0 & 0 & D_{1233} \end{pmatrix}_{(8)}. \tag{A8}$$

The unique axis of the point groups 2, m_h and $2/m_h$ is the x_3-axis.

(ii) For one-dimensional monoclinic quasicrystals of Laue class 3, the 10 independent tensor components are $D_{1133}, D_{2233}, D_{3333}, D_{1333}, D_{1131}, D_{2231}, D_{3331}, D_{1331}, D_{2332}, D_{1232}$.

$$
D_{ij3l} = \begin{pmatrix}
D_{1131} & 0 & D_{1133} \\
D_{2231} & 0 & D_{2233} \\
D_{3331} & 0 & D_{3333} \\
0 & D_{2332} & 0 \\
D_{1331} & 0 & D_{1333} \\
0 & D_{1232} & 0
\end{pmatrix}_{(10)}.
\tag{A9}
$$

The unique axis lies in the horizontal plane, e.g., along the x_2-axis.

Appendix B.3. Orthorhombic

For one-dimensional orthorhombic quasicrystals of Laue class 4, the five independent tensor components are D_{1133}, D_{2233}, D_{3333}, D_{2332}, D_{1331}.

$$
D_{ij3l} = \begin{pmatrix}
0 & 0 & D_{1133} \\
0 & 0 & D_{2233} \\
0 & 0 & D_{3333} \\
0 & D_{2332} & 0 \\
D_{1331} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}_{(5)}.
\tag{A10}
$$

Appendix B.4. Tetragonal, Hexagonal

(i)   For one-dimensional tetragonal quasicrystals of Laue class 5; and for one-dimensional hexagonal quasicrystals of Laue class 9, the four independent tensor components are $D_{1133} = D_{2233}$, D_{3333}, $D_{2332} = D_{1331}$, $D_{2331} = -D_{1332}$.

$$
D_{ij3l} = \begin{pmatrix}
0 & 0 & D_{1133} \\
0 & 0 & D_{1133} \\
0 & 0 & D_{3333} \\
D_{2331} & D_{2332} & 0 \\
D_{2332} & -D_{2331} & 0 \\
0 & 0 & 0
\end{pmatrix}_{(4)}.
\tag{A11}
$$

(ii)   For one-dimensional tetragonal quasicrystals of Laue class 6; and for one-dimensional hexagonal quasicrystals of Laue class 10, the three independent tensor components are $D_{1133} = D_{2233}$, D_{3333}, $D_{2332} = D_{1331}$.

$$
D_{ij3l} = \begin{pmatrix}
0 & 0 & D_{1133} \\
0 & 0 & D_{1133} \\
0 & 0 & D_{3333} \\
0 & D_{2332} & 0 \\
D_{2332} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}_{(3)}.
\tag{A12}
$$

Appendix B.5. Trigonal

(i)   For one-dimensional trigonal quasicrystals of Laue class 7, the six independent tensor components are $D_{1133} = D_{2233}$, D_{3333}, $D_{2332} = D_{1331}$, $D_{2331} = -D_{1332}$, $D_{1131} = -D_{2231} = -D_{1232}$, $D_{1132} = -D_{2232} = D_{1231}$.

$$
D_{ij3l} = \begin{pmatrix}
D_{1131} & D_{1132} & D_{1133} \\
-D_{1131} & -D_{1132} & D_{1133} \\
0 & 0 & D_{3333} \\
D_{2331} & D_{2332} & 0 \\
D_{2332} & -D_{2331} & 0 \\
D_{1132} & -D_{1131} & 0
\end{pmatrix}_{(6)}.
\tag{A13}
$$

(ii) For one-dimensional trigonal quasicrystals of Laue class 8, the four independent tensor components are $D_{1133} = D_{2233}$, D_{3333}, $D_{2332} = D_{1331}$, $D_{1132} = -D_{2232} = D_{1231}$.

$$
D_{ij3l} = \begin{pmatrix} 0 & D_{1132} & D_{1133} \\ 0 & -D_{1132} & D_{1133} \\ 0 & 0 & D_{3333} \\ 0 & D_{2332} & 0 \\ D_{2332} & 0 & 0 \\ D_{1132} & 0 & 0 \end{pmatrix}_{(4)}. \tag{A14}
$$

Appendix C. Classification of the Tensor f_{j3l} for One-Dimensional Piezoelectric Quasicrystals

In this Appendix, we give the classification of the asymmetric tensor of rank 2, f_{j3l}, in a 3×3 matrix representation for different Laue classes.

Appendix C.1. Triclinic

For one-dimensional triclinic piezoelectric quasicrystals of Laue class 1, there are nine independent components:

$$
f_{j3l} = \begin{pmatrix} f_{131} & f_{132} & f_{133} \\ f_{231} & f_{232} & f_{233} \\ f_{331} & f_{332} & f_{333} \end{pmatrix}_{(9)}. \tag{A15}
$$

Appendix C.2. Monoclinic

(i) For one-dimensional monoclinic piezoelectric quasicrystals of Laue class 2, there are five independent components:

$$
f_{j3l} = \begin{pmatrix} f_{131} & f_{132} & 0 \\ f_{231} & f_{232} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(5)}. \tag{A16}
$$

The unique axis of the point groups 2, m_h and $2/m_h$ is the x_3-axis.

(ii) For one-dimensional monoclinic piezoelectric quasicrystals of Laue class 3, there are five independent components:

$$
f_{j3l} = \begin{pmatrix} f_{131} & 0 & f_{133} \\ 0 & f_{232} & 0 \\ f_{331} & 0 & f_{333} \end{pmatrix}_{(5)}. \tag{A17}
$$

The unique axis lies in the horizontal plane, e.g., along the x_2-axis.

Appendix C.3. Orthorhombic

For one-dimensional orthorhombic piezoelectric quasicrystals of Laue class 4, there are three independent components:

$$
f_{j3l} = \begin{pmatrix} f_{131} & 0 & 0 \\ 0 & f_{232} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(3)}. \tag{A18}
$$

Appendix C.4. Tetragonal, Trigonal, Hexagonal

(i) For one-dimensional tetragonal piezoelectric quasicrystals of Laue class 5; for one-dimensional trigonal piezoelectric quasicrystals of Laue class 7; and for one-dimensional

hexagonal piezoelectric quasicrystals of Laue class 9, there are three independent components:

$$f_{j3l} = \begin{pmatrix} f_{131} & f_{132} & 0 \\ -f_{132} & f_{131} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(3)}. \tag{A19}$$

(ii) For one-dimensional tetragonal piezoelectric quasicrystals of Laue class 6; for one-dimensional trigonal piezoelectric quasicrystals of Laue class 8; and for one-dimensional hexagonal piezoelectric quasicrystals of Laue class 10, there are two independent components:

$$f_{j3l} = \begin{pmatrix} f_{131} & 0 & 0 \\ 0 & f_{131} & 0 \\ 0 & 0 & f_{333} \end{pmatrix}_{(2)}. \tag{A20}$$

References

1. Schouten, J.-A. *Tensor Analysis for Physicists*; Oxford University Press: Oxford, UK, 1951.
2. Nye, J.-F. *Physical Properties of Crystals*; Oxford University Press: Oxford, UK, 1957.
3. Landau, L.-D.; Lifshitz, E.-M. *Electrodynamics of Continuous Media*; Pergamon Press: Oxford, UK, 1960.
4. Shechtman, D.; Blech, I.; Gratias, D.; Cahn, J.-W. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* **1984**, *53*, 1951–1953. [CrossRef]
5. Bak, P. Symmetry, stability, and elastic properties of icosahedral incommensurate crystals. *Phys. Rev. B* **1985**, *32*, 5764–5772. [CrossRef] [PubMed]
6. Levine, D.; Lubensky, T.-C.; Ostlund, S.; Ramaswamy, S.; Steinhardt, P.-J.; Toner, J. Elasticity and dislocations in pentagonal and icosahedral quasicrystals. *Phys. Rev. Lett.* **1985**, *54*, 1520–1523. [CrossRef] [PubMed]
7. Wang, R.; Yang, W.; Hu, C.; Ding, D.-H. Point and space groups and elastic behaviours of one-dimensional quasicrystals. *J. Phys. Condens. Matter* **1997**, *9*, 2411–2422. [CrossRef]
8. Hu, C.; Wang, R.; Yang, W.; Ding, D. Point groups and elastic properties of two-dimensional quasicrystals. *Acta Cryst. A* **1996**, *52*, 251–256. [CrossRef]
9. Fan, T.-Y. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*; Science Press: Beijing, China; Springer: Berlin/Heidelberg, Germany, 2011.
10. Socolar, J.-E.-S. Simple octagonal and dodecagonal quasicrystals. *Phys. Rev. B* **1989**, *39*, 10519–10551. [CrossRef] [PubMed]
11. Ding, D.-H.; Yang, W.; Hu, C.; Wang, R. Generalized elasticity theory of quasicrystals. *Phys. Rev. B* **1993**, *48*, 7003–7010. [CrossRef]
12. Yang, W.; Wang, R.; Ding, D.-H.; Hu, C. Linear elasticity theory of cubic quasicrystals. *Phys. Rev. B* **1993**, *48*, 6999–7003. [CrossRef]
13. Yang, W.; Wang, R.; Ding, D.-H.; Hu, C. Elastic strains induced by electric fields in quasicrystals. *J. Phys. Condens. Matter* **1995**, *7*, L499–L502. [CrossRef]
14. Yang, W.; Ding, D.-H.; Wang, R.; Hu, C. Thermodynamics of equilibrium properties of quasicrystals. *Z. Phys. B* **1996**, *10*, 447–454. [CrossRef]
15. Hu, C.; Wang, R.; Ding, D.-H.; Yang, W. Piezoelectric effects in quasicrystals. *Phys. Rev. B* **1997**, *56*, 2463–2468. [CrossRef]
16. Hu, C.; Wang, R.; Ding, D.-H. Symmetry groups, physical property tensors, elasticity and dislocations in quasicrystals. *Rep. Prog. Phys.* **2000**, *63*, 1–39. [CrossRef]
17. Gong, P.; Hu, C.-Z.; Zhou, X.; Miao, L.; Wang, X. Isotropic and anisotropic properties of quasicrystals. *Eur. Phys. J. B* **2006**, *52*, 477–481. [CrossRef]
18. Wang, X.; Pan, E. Analytical solutions for some defect problems in 1D hexagonal and 2D octagonal quasicrystals. *Pramana J. Phys.* **2008**, *70*, 911–933. [CrossRef]
19. Yang, L.-Z.; Gao, Y.; Pan, E.; Waksmanski, N. Electric-elastic field induced by a straight dislocation in one-dimensional quasicrystals. *Acta Phys. Pol. A* **2014**, *126*, 467–470. [CrossRef]
20. Jbaily, A.; Yeung, R.-W. Piezoelectric devices for ocean energy: A brief survey. *J. Ocean Eng. Mar. Energy* **2015**, *1*, 101–118. [CrossRef]
21. Yang, W.; Hu, C.; Ding, D.-H.; Wang, R. Differences in elastic behavior between pentagonal and decagonal quasicrystals. *Phys. Rev. B* **1995**, *51*, 3906–3909. [CrossRef]
22. Chen, W.-Q.; Ma, Y.-L.; Ding, H.-J. On three-dimensional elastic problems of one-dimensional hexagonal quasicrystal bodies. *Mech. Res. Commun.* **2004**, *31*, 633–641. [CrossRef]
23. Guo, J.-H.; Yu, J.; Xing, Y.-M. Anti-plane analysis on a finite crack in a one-dimensional hexagonal quasicrystal strip. *Mech. Res. Commun.* **2013**, *52*, 40–45. [CrossRef]

24. Li, X.-Y.; Li, P.D.; Wu, T.-H.; Shi, M.-X.; Zhu, Z.-W. Three-dimensional fundamental solutions for one-dimensional hexagonal quasicrystal with piezoelectric effect. *Phys. Lett. A* **2014**, *378*, 826–834. [CrossRef]
25. Yu, J.; Guo, J.; Xing, Y. Complex variable method for an anti-plane elliptical cavity of one-dimensional hexagonal piezoelectric quasicrystals. *Chin. J. Aero.* **2015**, *28*, 1287–1295. [CrossRef]
26. Yang, J.; Zhou, Y.-T.; Ma, H.-L.; Ding, S.-H.; Li, X. The fracture behavior of two asymmetrical limited permeable cracks emanating from an elliptical hole in one-dimensional hexagonal quasicrystals with piezoelectric effect. *Int. J. Solids Struct.* **2017**, *108*, 175–185. [CrossRef]
27. Jiang, L.-J.; Liu, G.-T. The interaction between a screw dislocation and a wedge-shaped crack in one-dimensional hexagonal piezoelectric quasicrystals. *Chin. Phys. B* **2017**, *26*, 044601. [CrossRef]
28. Li, C.-L.; Liu, Y.-Y. The physical property tensors of one-dimensional quasicrystals. *Chin. Phys.* **2004**, *13*, 924–931.
29. Altay, G.; Dökmeci, M.-C. On the fundamental equations of piezoelasticity of quasicrystal media. *Int. J. Solids Struct.* **2012**, *49*, 3255–3262. [CrossRef]
30. Yang, W.; Ding, D.-H.; Hu, C.; Wang, R. Group-theoretical derivation of the numbers of independent physical constants of quasicrystals. *Phys. Rev. B* **1994**, *49*, 12656. [CrossRef] [PubMed]
31. Lazar, M.; Agiasofitou, E. Eshelbian mechanics of novel materials: Quasicrystals. *J. Micromech. Mol. Phys.* **2016**, *1*, 1640008. [CrossRef]
32. Jackson, J.-D. *Classical Electrodynamics*, 2nd ed.; Wiley: New York, NY, USA, 1975.
33. Kienzler, R.; Herrmann, G. *Mechanics in Material Space with Applications to Defect and Fracture Mechanics*; Springer: Berlin/Heidelberg, Germany, 2000.
34. Tiersten, H.-F. *Linear Piezoelectric Plate Vibrations*; Plenum Press: New York, NY, USA, 1969.
35. Agiasofitou, E.; Lazar, M. Electro-elastic dislocations in piezoelectric materials. *Phil. Mag.* **2020**, *100*, 1059–1101. [CrossRef]
36. Podio-Guidugli, P. A primer in elasticity. *J. Elast.* **2000**, *58*, 1–104. [CrossRef]
37. Yu, J.; Guo, J.; Pan, E.; Xing, Y. General solutions of plane problem in one-dimensional quasicrystal piezoelectric materials and its application on fracture mechanics. *Appl. Math. Mech.-Engl.* **2015**, *36*, 793–814. [CrossRef]
38. Lazar, M.; Agiasofitou, E. Fundamentals in generalized elasticity and dislocation theory of quasicrystals: Green tensor, dislocation key-formulas and dislocation loops. *Phil. Mag.* **2014**, *94*, 4080–4101. [CrossRef]
39. Agiasofitou, E.; Lazar, M.; Kirchner, H. Generalized dynamics of moving dislocations in quasicrystals. *J. Phys. Condens. Matter* **2010**, *22*, 495401. [CrossRef] [PubMed]
40. Nowick, A.-S. *Crystal Properties via Group Theory*; Cambridge University Press: Cambridge, UK, 1995.