# Expected Utility Maximization for Competitive Agents

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*Meinen Eltern gewidmet – Für alles.*

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# Prior Publications

Parts of Chapters 3, 4, 5, and 7 are direct quotes from the following prior publications:

- Bäuerle and Göll (2023a). Nash equilibria for relative investors via no-arbitrage arguments. *Mathematical Methods of Operations Research* **97**: 1–23.
- Bäuerle and Göll (2023b). Nash equilibria for relative investors with (non)linear price impact. arxiv:2303.18161.

# Contents

# Chapter 1

# Introduction

The general idea behind portfolio optimization problems is the following: An investor, endowed with some fixed initial capital $x \in \mathbb{R}$, has to decide *how many* shares of *which assets* she should hold at *which time* in order to maximize her optimality criterion in terms of her wealth at some fixed time $T > 0$ (Korn, 1997, p. 1).

The most intuitive approach to defining such a problem is probably to maximize the expected value of the investor's terminal wealth. In general, however, this does not reflect an investor's attitude towards risk. Indeed, most investors prefer a sure return of 2 percent over some risky investment with an expected return of 2.2 percent (Eberlein and Kallsen, 2019, p. 461). In the early stages of portfolio optimization problems, there were two approaches to describe the preferences of investors. Von Neumann and Morgenstern (1947) proposed axioms of rational behavior and proved that an investor accepts and acts according to these axioms if, and only if, she measures her preferences in terms of the expectation of some utility function applied to her terminal wealth. A utility function is a strictly increasing, strictly concave function capturing that most investors are risk-averse and prefer larger amounts of money over smaller amounts. The second approach was established by Markowitz (1952). He used a mean-variance criterion, i.e., he considered investors trying to achieve a high expected return while keeping the variance of the portfolio small. More specifically, the expected return is maximized under the constraint that the variance is bounded by some constant. While Von Neumann and Morgenstern (1947) only considered gambles or lotteries, Markowitz (1952) was the first to consider a problem of optimal investment. At that time, the mean-variance approach was only suited for one-period problems (Korn, 1997, p. 1). The starting point for continuous-time portfolio optimization problems using expected utility was provided by Merton (1969). He considered stock prices in continuous time governed by geometric Brownian motions. He also showed that the problem of optimal investment (and consumption) can be reduced to solving the so-called Hamilton-Jacobi-Bellman equation, a nonlinear partial differential equation.

Later, Karatzas et al. (1987) introduced a different method for solving continuous-time portfolio optimization problems – namely, the martingale approach.

Note that many authors, including Merton (1969) and Karatzas et al. (1987), consider optimal investment-consumption pairs and maximize the sum of cumulated (over time) utility from consumption and terminal utility of wealth. In this thesis, we are interested exclusively in optimal portfolios and do not consider the problem of optimal consumption any further.

Soon after its introduction, expected utility theory attracted substantial criticism. One famous example is the Allais paradox (Allais, 1953; Karmarkar, 1979). Further criticism followed from Kahneman and Tversky (1979) (see also Tversky and Kahneman, 1992), whose (cumulative) prospect theory presented an alternative to the classical terminal utility maximization. In spite of all criticism, expected utility maximization is still the most widely used objective for portfolio optimization. However, the method has since been further developed in many different directions. Let us mention a few examples of modifications of the original problem. One direction includes financial markets beyond geometric Brownian motion, including stochastic volatility, jumps, or more general semimartingale stock dynamics (see, for example, Kallsen and Muhle-Karbe, 2010, for utility maximization in a stochastic volatility model; Bäuerle and Blatter, 2011, for optimal investment and reinsurance in a jump-diffusion model; and Černỳ and Kallsen, 2007, for the mean-variance problem in a general semimartingale model). Other generalizations of the original problem include the use of adaptive utility functions (so-called forward utilities), additional constraints, or separate utility functions for gains and losses (see Musiela and Zariphopoulou, 2006, for the introduction of forward utilities; Basak and Shapiro, 2001, for an example of an additional constraint; and Tversky and Kahneman, 1992, for the cumulative prospect theory in which gains and losses are treated separately).

A relatively recent development in the field of portfolio optimization is the inclusion of competition between investors. This feature is motivated by the large variety of empirical evidence for competition between managers in the mutual and hedge fund industry. Indeed, Lacker and Soret (2020) state that a „particularly important and by now well-established point is that mutual fund choice is highly influenced by relative performance“. Moreover, Bielagk et al. (2017) give the following intuitive justification of relative performance concerns: „Making a 1 Euro profit while everyone else made 2 Euro *feels* distinctly different had everyone lost 2 Euro“. The empirical literature on relative performance is vast and we can only cover a fraction here. Some frequently mentioned articles are Chevalier and Ellison (1997) as well as Sirri and Tufano (1998) for competition between mutual fund managers, Agarwal et al. (2004) for competition between hedge fund managers, and Brown et al. (2001) as well as Kempf and Ruenzi (2008) for competition motivated by career motives. Further, we refer to the introductions of Basak and Makarov (2015) as well as Lacker and Zariphopoulou (2019) for a more detailed overview. Relative concerns can be motivated by different factors. On one hand, there are career concerns (Basak and Makarov, 2015) like promotion schedules for managers (Anthropelos et al., 2022) or peer-based underperformance penalties (Bielagk et al., 2017). On the other hand, companies are concerned with their reputation as well as their desire to attract new clients (Anthropelos et al., 2022) and to generate higher money inflows (Lacker and Zariphopoulou, 2019). Taking the argument a step further, Dos Reis and Platonov (2022) argue that „benchmarking is a feature of human nature“.

The empirical evidence on competition between managers gave motivation to the new research area of *competitive portfolio optimization problems* in continuous time which has grown rapidly over the last decade. Adding to the introductory work of Espinosa (2010), Basak and Makarov (2014), and their many successors, this thesis is focused on portfolio optimization problems in continuous time for some (infinitely) large number of agents that base their decisions on relative performance concerns. In five main chapters, we take a look at such problems from various perspectives and on different levels of generality. The competitive feature is incorporated in three different ways – via the objective function, in a stochastic constraint, and via cumulative price impact. In each of these situations, we are able to find explicit solutions to the emerging multi-objective optimization problems applying two different notions of optimality.

Each of the main chapters (Chapters 3 and 5 – 8) includes a comprehensive introduction containing a literature overview of the corresponding research area and a general motivation of the problem. More precisely, this thesis is structured as follows. Chapter 2 contains basic tools and definitions from stochastic calculus, mathematical finance, and multi-objective optimization, which are used throughout this thesis. It also contains the specification of a very general semimartingale financial market which forms the basis of most problems considered in this thesis. In Chapter 3, we display a portfolio optimization problem for a finite number of competitive agents. We use the notion of competitive utility functions, which incorporate relative performance concerns into the classical portfolio optimization problem. Without any specific assumptions on the model and the utility functions, we explain a method for determining Nash equilibria for $n$-player problems by decomposing the multi-objective optimization into a single-agent problem and a system of linear equations, which turns out to be uniquely solvable. This method is applied to various examples in Chapter 4, including models with jumps or stochastic volatility, and the more generalized cumulative prospect theory. In the subsequent Chapter 5, we consider the corresponding mean field game to the $n$-agent game solved in Chapter 3. First, we motivate the problem by analyzing the Nash equilibrium in the limit as the number of agents tends to infinity. Afterwards, we define the mean field game properly and provide a solution method similar to the one displayed in Chapter 3. In Chapter 6, we take a look at a different notion of optimality, namely, Pareto optimality. In a general model, without specific restrictions on the financial market or the utility functions, we explain a suitable scalarization of the $n$-player game and use it to find a Pareto optimum in terms of terminal wealth. It turns out that the Pareto optimum coincides with the Nash equilibrium from Chapter 3. In the next Chapter 7, we consider a financial market which differs strongly from the previous chapters as it consists of agents whose investments influence the stock prices. Thus, the competition between the agents now originates from two different sources. One source remains the competitive utility function from Chapter 3. However, as the agents have a cumulative impact on the stock price, they undergo a different, more subtle, source of competition. In a linear price impact model, we are able to determine the unique constant Nash equilibrium for both exponential and power utility functions. In the case of exponential utility, we also consider nonlinear price impact and analyze whether the resulting optimization problem has a finite optimal solution. In the final Chapter 8, we apply the relative concerns through a stochastic constraint instead of the competitive utility function. The constraint is motivated by the value at risk based approach introduced by Basak and Shapiro (2001). In a general financial market, we find the optimal terminal wealth for a finite number of agents using general utility functions. Due to the complicated structure of the solution, we consider a different terminal wealth profile similar to the optimal solution found by Basak and Shapiro (2001). In this more restrictive class of wealth profiles, we find the unique Nash equilibrium in terms of terminal wealth and discuss some of its properties. Finally, Chapters A–C in the Appendix contain additional material for Chapters 6–8.

Last but not least, it should be noted that the terms *agent*, *player*, *investor*, and *trader* are used interchangeably throughout this thesis.

# Chapter 2

# Fundamentals

The following chapter contains some mathematical basics used throughout this thesis. We assume that the reader is familiar with basic topics in probability theory, mathematical finance, and stochastic optimization. More specifically, we expect a basic understanding of stochastic processes, Itô calculus, no-arbitrage pricing theory, and portfolio optimization. For the mentioned topics, we refer to Protter (2005) for a rigorous discussion of Itô calculus, to Karatzas and Shreve (1998) as well as Eberlein and Kallsen (2019) for a widespread overview of topics regarding mathematical finance, and to Pham (2009) for an introduction to stochastic optimization. Nevertheless, we state below some classical results which are used throughout this thesis.

The remainder of this chapter is based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $T > 0$ is some arbitrary but fixed finite time horizon. We assume that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, i.e., it is right-continuous and complete.

## 2.1. Basic tools from stochastic calculus

This section contains important tools from stochastic calculus, which are used frequently throughout this thesis. The first and probably most important tool from stochastic calculus is the Itô-Doeblin formula.

**Theorem 2.1** (Itô-Doeblin formula; Protter, 2005, pp. 81-82)**.** *Let $X = (X^1, \dots, X^d)$ be a continuous $d$-dimensional semimartingale (i.e., each $X^j$, $j = 1, \dots, d$, is a semimartingale) and $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, i.e., all second order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $i, j \in \{1, \dots, d\}$, exist and are continuous. Then the process $f(X)$ is also a semimartingale and the following formula holds*

$$f(X_t) - f(X_0) = \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_s)\,\mathrm{d}X_s^i + \frac{1}{2} \sum_{1\le i,j\le d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,\mathrm{d}\langle X^i, X^j\rangle_s, \quad t \in [0,T].$$

According to Protter (2005, p. 66), the quadratic (co)variation (or bracket) process $\langle \cdot, \cdot \rangle$ is defined by

$$\langle X, Y\rangle_t = X_t Y_t - \int_0^t X_s \mathrm{d}Y_s - \int_0^t Y_s \mathrm{d}X_s, \quad \langle X\rangle_t = \langle X, X\rangle_t, \quad t \in [0, T],$$

where $X$ and $Y$ are continuous semimartingales. Protter (2005) provides a formula for the quadratic (co)variation of Itô integrals, which is often a helpful tool when dealing with stochastic differential equations. The formula is given in the following lemma.

**Lemma 2.2** (Protter, 2005, p. 167)**.** *Let $X$ and $Y$ be semimartingales and let $H \in L(X)$, $G \in L(Y)$, where $L(X)$ describes the set of predictable, $X$-integrable processes (see* (2.3) *in Subsection 2.3.1 below). Then*

$$\langle H \bullet X, G \bullet Y\rangle_t = \int_0^t H_s G_s \,\mathrm{d}\langle X, Y\rangle_s, \quad t \in [0, T]. \tag{2.1}$$

In (2.1), we used the abbreviation $(H \bullet X)_t = \int_0^t H_s \,\mathrm{d}X_s$, $t \in [0, T]$.

The Itô-Doeblin formula has many useful applications. Below are two results that can be deduced directly from the Itô-Doeblin formula.

**Corollary 2.3** (Integration by Parts, Protter, 2005, p. 68)**.** *Let $X$ and $Y$ be continuous semimartingales. Then $XY$ is a semimartingale and*

$$X_t Y_t = \int_0^t X_s \mathrm{d}Y_s + \int_0^t Y_s \mathrm{d}X_s + \langle X, Y\rangle_t, \quad t \in [0, T].$$

Another important consequence of the Itô-Doeblin formula is the stochastic exponential displayed in Theorem 2.4 below. Note that Theorem 2.4 is stated for a general semimartingale which is not necessarily continuous. Thus, a generalized version of the Itô-Doeblin formula is necessary to prove the assertion of the theorem. The general Itô-Doeblin formula can be found in Protter (2005, pp. 81–82).

**Theorem 2.4** (Stochastic Exponential; Protter, 2005, p. 84)**.** *Let $X$ be a semimartingale with $X_0 = 0$. Then there exists a unique semimartingale $Z$ that satisfies the equation $Z_t = 1 + \int_0^t Z_{s-} \mathrm{d}X_s$, given by*

$$Z_t = \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right) \prod_{0<s\leq t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right), \quad t \in [0, T], \tag{2.2}$$

*where the infinite product converges.*

If $X$ is a continuous semimartingale, the representation (2.2) of $Z_t$ simplifies to (see Protter, 2005, p. 85)

$$Z_t = \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right).$$

The stochastic exponential $Z$ corresponding to a semimartingale $X$ is sometimes denoted by $\mathcal{E}(X)$, i.e.,

$$\mathcal{E}(X) = \exp\left(X - \frac{1}{2}\langle X\rangle\right).$$

We conclude this section with a stochastic version of Fubini's theorem. The theorem uses the the *predictable $\sigma$-algebra $\mathcal{P}$*, which is generated by the class of adapted processes with càglàd paths (i.e., the paths are left continuous with existing right limits). We refer to Definition 2.6 below for a formal definition and a brief discussion of predictable processes.

**Theorem 2.5** (Stochastic version of Fubini's theorem; Protter, 2005, pp. 211–212)**.** *Let $(A, \mathcal{A}, \mu)$ be a measure space, where $\mu$ is a positive, finite measure on $\mathcal{A}$. Further, let $X$ be a semimartingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(H_t^a)_{t\geq 0}$ be an $\mathcal{A}\otimes\mathcal{P}$-measurable stochastic process. Further, assume that*

$$\left(\int_A (H_t^a)^2 \mu(\mathrm{d}a)\right)^{\frac{1}{2}} \in L(X),$$

*where $L(X)$ describes the set of predictable processes that are integrable with respect to $X$. Finally, assume that $(a, t, \omega) \mapsto Z_t^a(\omega)$, where $Z_t^a := \int_0^t H_s^a \mathrm{d}X_s$, defines an $\mathcal{A} \otimes \mathcal{B}((0,\infty)) \otimes \mathcal{F}$-measurable process and that $Z^a$ is càdlàg (i.e., the paths are right continuous with existing left limits) for any $a \in A$. Then the integral*

$$\int_A Z_t^a \mu(\mathrm{d}a) = \int_A \int_0^t H_s^a \mathrm{d}X_s \mu(\mathrm{d}a)$$

*exists and*

$$\int_A \int_0^t H_s^a \mathrm{d}X_s \mu(\mathrm{d}a) = \int_0^t \int_A H_s^a \mu(\mathrm{d}a) \mathrm{d}X_s$$

*holds in the sense that the integral on the left-hand side is a càdlàg version of the integral on the right-hand side.*

## 2.2. Definition of $\sigma$-martingales

In what follows, we explain a generalization of local martingales, so-called *$\sigma$-martingales*. Such processes play an important role in the general semimartingale financial market explained in Subsection 2.3.1 below. First, we need to introduce the notion of predictable processes.

As it is the case in most applications, we associate the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with the flow of information in our model. To be coherent with this interpretation, we need to make sure that all considered processes are in some sense in line with the flow of information. Apart from adapted and progressively measurable processes, we also need the definition of a predictable process. The following definition is taken from Protter (2005, pp. 56, 102) and Eberlein and Kallsen (2019, p. 98).

**Definition 2.6.** a) The *predictable $\sigma$-algebra* $\mathcal{P}$ is the smallest $\sigma$-algebra on $\Omega \times [0, \infty)$ such that all adapted processes with càglàd paths (i.e., paths that are left continuous with existing right limits) are measurable with respect to $\mathcal{P}$.

b) A random set $D \subset \Omega \times [0, \infty)$ is called *predictable* if $D \in \mathcal{P}$.

c) An $\mathbb{R}^d$-valued stochastic process $X$ is called *predictable* if it is measurable with respect to $\mathcal{P}$ (interpreted as a mapping $X : \Omega \times [0, \infty) \to \mathbb{R}^d$).

♦

*Remark* 2.7. To understand the concept of predictable processes, it helps to consider the discrete-time analogue. A discrete-time process $(X_n)_{n\in\mathbb{N}_0}$ is called *predictable* if $X_0$ is $\mathcal{F}_0$–measurable and, for any $n \in \mathbb{N}$, $X_n$ is $\mathcal{F}_{n-1}$–measurable (Jacod and Shiryaev, 2003, Definition I.2.36). This definition can be interpreted as follows: A decision made at time $n$ can only use information available at previous times, i.e., up to time $n-1$. This justifies the term predictable. Measurability of processes with càglàd paths provides a continuous-time analogue to the definition in discrete time. ◊

According to Protter (2005, p. 103), a predictable stochastic process is also progressively measurable, but the reverse does not hold in general.

Now that we are familiar with predictable processes, we can generalize the concept of a local martingale. Recall that a local martingale is a stochastic process that behaves like a martingale if we truncate time with respect to an increasing sequence of stopping times (see, e.g., Jacod and Shiryaev, 2003, Definition I.1.45). If we not only truncate with respect to $t$, but also with respect to $\omega$, we obtain a $\sigma$-martingale.

**Definition 2.8** (Jacod and Shiryaev, 2003, Definition III.6.33)**.** An $\mathbb{R}$-valued semimartingale $X$ is called a *$\sigma$-martingale* if there exists an increasing sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets with $D_n \uparrow \Omega \times [0,\infty)$ (up to an evanescent[1] set) such that, for any $n \geq 1$, the process $\mathbb{1}_{D_n} X$ is a uniformly integrable martingale. An $\mathbb{R}^d$-valued semimartingale is called a $\sigma$-martingale if each component is an $\mathbb{R}$-valued $\sigma$-martingale. ♦

Using Proposition III.6.34 in Jacod and Shiryaev (2003), this is in fact a generalization of a local martingale as any local martingale is a $\sigma$-martingale while the reverse does not hold in general (see, e.g., Jacod and Shiryaev, 2003, Example III.6.40). Further, Delbaen and Schachermayer (1998) state that the notion of $\sigma$-martingales „relates to martingales similarly as $\sigma$-finite measures relate to finite measures“.

There is also a different definition of $\sigma$-martingales used throughout the literature (see, for example, Protter, 2005, p. 237; Delbaen and Schachermayer, 2006, Definition 14.2.1). There, an $\mathbb{R}^d$-valued semimartingale $X$ is called $\sigma$-martingale if there exist an $\mathbb{R}^d$-valued martingale $M$ and an $M$-integrable, predictable, $(0,\infty)$-valued process $\varphi$ such that $X$ can be written as[2] $X = X_0 + \varphi \bullet M$. However, it can be shown, using Theorem 89 in Protter (2005, pp. 237–238) and Theorem III.6.41 by Jacod and Shiryaev (2003), that the two definitions are equivalent.

The notion of $\sigma$-martingales is popular in the mathematical finance literature due to the paper by Delbaen and Schachermayer (1998). There, the equivalence of the „no free lunch with vanishing risk“ condition and the existence of an equivalent $\sigma$-martingale measure is shown, which provides a generalization of the classical fundamental theorem of asset pricing. We also use the absence of arbitrage in the presence of an equivalent $\sigma$-martingale measure in Subsection 2.3.1. However, we understand arbitrage strategies in a slightly different sense. Delbaen and Schachermayer (2006, p. 142) also argue that the notion of a $\sigma$-martingale is „tailormade“ for the purpose of excluding arbitrage (in some sense) in general semimartingale markets. Namely, it is first unavoidable to find a more general concept than local martingales if $S$ should be allowed to have jumps of unbounded size. Second, they argue that „for the purposes of hedging contingent claims the notion of a

[1] A set $D \subset \Omega \times [0,\infty)$ is called an *evanescent set* if its projection onto $\Omega$ is a $\mathbb{P}$-null set, i.e., if $\mathbb{P}(\{\omega \in \Omega \mid \exists\, t \in [0,\infty) \text{ with } (\omega,t) \in D\}) = 0$ (see, for example, Jacod and Shiryaev, 2003, p. 3).

[2] Note that the definition given by Protter (2005) does not include the initial value $X_0$. However, at the beginning of Chapter IV, Protter (2005) assumes, without loss of generality, that all processes $X$ start in 0.

$\sigma$-martingale is just as useful as the notion of a local martingale" (see Delbaen and Schachermayer, 2006, p. 142).

## 2.3. Basics from mathematical finance

In this section, we explain some basic concepts from the general area of mathematical finance. In Subsection 2.3.1, we explain a very general semimartingale market which forms the foundation of most problems considered in this thesis. Afterwards, we discuss the issue of completeness in Subsection 2.3.2. Finally, we give a short introduction to portfolio optimization problems and common methods to solve such problems in Subsection 2.3.3.

### 2.3.1. General semimartingale market

In the following, we describe a general semimartingale financial market. The market is taken from Černý and Kallsen (2007) and Delbaen and Schachermayer (1996). We want to emphasize at this point that we do not need any special requirements for our later analysis. The financial market is therefore as general as possible, while all stochastic integrals that arise are well-defined. Thus, the stock prices are taken as semimartingales with càdlàg paths (i.e., the paths are right continuous with existing left limits). Delbaen and Schachermayer (2006, p. 130) argue that semimartingales are „precisely the class of processes allowing for a satisfactory integration theory".

The financial market is assumed to be frictionless and perfectly elastic[3]. It consists of $d \in \mathbb{N}$ stocks with price processes $S_k = (S_k(t))_{t\in[0,T]}$, $k = 1, \dots, d$, (which are assumed to be not identical, i.e., there are no two distinct indices $j$, $k \in \{1, \dots, d\}$ such that $S_j$ and $S_k$ are modifications of another) collected in the $d$-dimensional process $S = (S_1(t), \dots, S_d(t))_{t\in[0,T]}$. Moreover, there exists a riskless bond which is, without loss of generality, assumed to be identical to 1. This is not a restriction since we could simply take the bond as numéraire[4] (see Delbaen and Schachermayer, 1995). Moreover, we assume that $S$ is an $L^2(\mathbb{P})$-semimartingale with càdlàg paths (i.e., the paths are right continuous with existing left limits). Following Delbaen and Schachermayer (1996), an $L^2(\mathbb{P})$-semimartingale is defined as follows.

**Definition 2.9.** An $\mathbb{R}^d$-valued semimartingale $S$ that satisfies

$$\sup\left\{\mathbb{E}\left[(S_k(\tau\wedge T))^2\right] : \tau \text{ is an } (\mathcal{F}_t)_{t\in[0,T]}\text{-stopping time, } k = 1, \dots, d\right\} < \infty$$

is called $L^2(\mathbb{P})$-*semimartingale* ♦

To ensure that our financial market does not contain arbitrage opportunities, we require the existence of an equivalent probability measure under which the stock price processes become

[3] A market is called frictionless if there are no transaction costs and no limitations on short selling (Karatzas and Shreve, 1998, p. 8). Moreover, the term perfectly elastic expresses that the stock prices are not changed by the agents' orders, i.e., investors are assumed to be „small". We refer to Chapter 7 for a more detailed discussion of this topic.

[4] According to Bingham and Kiesel (2004), a numéraire is a price process $(X_t)_{t\in[0,T]}$ that is strictly positive almost surely for any $t \in [0, T]$. It can be interpreted as a benchmark unit used in the financial market. The prices of different assets in the market are given in terms of (relative to) the numéraire.

$\sigma$-martingales. Thus, following Černý and Kallsen (2007), we make the following standing assumption.

**Assumption 2.10.** There exists an equivalent probability measure $\mathbb{Q}$ with square-integrable density, i.e., $\mathbb{Q} \sim \mathbb{P}$ with $\mathbb{E}\Big[\big(\frac{d\mathbb{Q}}{d\mathbb{P}}\big)^2\Big] < \infty$, such that $S$ is a $\sigma$-martingale with respect to $\mathbb{Q}$. Such a $\mathbb{Q}$ is called *$\sigma$-martingale measure (with respect to $S$)* (S$\sigma$MM) with square integrable density. ♦

For the aforementioned $S\sigma$MM $\mathbb{Q}$, we denote the associated density process by

$$Z_t^{\mathbb{Q}} := \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{F}_t\right], \ t \in [0,T].$$

Černý and Kallsen (2007) argue that Assumption 2.10 „can be interpreted as a natural no-free-lunch condition in the present quadratic context“. We can explain this statement in more detail, but we have to define the set of admissible strategies first. Instead of simply stating the set of admissible strategies, we give a more illustrative derivation of the set similar to Černý and Kallsen (2007).

First, we need to ensure that a strategy $\varphi$, i.e., an $\mathbb{R}^d$–valued stochastic process, where $\varphi_k(t)$ represents the number of shares of stock $k$ held at time $t$, is integrable with respect to the stock price process $S$. Hence, we require that $\varphi \in L(S)$, where

$$L(S) := \Big\{\varphi : \varphi \ \text{ is } (\mathcal{F}_t)\text{-predictable and } S\text{-integrable}\Big\}. \tag{2.3}$$

The term „$S$-integrable“ needs to be defined in more detail. Let

$$S = S_0 + M + A$$

describe the unique (up to modifications) semimartingale decomposition of $S$, i.e., $S_0$ is $\mathcal{F}_0$-measurable, $M$ is a local martingale with $M_0 = 0$, and $A$ is a finite variation process with $A_0 = 0$ (see, for example, Protter, 2005, Theorem 1 and 2, pp. 102–103). Then we say that $\varphi$ is $S$-integrable if, and only if, $\varphi$ satisfies (see Jacod and Shiryaev, 2003, Definition III.6.17)

$$\mathbb{P}\left(\int_0^T (\varphi_k(t))^2 d\langle S_k\rangle_t < \infty\right) = 1, \qquad \mathbb{P}\left(\int_0^T \varphi_k(t) dA_k(t) < \infty\right) = 1,$$

where the integrals with respect to $\langle S_k\rangle$ and $A$ are defined as pathwise Lebesgue-Stieltjes integrals (see, for example, Protter, 2005, p. 39). For $\varphi \in L(S)$, we denote the stochastic integral of $\varphi$ with respect to $S$ by

$$(\varphi \bullet S)_t := \sum_{k=1}^d \int_0^t \varphi_k(u) dS_k(u), \ t \in [0,T].$$

The set $\mathcal{A}$ of admissible strategies will be constructed from the $L^2(\mathbb{P})$-closure (in some sense) of the set of simple strategies. More specifically, strategies are called admissible if their integral with respect to $S$ can be approximated in $L^2(\mathbb{P})$ using a sequence of simple strategies. Let us give a more detailed explanation of this choice. A stochastic process $\varphi$ is called *simple* (or *elementary*) if there exist finite stopping times $\tau_1 \leq \cdots \leq \tau_n$ and bounded, $\mathcal{F}_{\tau_k}$-measurable random variables $Y_k$

such that $\varphi$ can be written as

$$\varphi_t(\omega) = \sum_{k=1}^{n} Y_k(\omega)\mathbb{1}_{(\tau_1(\omega),\tau_2(\omega)]}(t),\ t \in [0,T],\ \omega \in \Omega$$

(see, e.g., Černý and Kallsen, 2007; Delbaen and Schachermayer, 1996). Eberlein and Kallsen (2019, p. 584) argue that simple strategies, although not very interesting mathematically, are still important because they are the only ones that are feasible in real life.

Now a stochastic process $\varphi \in L(S)$ is called *admissible* if the stochastic integral of $\varphi$ with respect to $S$ can be approximated by a sequence of stochastic integrals of simple strategies $\varphi^{(n)}$ with respect to $S$. More specifically, $\varphi \in L(S)$ is called admissible if there exists a sequence $(\varphi^{(n)})_{n\in\mathbb{N}}$ of simple strategies such that[5]

$$\begin{aligned}
\left(\varphi^{(n)} \bullet S\right)_t &\xrightarrow{\mathbb{P}} (\varphi \bullet S)_t \text{ for any } t \in [0,T],\\
\left(\varphi^{(n)} \bullet S\right)_T &\xrightarrow{L^2(\mathbb{P})} (\varphi \bullet S)_T,\ n \to \infty.
\end{aligned}$$

While this definition of admissible strategies allows for a straightforward interpretation as limits of strategies which are feasible in real life, the above condition is impractical in general. Therefore, Černý and Kallsen (2007) presented an equivalent characterization of this condition which is more convenient to use. They showed that some $\mathbb{R}^d$-valued process $\varphi$ is admissible (regarding the above definition) if, and only if, $\varphi$ is in $L(S)$, $(\varphi \bullet S)_T \in L^2(\mathbb{P})$, and $(\varphi \bullet S)Z^{\mathbb{Q}}$ is a $\mathbb{P}$-martingale for any S$\sigma$MM $\mathbb{Q}$ with square integrable density. Thus, the set of admissible strategies reads as

$$\begin{aligned}
\mathcal{A} = \Big\{\varphi \in L(S) :\ &(\varphi \bullet S)_T \in L^2(\mathbb{P}),\ (\varphi \bullet S)Z^{\mathbb{Q}} \text{ is a } \mathbb{P}\text{-martingale for all S}\sigma\text{MM } \mathbb{Q}\\
&\text{with density process } Z^{\mathbb{Q}} \text{ and square integrable density}\Big\}.
\end{aligned} \tag{2.4}$$

The wealth process $(X_t^{\varphi})_{t\in[0,T]}$ associated to some $\varphi \in \mathcal{A}$ is given by

$$X_t^{\varphi} = x_0 + (\varphi \bullet S)_t,\ t \in [0,T], \tag{2.5}$$

for the initial capital $x_0 \in \mathbb{R}$. In the previously described setting, the time-zero price of any claim $X \in L^2(\mathbb{P})$ is given by (see, for example, Bingham and Kiesel, 2004, Theorem 6.1.4)

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}\left[Z_T^{\mathbb{Q}} X\right] \tag{2.6}$$

for all $S\sigma$MM $\mathbb{Q}$ with square integrable density. Note that, in general, the price depends on the choice of $\mathbb{Q}$. Nevertheless, for $X_T^{\varphi}$ from (2.5), we obtain $\mathbb{E}_{\mathbb{Q}}[X_T^{\varphi}] = x_0$ for any choice of $\mathbb{Q}$ (see Eberlein and Kallsen, 2019, p. 540).

Now that the set of admissible strategies is fixed, we can justify that Assumption 2.10 implies the absence of arbitrage strategies contained in $\mathcal{A}$. Černý and Kallsen (2007) explain that Assumption 2.10 implies

$$\overline{K_2^S(0) - L_+^2} \cap L_+^2 = \{0\}, \tag{2.7}$$

[5] By $\xrightarrow{\mathbb{P}}$ we denote convergence in probability.

where $L^2_+$ is the set of non-negative, square integrable random variables. The closure needs to be understood in the $L^2(\mathbb{P})$-sense. In their notation, the set $K_2^S(0)$ describes the $L^2(\mathbb{P})$-closure of the set of all claims that are attainable from simple strategies with zero initial endowment, i.e., claims $H$ that can be written as $H = (\varphi \bullet S)_T$ for a simple strategy $\varphi$. Those are exactly the wealth processes attainable from admissible strategies $\varphi \in \mathcal{A}$ with an initial capital of 0. Then (2.7) implies that the only non-negative, square integrable random variable obtainable by a reduction of some attainable claim is constantly equal to zero. Hence, $\mathcal{A}$ does not contain arbitrage strategies.

### 2.3.2. A note on completeness

In addition to the absence of arbitrage, an often desirable property of financial markets is its completeness. Without going into detail, a financial market is called *complete* if any claim $H$ is attainable by some admissible strategy $\varphi$ (see, for example, Eberlein and Kallsen, 2019, p. 540). In general, a *claim* is an $\mathcal{F}_T$-measurable random variable $H$. In most cases, $H$ is required to satisfy additional properties like non-negativity, integrability or boundedness by a suitable random variable (see, e.g., Karatzas and Shreve, 1998, pp. 21-22; Eberlein and Kallsen, 2019, p. 587; Jeanblanc et al., 2009, p. 87). A claim is called *attainable* if there exist an admissible strategy $\varphi$ and an initial capital $x_0$ such that $H = X_T^\varphi = x_0 + (\varphi \bullet S)_T$. Apart from requiring that $\varphi$ is self financing, the conditions imposed on admissible strategies depend strongly on the specific model. If one is interested in defining the unique arbitrage free price of a claim $H$, being able to hedge this claim by some replicating strategy is of utmost importance.

In many cases, the so-called *second fundamental theorem of asset pricing* gives an equivalent characterization of completeness of an arbitrage free financial market. The statement of this theorem is (more or less) the same in different market models. While absence of arbitrage is associated with the existence of an equivalent ($\sigma$-/local) martingale measure, completeness is related to the uniqueness of such a measure. This type of assertion can be found in many cases varying from discrete-time models, considered, for example, by Föllmer and Schied (2016, Theorem 5.37), to general semimartingale models like the one used by Eberlein and Kallsen (2019, Theorem 11.54). Similar to the characterization of absence of arbitrage, there is neither *the* perfect model choice nor *the* optimal characterization of completeness. That is why we decided not to give a general characterization of completeness in this thesis. To the best of our knowledge, there is no equivalent characterization of completeness that fits the semimartingale market from Subsection 2.3.1. For the characterization that comes closest to the model considered above, we refer to Theorem 11.54 in Eberlein and Kallsen (2019).

Let us conclude this discussion with a popular example of a complete financial market. Assume that there exist a riskless bond with zero interest rate and $d$ stocks. Let the $d$ stock price processes be geometric Brownian motions, i.e., they take the form

$$S_k(t) = S_k(0) \exp\left(\left(\mu_k - \frac{1}{2}\sum_{\ell=1}^{d} \sigma_{k\ell}^2\right) t + \sum_{\ell=1}^{d} \sigma_{k\ell} W_\ell(t)\right),\ t \in [0,T],$$

$k = 1, \dots, d$, where $W_1, \dots, W_d$ are independent Brownian motions. The drift vector $\mu$ and the volatility matrix $\sigma$ are assumed to be deterministic and constant in time. Then the market is complete if, and only if, the volatility matrix $\sigma$ is regular (see, for example, Karatzas and Shreve, 1998, Theorems 4.2 and 6.6 in Chapter 1). Here, completeness means that any $\mathcal{F}_T$-measurable random variable $X$, which is bounded from below, is attainable by a square integrable, progressively measurable portfolio process.

### 2.3.3. Introduction to portfolio optimization

The basic idea behind portfolio optimization is the following. Given some initial capital $x_0$ and fixed time horizon $T > 0$, an investor aims to maximize her expected terminal wealth via optimal investment. Maximizing the expected terminal wealth usually results in an unbounded optimization problem. Thus, instead of maximizing expected wealth itself, it is usually assumed that investors measure their preferences by some (Inada) utility function applied to their terminal wealth. A utility function is defined as follows (see, for example, Korn, 1997, p. 38; Eberlein and Kallsen, 2019, pp. 462–463).

**Definition 2.11.** A strictly increasing, continuous function $U : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \in \{(0, \infty), \mathbb{R}\}$, is called *utility function.* Additionally, if $U$ is strictly concave and continuously differentiable, and satisfies the *Inada conditions*

$$\lim_{x \to \inf \mathcal{D}} U'(x) = \infty, \quad \lim_{x \to \infty} U'(x) = 0,$$

$U$ is called *Inada utility function.* ♦

Some common examples of Inada utility functions are the natural logarithm, the power utility function

$$U : (0, \infty) \to \mathbb{R}, x \mapsto \left(1 - \frac{1}{\delta}\right)^{-1} x^{1-\frac{1}{\delta}} \tag{2.8}$$

for a parameter $\delta > 0$, $\delta \neq 1$, or the exponential utility function

$$U : \mathbb{R} \to \mathbb{R},\ x \mapsto -\exp\left(-\frac{1}{\delta}x\right) \tag{2.9}$$

for a parameter $\delta > 0$ (see, for example, Eberlein and Kallsen, 2019, p. 463). Utility functions are often characterized in terms of their absolute or relative risk aversion. The notion was introduced by Pratt (1964) and Arrow (1974) and is defined as follows (see also Föllmer and Schied, 2016, pp. 82-83).

**Definition 2.12.** Let $U : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \in \{(0, \infty), \mathbb{R}\}$ be a twice continuously differentiable Inada utility function. Then

$$A(x) := -\frac{U''(x)}{U'(x)},\ x \in \mathcal{D},$$

defines the *absolute* and

$$R(x) := -\frac{xU''(x)}{U'(x)},\ x \in \mathcal{D},$$

the *relative Arrow-Pratt risk aversion coefficient.* ♦

The absolute Arrow-Pratt risk aversion coefficient of the exponential utility function (2.9) is given by $\delta^{-1}$. Thus, the exponential utility function belongs to the so-called CARA (constant absolute risk aversion) utility functions. Moreover, the relative risk aversion coefficient of the logarithm and the power utility function (2.8) is given by $\delta^{-1}$ with $\delta = 1$ for the logarithm. Hence, power and logarithmic utility belong to the class of CRRA (constant relative risk aversion) utility functions. This explains why we refer to $\delta^{-1}$ as the risk aversion and to $\delta$ as the risk tolerance parameter.

For an investor with initial capital $x_0$ and utility function $U : \mathcal{D} \to \mathbb{R}$, the optimization problem of expected utility maximization reads as

$$\begin{cases} & \max_{\varphi \in \mathcal{A}} \mathbb{E}[U(X_T^\varphi)], \\ \text{s.t.} & X_T^\varphi = x_0 + (\varphi \bullet S)_T. \end{cases} \tag{2.10}$$

The set of admissible portfolio strategies $\mathcal{A}$ depends on the specific financial market. It usually contains some measurability and integrability conditions imposed on the portfolio process $\varphi : [0, T] \times \Omega \to \mathbb{R}^d$, where $d$ describes the number of stocks in the underlying financial market. As the control of an agent is a stochastic process, (2.10) describes a special *dynamic optimization problem*. For portfolio optimization problems like (2.10), there are two solution methods used in the literature - the martingale method and the dynamic programming approach. We do not explain these methods in detail, but we give a short idea on the methods as they are both used in this thesis.

The martingale method (see, for example, Section 3.4 in Korn, 1997) consists of two sub-problems. The first one is the static optimization problem in which the optimal terminal wealth is determined. The second one is the representation problem that comprises the search for a replicating strategy for the optimal terminal wealth from the first step. The martingale approach is usually applied in complete markets since, in that case, it can be guaranteed that the representation problem has a solution. The static optimization problem is often uniquely solvable due to the concave objective function. If the underlying financial market has a unique ($\sigma$-/local) martingale measure $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ (see Subsection 2.3.1), the unique solution to the static optimization problem is given by

$$X^* = I(\lambda^* Z_T)$$

(see, for example, Kramkov and Schachermayer, 1999). The function $I = (U')^{-1}$ describes the inverse of the first order derivative of $U$ and $\lambda^* > 0$ denotes the Lagrange multiplier which is determined by the budget constraint $\mathbb{E}[Z_T X^*] = x_0$. For a more in-depth explanation of the martingale method, we refer to Kramkov and Schachermayer (1999) as well as Section 3.4 in Korn (1997).

Let us now explain the second method. The idea behind the dynamic programming approach is to derive a partial differential equation for the value function

$$J(t, x) := \sup_{\varphi \in \mathcal{A}} \mathbb{E}\left[U(X_T^\varphi) \middle| X_t^\varphi = x\right], \, t \in [0, T], \, x \in \mathcal{D},$$

with the terminal constraint $J(T, x) = U(x)$, $x \in \mathcal{D}$. The derivation of the partial differential equation is based on the so-called *Bellman principle* (or sometimes called *dynamic programming principle*). It states that

$$J(t, x) = \sup_{\varphi} \mathbb{E}\left[J(t', X_{t'}^\varphi) \middle| X_t^\varphi = x\right]$$

holds for all $0 \leq t \leq t' \leq T$ (see, for example, Equation (3.20) in Pham, 2009). Together with the Itô-Doeblin formula (see Theorem 2.1), one can derive a partial differential equation, the so-called Hamilton-Jacobi-Bellman (HJB) equation, which is then used to find a candidate for the value function $J$. From a solution $G$ to the HJB equation, one can infer a candidate for the optimal control. Finally, a verification theorem is necessary to ensure that the candidate control is in fact optimal and that the solution to the HJB equation coincides with the value function. For a more detailed explanation of the method, we refer to Chapter 3 in Pham (2009).

## 2.4. Introduction to multi-objective optimization

Generally speaking, a multi-objective optimization problem is an optimization problem that involves multiple (coupled) objective functions. We want to take a game theoretic approach to multi-objective optimization problems. Thus, we consider $n$ agents with objective functions $J_1, \dots, J_n$. Each agent is able to choose some control $\varphi^i \in \mathcal{A}_i$, $i = 1, \dots, n$, where $\mathcal{A}_i$ is the set of admissible strategies of agent $i$, i.e., the set of controls agent $i$ is allowed to choose from. In the problems considered throughout this thesis, controls $\varphi^i$ are $\mathbb{R}^d$-valued stochastic processes, defined on a finite time interval $[0, T]$. The set of admissible strategies usually contains some measurability and integrability conditions that depend on the specific model. Moreover, agent $i$ aims to maximize $J_i(\varphi^1, \dots, \varphi^n)$, i.e., each objective function depends on all $n$ strategies. Thus, there are $n$ objective functions that need to be maximized simultaneously. As the resulting objective function is $\mathbb{R}^n$-valued, it is not clear a priori under which conditions a vector of admissible strategies should be called optimal. Therefore, we need to clarify when a vector $(\varphi^1, \dots, \varphi^n)$ with $\varphi^i \in \mathcal{A}_i$ for all $i \in \{1, \dots, n\}$ is considered optimal. Throughout this thesis, we apply two different notions of optimality. The first one is the so-called *Nash equilibrium*, in which agents act competitive whereas the second one, a so-called *Pareto optimum*, is obtained when the public good of all $n$ players is optimized. In the following, we explain these two concepts in more depth. First, we consider Nash equilibria. The definition dates back to Nash (1951).

**Definition 2.13.** Let $J_i : \bigtimes_{j=1}^n \mathcal{A}_j \to \mathbb{R}$ be the objective function of agent $i$, $i = 1, \dots, n$. A vector $(\varphi^{1,*}, \dots, \varphi^{n,*})$ of admissible strategies is called a *Nash equilibrium* if, for all admissible $\varphi^i \in \mathcal{A}_i$ and $i \in \{1, \dots, n\}$,

$$J_i(\varphi^{1,*}, \dots, \varphi^{i,*}, \dots, \varphi^{n,*}) \geq J_i(\varphi^{1,*}, \dots, \varphi^{i-1,*}, \varphi^i, \varphi^{i+1,*}, \dots, \varphi^{n,*}). \tag{2.11}$$

Further, $(\varphi^{1,*}, \dots, \varphi^{n,*})$ is called a *constant Nash equilibrium*, if $\varphi_t^{i,*} = \varphi_0^{i,*}$ holds for each $t \in [0, T]$ and each $i \in \{1, \dots, n\}$. ♦

*Remark* 2.14. For a vector $(\varphi^{1,*}, \dots, \varphi^{n,*})$ of constant strategies, i.e., $\varphi^{i,*}(t) = \varphi^{i,*}(0)$ for all $i \in \{1, \dots, n\}$, to be a constant Nash equilibrium, condition (2.11) still needs to be satisfied *for any* $\varphi^i \in \mathcal{A}_i$, not just for constant strategies. ◊

Definition 2.13 shows that a Nash equilibrium can be interpreted as a vector of admissible strategies, chosen by the $n$ players, such that none of the agents would benefit from deviating from the equilibrium strategy unilaterally. Thus, Nash equilibria provide a concept of optimality for competitive agents, where each agent is focused on maximizing her own objective and takes the controls of the other agents as given. In the literature on $n$-agent games, there are two different kinds of Nash equilibria which need to be distinguished – *open-loop* and *closed-loop Nash equilibria.* Carmona and Delarue (2018a, p. 72) argue that the above definition of a Nash equilibrium only makes sense once it is properly defined how the strategies of the players $j \neq i$ are „frozen" from the perspective of agent $i$. To be more specific, it is important to explain how the players update their strategies. In an open-loop equilibrium, agent $i$ treats the strategies of the other agents as fixed in the sense that the strategies do not change when agent $i$ changes her strategy in order to find the optimal one. On the other hand, when looking for a closed-loop equilibrium, agent $i$ considers the strategies of the other players in feedback form. Thus, if agent $i$ changes her strategy, the strategies of the other players change as well. Carmona and Delarue (2018a, pp. 72-75) give rigorous definitions of open-loop and closed-loop equilibria.

**Definition 2.15** (see Carmona and Delarue, 2018a, Definition 2.4)**.** Let $(\varphi^{1,*}, \dots, \varphi^{n,*})$ be a vector of admissible strategies such that (2.11) holds for all $i \in \{1, \dots, n\}$. If, for all $i \in \{1, \dots, n\}$, the vector $(\varphi^{1,*}, \dots, \varphi^{i-1,*}, \varphi^{i+1,*}, \dots, \varphi^{n,*})$ remains the same if player $i$ changes her strategy from $\varphi^{i,*}$ to a different strategy $\varphi^i$, $(\varphi^{1,*}, \dots, \varphi^{n,*})$ is called an *open-loop Nash equilibrium.* ♦

**Definition 2.16** (see Carmona and Delarue, 2018a, Definition 2.6)**.** Let $(\varphi^{1,*}, \dots, \varphi^{n,*})$ be a vector of admissible strategies of the form $\varphi^{i,*} = \phi^{i,*}(t, X_{[0,t]}^{1,\varphi^{1,*}}, \dots, X_{[0,t]}^{n,\varphi^{n,*}})$. If

$$\begin{aligned} &J_i\left(\phi^{1,*}(t, X_{[0,t]}), \dots, \phi^{n,*}(t, X_{[0,t]})\right) \\ &\quad \geq J_i\left(\phi^{1,*}(t, \widetilde{X}_{[0,t]}), \dots, \phi^{i-1,*}(t, \widetilde{X}_{[0,t]}), \phi^i(t, \widetilde{X}_{[0,t]}), \phi^{i+1,*}(t, \widetilde{X}_{[0,t]}), \dots, \phi^{n,*}(t, \widetilde{X}_{[0,t]})\right) \end{aligned}$$

holds for all $i \in \{1, \dots, n\}$, where $\varphi^i = \phi^i(t, \widetilde{X}_{[0,t]})$, $X_t = \left(X_t^{1,\varphi^{1,*}}, \dots, X_t^{i,\varphi^{i,*}}, \dots, X_t^{n,\varphi^{n,*}}\right)$, and $\widetilde{X}_t = \left(X_t^{1,\varphi^{1,*}}, \dots, X_t^{i-1,\varphi^{i-1,*}}, X_t^{i,\varphi^i}, X_t^{i+1,\varphi^{i+1,*}}, \dots, X_t^{n,\varphi^{n,*}}\right)$, then $(\varphi^{1,*}, \dots, \varphi^{n,*})$ is called a *closed-loop Nash equilibrium.* ♦

In general, closed-loop equilibria are preferable in terms of a more realistic interpretation, whereas open-loop equilibria are mathematically more tractable. Thus, open-loop equilibria are often preferred in the literature on dynamic optimization for $n$-agent games (Carmona and Delarue, 2018a, Remark 2.8). Throughout this thesis, we only consider open-loop Nash equilibria and thus, refer to them simply as Nash equilibria. It should be noted that for constant Nash equilibria open-loop and closed-loop Nash equilibria coincide (see, e.g., Lacker and Zariphopoulou (2019)).

As the games considered in this thesis are motivated by competitive investors, Nash equilibria are the most used notion of optimality. However, we also want to consider a different type of optimality, namely, *Pareto optimality.* A Pareto optimum is defined as follows.

**Definition 2.17** (Miettinen, 1999, Definition 2.2.1)**.** A vector $(\varphi^{1,*}, \dots, \varphi^{n,*})$ of admissible strategies $\varphi^{i,*} \in \mathcal{A}_i$, $i = 1, \dots, n$, is called *Pareto optimal* if there is no vector $(\varphi^1, \dots, \varphi^n)$ of admissible strategies such that

$$J_i(\varphi^1, \dots, \varphi^n) \geq J_i(\varphi^{1,*}, \dots, \varphi^{n,*}) \text{ for all } i = 1, \dots, n$$

and

$$J_i(\varphi^1, \dots, \varphi^n) > J_i(\varphi^{1,*}, \dots, \varphi^{n,*}) \text{ for at least one } i \in \{1, \dots, n\}.$$ ♦

In contrast to Nash equilibria, Pareto optima are related to optimization of the public good instead of each individuals objective. A Pareto optimum is attained if no player can increase her objective without decreasing the objective of another player. One could also imagine some kind of central planner or manager that controls the state processes of $n$ clients simultaneously. Thus, the manager wants to satisfy all clients at the same time, and therefore aims to achieve a Pareto optimum. We revisit this interpretation in Chapter 6.

# CHAPTER 3

## Nash equilibria for relative investors

Dating back to the pioneering work of Von Neumann and Morgenstern (1947) and Markowitz (1952), portfolio optimization problems have been treated extensively in the mathematical finance literature. It is usually assumed that one single investor aims to maximize an objective function applied to her terminal wealth, possibly under additional constraints. Such problems include expected utility maximization, mean-variance optimization, and many others. Although these problems might look very different, they have one thing in common: all of them consider only a single investor. However, due to the widespread empirical evidence on competition between fund managers (briefly presented in the introduction of this thesis), there is a rapidly growing strand of literature on competitive optimal investment and related issues.

In general, there are two different types of competitive optimization problems – zero-sum[1] and non-zero-sum games. Our focus lies solely on non-zero-sum games, for an example of a zero-sum investment game between two players we refer to Browne (2000).

Espinosa and Touzi (2015) argue that the most natural approach to modeling the interaction between competing investors is a general equilibrium model, where the behavior of the agents is coupled via market equilibrium[2] constraints. However, this leads to intractable calculations, which create the need for a different method to include interactions. It turns out that a different kind of equilibrium is better suited to treat the model mathematically. Instead of searching for market equilibria, portfolio optimization problems can be transformed to multi-objective portfolio optimization problems in which the search for Nash equilibria becomes the main goal. Most of the literature on many player games of wealth optimization includes the competitive feature into the problem by changing the argument of the objective function of a classical portfolio optimization

[1] Zero-sum games are competitive games in which the objective functions of all participants add up to zero (see, e.g., Carmona, 2016, p. 177).

[2] A financial market is in equilibrium if market-clearing is realized. That is, if the supply and demand for assets in the market coincide (see, e.g., Yan, 2018, p. 47).

problem. Instead of the (terminal) wealth of a single investor, a so-called *relative performance metric* is used inside the objective function (see, e.g., Geng and Zariphopoulou, 2017; Dos Reis and Platonov, 2021, 2022). With the relative performance metric, each agent places a certain weight on the optimization of her performance compared to competitors in the same market. Hence, maximizing the relative performance metric allows the agent to simultaneously maximize her own state variable (e.g., her terminal wealth) while also comparing her outcome to that of her competitors. Using a relative performance metric transforms the single-agent portfolio optimization problem into a multi-objective portfolio optimization problem.

Most competitive portfolio optimization problems use either the additive or the multiplicative relative performance metric. For $n$ agents with state variables $X^1, \dots, X^n$, the *additive relative performance metric*

$$X^i - \frac{\theta_i}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} X^j \tag{3.1}$$

of agent $i \in \{1, \dots, n\}$ is given as the difference of her own state variable and a weighted arithmetic mean of the other agents' state variables. Alternatively, the *multiplicative relative performance metric*

$$X^i \Bigg( \prod_{j=1,\, j \neq i}^{n} X^j \Bigg)^{-\frac{\theta_i}{n}} \tag{3.2}$$

of agent $i \in \{1, \dots, n\}$ describes the quotient of her own state variable and a weighted geometric mean of those of the other agents. In most cases, the state variables $X^1, \dots, X^n$ describe the wealth of $n$ agents at the end of some time period $[0, T]$, where $T > 0$ is fixed beforehand. In both expressions, the parameters $\theta_i \in [0, 1]$, $i = 1, \dots, n$, indicate how much agent $i$ cares about her performance with respect to her competitors. A larger choice of $\theta_i$ implies that the agent is more concerned with her relative performance. Hence, we refer to $\theta_i$ as the *competition weight* of agent $i$. Sometimes $\theta_i$ is also called concern rate or jealousy factor (see Bielagk et al., 2017). Throughout this thesis, we refer to the composition of a utility function and one of the relative performance metrics as the *competitive utility function*.

To the best of our knowledge, the additive relative performance metric first appeared in Palomino (2005) in a non-zero-sum game of $n$ fund managers competing to maximize their compensation. As the argument of an expected utility criterion, it first appeared in the PhD thesis of Espinosa (2010). The results were later published by Espinosa and Touzi (2015). The linear structure of the additive relative performance metric makes the resulting optimization particularly tractable, especially when combined with an expected utility of CARA type. The combination of the additive relative performance metric and CARA utility was, for instance, used by Frei and Dos Reis (2011) as well as Lacker and Zariphopoulou (2019) in stock markets modeled by Itô diffusions with constant parameters, and by Deng et al. (2020) in a model with unobservable drift of the risky stock. Moreover, Hu and Zariphopoulou (2022) used CARA utility functions combined with the additive relative performance metric in a market model with stock price dynamics that contain possibly non-Markovian drift and volatility processes. Fu et al. (2020) modified (3.1) by allowing for a random competition weight parameter. Tangpi and Zhou (2022) used expected CARA utility in combination with a modified version of (3.1) as well. In their model, each agent $i$ assigns a specific competition weight $\theta_{ij}$ to each competitor $j \neq i$. Apart from CARA utility, the additive relative performance metric was also used in combination with a mean-variance criterion by Guan and Hu (2022) as well as Yang and Chen (2022), or a convex, monetary risk measure by Bielagk et al. (2017).

The multiplicative relative performance metric was (seemingly) first applied by Basak and Makarov (2014) in a model that contains $n$ competing agents aiming to maximize expected (kinked) CRRA utility applied to (3.2) for the terminal wealth of the $n$ investors. The multiplicative structure of (3.2) works well combined with CRRA utility and hence, this combination is very popular in the literature on many player games of wealth optimization. Basak and Makarov (2015) continued to work with expected CRRA utility applied to the multiplicative relative performance metric of two agents including an asset specialization incentive. Their work was later extended to an $n$-agent game by Whitmeyer (2019), allowing for competition weights $\theta_{ij}$ specifically assigned by agent $i$ to competitor $j$. Similar problems were also considered by Lacker and Zariphopoulou (2019) as well as Lacker and Soret (2020), where the latter were focused on finding optimal investment-consumption strategies for $n$ competing agents. The problem of optimal investment and consumption was also considered by Bo and Li (2022) in a stochastic growth model with jumps. In a generalized Heston model, Kraft et al. (2020) solved the problem of optimal investment for two players with CRRA utility combined with the multiplicative relative performance metric. Moreover, Wang and Ye (2023) maximized the running reward of $n$ competing agents, where the objective function consists of the accumulated, exponentially discounted CRRA utility applied to a modified version of (3.2). A similar objective function, including a running as well as a terminal reward, was also used in the previously mentioned article by Bo and Li (2022).

Although the problem of (classical) expected utility maximization involving some kind of relative performance metric takes up a large portion of the competitive portfolio optimization literature, there are different strands of research that emerged from the seminal works of Espinosa (2010) and Basak and Makarov (2014). For example, Geng and Zariphopoulou (2017) combine the multiplicative relative performance metric with forward utilities. Their work has later been taken up and extended by Dos Reis and Platonov (2021, 2022) as well as Anthropelos et al. (2022). Another application of the relative performance metric lies in the study of competing insurance companies. This problem was considered by Deng et al. (2018), Guan and Hu (2022), and Yang and Chen (2022). A third strand of literature is focused on so-called market impact games in which competing agents are interested in the optimal execution or liquidation of a fixed position. Such issues arise in financial markets where the agents are „large“ in the sense that their investment has an impact on the stock price movements. We give an explanation of such models and a short literature overview in Chapter 7.

In what follows, we are able to solve an $n$-player competitive portfolio optimization problem in which the agents maximize an expected utility function applied to the *additive* relative performance metric of terminal wealth. In contrast to most of the articles mentioned above, we do not need to specify the utility function. Moreover, we are able to solve the problem in a very general semimartingale financial market that covers most settings in the existing literature. The foundation lies in our ability to reduce the $n$-player problem to a classical single-investor portfolio optimization problem. We can therefore rely on the wide variety of literature on single-investor portfolio optimization problems. However, we are not able to relax all assumptions used in the previously mentioned articles. In the following, we discuss the restrictions used throughout this chapter. Note that these limitations are present in the other chapters as well.

For one thing, we pose the assumption of full diversification for all $n$ agents. This means that the assets traded in the financial market are available to each of the $n$ investors. This restriction stands in contrast to the asset specialization models which are used frequently in the literature. In these settings, each agents specializes to a specific stock (or a set of stocks) and does not consider the other assets available in the market. The assumption of asset specialization was, for instance, used by Basak and Makarov (2015), Lacker and Zariphopoulou (2019), Lacker and Soret (2020), and Dos Reis and Platonov (2021, 2022), to name just a few. The main justification is that investors tend to invest into stocks they are familiar with (see Basak and Makarov, 2015, and their references). Other reasons include the reduction of trading or learning cost (see, e.g., Dos Reis and Platonov, 2022, and references therein). However, there is also an increasing number of research papers that consider a model of full diversification. Some examples are Frei and Dos Reis (2011), Espinosa and Touzi (2015), and Geng and Zariphopoulou (2017). The article by Basak and Makarov (2015) treats both asset specialization and diversification and provides a comparison of the optimal values in the two settings. It turns out that choosing a fully diverse portfolio is beneficial for risk-averse investors. Since our focus lies solely on risk-averse agents, the assumption of full diversification instead of asset specialization is reasonable. Moreover, full diversification has the additional advantage of being mathematically more manageable than problems where agents specialize to a specific stock.

Additionally, we need to assume that the agents agree upon their beliefs on the stock price dynamics. This assumption is of purely technical nature, a model allowing for heterogeneous beliefs on the stock price dynamics would, apparently, be more realistic. There are a few articles allowing for heterogeneous beliefs, like the market impact game considered by Evangelista and Thamsten (2021). However, the literature on $n$-agent games under heterogeneous beliefs on the market dynamics is still very sparse.

Another rather restrictive assumption is that each agent faces the same time horizon $T > 0$. This assumption is solely for mathematical purposes, since the empirical evidence suggests the exact opposite. Geng and Zariphopoulou (2017) and Anthropelos et al. (2022) discuss this restriction in more detail. We merely give a short summary of their arguments. While it is typical for managers to report their performance within standardized time frames (e.g., quarterly or annually), agents do not act in a single trading horizon but consider their past performance. Hence, we cannot divide the optimization problems, which they face over a longer time period, into a sequence of decisions over standardized time intervals. Moreover, horizons of performance evaluation are company specific factors and cannot be generalized for a large number of agents. However, as explained at the beginning of this paragraph, we need to make the assumption of a common time horizon for tractability reasons. It should be noted that the assumption is very common throughout the literature. To the best of our knowledge, Anthropelos et al. (2022) were the only authors to allow for different time horizons among the competing agents.

Finally, we need to make the assumption of each agent having full information about their competitors' model parameters (e.g., initial capital, risk tolerance, and competition weight). We can refer to Basak and Makarov (2015), Wang and Ye (2023), and the references therein for how these parameters can be estimated from observed data.

To conclude this introduction, we give a short overview of the current chapter. In Section 3.1, we state the $n$-player portfolio optimization problem based on the semimartingale financial market presented in Subsection 2.3.1. The statement of the problem is followed by Section 3.2, where we find the unique Nash equilibrium for the $n$-player optimization problem. More specifically, questions of existence and uniqueness of a Nash equilibrium are reduced to asking for the existence and uniqueness of an optimal solution to the corresponding single-investor problem. At the end of this chapter, we provide a brief discussion of the general equilibrium, which is taken up and extended in Chapter 4 in a variety of examples covered by the semimartingale model from Subsection 2.3.1.

## 3.1. Relative performance problem under general utility

We assume that there are $n$ agents trading in the semimartingale market from Subsection 2.3.1. To summarize, there are $d+1$ assets – a riskless bond with zero interest rate and $d$ risky stocks with price process $(S(t))_{t\in[0,T]} = (S_1(t),\dots,S_d(t))_{t\in[0,T]}$, defined on the finite time interval $[0,T]$. The stock prices are given as $L^2(\mathbb{P})$–semimartingales with càdlàg paths. To exclude arbitrage strategies, we require the existence of an equivalent $\sigma$–martingale measure $\mathbb{Q}$.

Each of the $n$ agents has the same investment opportunities provided by the semimartingale financial market. Thus, we work in the framework of full diversification. To include interaction into the optimization problem, we combine the additive relative performance metric with the expected utility of terminal wealth. The $n$ agents are allowed to be heterogeneous in the sense that their initial wealth, their preference parameters, and their utility functions are allowed to be distinct.

To be more specific, we consider the following setting. Each investor $i$ is endowed with an initial capital $x_0^i \in \mathbb{R}$. The preferences of agent $i$ are measured with respect to a utility function $U_i : \mathcal{D}_i \to \mathbb{R}$, $\mathcal{D}_i \in \{(0,\infty),\mathbb{R}\}$, and a competition weight $\theta_i \in [0,1]$, $i = 1,\dots,n$. The utility function can be rather general here. We only require $U_i$ to be continuous and strictly increasing (see also Definition 2.11). For convenience, we extend all utility functions to functions on $\mathbb{R}$ by setting $U_i(x) = -\infty$ if $x \notin \mathcal{D}$. A similar convention was used by Kramkov and Schachermayer (1999). Note that this does not affect the optimal value of the problem. The objective function of agent $i \in \{1,\dots,n\}$ consists of her expected utility applied to the additive relative performance metric of her own as well as the other agents' terminal wealth. Moreover, she is limited to her initial capital. Hence, agent $i$ aims to solve the following optimization problem

$$\begin{cases} & \sup_{\varphi^i\in\mathcal{A}} \mathbb{E}\left[U_i\left(X_T^{i,\varphi^i} - \theta_i \bar{X}_T^{-i,\varphi}\right)\right], \\ \text{s.t.} & X_T^{i,\varphi^i} = x_0^i + (\varphi^i \bullet S)_T, \end{cases} \qquad i = 1,\dots,n, \tag{3.3}$$

where $\bar{X}_T^{-i,\varphi} = \frac{1}{n}\sum_{j\neq i} X_T^{j,\varphi^j}$ and $\varphi^j \in \mathcal{A}$, $j \neq i$, are fixed admissible strategies and $\mathcal{A}$ denotes the set of admissible strategies from (2.4). For the later analysis, it is slightly more convenient to scale the sum by $n$ instead of $n-1$.

Using a weighted difference of the terminal wealth of all $n$ agents instead of only her own terminal wealth incorporates the fact that agent $i$ wants to maximize her own terminal wealth, while also making a good performance with respect to the other agents. We sometimes refer to the function $X \mapsto U_i\left(X - \theta_i \bar{X}_T^{-i,\varphi}\right)$ as the *competitive utility function* of agent $i$.

*Remark* 3.1. We need to ensure that agent $i$ is able to attain a terminal wealth $X_T^{i,\varphi^i}$ such that $X_T^{i,\varphi^i} - \theta_i \bar{X}_T^{-i,\varphi} \in \mathcal{D}_i$ $\mathbb{P}$-almost surely. Otherwise, the problem is trivial. We will later see that this condition is satisfied when we choose the competition weight $\theta_i \in [0,1]$ under the constraint $x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j \in \mathcal{D}_i$. Obviously, this constraint is only relevant if $\mathcal{D}_i = (0,\infty)$. In this case, we need to make sure that

$$x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j > 0$$

is satisfied. This is equivalent to

$$\left(1 + \frac{\theta_i}{n}\right) x_0^i - \frac{\theta_i}{n}\sum_{j=1}^{n} x_0^j > 0 \Longleftrightarrow \frac{\frac{\theta_i}{n}}{1+\frac{\theta_i}{n}} < \frac{x_0^i}{\sum_{j=1}^n x_0^j} \Longleftrightarrow \frac{\theta_i}{n} < \frac{\alpha_i}{1-\alpha_i},$$

where

$$\alpha_i := \frac{x_0^i}{\sum_{j=1}^n x_0^j}$$

describes the fraction of the collective initial capital originating from investor $i$. Therefore, the constraint $x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j > 0$ implies an upper bound on the choice of $\theta_i$, which is increasing in the fraction $\alpha_i$. Hence, the upper bound is increasing in terms of the $i$-th agent's initial capital and decreasing in terms of the other $n-1$ agents' initial investment. We can interpret this observation as follows: The more an investor contributes in the beginning, the more she may care about the other agents' investment behavior. Moreover, the upper bound on $\theta_i$ is strictly smaller than 1 if, and only if, $\alpha_i < (n+1)^{-1}$, i.e., if the fraction of initial wealth originating from agent $i$ is smaller than $(n+1)^{-1}$. If, for example, each agent invests the same amount in the beginning, there are no restrictions on the competition weight parameters. $\diamondsuit$

The optimization problem (3.3) contains $n$ objective functions, where each objective function depends on all $n$ strategies $\varphi^j$, $j = 1, \dots, n$. Since these objective functions need to be maximized simultaneously, (3.3) is a multi-objective optimization problem. There is no a priori given notion of optimality for such problems. We work with the concept of *Nash equilibria*, i.e., each investor tries to maximize her own objective function while the strategies of the other investors are assumed to be fixed. For the definition of a Nash equilibrium, we refer to Definition 2.13 as well as Definition 2.15. To summarize, a Nash equilibrium is a vector of admissible strategies, chosen by the players, such that no agent would benefit from deviating unilaterally from her equilibrium strategy. To be more specific, we search for open-loop Nash equilibria. For a differentiation between open-loop and closed-loop Nash equilibria, we refer to Section 2.4.

## 3.2. Solution method via problem reduction

Next, we explain how to find (open-loop) Nash equilibria for (3.3). Thus, each investor tries to maximize her own objective function while the strategies of the other investors are assumed to be fixed. This maximization, often called *best response problem*, results in the optimal strategy of agent $i$ in terms of the strategies of her competitors. The second step of the optimization process is a fixed point problem in order to find the $n$-tuple of admissible strategies to satisfy each investors' preference determined in the first step.

Generally, one would proceed in the previously described way by fixing some investor $i$, fixing the other agents' strategies, maximizing the $i$-th objective function, and solving the fixed point problem afterwards. While being very intuitive, the described approach brings some disadvantages. First, the fixed point problem in the second step can be nonlinear and hence, hard or even impossible to solve. Moreover, the method depends strongly on the explicit model and often requires a very restrictive mathematical environment. For example, Lacker and Zariphopoulou (2019) were only able to consider constant Nash equilibria and a Black-Scholes type market with constant market parameters. Note that the restrictive environment is also partly caused by the asset specialization framework in their paper.

To avoid these technical difficulties, we choose a different approach to find a Nash equilibrium and discuss its uniqueness, which enables us to solve the problem in a very general financial market without any specific assumptions on the utility function. Moreover, the method can even be extended to problems beyond utility maximization (see Remark 3.4 and Section 4.5).

First, we consider the expression inside the utility function in (3.3). Since agent $i$ can only control her own strategy, the random variable $\theta_i \bar{X}_T^{-i,\varphi}$ can be understood as a fixed asset claim. An arbitrary strategy $\varphi \in \mathcal{A}$ can due to linearity always be decomposed into

$$X_T^{\varphi} = X_T^{\varphi} - X_T^{\varphi'} + X_T^{\varphi'} = X_T^{\varphi - \varphi'} + X_T^{\varphi'},$$

for arbitrary $\varphi' \in \mathcal{A}$. Investor $i$ can, without loss of generality, invest some fraction of her initial capital in order to hedge the claim $\theta_i \bar{X}_T^{-i,\varphi}$. The remaining part of her initial capital can then be used to maximize her own terminal wealth. This idea leads to a much simpler method of determining Nash equilibria in the given context.

The first step is to determine the price of the claim $\theta_i \bar{X}_T^{-i,\varphi}$. Each investor $j$, where $j \neq i$, has some initial capital $x_0^j$. Hence, the time zero price of $X_T^{j,\varphi^j}$ equals the initial capital $x_0^j$ (see (2.6) in Subsection 2.3.1). By linearity, the price of the claim $\theta_i \bar{X}_T^{-i,\varphi}$ is simply given by

$$\theta_i \bar{x}_0^{-i} := \frac{\theta_i}{n} \sum_{j \neq i} x_0^j.$$

This shows that the time zero price is independent of the strategies $\varphi^j$, $j \neq i$, chosen by the other investors. Hence, the maximization problem in the second step does not depend on the other $n-1$ agents' strategies.

In the second step, investor $i$ needs to solve a classical portfolio optimization problem, using the utility function $U_i$ and the reduced initial capital $\widetilde{x}_0^i := x_0^i - \theta_i \bar{x}_0^{-i}$. The portfolio optimization problem

$$\begin{cases} \sup\limits_{\psi^i \in \mathcal{A}} \mathbb{E}\Big[U_i\Big(Y_T^{i,\psi^i}\Big)\Big], \\ \text{s.t.} \quad Y_T^{i,\psi^i} = \widetilde{x}_0^i + (\psi^i \bullet S)_T, \end{cases} \tag{3.4}$$

can be solved using standard methods. Existence and uniqueness of a solution does of course depend on the specific choice of the utility function and underlying financial market. We assume here that there exists a unique optimal strategy $\psi^{i,*} \in \mathcal{A}$ for (3.4).

In the last step, the Nash equilibria are determined using the linearity of the price process. By construction, the process $(Y_t^{i,\psi^{i,*}})_{t\in[0,T]}$ can be written as

$$Y_t^{i,\psi^{i,*}} = X_t^{i,\varphi^i} - \frac{\theta_i}{n} \sum_{j \neq i} X_t^{j,\varphi^j}$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$. Then, using the linearity of the wealth process (recall that we assumed in Subsection 2.3.1 that stock prices are not identical) and the uniqueness of $\psi^{i,*}$, the strategies $\varphi^j \in \mathcal{A}$ are obtained from

$$\psi_k^{i,*}(t) = \varphi_k^i(t) - \frac{\theta_i}{n} \sum_{j \neq i} \varphi_k^j(t), \quad k = 1, \dots, d, \; i = 1, \dots, n, \tag{3.5}$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$. Hence, the Nash equilibria can be determined as the solution to the system of linear equations (3.5), where the solutions $\psi_k^{i,*}$ to the classical problems are given. If (3.4) and (3.5) have a unique solution, the resulting Nash equilibrium is unique as well. The question of uniqueness has been posed as an open problem in Lacker and Zariphopoulou (2019). Our setting gives a partial answer to their question due to the asset specialization feature in their model.

**Theorem 3.2.** *If* (3.4) *has a unique (up to modifications) optimal portfolio strategy* $\psi^{i,*}$ *for all* $i \in \{1, \dots, n\}$*, then there exists a unique Nash equilibrium for* (3.3) *given by*

$$\varphi_k^i(t) := \frac{n}{n+\theta_i} \psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})} \sum_{j=1}^n \frac{n}{n+\theta_j} \psi_k^{j,*}(t), \tag{3.6}$$

$k = 1, \dots, d$, $i = 1, \dots, n$, $\mathbb{P}$*-almost surely for all* $t \in [0,T]$*, where* $\hat{\theta} := \sum_{i=1}^n \frac{\theta_i}{n+\theta_i}$.

The proof of Theorem 3.2 uses the following lemma.

**Lemma 3.3.** *Let* $\theta_i \in [0,1]$*,* $i = 1, \dots, n$*. Then*

$$\hat{\theta} = \sum_{i=1}^n \frac{\theta_i}{n+\theta_i} \in [0,1).$$

*Proof.* Since $\theta_i \in [0,1]$, $i = 1, \dots, n$, we obtain

$$0 \leq \hat{\theta} = \sum_{i=1}^n \frac{\theta_i}{n+\theta_i} = \sum_{i=1}^n \left(1 + \frac{n}{\theta_i}\right)^{-1} \leq \sum_{i=1}^n (1+n)^{-1} = \frac{n}{n+1} < 1. \qquad \square$$

*Proof (Theorem 3.2).* Without knowing $\psi^{i,*}$ explicitly, we can determine the solution to (3.5) in terms of $\psi^{i,*}$, $i = 1, \ldots, n$, for all $t \in [0, T]$ and $\mathbb{P}$-almost all $\omega \in \Omega$. Therefore, we fix some arbitrary $t \in [0, T]$ and $\omega \in \Omega$ (we omit the argument $\omega$ in the following calculations), and define

$$\widehat{\varphi}_k(t) := \sum_{j=1}^{n} \varphi_k^j(t), \quad k = 1, \ldots, d. \tag{3.7}$$

Hence, (3.5) is equivalent to

$$\psi_k^{i,*}(t) = \frac{n + \theta_i}{n} \varphi_k^i(t) - \frac{\theta_i}{n} \widehat{\varphi}_k(t)$$

and therefore, $\varphi_k^i(t)$ is implicitly given by

$$\varphi_k^i(t) = \frac{n}{n + \theta_i} \left( \psi_k^{i,*}(t) + \frac{\theta_i}{n} \widehat{\varphi}_k(t) \right). \tag{3.8}$$

Inserting (3.8) into (3.7) yields

$$\begin{aligned} \widehat{\varphi}_k(t) = \sum_{i=1}^{n} \varphi_k^i(t) &= \sum_{i=1}^{n} \frac{n}{n + \theta_i} \psi_k^{i,*}(t) + \widehat{\varphi}_k(t) \sum_{i=1}^{n} \frac{\theta_i}{n + \theta_i} \\ &= \sum_{i=1}^{n} \frac{n}{n + \theta_i} \psi_k^{i,*}(t) + \hat{\theta} \widehat{\varphi}_k(t), \end{aligned}$$

which is equivalent to

$$(1 - \hat{\theta}) \widehat{\varphi}_k(t) = \sum_{i=1}^{n} \frac{n}{n + \theta_i} \psi_k^{i,*}(t),$$

where we used the abbreviation of $\hat{\theta}$. Therefore, using Lemma 3.3, we can deduce further that

$$\widehat{\varphi}_k(t) = \frac{1}{1 - \hat{\theta}} \sum_{i=1}^{n} \frac{n}{n + \theta_i} \psi_k^{i,*}(t) \tag{3.9}$$

holds $\mathbb{P}$-almost surely for all $t \in [0, T]$. Finally, inserting (3.9) into (3.8) yields the stated expression for the Nash equilibrium. The line of arguments implies that there exists a unique (up to modifications) Nash equilibrium given by (3.6) if, and only if, there exists a unique (up to modifications) optimal portfolio strategy $\psi^{i,*}$ to the auxiliary problem (3.4). □

*Remark* 3.4. The described method is not limited to classical expected utility maximization. Some examples of other types of optimization problems that can be treated with the described method are the VaR-based optimization by Basak and Shapiro (2001), the rank-dependent utility model with a VaR-based constraint by He and Zhou (2016), or general mean-variance problems that can be found for example in Korn (1997). Another example is the cumulative prospect theory (CPT) by Tversky and Kahneman (1992), that is further analyzed in Section 4.5. Moreover, the arguments presented in the current section also apply for financial markets in discrete time. In order to keep the setting as simple, but also as general as possible, we consider a setting in continuous time that covers many important market models (see Chapter 4 for some examples).

However, the arguments in the current section do not depend at all on the underlying financial market, so that one could also have a general arbitrage-free financial market in discrete time in mind. Later, we take a closer look at a discrete,time financial market in Section 4.3.
Additionally, we want to point out that the use of the *additive* relative performance metric is crucial here. One could argue that a similar approach might work for the multiplicative relative performance metric if one uses the invested fraction instead of the number of shares. In that case, the wealth of an agent is a stochastic exponential and thus, the multiplicative relative performance metric applied to the terminal wealth of the agents has an exponential structure as well. However, the expression inside the exponential function is not linear since, for general semimartingales $X$ and $Y$,

$$\mathcal{E}(X) \cdot \mathcal{E}(Y) = \mathcal{E}(X + Y + \langle X, Y \rangle) \neq \mathcal{E}(X + Y)$$

holds in the (interesting) cases where $\langle X, Y \rangle \neq 0$. Recall that $\mathcal{E}(X)$ denotes the stochastic exponential of $X$ (see Theorem 2.4). For the first equality, we refer to Theorem 38 in Protter (2005, p. 86). $\diamondsuit$

*Remark* 3.5. Since the Nash equilibrium given in Theorem 3.2 explicitly contains the optimal solutions to the associated classical portfolio optimization problems, we can compare the optimal solution $\psi_k^{i,*}$ of the classical problem to the associated component $\varphi_k^i$ of the Nash equilibrium. Indeed, if we set $\theta_i = 0$, the agent does not care about the other agents and just solves the classical portfolio problem, i.e., $\varphi_k^i = \psi_k^{i,*}$. If $\psi_k^{i,*} > 0\,(< 0)$ and $\sum_{j \neq i} \frac{n}{n+\theta_j} \psi_k^{j,*} > 0\,(< 0)$, $\varphi_k^i(t)$ is increasing (decreasing) in terms of $\theta_i$, which can be seen directly considering the partial derivative

$$\begin{aligned}\frac{\partial}{\partial \theta_i} \varphi_k^i(t) =& \frac{n}{(n+\theta_i)^2} \left(1 + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\right) \left(\frac{n}{(n+\theta_i)(1-\hat{\theta})} - 1\right) \psi_k^{i,*}(t) \\ &+ \frac{n}{(n+\theta_i)^2(1-\hat{\theta})} \left(1 + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\right) \sum_{j \neq i} \frac{n}{n+\theta_j} \psi_k^{j,*}(t).\end{aligned}$$

Except from the third factor in the first summand, all factors are obviously non-negative. Moreover,

$$\frac{n}{(n+\theta_i)(1-\hat{\theta})} \geq \frac{n}{(n+\theta_i)\left(1 - \frac{\theta_i}{n+\theta_i}\right)} = 1, \tag{3.10}$$

using the definition of $\hat{\theta}$. Hence, the remaining factor is non-negative as well. Thus, under these conditions in a competitive environment ($\theta_i > 0$), agent $i$ invests more in the stocks than in a classical non-competitive environment ($\theta_i = 0$). A similar result was obtained by Espinosa and Touzi (2015) (in a less general setting). $\diamondsuit$

# CHAPTER 4

# Application of the method from Section 3.2

In the following, we apply the method described in Section 3.2 to different examples from the literature, including a discrete-time market, a market with jumps, and a stochastic volatility model, as well as a problem that goes beyond expected utility maximization. In Section 4.1, we consider agents applying exponential utility functions in a Lévy market with jumps. A special case of the Lévy market, a Black-Scholes market, is considered in Section 4.2. There, we consider exponential utilities in Subsection 4.2.1, as well as power and logarithmic utility functions in Subsection 4.2.2. Section 4.3 covers a financial market in discrete time, the Cox-Ross-Rubinstein market, in which agents use exponential utilities. In Section 4.4, we consider a model with stochastic volatility, and the final Section 4.5 contains an example that goes beyond expected utility maximization, namely, cumulative prospect theory.

It should be noted that the definition of the set $\mathcal{A}$ of admissible strategies changes throughout this chapter. In each example, we take $\mathcal{A}$ to be the set of strategies satisfying the necessary assumptions for the specific example according to the paper from which the example is taken.

## 4.1. Lévy market

The first example is a Lévy market taken from Bäuerle and Blatter (2011). The market consists of a riskless bond with interest rate $r = 0$ and $d$ stocks. Let $W$ be a $d$-dimensional Brownian motion and $N$ a Poisson random measure on $[0,T] \times (-1,\infty)^d$, i.e., $N([0,t] \times B)$ is the number of all jumps taking values in the set $B \in \mathcal{B}((-1,\infty)^d)$ up to the time $t \in [0,T]$, where $\mathcal{B}((-1,\infty)^d)$ denotes the Borel $\sigma$-algebra on $(-1,\infty)^d$. For the definition of a Poisson random measure (or Poisson point process), we refer to Definition 19.1 of Sato (1999) as well as Last and Penrose (2017, p. 19). We denote the associated Lévy measure by $\nu$, i.e., $\nu(B) = \mathbb{E}[N([0,1] \times B)]$ gives the expected number of jumps per unit time whose size belongs to $B$ (see, e.g., Cont and Tankov, 2004, Definition 3.4). For notational convenience, let us define $\bar{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \mathbb{1}_{\{\|z\|<1\}}\nu(\mathrm{d}z)\mathrm{d}t$.

The price processes for $k \in \{1,\dots,d\}$ are given by the stochastic differential equation

$$\mathrm{d}S_k(t) = S_k(t_-)\left(\mu_k\,\mathrm{d}t + \sum_{\ell=1}^{d}\sigma_{k\ell}\,\mathrm{d}W_\ell(t) + \int_{(-1,\infty)^d} z_k\,\bar{N}(\mathrm{d}t,\mathrm{d}z)\right), \tag{4.1}$$

where $S_k(0)=1$, $\sigma_{k\ell}\geq 0$, $k,\ell=1,\dots,d$. By $\mu=(\mu_1,\dots,\mu_d)$ we denote the drift vector and by $\sigma=(\sigma_{k\ell})_{1\leq k,\ell\leq d}$ the volatility matrix, which is assumed to be regular. The restriction of $N$ onto $[0,T]\times(-1,\infty)^d$, which has the interpretation that the size of any jump is strictly larger than $-1$, ensures that the stock price processes (4.1) stay positive (see, e.g., Proposition 8.21 in Cont and Tankov, 2004, or Theorem 2.4).

There are $n$ investors trading in the Lévy market, each endowed with an initial capital of $x_0^i\in\mathbb{R}$, $i=1,\dots,n$. We assume that each investor uses an exponential utility function

$$U_i:\mathbb{R}\to\mathbb{R},\ U_i(x) = -\exp\left(-\frac{1}{\delta_i}x\right),$$

for parameters $\delta_i>0$, $i=1,\dots,n$. Hence, the objective function of agent $i$ is given by

$$\mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left(X_T^{i,\varphi^i}-\theta_i\bar{X}_T^{-i,\varphi}\right)\right)\right],$$

for the competition weight parameters $\theta_i\in[0,1]$, $i=1,\dots,n$. We assume that the market is free of arbitrage for an appropriate strategy class and that

$$\int\limits_{\|z\|>1}\|z\|\exp\left(\frac{1}{\delta_i}\Lambda_i\|z\|\right)\nu(\mathrm{d}z)<\infty \tag{4.2}$$

for constants $0<\Lambda_i<\infty$, $i=1,\dots,n$. The integrability condition (4.2) is used in the proof of Theorem 2.1 by Bäuerle and Blatter (2011), which provides the unique optimal solution to the problem of maximizing exponential utility shown below.

Now we use the method from Section 3.2 to determine the Nash equilibrium in the given situation. First, the unique (up to modifications) optimal solution to the optimization problem

$$\begin{cases} & \sup_{\psi^i\in\mathcal{A}}\mathbb{E}\left[-\exp\left(-\delta_i^{-1}Y_T^{i,\psi^i}\right)\right],\\ \text{s.t.} & Y_T^{i,\psi^i}=\widetilde{x}_0^i+(\psi^i\bullet S)_T,\end{cases}$$

for $\widetilde{x}_0^i = x_0^i-\frac{\theta_i}{n}\sum_{j\neq i}x_0^j$, is given by

$$\psi_k^{i,*}(t)=\frac{\pi_k^{i,*}}{S_k(t)}.$$

Here, $(\pi_1^{i,*},\dots,\pi_d^{i,*})$ is the (deterministic and constant) solution of

$$0=\mu_k-\frac{1}{\delta_i}\sum_{\ell=1}^{d}\sum_{r=1}^{d}\pi_r^{i,*}\sigma_{k\ell}\sigma_{r\ell}+\int_{(-1,\infty)^d} z_k\left(\exp\left(-\frac{1}{\delta_i}\sum_{r=1}^{d}\pi_r^{i,*}z_r\right)-\mathbb{1}_{\{\|z\|<1\}}\right)\nu(\mathrm{d}z), \tag{4.3}$$

$k = 1, \dots, d$, which we assume to exist and be unique (see Bäuerle and Blatter, 2011). Regarding the existence of a solution to (4.3), Blatter (2009) states that one could apply a fixed point theorem which can be found, for example, in Schäfer (2007). If the parameters of the model are chosen such that the conditions of the fixed point theorem are satisfied, we can guarantee the existence of a solution to (4.3). However, according to Blatter (2009), it is not possible to derive explicit conditions on the model parameters.

Using Theorem 3.2, we know that the unique (up to modifications) Nash equilibrium is then given by

$$\varphi_k^{i,*}(t) = \frac{n}{n+\theta_i}\psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^{n}\frac{n}{n+\theta_j}\psi_k^{j,*}(t)$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$ and therefore, the Nash equilibrium $\pi_k^i(t) := \varphi_k^{i,*}(t)S_k(t)$ for $k = 1, \dots, d$, $i = 1, \dots, n$, in terms of invested amounts, is deterministic and constant.

## 4.2. Black-Scholes market

Our next example is a special case of the setting used in Section 4.1. We consider a Black-Scholes market consisting of a riskless bond with interest rate $r = 0$ and $d$ stocks with price processes

$$\mathrm{d}S_k(t) = S_k(t)\left(\mu_k \mathrm{d}t + \sum_{\ell=1}^{d}\sigma_{k\ell}\mathrm{d}W_\ell(t)\right),\ S_k(0) = 1,\ k = 1, \dots, d,$$

driven by a $d$-dimensional Brownian motion $W = (W_1, \dots, W_d)$. By $\mu = (\mu_1, \dots, \mu_d)$ we denote the drift vector and by $\sigma = (\sigma_{jk})_{1\le j,k\le d}$ the volatility matrix, which is again assumed to be regular. This model can be obtained from the Lévy market described in Section 4.1 by omitting the jump component.

### 4.2.1. Optimization under exponential utility

First, we assume that agents use exponential utility functions with risk tolerance parameters $\delta_i > 0$, $i = 1, \dots, n$. The method from Section 3.2 can be used to find all Nash equilibria for

$$\mathbb{E}\Big[-\exp\Big(-\frac{1}{\delta_i}\Big(X_T^{i,\varphi^i} - \theta_i\bar{X}_T^{-i,\varphi}\Big)\Big)\Big],\ i = 1, \dots, n.$$

The unique optimal solution to the auxiliary problem

$$\begin{cases} & \sup_{\psi^i\in\mathcal{A}}\mathbb{E}\Big[-\exp\Big(-\delta_i^{-1}Y_T^{i,\psi^i}\Big)\Big], \\ \text{s.t.} & Y_T^{i,\psi^i} = \widetilde{x}_0^i + (\psi^i \bullet S)_T, \end{cases}$$

is given by

$$\psi_k^i(t)S_k(t) = \delta_i\big(\sigma\sigma^\top\big)^{-1}\mu,\ k = 1, \dots, d,\ i = 1, \dots, n$$

(see, e.g., Eberlein and Kallsen, 2019, Example 10.20, for one stock – the extension to $d$ stocks is straightforward). Therefore, the amount invested into the $d$ stocks is constant in time and deterministic. Hence, Theorem 3.2 implies that the unique (up to modifications) Nash equilibrium in terms of invested amounts is given by

$$\pi^{i,*}(t) = \left(\frac{n\delta_i}{n+\theta_i} + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^{n}\frac{n\delta_j}{n+\theta_j}\right)\cdot\left(\sigma\sigma^\top\right)^{-1}\mu,\ t\in[0,T],\ i=1,\ldots,n. \tag{4.4}$$

Thus, the Nash equilibrium in terms of invested amounts is deterministic and constant in time.

*Remark* 4.1. If we set $d=1$, $\mu_1=\mu$, and $\sigma_1=\sigma>0$, we obtain the constant Nash equilibrium found by Lacker and Zariphopoulou (2019, Corollary 2.4). They use the slightly different objective function

$$\mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left(X_T^{i,\varphi^i} - \frac{\theta_i}{n}\sum_{j=1}^{n}X_T^{j,\varphi^j}\right)\right)\right],$$

which can be rewritten as

$$\mathbb{E}\left[-\exp\left(-\frac{1}{\tilde{\delta}_i}\left(X_T^{i,\varphi^i} - \tilde{\theta}_i\bar{X}_T^{-i,\varphi}\right)\right)\right] \tag{4.5}$$

by introducing the parameters $\tilde{\delta}_i = \frac{\delta_i}{1-\frac{\theta_i}{n}}$ and $\tilde{\theta}_i = \frac{\theta_i}{1-\frac{\theta_i}{n}}$. Hence, solving (4.5) for $i=1,\ldots,n$ yields the (one-dimensional) Nash equilibrium

$$\pi_i^* = \left(\delta_i + \theta_i\frac{\bar{\delta}_n}{1-\bar{\theta}_n}\right)\frac{\mu}{\sigma^2},$$

where we used the auxiliary calculations

$$\frac{n\tilde{\delta}_i}{n+\tilde{\theta}_i} = \delta_i,\ \sum_{j=1}^{n}\frac{\tilde{\theta}_j}{n+\tilde{\theta}_j} = \bar{\theta}_n,$$

and the abbreviations $\bar{\theta}_n := \frac{1}{n}\sum_{j=1}^{n}\theta_j$, $\bar{\delta}_n = \frac{1}{n}\sum_{j=1}^{n}\delta_j$. It is important to notice here that Lacker and Zariphopoulou (2019) had to restrict their problem to finding *constant* Nash equilibria while we are able to consider a more general Nash equilibrium which turns out to be constant. ◊

### Comparison to the single-agent optimization problem

In the single-agent problem of maximizing the expected exponential utility of the terminal wealth in the underlying Black-Scholes market, the optimally invested amount is known to be given by

$$\pi^{i,*} = \delta_i\left(\sigma\sigma^\top\right)^{-1}\mu. \tag{4.6}$$

A comparison of the optimal portfolio (4.6) in the single-agent problem and the Nash equilibrium (4.4) shows that the overall structure is the same. In both cases, the optimal amount invested into the $d$ stocks is constant over time and deterministic. Moreover, the optimally invested amount is given as $(\sigma\sigma^\top)^{-1}\mu$ multiplied by a constant. We define the constant in (4.4) by $C_i$, i.e.,

$$C_i := \frac{n\delta_i}{n+\theta_i} + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^{n}\frac{n\delta_j}{n+\theta_j}. \tag{4.7}$$

In the single-agent scenario ($\theta_i = 0$), the constant is simply given by $\delta_i$.

Let us discuss some properties of $C_i$. First, we rewrite $C_i$ as

$$C_i = \frac{n\delta_i}{n+\theta_i}\left(1 + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\right) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j\neq i}\frac{n\delta_j}{n+\theta_j}.$$

Regarding the monotonicity of $C_i$ in terms of $\theta_i$, we take the first order partial derivative with respect to $\theta_i$. Thus,

$$\begin{aligned}
\frac{\partial}{\partial\theta_i}C_i &= -\frac{n\delta_i}{(n+\theta_i)^2}\left(1 + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\right) \\
&\quad + \frac{n\delta_i}{n+\theta_i}\left(\frac{n}{(n+\theta_i)^2}\frac{1}{1-\hat{\theta}} + \frac{\theta_i}{n+\theta_i}\frac{1}{(1-\hat{\theta})^2}\frac{n}{(n+\theta_i)^2}\right) \\
&\quad + \sum_{j\neq i}\frac{n\delta_j}{n+\theta_j}\left(\frac{n}{(n+\theta_i)^2}\frac{1}{1-\hat{\theta}} + \frac{\theta_i}{n+\theta_i}\frac{1}{(1-\hat{\theta})^2}\frac{n}{(n+\theta_i)^2}\right) \\
&= \frac{n\delta_i}{(n+\theta_i)^2}\left(\frac{n}{n+\theta_i}\frac{1}{1-\hat{\theta}} - 1\right)\left(1 + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\right) &(4.8)\\
&\quad + \sum_{j\neq i}\frac{n\delta_j}{n+\theta_j}\left(\frac{n}{(n+\theta_i)^2}\frac{1}{1-\hat{\theta}} + \frac{\theta_i}{n+\theta_i}\frac{1}{(1-\hat{\theta})^2}\frac{n}{(n+\theta_i)^2}\right). &(4.9)
\end{aligned}$$

The second summand (4.9) is strictly positive since $\hat{\theta} < 1$ for every admissible choice of $\theta_1, \ldots, \theta_n$ (see Lemma 3.3). Moreover, all factors in the first summand (4.8) are non-negative (see (3.10) for the second factor). Hence, the constant $C_i$ is strictly increasing in terms of $\theta_i$. Moreover, the factors $C_i$ and $\delta_i$ coincide if, and only if, $\theta_i = 0$.

Let us now analyze the influence of $\theta_i$ on the optimal terminal wealth of agent $i$. As the optimally invested amount for agent $i$ is deterministic and constant in time, the optimal terminal wealth is given by

$$\begin{aligned}
X_T^{i,\varphi^{i,*}} &= x_0^i + C_i\mu^\top(\sigma\sigma^\top)^{-1}\mu T + C_i\mu^\top(\sigma\sigma^\top)^{-1}\sigma W(T) \\
&= x_0^i + C_i\|\sigma^{-1}\mu\|^2 T + C_i(\sigma^{-1}\mu)^\top W(T).
\end{aligned}$$

Therefore, the expected optimal terminal wealth is given by

$$\mathbb{E}\left[X_T^{i,\varphi^{i,*}}\right] = x_0^i + C_i\|\sigma^{-1}\mu\|^2 T \tag{4.10}$$

and thus, strictly increasing in terms of $\theta_i$. Hence, in order to maximize the expected terminal wealth, agent $i$ should choose the competition weight $\theta_i = 1$.

However, choosing a high competition weight also brings the disadvantage of increasing the probability of a loss (with respect to the initial capital $x_0^i$). To justify this assertion, we choose a constant $K < x_0^i$ and consider the probability that the optimal terminal wealth $X_T^{i,\varphi^{i,*}}$ is less or equal than $K$. Since $W(T)$ has a $d$-dimensional normal distribution, we obtain

$$(\sigma^{-1}\mu)^\top W(T) \sim \mathcal{N}\left(0,\, \|\sigma^{-1}\mu\|^2 T\right),$$

and therefore, it follows

$$\begin{aligned}
\mathbb{P}\Big(X_T^{i,\varphi^{i,*}} \leq K\Big) &= \mathbb{P}\left(x_0^i + C_i\|\sigma^{-1}\mu\|^2 T + C_i(\sigma^{-1}\mu)^\top W(T) \leq K\right) \\
&= \mathbb{P}\left(\frac{(\sigma^{-1}\mu)^\top W(T)}{\|\sigma^{-1}\mu\|\sqrt{T}} \leq \frac{K - x_0^i}{C_i\|\sigma^{-1}\mu\|\sqrt{T}} - \|\sigma^{-1}\mu\|\sqrt{T}\right) \\
&= \Phi\left(\frac{K - x_0^i}{C_i\|\sigma^{-1}\mu\|\sqrt{T}} - \|\sigma^{-1}\mu\|\sqrt{T}\right),
\end{aligned} \tag{4.11}$$

where $\Phi$ denotes the distribution function of the standard normal distribution. The expression inside $\Phi$ is increasing in $\theta_i$ since $K - x_0^i < 0$. Hence, the probability of the terminal wealth being significantly smaller than the initial wealth is strictly increasing in terms of $\theta_i$. Thus, a larger choice of $\theta_i$ results in a riskier strategy.

The question remains which of the two factors is more significant - the expected optimal terminal wealth or the probability of a loss. More precisely, we are interested in how the parameter $\theta_i$ should be optimally chosen (depending on the investor's risk tolerance). We discuss this question based on numerical results obtained from the explicit representations (4.10) and (4.11). For this purpose, let $d = 1$ (one stock), $n = 12$, and consider the expected optimal terminal wealth of agent 1 while the other agents use risk tolerance parameters increasing from 0.5 to 2.7 in steps of size 0.2, and competition weights increasing from 0 to 1 in steps of size 0.1. Further, let $T = 3$, $x_0^1 = 10$, $\mu = 0.03$, and $\sigma = 0.2$. To simplify the representation of our numerical results, define $\varepsilon(\theta_1) \coloneqq \mathbb{E}\left[X_T^{1,\varphi^{1,*}}\right]$ and $\rho(\theta_1) = \rho(\theta_1, K) \coloneqq \mathbb{P}\Big(X_T^{1,\varphi^{1,*}} \leq K\Big)$, given that agent 1 uses the competition weight $\theta_1$. The results are displayed in Figure 4.2.1 and Table 4.1.

Figure 4.2.1 illustrates a comparison of the expected optimal terminal wealth and the probability of a loss for agent 1 in terms of $\theta_1 \in [0, 1]$. The underlying data was generated using $K = 0.99x_0^1$ and three different choices for the risk tolerance parameter $\delta_1 \in \{0.5, 1, 2\}$. Note that the expected terminal wealth $\varepsilon(\theta_1)$ was shifted appropriately (scaled by $x_0^1$ and moved down by the value 0.75) to simplify the comparison between the expected optimal terminal wealth and the probability of a loss. Of course, the shift does not change the overall growth behavior of $\varepsilon$. Figure 4.2.1 shows that, as explained earlier, both expressions are strictly increasing in terms of $\theta_1$. However, the probability of a loss $\rho$ grows steeper than the expected terminal wealth $\varepsilon$. The difference in slope between $\varepsilon$ and $\rho$ decreases as the risk tolerance parameter $\delta_1$ increases.

A similar observation can be made in Table 4.1. The table shows the change of $\varepsilon$ and $\rho$ between $\theta_1 = 0$ and $\theta_1 = 1$ for five different values of $\delta_1$. For both $\varepsilon$ and $\rho$, we displayed the value at $\theta_1 = 0$, and the total and relative change between $\theta_1 = 0$ and $\theta_1 = 1$. We notice again that $\varepsilon$ and $\rho$ are increasing in terms of $\theta_1$ since, for both $\varepsilon$ and $\rho$, the total difference is larger than 0 and the relative change is larger than 1 for each parameter choice. Moreover, we observe an increase in both $\varepsilon(0)$ and $\rho(0)$ as the risk tolerance $\delta_1$ increases. However, we also notice that the total and relative change in $\varepsilon$ is increasing in $\delta_1$, whereas the total and relative change in $\rho$ is decreasing in terms of $\delta_1$. We can interpret this observation as follows. If an agent is more careful due to her strong aversion against risk, the change between $\theta_1 = 0$ and $\theta_1 = 1$ has a relatively small effect on the expected optimal terminal wealth and a relatively large effect on the probability of a loss. It should also be noted that the loss probability $\rho(0)$ is much smaller for a highly risk-averse agent

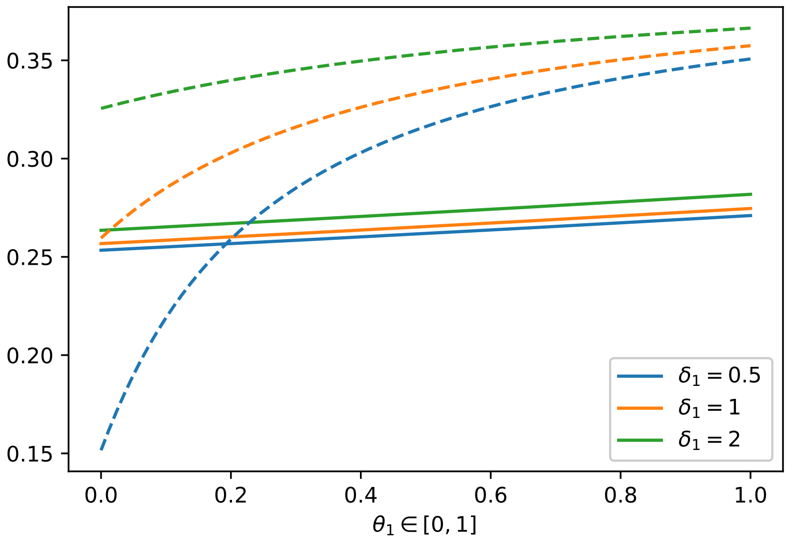



Figure 4.2.1.: Illustration of (4.10) and (4.11) for the first of $n = 12$ agents for market parameters $d = 1$, $T = 3$, $\mu = 0.03$, and $\sigma = 0.2$. The agents $j \geq 2$ use $\theta_j$ and $\delta_j$, $j \geq 2$, increasing from 0 to 1 with step size 0.1, and from 0.5 to 2.7 by step size 0.2, respectively. The figure shows $\varepsilon(\theta_1)$ (shifted appropriately) in a solid and $\rho(\theta_1)$ in a dashed line, in terms of $\theta_1$ with $\delta_1 \in \{0.5, 1, 2\}$ and fixed $K = 0.99x_0^1$.

| | $\delta_1 = 0.5$ | $\delta_1 = 1$ | $\delta_1 = 2$ | $\delta_1 = 5$ | $\delta_1 = 10$ |
|---|---|---|---|---|---|
| $\varepsilon(0)$ | 10.0338 | 10.0675 | 10.1350 | 10.3375 | 10.6750 |
| $\varepsilon(1) - \varepsilon(0)$ | 0.1764 | 0.1787 | 0.1833 | 0.1971 | 0.2201 |
| $\varepsilon(1)/\varepsilon(0)$ | 1.0176 | 1.0178 | 1.0181 | 1.0191 | 1.0206 |
| $\rho(0)$ | 0.1516 | 0.2596 | 0.3255 | 0.3681 | 0.3827 |
| $\rho(1) - \rho(0)$ | 0.1991 | 0.0979 | 0.0408 | 0.0107 | 0.0036 |
| $\rho(1)/\rho(0)$ | 2.3134 | 1.3771 | 1.1255 | 1.0292 | 1.0094 |

Table 4.1.: Comparison of (4.10) and (4.11) for the first of $n = 12$ agents. Agent 1 uses $\theta_1 \in \{0, 1\}$ and $\delta_1 \in \{0.5, 1, 2, 5, 10\}$ for comparison while the other agents use $\theta_j$ and $\delta_j$, $j \geq 2$, increasing from 0 to 1 with step size 0.1, and from 0.5 to 2.7 by step size 0.2, respectively. The market parameters are $d = 1$, $T = 3$, $\mu = 0.03$, and $\sigma = 0.2$.

($\delta_1 = 0.5$) than for a more risk-tolerant agent ($\delta_1 \in \{5, 10\}$). If, in contrast, the agent is more tolerant towards risk, the change in the expected optimal terminal wealth is relatively large and the change in the loss probability is a lot smaller than the change observed for a highly risk-averse agent. It should be noted that the more risk-tolerant agent faces a high loss probability even for $\theta_1 = 0$.

To summarize, in the current example a highly risk-averse agent should choose a value of $\theta_1$ close to 0 as the benefit from an increase in expected return does not outweigh the increase in the probability of a loss. If, however, the agent is more tolerant towards risk, a choice of $\theta_1$ close to one seems more appropriate since the increase in the loss probability is small while the increase in expected return is relatively high. We note at this point that, due to the strictly concave utility function chosen in this example, all investors are assumed to be risk-averse. However, some agents are more tolerant towards risk (larger value of $\delta_1$) than others (smaller value of $\delta_1$).

### 4.2.2. OPTIMIZATION UNDER POWER UTILITY

Let the underlying financial market again be given by the Black-Scholes market introduced at the beginning of Section 4.2. Further, we assume that each agent uses a power utility function of the form

$$U_i : (0,\infty) \to \mathbb{R},\ U_i(x) = \left(1 - \frac{1}{\delta_i}\right)^{-1} x^{1-\frac{1}{\delta_i}},$$

for risk tolerance parameters $\delta_i > 0$, $\delta_i \neq 1$, $i = 1, \ldots, n$. The objective function of agent $i$ is then given by

$$\mathbb{E}\bigg[\left(1 - \frac{1}{\delta_i}\right)^{-1} \left(X_T^{i,\varphi^i} - \theta_i \bar{X}_T^{-i,\varphi}\right)^{1-\frac{1}{\delta_i}}\bigg].$$

The competition weights $\theta_i \in [0,1]$ are chosen with respect to the condition $x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j > 0$ for the initial capitals $x_0^i > 0$, $i = 1, \ldots, n$. The unique optimal solution to the auxiliary problem

$$\begin{cases} & \sup_{\psi^i \in \mathcal{A}} \mathbb{E}\Big[\left(1 - \frac{1}{\delta_i}\right)^{-1} \left(Y_T^{i,\psi^i}\right)^{1-\frac{1}{\delta_i}}\Big] \\ \text{s.t.} & Y_T^{i,\psi^i} = \widetilde{x}_0^i + (\psi^i \bullet S)_T, \end{cases}$$

is given by

$$\psi_k^i(t) S_k(t) = \delta_i \widetilde{x}_0^i \Big(\left(\sigma\sigma^\top\right)^{-1}\mu\Big)_k \exp\left(\delta_i (\sigma^{-1}\mu)^\top W(t) + \left(\delta_i - \frac{\delta_i^2}{2}\right)\mu^\top \left(\sigma\sigma^\top\right)^{-1}\mu t\right),$$

$k = 1, \ldots, d$, $i = 1, \ldots, n$ (using Korn, 1997, p. 58), i.e., the optimally invested fraction of wealth is constant. Hence, using Theorem 3.2, the unique (up to modifications) Nash equilibrium is given by

$$\varphi_k^i(t) := \frac{n}{n+\theta_i}\psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})} \sum_{j=1}^n \frac{n}{n+\theta_j}\psi_k^{j,*}(t),$$

for $t \in [0,T]$, $k = 1, \ldots, d$, and $i = 1, \ldots, n$.

*Remark* 4.2. Lacker and Zariphopoulou (2019) suggest that the multiplicative relative performance metric is better suited for CRRA utility functions. Consequently, they use objective functions of the form

$$\mathbb{E}\left[\left(1 - \frac{1}{\delta_i}\right)^{-1} \left(X_T^{i,\varphi^i} \Big(\prod_{j=1}^n X_T^{j,\varphi^j}\Big)^{-\frac{\theta_i}{n}}\right)^{1-\frac{1}{\delta_i}}\right],\ i = 1, \ldots, n.$$

Indeed, the multiplicative structure of the argument does simplify some calculations and is also used in this thesis (see Section 7.4). Moreover, the *constant* Nash equilibrium (in terms of invested fractions) has a shorter and overall „nicer“ representation (see Lacker and Zariphopoulou, 2019, Corollary 3.2). However, the solution is less general since the method described in Section 3.2 cannot be applied here (see Remark 3.4) and thus, only constant Nash equilibria are considered. ◊

## 4.3. Binomial model in discrete time

The next example is a Cox-Ross-Rubinstein market in discrete time (see, for example, Bäuerle and Rieder, 2011, p. 60). The model brings the advantage of being very simple, but also quite popular among financial markets in discrete time. Moreover, the Cox-Ross-Rubinstein model is a discrete-time approximation of the Black-Scholes market. Hence, it does not come as a surprise that the overall structures of the Nash equilibria in the Black-Scholes market and the Cox-Ross-Rubinstein market coincide.

The market consists of one riskless bond with zero interest rate and one stock ($d = 1$) with price process $(S(t_k))_{k=0,\dots,N}$, where $t_k = k \cdot T/N$, $k = 0, \dots, N$. The stock price process is given by

$$S(t_n) = S(0) \prod_{k=1}^{n} R_k, \ n = 0, \dots, N.$$

The random variables $R_k$ for $k \in \{1, \dots, N\}$ are independent and identically distributed with $\mathbb{P}(R_n = u) = p = 1 - \mathbb{P}(R_n = d)$ for $0 < d < u$ and $p \in (0, 1)$. We assume that $d < 1 < u$ to exclude arbitrage (see, e.g., Bäuerle and Rieder, 2011, Example 3.1.7). For a self financing strategy $\varphi = (\varphi(t_k))_{k=0,\dots,N-1}$ and some initial capital $x_0 \in \mathbb{R}$, the wealth process of an investor is given by (see, e.g., Bäuerle and Rieder, 2017, Lemma 2.2)

$$X_{t_k}^{\varphi} = x_0 + \sum_{\ell=1}^{k} \varphi(t_{\ell-1})(S(t_\ell) - S(t_{\ell-1})).$$

Thus, the wealth process is linear in $\varphi$. Moreover, we assume that investors use exponential utility functions given by

$$U_i : \mathbb{R} \to \mathbb{R}, \, U_i(x) = -\exp\left(-\frac{1}{\delta_i} x\right),$$

for risk tolerance parameters $\delta_i > 0$, $i = 1, \dots, n$. The unique optimal solution to the classical optimization problem

$$\begin{cases} & \sup_{\psi^i \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i} Y_T^{i,\psi^i}\right)\right], \\ \text{s.t.} & Y_T^{i,\psi^i} = \widetilde{x}_0^i + \sum_{\ell=1}^{N} \psi^i(t_{\ell-1})(S(t_\ell) - S(t_{\ell-1})), \end{cases}$$

is given by

$$\psi^{i,*}(t_k) = \frac{\delta_i}{S(t_k)} \frac{\log\left(\frac{1-q}{1-p}\right) - \log\left(\frac{q}{p}\right)}{u - d},$$

where $q = \frac{1-d}{u-d}$ (see Bäuerle and Rieder, 2011, p. 92). Theorem 3.2 yields the unique (up to modifications) Nash equilibrium $\pi^{i,*}$, $i = 1, \dots, n$, where

$$\pi_i^* = \left(\frac{n}{n+\theta_i}\delta_i + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})} \sum_{j=1}^{n} \frac{n}{n+\theta_j}\delta_j\right) \frac{\log\left(\frac{1-q}{1-p}\right) - \log\left(\frac{q}{p}\right)}{u - d}$$

describes the invested amount. Similar to the Nash equilibrium (4.4) in Subsection 4.2.1, the invested amount is constant in time and given as the constant $C_i$ from (4.7) multiplied by some expression depending only on the market parameters. If the expression $\frac{\log\left(\frac{1-q}{1-p}\right)-\log\left(\frac{q}{p}\right)}{u-d}$ is strictly positive (this is equivalent to $p > q$), we can use the same argumentation regarding the monotonicity of $\pi^{i,*}$ in terms of $\theta_i$ as in Subsection 4.2.1.

## 4.4. MARKET WITH STOCHASTIC VOLATILITY

In this section, we consider a stochastic volatility model, the so-called *Heston model* (see Heston, 1993; Kallsen and Muhle-Karbe, 2010). The model consists of one risky asset (a stock) and a riskless bond with zero interest rate. There are two correlated Brownian motions $W^S$ and $W^Z$ with correlation $\rho$. The price process $S$ of the risky asset is described by the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}S_t &= S_t\left(\lambda Z_t \mathrm{d}t + \sqrt{Z_t}\mathrm{d}W_t^S\right),\ S_0 > 0,\\ \mathrm{d}Z_t &= \kappa(\vartheta - Z_t)\mathrm{d}t + \sigma\sqrt{Z_t}\mathrm{d}W_t^Z,\ Z_0 > 0, \end{aligned} \tag{4.12}$$

$t \in [0,T]$. The constants $\lambda, \kappa, \vartheta$, and $\sigma$ are assumed to be positive and satisfy the Feller condition $2\kappa\vartheta \geq \sigma^2$ to ensure that $Z$ is strictly positive (see, e.g., Jeanblanc et al., 2009, p. 357).

*Remark* 4.3. The volatility process $(Z_t)_{t\in[0,T]}$ describes a so-called square root diffusion. The process originates from the idea to model the square root of the volatility process as an Ornstein-Uhlenbeck process, i.e.,

$$\mathrm{d}\sqrt{Z_t} = -\beta\sqrt{Z_t}\mathrm{d}t + \delta\mathrm{d}W_t^Z$$

for constants $\beta$, $\delta > 0$ (see Heston, 1993). Using the Itô-Doeblin formula (Theorem 2.1), it can be shown that the process $Z$ solves (4.12) if the constants are chosen appropriately. $\diamondsuit$

We assume here that each agent uses a power utility function of the form

$$U_i : (0,\infty) \to \mathbb{R},\ U_i(x) = \left(1 - \frac{1}{\delta_i}\right)^{-1} x^{1-\frac{1}{\delta_i}},$$

for risk tolerance parameters $\delta_i > 0$, $\delta_i \neq 1$, $i = 1, \ldots, n$. The objective functions are given by

$$\mathbb{E}\left[\left(1 - \frac{1}{\delta_i}\right)^{-1}\left(X_T^{i,\varphi^i} - \theta_i \bar{X}_T^{-i,\varphi}\right)^{1-\frac{1}{\delta_i}}\right],\ i = 1, \ldots, n.$$

The competition weights $\theta_i \in [0,1]$ are chosen with respect to the condition $\widetilde{x}_0^i = x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j > 0$ for the initial capital $x_0^i > 0$, $i = 1, \ldots, n$. The unique (up to modifications) optimal solution of the classical portfolio optimization problem

$$\begin{cases} & \sup_{\psi^i\in\mathcal{A}} \mathbb{E}\left[\left(1-\frac{1}{\delta_i}\right)^{-1}\left(Y_T^{i,\psi^i}\right)^{1-\frac{1}{\delta_i}}\right], \\ \text{s.t.} & Y_T^{i,\psi^i} = \widetilde{x}_0^i + (\psi^i \bullet S)_T, \end{cases} \tag{4.13}$$

is given by

$$\frac{\psi^{i,*}(t)S(t)}{Y_t^{i,*}} = \delta_i\lambda + f_i(t) \tag{4.14}$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$, where the deterministic function $f_i$ can be given explicitly (see Kallsen and Muhle-Karbe, 2010). By $(Y_t^{i,*})_{t\in[0,T]}$, we denote the optimal wealth process for (4.13), i.e., the expression on the right-hand side of (4.14) describes the optimally invested fraction of wealth. Finally, $\varphi^{i,*}$, $i = 1, \ldots, n$, given by

$$\varphi^{i,*}(t) = \frac{n}{n+\theta_i}\psi^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^{n}\frac{n}{n+\theta_j}\psi^{i,*}(t)$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$, is the unique (up to modifications) Nash equilibrium.

## 4.5. Cumulative prospect theory

In the cumulative prospect theory (CPT), introduced by Tversky and Kahneman (1992), investors evaluate their wealth relative to some reference point $\xi > 0$. Values smaller than $\xi$ are treated as losses while values larger than $\xi$ are seen as gains. Studies have shown that people tend to act risk-seeking when dealing with losses and risk-averse when dealing with gains (see Tversky and Kahneman, 1992, and references therein). This effect is captured by S-shaped utility functions $U : (0,\infty) \to \mathbb{R}$, for example of the form

$$U(x) = \begin{cases} -a\cdot(\xi - x)^{\delta}, & x \le \xi, \\ b\cdot(x-\xi)^{\gamma}, & x > \xi, \end{cases} \tag{4.15}$$

for $0 < \delta \le 1$, $0 < \gamma < 1$ and $a > b > 0$. Figure 4.5.1 shows the S-shaped utility function (4.15) for the parameters[1] $a = 2.25$, $b = 1$, and $\delta = \gamma = 0.5$. The reference point is chosen as $\xi = 1$ which could, for example, describe the initial capital of some investor.

Berkelaar et al. (2004) found the unique optimal solution to the associated single investor portfolio optimization problem in a Black-Scholes market with constant market parameters. The stock price processes are hence given by

$$\mathrm{d}S_k(t) = S_k(t)\left(\mu_k \mathrm{d}t + \sum_{\ell=1}^{d}\sigma_{k\ell}\mathrm{d}W_\ell(t)\right),\ t\in[0,T],\ k = 1,\ldots,d.$$

Moreover, there exists a riskless bond with zero interest rate. Since this thesis is focused on investors evaluating their wealth with respect to the wealth of their competitors, we use a reference point in terms of the weighted arithmetic mean of the other investors' wealth. Therefore, the objective function of agent $i$ is given by

$$\begin{aligned}\mathbb{E}\Bigg[ &-a_i\cdot\left(\frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\varphi^j} - X_T^{i,\varphi^i}\right)^{\delta_i}\mathbb{1}\left\{X_T^{i,\varphi^i} \le \frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\varphi^j}\right\} \\ &+ b_i\cdot\left(X_T^{i,\varphi^i} - \frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\varphi^j}\right)^{\gamma_i}\mathbb{1}\left\{X_T^{i,\varphi^i} > \frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\varphi^j}\right\}\Bigg]\end{aligned}$$

[1] Berkelaar et al. (2004) used the similar parameter choice $a = 2.25$, $b = 1$, $\xi = 1$, and $\delta = \gamma = 0.88$. However, we use $\delta = \gamma = 0.5$ for a better illustration of the S-shape of $U$.

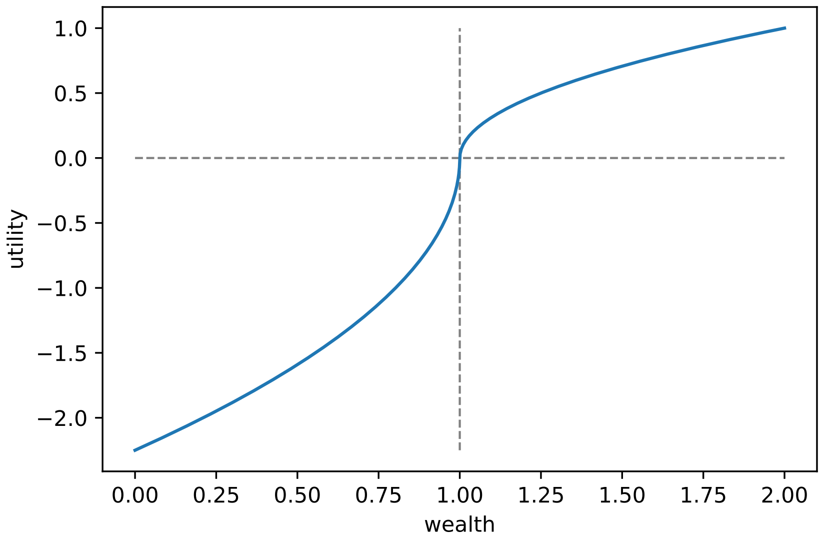



Figure 4.5.1.: Illustration of the S-shaped utility function (4.15) for $\xi = 1$, $a = 2.25$, $b = 1$, and $\delta = \gamma = 0.5$.

for $0 < \delta_i \leq 1$, $0 < \gamma_i < 1$, and $a_i > b_i > 0$, $i = 1, \dots, n$. We further introduce the constraint $X_T^{i,\varphi^i} - \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\varphi^j} \geq -\xi_i$ for some $\xi_i > 0$. Economically, this means that the investor only accepts a downward deviation from the weighted average wealth of the other investors by a constant $\xi_i$. The introduction of $\xi_i$ ensures that the corresponding classical problem is similar to the optimization considered by Berkelaar et al. (2004). Hence, the optimization problem reads as

$$\begin{cases} \sup_{\varphi^i \in \mathcal{A}} & \mathbb{E}\Big[ - a_i \cdot \Big( \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\varphi^j} - X_T^{i,\varphi^i} \Big)^{\delta_i} \mathbb{1}\Big\{ X_T^{i,\varphi^i} \leq \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\varphi^j} \Big\} \\ & \qquad + b_i \cdot \Big( X_T^{i,\varphi^i} - \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\varphi^j} \Big)^{\gamma_i} \mathbb{1}\Big\{ X_T^{i,\varphi^i} > \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\varphi^j} \Big\} \Big], \\ \text{s.t.} & X_T^{i,\varphi^i} = x_0^i + (\varphi^i \bullet S)_T,\ X_T^{i,\varphi^i} - \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\varphi^j} \geq -\xi_i, \end{cases}$$

for $i = 1, \dots, n$. The unique solution to the associated classical problem

$$\begin{cases} & \sup_{\psi^i \in \mathcal{A}} \mathbb{E}\left[ -a_i \left( \xi_i - Y_T^{i,\psi^i} \right)^{\delta_i} \mathbb{1}\left\{ Y_T^{i,\psi^i} \leq \xi_i \right\} + b_i \left( Y_T^{i,\psi^i} - \xi_i \right)^{\gamma_i} \mathbb{1}\left\{ Y_T^{i,\psi^i} > \xi_i \right\} \right], \\ \text{s.t.} & Y_T^{i,\psi^i} = \tilde{x}_0^i + \xi_i + (\psi^i \bullet S)_T,\ Y_T^{i,\psi^i} \geq 0, \end{cases}$$

is then given by (Berkelaar et al., 2004, Proposition 6)

$$\begin{aligned} \psi_k^{i,*}(t) S_k(t) = & \left( \left( \sigma \sigma^\top \right)^{-1} \mu \right)_k \cdot \left\{ \frac{\xi_i \phi(g(t, \bar{Z}_i))}{\|\kappa\| \sqrt{T-t}} \right. \\ & \left. + \left( \frac{b_i \gamma_i}{\lambda_i Z(t)} \right)^{\frac{1}{1-\gamma_i}} \mathrm{e}^{\Gamma_i(t)} \left( \frac{\phi\Big( g(t, \bar{Z}_i) + \frac{\|\kappa\| \sqrt{T-t}}{1-\gamma_i} \Big)}{\|\kappa\| \sqrt{T-t}} + \frac{\Phi\Big( g(t, \bar{Z}_i) + \frac{\|\kappa\| \sqrt{T-t}}{1-\gamma_i} \Big)}{1-\gamma_i} \right) \right\}, \end{aligned}$$

where $\kappa$ and $(Z(t))_{t \in [0,T]}$ are the market price of risk and state price density process in the given Black-Scholes market, i.e., $\kappa = \sigma^{-1} \mu$ and $Z(t) = \mathcal{E}(-\kappa \bullet W)_t$. By $\phi$ and $\Phi$ we denote the density and cumulative distribution function of the standard normal distribution. Moreover, $g$ and $\Gamma_i$ are functions in terms of the market parameters, where $g(t,\cdot)$ also depends on $Z(t)$. Finally, $\lambda_i > 0$ is the Lagrange multiplier to the constraint $\mathbb{E}[Z(T)Y_T^{i,\psi^i}] = \widetilde{x}_0^i + \xi_i$, and $\bar{Z}_i$ is the unique root of some additional function $f_i$, which depends only on the market and preference parameters of the problem. Explicit representations of the mentioned functions can be found in Proposition 6 in Berkelaar et al. (2004).

Finally, $\varphi_k^{i,*}$, $k = 1,\dots,d$, $i = 1,\dots,n$, given by

$$\varphi_k^{i,*}(t) = \frac{n}{n+\theta_i}\psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^{n}\frac{n}{n+\theta_j}\psi_k^{i,*}(t)$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$ and $i = 1,\dots,n$, is the unique (up to modifications) Nash equilibrium. Apparently, $\psi_k^{i,*}(t) > 0$ holds for all $t \in [0,T]$ if, and only if, $\left((\sigma\sigma^\top)^{-1}\mu\right)_k > 0$, $k = 1,\dots,d$. If this is the case, it follows from Remark 3.5 that the associated component $\varphi_k^{i,*}(t)$ of the Nash equilibrium is increasing in terms of $\theta_i$.

# CHAPTER 5

## Mean field equilibria for relative investors

In the following, we give a short introduction to the extensive field of research on mean field games. Inspired by the mean field theory from physics for large particle systems, mean field games were introduced independently by Huang et al. (2006) and Lasry and Lions (2007) to simplify the solution to $n$-player games where the number of agents is large. We refer to Lacker (2018) for a general introduction and motivation of the topic, and to Carmona and Delarue (2018a,b) for a thorough treatment of mean field game theory and its numerous applications. Still, we give a brief motivation and highlight the key assumptions and features of mean field games considered in the literature. After the discussion of general mean field games, we take a look at the literature on mean field portfolio optimization for relative investors, which is closely related to the results presented in this chapter.

The basic motivation to introducing mean field games was to find a mathematically tractable approximation of $n$-player games with a large number of players. As Carmona and Delarue (2013) state, „large stochastic differential games are notoriously untractable". Similar to the mean field theory used in physics, where infinite particle systems are used to approximate the finite particle setting, a model containing an infinite number of players (a continuum) is used to approximate the corresponding $n$-player game. Heuristically, due to the law of large numbers, some form of averaging is expected when the number of agents tends to infinity (see Carmona and Delarue, 2013; Lacker, 2016). To provide a mathematically rigorous explanation of this heuristic idea, one has to make two fundamental assumptions on the model. The first is that the agents are „small" in the sense that each player has very little influence on the overall system and the influence becomes negligible for large $n$ (see Lacker, 2016; Carmona and Delarue, 2013). The second basic assumption is that the agents are indistinguishable or, as Carmona and Delarue (2013) describe it, „statistically identical". Lacker (2018) phrases this assumption by asserting that players are „interchangeable". The symmetry among the players implies that the system does not depend on the individual states of the players, but only on their collective empirical distribution. Roughly speaking, we are only interested in which states are present, but not to whom each state belongs. If these basic assumptions are satisfied, we can expect the mean field game to provide a sufficiently accurate approximation of the underlying $n$-player game.

Generally, the mean field game is more tractable than its $n$-player counterpart since, instead of solving a system of $n$ coupled problems, we only need to deal with a standard control problem faced by a single representative agent who interacts with the environment (Carmona et al., 2016). By the term „environment“ we describe the state of the continuum of agents, which can be assumed to be fixed since the influence of the representative agent on the system is negligible. After solving the control problem for a single agent, the next step is to find a fixed point of the so-called consistency condition. A rough intuition for this condition is that one has to make sure that the representative agent is, in fact, representative for the whole population.

There are two different approaches to mean field games in the literature. The first is to solve the problem with a continuum of agents and then construct an approximate solution for the corresponding $n$-player game (see, among others, Huang et al., 2006). The second approach is to solve the $n$-player game first and then consider the limit as $n$ tends to infinity (see, e.g., Lasry and Lions, 2007). The relationship between the $n$-player and the mean field game is one of the three key questions tackled by the literature. More precisely, it contains two subquestions: Does each $n$-player solution converge to a mean field solution and, conversely, can any mean field solution be obtained as the limit of some (approximate) $n$-player solution? Under fairly general assumptions, it is possible to show that the answer to both subquestions is affirmative (see Lacker, 2016). The other main questions considered in the context of mean field games comprise the existence and uniqueness of solutions, both for the $n$-player and the mean field game. Of course, if there is a one-to-one correspondence between mean field solutions and sequences of $n$-player solutions, existence and uniqueness of solutions for the $n$-player and the mean field game are equivalent. It is possible to prove under general assumptions on the model parameters, that both the $n$-player and the mean field game admit a unique solution (see, e.g., Lacker, 2018; Lasry and Lions, 2007).

Furthermore, one key feature of a mean field game model is whether or not it contains common noise. In the early literature on the topic, the state diffusions of individual agents were assumed to be governed by independent Brownian motions. To include the more realistic scenario of random perturbations affecting each agent simultaneously, Carmona et al. (2016) introduced mean field games with common noise by adding an additional independent Brownian motion to the state diffusion of each agent. Later in this chapter, we consider a problem of competitive investment in a financial market common to all agents. Thus, in contrast to the majority of literature on mean field games, we consider common noise only.

Finally, let us mention the two different ways of solving mean field games as they appear in the literature (see Lacker, 2015). The first is an analytic approach based on two tightly coupled partial differential equations. One of the partial differential equations has a terminal condition and has to be solved backward in time while the second has an initial condition and needs to be solved forward in time. This approach was, for instance, used by Lasry and Lions (2007) as well as by Guéant et al. (2011). Carmona and Delarue (2013) chose a different, probabilistic approach to tackle mean field games. Their method, based on the stochastic maximum principle, results in coupled forward-backward stochastic differential equations.

Later in this chapter, we consider a special mean field game which is not solved using the classical methods described above. Thus, we do not go into any more detail on the general theory on mean field games. Let us instead consider the special case of mean field portfolio problems for relative investors, which are closely related to the problem solved in this chapter. To be more specific, we focus on portfolio optimization problems where agents measure their preferences with respect to the expected utility of an additive relative performance metric applied to their own as well as their competitors' terminal wealth. For the definition of relative performance metrics in $n$-player games, we refer to the introduction of Chapter 3.

Lacker and Zariphopoulou (2019) were the initiators of the study of mean field games for relative investors. In a continuous-time financial market with constant market parameters, they consider agents which specialize in a single stock each, and they incorporate both an independent as well as a common noise term. After finding explicit representations of unique constant Nash equilibria for CARA and CRRA utility functions, they consider the (heuristic) limit of the $n$-player equilibria as the number of agents tends to infinity. Following this informal derivation, they define the mean field game rigorously for both the additive and the multiplicative relative performance metric. Finally, they give explicit representations of the unique mean field equilibrium for both relative performance metrics which turn out to coincide with the previously derived informal limits.

The mean field problem introduced by Lacker and Zariphopoulou (2019) was soon extended in various ways. For reasons of tractability, each of the subsequent authors used either the additive relative performance metric in combination with CARA utility, or the multiplicative relative performance metric together with CRRA utility. The setting closest to the original was considered by Lacker and Soret (2020). They added consumption to the CRRA problem with a multiplicative relative performance metric. The assumption of constant market parameters was lifted by Fu et al. (2020) and Fu and Zhou (2023), who consider a model with random coefficients for CARA and CRRA utility, respectively. Hu and Zariphopoulou (2022) allowed for the market coefficients of the stock, which is common to all agents, to depend on an additional independent Brownian motion. Bo, Wang and Yu (2023) extended the financial market by allowing for jumps in the stock price process.

Instead of classical utility functions, some authors consider forward utilities. For example, Dos Reis and Platonov (2021, 2022) used forward utilities with CARA and CRRA priors, respectively. The underlying stock price processes coincide with those used by Lacker and Zariphopoulou (2019). Park (2022) considered forward utilities with CARA priors as well, allowing the stock price coefficients to be suitable stochastic processes. In contrast to the articles mentioned up to this point, which feature only risk-averse agents, Nguyen (2022) considered a system consisting of both risk-averse and risk-seeking investors. Moreover, he allowed for competition weights taking values between $-1$ and 1, which previously had to be chosen between 0 and 1.

Finally, we want to mention an extension of the original problem which considers competing insurance companies instead of investors. For example, Guan and Hu (2022), Bo, Wang and Zhou

(2023), and He et al. (2023) searched for optimal investment and (re-)insurance strategies for competing insurance companies.

Apart from Hu and Zariphopoulou (2022), all of the previously mentioned authors assumed that investors specialize in a particular stock. Furthermore, each of the above articles requires specific assumptions regarding the utility function employed by the agents and the underlying financial market. In contrast to these models, we consider a mean field game for relative investors using a general utility function to evaluate their terminal wealth with respect to the continuum of competitors in the market. The relative concerns of an investor are measured with respect to the additive relative performance metric. Moreover, the underlying financial market, which is common to all investors, takes a very general form. We use the semimartingale financial market introduced in Subsection 2.3.1, but assume that the probability space now contains three additional random variables. These are assumed to be independent of the stock price processes and represent the initial capital, the risk tolerance parameter, and the competition weight of a representative agent taken from the continuum of infinitely many investors. Similar to Section 3.1, we are able to reduce the mean field game to a single-agent portfolio optimization problem and to express the mean field equilibrium in terms of the unique solution of the corresponding classical problem. Although the setting is rather general (especially compared to the existing literature), we need to assume that each agent in the continuum uses the same utility function $U$. Only the risk tolerance parameter is allowed to differ between agents. In the corresponding $n$-agent game in Section 3.1, we did not require this assumption. However, mean field games call for a lot of symmetry among the agents. At least, the specific choice of $U$ is not relevant for the proof displayed below.

The chapter is organized as follows. In Section 5.1, we provide an informal derivation of the mean field equilibrium as the limit of the $n$-player Nash equilibrium from Theorem 3.2 when the number of agents tends to infinity. Afterwards, we define the mean field game properly, similar to Lacker and Zariphopoulou (2019). Finally, we reduce the mean field game to a suitable single-agent problem and display the unique mean field equilibrium in terms of the solution of the corresponding single-agent problem. It turns out that the mean field equilibrium coincides with the informal limit derived earlier.

## 5.1. Motivation and heuristic derivation of the mean field game

In Chapter 3, we modeled $n$ agents through their *type vector* $\zeta_i = (x_0^i, \delta_i, \theta_i)$ which contains the initial wealth and preference parameters of agent $i \in \{1, \ldots, n\}$. In the mean field game, the type vector of some representative agent will be given as the realization of a suitable random vector $\zeta = (\xi, \delta, \theta)$, which is independent of the price processes in the underlying financial market. Before we properly define the mean field game, we provide a heuristic derivation of its solution. More specifically, we provide an informal derivation of the limit of the Nash equilibrium (3.6) as $n$ tends to infinity. A similar heuristic can be found in Lacker and Zariphopoulou (2019).

Recall that the $i$-th entry of the Nash equilibrium (3.6) is given by

$$\varphi_k^{i,*}(t) := \frac{n}{n+\theta_i}\psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^n \frac{n}{n+\theta_j}\psi_k^{j,*}(t),\ k=1,\dots,d,\ t\in[0,T],$$

where $\psi^{j,*}$, $j \in \{1,\dots,n\}$, describes the unique optimal portfolio for a suitable auxiliary problem. First, we obtain

$$\lim_{n\to\infty}\frac{n}{n+\theta_i} = 1, \quad \lim_{n\to\infty}\frac{n\theta_i}{n+\theta_i} = \theta_i.$$

Moreover, if we assume that $\theta_1, \theta_2, \dots$ are independent and identically distributed random variables, independent of the underlying filtration $(\mathcal{F}_t)_{t\in[0,T]}$, we obtain

$$\hat{\theta} = \sum_{j=1}^n \frac{\theta_j}{n+\theta_j} \xrightarrow{\text{a.s.}} \mathbb{E}[\theta_1],\ n\to\infty,$$

since, using the law of large numbers[1],

$$\hat{\theta} = \sum_{j=1}^n \frac{\theta_j}{n}\cdot\frac{n}{n+\theta_j} \le \sum_{j=1}^n \frac{\theta_j}{n} \xrightarrow{\text{a.s.}} \mathbb{E}[\theta_1],\ n\to\infty,$$
$$\hat{\theta} = \sum_{j=1}^n \frac{\theta_j}{n}\cdot\frac{n}{n+\theta_j} \ge \frac{n}{n+1}\sum_{j=1}^n \frac{\theta_j}{n} \xrightarrow{\text{a.s.}} \mathbb{E}[\theta_1],\ n\to\infty.$$

Finally, if we assume that, conditional on $\mathcal{F}_T$, $\psi_k^{1,*}(t), \psi_k^{2,*}(t), \dots$ are independent and identically distributed random variables (for any $t\in[0,T]$, $k\in\{1,\dots,d\}$) we obtain (analogously, using the law of large numbers and a sandwich argument)

$$\sum_{j=1}^n \frac{1}{n+\theta_j}\psi_k^{j,*}(t) \xrightarrow{\text{a.s.}} \mathbb{E}\big[\psi_k^{1,*}(t)|\mathcal{F}_T\big],\ n\to\infty.$$

Note that we can only assume that $\psi_k^{j,*}(t)$ are independent and identically distributed *given* $\mathcal{F}_T$ as they are solutions to portfolio optimization problems at time $T$.

Hence, we expect that the components $\varphi_k^{i,*}(t)$ of the Nash equilibrium (3.6) converge to

$$\psi_k^*(t) + \frac{\theta}{1-\mathbb{E}[\theta]}\mathbb{E}[\psi_k^*(t)|\mathcal{F}_T], \quad k=1,\dots,d,$$

as $n\to\infty$, where $\theta \stackrel{\mathcal{D}}{=} \theta_1$ and $\psi_k^*(t) \stackrel{\mathcal{D}}{=} \psi_k^{1,*}(t)$ given $\mathcal{F}_T$. By $\stackrel{\mathcal{D}}{=}$ we denote equality in distribution.

The convergence of the Nash equilibrium as the number $n$ of agents tends to infinity can also be observed in Figure 5.1.1. There, the first component of the Nash equilibrium (in terms of the number of stocks) for $n$ agents using exponential utility functions is displayed in terms of the number of agents. The underlying financial market is a one-dimensional Black-Scholes market with zero interest rate, constant drift $\mu = 0.03$, and constant volatility $\sigma = 0.2$. Moreover, the competition weights $\theta_i$ and risk tolerance parameters $\delta_i$, $i=1,\dots,n$, are realizations of independent and identically $\mathcal{U}(0,1)$- and $\mathcal{U}(0.5,3)$-distributed random variables, respectively.

[1]By $\xrightarrow{\text{a.s.}}$ we denote $\mathbb{P}$-almost sure convergence.

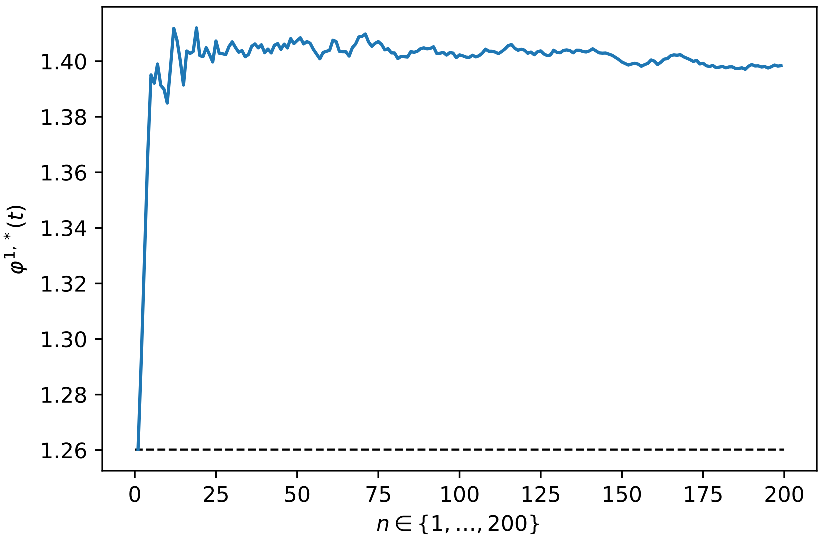



Figure 5.1.1.: Illustration of $\varphi^{1,*}(t)$ in a Black-Scholes market with $d = 1$, $r = 0$, $\mu = 0.03$, and $\sigma = 0.2$, at $t = 1$ for $n \in \{1, \dots, 200\}$ under CARA utility, where $\theta_i$ and $\delta_i$ are realizations of i.i.d. $\mathcal{U}(0,1)$- and $\mathcal{U}(0.5,3)$-distributed random variables, respectively. The dashed line marks the optimal solution to the single-agent problem ($\theta_1 = 0$) for the realization $\delta_1$ of the $\mathcal{U}(0.5,3)$-distributed random variable used for agent 1.

Considering Figure 5.1.1, we notice that for small values of $n$, $\varphi^{1,*}$ grows very fast compared to the behavior for larger values of $n$. Moreover, Figure 5.1.1 indicates that, for $n \geq 2$, the first component of the Nash equilibrium is always larger than the strategy obtained for $n = 1$, which corresponds to the optimal strategy without competition. This observation can be explained as follows

$$\begin{aligned}\varphi_k^{i,*}(t) &= \left(\frac{n}{n+\theta_i} + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\right)\psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{n}{n+\theta_j}\psi_k^{j,*}(t) \\ &\geq \psi_k^{i,*}(t) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{n}{n+\theta_j}\psi_k^{j,*}(t) \geq \psi_k^{i,*}(t) \end{aligned} \tag{5.1}$$

where $\varphi_k^{i,*}(t)$ is taken from (3.6). The chain of inequalities in (5.1) holds if $\psi_k^{j,*}(t) \geq 0$ for all $j = 1, \dots, n$, which is the case in the framework used to generate Figure 5.1.1. Moreover, we also used that $\hat{\theta} \in [0, 1)$ (see Lemma 3.3).

*Remark* 5.1. One might argue that it is not necessary to consider the mean field game associated to problem (3.3) since there exists a unique explicit solution to the $n$-player game given in Theorem 3.2. However, as Lacker and Zariphopoulou (2019) argue, the mean field model presented here might be generalized to more involved models in which the mean field game formulation is mathematically tractable whereas the $n$-player game brings some serious difficulties. We do not elaborate this any further and refer to Lacker and Zariphopoulou (2019) for examples on models with difficult $n$-player games but promising corresponding mean field game formulations. $\Diamond$

## 5.2. Formal definition of the mean field game

Let us now give a formal definition of the mean field game and a mean field equilibrium. Further, we prove that the mean field equilibrium coincides with the informally derived limit above. The structure of the mean field game is similar to the problem introduced by Lacker and Zariphopoulou (2019, Section 2.2). We use the semimartingale financial market described in Subsection 2.3.1 again. To summarize, the market consists of a riskless bond with zero interest rate and $d$ stocks. The stock price processes $(S_k(t))_{t\in[0,T]}$, $k=1,\dots,d$, are assumed to be semimartingales with càdlàg paths (i.e., the paths are right continuous with existing left limits). Further, it is assumed that there exists an equivalent $\sigma$-martingale measure $\mathbb{Q}$. Additionally, we assume that the underlying probability space contains a $(0,\infty)\times(0,\Delta]\times[0,1]$-valued random vector $\zeta=(\xi,\delta,\theta)$ independent of the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with $\Delta>0$. We assume that $\mathbb{E}[\xi^2]<\infty$. Finally, we define an additional filtration $(\mathcal{F}_t^{\mathrm{MF}})_{t\in[0,T]}$ given by

$$\mathcal{F}_t^{\mathrm{MF}} \coloneqq \sigma\left(\mathcal{F}_t,\ \zeta\right),\ t\in[0,T].$$

The random variables $\xi$, $\delta$, and $\theta$ denote the initial capital and preference parameters of a representative investor. In this setting, the wealth process of a representative agent is given by

$$X_t^{\varphi} = \xi + (\varphi \bullet S)_t,\ \ t\in[0,T],$$

where $\varphi$ is an admissible strategy representing the number of stocks held at time $t\in[0,T]$. We say that $\varphi$ is an admissible strategy if $\varphi\in\mathcal{A}^{\mathrm{MF}}$, where

$$\mathcal{A}^{\mathrm{MF}} \coloneqq \Bigg\{\varphi\in L^{\mathrm{MF}}(S):\ (\varphi\bullet S)_T\in L^2(\mathbb{P}),\ (\varphi\bullet S)Z^{\mathbb{Q}} \text{ is a } \mathbb{P}\text{-martingale for all S}\sigma\text{MM } \mathbb{Q}$$
$$\text{with density process } Z^{\mathbb{Q}} \text{ and } \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\in L^2(\mathbb{P})\Bigg\}.$$

By $L^{\mathrm{MF}}(S)$ we denote the set of $(\mathcal{F}_t^{\mathrm{MF}})$-predictable, $S$-integrable stochastic processes.

Further, we assume that $U:\mathcal{D}\to\mathbb{R}$ is a utility function, in the sense of Definition 2.11, defined on a domain $\mathcal{D}\in\{\mathbb{R},\ (0,\infty)\}$ including some parameter $\delta>0$. We again extend the definition of $U$ to the whole real line by setting $U(x)=-\infty$ if $x\notin\mathcal{D}$. Now assume that $\delta$ is part of the characterization of the representative investor. Then the combination of $U$ and $\delta$ (by inserting $\delta$ as the parameter in $U$) yields a (stochastic) utility function denoted by $U_\delta$. Now a representative investor faces the following optimization problem

$$\begin{cases} & \sup_{\varphi\in\mathcal{A}^{\mathrm{MF}}} \mathbb{E}\left[U_\delta\left(X_T^{\varphi}-\theta\bar{X}\right)\right], \\ \text{s.t.} & X_T^{\varphi}=\xi+(\varphi\bullet S)_T,\ \bar{X}=\mathbb{E}[X_T^{\varphi}|\mathcal{F}_T]. \end{cases} \tag{5.2}$$

The *consistency condition* $\bar{X}=\mathbb{E}[X_T^{\varphi}|\mathcal{F}_T]$ (see Lacker and Zariphopoulou, 2019) models the feature that each agent in the continuum faces the same type vector $\zeta$ and the same randomness from the financial market. Hence, if we condition on the information $\mathcal{F}_T$ provided by the financial market at time $T$, each agent faces an independent and identically distributed copy of the same optimization problem. Roughly speaking, the consistency condition ensures that the representative agent is indeed representative for the whole population of agents.

*Remark* 5.2. We need to ensure that there is at least one strategy $\varphi \in \mathcal{A}^{\mathrm{MF}}$ such that $X_T^{\varphi} - \theta \bar{X} \in \mathcal{D}$ $\mathbb{P}$-almost surely. If $\mathcal{D} = (0, \infty)$, we can therefore only allow choices of $\xi$ and $\theta$ that satisfy $\xi - \theta \bar{\xi} > 0$ $\mathbb{P}$-almost surely, where $\bar{\xi} := \mathbb{E}[\xi]$. ♢

The optimal solution to (5.2) is called *mean field equilibrium.*

**Definition 5.3** (Lacker and Zariphopoulou, 2019, Definition 2.9)**.** A strategy $\varphi^* \in \mathcal{A}^{\mathrm{MF}}$ is called a *mean field equilibrium*, if it is an optimal solution to the optimization problem (5.2). This means in particular that $\varphi^*$ needs to satisfy the consistency condition $\bar{X} = \mathbb{E}[X_T^{\varphi^*} | \mathcal{F}_T]$. ♦

## 5.3. Solution method via problem reduction

The optimization problem (5.2) can be solved similarly to the $n$-agent problem. Therefore, we define the auxiliary problem

$$\begin{cases} & \sup_{\psi \in \mathcal{A}^{\mathrm{MF}}} \mathbb{E}\left[U_\delta(Z_T^\psi)\right], \\ \text{s.t.} & Z_T^\psi = \xi - \theta \bar{\xi} + (\psi \bullet S)_T. \end{cases} \tag{5.3}$$

Then the mean field equilibrium for (5.2) is given in the following theorem in terms of the optimal solution to the auxiliary problem (5.3). Note that we obtain exactly the representation we derived heuristically in Section 5.1.

**Theorem 5.4.** *Let $\mathbb{E}[\theta] =: \bar{\theta} < 1$. If* (5.3) *has a unique (up to modifications) optimal portfolio strategy $\psi^*$, then there exists a unique mean field equilibrium for* (5.2) *given by*

$$\varphi_k^*(t) = \psi_k^*(t) + \frac{\theta}{1 - \bar{\theta}} \mathbb{E}\left[\psi_k^*(t) | \mathcal{F}_T\right], \; k = 1, \dots, d, \tag{5.4}$$

$\mathbb{P}$*-almost surely for all $t \in [0, T]$.*

*Proof.* In order to solve the optimization problem (5.2), we assume that $\bar{X}$ is an $\mathcal{F}_T$-measurable random variable of the form

$$\bar{X} = \mathbb{E}\left[X_T^\alpha | \mathcal{F}_T\right]$$

for an admissible strategy $\alpha \in \mathcal{A}^{\mathrm{MF}}$ with $X_0^\alpha = \xi$. Moreover, we define the process

$$\bar{X}_t^\alpha := \mathbb{E}\left[X_t^\alpha | \mathcal{F}_T\right], \; t \in [0, T].$$

Since $X_t^\alpha$ can be written as

$$X_t^\alpha = \xi + \sum_{k=1}^d \int_0^t \alpha_k(u) \mathrm{d}S_k(u),$$

we obtain

$$\bar{X}_t^\alpha = \bar{\xi} + \sum_{k=1}^d \int_0^t \bar{\alpha}_k(u) \mathrm{d}S_k(u), \tag{5.5}$$

where $\bar{\alpha}_k(u) := \mathbb{E}[\alpha_k(u)|\mathcal{F}_T]$, $k = 1, \dots, d$, and $\bar{\xi} = \mathbb{E}[\xi]$. The representation (5.5) of $\bar{X}_t^\alpha$ requires more explanation. As a first step, we used the independence of $\xi$ and $(\mathcal{F}_t)_{t\in[0,T]}$, and the linearity of the conditional expectation. To obtain

$$\mathbb{E}\left[\int_0^t \alpha_k(u)\mathrm{d}S_k(u)\Big|\mathcal{F}_T\right] = \int_0^t \bar{\alpha}_k(u)\mathrm{d}S_k(u), \tag{5.6}$$

we first observe that the sample space can be written as $\Omega = \Omega_1 \times \Omega_2$ with $\sigma$-algebras $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ on $\Omega_1$, $\Omega_2$, respectively, and probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$, satisfying $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$, due to the independence of $(\mathcal{F}_t)$ and $\zeta$. Hence, $\Omega_1$ and $\Omega_2$ are associated to $(\mathcal{F}_t)$ and $\zeta$, respectively. It follows that for any $F \in \mathcal{F}_T$, there exists a unique $G \in \mathcal{F}^{(1)}$ such that $F = G \times \Omega_2$. Now we prove that the conditional expectation with respect to $\mathcal{F}_T$ of some $\mathcal{F}_T^{\mathrm{MF}}$-measurable, $\mathbb{P}$-integrable random variable $Y = Y(\omega_1, \omega_2)$ can be written as an integral over $\Omega_2$ with respect to $\mathbb{P}_2$. First, we notice that $\mathbb{E}[Y|\mathcal{F}_T]$ is a random variable that can, due to the $\mathcal{F}_T$-measurability, be written in terms of $\omega_1$ only. Now let $F \in \mathcal{F}_T$ with decomposition $F = G \times \Omega_2$. It follows (using Fubini's theorem and the definition of conditional expectation[2])

$$\begin{aligned}
\int_G \int_{\Omega_2} Y(\omega_1, \omega_2)\mathrm{d}\mathbb{P}_2(\omega_2)\mathrm{d}\mathbb{P}_1(\omega_1) &= \int_F Y(\omega_1, \omega_2)\mathrm{d}\mathbb{P}(\omega_1, \omega_2) \\
&= \int_F \mathbb{E}[Y|\mathcal{F}_T](\omega_1, \omega_2)\mathrm{d}\mathbb{P}(\omega_1, \omega_2) \\
&= \int_G \int_{\Omega_2} \mathbb{E}[Y|\mathcal{F}_T](\omega_1, \omega_2)\mathrm{d}\mathbb{P}_2(\omega_2)\mathrm{d}\mathbb{P}_1(\omega_1) \\
&= \int_G \int_{\Omega_2} \mathbb{E}[Y|\mathcal{F}_T](\omega_1)\mathrm{d}\mathbb{P}_2(\omega_2)\mathrm{d}\mathbb{P}_1(\omega_1) \\
&= \int_G \mathbb{E}[Y|\mathcal{F}_T](\omega_1)\mathrm{d}\mathbb{P}_1(\omega_1).
\end{aligned}$$

Therefore, we obtain

$$\mathbb{E}[Y|\mathcal{F}_T](\omega_1) = \int_{\Omega_2} Y(\omega_1, \omega_2)\mathrm{d}\mathbb{P}_2(\omega_2)$$

for $\mathbb{P}_1$-almost all $\omega_1 \in \Omega_1$ (following the arguments of the proof of Theorem 8.12 in Klenke, 2020). Now we can use this result to prove (5.6)

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \alpha_k(u)\mathrm{d}S_k(u)\Big|\mathcal{F}_T\right] &= \int_{\Omega_2} \int_0^t \alpha_k(u)\mathrm{d}S_k(u)\mathrm{d}\mathbb{P}_2(\omega_2) \\
&= \int_0^t \int_{\Omega_2} \alpha_k(u)\mathrm{d}\mathbb{P}_2(\omega_2)\mathrm{d}S_k(u) \\
&= \int_0^t \mathbb{E}[\alpha_k(u)|\mathcal{F}_T]\mathrm{d}S_k(u) \\
&= \int_0^t \bar{\alpha}_k(u)\mathrm{d}S_k(u).
\end{aligned}$$

The second equality holds due to a stochastic version of Fubini's theorem by Protter (2005) (see Theorem 2.5). Hence, the representation (5.5) is in fact correct and we can proceed with the

[2] If $X$ is an integrable random variable and $\mathcal{G}$ a sub $\sigma$-algebra of $\mathcal{F}$, the conditional expectation of $X$ given $\mathcal{G}$ is the almost surely unique, $\mathcal{G}$-measurable random variable $Y$ such that $\int_G X\,\mathrm{d}\mathbb{P} = \int_G Y\,\mathrm{d}\mathbb{P}$ for all $G \in \mathcal{G}$ (see, for example, Klenke, 2020, Definition 8.11 and Theorem 8.12).

solution of problem (5.2). By construction, the equation $\bar{X}_T^\alpha = \bar{X}$ holds. Using the previously defined process $(\bar{X}_t^\alpha)$, we define another process $(Z_t)_{t\in[0,T]}$ for $\varphi \in \mathcal{A}^{\mathrm{MF}}$ and $X_0^\varphi = \xi$ by

$$Z_t \coloneqq X_t^\varphi - \theta \bar{X}_t^\alpha.$$

Thus, $Z_t$ can be written as

$$Z_t = \xi - \theta\bar{\xi} + \sum_{k=1}^d \int_0^t \underbrace{(\varphi_k(u) - \theta\bar{\alpha}_k(u))}_{=:\psi_k(u)} \mathrm{d}S_k(u) =: Z_t^\psi.$$

The random variable $Z_T^\psi$ coincides with the argument of the objective function in (5.2). Therefore, we consider the auxiliary problem (5.3)

$$\begin{cases} & \sup_{\psi\in\mathcal{A}^{\mathrm{MF}}} \mathbb{E}\left[U_\delta(Z_T^\psi)\right], \\ \text{s.t.} & Z_T^\psi = \xi - \theta\bar{\xi} + (\psi \bullet S)_T. \end{cases}$$

If $\psi^*$ is the unique optimal portfolio strategy to (5.3), we can determine the solution to (5.2) as follows. By definition, we have $Z_T^{\psi^*} = X_T^\varphi - \theta\bar{X}$ or equivalently $X_T^\varphi = Z_T^{\psi^*} + \theta\bar{X}$. Moreover, the random variable $\bar{X}$ needs to satisfy the consistency condition $\bar{X} = \mathbb{E}\left[X_T^\varphi|\mathcal{F}_T\right]$. Hence, it follows

$$\bar{X} = \mathbb{E}[X_T^\varphi|\mathcal{F}_T] = \mathbb{E}\left[Z_T^{\psi^*} + \theta\bar{X}|\mathcal{F}_T\right] = \mathbb{E}\left[Z_T^{\psi^*}|\mathcal{F}_T\right] + \bar{X}\mathbb{E}[\theta|\mathcal{F}_T] = \mathbb{E}\left[Z_T^{\psi^*}|\mathcal{F}_T\right] + \bar{\theta}\bar{X},$$

where we used that $\bar{X}$ is $\mathcal{F}_T$-measurable and that $\theta$ is independent of $(\mathcal{F}_t)$. Moreover, we introduced the notation $\bar{\theta} \coloneqq \mathbb{E}[\theta]$. Under the assumption that $\bar{\theta} < 1$, we obtain

$$\bar{X} = \frac{1}{1-\bar{\theta}}\mathbb{E}\left[Z_T^{\psi^*}|\mathcal{F}_T\right].$$

Therefore, the optimal wealth $X_T^\varphi$ is given by

$$X_T^\varphi = Z_T^{\psi^*} + \theta\bar{X} = Z_T^{\psi^*} + \frac{\theta}{1-\bar{\theta}}\mathbb{E}\left[Z_T^{\psi^*}|\mathcal{F}_T\right].$$

Since the wealth process is linear in terms of the strategy and the solution $\psi^*$ is unique, it follows that

$$\varphi(t) = \psi^*(t) + \frac{\theta}{1-\bar{\theta}}\mathbb{E}\left[\psi^*(t)|\mathcal{F}_T\right]$$

componentwise $\mathbb{P}$-almost surely for all $t \in [0,T]$. The line of arguments implies that there exists a unique Nash equilibrium given by (5.4) if, and only if, the auxiliary problem (5.3) is uniquely solvable. □

*Remark* 5.5. The mean field equilibrium (5.4) shows, similar to Remark 3.5, that a larger value of $\theta$ results in a more risky investment behavior of a representative agent. If we substitute $\theta$ by a different, $[0,1]$-valued random variable $\tilde{\theta}$ with $\mathbb{E}[\tilde{\theta}] < 1$ and $\tilde{\theta} > \theta$ $\mathbb{P}$-almost surely, the resulting Nash equilibrium becomes more risky in the sense that more shares of the risky asset are purchased or sold short depending on whether the realization of $\psi_k^*(t)$ is positive or negative. ◊

Finally, we present an application of Theorem 5.4 which describes a mean field counterpart of the model in Subsection 4.2.1.

*Example* 5.6. We consider a one-dimensional Black-Scholes financial market with constant drift $\mu > 0$ and volatility $\sigma > 0$. Moreover, let $U_\delta(x) = -\exp\left(-\delta^{-1}x\right)$, $x \in \mathbb{R}$. Then the solution to the auxiliary problem (5.3) in terms of amounts is known to be given by

$$\pi^Z = \delta \cdot \frac{\mu}{\sigma^2}.$$

Therefore, the mean field equilibrium to (5.2) in terms of amounts is given by

$$\pi = \delta\frac{\mu}{\sigma^2} + \frac{\theta}{1-\bar{\theta}}\mathbb{E}\left[\delta\frac{\mu}{\sigma^2}\Big|\mathcal{F}_T\right] = \left(\delta + \bar{\delta}\frac{\theta}{1-\bar{\theta}}\right)\frac{\mu}{\sigma^2}. \qquad \Diamond$$

# Chapter 6

# Pareto optima for relative investors

In general, there are two main optimality criteria used in $n$-person stochastic games: Nash equilibria and Pareto optimality. Until now, we have only looked for Nash equilibria in the multi-objective optimization problem (3.3) and the corresponding mean field game. In this section, we search for Pareto optimal strategies. The criterion is named after Vilfredo Pareto who developed the concept further in 1896 after it had already been used in 1881 (see Miettinen, 1999, pp. 10-11, and the references therein). In contrast to a Nash equilibrium, where $n$ agents maximize their objectives simultaneously, each assuming that the strategies of the other players are fixed, a Pareto optimal vector of strategies is found if no player can improve her objective without worsening the objective of another player. Thus, Pareto optima are related to maximizing the common good of all players as opposed to the more self-centered Nash equilibria.

The literature on Pareto optimal strategies for $n$-player stochastic games is rather sparse and, to the best of our knowledge, the problem of optimal investment for competing agents has not yet been solved using the concept of Pareto optimality. Of course, as the agents are assumed to be competitive, it is more intuitive to search for Nash equilibria. However, instead of interpreting Pareto optimality as optimizing the common good, we can have some central planner in mind that manages the portfolios of $n$ competing agents. Hence, the central planner aims to maximize the common good of her clients without this behavior contradicting the competitive incentive of the $n$ agents.

Closest to our work is the article by Branger et al. (2023). They considered $n$ non-competitive agents aiming to maximize the expected utility of their terminal wealth given some fixed initial capital. While the solution to such problems for a single investor is well known in many interesting special cases, Branger et al. (2023) considered the modified problem of collective asset allocation and searched for Pareto optimal vectors of strategies. They defined a novel type of utility function, the so-called collective utility function, which is used to find a Pareto optimal vector of strategies for the $n$ agents. An adapted version of their collective utility function is used in this chapter. The collective utility function has the advantage of containing only one state variable, as opposed to the „classical“ approach of maximizing a weighted sum of the $n$ players' objectives, which results in an objective function that depends on all $n$ state variables. The classical approach of maximizing a weighted sum of the $n$ agents' objective functions was, e.g., used by Ferrari et al. (2017) and Guo and Xu (2020) in problems other than portfolio optimization. The paper by Ferrari et al. (2017) considers the optimal allocation of the initial wealth of $n$ agents between personal consumption and irreversible contributions to increase the level of some public good. The paper by Guo and Xu (2020) considers a central planner problem for $n$ players aiming to minimize the running cost corresponding to some general diffusion state process. We refer to Branger et al. (2023) for a more detailed literature overview of collective optimal investment and related topics.

We conclude this introduction with a summary of this chapter. The underlying financial market and the optimization problem are explained in Section 6.1. In Section 6.2, we adapt the scalarization method introduced by Branger et al. (2023) to competitive utility functions (see Section 3.1 for the definition of competitive utility functions). In Section 6.3, we determine a Pareto optimum in terms of optimal terminal wealth. In order to achieve this, we determine the optimal collective terminal wealth first and then deduce the optimal terminal wealth for the individual agents. In the final Section 6.4, we compare the Pareto optimum to the Nash equilibrium from Chapter 3.

## 6.1. PROBLEM FORMULATION

We base our analysis on the semimartingale financial market from Subsection 2.3.1. In summary, there are $d+1$ assets in which $n$ agents can invest. The assets consist of one riskless bond with zero interest rate and $d$ risky stocks. The stock price processes $(S_k(t))_{t\in[0,T]}$, $k=1,\dots,d$, are $L^2(\mathbb{P})$-semimartingales with càdlàg paths. To exclude arbitrage, we require the existence of an equivalent $\sigma$-martingale measure $\mathbb{Q}$. In contrast to Subsection 2.3.1, we make the additional assumption that $\mathbb{Q}$ is the *unique* equivalent $\sigma$-martingale measure. Under this assumption, the associated density process

$$Z_t^{\mathbb{Q}} = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|\mathcal{F}_t\right],\ t\in[0,T],$$

is unique as well. Thus, we write $Z_t := Z_t^{\mathbb{Q}}$, $t\in[0,T]$, throughout the present chapter. The assumption that $\mathbb{Q}$ is the unique equivalent $\sigma$-martingale measure has the important consequence that each claim $X\in L^2(\mathbb{P})$ has a *unique* fair price given by (cf. (2.6))

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}\left[Z_T X\right].$$

Moreover, the existence of a unique equivalent ($\sigma$-/local) martingale measure is usually related to a complete financial market (see Subsection 2.3.2). Although at this point we cannot deduce that the financial market is complete, we conjecture that this is the case. However, the possible lack of completeness is not a problem for the later analysis. We refer to Remark 6.10 in which we revisit the issue of completeness.

Let us now state the central optimization problem of this chapter. Similar to Chapter 3, we consider $n$ agents with objectives

$$\max_{\pi^i \in \mathcal{A}} \mathbb{E}\left[U_i\left(X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_T^{j,\pi^j}\right)\right], \; i = 1, \dots, n. \tag{6.1}$$

The functions $U_i : \mathcal{D}_i \to \mathbb{R}$, defined on $\mathcal{D}_i \in \{(0,\infty), \mathbb{R}\}$, $i = 1, \dots, n$, are assumed to be Inada utility functions (see Definition 2.11). Note that we did not have to impose the Inada conditions in Chapter 3, but it is crucial for the proofs displayed later in the current chapter. The assumption is not a severe restriction, however, since many classical examples such as the natural logarithm and the exponential or power utility satisfy the Inada conditions.

There are two main differences between the analysis in Chapter 3 and the current chapter. The first one is that in Chapter 3 we were interested in finding the optimal strategy $\pi^{i,*}$, while in this chapter we only search for the optimal terminal wealth. We adapt the solution method proposed by Branger et al. (2023). They use the martingale approach (see, for example, Korn, 1997, Section 3.4) to solve the problem of finding the optimal collective terminal wealth. Thus, they search for the optimal terminal wealth first and find the replicating strategy afterwards. If the financial market is complete, it is always possible to hedge the optimal terminal wealth. However, we do not make the assumption of a complete financial market.

Thus, the objective of agent $i$ is to solve the static optimization problem corresponding to the dynamic problem (6.1). The objective of agent $i \in \{1, \dots, n\}$ then reads as

$$\begin{cases} & \max_{X_i} J_i(X_1, \dots, X_n) := \mathbb{E}\left[U_i\left(X_i - \frac{\theta_i}{n}\sum_{j\neq i} X_j\right)\right], \\ \text{s.t.} & X_i \text{ is } \mathcal{F}_T - \text{measurable}, \; \mathbb{E}[Z_T X_i] \le x_0^i, \end{cases} \tag{6.2}$$

where $x_0^i$ denotes the initial capital and $X_i$ the terminal wealth of agent $i \in \{1, \dots, n\}$.

The second major difference to Chapter 3 is that the goal in the current chapter is to find vectors of Pareto optimal strategies instead of Nash equilibria. We already gave a definition of a Pareto optimal vector of *strategies* in Section 2.4. However, as we optimize the terminal wealth $X_i$ instead of the portfolio strategy, we repeat the definition in terms of terminal wealth.

**Definition 6.1** (Miettinen, 1999, Definition 2.2.1)**.** A vector $(X_1^*, \dots, X_n^*)$ of admissible states $X_i^*$, $i = 1, \dots, n$, such that there is no admissible vector $(X_1, \dots, X_n)$ with

$$J_i(X_1, \dots, X_n) \ge J_i(X_1^*, \dots, X_n^*) \quad \text{for all } i = 1, \dots, n$$

and

$$J_i(X_1, \dots, X_n) > J_i(X_1^*, \dots, X_n^*) \quad \text{for at least one } i \in \{1, \dots, n\},$$

is called *Pareto optimal.* A random variable $X_i$ is considered admissible if it satisfies the conditions of (6.2). ♦

Let us now describe how to find a Pareto optimum for (6.2). The definition of Pareto optimality introduces an optimization problem including an $n$-dimensional objective function which consists of the $n$ agents' expected competitive utilities. In general, the first step in solving such problems is to scalarize the objective function. A common and fairly intuitive method of scalarization is to maximize a weighted sum of the $n$ objective functions. More specifically, let $\beta_1, \dots, \beta_n \in (0,1)$ with $\sum_{i=1}^n \beta_i = 1$ be weights assigned to the $n$ agents. Then the scalarized objective function takes the form

$$\sum_{i=1}^n \beta_i J_i(X_1, \dots, X_n).$$

For any choice of weights $\beta_1, \dots, \beta_n$, the resulting solution is Pareto optimal (see, e.g., Aubin, 2003, p. 193). Although this method seems simple at first, using it to search for an analytical solution to the problem (6.2) results in some serious difficulties. The main disadvantage we discovered was a system of nonlinear equations emerging in the search for the maximum. Therefore, we use a different scalarization approach which is carried out in the next section.

## 6.2. Scalarization of the objective function

In the following, we apply a scalarization method introduced by Branger et al. (2023) to the competitive utility functions

$$U_i\left(x_i - \theta_i \bar{x}^{-i}\right)$$

for Inada utility functions $U_i : \mathcal{D}_i \to \mathbb{R}$, $\mathcal{D}_i \in \{(0,\infty), \mathbb{R}\}$, and competition weights $\theta_i \in [0,1]$, $i = 1, \dots, n$, where $\bar{x}^{-i} = n^{-1} \sum_{j \neq i} x_j$. The scalarized utility function, denoted by $\widetilde{U}_\beta$, is determined as the optimal value of the optimization problem

$$\begin{cases} & \widetilde{U}_\beta(x) = \max\limits_{x_1,\dots,x_n} \sum_{i=1}^n \beta_i U_i\left(x_i - \theta_i \bar{x}^{-i}\right), \\ \text{s.t.} & \sum_{i=1}^n x_i = x, \ x_i - \theta_i \bar{x}^{-i} \in \mathcal{D}_i, \ i = 1, \dots, n, \end{cases} \tag{6.3}$$

where $\beta_1, \dots, \beta_n \in (0,1)$ describe weights, assigned to the $n$ agents, with $\sum_{i=1}^n \beta_i = 1$.

Note that it is not necessary that the utility functions are of the same „type". For example, it would be possible that agent 1 uses an exponential utility function while agent 2 uses the natural logarithm. Moreover, they do not need to be defined on the same domain. We just need to assume that the domain is either the whole real numbers or the strictly positive real numbers, which is a reasonable requirement in the context of utility maximization.

Now we can state the following lemma which gives the optimal solution to (6.3) and the resulting scalarized objective function $\widetilde{U}_\beta$. It turns out that $\widetilde{U}_\beta$ is an Inada utility function (see Definition 2.11).

**Lemma 6.2.** *For $n \in \mathbb{N}$, let $U_i : \mathcal{D}_i \to \mathbb{R}$, $i = 1, \dots, n$, be Inada utility functions defined on domains $\mathcal{D}_i \in \{(0,\infty), \mathbb{R}\}$. Let $I_i : (0,\infty) \to \mathcal{D}_i$ denote the inverse of the first order derivative $U_i'$ of $U_i$. Further, let $\theta_i \in [0,1]$ and $\beta_i \in (0,1)$, $i = 1, \dots, n$, be chosen so that $\sum_{j=1}^n \beta_j = 1$. Finally, define a function $\tilde{I} : (0,\infty) \to \bigcup_{j=1}^n \mathcal{D}_j$ by*

$$\tilde{I}(y) = \frac{1}{1-\hat{\theta}} \sum_{j=1}^n \frac{n}{n+\theta_j} I_j\left(\frac{ny}{\beta_j(n+\theta_j)(1-\hat{\theta})}\right),$$

*where* $\hat{\theta} = \sum_{j=1}^{n} \frac{\theta_j}{n+\theta_j}$*. Then, for any* $x \in \bigcup_{j=1}^{n} \mathcal{D}_j$*, the optimization problem* (6.3) *has a unique solution* $x_1^*, \ldots, x_n^*$ *given by*

$$x_i^* = f_i(x) := \frac{\theta_i}{n+\theta_i} \cdot x + \frac{n}{n+\theta_i} I_i\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i(n+\theta_i)(1-\hat{\theta})}\right),\ i = 1, \ldots, n, \tag{6.4}$$

*in terms of the inverse function* $\tilde{I}^{-1} : \bigcup_{j=1}^{n} \mathcal{D}_j \to (0,\infty)$ *of* $\tilde{I}$*. The functions* $f_i$*,* $i = 1, \ldots, n$*, are bijections from* $\bigcup_{j=1}^{n} \mathcal{D}_j$ *to* $\bigcup_{j=1}^{n} \mathcal{D}_j$*. Finally, for* $x \in \bigcup_{j=1}^{n} \mathcal{D}_j$*, the optimal value of* (6.3) *is given by*

$$\widetilde{U}_\beta(x) := \sum_{j=1}^{n} \beta_j U_j\left(I_j\left(\frac{n\tilde{I}^{-1}(x)}{\beta_j(n+\theta_j)(1-\hat{\theta})}\right)\right),$$

*which defines an Inada utility function* $\widetilde{U}_\beta : \bigcup_{j=1}^{n} \mathcal{D}_j \to \mathbb{R}$*.*

*Proof.* The optimization problem is solved using the Lagrange dual method. The constraints $x_i - \theta_i \bar{x}^{-i} \in \mathcal{D}_i$, $i = 1, \ldots, n$, are only relevant if $\mathcal{D}_i = (0,\infty)$. We solve the optimization problem without the constraints and prove that the optimal solution satisfies them afterwards. Hence, we consider the optimization problem

$$\begin{cases} & \max\limits_{x_1,\ldots,x_n} \sum_{i=1}^{n} \beta_i U_i\left(x_i - \theta_i \bar{x}^{-i}\right), \\ \text{s.t.} & \sum_{i=1}^{n} x_i = x. \end{cases} \tag{6.5}$$

The associated Lagrangian function is given by

$$\mathcal{L}(x_1, \ldots, x_n; \lambda) = \sum_{i=1}^{n} \beta_i U_i\left(x_i - \theta_i \bar{x}^{-i}\right) - \lambda\left(\sum_{i=1}^{n} x_i - x\right)$$

for the Lagrange multiplier $\lambda > 0$. Note that the objective function is concave and the constraint is linear. Thus, to find a solution to (6.5), it is sufficient to determine $x_1, \ldots, x_n$ such that $\frac{\partial}{\partial x_j}\mathcal{L}(x_1, \ldots, x_n; \lambda) = 0$ for all $j \in \{1, \ldots, n\}$, and such that $\sum_{j=1}^{n} x_j = x$. The first order partial derivative of $\mathcal{L}$ with respect to $x_j$ for some $j \in \{1, \ldots, n\}$ is given by

$$\begin{aligned} \frac{\partial}{\partial x_j}\mathcal{L}(x_1, \ldots, x_n; \lambda) &= \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_j} U_i\left(x_i - \theta_i \bar{x}^{-i}\right) - \lambda \\ &= \beta_j U_j'\left(x_j - \theta_j \bar{x}^{-j}\right) - \sum_{i \neq j} \frac{\beta_i \theta_i}{n} U_i'\left(x_i - \theta_i \bar{x}^{-i}\right) - \lambda \stackrel{!}{=} 0. \end{aligned}$$

If we set

$$u_i := U_i'\left(x_i - \theta_i \bar{x}^{-i}\right), \quad \hat{u} := \sum_{i=1}^{n} \frac{\beta_i \theta_i}{n} u_i,$$

the resulting system of linear equations can be written as

$$\beta_i\left(1 + \frac{\theta_i}{n}\right) u_i - \hat{u} = \lambda, \quad i = 1, \ldots, n.$$

The unique (implicit) solution to this system of linear equations is given by

$$u_i = \frac{n(\lambda + \hat{u})}{\beta_i\,(n + \theta_i)}. \tag{6.6}$$

Inserting (6.6) into the definition of $\hat{u}$ yields

$$\hat{u} = \sum_{i=1}^{n} \frac{\beta_i \theta_i}{n} u_i = (\lambda + \hat{u}) \sum_{i=1}^{n} \frac{\theta_i}{n + \theta_i} = \hat{\theta}(\lambda + \hat{u}).$$

Since $0 \leq \hat{\theta} < 1$ (see Lemma 3.3), it follows

$$\hat{u} = \frac{\lambda \hat{\theta}}{1 - \hat{\theta}}.$$

Therefore, using (6.6), an explicit representation of $u_i$ is given by

$$u_i = \frac{n}{\beta_i\,(n + \theta_i)} \left( \lambda + \frac{\hat{\theta}}{1 - \hat{\theta}} \lambda \right) = \frac{n\lambda}{\beta_i\,(n + \theta_i)\,(1 - \hat{\theta})}.$$

Thus, using the definition of $u_i$ and the inverse $I_i$ of $U_i'$,

$$\left(1 + \frac{\theta_i}{n}\right) x_i - \frac{\theta_i}{n} \hat{x} \coloneqq x_i - \theta_i \bar{x}^{-i} = I_i \left( \frac{n\lambda}{\beta_i\,(n + \theta_i)\,(1 - \hat{\theta})} \right),$$

which yields the following implicit representation of $x_i$

$$x_i = \frac{\theta_i}{n + \theta_i} \hat{x} + \frac{n}{n + \theta_i} I_i \left( \frac{n\lambda}{\beta_i\,(n + \theta_i)\,(1 - \hat{\theta})} \right), \tag{6.7}$$

where we defined $\hat{x} \coloneqq \sum_{i=1}^{n} x_i$. Inserting (6.7) into the definition of $\hat{x}$ yields

$$\hat{x} = \sum_{i=1}^{n} x_i = \hat{\theta}\hat{x} + \sum_{i=1}^{n} \frac{n}{n + \theta_i} I_i \left( \frac{n\lambda}{\beta_i\,(n + \theta_i)\,(1 - \hat{\theta})} \right).$$

Since $\hat{\theta} < 1$ (see Lemma 3.3), it follows

$$\hat{x} = \frac{1}{1 - \hat{\theta}} \sum_{i=1}^{n} \frac{n}{n + \theta_i} I_i \left( \frac{n\lambda}{\beta_i\,(n + \theta_i)\,(1 - \hat{\theta})} \right),$$

and therefore,

$$x_i = \frac{\theta_i}{(n + \theta_i)\,(1 - \hat{\theta})} \sum_{j=1}^{n} \frac{n}{n + \theta_j} I_j \left( \frac{n\lambda}{\beta_j\,(n + \theta_j)\,(1 - \hat{\theta})} \right) + \frac{n}{n + \theta_i} I_i \left( \frac{n\lambda}{\beta_i\,(n + \theta_i)\,(1 - \hat{\theta})} \right).$$

Finally, the Lagrange multiplier $\lambda$ needs to be chosen such that the constraint $\sum_{i=1}^{n} x_i = x$ is satisfied. Therefore, we define the function $\tilde{I} : (0,\infty) \to \bigcup_{j=1}^{n} \mathcal{D}_j$ by

$$\tilde{I}(\lambda) = \frac{1}{1-\hat{\theta}} \sum_{j=1}^{n} \frac{n}{n+\theta_j} I_j \left( \frac{n\lambda}{\beta_j \left(n+\theta_j\right)\left(1-\hat{\theta}\right)} \right) (= \hat{x}).$$

Since the functions $I_i$ take values in $\mathcal{D}_i \in \{(0,\infty), \mathbb{R}\}$, $i = 1,\dots,n$, and all constant factors are strictly positive, the function $\tilde{I}$ takes values in $\bigcup_{j=1}^{n} \mathcal{D}_j$. The functions $I_i$ are strictly decreasing and continuously differentiable. Thus, $\tilde{I}$ also has these properties. Moreover, $U_i$ satisfies the Inada conditions, which implies

$$\lim_{x \searrow 0} I_i(x) = \infty, \quad \lim_{x\to\infty} I_i(x) = \inf \mathcal{D}_i \in \{0, -\infty\}$$

for all $i \in \{1,\dots,n\}$. Hence, it follows that $\lim\limits_{\lambda \searrow 0} \tilde{I}(\lambda) = \infty$.

When $\lambda$ tends to $\infty$, there are two possibilities we need to consider. If $\mathcal{D}_i = (0,\infty)$ for all $i \in \{1,\dots,n\}$, then each summand in the definition of $\tilde{I}$ tends to 0 and therefore, $\tilde{I}$ converges to 0 as $\lambda$ tends to $\infty$. If at least one of the utility functions is defined on the whole real numbers, at least one summand in the definition of $\tilde{I}$ tends to $-\infty$, while the other summands tend to 0 or to $-\infty$ as well. Hence, in this case $\tilde{I}$ converges to $-\infty$ as $\lambda$ tends to $\infty$.

In summary, the asymptotic behavior of $\tilde{I}$ is given as follows

$$\lim_{\lambda \searrow 0} \tilde{I}(\lambda) = \infty, \ \lim_{\lambda\to\infty} \tilde{I}(\lambda) = \inf \bigcup_{j=1}^{n} \mathcal{D}_j \in \{0, -\infty\}.$$

The previous observations imply that $\tilde{I}$ is a bijection from $(0,\infty)$ to $\bigcup_{j=1}^{n} \mathcal{D}_j$ and therefore, we can consider the inverse of $\tilde{I}$ denoted by $\tilde{I}^{-1} : \bigcup_{j=1}^{n} \mathcal{D}_j \to (0,\infty)$. Since $\tilde{I}$ is strictly decreasing and continuously differentiable, $\tilde{I}^{-1}$ inherits those properties.

Finally, since $\tilde{I}^{-1}$ is a bijection from $\bigcup_{j=1}^{n} \mathcal{D}_j$ to $(0,\infty)$, we can conclude that there exists a unique $\lambda^* \in (0,\infty)$ such that $\tilde{I}(\lambda^*) = x$, which is given by $\lambda^* = \tilde{I}^{-1}(x)$. Hence, a candidate for the optimal solution is given by

$$\begin{aligned} x_i &= \frac{\theta_i}{\left(n+\theta_i\right)\left(1-\hat{\theta}\right)} \sum_{j=1}^{n} \frac{n}{n+\theta_j} I_j \left( \frac{n\tilde{I}^{-1}(x)}{\beta_j \left(n+\theta_j\right)\left(1-\hat{\theta}\right)} \right) + \frac{n}{n+\theta_i} I_i \left( \frac{n\tilde{I}^{-1}(x)}{\beta_i \left(n+\theta_i\right)\left(1-\hat{\theta}\right)} \right) \\ &= \frac{\theta_i}{n+\theta_i} x + \frac{n}{n+\theta_i} I_i \left( \frac{n\tilde{I}^{-1}(x)}{\beta_i \left(n+\theta_i\right)\left(1-\hat{\theta}\right)} \right). \end{aligned} \tag{6.8}$$

By construction, the values $x_i$, $i = 1,\dots,n$, maximize the objective function and satisfy the constraint $\sum_{i=1}^{n} x_i = x$. Therefore, the only thing left to show is that the remaining constraint $x_i - \theta_i \bar{x}^{-i} \in \mathcal{D}_i$ is also satisfied. It follows

$$x_i - \theta_i \bar{x}^{-i} = \left(1 + \frac{\theta_i}{n}\right) x_i - \frac{\theta_i}{n} \sum_{j=1}^{n} x_j = \left(1 + \frac{\theta_i}{n}\right) x_i - \frac{\theta_i}{n} x = I_i \left( \frac{n\tilde{I}^{-1}(x)}{\beta_i \left(n+\theta_i\right)\left(1-\hat{\theta}\right)} \right),$$

where we used (6.8) in the last step. Since $I_i$ takes values in $\mathcal{D}_i$, the constraint $x_i - \theta_i \bar{x}^{-i} \in \mathcal{D}_i$ is satisfied.

Therefore, the optimal solution to the optimization problem is indeed given by

$$x_i^* = \frac{\theta_i}{n+\theta_i} x + \frac{n}{n+\theta_i} I_i\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right).$$

The optimal value of the objective function is then given by

$$\widetilde{U}_\beta(x) := \sum_{i=1}^{n} \beta_i U_i\left(I_i\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right),$$

defining a function $\widetilde{U}_\beta : \bigcup_{i=1}^{n} \mathcal{D}_i \to \mathbb{R}$. In the following, we prove that $\widetilde{U}_\beta$ satisfies the properties of an Inada utility function. To begin with, we determine the first and second order derivative of $\widetilde{U}_\beta$:

$$\begin{aligned}\widetilde{U}_\beta'(x) &= \sum_{i=1}^{n}\left\{\beta_i U_i'\left(I_i\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right) I_i'\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\cdot\frac{n\left(\tilde{I}^{-1}\right)'(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right\}\\ &= \tilde{I}^{-1}(x)\left(\tilde{I}^{-1}\right)'(x)\sum_{i=1}^{n}\frac{n^2}{\beta_i\,(n+\theta_i)^2\,(1-\hat{\theta})^2} I_i'\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right) = \tilde{I}^{-1}(x).\end{aligned}$$

In the last step, we used the following auxiliary calculation

$$\begin{aligned}\left(\tilde{I}^{-1}\right)'(x) = \frac{1}{\tilde{I}'(\tilde{I}^{-1}(x))} &= \left(\frac{\mathrm{d}}{\mathrm{d}y}\sum_{i=1}^{n}\frac{n}{(n+\theta_i)\,(1-\hat{\theta})} I_i\left(\frac{ny}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\bigg|_{y=\tilde{I}^{-1}(x)}\right)^{-1}\\ &= \left(\sum_{i=1}^{n}\frac{n^2}{\beta_i\,(n+\theta_i)^2\,(1-\hat{\theta})^2} I_i'\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right)^{-1}.\end{aligned}$$

Hence, the second order derivative of $\widetilde{U}_\beta$ is simply given by

$$\widetilde{U}_\beta''(x) = \left(\tilde{I}^{-1}\right)'(x) = \left(\sum_{i=1}^{n}\frac{n^2}{\beta_i\,(n+\theta_i)^2\,(1-\hat{\theta})^2} I_i'\left(\frac{n\tilde{I}^{-1}(x)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right)^{-1}. \tag{6.9}$$

Since the utility functions $U_i$ were assumed to be twice continuously differentiable, the previous representations of the first and second order derivative of $\widetilde{U}_\beta$ imply that $\widetilde{U}_\beta$ is twice continuously differentiable as well. Additionally, as $\tilde{I}^{-1}$ takes values in $(0,\infty)$, the first order derivative of $\widetilde{U}_\beta$ is strictly positive and $\widetilde{U}_\beta$ is strictly increasing. Moreover, the functions $I_i$ are strictly decreasing by the strict concavity of $U_i$ and therefore, (6.9) implies that $\widetilde{U}_\beta$ is strictly concave.

Finally, we need to examine the asymptotic behavior of $\widetilde{U}_\beta' = \tilde{I}^{-1}$. The Inada conditions of $U_i$, $i = 1, \ldots, n$, imply

$$\begin{aligned}\lim_{y\searrow 0}\tilde{I}(y) &= \sum_{i=1}^{n}\frac{n}{(n+\theta_i)\,(1-\hat{\theta})}\underbrace{\lim_{y\searrow 0} I_i\left(\frac{ny}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)}_{=\infty} = \infty,\\ \lim_{y\to\infty}\tilde{I}(y) &= \sum_{i=1}^{n}\frac{n}{(n+\theta_i)\,(1-\hat{\theta})}\underbrace{\lim_{y\to\infty} I_i\left(\frac{ny}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)}_{=\inf\mathcal{D}_i} = \inf\bigcup_{i=1}^{n}\mathcal{D}_i.\end{aligned} \tag{6.10}$$

The last equation (6.10) follows from the assumption $\mathcal{D}_i \in \{(0,\infty), \mathbb{R}\}$ and the fact that the constant factors are strictly positive. Hence, the inverse of $\tilde{I}$ satisfies

$$\lim_{x\to\infty} \tilde{I}^{-1}(x) = 0, \quad \lim_{x\to\inf \bigcup_{i=1}^n \mathcal{D}_i} \tilde{I}^{-1}(x) = \infty.$$

In summary, it follows that $\widetilde{U}_\beta$ is an Inada utility function.

Finally, it remains to show that the functions $f_i$ from (6.4) are bijections. The function $f_i$ is strictly increasing in $x$, since $I_i$ and $\tilde{I}^{-1}$ are both strictly decreasing and all constant factors are strictly positive. Moreover, $f_i$ is continuous as a composition of continuous functions. Finally, using the asymptotic behavior of $I_i$ and $\tilde{I}^{-1}$,

$$\lim_{x\to\infty} f_i(x) = \infty, \quad \lim_{x\to\inf \bigcup_{j=1}^n \mathcal{D}_j} f_i(x) = \inf \bigcup_{j=1}^n \mathcal{D}_j.$$

Hence, $f_i$ is a bijection from $\bigcup_{j=1}^n \mathcal{D}_j$ onto $\bigcup_{j=1}^n \mathcal{D}_j$ for each $i \in \{1,\dots,n\}$. This concludes our proof. □

The collective competitive utility function $\widetilde{U}_\beta$ can be used to find a Pareto optimum $(X_1^*,\dots,X_n^*)$, where $X_i^*$ represents the optimal terminal wealth of agent $i$. First, the optimal collective wealth $X^*$ is determined with respect to some classical portfolio optimization problem under the utility function $\widetilde{U}_\beta$. Afterwards, choosing $X_i^* = f_i(X^*)$, $i = 1,\dots,n$, gives a Pareto optimum with respect to the original objective functions $\mathbb{E}[U_i(X_i - \theta_i \bar{X}^{-i})]$. To verify that this yields a Pareto optimal solution to the original problem, let $X^*$ be the optimal solution to

$$\begin{cases} & \max_X \mathbb{E}\left[\widetilde{U}_\beta(X)\right], \\ \text{s.t.} & X \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E}[Z_T X] \le x_0 = \sum_{i=1}^n x_0^i. \end{cases} \tag{6.11}$$

Lemma 6.2 provides bijections $f_1,\dots,f_n$ for which the random variables $X_i^* := f_i(X^*)$, $i = 1,\dots,n$, satisfy

$$\widetilde{U}_\beta(X^*) = \max_{\sum_{i=1}^n X_i = X^*} \sum_{i=1}^n \beta_i U_i(X_1,\dots,X_n) = \sum_{i=1}^n \beta_i U_i(X_1^*,\dots,X_n^*).$$

The expressions $U_i(X_1,\dots,X_n)$ denote the competitive utility functions

$$U_i(X_1,\dots,X_n) := U_i\left(X_i - \frac{\theta_i}{n}\sum_{j\neq i} X_j\right), \; i = 1,\dots,n.$$

Further, assume that there exist admissible $\widetilde{X}_1,\dots,\widetilde{X}_n$ (i.e., $\widetilde{X}_j$ satisfies the constraints of (6.2), $j = 1,\dots,n$) such that

$$\begin{aligned} &\mathbb{E}[U_i(X_1^*,\dots,X_n^*)] \le \mathbb{E}[U_i(\widetilde{X}_1,\dots,\widetilde{X}_n)] \text{ for all } i = 1,\dots,n, \\ &\mathbb{E}[U_j(X_1^*,\dots,X_n^*)] < \mathbb{E}[U_j(\widetilde{X}_1,\dots,\widetilde{X}_n)] \text{ for at least one } j \in \{1,\dots,n\}. \end{aligned}$$

Moreover, define $\widetilde{X} := \sum_{i=1}^n \widetilde{X}_i$. Then, using that $\beta_i > 0$ for all $i = 1, \dots, n$,

$$\begin{aligned}
\mathbb{E}\left[\widetilde{U}_\beta(X^*)\right] &= \sum_{i=1}^n \beta_i \mathbb{E}\left[U_i(X_1^*, \dots, X_n^*)\right] \\
&< \sum_{i=1}^n \beta_i \mathbb{E}\left[U_i(\widetilde{X}_1, \dots, \widetilde{X}_n)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n \beta_i U_i(\widetilde{X}_1, \dots, \widetilde{X}_n)\right] \\
&\leq \mathbb{E}\left[\max_{\sum_{i=1}^n X_i = \widetilde{X}} \sum_{i=1}^n \beta_i U_i(X_1, \dots, X_n)\right] = \mathbb{E}\left[\widetilde{U}_\beta(\widetilde{X})\right].
\end{aligned}$$

This contradicts the assumption that $X^*$ is an optimal solution to (6.11). Hence, $(X_1^*, \dots, X_n^*)$ is Pareto optimal.

*Remark* 6.3. If we replace the constraint $\sum_{i=1}^n x_i = x$ in (6.3) by $\sum_{i=1}^n \alpha_i x_i = x$, where $\alpha_i \in (0,1)$ are weights with $\sum_{i=1}^n \alpha_i = 1$, it follows

$$\widetilde{U}_\beta(x) = \sum_{i=1}^n \beta_i U_i \left( I_i \left( \frac{n\tilde{I}^{-1}(x)}{\beta_i(n+\theta_i)} \left( \alpha_i + \frac{\widehat{\alpha\theta}}{1-\hat{\theta}} \right) \right) \right).$$

The function $\tilde{I}^{-1}$ denotes the inverse of $\tilde{I} : (0,\infty) \to \bigcup_{j=1}^n \mathcal{D}_j$ defined by

$$\tilde{I}(y) = \sum_{j=1}^n \frac{n}{n+\theta_j} \left( \alpha_j + \frac{\widehat{\alpha\theta}}{1-\hat{\theta}} \right) I_j \left( \frac{ny}{\beta_j(n+\theta_j)} \left( \alpha_j + \frac{\widehat{\alpha\theta}}{1-\hat{\theta}} \right) \right),$$

where $\widehat{\alpha\theta} := \sum_{j=1}^n \frac{\alpha_j\theta_j}{n+\theta_j}$. The introduction of $\alpha_1, \dots, \alpha_n$ extends the function $\widetilde{U}_\beta$ by $n$ additional free parameters that can be chosen with respect to some further conditions on the functions $f_i$ or the resulting terminal wealth $X_i = f_i(X)$, where $X$ denotes the collective terminal wealth. $\Diamond$

Before we use the scalarized objective function $\widetilde{U}_\beta$ to find a Pareto optimum for (6.2), let us determine $\widetilde{U}_\beta$ for two examples of utility functions $U_1, \dots, U_n$.

*Example* 6.4. Let us apply Lemma 6.2 to two special cases of Inada utility functions.

a) Let $n \in \mathbb{N}$ and $U_i(x) = \log(x)$, $i = 1, \dots, n$, with $\mathcal{D}_i = (0,\infty)$. Then the collective utility function $\widetilde{U}_\beta : (0,\infty) \to \mathbb{R}$ from Lemma 6.2 is given by

$$\widetilde{U}_\beta(x) = \log(x) + \sum_{i=1}^n \beta_i \log\left( \beta_i \left( 1 + \frac{\theta_i}{n} \right) (1-\hat{\theta}) \right).$$

b) Let $n \in \mathbb{N}$ and $U_i(x) = -\exp(-\delta_i^{-1}x)$ on $\mathcal{D}_i = \mathbb{R}$ for parameters $\delta_i > 0$, $i = 1, \dots, n$. Then the collective utility function $\widetilde{U}_\beta : \mathbb{R} \to \mathbb{R}$ from Lemma 6.2 is given by

$$\widetilde{U}_\beta(x) = -\frac{\widehat{\delta\theta}}{1-\hat{\theta}} \cdot \exp\left( -\frac{1}{\widehat{\delta\theta}} \sum_{j=1}^n \frac{n\delta_j}{n+\theta_j} \log\left( \frac{n\delta_j}{\beta_j\,(n+\theta_j)\,(1-\hat{\theta})} \right) \right) \cdot \exp\left( -\frac{1-\hat{\theta}}{\widehat{\delta\theta}} \cdot x \right),$$

where $\widehat{\delta\theta} := \sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}$. $\Diamond$

*Remark* 6.5. Branger et al. (2023) introduced the collective utility function as the optimal value of the optimization problem

$$\widetilde{U}_\beta(X) = \max_{\sum_{i=1}^n X_i = X} \sum_{i=1}^n \beta_i U_i(X_i).$$

They obtained

$$\widetilde{U}_\beta(X) = \sum_{i=1}^n \beta_i U_i\Big(I_i\Big(\frac{y}{\beta_i}\Big)\Big),$$

where $y$ is chosen such that

$$\sum_{i=1}^n I_i\Big(\frac{y}{\beta_i}\Big) = X.$$

Thus, up to constants depending only on the preference parameters of the agents, the optimal value coincides with our result displayed in Lemma 6.2. ♢

## 6.3. Pareto optimum for the relative performance problem

In the following, the scalarized utility function $\widetilde{U}_\beta$ is used to find a Pareto optimum for (6.2).

### 6.3.1. Maximization of the collective terminal wealth

The first step towards a Pareto optimum for (6.2) is to determine the collective optimal terminal wealth with respect to the collective competitive utility function $\widetilde{U}_\beta$. The optimal collective wealth is determined as the optimal solution to

$$\begin{cases} & \max_X \mathbb{E}\left[\widetilde{U}_\beta(X)\right], \\ \text{s.t.} & X \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E}\left[Z_T X\right] \le x_0 := \sum_{j=1}^n x_0^j. \end{cases} \tag{6.12}$$

The following lemma provides the unique optimal solution to (6.12).

**Lemma 6.6.** *Let $U_i : \mathcal{D}_i \to \mathbb{R}$ be Inada utility functions on $\mathcal{D}_i \in \{(0,\infty), \mathbb{R}\}$, $i = 1, \ldots, n$, and let $\widetilde{U}_\beta$ be the scalarized objective function from Lemma 6.2. Further, let $x_0 \in \bigcup_{i=1}^n \mathcal{D}_i$. Finally, assume that*

$$\mathbb{E}\left[\left|Z_T I_i\left(\frac{nyZ_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right|\right] < \infty \tag{6.13}$$

*for all $y \in (0,\infty)$ and $i = 1, \ldots, n$. Then the unique optimal solution to (6.12) is given by*

$$X^* = \tilde{I}\left(\lambda^* Z_T\right) = \sum_{i=1}^n \frac{n}{(n+\theta_i)\,(1-\hat{\theta})} I_i\left(\frac{n\lambda^* Z_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right),$$

*where $\lambda^* > 0$ is the unique value such that $\mathbb{E}[Z_T X^*] = x_0$.*

*Proof.* Using a standard result (see, e.g., Kramkov and Schachermayer, 1999) and Lemma 6.2, which provides that $\widetilde{U}_\beta$ is an Inada utility function, the unique optimal solution to (6.12) is given by

$$X^* = \left(\widetilde{U}_\beta'\right)^{-1}(\lambda Z_T).$$

The Lagrange multiplier $\lambda > 0$ needs to be chosen with respect to the constraint $\mathbb{E}\left[Z_T X^*\right] = x_0$. Using results from the proof of Lemma 6.2, the first order derivative of $\widetilde{U}_\beta$ coincides with the inverse $\tilde{I}^{-1}$ of

$$\tilde{I} : (0,\infty) \to \bigcup_{i=1}^{n} \mathcal{D}_i,\ y \mapsto \sum_{i=1}^{n} \frac{n}{(n+\theta_i)(1-\hat{\theta})} I_i\left(\frac{ny}{\beta_i (n+\theta_i)(1-\hat{\theta})}\right).$$

Therefore, $(\widetilde{U}_\beta)^{-1} = \tilde{I}$ and $X^*$ is given by

$$X^* = \tilde{I}(\lambda Z_T) = \sum_{i=1}^{n} \frac{n}{(n+\theta_i)(1-\hat{\theta})} I_i\left(\frac{n\lambda Z_T}{\beta_i (n+\theta_i)(1-\hat{\theta})}\right).$$

To find the Lagrange multiplier, define a function $\mathcal{H}$ on $(0,\infty)$ by

$$\mathcal{H}(\lambda) := \mathbb{E}\left[Z_T X^*\right] = \sum_{i=1}^{n} \frac{n}{(n+\theta_i)(1-\hat{\theta})} \mathbb{E}\left[Z_T \cdot I_i\left(\frac{n\lambda Z_T}{\beta_i (n+\theta_i)(1-\hat{\theta})}\right)\right].$$

Now $\lambda^* > 0$ needs to be chosen such that $\mathcal{H}(\lambda^*) = x_0$. Existence and uniqueness of $\lambda^*$ are shown using the intermediate value theorem. To verify that $\mathcal{H}$ is continuous, fix some arbitrary $\lambda > 0$ and a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $\lim_{k\to\infty} \lambda_k = \lambda$. Then it follows that

$$\begin{aligned}
\lim_{k\to\infty} \mathcal{H}(\lambda_k) &= \sum_{i=1}^{n} \frac{n}{(n+\theta_i)(1-\hat{\theta})} \lim_{k\to\infty} \mathbb{E}\left[Z_T \cdot I_i\left(\frac{n\lambda_k Z_T}{\beta_i (n+\theta_i)(1-\hat{\theta})}\right)\right] \\
&= \sum_{i=1}^{n} \frac{n}{(n+\theta_i)(1-\hat{\theta})} \mathbb{E}\left[Z_T \cdot I_i\left(\frac{n\lambda Z_T}{\beta_i (n+\theta_i)(1-\hat{\theta})}\right)\right] \qquad (6.14) \\
&= \mathcal{H}(\lambda).
\end{aligned}$$

Equation (6.14) holds since each function $I_i$, $i = 1,\dots,n$, is continuous and strictly decreasing, in combination with the dominated convergence theorem and the integrability assumption (6.13) (see Section A.1 in the appendix for the derivation of an integrable majorant). Thus, $\mathcal{H}$ is continuous.

Now we need to examine the asymptotic behavior of $\mathcal{H}$. Using a monotone convergence theorem for increasing and decreasing sequences of functions which are not necessarily non-negative (Theorem 11.1 in Schilling, 2005), we obtain

$$\lim_{\lambda\searrow 0} \mathcal{H}(\lambda) = \lim_{\lambda\searrow 0} \mathbb{E}\left[Z_T \tilde{I}\left(\lambda Z_T\right)\right] = \infty,$$

since $\tilde{I}$ is strictly decreasing with $\lim_{y\searrow 0} \tilde{I}(y) = \infty$ (which was shown in the proof of Lemma 6.2). If $\mathcal{D}_i = \mathbb{R}$ for at least one $i \in \{1,\dots,n\}$, we need to decompose the argument of the expectation into its positive and negative part. Analogously, it follows

$$\lim_{\lambda\to\infty} \mathcal{H}(\lambda) = \lim_{\lambda\to\infty} \mathbb{E}\left[Z_T \tilde{I}\left(\lambda Z_T\right)\right] = \inf \bigcup_{i=1}^{n} \mathcal{D}_i,$$

since $\lim_{y\to\infty} \tilde{I}(y) = \inf \bigcup_{j=1}^{n} \mathcal{D}_j$. Now the intermediate value theorem in combination with the strict monotonicity of $\mathcal{H}$ implies the existence of a unique value $\lambda^* \in (0,\infty)$ with $\mathcal{H}(\lambda^*) = x_0$.

In summary,

$$X^* = \sum_{i=1}^{n} \frac{n}{(n+\theta_i)\,(1-\hat{\theta})} I_i \left( \frac{n\lambda^* Z_T}{\beta_i (n+\theta_i)(1-\hat{\theta})} \right)$$

describes the unique optimal solution to (6.12). □

*Remark* 6.7. The integrability assumption (6.13) is, for example, satisfied in the important special cases of a lognormal state price density in combination with logarithmic, power or exponential utility. In the logarithmic and power case, this observation follows easily using the lognormal distribution of $Z_T$. The integrability under exponential utility follows from Lemma A.1 in the Appendix. ◊

### 6.3.2. Pareto optimal terminal wealth of the individual investors

The second step in the search for a Pareto optimum is to combine the optimal collective terminal wealth $X^*$ from Lemma 6.6 and the bijections $f_i$, $i = 1,\ldots,n$, from Lemma 6.2. However, we need to make sure that the budget constraint

$$\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i \tag{6.15}$$

is satisfied for each agent $i \in \{1,\ldots,n\}$. It turns out that this is only the case if the weights $\beta_1,\ldots,\beta_n$ are chosen correctly. The subsequent lemma provides the unique choice of weights $\beta_1,\ldots,\beta_n$ subject to (6.15).

**Lemma 6.8.** *Assume that the conditions of Lemmas 6.2 and 6.6 are satisfied. Moreover, let $\lambda^* > 0$ be the Lagrange multiplier chosen in Lemma 6.6, $X^*$ the associated optimal collective wealth at time $T$, and $f_i$, $i = 1,\ldots,n$, the bijections from Lemma 6.2. Then the unique solution $\beta_i^*$, $i = 1,\ldots,n$, to the system of equations*

$$\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i,\ i = 1,\ldots,n,\ \sum_{i=1}^{n} \beta_i = 1,$$

*is given by*

$$\beta_i^* = \frac{h_i\left(x_0^i - \theta_i \bar{x}_0^{-i}\right)}{\sum_{j=1}^{n} h_j\left(x_0^j - \theta_j \bar{x}_0^{-j}\right)}.$$

*The functions $h_i : \mathcal{D}_i \to (0,\infty)$ are defined as $h_i = \frac{1}{H_i^{-1}}$, where $H_i^{-1}$ denotes the inverse of $H_i : (0,\infty) \to \mathcal{D}_i$, where*

$$H_i(y) = \mathbb{E}\left[ Z_T I_i \left( \frac{nyZ_T}{(n+\theta_i)(1-\hat{\theta})} \right) \right],\ i = 1,\ldots,n.$$

*Proof.* To begin with, it is important to understand that the $n$-dimensional system of equations given by $\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i$, $i = 1,\ldots,n$, is underdetermined. Assume that $\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i$ is satisfied for all $i \in \{1, \dots, n-1\}$. Then it follows

$$\sum_{i=1}^{n} x_0^i = x_0 = \mathbb{E}\left[Z_T X^*\right] = \mathbb{E}\left[Z_T \sum_{i=1}^{n} f_i(X^*)\right] = \sum_{i=1}^{n-1} x_0^i + \mathbb{E}\left[Z_T f_n(X^*)\right]$$

and, thus, $\mathbb{E}\left[Z_T f_n(X^*)\right] = x_0 - \sum_{i=1}^{n-1} x_0^i = x_0^n$. Hence, solving the system of equations given by $\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i$, $i = 1, \dots, n$, leaves one free variable which can then be chosen with respect to the condition $\sum_{i=1}^{n} \beta_i = 1$.

Now let $X^* = \tilde{I}(\lambda^* Z_T)$ be the collective optimal terminal wealth displayed in Lemma 6.6 and, for $i \in \{1, \dots, n\}$, let $f_i$ be the bijection given in Lemma 6.2, i.e.,

$$\begin{aligned} f_i(X^*) &= \frac{\theta_i}{n+\theta_i} X^* + \frac{n}{n+\theta_i} I_i\left(\frac{n\tilde{I}^{-1}(X^*)}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right) \\ &= \frac{\theta_i}{n+\theta_i} \tilde{I}(\lambda^* Z_T) + \frac{n}{n+\theta_i} I_i\left(\frac{n\lambda^* Z_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right). \end{aligned}$$

The Lagrange multiplier $\lambda^* > 0$ was chosen in Lemma 6.6 subject to

$$\mathbb{E}\left[Z_T \tilde{I}\left(\lambda^* Z_T\right)\right] = x_0 = \sum_{i=1}^{n} x_0^i.$$

Hence, it follows

$$\mathbb{E}\left[Z_T f_i(X^*)\right] = \frac{\theta_i}{n+\theta_i} x_0 + \frac{n}{n+\theta_i} \mathbb{E}\left[Z_T I_i\left(\frac{n\lambda^* Z_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right] \stackrel{!}{=} x_0^i.$$

This can equivalently be written as

$$H_i\left(\frac{\lambda^*}{\beta_i}\right) := \mathbb{E}\left[Z_T I_i\left(\frac{n\lambda^* Z_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right] = x_0^i - \theta_i \bar{x}_0^{-i}, \tag{6.16}$$

where $\bar{x}_0^{-i} = \frac{1}{n}\sum_{j\neq i} x_0^j$, $i = 1, \dots, n$. The proof of Lemma 6.6 provides that the functions $H_i$ are continuous, strictly decreasing, and satisfy

$$\lim_{y \searrow 0} H_i(y) = \infty, \ \lim_{y\to\infty} H_i(y) = \inf \mathcal{D}_i, \ i = 1, \dots, n.$$

Hence, $H_i$ has an inverse $H_i^{-1} : \mathcal{D}_i \to (0, \infty)$ and we can equivalently rewrite (6.16) as

$$\frac{\lambda^*}{\beta_i} = H_i^{-1}(x_0^i - \theta_i \bar{x}_0^{-i}) \tag{6.17}$$

for all $i \in \{1, \dots, n\}$. Note that (6.17) only gives an implicit representation of $\beta_i$ since $\lambda^*$ depends on $\beta_1, \dots, \beta_n$. Since the system of equations $\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i$, $i = 1, \dots, n$, is underdetermined, we find $\beta_1, \dots, \beta_{n-1}$ in terms of $\beta_n$ and choose $\beta_n = 1 - \sum_{i=1}^{n-1} \beta_i$ afterwards. As (6.17) needs to hold for each $i \in \{1, \dots, n\}$, we deduce

$$\beta_i H_i^{-1}(x_0^i - \theta_i \bar{x}_0^{-i}) = \lambda^* = \beta_n H_n^{-1}(x_0^n - \theta_n \bar{x}_0^{-n})$$

for all $i \in \{1, \dots, n\}$. Equivalently, $\beta_i$ is given by

$$\beta_i = \beta_n \frac{H_n^{-1}(x_0^n - \theta_n \bar{x}_0^{-n})}{H_i^{-1}(x_0^i - \theta_i \bar{x}_0^{-i})}, \; i = 1, \dots, n-1, \tag{6.18}$$

in terms of $\beta_n$. Now the remaining condition $\sum_{i=1}^n \beta_i = 1$ implies

$$\beta_n = 1 - \sum_{i=1}^{n-1} \beta_i = 1 - \beta_n H_n^{-1}(x_0^n - \theta_n \bar{x}_0^{-n}) \sum_{i=1}^{n-1} \frac{1}{H_i^{-1}(x_0^i - \theta_i \bar{x}_0^{-i})}, \quad i = 1, \dots, n. \tag{6.19}$$

The functions $H_i^{-1}$ take strictly positive values. Thus, we can rewrite (6.19) as

$$\begin{aligned} \beta_n &= \left( 1 + H_n^{-1}(x_0^n - \theta_n \bar{x}_0^{-n}) \sum_{i=1}^{n-1} \frac{1}{H_i^{-1}(x_0^i - \theta_i \bar{x}_0^{-i})} \right)^{-1} \\ &= \left( H_n^{-1}(x_0^n - \theta_n \bar{x}_0^{-n}) \sum_{i=1}^{n} \frac{1}{H_i^{-1}(x_0^i - \theta_i \bar{x}_0^{-i})} \right)^{-1} = \frac{h_n(x_0^n - \theta_n \bar{x}_0^{-n})}{\sum_{i=1}^n h_i(x_0^i - \theta_i \bar{x}_0^{-i})}, \end{aligned}$$

where we introduced $h_i := \frac{1}{H_i^{-1}}$, $i = 1, \dots, n$. Inserting $\beta_n$ into (6.18) yields

$$\beta_i = \beta_n \frac{h_i\left(x_0^i - \theta_i \bar{x}_0^{-i}\right)}{h_n\left(x_0^n - \theta_n \bar{x}_0^{-n}\right)} = \frac{h_i\left(x_0^i - \theta_i \bar{x}_0^{-i}\right)}{\sum_{j=1}^n h_j\left(x_0^j - \theta_j \bar{x}_0^{-j}\right)}. \tag{6.20}$$

Since $\beta_i \in (0,1)$, $i = 1, \dots, n$, is obviously satisfied, the unique choice of $\beta_i$, $i = 1, \dots, n$, with the desired properties is given by (6.20). □

If the weights $\beta_i$, $i = 1, \dots, n$, are chosen with respect to Lemma 6.8, the budget constraints

$$\mathbb{E}\left[Z_T f_i(X^*)\right] = x_0^i, \quad i = 1, \dots, n,$$

are satisfied by construction. Hence, the following theorem gives a Pareto optimum for the multi-objective optimization problem

$$\begin{cases} \quad \max_{X_i} \mathbb{E}\left[U_i\left(X_i - \theta_i \bar{X}^{-i}\right)\right], & \\ \text{s.t.} \quad X_i \text{ is } \mathcal{F}_T - \text{measurable}, \ \mathbb{E}\left[Z_T X_i\right] \le x_0^i, & \end{cases} \qquad i = 1, \dots, n.$$

**Theorem 6.9.** *Assume that the conditions of Lemmas 6.2 and 6.6 are satisfied. Moreover, let $\lambda^* > 0$ be the unique Lagrange multiplier chosen in the proof of Lemma 6.6 and $\beta_i^*$, $i = 1, \dots, n$, the weights from Lemma 6.8. Then*

$$X_i^* = \frac{n}{n+\theta_i} I_i\left(\frac{n\lambda_i^* Z_T}{(n+\theta_i)(1-\hat{\theta})}\right) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})} \sum_{j=1}^n \frac{n}{n+\theta_j} I_j\left(\frac{n\lambda_j^* Z_T}{(n+\theta_j)(1-\hat{\theta})}\right), \tag{6.21}$$

*$i = 1, \dots, n$, is a Pareto optimum for (6.2). The constants $\lambda_i^*$, $i = 1, \dots, n$, are given by $\lambda_i^* = H_i^{-1}\left(x_0^i - \theta_i \bar{x}_0^{-i}\right)$ for the function $H_i^{-1}$ introduced in Lemma 6.8.*

*Proof.* The assertion is a direct consequence of the statements of Lemmas 6.2, 6.6, and 6.8. For the introduction of the constants $\lambda_i^*$, we used that (6.17) holds for all $i \in \{1, \dots, n\}$. □

*Remark* 6.10. In Section 6.1, we explained that the search for the optimal terminal wealth instead of the optimal investment strategy originates from the martingale approach, a commonly used method to solve portfolio optimization problems (see, for example, Section 3.4 in Korn, 1997, Kramkov and Schachermayer, 1999, or Subsection 2.3.3). The martingale approach consists of two subproblems. The first one is the static optimization problem in which the optimal terminal wealth $X^*$ is determined. The second one is the representation problem in which a replicating strategy is determined, i.e., an admissible strategy $\varphi$ for which $X^* = X_T^\varphi = x_0 + (\varphi \bullet S)_T$ holds. By $x_0$ we denote the corresponding initial capital. The martingale approach is usually applied in complete financial markets because in such markets, it can be guaranteed that the representation problem is solvable. We do not explicitly assume here that the financial market is complete. Under the additional assumption of completeness, we could ensure that the representation problem has a solution. Of course, even without completeness it might be the case that the specific optimal terminal wealth obtained in Theorem 6.9 is attainable by some admissible strategy. However, we leave this problem open for future research. ◊

*Example* 6.11. Let us apply Theorem 6.9 to the two utility functions used in Example 6.4.

a) First, assume that all $n$ agents use the natural logarithm as their utility function. We do not have to make any further assumptions regarding the underlying financial market. The reason is that the inverse of the first order derivative of the natural logarithm is given by $I(x) = x^{-1}$, so that the expected value $\mathbb{E}[Z_T I(\lambda Z_T)] = \lambda^{-1}$ can be determined directly without specifying the distribution of $Z_T$. Thus, $H_i(y) = \frac{(n+\theta_i)(1-\hat{\theta})}{ny} = H_i^{-1}(y)$ and therefore,

$$\lambda_i^* = \frac{(n+\theta_i)(1-\hat{\theta})}{n(x_0^i - \theta_i \bar{x}_0^{-i})}, \quad i = 1, \dots, n.$$

Inserting $I_i(x) = x^{-1}$ and $\lambda_i^*$ into (6.21) yields

$$X_i^* = \frac{x_0^i}{Z_T}, \; i = 1, \dots, n, \text{ and } X^* = \frac{x_0}{Z_T}.$$

Hence, the optimal terminal wealth for the individual agents coincides with the optimal terminal wealth for a single agent (without interaction) using the natural logarithm as her utility function and the initial capital $x_0^i$, $i = 1, \dots, n$.

b) Now assume that all agents use exponential utility functions of the form $U_i(x) = -\exp(-\frac{1}{\delta_i}x)$ for risk tolerance parameters $\delta_i > 0$, $i = 1, \dots, n$. Moreover, in order to determine the Lagrange multipliers $\lambda^*$ and $\lambda_1^*, \dots, \lambda_n^*$ explicitly, we need to specify the underlying financial market. We consider a one-dimensional Black-Scholes market with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and interest rate $r = 0$. Hence, the discounted state price density is given by

$$Z_T = \exp\left(-\frac{\mu}{\sigma}W_T - \frac{\mu^2}{2\sigma^2}T\right),$$

where $W$ is a one-dimensional Brownian motion (see, e.g., Eberlein and Kallsen, 2019, Example 9.17). Theorem 6.9 implies

$$X_i^* = x_0^i + \left(\frac{n\delta_i}{n+\theta_i} + \frac{\theta_i \widehat{\delta\theta}}{(n+\theta_i)(1-\hat{\theta})}\right)\left(\frac{\mu}{\sigma}W_T + \frac{\mu^2}{\sigma^2}T\right)$$

and

$$X^* = x_0 + \frac{\widehat{\delta\theta}}{1-\hat{\theta}}\left(\frac{\mu}{\sigma}W_T + \frac{\mu^2}{\sigma^2}T\right),$$

where $\widehat{\delta\theta} = \sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}$. Note that we used the following auxiliary result to determine the Lagrange multipliers: If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X \cdot \exp(X)] = (\mu + \sigma^2)\exp(\mu + \sigma^2/2)$. The auxiliary result can be shown through a direct calculation (see Lemma A.1 in the Appendix).

$\Diamond$

## 6.4. Comparison with the Nash equilibrium from Section 3.2

In the previous chapters, we searched for Nash equilibria. In general, Nash equilibria are not Pareto optimal (see, e.g., Carmona, 2016, p. 169). In the underlying situation however, the Nash equilibrium determined in Chapter 3 coincides with the Pareto optimum given in Theorem 6.9. In the following, we justify this observation.

Theorem 6.9 displays the optimal terminal wealth of agent $i \in \{1, \dots, n\}$ in the Pareto optimum as

$$X_i^{*,P} = \frac{n}{n+\theta_i} I_i\left(\frac{n\lambda_i^* Z_T}{(n+\theta_i)(1-\hat{\theta})}\right) + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^n \frac{n}{n+\theta_j} I_j\left(\frac{n\lambda_j^* Z_T}{(n+\theta_j)(1-\hat{\theta})}\right),$$

where $\lambda_i^* = H_i^{-1}\left(x_0^i - \theta_i \bar{x}_0^{-i}\right)$. The function $H_i^{-1}$ was defined in Lemma 6.8 as the inverse of $H_i$, where

$$H_i(y) = \mathbb{E}\left[Z_T I_i\left(\frac{nyZ_T}{(n+\theta_i)(1-\hat{\theta})}\right)\right], \; y \in (0, \infty).$$

Now we compare $X_i^{*,P}$ to the optimal terminal wealth $X_i^{*,N}$ in the Nash equilibrium. Thus, Theorem 3.2 implies

$$X_i^{*,N} = \frac{n}{n+\theta_i} Y_i^* + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^n \frac{n}{n+\theta_j} Y_j^*,$$

where $Y_i^*$ is the unique optimal solution to

$$\begin{cases} & \max_{Y_i} \mathbb{E}\left[U_i(Y_i)\right], \\ \text{s.t.} & Y_i \text{ is } \mathcal{F}_T - \text{measurable}, \; \mathbb{E}\left[Z_T Y_i\right] \le \widetilde{x}_0^i, \end{cases}$$

for $\widetilde{x}_0^i = x_0^i - \frac{\theta_i}{n}\sum_{j \neq i} x_0^j$. Using standard arguments (see Kramkov and Schachermayer, 1999), $Y_i^*$ is given by

$$Y_i^* = I_i\left(G_i^{-1}(\widetilde{x}_0^i) Z_T\right),$$

where $G_i^{-1}$ is defined as the inverse of $G_i : (0, \infty) \to \mathbb{R}$ given by

$$G_i(y) = \mathbb{E}\left[Z_T I_i(yZ_T)\right] = H_i\left(y\left(1 + \frac{\theta_i}{n}\right)(1 - \hat{\theta})\right).$$

Thus,

$$X_i^{*,N} = \frac{n}{n + \theta_i} I_i\left(G_i^{-1}(\widetilde{x}_0^i) Z_T\right) + \frac{\theta_i}{(n + \theta_i)(1 - \hat{\theta})} \sum_{j=1}^{n} \frac{n}{n + \theta_j} I_j\left(G_j^{-1}(\widetilde{x}_0^j) Z_T\right).$$

Since[1]

$$G_i^{-1}(\cdot) = \frac{n H_i^{-1}(\cdot)}{(n + \theta_i)(1 - \hat{\theta})},$$

it follows that $X_i^{*,N} = X_i^{*,P}$. Thus, the optimal terminal wealth in the Nash equilibrium and the Pareto optimum are identical.

[1] For functions $f$ and $g$ with inverse functions $f^{-1}$ and $g^{-1}$, and a constant $\alpha \neq 0$ with $f(x) = g(\alpha x)$ for all $x$, it follows that $f^{-1} = \frac{g^{-1}}{\alpha}$

# Chapter 7

# Nash equilibria for relative investors under (non)linear price impact

In the classical mathematical finance literature, it is usually assumed that the underlying financial markets are perfectly elastic, meaning that asset prices are not affected by trades (see, among others, Merton, 1969, 1975; Black and Scholes, 1973). While this assumption is reasonable for small investors, there is plenty of empirical evidence for large traders having a significant impact on asset prices (see, for example, Bouchaud, 2009; Cronqvist and Fahlenbrach, 2009). Thus, there is a rapidly growing strand of mathematical literature regarding financial markets without perfect elasticity. Webster (2023) gives a historical overview of the literature on price impact dating back to the seminal work of Kyle (1985). According to Webster (2023, p. 5), the general economic idea behind price impact is that „trading of [a] stock cause[s] price moves for the stock that otherwise would not have happened". The literature lists several different explanations for the appearance of price impact. Jarrow (1992, 1994) explains that the reason a large trader (i.e., an investor that affects prices through her trades) is „large" might either be significant wealth or that other traders think she has private information. DeMarzo and Urošević (2006) argue that large investors are often institutions or companies, but it is also possible for a private individual to be a large trader. Bouchaud (2009) lists additional reasons for price impact and gives a short review on the different possibilities for mathematical market models to incorporate price impact. A survey by Gatheral and Schied (2013) gives a more detailed overview of the various price impact models covered by the literature. In the following, we explain some features of these models, but our introduction cannot claim to give an exhaustive overview of the diverse literature on price impact.

As stated by Schöneborn and Schied (2009), the literature on price impact follows two separate paths. The first is to derive models that describe empirical findings for price impact appropriately. The second line of research takes exogenously given models and solves classical mathematical finance problems such as absence of arbitrage, asset pricing, and optimal trading. As we aim to find optimal investment strategies for risk-averse agents in a competitive price impact model, our focus lies solely on the second line of research. For literature on the optimal model choice based on empirical data, we refer to Schöneborn and Schied (2009) and references therein. Moreover, Webster (2023) includes empirical data and compares it to different models from the mathematical literature.

It is important to understand that there are two different types of price impact – temporary and permanent price impact. Almgren and Chriss (2001) explain how temporary price impact is caused by a large order being placed, leading to a temporary imbalance in supply. Hence, the asset price deviates from equilibrium[1] for a short period of time and then returns. In contrast, permanent price impact describes a change in the equilibrium price which lasts over the whole time span covered by the model. There is a wide variety of articles displaying models which contain both temporary and permanent price impact (Almgren and Chriss, 2001; Schöneborn and Schied, 2009; Horst and Naujokat, 2010; Schied and Zhang, 2017, 2019; Schied et al., 2017; Luo and Schied, 2019, to name a few). The model used throughout this chapter includes only permanent price impact. Nevertheless, let us explain the difference between these two features and the resulting interpretations.

In general, temporary price impact is unfavorable for the large trader as she has to pay the price *after* it has been changed by her order. As Bank and Baum (2004) argue, the large trader „always has to trade on the bad side“. Hence, in such market models, questions of optimal execution of an order or optimal liquidation of a fixed position play an important role. Such problems have been considered by, among others, He and Mamaysky (2005), Schied et al. (2017), Schied and Zhang (2017, 2019), and Schöneborn and Schied (2009). Moreover, we should also mention the popular linear price impact model by Obizhaeva and Wang (2013), in which the authors find the optimal order execution strategy that minimizes execution cost. Finally, in the model of Bank and Dolinsky (2023), the large trader has additional information on the course of the future stock price. Specifically, they introduce price impact as a penalty for the large trader in order to exclude arbitrage strategies arising from the additional information available to the trader.

For permanent price impact, it is generally not possible to tell whether or not the price impact is beneficial for the large trader. In this case, it depends on a particular feature of the model. If the market is constructed such that prices change immediately and the large investor trades on the bad side of the price, permanent price impact admits the same interpretation as temporary price impact and can be considered a disadvantage for the large trader. This is often the case if the stock price is affected by the large trader's order directly through some reaction function (see, for example, Bank and Baum, 2004; Jarrow, 1992, 1994). In these models, so-called market manipulation strategies are often of interest. A market manipulation strategy is an arbitrage strategy with respect to real wealth instead of paper wealth, i.e., the amount of money attained after liquidating the current position (Jarrow, 1992, 1994). Identifying under what conditions such strategies do (not) exist is a central topic in the literature on price impact. If prices do not change immediately, the large trader could use her impact to her own benefit. In such models, finding criteria for the absence of arbitrage portfolios is of significant importance. Cvitanić and Ma (1996)

[1]A financial market is in equilibrium if market-clearing is realized. That is, if the supply and demand for assets in the market coincide (see, e.g., Yan, 2018, p. 47).

and Cuoco and Cvitanić (1998) defined a market model with indirect price impact, where the large trader's orders only affect the model parameters, i.e., the interest rate of the riskless asset, and the drift and volatility process of the risky stocks. A special case of their model was later used by Kraft and Kühn (2011). Eksi and Ku (2017) and Ku and Zhang (2018) described similar models where the large trader only affects the drift process of the risky asset. The model specified in Section 7.1 provides a special case of the aforementioned models. A different approach to define an indirect price impact model that reacts slowly to the large trader's orders was given by Busch et al. (2013). There, the large trader only affects the regime switching intensity of the risky asset's price process. Eksi and Ku (2017) argue that in models where price impact is beneficial for the large trader, it is counterintuitive to pose questions of optimal order execution. Thus, typical issues considered in such models are the pricing of asset claims as well as optimal investment and consumption over a fixed time interval.

Although this feature is not relevant to the analysis in this thesis, let us mention a third type of price impact, namely, transient price impact. Under transient price impact, the effect of a large order is strongest immediately after the order has been placed and vanishes over time. Common ways to model this feature are exponential decay as used by Schied et al. (2017), or a more general decay kernel as used by Luo and Schied (2019) as well as Schied and Zhang (2019).

Throughout this thesis, we are interested in problems of optimal portfolio choice for strategically interacting agents. Competition between agents can be seen as an effect of price impact if the agents are aware of their own as well as the others' impact on asset prices. Problems of strategically interacting large traders, often called *market impact games*, have been considered by Carlin et al. (2007), Horst and Naujokat (2010), Schöneborn and Schied (2009), Schied et al. (2017), and Schied and Zhang (2019). In each of these papers, a finite number of agents aims to minimize liquidation/execution cost in a competitive environment. Risk-averse investors competing to maximize expected utility of terminal wealth have, for instance, been considered by Curatola (2019, 2022) and Schied and Zhang (2017). In the aforementioned articles, the authors search for open-loop Nash equilibria for agents interacting strategically via price impact. In contrast, Micheli et al. (2021) find closed-loop Nash equilibria for strategic agents who cumulatively affect the price process of a risky stock. They also elaborate the difference between open-loop and closed-loop Nash equilibria (see also Section 2.4).

To the best of our knowledge, the existing literature only considers strategic interaction between competing investors caused by either relative concerns included into the objective function or by their joint impact on asset prices. In the current chapter, we combine these features in order to display both the effect of large investments on the stock price (caused by a large number of small investors) and the investors' desire to outperform their opponents. Thus, we find the unique solution to a multi-objective portfolio optimization problem, similar to the one introduced in Chapter 3, in a financial market where the $n$ agents' investment affects the stock price dynamics linearly. Our model provides a generalization of the financial market used by Kraft and Kühn (2011) by including multiple risky assets instead of just one. First, we determine the unique constant Nash equilibrium if investors use exponential utility functions. Afterwards, we study the influence of the parameter $\alpha$, which measures the sensitivity of the stock price to the agents' investment, on the Nash equilibrium and the stock price process. Furthermore, as empirical data suggests that price impact is *not* linear, we also consider the nonlinear case. We prove that, as long as the price impact grows sublinearly, the emerging best response problems are solvable. If the price impact grows superlinearly, there is no optimal solution to these best response problems. Finally, we switch from exponential to power utility and introduce a different type of objective function including the multiplicative relative performance metric (3.2). This criterion was previously used by Basak and Makarov (2015) and Lacker and Zariphopoulou (2019), among others. In that case, using the linear price impact model again, we are able to find the unique constant Nash equilibrium.

## 7.1. Introduction of the price impact market

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and $T > 0$ a finite time horizon. Further, let $W = (W_1, \dots, W_d)$ be a $d$-dimensional Brownian motion. The underlying financial market contains a riskless bond, which is for simplicity assumed to be identical to 1. Moreover, there are $d$ stocks. Our goal is to define their price processes as the solution to the following stochastic differential equation

$$\mathrm{d}S_k(t) = S_k(t)\left((\mu_k(t) + \alpha_k\bar{\pi}_k(t))\,\mathrm{d}t + \sum_{\ell=1}^{d}\sigma_{k\ell}(t)\mathrm{d}W_\ell(t)\right),\ S_k(0) = 1. \tag{7.1}$$

The expression $\bar{\pi}_k(t) = \frac{1}{n}\sum_{j=1}^{n}\pi_k^j(t)$ describes the arithmetic mean of the $n$ agents investment into the $k$-th stock at time $t \in [0, T]$.

Before we discuss the requirements we need to impose on the components of (7.1), let us discuss the structure of the stock price and its interpretations in general. First, we notice that the price impact of the $n$ agents is homogeneous in the sense that the weight $\frac{\alpha_k}{n}$ of the investment into the $k$-th stock is the same for each of the $n$ agents. This feature of our model is based on the idea that the $n$ agents are small if we consider them by themselves. However, as $n$ is supposed to be large, we can treat the group of $n$ agents like one large agent. Hence, it is reasonable to assume that the influence of each agent's investment on the stock price is assigned the same weight. Moreover, it makes the model mathematically more tractable. Second, we assume that the agents' investment only affects the drift and not the volatility of the stock price. Cuoco and Cvitanić (1998) give a detailed explanation of this assumption. To summarize, they base the assumption on two different factors. On one hand, they argue that market equilibrium conditions affect the stock price as a whole, and not the drift and volatility processes separately. Hence, one could fix the volatility matrix and only manipulate the drift in order to clear the market. On the other hand, they refer to empirical data in which the volatility is independent of the quantities invested into the asset. A third feature of our model is the affine linear structure of the drift process, which was also used by Kraft and Kühn (2011). They argue that, additionally to resulting in a more tractable model, the affine linear structure is a typical outcome of equilibrium models and hence, a reasonable assumption. Moreover, they explain that the affine linear structure can be interpreted as a first-order approximation of a more general price impact function.

Now we proceed with the conditions we need to impose on the processes contained in the stock price dynamics (7.1). The conditions are taken from Section 1.1 in Karatzas and Shreve (1998). The drift process $\mu = (\mu_1, \ldots, \mu_d)$ is an $(\mathcal{F}_t)$-progressively measurable, $d$-dimensional stochastic process satisfying

$$\mathbb{P}\left(\int_0^T \|\mu(t)\|\mathrm{d}t < \infty\right) = 1. \tag{7.2}$$

The volatility process $\sigma = (\sigma_{k\ell})_{1\leq k,\ell\leq d}$ is an $\mathbb{R}^{d\times d}$-valued stochastic process such that $\sigma_{k\ell}$ is $(\mathcal{F}_t)$-progressively measurable with

$$\mathbb{P}\left(\int_0^T \sigma_{k\ell}(t)^2\mathrm{d}t < \infty\right) = 1$$

for all $k, \ell \in \{1, \ldots, d\}$. Moreover, we assume that $\sigma(t)$ is regular $\mathbb{P}$-almost surely for all $t \in [0, T]$. The expression $\bar{\pi}_k(t)$ denotes the arithmetic mean of the investment of $n$ agents into the $k$-th stock at time $t \in [0, T]$, i.e.,

$$\bar{\pi}_k(t) = \frac{1}{n}\sum_{i=1}^{n} \pi_k^i(t),$$

where $\pi_k^i(t)$ describes either the amount or the fraction of wealth agent $i$ invests into the $k$-th stock at time $t \in [0, T]$.

Finally, $\alpha_k \in \mathbb{R}$, $k = 1, \ldots, d$, are constants that describe the impact of the $n$ agents' investment into the $k$-th stock. Hence, we refer to them as *price impact parameters*. Some authors argue that $\alpha_k$ should take both positive and negative values due to the fact that (large) investors may have both positive and negative impact on stock returns (see, e.g., Cronqvist and Fahlenbrach, 2009; Curatola, 2019). On the other hand, Bank and Baum (2004) prove that stock prices need to be increasing in terms of a large trader's investment. Otherwise, it would be possible to construct some „In & Out“ arbitrage strategy. However, such arbitrage strategies arise due to the direct change in the share price in their model and are therefore not an issue in our case. Moreover, since the optimization problems considered in the linear price impact market have finite optimal solutions, our market is free of arbitrage. Hence, we allow for general $\alpha_k \in \mathbb{R}$, $k = 1, \ldots, d$. The constants $\alpha_1, \ldots, \alpha_n$ are collected in the diagonal matrix

$$A \coloneqq \operatorname{diag}\left(\alpha_1, \ldots, \alpha_d\right),$$

which will be used later on.

Before we can use the stock price processes (7.1), we need to pose some assumptions on the investment processes $\pi^i$, $i = 1, \ldots, n$, of the $n$ investors. The assumptions are again based on the financial market in Section 1.1 of Karatzas and Shreve (1998). Since the drift in (7.1) needs to satisfy (7.2), we obtain the condition

$$\mathbb{P}\left(\int_0^T \|\mu(t) + A\bar{\pi}_t\|\mathrm{d}t < \infty\right) = 1.$$

However, using the triangle inequality, it suffices[2] to request that the strategies $\pi^i$ satisfy

$$\mathbb{P}\left(\int_0^T \|\pi^i(t)\|^2 \mathrm{d}t < \infty\right) = 1,\ i = 1, \dots, n,$$

since $\mu$ satisfies (7.2) and $A$ is a deterministic, constant diagonal matrix.

Regarding the investment strategies, Karatzas and Shreve (1998) request, additional to progressive measurability, the following conditions for a portfolio process $\pi$

$$\mathbb{P}\left(\int_0^T \left|\pi(t)^\top \mu(t)\right| \mathrm{d}t < \infty\right) = 1, \tag{7.3}$$

$$\mathbb{P}\left(\int_0^T \left\|\sigma(t)^\top \pi(t)\right\|^2 \mathrm{d}t < \infty\right) = 1. \tag{7.4}$$

The second assumption (7.4) is the same in our setting, whereas the first assumption (7.3) translates to

$$\mathbb{P}\left(\int_0^T \left|\pi^i(t)^\top \left(\mu(t) + A\bar{\pi}_t\right)\right| \mathrm{d}t < \infty\right) = 1.$$

Using the triangle and Cauchy-Schwarz inequality, we can request instead that

$$\mathbb{P}\left(\int_0^T \left|\mu(t)^\top \pi^i(t)\right| \mathrm{d}t < \infty\right) = 1,$$

$$\mathbb{P}\left(\int_0^T \left\|\pi^i(t)\right\|^2 \mathrm{d}t < \infty\right) = 1$$

hold for all $i \in \{1, \dots, n\}$. To summarize, the sufficient integrability assumptions for a strategy $\pi^i$ are

$$\mathbb{P}\left(\int_0^T \|\pi^i(t)\|^2 \mathrm{d}t < \infty\right) = 1, \tag{7.5}$$

$$\mathbb{P}\left(\int_0^T \left|\mu(t)^\top \pi^i(t)\right| \mathrm{d}t < \infty\right) = 1, \tag{7.6}$$

$$\mathbb{P}\left(\int_0^T \left\|\sigma(t)^\top \pi^i(t)\right\|^2 \mathrm{d}t < \infty\right) = 1. \tag{7.7}$$

The set $\mathcal{A}$ of admissible strategies is, thus, given by

$$\mathcal{A} := \Big\{\pi : \pi \text{ is an } \mathbb{R}^d\text{-valued, } (\mathcal{F}_t)_{t\in[0,T]}\text{-progressively measurable process that satisfies the integrability conditions (7.5)} - \text{(7.7)}\Big\}. \tag{7.8}$$

*Remark* 7.1. If $\mu$ and $\sigma$ are constant, the set $\mathcal{A}$ is simply given by the set of $\mathbb{R}^d$-valued, progressively measurable processes $\pi$ such that $\mathbb{P}\left(\int_0^T \|\pi(t)\|^2 \mathrm{d}t < \infty\right) = 1$. ◊

[2]Note that it would be sufficient to assume that $\mathbb{P}(\int_0^T \|\pi^i(t)\| \mathrm{d}t < \infty) = 1$, but since we need the stronger quadratic assumption later, we already pose it at this point.

For admissible strategies $\pi^j$, $j = 1, \dots, n$, we denote the wealth process of investor $i \in \{1, \dots, n\}$ by $(X_t^{i,\pi^i})_{t\in[0,T]}$. It should be noted that $X^{i,\pi^i}$ depends on all $n$ strategies $\pi^1, \dots, \pi^n$. The expression $\pi_k^j(t)$ will be interpreted as either the amount or the fraction of wealth invested by investor $j$ into stock $k$ at time $t \in [0, T]$. This depends on the particular example since, for CARA utilities, it is more convenient to optimize the invested amount, while the invested fraction works better in combination with CRRA utility functions.

Now let $x_0^i$ denote the initial capital of agent $i \in \{1, \dots, n\}$. If $\pi_k^i(t)$ describes the amount investor $i$ invests into stock $k$ at time $t$, the associated wealth process is given by

$$\begin{aligned} X_t^{i,\pi^i} &= x_0^i + \sum_{k=1}^d \int_0^t \pi_k^i(s) \left( (\mu_k(s) + \alpha_k \bar{\pi}_k(s))\, \mathrm{d}s + \sum_{\ell=1}^d \sigma_{k\ell}(s) \mathrm{d}W_\ell(s) \right) \\ &= x_0^i + \int_0^t \pi^i(s)^\top \left( (\mu(s) + A\bar{\pi}(s)) \mathrm{d}s + \sigma \mathrm{d}W(s) \right), \ t \in [0, T]. \end{aligned}$$

In contrast, if $\pi_k^i(t)$ describes the fraction of investor $i$'s wealth invested into stock $k$ at time $t$, the wealth process is given by

$$\begin{aligned} X_t^{i,\pi^i} &= x_0^i \exp\Bigg( \int_0^t \sum_{k=1}^d \pi_k^i(s) \Big( \mu_k(s) + \alpha_k \bar{\pi}_k(s) - \frac{1}{2} \sum_{\ell,p=1}^d \pi_p^i(s) \sigma_{k\ell}(s) \sigma_{p\ell}(s) \Big) \mathrm{d}s \\ &\qquad + \int_0^t \sum_{k,\ell=1}^d \pi_k^i(s) \sigma_{k\ell}(s) \mathrm{d}W_\ell(s) \Bigg) \\ &= x_0^i \exp\left( \int_0^t \pi^i(s)^\top \left( \mu(s) + A\bar{\pi}(s) - \frac{1}{2} \sigma(s)\sigma(s)^\top \pi^i(s) \right) \mathrm{d}s + \int_0^t \pi^i(s)^\top \sigma(s) \mathrm{d}W(s) \right), \ t \in [0, T]. \end{aligned}$$

## 7.2. Relative performance problem under general utility

As explained at the beginning of this chapter, our goal is to solve a problem similar to (3.3) in the previously described price impact market. In the following, we describe a method to find Nash equilibria in terms of invested amounts for objective functions including the additive relative performance metric. Hence, our goal is to find all Nash equilibria to the multi-objective optimization problem

$$\begin{cases} & \sup_{\pi^i \in \mathcal{A}} \mathbb{E}\left[ U_i\left( X_T^{i,\pi^i} - \frac{\theta_i}{n} \sum_{j \neq i} X_T^{j,\pi^j} \right) \right], \\ \text{s.t.} & X_T^{i,\pi^i} = x_0^i + \displaystyle\int_0^T \pi^i(t)^\top \left( (\mu(t) + A\bar{\pi}(t))\, \mathrm{d}t + \sigma(t) \mathrm{d}W(t) \right), \end{cases} \tag{7.9}$$

$i = 1, \dots, n$, where $U_i : \mathcal{D}_i \to \mathbb{R}$ describes a general utility function defined on some domain $\mathcal{D}_i \in \{(0, \infty), \mathbb{R}\}$ (see Definition 2.11). Since the difference in the argument of $U_i$ may become negative, we extend $U_i$ to the whole real line by setting $U_i(x) = -\infty$ if $x \notin \mathcal{D}_i$. Obviously, this does not change the optimal value in (7.9). Nevertheless, we assume that $x_0^i - \frac{\theta_i}{n} \sum_{j \neq i} x_0^j \in \mathcal{D}_i$ for all $i \in \{1, \dots, n\}$ (see also Remark 3.1).

In order to solve (7.9), we choose some investor $i \in \{1, \dots, n\}$ and assume that the strategies $\pi^j$, $j \neq i$, of the other agents are given. Under these conditions, we can rewrite the optimization problem (7.9) as a classical portfolio optimization problem in a different price impact market.

Afterwards, Nash equilibria can be determined using the solution to the classical problem. First, define the process $(Y_t^{i,\varphi^i})_{t\in[0,T]}$ by

$$Y_t^{i,\varphi^i} = X_t^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_t^{j,\pi^j},\ t\in[0,T],\ i=1,\dots,n, \tag{7.10}$$

where we further defined the strategy $\varphi^i$ by

$$\varphi_k^i(t) = \pi_k^i(t) - \frac{\theta_i}{n}\sum_{j\neq i}\pi_k^j(t),\ t\in[0,T],\ k=1,\dots,d,\ i=1,\dots,n.$$

Note that $\varphi^i$ is admissible since the conditions imposed in the definition of $\mathcal{A}$ are preserved under linear combinations of elements in $\mathcal{A}$. Then we can write $Y_T^{i,\varphi^i}$ as

$$\begin{aligned}
Y_T^{i,\varphi^i} &= X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_T^{j,\pi^j}\\
&= \underbrace{x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j}_{=\widetilde{x}_0^i} + \sum_{k=1}^d\int_0^T \Bigg(\underbrace{\pi_k^i(t) - \frac{\theta_i}{n}\sum_{j\neq i}\pi_k^j(t)}_{=\varphi_k^i(t)}\Bigg)\Bigg(\left(\mu_k(t)+\alpha_k\bar{\pi}_k(t)\right)\mathrm{d}t + \sum_{\ell=1}^d \sigma_{k\ell}(t)\mathrm{d}W_\ell(t)\Bigg)\\
&=: \widetilde{x}_0^i + \sum_{k=1}^d\int_0^T \varphi_k^i(t)\left(\left(\mu_k(t) + \frac{\alpha_k}{n}\varphi_k^i(t) + \alpha_k\frac{n+\theta_i}{n}\bar{\pi}_k^{-i}(t)\right)\mathrm{d}t + \sum_{\ell=1}^d \sigma_{k\ell}(t)\mathrm{d}W_\ell(t)\right)\\
&=: \widetilde{x}_0^i + \sum_{k=1}^d\int_0^T \varphi_k^i(t)\left(\left(\widetilde{\mu}_k^{-i}(t) + \frac{\alpha_k}{n}\varphi_k^i(t)\right)\mathrm{d}t + \sum_{\ell=1}^d \sigma_{k\ell}(t)\mathrm{d}W_\ell(t)\right),
\end{aligned}$$

where we introduced the following notation

$$\bar{\pi}_k^{-i}(t) := \frac{1}{n}\sum_{j\neq i}\pi_k^j(t),\quad \widetilde{\mu}_k^{-i}(t) := \mu_k(t) + \alpha_k\frac{n+\theta_i}{n}\bar{\pi}_k^{-i}(t).$$

Hence, in order to solve the best response problem associated to (7.9), we can solve the single investor portfolio optimization problem

$$\begin{cases} & \sup_{\varphi^i\in\mathcal{A}}\mathbb{E}\left[U_i\left(Y_T^{i,\varphi^i}\right)\right],\\ \text{s.t.} & Y_T^{i,\varphi^i} = \widetilde{x}_0^i + \int_0^T \varphi^i(t)^\top\left((\widetilde{\mu}^{-i}(t) + \frac{1}{n}A\varphi^i(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\right),\end{cases} \tag{7.11}$$

in a different price impact market. Now assume that $\varphi^{i,*} = \varphi^{i,*}(\widetilde{\mu}^{-i})$ is the unique optimal solution to (7.11) depending on the drift process $\widetilde{\mu}^{-i}$. Then the optimal solution to the best response problem with respect to (7.9) is uniquely determined by

$$\pi_k^i = \varphi_k^{i,*}(\widetilde{\mu}^{-i}) + \frac{\theta_i}{n}\sum_{j\neq i}\pi_k^j,\ k=1,\dots,d,\ i=1,\dots,n. \tag{7.12}$$

Note that we can find a unique Nash equilibrium if, and only if, problem (7.11) and the system of equations (7.12) are uniquely solvable.

## 7.3. Optimization under exponential utility

Given the situation from Sections 7.1 and 7.2, suppose that the $n$ investors use exponential utility functions of the form

$$U_i : \mathbb{R} \to \mathbb{R},\ x \mapsto -\exp\left(-\delta_i^{-1} x\right)$$

for parameters $\delta_i > 0$, $i = 1, \dots, n$. Moreover, let the market parameters $\sigma$ and $\mu$ be constant.

In order to solve the auxiliary problem (7.11), we need to restrict ourselves to constant Nash equilibria. Hence, for the best response problem of agent $i$, we assume that the strategies of the other agents $j \neq i$ are constant while we still need to allow agent $i$ to consider any admissible strategy in $\mathcal{A}$ (see Remark 2.14). If the optimal strategy of agent $i$ turns out to be constant as well, we found a constant Nash equilibrium.

Lacker and Zariphopoulou (2019) give a justification of the restriction to constant strategies. They argue that the assumption is natural in the case of lognormal stock prices in combination with CARA or CRRA utilities. Indeed, Merton (1969) considered lognormal stock prices and obtained constant expressions for the optimally invested amount and fraction in the cases of CARA and CRRA utility, respectively. Thus, constant strategies are somehow expected and the restriction to such strategies is not too severe.

### 7.3.1. Unique constant Nash equilibrium

Let us now search for constant Nash equilibria for (7.9). Each agent $i$ aims to solve an optimization problem of the form

$$\begin{cases} & \sup_{\pi^i \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left(X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_T^{j,\pi^j}\right)\right)\right], \\ \text{s.t.} & X_T^{i,\pi^i} = x_0^i + \int_0^T \pi^i(t)^\top \left((\mu + A\bar{\pi}_t)\,\mathrm{d}t + \sigma \mathrm{d}W(t)\right), \end{cases} \tag{7.13}$$

where $\delta_i > 0$ and $\theta_i \in [0,1]$ denote the risk tolerance parameter and competition weight of agent $i \in \{1, \dots, n\}$, respectively. Then the following theorem displays the unique constant Nash equilibrium for (7.13).

**Theorem 7.2.** *Assume that the following assumptions hold*

*a)* $\sigma\sigma^\top - \frac{2\delta_i}{n} A$ *is positive-definite for all* $i \in \{1, \dots, n\}$,

*b)* $(1-\hat{\theta})I_d - \sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1} A$ *is regular,*

*where* $\hat{\theta} = \sum_{i=1}^n \frac{\theta_i}{n+\theta_i} < 1$ *(see Lemma 3.3). Then the unique constant Nash equilibrium to* (7.13) *is given by*

$$\begin{aligned} \pi^{i,*} =& \frac{n\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}\mu + \left(\frac{\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}A + \frac{\theta_i}{n+\theta_i}I_d\right) \\ & \cdot \left((1-\hat{\theta})I_d - \sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}A\right)^{-1} \sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}\mu \end{aligned} \tag{7.14}$$

*for* $i \in \{1, \dots, n\}$.

*Remark* 7.3. The first condition of the theorem implies that the matrix $\sigma\sigma^\top - \frac{\delta_i}{n}A$ is also positive-definite since

$$\frac{n}{\delta_i}\sigma\sigma^\top - A >_L \frac{n}{2\delta_i}\sigma\sigma^\top - A >_L 0,$$

where $>_L$ describes the Loewner order for Hermitian matrices (see, for example, Horn and Johnson, 2013, Section 7.7). The first inequality is valid since $\sigma$ was assumed to be regular and hence, $\sigma\sigma^\top$ is positive-definite. The second inequality is equivalent to condition a) from Theorem 7.2. ◊

The following example provides a class of matrices that satisfy the conditions of Theorem 7.2. Roughly speaking, the conditions of Theorem 7.2 state that the price impact parameters $\alpha_k$ cannot be too large. Similar requirements were, for example, used by Kraft and Kühn (2011).

*Example* 7.4. In the previous theorem, assume that $A = \alpha \cdot I_d$ for some constant $\alpha \in \mathbb{R}$. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $\sigma\sigma^\top$ which are strictly positive since $\sigma\sigma^\top$ is positive-definite. Then the eigenvalues of the matrix in condition a) of Theorem 7.2 are given by $\lambda_\ell - \frac{2\delta_i}{n}\alpha$, $\ell = 1, \ldots, d$ (Bernstein, 2009, Proposition 4.4.5). Hence, condition a) is satisfied if, and only if,

$$\alpha < \frac{n\lambda_{\min}}{2\delta_{\max}}, \tag{7.15}$$

where $\lambda_{\min} \coloneqq \min\{\lambda_1, \ldots, \lambda_d\}$ and $\delta_{\max} \coloneqq \max\{\delta_1, \ldots, \delta_n\}$. If $n$ is sufficiently large, (7.15) is satisfied and the matrix in a) is positive-definite.

Now we need to consider condition b) of Theorem 7.2. If $\alpha$ is chosen with respect to (7.15), the matrix

$$\sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}$$

is positive-definite (see Remark 7.3). Hence, its eigenvalues $\mu_1, \ldots, \mu_d$ are strictly positive. Moreover, using $A = \alpha \cdot I_d$ and Proposition 4.4.5 in Bernstein (2009), the eigenvalues of the matrix in b) are given by

$$-\alpha\mu_\ell + 1 - \hat{\theta},\ \ell = 1, \ldots, d.$$

We can now ensure that the conditions of the theorem are satisfied if we choose $\alpha$ sufficiently small (with respect to (7.15)) and so that $\alpha \neq \frac{1-\hat{\theta}}{\mu_\ell}$ for all $\ell = 1, \ldots, d$. ◊

Now we proceed with the proof of Theorem 7.2 using the auxiliary problem (7.11).

*Proof (Theorem 7.2).* We solve the auxiliary problem (7.11) for exponential utility to find a constant Nash equilibrium for the multi-objective optimization problem (7.13). Therefore, let $i \in \{1, \ldots, n\}$ be arbitrary but fixed and assume that the strategies $\pi^j$ of the investors $j \neq i$ are fixed and constant. Then we determine the optimal strategy for investor $i$, which turns out to be constant as well. Afterwards, we can solve a system of linear equations to determine a Nash equilibrium for the original problem.

Since the strategies $\pi^j$, $j \neq i$, are constant, the drift $\widetilde{\mu}^{-i}$ is also constant. The process $Y^{i,\varphi^i}$ from (7.10) is therefore given by

$$Y_t^{i,\varphi^i} = \widetilde{x}_0^i + \int_0^t \varphi^i(s)^\top \left(\left(\widetilde{\mu}^{-i} + \frac{1}{n}A\varphi^i(s)\right)\mathrm{d}s + \sigma \mathrm{d}W(s)\right),\ t \in [0,T]. \tag{7.16}$$

In order to derive an HJB equation, we introduce the value functions

$$J(t,y;\varphi^i) := \mathbb{E}^{t,y}\left[-\exp\left(-\frac{1}{\delta_i}Y_T^{i,\varphi^i}\right)\right], \quad J(t,y) := \sup_{\varphi^i\in\mathcal{A}} J(t,y;\varphi^i),$$

$t \in [0,T]$, $y \in \mathbb{R}$. By $\mathbb{E}^{t,y}$ we denote the conditional expectation given that $Y_t^{i,\varphi^i} = y$. Since the parameters of the adapted price impact market are constant, we can use the *Bellman principle* (see, for example, Equation (3.20) in Pham, 2009)

$$J(t,y) = \sup_{\varphi^i\in\mathcal{A}} \mathbb{E}^{t,y}\left[J\left(t', Y_{t'}^{i,\varphi^i}\right)\right], \tag{7.17}$$

that holds for each $t \in [0,T]$ and $t' \in [t,T]$.

Now let $t \le t' \le T$ and let $J_t$, $J_y$, $J_{yy}$ denote the first and second order partial derivatives of $J$ in terms of $t$ and $y$. We assume here that $J \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ for the heuristic derivation of the HJB equation. Using the Itô-Doeblin formula (Theorem 2.1) on the interval $[t,t']$ in combination with Lemma 2.2 implies

$$\begin{aligned}
&J\left(t', Y_{t'}^{i,\varphi^i}\right)\\
&= J\left(t, Y_t^{i,\varphi^i}\right) + \int_t^{t'} J_t\left(s, Y_s^{i,\varphi^i}\right)\mathrm{d}s + \int_t^{t'} J_y\left(s, Y_s^{i,\varphi^i}\right)\mathrm{d}Y_s^{i,\varphi^i} + \frac{1}{2}\int_t^{t'} J_{yy}\left(s, Y_s^{i,\varphi^i}\right)\mathrm{d}\langle Y^{i,\varphi^i}\rangle_s\\
&= J\left(t, Y_t^{i,\varphi^i}\right) + \int_t^{t'} J_t\left(s, Y_s^{i,\varphi^i}\right)\mathrm{d}s + \int_t^{t'} J_y\left(s, Y_s^{i,\varphi^i}\right)\varphi^i(s)^\top\left(\widetilde{\mu}^{-i} + \frac{1}{n}A\varphi^i(s)\right)\mathrm{d}s\\
&\quad + \int_t^{t'} J_y\left(s, Y_s^{i,\varphi^i}\right)\varphi^i(s)^\top\sigma\mathrm{d}W(s) + \frac{1}{2}\int_t^{t'} J_{yy}\left(s, Y_s^{i,\varphi^i}\right)\varphi^i(s)^\top\sigma\sigma^\top\varphi^i(s)\,\mathrm{d}s.
\end{aligned} \tag{7.18}$$

Now we can apply the Bellman principle (7.17), divide by $t'-t$, and take the limit $t' \to t$ to obtain the following HJB equation

$$0 = G_t + \sup_{\varphi^i\in\mathbb{R}^d}\left\{G_y\cdot(\varphi^i)^\top\left(\widetilde{\mu}^{-i} + \frac{1}{n}A\varphi^i\right) + \frac{1}{2}G_{yy}\cdot(\varphi^i)^\top\sigma\sigma^\top\varphi^i\right\}, \tag{7.19}$$

with terminal condition $G(T,y) = -\exp\left(-\frac{1}{\delta_i}y\right)$, $y \in \mathbb{R}$. We omitted the arguments of the partial derivatives of $G$ to simplify notation. In the previous step, we assumed that the stochastic integral in (7.18) is a martingale and hence, the conditional expectation vanishes. Further, we assumed that the limit $t' \to t$ and the conditional expectation can be interchanged. These assumptions are no limitation since we only use them in the heuristic derivation of the HJB equation. Lemma B.1 in the Appendix verifies that a solution to the HJB equation does, in fact, provide a unique optimal portfolio strategy for (7.11). In the proof of Lemma B.1, it is also shown that the stochastic integral in (7.18) is a martingale.

Let $h(\varphi^i)$ describe the argument inside the supremum, i.e.,

$$h(\varphi^i) = G_y\cdot(\varphi^i)^\top\widetilde{\mu}^{-i} + (\varphi^i)^\top\left(\frac{1}{n}G_yA + \frac{1}{2}G_{yy}\sigma\sigma^\top\right)\varphi^i.$$

To find the maximizer of $h$, we determine the gradient. Thus,

$$\nabla h(\varphi^i) = G_y \widetilde{\mu}^{-i} + 2\left(\frac{1}{n} G_y A + \frac{1}{2} G_{yy} \sigma\sigma^\top\right) \varphi^i \stackrel{!}{=} 0.$$

Solving the previous equation for $\varphi^i$ would yield a possible maximizer of $h$. However, since we cannot ensure that the matrix $\frac{1}{n} G_y A + \frac{1}{2} G_{yy} \sigma\sigma^\top$ is regular, we first insert an ansatz for $G$ and then solve for $\varphi^i$ using the assumptions of the theorem. Let

$$G(t, y) = f(t) \widetilde{U}(y) := f(t) \cdot \left(-\exp\left(-\frac{1}{\delta_i} y\right)\right),\ t \in [0, T],\ y \in \mathbb{R},$$

for some continuously differentiable function $f : [0, T] \to (0, \infty)$ with $f(T) = 1$. Then $G$ is sufficiently differentiable, i.e., $G \in \mathcal{C}^{1,2}([0, T], \mathbb{R})$, and the partial derivatives of $G$ with respect to $t$ and $y$ are given by

$$G_t(t, y) = f'(t) \widetilde{U}(y), \quad G_y(t, y) = -\frac{1}{\delta_i} \cdot f(t) \widetilde{U}(y), \quad G_{yy}(t, y) = \frac{1}{\delta_i^2} \cdot f(t) \widetilde{U}(y).$$

Thus, $\nabla h$ simplifies to

$$\nabla h(\varphi^i) = \frac{f(t) \widetilde{U}(y)}{\delta_i^2} \left(-\delta_i \widetilde{\mu}^{-i} + \left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right) \varphi^i\right) \stackrel{!}{=} 0,$$

which we can equivalently rewrite as

$$\left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right) \varphi^{i,*} := \left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right) \varphi^i = \delta_i \widetilde{\mu}^{-i}, \tag{7.20}$$

since $f(t) \widetilde{U}(y) \neq 0$ for all $(t, y) \in [0, T] \times \mathbb{R}$. Moreover, the Hessian matrix of $h$ is given by

$$H_h(\varphi^i) = \frac{f(t) \widetilde{U}(y)}{\delta_i^2} \left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right),$$

which is negative-definite since $f > 0$, $\widetilde{U} < 0$, and $\sigma\sigma^\top - \frac{2\delta_i}{n} A$ is positive-definite by assumption. Hence, $\varphi^{i,*}$ from (7.20) is the unique maximizer of $h$. Further, we can solve (7.20) for $\varphi^i$ since $\sigma\sigma^\top - \frac{2\delta_i}{n} A$ is regular by assumption.

It remains to show that the ansatz for $G$ solves (7.19). Inserting the ansatz into the definition of $h$ implies (where we omitted the arguments of $f$ and $\widetilde{U}$ to simplify notation)

$$h(\varphi) = \frac{f \widetilde{U}}{\delta_i^2} \left(\frac{1}{2} \varphi^\top \left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right) \varphi - \delta_i \varphi^\top \widetilde{\mu}^{-i}\right).$$

Therefore, $h(\varphi^{i,*})$ is given by

$$h(\varphi^{i,*}) = -\frac{1}{2} f \widetilde{U} \cdot \left(\widetilde{\mu}^{-i}\right)^\top \left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right)^{-1} \widetilde{\mu}^{-i},$$

where we used that $\sigma\sigma^\top - \frac{2\delta_i}{n} A$ is symmetric. Now we can finally insert the ansatz for $G$ into the

HJB equation (7.19), which yields

$$\begin{aligned} 0 &= f'(t)\widetilde{U}(y) - \frac{1}{2} f(t)\widetilde{U}(y)(\widetilde{\mu}^{-i})^\top \left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right)^{-1} \widetilde{\mu}^{-i} \\ &=: \widetilde{U}(y)\left(f'(t) - \rho f(t)\right) \end{aligned} \tag{7.21}$$

with terminal condition $f(T) = 1$, where $\rho = \frac{1}{2}\left(\widetilde{\mu}^{-i}\right)^\top \left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right)^{-1} \widetilde{\mu}^{-i}$. Since $\widetilde{U}(y) \neq 0$ for all $y \in \mathbb{R}$, $f$ needs to solve

$$f'(t) = \rho f(t),\ t \in [0,T], \quad f(T) = 1.$$

The unique solution to this ordinary differential equation is given by

$$f(t) = \mathrm{e}^{-\rho(T-t)},\ t \in [0,T].$$

Hence, the function $G : [0,T] \times \mathbb{R} \to \mathbb{R}$ with $G(t,y) = -\exp\left(-\frac{1}{\delta_i}y - \rho(T-t)\right)$ solves (7.19) and

$$\varphi^{i,*} = \delta_i \left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right)^{-1} \widetilde{\mu}^{-i}$$

is a candidate for the optimal solution to the auxiliary problem. Lemma B.1 in the Appendix verifies that $\varphi^{i,*}$ is, in fact, an optimal solution to the auxiliary problem. Moreover, Lemma B.1 also states that the above solution $G$ and the value function $J$ are equal, which also implies that the solution $\varphi^{i,*}$ is unique (up to modifications).

Now we can reinsert the definition of $\widetilde{\mu}^{-i}$ and $\varphi^{i,*}$ to determine the Nash equilibrium. Recall that

$$\widetilde{\mu}^{-i} = \mu + \frac{n+\theta_i}{n^2} A \sum_{j\neq i} \pi^j,\ \varphi^i = \pi^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j.$$

Therefore, we obtain the system of linear equations

$$\left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right) \cdot \left(\pi^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\right) = \delta_i \widetilde{\mu}^{-i} = \delta_i\mu + \frac{n+\theta_i}{n} \cdot \frac{\delta_i}{n} A \sum_{j\neq i}\pi^j,$$

which can be equivalently rewritten as

$$\left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right)\pi^i = \delta_i\mu + \left(\frac{\delta_i}{n}A + \frac{\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)\right)\sum_{j\neq i}\pi^j. \tag{7.22}$$

In order to solve (7.22), we add $\left(\frac{\delta_i}{n}A + \frac{\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)\right)\pi^i$ on both sides to obtain

$$\frac{n+\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)\pi^i = \delta_i\mu + \left(\frac{\delta_i}{n}A + \frac{\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)\right)\sum_{j=1}^n \pi^j.$$

Since the matrix $\sigma\sigma^\top - \frac{\delta_i}{n}A$ is regular by assumption and the arguments in Remark 7.3, it follows that

$$\begin{aligned}\pi^i &= \frac{n\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}\mu + \frac{n}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}\left(\frac{\delta_i}{n}A + \frac{\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)\right)\sum_{j=1}^n \pi^j \\ &= \frac{n\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}\mu + \left(\frac{\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}A + \frac{\theta_i}{n+\theta_i}I_d\right)\sum_{j=1}^n \pi^j. \qquad (7.23)\end{aligned}$$

Taking the sum over all $i \in \{1, \dots, n\}$ on both sides yields

$$\sum_{j=1}^n \pi^j = \sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}\mu + \left(\sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}A + \hat{\theta}\cdot I_d\right)\sum_{j=1}^n \pi^j.$$

This can equivalently be rewritten as

$$\left((1-\hat{\theta})I_d - \sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}A\right)\sum_{j=1}^n \pi^j = \sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}\mu.$$

The matrix on the left-hand side is regular by assumption, so we can further deduce

$$\sum_{j=1}^n \pi^j = \left((1-\hat{\theta})I_d - \sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}A\right)^{-1}\sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}\mu.$$

Finally, we insert $\sum_{j=1}^n \pi^j$ into (7.23) to obtain the asserted constant Nash equilibrium given by

$$\begin{aligned}\pi^{i,*} &= \frac{n\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}\mu + \left(\frac{\delta_i}{n+\theta_i}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right)^{-1}A + \frac{\theta_i}{n+\theta_i}I_d\right) \\ &\quad \cdot \left((1-\hat{\theta})I_d - \sum_{j=1}^n \frac{\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}A\right)^{-1}\sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right)^{-1}\mu. \qquad \square\end{aligned}$$

*Remark* 7.5. a) In the previous proof, we used condition a) of Theorem 7.2 to ensure that the local extremum of the function $h$ is, in fact, a maximum. Roughly speaking, condition a) prevents the price impact of the agents from becoming too large. Otherwise, they might be able to exploit their price impact by investing an infinite amount into the stock and gaining an infinite amount of utility. A similar assumption has also been used by Kraft and Kühn (2011). They explain that without the assumption that the price impact parameter is sufficiently small, „the demand for stocks is infinite“.

b) If assumption b) in Theorem 7.2 is not satisfied, there is no constant Nash equilibrium (in most cases) since, in general, the system of linear equations (7.22) is not solvable without condition b). ◊

*Remark* 7.6. Inserting $A = 0$ into the Nash equilibrium (7.14) yields

$$\pi^{i,*} = \left(\frac{n\delta_i}{n+\theta_i} + \frac{\theta_i}{(n+\theta_i)(1-\hat{\theta})}\sum_{j=1}^n \frac{n\delta_j}{n+\theta_j}\right)\left(\sigma\sigma^\top\right)^{-1}\mu,$$

which is exactly the solution to the problem without price impact (see Subsection 4.2.1). This does not come as a surprise since the matrix $A$ contains the coefficients $\alpha_k$, which model the impact of the arithmetic mean of the $n$ agents' investment on the price process of the $k$-th stock, $k = 1, \dots, d$. Moreover, if we let $A$ be arbitrary and set $\theta_i = 0$, we can deduce the optimal solution to the optimization problem without relative concerns. In this case, interaction is caused solely by the cumulative price impact. Then

$$\begin{aligned}\pi^{i,*} =& \delta_i\Big(\sigma\sigma^\top - \frac{\delta_i}{n}A\Big)^{-1}\mu + \frac{\delta_i}{n}\Big(\sigma\sigma^\top - \frac{\delta_i}{n}A\Big)^{-1}A \\ &\cdot \left(I_d - \sum_{j=1}^n \frac{\delta_j}{n}\Big(\sigma\sigma^\top - \frac{\delta_j}{n}A\Big)^{-1}A\right)^{-1} \sum_{j=1}^n \delta_j\Big(\sigma\sigma^\top - \frac{\delta_j}{n}A\Big)^{-1}\mu\end{aligned}$$

describes the unique Nash equilibrium. ◊

We conclude this subsection with a version of Theorem 7.2 for a market with only one stock.

**Corollary 7.7.** *Let $d = 1$, $A = \alpha$, and assume that*

$$n\sigma^2 - 2\alpha\delta_j > 0,\ j = 1, \dots, n, \quad \sum_{j=1}^n \frac{n\alpha\delta_j}{(n+\theta_j)(n\sigma^2 - \delta_j\alpha)} \neq 1 - \hat{\theta}.$$

*Then the unique constant Nash equilibrium for* (7.13) *is given by*

$$\pi^{i,*} = \frac{n^2\delta_i\mu}{(n+\theta_i)\,(n\sigma^2 - \delta_i\alpha)} + \left(\frac{\theta_i}{n+\theta_i} + \frac{n\alpha\delta_i}{(n+\theta_i)\,(n\sigma^2 - \delta_i\alpha)}\right) \cdot \frac{\sum_{j=1}^n \frac{n^2\delta_j}{(n+\theta_j)(n\sigma^2-\delta_j\alpha)} \cdot \mu}{1 - \hat{\theta} - \sum_{j=1}^n \frac{n\alpha\delta_j}{(n+\theta_j)(n\sigma^2-\delta_j\alpha)}}$$

*for $i \in \{1, \dots, n\}$.*

### 7.3.2. Influence of the price impact parameter

The goal of this subsection is to study the influence of the price impact parameter $\alpha$ on the components of the constant Nash equilibrium and on the stock price. To do this, assume that there is only one stock ($d = 1$) with constant and deterministic parameters $\mu_1 \equiv \mu > 0$, $\sigma_{11} \equiv \sigma > 0$, and $\alpha_1 = \alpha$, driven by a one-dimensional Brownian motion.

The first step is to take a closer look at the assumptions of Corollary 7.7. The first assumption can be written as

$$\alpha < \alpha_{\max} := \frac{n\sigma^2}{2\delta_{\max}},$$

where $\delta_{\max} := \max\{\delta_1, \dots, \delta_n\}$. For the second assumption, we define $s : (-\infty, \alpha_{\max}] \to \mathbb{R}$ by

$$s(\alpha) := \sum_{j=1}^n \frac{n\alpha\delta_j}{(n+\theta_j)\,(n\sigma^2 - \alpha\delta_j)}. \tag{7.24}$$

Then we can deduce that there exists a unique value $\alpha_0$ such that $s(\alpha_0) = 1 - \hat{\theta}$. Moreover, we have $\alpha_0 \in (0, \alpha_{\max})$. To verify this observation, we introduce functions $f_j : (-\infty, \alpha_{\max}] \to \mathbb{R}$,

$j = 1, \dots, n$, given by

$$f_j(\alpha) := \frac{n\delta_j}{(n+\theta_j)(n\sigma^2 - \alpha\delta_j)}.$$

Note that, for any $j \in \{1, \dots, n\}$, $f_j$ is strictly positive and continuously differentiable in $\alpha$ for $\alpha \in (-\infty, \alpha_{\max}]$. Now we can display the components of the Nash equilibrium from Corollary 7.7 in terms of the functions $f_1, \dots, f_n$ as

$$\pi^{i,*} = \left( n f_i(\alpha) + \left( \frac{\theta_i}{n+\theta_i} + \alpha f_i(\alpha) \right) \cdot \frac{\sum_{j=1}^n n f_j(\alpha)}{1 - \hat{\theta} - s(\alpha)} \right) \cdot \mu, \tag{7.25}$$

and the function $s$ from (7.24) can be written as

$$s(\alpha) = \alpha \sum_{j=1}^n f_j(\alpha), \quad \alpha \in (-\infty, \alpha_{\max}].$$

The functions $f_j$ are strictly increasing in terms of $\alpha$ since

$$\frac{\partial}{\partial \alpha} f_j(\alpha) = f_j'(\alpha) = \frac{n\delta_j^2}{(n+\theta_j)(n\sigma^2 - \alpha\delta_j)^2} > 0.$$

Moreover, the functions $\alpha \mapsto \alpha f_j(\alpha)$, $j = 1, \dots, n$, are strictly increasing in $\alpha$ since

$$\begin{aligned} \frac{\partial}{\partial \alpha}(\alpha f_j(\alpha)) &= \frac{n\delta_j}{(n+\theta_j)(n\sigma^2 - \alpha\delta_j)} + \alpha \cdot \frac{n\delta_j^2}{(n+\theta_j)(n\sigma^2 - \alpha\delta_j)^2} \\ &= \frac{n^2\sigma^2\delta_j}{(n+\theta_j)(n\sigma^2 - \alpha\delta_j)^2} > 0 \end{aligned}$$

holds for any $\alpha \leq \alpha_{\max}$. Hence, the function $s$ is strictly increasing in $\alpha$ as well. Moreover, $s(0) = 0$ and

$$\begin{aligned} s(\alpha_{\max}) + \hat{\theta} &= \sum_{j=1}^n \left( \frac{n\delta_j}{(n+\theta_j)(2\delta_{\max} - \delta_j)} + \frac{\theta_j}{n+\theta_j} \right) \\ &= \sum_{j=1}^n \frac{n\delta_j + \theta_j(2\delta_{\max} - \delta_j)}{(n+\theta_j)(2\delta_{\max} - \delta_j)} \\ &> \frac{n\delta_{\max} + \theta_k(2\delta_{\max} - \delta_{\max})}{(n+\theta_k)(2\delta_{\max} - \delta_{\max})} = 1, \end{aligned} \tag{7.26}$$

where $k \in \{1, \dots, n\}$ is chosen so that $\delta_k = \delta_{\max}$. Hence, $s(\alpha_{\max}) > 1 - \hat{\theta}$. Finally, since $s$ is continuous on $(-\infty, \alpha_{\max}]$, the intermediate value theorem implies that there exists a unique point $\alpha_0 \in (0, \alpha_{\max})$ such that $s(\alpha_0) = 1 - \hat{\theta}$.

To summarize, $\alpha \in \mathbb{R}$ is an admissible price impact parameter if, and only if, $\alpha < \alpha_{\max}$ and $\alpha \neq \alpha_0$, where admissible means that $\alpha$ satisfies the conditions of Corollary 7.7. This can now be used to analyze the influence of the price impact parameter.

### Influence on the Nash equilibrium

First, we consider the influence of the price impact parameter $\alpha$ on the components of the Nash equilibrium from Corollary 7.7. The representation of $\pi^{i,*}$ in (7.25) implies

$$\begin{aligned}(1-\hat{\theta}-s(\alpha))\frac{\pi^{i,*}}{\mu} &= nf_i(\alpha)(1-\hat{\theta}-s(\alpha)) + \left(\frac{\theta_i}{n+\theta_i}+\alpha f_i(\alpha)\right)\sum_{j=1}^n nf_j(\alpha) \\ &= nf_i(\alpha)(1-\hat{\theta}-s(\alpha)) + \frac{n\theta_i}{n+\theta_i}\sum_{j=1}^n f_j(\alpha) + nf_i(\alpha)s(\alpha) \\ &= n(1-\hat{\theta})f_i(\alpha) + \frac{n\theta_i}{n+\theta_i}\sum_{j=1}^n f_j(\alpha) > 0,\end{aligned}$$

since $\hat{\theta}\in[0,1)$ (see Lemma 3.3) and $f_j(\alpha)>0$ for all $\alpha<\alpha_{\max}$ and all $j\in\{1,\ldots,n\}$. Hence, since $\mu>0$, the sign of $\pi^{i,*}$ equals the sign of $1-\hat{\theta}-s(\alpha)$. By definition of $\alpha_0$, $1-\hat{\theta}-s(\alpha)>0$ holds if, and only if, $\alpha<\alpha_0$. Thus, $\pi^{i,*}>0$ if, and only if, $\alpha<\alpha_0$. The behavior of $\pi^{i,*}$ at the discontinuity $\alpha_0$ and the left boundary ($\alpha\to-\infty$) can be further specified. If $\alpha$ becomes increasingly small, $\pi^{i,*}$ tends to zero. To verify this, we consider the asymptotic behavior of the functions $f_j$, $j=1,\ldots,n$, and $s$. It follows

$$\begin{aligned}\lim_{\alpha\to-\infty} f_j(\alpha) &= \lim_{\alpha\to-\infty}\frac{n\delta_j}{(n+\theta_j)(n\sigma^2-\alpha\delta_j)} = 0, \\ \lim_{\alpha\to-\infty}\alpha f_j(\alpha) &= \lim_{\alpha\to-\infty}\frac{n\delta_j\alpha}{(n+\theta_j)(n\sigma^2-\alpha\delta_j)} = \frac{n\delta_j}{n+\theta_j}\lim_{\alpha\to-\infty}\left(\frac{n\sigma^2}{\alpha}-\delta_j\right)^{-1} = -\frac{n}{n+\theta_j}, \\ \lim_{\alpha\to-\infty} s(\alpha) &= -\sum_{j=1}^n\frac{n}{n+\theta_j}. \end{aligned} \tag{7.27}$$

Hence,

$$\lim_{\alpha\to-\infty}\pi^{i,*} = \lim_{\alpha\to-\infty}\left(nf_i(\alpha)+\left(\frac{\theta_i}{n+\theta_i}+\alpha f_i(\alpha)\right)\cdot\frac{\sum_{j=1}^n nf_j(\alpha)}{1-\hat{\theta}-s(\alpha)}\right)\cdot\mu = 0.$$

Moreover, since $\lim_{\alpha\searrow\alpha_0}1-\hat{\theta}-s(\alpha)=0=\lim_{\alpha\nearrow\alpha_0}1-\hat{\theta}-s(\alpha)$ and $\pi^{i,*}>0$ if, and only if, $\alpha<\alpha_0$, it follows

$$\lim_{\alpha\nearrow\alpha_0}\pi^{i,*}=\infty, \qquad \lim_{\alpha\searrow\alpha_0}\pi^{i,*}=-\infty.$$

Furthermore, we can consider the monotonicity of $\pi^{i,*}$ in terms of $\alpha$. The first order derivative of $\pi^{i,*}$ from (7.25) is given by

$$\begin{aligned}\frac{\partial}{\partial\alpha}\frac{\pi^{i,*}}{\mu} &= nf_i'(\alpha) + (f_i(\alpha)+\alpha f_i'(\alpha))\cdot\frac{n\sum_{j=1}^n f_j(\alpha)}{1-\hat{\theta}-\sum_{j=1}^n\alpha f_j(\alpha)} + \left(\frac{\theta_i}{n+\theta_i}+\alpha f_i(\alpha)\right) \\ &\quad\cdot\frac{n\sum_{j=1}^n f_j'(\alpha)\left(1-\hat{\theta}-\sum_{j=1}^n\alpha f_j(\alpha)\right)+n\sum_{j=1}^n f_j(\alpha)\left(\sum_{j=1}^n f_j(\alpha)+\alpha f_j'(\alpha)\right)}{\left(1-\hat{\theta}-\sum_{j=1}^n\alpha f_j(\alpha)\right)^2},\end{aligned}$$

which can be rewritten as

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\frac{\pi^{i,*}}{\mu} =& \Big(1-\hat\theta-\alpha\sum_{j=1}^n f_j(\alpha)\Big)^{-2}\Big\{n(1-\hat\theta)^2 f_i'(\alpha) - n\alpha(1-\hat\theta)f_i'(\alpha)\sum_{j=1}^n f_j(\alpha)\\
&+ n(1-\hat\theta)f_i(\alpha)\sum_{j=1}^n f_j(\alpha) + n\alpha(1-\hat\theta)f_i(\alpha)\sum_{j=1}^n f_j'(\alpha)\\
&+ \frac{\theta_i}{n+\theta_i}\Big(n(1-\hat\theta)\sum_{j=1}^n f_j'(\alpha) + n\Big(\sum_{j=1}^n f_j(\alpha)\Big)^2\Big)\Big\} \qquad (7.28)\\
=& \Big(1-\hat\theta-\alpha\sum_{j=1}^n f_j(\alpha)\Big)^{-2}\Big\{n(1-\hat\theta)f_i'(\alpha)\Big(1-\hat\theta-\alpha\sum_{j=1}^n f_j(\alpha)\Big)\\
&+ n(1-\hat\theta)f_i(\alpha)\Big(\sum_{j=1}^n f_j(\alpha)+\alpha\sum_{j=1}^n f_j'(\alpha)\Big)\\
&+ \frac{\theta_i}{n+\theta_i}\Big(n(1-\hat\theta)\sum_{j=1}^n f_j'(\alpha) + n\Big(\sum_{j=1}^n f_j(\alpha)\Big)^2\Big)\Big\}. \qquad (7.29)
\end{aligned}
$$

If $\alpha < \alpha_0$, all summands in (7.29) are strictly positive. If $\alpha_0 < \alpha < \alpha_{\max}$, it follows

$$
\begin{aligned}
f_i(\alpha) - \alpha f_i'(\alpha) &= \frac{n\delta_i}{(n+\theta_i)(n\sigma^2-\alpha\delta_i)} - \alpha\frac{n\delta_i^2}{(n+\theta_i)(n\sigma^2-\alpha\delta_i)^2}\\
&= \frac{n\delta_i}{(n+\theta_i)(n\sigma^2-\alpha\delta_i)}\cdot\frac{n\sigma^2-2\alpha\delta_i}{n\sigma^2-\alpha\delta_i} > 0
\end{aligned}
$$

since $\alpha < \alpha_{\max}$. Hence, we obtain the following for the second and third summand in (7.28)

$$
-n\alpha(1-\hat\theta)f_i'(\alpha)\sum_{j=1}^n f_j(\alpha) + n(1-\hat\theta)f_i(\alpha)\sum_{j=1}^n f_j(\alpha) = n(1-\hat\theta)\sum_{j=1}^n f_j(\alpha)(f_i(\alpha)-\alpha f_i'(\alpha)) > 0.
$$

Since the other summands in (7.28) are strictly positive as well, the derivative of $\pi^{i,*}$ is strictly positive for $\alpha \in (\alpha_0, \alpha_{\max})$ and thus, for all admissible $\alpha$. Note that, due to the jump located at $\alpha_0$, $\pi^{i,*}$ is only piecewise strictly increasing in terms of $\alpha$ on $(-\infty, \alpha_0)$ and $(\alpha_0, \alpha_{\max})$.

The previously derived properties regarding the behavior of $\pi^{i,*}$ in terms of $\alpha$ can also be observed in Figure 7.3.1. The vertical lines (dotted) show the discontinuity $\alpha_0$ for the different parameter choices. The gray horizontal line (dashed) marks the value 0 while the orange and blue horizontal lines (dashed) display the optimal solution to the classical problem of maximizing expected terminal wealth under exponential utility, without price impact and relative concerns, given by $\delta_1\mu\sigma^{-2}$ (Merton ratio). There are two ways the agents may try to influence the stock price to their advantage. By buying the stock, they may jointly increase the stock value and thus, raise their utility, or by jointly short-selling the stock and thus, decrease its value. Our analysis shows that, in case of a small price impact, the agents go for the first option and, in case of a larger price impact, they go for the latter option. Of course, this is only true under the exponential utility where short-selling is no problem. Under an increasingly negative price impact, the investors engage less in the financial market.

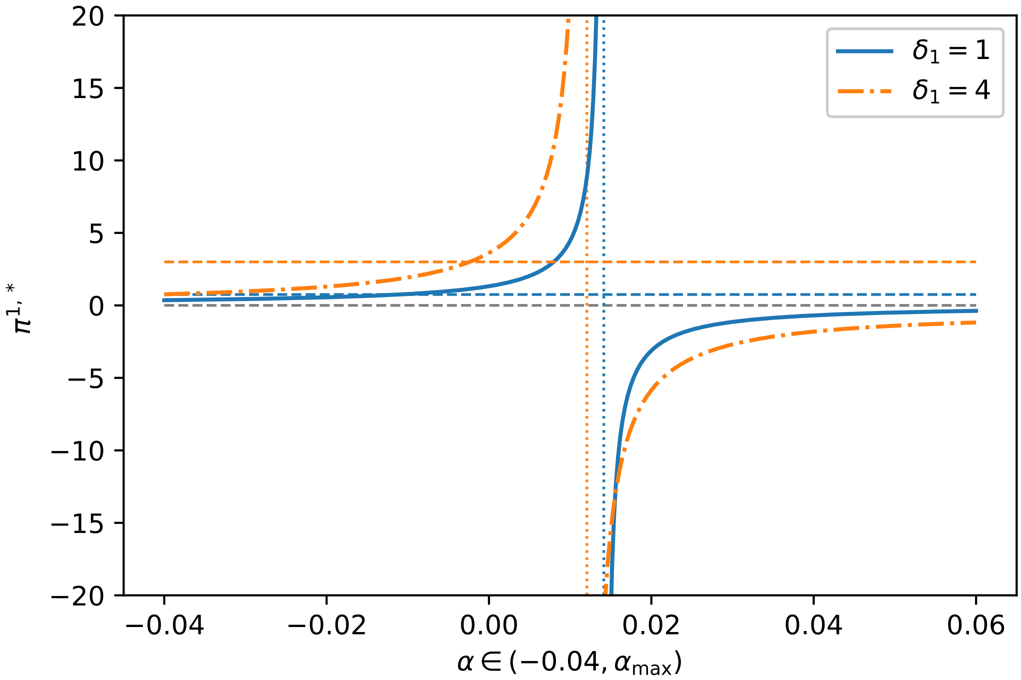



Figure 7.3.1.: Illustration of $\pi^{1,*}$ from Corollary 7.7 in terms of $\alpha \in (-0.04, \alpha_{\max})$ for $n = 12$, $\mu = 0.03$, $\sigma = 0.2$, $\alpha_{\max} = n\sigma^2/8$. The parameters for agent 1 are $\theta_1 = 0.3$, $\delta_1 \in \{1, 4\}$, and the parameters $\theta_j$ and $\delta_j$, $j \geq 2$ are increasing from 0 to 1 with step size 0.1, and from 0.5 to 2.7 by step size 0.2, respectively.

### Influence on the stock price

In the second part of this subsection, we consider the influence of the price impact parameter $\alpha$ on the stock price. Recall that the dynamics of the stock price are given by

$$\mathrm{d}S_t = S_t\big((\mu + \alpha\bar{\pi})\mathrm{d}t + \sigma \mathrm{d}W_t\big),\ t \in [0,T], \quad S_0 = 1.$$

If we insert the constant Nash equilibrium from (7.25) and use that $\mu$ and $\sigma$ are constants, Theorem 2.4 implies

$$S_t^*(\alpha) := S_t = \exp\left(\left(\mu + \alpha \cdot \frac{\sum_{j=1}^n f_j(\alpha)}{1 - \hat{\theta} - s(\alpha)} \cdot \mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right),\ t \in [0,T]. \tag{7.30}$$

Since the functions $\alpha \mapsto \alpha f_j(\alpha)$, $j = 1, \dots, n$, and $s$ are strictly increasing in terms of $\alpha$, the stock price $S_t^*$ is piecewise increasing on $(-\infty, \alpha_0)$ and $(\alpha_0, \alpha_{\max})$ in terms of $\alpha$ with a jump located at $\alpha_0$. Moreover, the summand $\frac{\alpha \sum_{j=1}^n f_j(\alpha)}{1-\hat{\theta}-s(\alpha)} \cdot \mu$ is strictly positive if, and only if, $\alpha \in (0, \alpha_0)$. Thus, $S_t^*(\alpha) > S_t^*(0)$ if, and only if, $\alpha \in (0, \alpha_0)$. More specifically, we obtain the following ordering for any choice of $\alpha_1 < 0$, $\alpha_2 \in (0, \alpha_0)$, and $\alpha_3 \in (\alpha_0, \alpha_{\max})$

$$S_t^*(\alpha_3) < S_t^*(\alpha_1) < S_t^*(0) < S_t^*(\alpha_2). \tag{7.31}$$

The second and third inequality follow directly from the previous observations. For the first inequality, we consider the limit of $S_t^*$ as $\alpha$ tends to $-\infty$ and $\alpha_{\max}$. Obviously, it suffices to consider the expression $\frac{s(\alpha)}{1-\hat{\theta}-s(\alpha)} = \frac{\alpha\sum_{j=1}^n f_j(\alpha)}{1-\hat{\theta}-s(\alpha)}$. If $\alpha$ tends to $\alpha_{\max}$, it follows

$$\lim_{\alpha\to\alpha_{\max}} \frac{s(\alpha)}{1-\hat{\theta}-s(\alpha)} = \frac{\sum_{j=1}^n \frac{n\delta_j}{(n+\theta_j)(2\delta_{\max}-\delta_j)}}{1-\hat{\theta}-\sum_{j=1}^n \frac{n\delta_j}{(n+\theta_j)(2\delta_{\max}-\delta_j)}}. \tag{7.32}$$

Further, recall that $s(\alpha) \to -\sum_{j=1}^n \frac{n}{n+\theta_j}$ as $\alpha \to -\infty$ (see (7.27)). Now consider the function $r : \mathbb{R}\backslash\{1-\hat{\theta}\} \to \mathbb{R}$, $x \mapsto \frac{x}{1-\hat{\theta}-x}$. Then

$$\lim_{x\to\pm\infty} r(x) = -1$$

and $r$ is piecewise increasing on $(-\infty, 1-\hat{\theta})$ and $(1-\hat{\theta},\infty)$ with a jump from positive to negative values located at $1-\hat{\theta}$. Hence, $r(x) > -1$ on $(-\infty, 1-\hat{\theta})$ and $r(x) < -1$ on $(1-\hat{\theta},\infty)$ and, in particular,

$$r(x) > -1 > r(y) \quad \text{for all} \quad x < 1-\hat{\theta} < y. \tag{7.33}$$

Since $-\sum_{j=1}^n \frac{n}{n+\theta_j} < 0 < 1-\hat{\theta}$ and $\sum_{j=1}^n \frac{n\delta_j}{(n+\theta_j)(2\delta_{\max}-\delta_j)} > 1-\hat{\theta}$ (see (7.26)), (7.27), (7.32) and (7.33) imply

$$\begin{aligned}\lim_{\alpha\to-\infty} \frac{s(\alpha)}{1-\hat{\theta}-s(\alpha)} &= r\left(-\sum_{j=1}^n \frac{n}{n+\theta_j}\right)\\ &> r\left(\sum_{j=1}^n \frac{n\delta_j}{(n+\theta_j)(2\delta_{\max}-\delta_j)}\right) = \lim_{\alpha\to\alpha_{\max}} \frac{s(\alpha)}{1-\hat{\theta}-s(\alpha)}.\end{aligned}$$

Since $\frac{s(\alpha)}{1-\hat{\theta}-s(\alpha)}$ is piecewise increasing in $\alpha$ on $(-\infty,\alpha_0)$ and $(\alpha_0,\alpha_{\max})$, the first inequality in (7.31) follows as well.

The behavior of $S_T^*$ in terms of $\alpha$ can also be observed in Figure 7.3.2. The horizontal dashed lines represent the stock price $S_T^*(0)$ without price impact for three possible realizations of $W_T$. The vertical dotted lines mark the values $\alpha = 0$ and $\alpha = \alpha_0$. The figure shows the monotonicity of $S_T^*$ in terms of $\alpha$ as well as the ordering discussed in (7.31).

### 7.3.3. Discussion of nonlinear price impact

At the beginning of Section 7.1, we assumed that the price impact of the $n$ investors is incorporated into the drift of the stock as a linear function in terms of the arithmetic mean of the investors' strategies. While the use of the arithmetic mean seems intuitive and reasonable since we assumed that the single investors are „small“, one could ask whether using a different function than a linear one would lead to a different optimization problem and hence, also a different optimal investment. Moreover, Muhle-Karbe et al. (2022) (and references therein) discuss empirical data which suggests that price impact is not linear but concave in order size. Further, Bouchaud (2009) argues that the volume dependence of price impact is sublinear and behaves like a power function with an exponent approximately between 0.1 and 0.3.

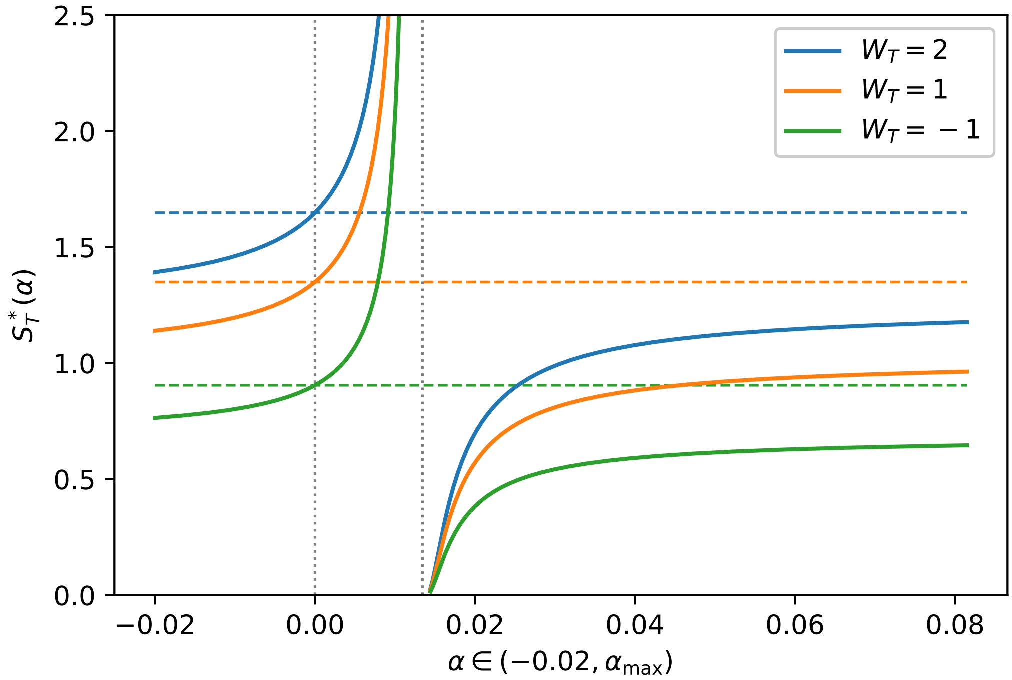



Figure 7.3.2.: Illustration of $S_T^*(\alpha)$ (solid) from (7.30) in terms of $\alpha \in (-0.02, \alpha_{\max})$ for three different realizations of $W_T$, where $T = 10$, $n = 11$, $\mu = 0.03$, $\sigma = 0.2$. The parameters $\theta_j$ and $\delta_j$, $j \in \{1, \ldots, 11\}$, are increasing from 0 to 1 with step size 0.1, and from 0.5 to 2.7 by step size 0.2, respectively. The horizontal dashed lines mark the stock price for $\alpha = 0$.

In Theorem 7.2 and the subsequent Corollary 7.7, we were able to find an explicit constant Nash equilibrium to the associated multi-objective portfolio optimization problem using exponential utility (if the parameters are chosen accordingly). The proof depends strongly on the linearity of the price impact. Thus, we are not be able to give an explicit solution to the optimization problem if the price impact is included into the drift via a nonlinear function $g$. However, we can show that using a function $g$ that grows superlinearly results in a problem that does not have a finite optimal solution, whereas a function $g$ that grows sublinearly yields a finite optimal solution. If $g$ is a linear function, it depends on the parameter choices whether or not there exists a finite optimal solution (see Corollary 7.7).

In the case of linear price impact, we allowed the price impact parameter $\alpha$ to take both positive and negative values. However, the considerations in Subsection 7.3.2 showed that the $n$ agents' investment into the stock becomes negligible for increasingly negative price impact. Hence, in the nonlinear case, we only consider price impact which is increasing in order size. Therefore, the price impact is now modeled by a continuous and strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$. Thus, the stock price process is given as the solution to the stochastic differential equation

$$\mathrm{d}S_t = S_t\big( (\mu + g(\bar{\pi}_t))\,\mathrm{d}t + \sigma \mathrm{d}W_t \big),\ S_0 = 1.$$

We have to restrict the problem to constant Nash equilibria again. Therefore, from the perspective of investor $i \in \{1,\dots,n\}$, we can rewrite the expression $g(\bar{\pi}_t)$ in the previous stochastic differential equation as

$$g(\bar{\pi}_t) = g\Big(\frac{1}{n}\sum_{j=1}^n \pi_t^j\Big) = g\Big(\frac{1}{n}\pi_t^i + \frac{1}{n}\sum_{j\neq i}\pi^j\Big) =: \widetilde{g}(\pi_t^i),$$

where $\widetilde{g}(p) := g\left(\frac{p}{n} + \frac{1}{n}\sum_{j\neq i}\pi^j\right)$, $p \in \mathbb{R}$. We assume that the strategies $\pi^j$, $j \neq i$, of the other investors are fixed, constant, and deterministic. Moreover, we require $\pi^i \in \mathcal{A}$ for the set $\mathcal{A}$ of admissible strategies from (7.8). From the definition of $\widetilde{g}$, it follows directly that $\widetilde{g}$ is still continuous and strictly increasing and satisfies $\widetilde{g}\left(-\sum_{j\neq i}\pi^j\right) = 0$. The wealth process of investor $i$ is given by

$$X_t^{i,\pi^i} = x_0^i + \int_0^t \pi_s^i\left(\left(\mu + \widetilde{g}(\pi_s^i)\right)\mathrm{d}s + \sigma \mathrm{d}W_s\right),\ t \in [0,T].$$

In the following, we prove that the best response problem for

$$\begin{cases} & \sup_{\pi^i\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left(X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_T^{j,\pi^j}\right)\right)\right], \\ \text{s.t.} & X_T^{i,\pi^i} = x_0^i + \int_0^T \pi_t^i\left((\mu + g(\bar{\pi}_t))\,\mathrm{d}t + \sigma\mathrm{d}W_t\right), \end{cases} \quad i = 1,\dots,n, \tag{7.34}$$

has a finite optimal solution if $g$ grows sublinearly, and is not solvable if $g$ grows superlinearly. The following proposition summarizes the first assertion of this subsection which treats the case of sublinear growth.

**Proposition 7.8.** *Let $i \in \{1,\dots,n\}$ arbitrary but fixed, and $\pi^j$, $j \neq i$, deterministic and constant. If $\lim_{x\to\pm\infty}\frac{g(x)}{x} = 0$, the best response problem for constant Nash equilibria associated to* (7.34) *has a finite, constant optimal solution. Moreover, each optimal solution is a global maximum point of the function $h : \mathbb{R} \to \mathbb{R}$ given by*

$$h(a) = \Big(a - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)\Big(\mu + \widetilde{g}(a)\Big) - \frac{\sigma^2}{2\delta_i}\Big(a - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^2.$$

*Remark* 7.9. A similar observation was made by Eksi and Ku (2017). They showed that their price impact market does not contain arbitrage strategies if the price impact function is sublinear. However, they had to restrict their set of admissible strategies to bounded strategies. $\Diamond$

*Proof.* For the moment, we restrict the set of admissible strategies to bounded strategies $(\pi_t^i)_{t\in[0,T]}$, i.e. there exists a constant $K > 0$ such that $|\pi_t^i| \leq K$ $\mathbb{P}$-almost surely for all $t \in [0,T]$. For constants $\pi^j$, $j \neq i$, we obtain

$$\begin{aligned} &-\frac{1}{\delta_i}\Big(X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\pi^j}\Big) \\ =& -\frac{1}{\delta_i}\Big(x_0^i - \frac{\theta_i}{n}\sum_{j\neq i}x_0^j\Big) - \frac{1}{\delta_i}\Big(\int_0^T\Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)(\mu + \widetilde{g}(\pi_t^i))\mathrm{d}t + \sigma\int_0^T\Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)\mathrm{d}W_t\Big) \\ &- \frac{\sigma^2}{2\delta_i^2}\int_0^T\Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^2\mathrm{d}t + \frac{\sigma^2}{2\delta_i^2}\int_0^T\Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^2\mathrm{d}t. \end{aligned}$$

Now define a new probability measure $\mathbb{Q}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\bigg( -\frac{\sigma^2}{2\delta_i^2}\int_0^T \Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^2 \mathrm{d}t - \frac{\sigma}{\delta_i}\int_0^T \Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)\mathrm{d}W_t\bigg).$$

Note that the expression on the right-hand side is a density since $(\pi_t^i)_{t\in[0,T]}$ is bounded. Thus, we can write the objective function of (7.34) as

$$\begin{aligned}
&\mathbb{E}\bigg[ -\exp\bigg( -\frac{1}{\delta_i}\Big( X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_T^{j,\pi^j}\Big)\bigg)\bigg] \\
&= -\exp\Big( -\frac{\widetilde{x}_0^i}{\delta_i}\Big)\cdot \mathbb{E}_{\mathbb{Q}}\bigg[\exp\bigg( -\frac{1}{\delta_i}\int_0^T \Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)(\mu + \widetilde{g}(\pi_t^i)) - \frac{\sigma^2}{2\delta_i}\Big(\pi_t^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^2 \mathrm{d}t\bigg)\bigg] \\
&= -\exp\Big( -\frac{\widetilde{x}_0^i}{\delta_i}\Big)\cdot \mathbb{E}_{\mathbb{Q}}\bigg[\exp\bigg( -\frac{1}{\delta_i}\int_0^T h(\pi_t^i)\,\mathrm{d}t\bigg)\bigg],
\end{aligned}$$

where we used the abbreviation of $\widetilde{x}_0^i = x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j$ and the function $h$ from the proposition. But now in order to maximize the expectation, we can do this pointwise under the integral which leads to maximizing the function $h$. If $a$ tends to $\pm\infty$, the function $h$ converges to $-\infty$ since $g$, and thus $\widetilde{g}$, grows sublinearly. Hence, the maximizing point of $h$ is not at the boundary and the assumption of bounded strategies is no restriction. Moreover, $h$ is continuous since $\widetilde{g}$ is continuous and hence, $h$ takes its maximum. Thus, an optimal strategy to (7.34) exists and is finite. This concludes our proof. □

*Remark* 7.10. The structure of the function $h$ in Proposition 7.8 suggests that there are at least one and at most two global maxima. To verify this, we consider the components of $h$. The first factor $a - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j$ is strictly increasing in $a$ for all $a\in\mathbb{R}$ and strictly positive if, and only if, $a > \frac{\theta_i}{n}\sum_{j\neq i}\pi^j$. The second factor $\mu + \widetilde{g}(a)$ is also strictly increasing for all $a\in\mathbb{R}$ and strictly positive if, and only if, $\widetilde{g}(a) > -\mu$. Given the continuity and monotonicity of $\widetilde{g}$, there exists a unique point $a' \in [-\infty, -\sum_{j\neq i}\pi^j)$ such that $\widetilde{g}(a) > -\mu$ if, and only if, $a > a'$. Note that $a' = -\infty$ is possible since $\widetilde{g}$ might be bounded from below by a constant strictly larger than $-\mu$. The last component of $h$, given by

$$-\frac{\sigma^2}{2\delta_i}\Big(a - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^2,$$

is strictly negative for all $a\in\mathbb{R}$, strictly increasing for $a < \frac{\theta_i}{n}\sum_{j\neq i}\pi^j$, and strictly decreasing, otherwise. A combination of these observations, together with the asymptotic behavior of $h$, yields that $h$ can have at most three local extrema, as there are only three points at which the monotonicity behavior of $h$ might change. More specifically, there is either only one local maximum, which is also the global maximum of $h$, or there are two local maxima and one local minimum. ◊

A combination of Corollary 7.7 and Proposition 7.8 yields that, as long as the asymptotic growth of $g$ is at most linear, there exists an optimal solution to the best response problem. The next proposition shows that the opposite holds if $g$ grows superlinearly.

**Proposition 7.11.** *Let $i \in \{1,\dots,n\}$ arbitrary but fixed, and $\pi^j$, $j \neq i$, deterministic and constant. If $\lim_{x\to\pm\infty} \frac{g(x)}{x} = \infty$, the best response problem for constant Nash equilibria associated to* (7.34) *does not have an optimal solution.*

*Proof.* To prove that the best response problem for (7.34) does not have a finite optimal solution, we show that even if we consider only constant strategies for agent $i$, the optimal value is 0 and the associated strategy is infinite. If $\pi^j$ is constant for all $j \in \{1,\dots,n\}$, it follows

$$\begin{aligned}
&X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i} X_T^{j,\pi^j} \\
&= x_0^i - \frac{\theta_i}{n}\sum_{j\neq i} x_0^j + \Big(\pi^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)(\mu + \widetilde{g}(\pi^i))T + \Big(\pi^i - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)\sigma W_T \\
&=: \widetilde{x}_0^i + \mu(\pi^i)T + \sigma(\pi^i)W_T.
\end{aligned} \tag{7.35}$$

The random variable in (7.35) has a normal distribution with mean $\mu(\pi^i)$ and variance $\sigma(\pi^i)^2T$. Therefore, if we apply the exponential utility function to (7.35), we obtain a lognormally distributed random variable and the expectation can be determined explicitly. Hence, for constant $\pi^j$, $j = 1,\dots,n$, the value of the objective function in (7.34) is given by (see Johnson et al., 1994, p. 212)

$$\mathbb{E}\Big[-\exp\Big(-\frac{1}{\delta_i}\Big(\widetilde{x}_0^i + \mu(\pi^i)T + \sigma(\pi^i)W_T\Big)\Big)\Big] = -\exp\Big(-\frac{1}{\delta_i}\widetilde{x}_0^i\Big)\cdot\exp\Big(-\frac{1}{\delta_i}\Big(\mu(\pi^i) - \frac{\sigma(\pi^i)^2}{2\delta_i}\Big)T\Big).$$

Thus, maximizing the objective function of (7.34) with respect to constant strategies $\pi^i$ is equivalent to maximizing $\mu(\pi^i) - \frac{\sigma(\pi^i)^2}{2\delta_i}$. Reinserting the definition of $\mu(\pi^i)$ and $\sigma(\pi^i)$ yields

$$\begin{aligned}
&\mu(\pi^i) - \frac{\sigma(\pi^i)^2}{2\delta_i} \\
&= \pi^i\widetilde{g}(\pi^i) - \frac{\sigma^2}{2\delta_i}(\pi^i)^2 + \pi^i\Big(\mu + \frac{\sigma^2\theta_i}{n\delta_i}\sum_{j\neq i}\pi^j\Big) - \frac{\theta_i}{n}\widetilde{g}(\pi^i)\sum_{j\neq i}\pi^j - \frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big(\mu + \frac{\sigma^2\theta_i}{2n\delta_i}\sum_{j\neq i}\pi^j\Big),
\end{aligned}$$

which converges to $\infty$ if $\pi^i$ converges to $\pm\infty$ using the assumption that $g$ (and thus, $\widetilde{g}$) grows superlinearly. Therefore,

$$\begin{aligned}
0 &\geq \sup_{\pi^i\in\mathcal{A}} \mathbb{E}\Big[-\exp\Big(-\frac{1}{\delta_i}\Big(X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\pi^j}\Big)\Big)\Big] \\
&\geq \sup_{\substack{\pi^i\in\mathcal{A}\\ \pi^i \text{ constant}}} \mathbb{E}\Big[-\exp\Big(-\frac{1}{\delta_i}\Big(X_T^{i,\pi^i} - \frac{\theta_i}{n}\sum_{j\neq i}X_T^{j,\pi^j}\Big)\Big)\Big] = 0.
\end{aligned}$$

Hence, the optimal value of (7.34) is 0 which implies that the argument inside the exponential function needs to be infinite. Thus, the problem does not have a finite optimal solution. ☐

## 7.4. Optimization under power utility

In the previous two sections, we used the additive relative performance metric to measure the relative concerns of $n$ agents. For the definition of relative performance metrics, we refer to the introduction of Chapter 3. Afterwards, we found the unique constant Nash equilibrium in a more specific setting, where the market parameters were assumed to be constant and the investors used exponential utility functions. In combination with exponential utility, it was convenient to use the arithmetic mean in the objective function and to optimize the amount invested into each of the $d$ stocks. However, when using power utility functions, it is more convenient to optimize the invested *fraction* of wealth and use the *geometric* mean in the objective function (see, among others, Lacker and Zariphopoulou, 2019). Indeed, it is possible to determine a constant Nash equilibrium to the resulting portfolio optimization problem explicitly. We refer to the beginning of Section 7.3 for a justification of the restriction to constant strategies. Another benefit of the geometric mean (compared to the arithmetic mean) is that we do not need any restrictions on the competition weight $\theta_i$, since the argument inside the utility function is strictly positive if, and only if, the initial capital $x_0^i$ of each agent $i \in \{1, \dots, n\}$ is strictly positive.

Let us assume that the agents use power utility functions of the form

$$U_i : (0, \infty) \to \mathbb{R},\, x \mapsto \left(1 - \frac{1}{\delta_i}\right)^{-1} x^{1-\frac{1}{\delta_i}} = \frac{\delta_i}{\delta_i - 1} x^{\frac{\delta_i - 1}{\delta_i}},$$

for risk tolerance parameters $\delta_i > 0$, $\delta_i \neq 1$, $i = 1, \dots, n$. The underlying financial market again consists of one riskless bond, without loss of generality assumed to be identical to 1, and $d$ stocks with price processes determined by

$$\mathrm{d}S_k(t) = S_k(t) \left( (\mu_k + \alpha_k \bar{\pi}_k(t))\, \mathrm{d}t + \sum_{\ell=1}^{d} \sigma_{k\ell} \mathrm{d}W_\ell(t) \right),\, t \in [0, T],\, S_k(0) = 1,\, k = 1, \dots, d.$$

For tractability reasons, we assume that the market parameters $\mu$ and $\sigma$ are constant and deterministic. The volatility matrix $\sigma$ is again assumed to be regular and the strategies $\pi^j$, $j = 1, \dots, n$, are taken from the set $\mathcal{A}$ of admissible strategies defined in (7.8).

In contrast to the model presented in Section 7.1, the process $(\pi^j(t))_{t \in [0,T]} = (\pi_1^j(t), \dots, \pi_d^j(t))_{t \in [0,T]}$ now represents the fractions of agent $j$'s wealth invested into the $d$ stocks. Nevertheless, $\bar{\pi}$ still describes the arithmetic mean of the $n$ investors' strategies. The wealth process of agent $i \in \{1, \dots, n\}$ solves the stochastic differential equation

$$\mathrm{d}X_t^{i,\pi^i} = X_t^{i,\pi^i} \sum_{k=1}^{d} \pi_k^i(t) \left( (\mu_k + \alpha_k \bar{\pi}_k(t))\, \mathrm{d}t + \sum_{\ell=1}^{d} \sigma_{k\ell} \mathrm{d}W_\ell(t) \right),\, X_0^{i,\pi^i} = x_0^i.$$

An explicit representation of $X^{i,\pi^i}$ is thus given by (see Theorem 2.4 and Lemma 2.2)

$$X_t^{i,\pi^i} = x_0^i \exp\left( \int_0^t \pi^i(s)^\top \left( \mu + A\bar{\pi}(s) - \frac{1}{2}\sigma\sigma^\top \pi^i(s) \right) \mathrm{d}s + \int_0^t \pi^i(s)^\top \sigma \mathrm{d}W(s) \right), \tag{7.36}$$

$t \in [0,T]$. Hence, the portfolio optimization problem of agent $i$ is given by

$$\begin{cases} & \max_{\pi^i\in\mathcal{A}} \mathbb{E}\left[\frac{\delta_i}{\delta_i-1}\left(X_T^{i,\pi^i}\left(\prod_{j\neq i} X_T^{j,\pi^j}\right)^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i-1}{\delta_i}}\right], \\ \text{s.t.} & \mathrm{d}X_t^{i,\pi^i} = X_t^{i,\pi^i}\pi^i(t)^\top\left((\mu + A\bar{\pi}(t))\mathrm{d}t + \sigma\mathrm{d}W(t)\right), X_0^{i,\pi^i} = x_0^i. \end{cases} \tag{7.37}$$

In order to determine a Nash equilibrium explicitly, we need to restrict the problem to constant Nash equilibria again. Then the unique constant Nash equilibrium is given in the following theorem.

**Theorem 7.12.** *Assume that the following assumptions hold*

*a)* $\frac{n+\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right) - \frac{\theta_i\delta_i}{n}\sigma\sigma^\top$ *is regular for all* $i \in \{1,\dots,n\}$,

*b)* $\sigma\sigma^\top - \frac{2\delta_i}{n}A$ *is positive-definite for all* $i \in \{1,\dots,n\}$,

*c)* $I_d - \sum_{j=1}^n\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right) - \frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\left(1-\frac{\theta_i}{n}\right)A - \frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right)$ *is regular.*

*Then the unique constant Nash equilibrium for* (7.37) *in terms of invested fractions is given by*

$$\begin{aligned}
\pi^{i,*} = {}&\delta_i\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right) - \frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)^{-1}\mu \\
&+\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top - \frac{\delta_i}{n}A\right) - \frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\left(1-\frac{\theta_i}{n}\right)A - \frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right) \\
&\cdot\left(I_d - \sum_{j=1}^n\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right) - \frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\left(1-\frac{\theta_i}{n}\right)A - \frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right)\right)^{-1} \\
&\cdot\sum_{j=1}^n\delta_j\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right) - \frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\mu, \quad i = 1,\dots,n.
\end{aligned}$$

Before we display the proof of Theorem 7.12, we provide an example of a class of matrices $A$ that satisfy the conditions of the theorem. Roughly speaking, the conditions a) – c) of Theorem 7.12 are satisfied if the price impact parameters $\alpha_k$, $k = 1,\dots,d$, are not too large.

*Example* 7.13. Let $A = \alpha I_d$ for some constant $\alpha \in \mathbb{R}$. In order to ensure that $A$ satisfies the conditions of Theorem 7.12, we can find sufficient conditions on $\alpha$.

Since the volatility matrix $\sigma$ is regular by assumption, $\sigma\sigma^\top$ is positive-definite. Let $\lambda_1,\dots,\lambda_d$ denote the (strictly positive) eigenvalues of $\sigma\sigma^\top$. It turns out that the condition

$$\alpha < \frac{n\theta_j(\delta_j-1)}{\delta_j(n-\theta_j)}\lambda_\ell \quad \text{for all } \ell \in \{1,\dots,d\},\ j \in \{1,\dots,n\}, \tag{7.38}$$

is sufficient to ensure that $A = \alpha I_d$ satisfies the assumptions of Theorem 7.12, given that $n$ is sufficiently large. The following shows why this is the case.

For $j \in \{1,\dots,n\}$, the eigenvalues of the matrix

$$\frac{\theta_j(\delta_j-1)}{n}\sigma\sigma^\top - \frac{\delta_j(n-\theta_j)}{n^2}A$$

(that appears in condition c) of the previous theorem) are given by

$$\frac{\theta_j(\delta_j - 1)}{n}\lambda_\ell - \frac{\delta_j(n-\theta_j)}{n^2}\alpha, \tag{7.39}$$

$\ell = 1, \dots, d$ (see Proposition 4.4.5 in Bernstein, 2009). The expression in (7.39) is strictly positive if, and only if,

$$\alpha < \frac{n\theta_j(\delta_j - 1)}{\delta_j(n-\theta_j)}\lambda_\ell$$

for all $\ell \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$, i.e., if the asserted sufficient condition (7.38) is satisfied.

Using Proposition 4.4.5 by Bernstein (2009) again, the eigenvalues of the matrix in assumption b) are given by

$$\lambda_\ell - \frac{2\delta_j}{n}\alpha,\ \ell = 1, \dots, d,$$

for $j \in \{1, \dots, n\}$. Thus, the eigenvalues are strictly positive if, and only if,

$$\alpha < \frac{n\lambda_\ell}{2\delta_j} \text{ for all } \ell = 1, \dots, d,\ j = 1, \dots, n.$$

If $n$ is chosen large enough, i.e., if $n > \theta_j(2\delta_j - 1)$ for all $j \in \{1, \dots, n\}$, the following holds

$$\frac{n\theta_j(\delta_j - 1)}{\delta_j(n-\theta_j)} < \frac{n}{2\delta_j}.$$

Therefore, condition (7.38) implies that $A$ satisfies condition b) of Theorem 7.12 if $n$ is sufficiently large. Moreover, the eigenvalues of the matrix in a) are given by

$$\frac{n+\theta_j(1-\delta_j)}{n}\lambda_\ell - \frac{\delta_j(n+\theta_j)}{n^2}\alpha, \quad \ell = 1, \dots, d, \tag{7.40}$$

for $j \in \{1, \dots, n\}$. The expression in (7.40) is strictly positive if, and only if,

$$\alpha < \frac{n(n+\theta_j(1-\delta_j))}{\delta_j(n+\theta_j)}\lambda_\ell$$

for all $\ell = 1, \dots, d$, $j = 1, \dots, n$. Note that

$$\frac{\theta_j(\delta_j - 1)}{n-\theta_j} < \frac{n+\theta_j(1-\delta_j)}{n+\theta_j}$$

holds if, and only if, $n > \theta_j(2\delta_j - 1)$, which we already assumed earlier. Hence, the matrix in a) is positive-definite and thus, regular, if (7.38) holds. Finally, this implies that the matrix in condition c) of Theorem 7.12, which can be rewritten as

$$I_d + \sum_{j=1}^{n}\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top - \frac{\delta_j}{n}A\right) - \frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\theta_j(\delta_j-1)}{n}\sigma\sigma^\top - \frac{\delta_j(n-\theta_j)}{n^2}A\right),$$

is a sum of positive-definite matrices. Thus, the matrix is positive-definite as well and hence, regular. Altogether, assumption (7.38) ensures that $A = \alpha I_d$ satisfies the requirements of Theorem 7.12 as long as $n > \theta_j(2\delta_j - 1)$ for all $j \in \{1, \dots, n\}$. This is not a restriction in our case considering that

$n$ represents a (large) number of agents. It should be noted that condition (7.38) is sufficient, but not necessary, to ensure that the requirements of Theorem 7.12 are satisfied. Indeed, we chose (7.38) to ensure that the matrices apparent in the sum in condition c) are positive-definite while the theorem only requires that the sum of matrices in c) is regular. However, it simplifies the arguments to consider positive-definite summands instead of regularity for the whole matrix. $\Diamond$

Now we proceed with the proof of Theorem 7.12.

*Proof (Theorem 7.12).* Choose some arbitrary $i \in \{1, \dots, n\}$ and assume that the agents $j \neq i$ use constant strategies $\pi^j$, $j \neq i$, which are assumed to be arbitrary but fixed. Now define a stochastic process $(Y_t^{-i})_{t\in[0,T]}$ by $Y_t^{-i} = \prod_{j\neq i} X_t^{j,\pi^j}$, $t \in [0, T]$. Using (7.36), it follows

$$Y_t^{-i} = y_0^i \exp\left( \int_0^t \Big( \sum_{j\neq i} \pi^j \Big)^\top \Big( (\mu + A\bar{\pi}(s))\mathrm{d}s + \sigma \mathrm{d}W(s) \Big) - \frac{1}{2} \int_0^t \sum_{j\neq i} (\pi^j)^\top \sigma\sigma^\top \pi^j \mathrm{d}s \right),$$

where $y_0^i := \prod_{j\neq i} x_0^j$. Thus, the argument of the expected utility in (7.37) can be written as follows

$$\begin{aligned}
&X_t^{i,\pi^i} \left(Y_t^{-i}\right)^{-\frac{\theta_i}{n}} \\
&= x_0^i \, (y_0^i)^{-\frac{\theta_i}{n}} \exp\Bigg( \int_0^t \pi^i(s)^\top \Big( \mu + A\bar{\pi}(s) - \frac{1}{2}\sigma\sigma^\top \pi^i(s) \Big) \mathrm{d}s + \int_0^t \pi^i(s)^\top \sigma \mathrm{d}W(s) \\
&\quad - \frac{\theta_i}{n} \bigg\{ \int_0^t \Big( \sum_{j\neq i} \pi^j \Big)^\top (\mu + A\bar{\pi}(s))\mathrm{d}s + \int_0^t \Big( \sum_{j\neq i} \pi^j \Big)^\top \sigma \mathrm{d}W(s) - \frac{1}{2} \int_0^t \sum_{j\neq i} (\pi^j)^\top \sigma\sigma^\top \pi^j \mathrm{d}s \bigg\} \Bigg) \\
&= x_0^i \exp\Bigg( \int_0^t \pi^i(s)^\top \Big( \mu + \Big( \frac{1}{n} A - \frac{1}{2} \sigma\sigma^\top \Big) \pi^i(s) + \frac{1}{n} \Big( 1 - \frac{\theta_i}{n} \Big) A \sum_{j\neq i} \pi^j \Big) \mathrm{d}s + \int_0^t \pi^i(s)^\top \sigma \mathrm{d}W(s) \Bigg) \\
&\quad \cdot (y_0^i)^{-\frac{\theta_i}{n}} \exp\Bigg( - \frac{\theta_i}{n} \bigg\{ \int_0^t \Big( \sum_{j\neq i} \pi^j \Big)^\top \Big( \mu + \frac{1}{n} A \sum_{j\neq i} \pi^j \Big) \mathrm{d}s + \int_0^t \Big( \sum_{j\neq i} \pi^j \Big)^\top \sigma \mathrm{d}W(s) \\
&\qquad\qquad - \frac{1}{2} \int_0^t \sum_{j\neq i} (\pi^j)^\top \sigma\sigma^\top \pi^j \mathrm{d}s \bigg\} \Bigg) \\
&=: \widetilde{X}_t^{i,\pi^i} \left(\widetilde{Y}_t^{-i}\right)^{-\frac{\theta_i}{n}},
\end{aligned}$$

where the process $\widetilde{Y}^{-i}$ does not depend on $\pi^i$. More specifically, the processes $\widetilde{X}^{i,\pi^i}$ and $\widetilde{Y}^{-i}$ are the unique solutions to the following stochastic differential equations (using Theorem 2.4 and Lemma 2.2)

$$\begin{aligned}
\mathrm{d}\widetilde{X}_t^{i,\pi^i} &= \widetilde{X}_t^{i,\pi^i} \pi^i(t)^\top \Bigg( \Big( \mu + \frac{1}{n} A\pi^i(t) + \frac{1}{n} \Big( 1 - \frac{\theta_i}{n} \Big) A \sum_{j\neq i} \pi^j \Big) \mathrm{d}t + \sigma \mathrm{d}W(t) \Bigg), \\
\mathrm{d}\widetilde{Y}_t^{-i} &= \widetilde{Y}_t^{-i} \Bigg( \Big( \sum_{j\neq i} \pi^j \Big)^\top \Bigg( \Big( \mu + \frac{1}{n} A \sum_{j\neq i} \pi^j \Big) \mathrm{d}t + \sigma \mathrm{d}W(t) \Bigg) + \frac{1}{2} \sum_{\substack{1\le h\neq j\le n \\ h,j\neq i}} (\pi^h)^\top \sigma\sigma^\top \pi^j \mathrm{d}t \Bigg)
\end{aligned}$$

with initial values $\widetilde{X}_0^{i,\pi^i} = x_0^i$ and $\widetilde{Y}_0^{-i} = y_0^i$. The introduction of $\widetilde{X}^{i,\pi^i}$ and $\widetilde{Y}^{-i}$ simplifies the derivation of the HJB equation below. First, we notice that the quadratic (co)variations of $\widetilde{X}^{i,\pi^i}$

and $\widetilde{Y}^{-i}$ are given as follows (see Lemma 2.2)

$$\mathrm{d}\langle \widetilde{X}^{i,\pi^i}\rangle_t = \left(\widetilde{X}_t^{i,\pi^i}\right)^2 \pi^i(t)^\top\sigma\sigma^\top\pi^i(t)\mathrm{d}t, \quad \mathrm{d}\langle \widetilde{Y}^{-i}\rangle_t = \left(\widetilde{Y}_t^{-i}\right)^2 \Big(\sum_{j\neq i}\pi^j\Big)^\top\sigma\sigma^\top\sum_{j\neq i}\pi^j\mathrm{d}t,$$
$$\mathrm{d}\langle \widetilde{X}^{i,\pi^i},\widetilde{Y}^{-i}\rangle_t = \widetilde{X}_t^{i,\pi^i}\widetilde{Y}_t^{-i}\pi^i(t)^\top\sigma\sigma^\top\sum_{j\neq i}\pi^j\mathrm{d}t,$$

for $t\in[0,T]$. Let us now define the following value functions for $t\in[0,T]$, $x,y\in(0,\infty)$

$$\begin{aligned}
J(t,x,y;\pi^i) &:= \mathbb{E}\left[\frac{\delta_i}{\delta_i-1}\left(\widetilde{X}_T^{i,\pi^i}\left(\widetilde{Y}_T^{-i}\right)^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i-1}{\delta_i}}\middle|\widetilde{X}_t^{i,\pi^i}=x,\,\widetilde{Y}_t^{-i}=y\right]\\
&=: \mathbb{E}^{t,x,y}\left[\frac{\delta_i}{\delta_i-1}\left(\widetilde{X}_T^{i,\pi^i}\left(\widetilde{Y}_T^{-i}\right)^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i-1}{\delta_i}}\right],\\
J(t,x,y) &:= \sup_{\pi^i\in\mathcal{A}} J(t,x,y;\pi^i).
\end{aligned}$$

Then, for any $t\le t'\le T$, the *Bellman principle*

$$J(t,x,y) = \sup_{\pi^i\in\mathcal{A}}\mathbb{E}^{t,x,y}\left[J\left(t',\widetilde{X}_{t'}^{i,\pi^i},\widetilde{Y}_{t'}^{-i}\right)\right] \tag{7.41}$$

holds (see, for example, Equation (3.20) by Pham, 2009). For the following heuristic derivation of the HJB equation, we assume that $J\in\mathcal{C}^{1,2,2}([0,T]\times(0,\infty)^2)$. Now we can apply the Itô-Doeblin formula (Theorem 2.1) for $J$ on the interval $[t,t']$, in combination with Lemma 2.2, to obtain

$$\begin{aligned}
J\left(t',\widetilde{X}_{t'}^{i,\pi^i},\widetilde{Y}_{t'}^{-i}\right) &= J\left(t,\widetilde{X}_t^{i,\pi^i},\widetilde{Y}_t^{-i}\right) + \int_t^{t'} J_t\left(s,\widetilde{X}_s^{i,\pi^i},\widetilde{Y}_s^{-i}\right)\mathrm{d}s + \int_t^{t'} J_x\left(s,\widetilde{X}_s^{i,\pi^i},\widetilde{Y}_s^{-i}\right)\mathrm{d}\widetilde{X}_s^{i,\pi^i}\\
&+\int_t^{t'} J_y\left(s,\widetilde{X}_s^{i,\pi^i},\widetilde{Y}_s^{-i}\right)\mathrm{d}\widetilde{Y}_s^{-i} + \frac{1}{2}\int_t^{t'} J_{xx}\left(s,\widetilde{X}_s^{i,\pi^i},\widetilde{Y}_s^{-i}\right)\mathrm{d}\langle\widetilde{X}^{i,\pi^i}\rangle_s\\
&+\int_t^{t'} J_{xy}\left(s,\widetilde{X}_s^{i,\pi^i},\widetilde{Y}_s^{-i}\right)\mathrm{d}\langle\widetilde{X}^{i,\pi^i},\widetilde{Y}^{-i}\rangle_s + \frac{1}{2}\int_t^{t'} J_{yy}\left(s,\widetilde{X}_s^{i,\pi^i},\widetilde{Y}_s^{-i}\right)\mathrm{d}\langle\widetilde{Y}^{-i}\rangle_s.
\end{aligned} \tag{7.42}$$

By $J_x$ we denote the first order partial derivative of $J$ with respect to $x$. The other partial derivatives used above are denoted analogously. To simplify notation, the arguments of $J$ and its derivatives are omitted from now on. Continuing (7.42) yields

$$\begin{aligned}
\ldots = J\left(t,\widetilde{X}_t^{i,\pi^i},\widetilde{Y}_t^{-i}\right) &+ \int_t^{t'} J_t\,\mathrm{d}s + \int_t^{t'} J_x\widetilde{X}_s^{i,\pi^i}\pi^i(s)^\top\left(\mu+\frac{1}{n}A\pi^i(s)+\frac{1}{n}\left(1-\frac{\theta_i}{n}\right)A\sum_{j\neq i}\pi^j\right)\mathrm{d}s\\
&+\int_t^{t'} J_x\widetilde{X}_s^{i,\pi^i}\pi^i(s)^\top\sigma\mathrm{d}W(s) + \int_t^{t'} J_y\widetilde{Y}_s^{-i}\Big(\sum_{j\neq i}\pi^j\Big)^\top\sigma\mathrm{d}W(s)\\
&+\int_t^{t'} J_y\widetilde{Y}_s^{-i}\left(\Big(\sum_{j\neq i}\pi^j\Big)^\top\left(\mu+\frac{1}{n}A\sum_{j\neq i}\pi^j\right)+\frac{1}{2}\sum_{\substack{h\neq j\\ h,j\neq i}}(\pi^h)^\top\sigma\sigma^\top\pi^j\right)\mathrm{d}s\\
&+\frac{1}{2}\int_t^{t'} J_{xx}\left(\widetilde{X}_s^{i,\pi^i}\right)^2\pi^i(s)^\top\sigma\sigma^\top\pi^i(s)\mathrm{d}s + \int_t^{t'} J_{xy}\widetilde{X}_s^{i,\pi^i}\widetilde{Y}_s^{-i}\pi^i(s)^\top\sigma\sigma^\top\sum_{j\neq i}\pi^j\mathrm{d}s\\
&+\frac{1}{2}\int_t^{t'} J_{yy}\left(\widetilde{Y}_s^{-i}\right)^2\sum_{h,j\neq i}(\pi^h)^\top\sigma\sigma^\top\pi^j\mathrm{d}s.
\end{aligned} \tag{7.43}$$

Now we insert the previous representation of $J(t', \widetilde{X}_{t'}^{i,\pi^i}, \widetilde{Y}_{t'}^{-i})$, into the Bellman equation (7.41), divide by $t'-t$ and take the limit $t' \to t$, to obtain

$$
\begin{aligned}
0 = G_t + yG_y &\Bigg\{ \Big( \sum_{j\neq i} \pi^j \Big)^\top \Big( \mu + \frac{1}{n} A \sum_{j\neq i} \pi^j \Big) + \frac{1}{2} \sum_{\substack{h\neq j\\ h,j\neq i}} (\pi^h)^\top \sigma\sigma^\top \pi^j \Bigg\} + \frac{1}{2} y^2 G_{yy} \sum_{h,j\neq i} (\pi^h)^\top \sigma\sigma^\top \pi^j \\
&+ \sup_{\pi^i\in\mathbb{R}^d} \Bigg\{ xG_x \cdot (\pi^i)^\top \mu + \frac{1}{n} xG_x \cdot (\pi^i)^\top A \pi^i + xG_x \frac{1}{n}\Big(1-\frac{\theta_i}{n}\Big)(\pi^i)^\top A \sum_{j\neq i}\pi^j \\
&\qquad + \frac{1}{2} x^2 G_{xx} \cdot (\pi^i)^\top \sigma\sigma^\top \pi^i + xyG_{xy} \cdot (\pi^i)^\top \sigma\sigma^\top \sum_{j\neq i} \pi^j \Bigg\}
\end{aligned}
\tag{7.44}
$$

with terminal condition $G(T,x,y) = \Big(1-\frac{1}{\delta_i}\Big)^{-1}\Big(xy^{-\frac{\theta_i}{n}}\Big)^{1-\frac{1}{\delta_i}}$, $x,y\in(0,\infty)$. Note that we assumed that the limit $t'\to t$ and the conditional expectation $\mathbb{E}^{t,x,y}$ can be interchanged. Further, we assumed that the stochastic integrals in (7.43) are martingales so that the conditional expectation vanishes. This is not a restriction as the verification theorem (Lemma B.2 in the Appendix) ensures that a solution to the HJB equation coincides with the value function and provides a unique best response $\pi^{i,*}$.

To determine the supremum in the previous equation, we define the expression inside the supremum as $h(\pi^i)$, i.e.,

$$
\begin{aligned}
&h(\pi^i) \\
&= (\pi^i)^\top \left( xG_x\mu + \left( xG_x \frac{1}{n}\left(1-\frac{\theta_i}{n}\right) A + xyG_{xy}\sigma\sigma^\top \right) \sum_{j\neq i} \pi^j \right) + (\pi^i)^\top \left( \frac{1}{n} xG_x A + \frac{1}{2} x^2 G_{xx} \sigma\sigma^\top \right) \pi^i \\
&=: (\pi^i)^\top \left( xG_x\mu + C_2 \sum_{j\neq i} \pi^j \right) + (\pi^i)^\top C_1 \pi^i,
\end{aligned}
$$

where

$$
C_1 = \frac{1}{n} xG_x A + \frac{1}{2} x^2 G_{xx} \sigma\sigma^\top, \quad C_2 = xG_x \frac{1}{n}\left(1-\frac{\theta_i}{n}\right) A + xyG_{xy}\sigma\sigma^\top.
$$

The gradient of $h$ is then given by

$$
\nabla h(\pi^i) = xG_x\mu + C_2 \sum_{j\neq i}\pi^j + 2C_1\pi^i \stackrel{!}{=} 0.
$$

Moreover, the Hessian matrix of $h$ is given by

$$
H_h(\pi^i) = 2C_1.
$$

Hence, a candidate for the maximizer of $h$ is determined by

$$
C_1\pi^i = -\frac{1}{2} xG_x\mu - \frac{1}{2} C_2 \sum_{j\neq i}\pi^j. \tag{7.45}
$$

Since the matrix $C_1$ depends on the unknown function $G$, we cannot guarantee that $C_1$ is regular.

Therefore, we insert an ansatz for $G$ first. We choose

$$G(t,x,y) = f(t) \cdot \frac{\delta_i}{\delta_i - 1}\left(xy^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i - 1}{\delta_i}} =: f(t)\cdot \widetilde{U}(x,y),\ x,y \in (0,\infty),$$

where $f : [0,T] \to (0,\infty)$ is some continuously differentiable function with $f(T) = 1$. Then the terminal condition is obviously satisfied and $G \in \mathcal{C}^{1,2,2}([0,T],(0,\infty)^2)$. Moreover, we can directly deduce that $G(t,x,y) \neq 0$ for all $t \in [0,T]$ and $x,y \in (0,\infty)$. Let us now find a suitable choice for $f$ to ensure that $G$ solves the HJB equation (7.44). First, we determine the partial derivatives of $G$ with respect to $t$, $x$, and $y$:

$$\begin{aligned}
G_t(t,x,y) &= f'(t)\widetilde{U}(x,y), & G_{xx}(t,x,y) &= -\frac{\delta_i - 1}{\delta_i^2}x^{-2}G(t,x,y),\\
G_x(t,x,y) &= \frac{\delta_i - 1}{\delta_i}x^{-1}G(t,x,y), & G_{xy}(t,x,y) &= -\frac{\theta_i}{n}\left(\frac{\delta_i - 1}{\delta_i}\right)^2 (xy)^{-1}G(t,x,y),\\
G_y(t,x,y) &= -\frac{\theta_i}{n}\frac{\delta_i - 1}{\delta_i}y^{-1}G(t,x,y), & G_{yy}(t,x,y) &= \frac{\theta_i}{n}\frac{\delta_i - 1}{\delta_i}\left(1 + \frac{\theta_i}{n}\frac{\delta_i - 1}{\delta_i}\right)y^{-2}G(t,x,y).
\end{aligned}$$

The matrices $C_1$ and $C_2$ can then be simplified to

$$\begin{aligned}
C_1 &= \frac{1}{n}xG_xA + \frac{1}{2}x^2G_{xx}\sigma\sigma^\top = -\frac{1}{2}\frac{\delta_i - 1}{\delta_i^2}G\cdot\left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right),\\
C_2 &= xG_x\frac{1}{n}\left(1 - \frac{\theta_i}{n}\right)A + xyG_{xy}\sigma\sigma^\top = \frac{\delta_i - 1}{\delta_i}G\cdot\left(\left(1 - \frac{\theta_i}{n}\right)\frac{1}{n}A - \frac{\theta_i}{n}\frac{\delta_i - 1}{\delta_i}\sigma\sigma^\top\right).
\end{aligned}$$

Hence, $C_1$ is regular using assumption b) as well as $G \neq 0$ and $\delta_i \neq 1$. Thus, we can insert $C_1$ and $C_2$ into (7.45) to obtain

$$\begin{aligned}
\pi^{i,*} &= -\frac{1}{2}C_1^{-1}\left(xG_x\mu + C_2\sum_{j\neq i}\pi^j\right)\\
&= \left(\sigma\sigma^\top - \frac{2\delta_i}{n}A\right)^{-1}\left(\delta_i\mu + \left(\frac{\delta_i}{n}\left(1 - \frac{\theta_i}{n}\right)A - \frac{\theta_i}{n}(\delta_i - 1)\sigma\sigma^\top\right)\sum_{j\neq i}\pi^j\right).
\end{aligned}$$

Therefore, the candidate $\pi^{i,*}$ for the optimal strategy of investor $i$ appears to be constant as well. However, we need to verify that $G$ is in fact a solution to the HJB equation (7.44). The first step is to insert $\pi^{i,*}$ into the function $h$. This yields (since $C_1$ is symmetric by construction)

$$\begin{aligned}
h(\pi^{i,*}) &= (\pi^{i,*})^\top\Big(xG_x\mu + C_2\sum_{j\neq i}\pi^j\Big) + (\pi^{i,*})^\top C_1\pi^{i,*}\\
&= -\frac{1}{4}\Big(xG_x\mu + C_2\sum_{j\neq i}\pi^j\Big)^\top C_1^{-1}\Big(xG_x\mu + C_2\sum_{j\neq i}\pi^j\Big)\\
&= \frac{1}{2}(\delta_i - 1)G\cdot\left(\mu + \Big(\frac{1}{n}\Big(1 - \frac{\theta_i}{n}\Big)A - \frac{\theta_i}{n}\frac{\delta_i - 1}{\delta_i}\sigma\sigma^\top\Big)\sum_{j\neq i}\pi^j\right)^\top\Big(\sigma\sigma^\top - \frac{2\delta_i}{n}A\Big)^{-1}\\
&\quad\cdot\left(\mu + \Big(\frac{1}{n}\Big(1 - \frac{\theta_i}{n}\Big)A - \frac{\theta_i}{n}\frac{\delta_i - 1}{\delta_i}\sigma\sigma^\top\Big)\sum_{j\neq i}\pi^j\right)\\
&=: \rho_1\cdot G
\end{aligned}$$

for a constant $\rho_1 \in \mathbb{R}$, since $\pi^j$, $j \neq i$, are constant by assumption. It also follows that $\pi^{i,*}$ is indeed a maximizer of $h$ since the Hessian matrix $H_h(\pi^{i,*}) = 2C_1$ is negative-definite due to assumption b) of the theorem and the fact that

$$-\frac{\delta_i - 1}{\delta_i^2} G = -\frac{1}{\delta_i} f(t) \cdot \left(xy^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i - 1}{\delta_i}} < 0$$

for all $x, y \in (0, \infty)$, $t \in [0, T]$. Moreover, we can simplify the first three summands of (7.44) as follows

$$\begin{aligned}
&G_t = f' \cdot \widetilde{U}, \\
&yG_y \Big\{ \Big(\sum_{j \neq i} \pi^j\Big)^\top \Big(\mu + \frac{1}{n} A \sum_{j \neq i} \pi^j\Big) + \frac{1}{2} \sum_{\substack{h \neq j \\ h,j \neq i}} (\pi^h)^\top \sigma\sigma^\top \pi^j \Big\} = \rho_2 \cdot G, \\
&\frac{1}{2} y^2 G_{yy} \cdot \sum_{h,j \neq i} (\pi^h)^\top \sigma\sigma^\top \pi^j = \rho_3 \cdot G,
\end{aligned}$$

for constants $\rho_2$, $\rho_3 \in \mathbb{R}$ that can be given explicitly but are not of interest for the rest of the proof. Hence, after inserting the ansatz for $G$, the HJB equation (7.44) simplifies to (using $G = f \cdot \widetilde{U}$)

$$0 = f'\widetilde{U} + \rho_2 f\widetilde{U} + \rho_3 f\widetilde{U} + \rho_1 f\widetilde{U} =: \widetilde{U}\,(f' + \rho f) \tag{7.46}$$

for $\rho = \rho_1 + \rho_2 + \rho_3$. Since $\widetilde{U}(x, y) \neq 0$ for all $x, y \in (0, \infty)$, it follows that $f$ needs to solve

$$f'(t) = -\rho f(t),\ t \in [0, T], \tag{7.47}$$

with terminal condition $f(T) = 1$. The unique solution to this ordinary differential equation is given by

$$f(t) = \mathrm{e}^{\rho(T-t)}\, t \in [0, T].$$

Hence, the ansatz for $G$ does in fact yield a solution to the HJB equation if we choose $f$ as in (7.47). Therefore, the function $G : [0, T] \times (0, \infty)^2 \to \mathbb{R}$ given by

$$G(t, x, y) = \mathrm{e}^{\rho(T-t)} \frac{\delta_i}{\delta_i - 1} \left(xy^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i - 1}{\delta_i}}$$

is a solution to the HJB equation. Using Lemma B.2 in the Appendix and (7.45), the unique (up to modifications) optimal solution of the best response problem is given by $\pi^{i,*}$, where $\pi^{i,*}$ solves

$$\left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right) \pi^{i,*} = \delta_i \mu + \left(\frac{\delta_i}{n}\left(1 - \frac{\theta_i}{n}\right) A - \frac{\theta_i}{n}(\delta_i - 1)\sigma\sigma^\top\right) \sum_{j \neq i} \pi^j. \tag{7.48}$$

Moreover, Lemma B.2 implies that the solution $G$ and the value function $J$ are equal which also implies that $G$ is the unique solution to the HJB equation (7.44).

The only task left in order to find the Nash equilibrium is to solve the system of linear equations given by (7.48). First, we add $\left(\frac{\delta_i}{n}\left(1-\frac{\theta_i}{n}\right)A-\frac{\theta_i}{n}(\delta_i-1)\sigma\sigma^\top\right)\pi^i$ on both sides to obtain

$$\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top-\frac{\delta_i}{n}A\right)-\frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)\pi^i=\delta_i\mu+\left(\frac{\delta_i}{n}\left(1-\frac{\theta_i}{n}\right)A-\frac{\theta_i}{n}(\delta_i-1)\sigma\sigma^\top\right)\sum_{j=1}^n\pi^j.$$

By assumption a), the matrix on the left-hand side of the previous equation is regular, so it follows that

$$\begin{aligned}\pi^i=&\delta_i\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top-\frac{\delta_i}{n}A\right)-\frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)^{-1}\mu\\&+\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top-\frac{\delta_i}{n}A\right)-\frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_i}{n}\frac{n-\theta_i}{n}A-\frac{\theta_i}{n}(\delta_i-1)\sigma\sigma^\top\right)\sum_{j=1}^n\pi^j.\end{aligned}$$

Summing over all $i\in\{1,\dots,n\}$ on both sides of the previous equation implies

$$\begin{aligned}\sum_{j=1}^n\pi^j=&\sum_{j=1}^n\delta_j\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\cdot\mu\\&+\sum_{j=1}^n\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\frac{n-\theta_j}{n}A-\frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right)\sum_{j=1}^n\pi^j.\end{aligned}$$

By assumption c), the matrix

$$I_d-\sum_{j=1}^n\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\frac{n-\theta_j}{n}A-\frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right)$$

is regular. Hence, the sum of all $\pi^j$, $j=1,\dots,n$, is given by

$$\begin{aligned}\sum_{j=1}^n\pi^j=&\left(I_d-\sum_{j=1}^n\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\frac{n-\theta_j}{n}A-\frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right)\right)^{-1}\\&\cdot\sum_{j=1}^n\delta_j\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\cdot\mu.\end{aligned}$$

Finally, the unique constant Nash equilibrium is given by

$$\begin{aligned}\pi^{i,*}=&\,\delta_i\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top-\frac{\delta_i}{n}A\right)-\frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)^{-1}\mu\\&+\left(\frac{n+\theta_i}{n}\left(\sigma\sigma^\top-\frac{\delta_i}{n}A\right)-\frac{\theta_i\delta_i}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_i}{n}\frac{n-\theta_i}{n}A-\frac{\theta_i}{n}(\delta_i-1)\sigma\sigma^\top\right)\\&\cdot\left(I_d-\sum_{j=1}^n\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\left(\frac{\delta_j}{n}\frac{n-\theta_j}{n}A-\frac{\theta_j}{n}(\delta_j-1)\sigma\sigma^\top\right)\right)^{-1}\\&\cdot\sum_{j=1}^n\delta_j\left(\frac{n+\theta_j}{n}\left(\sigma\sigma^\top-\frac{\delta_j}{n}A\right)-\frac{\theta_j\delta_j}{n}\sigma\sigma^\top\right)^{-1}\cdot\mu.\end{aligned}$$ □

Similarly to Corollary 7.7, we can deduce the unique constant Nash equilibrium for the case of only one stock ($d=1$).

**Corollary 7.14.** *Let $d = 1$, $A = \alpha$, and assume that the following assumptions hold*

*a)* $(n + \theta_i)\left(n\sigma^2 - \delta_i\alpha\right) - n\theta_i\delta_i\sigma^2 \neq 0$ *for all* $i \in \{1, \dots, n\}$,

*b)* $n\sigma^2 - 2\delta_i\alpha > 0$ *for all* $i \in \{1, \dots, n\}$,

*c)* $\sum_{j=1}^{n} \frac{(n-\theta_j)\alpha\delta_j - n\theta_j(\delta_j-1)\sigma^2}{(n+\theta_j)(n\sigma^2-\alpha\delta_j)-n\theta_j\delta_j\sigma^2} \neq 1$.

*Then the unique constant Nash equilibrium to* (7.37) *in terms of invested fractions is given by*

$$\begin{aligned}\pi^{i,*} =& \frac{n^2\delta_i\mu}{(n+\theta_i)(n\sigma^2-\delta_i\alpha)-n\theta_i\delta_i\sigma^2} + \frac{(n-\theta_i)\alpha\delta_i - n\theta_i(\delta_i-1)\sigma^2}{(n+\theta_i)(n\sigma^2-\delta_i\alpha)-n\theta_i\delta_i\sigma^2} \\ &\cdot \left(1 - \sum_{j=1}^{n} \frac{(n-\theta_j)\alpha\delta_j - n\theta_j(\delta_j-1)\sigma^2}{(n+\theta_j)(n\sigma^2-\alpha\delta_j)-n\theta_j\delta_j\sigma^2}\right)^{-1} \sum_{j=1}^{n} \frac{n^2\delta_j\mu}{(n+\theta_j)(n\sigma^2-\delta_j\alpha)-n\theta_j\delta_j\sigma^2}\end{aligned}$$

*for* $i \in \{1, \dots, n\}$.

Subsequent to Corollary 7.7, we analyzed the influence of the price impact parameter $\alpha$ on the entries of the constant Nash equilibrium in terms of invested amounts under exponential utility. Due to the more complicated structure of the constant Nash equilibrium in Corollary 7.14, we do not discuss the influence of $\alpha$ as detailed as in Subsection 7.3.2. However, Corollary 7.14 enables us to compute the value of a component of the Nash equilibrium for specific parameter choices. The results are illustrated in Figure 7.4.1.

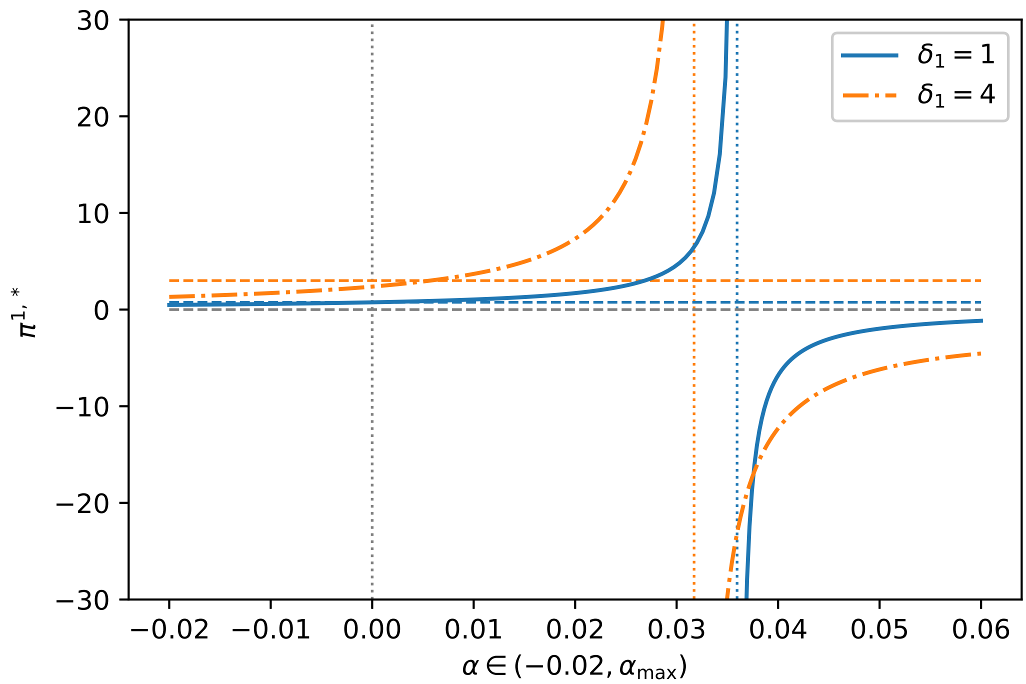



Figure 7.4.1.: Illustration of $\pi^{1,*}$ from Corollary 7.14 in terms of $\alpha \in (-0.02, \alpha_{\max})$ for $n = 12$, $\mu = 0.03$, $\sigma = 0.2$, and $\alpha_{\max} = n\sigma^2/8$. Further, $\theta_1 = 0.3$, $\delta_1 \in \{1, 4\}$ and the parameters $\theta_j$ and $\delta_j$, $j \geq 2$ are increasing from 0 to 1 with step size 0.1, and from 0.5 to 2.7 by step size 0.2, respectively.

Figure 7.4.1 displays the first component $\pi^{1,*}$ of the constant Nash equilibrium given in Corollary 7.14 in terms of $\alpha$ varying between $-0.02$ and $\alpha_{\max}$ for the two different risk tolerance parameters $\delta_1 = 1$ and $\delta_1 = 4$. The expression $\alpha_{\max}$ is defined analogously to Subsection 7.3.2 as

$$\alpha_{\max} = \frac{n\sigma^2}{2\delta_{\max}},$$

where $\delta_{\max} = \max\{\delta_1, \ldots, \delta_n\}$. In the example displayed in Figure 7.4.1, we used $\delta_{\max} = 4$. The market parameters are chosen as $\mu = 0.03$ and $\sigma = 0.2$. Note that all considered parameter combinations satisfy the conditions of Corollary 7.14. Similar to Figure 7.3.1, we observe a discontinuity of $\pi^{1,*}$. In Subsection 7.3.2, we provided a detailed discussion of the existence of a unique point of discontinuity. Here, we only give a short explanation regarding the discontinuity. For the specific parameter choices used in the example, conditions a) and b) of Corollary 7.14 are always satisfied. The discontinuity is due to condition c), i.e., for both parameter choices $\delta_1 \in \{1, 4\}$, there exists a unique value $\alpha_0 \in (-\infty, \alpha_{\max})$ such that the expression in condition c) is zero. In the figure, the value $\alpha_0$ is highlighted by a vertical dotted line for each of the two parameter choices $\delta_1 \in \{1, 4\}$. Moreover, the blue and orange horizontal dashed lines mark the Merton ratio $\delta_1\mu\sigma^{-2}$, i.e., the unique optimally invested fraction in the associated classical problem ($\alpha = 0$, $\theta_1 = 0$), for the two different values used for $\delta_1$. Finally, we highlighted the value zero on both axes by a grey line.

Considering the behavior of $\pi^{1,*}$ in terms of $\alpha$, we notice that $\pi^{1,*}$ is strictly positive for $\alpha < \alpha_0$ and strictly negative for $\alpha > \alpha_0$. Moreover, we observe that for larger price impact (i.e., if the absolute value of $\alpha$ increases), the agents engage less in the financial market which is represented by a decrease in the absolute value of $\pi^{1,*}$. Overall, we notice the same behavior of $\pi^{1,*}$ in terms of $\alpha$ as in the case of exponential utility which we considered in Subsection 7.3.2.

# CHAPTER 8

# Relative performance via a VaR-type constraint

In Chapters 3 – 7, we modeled the relative concerns of investors by including a relative performance metric into their utility function. In this chapter, we take a different approach by adapting a method introduced by Basak and Shapiro (2001). They incorporated risk constraints into the classical utility maximization problem. As a risk measure, they chose the value at risk (VaR) which is a standard criterion to assess risk in the financial industry (see, e.g., Berkelaar et al., 2002). The value at risk at level $1-\alpha \in (0,1)$ of some payoff $X$ describes the „smallest amount of capital which, if added to $X$ and invested in the risk-free asset, keeps the probability of a negative outcome below the level $1-\alpha$". Mathematically speaking, the value at risk at level $1-\alpha$ of a payoff $X$ is defined as

$$\mathrm{VaR}_{1-\alpha}(X) = \inf\Big\{m \in \mathbb{R} \,\big|\, \mathbb{P}(X+m<0) \leq 1-\alpha\Big\}.$$

Thus, $\mathrm{VaR}_{1-\alpha}(X)$ describes the lower $\alpha$–quantile[1] of $-X$ (see Föllmer and Schied, 2016, p. 231). Basak and Shapiro (2001) define the value at risk slightly different by considering the loss of wealth suffered by some portfolio over a fixed time period. They describe it as the worst loss over a given time interval under „normal market conditions". The term „normal market conditions" refers to the fact that the level $1-\alpha$ is usually chosen as a small percentage, for example 5% or 1%. Hence, the loss is bounded from above in a fraction of $\alpha$ of all possible market scenarios and, for small values of $1-\alpha$, this fraction is close to 1. They specify their definition by explaining the value at risk at level $1-\alpha$ for some wealth process $\big(X(t)\big)_{t\in[0,T]}$ over the time period $[0,T]$ as the real number $\mathrm{VaR}_{1-\alpha}\big(X(T)-X(0)\big)$ for which

$$\mathbb{P}\Big(X(0)-X(T) \leq \mathrm{VaR}_{1-\alpha}\big(X(T)-X(0)\big)\Big) = \alpha.$$

[1] Note that $\alpha$ is usually chosen as the level of the value at risk instead of $1-\alpha$. However, it simplifies the notation for the later analysis to choose $1-\alpha$ instead of $\alpha$ as the level of the value at risk.

If the cumulative distribution function of $X(T) - X(0)$ is strictly increasing and continuous on its support, this is simply the $\alpha$-quantile of $-\big(X(T) - X(0)\big)$. Note that if the cumulative distribution function is strictly increasing, the upper and lower quantiles[2] are equal and we simply refer to them as quantiles. Thus, the definition of Basak and Shapiro (2001) coincides with the standard definition for the payoff $X(T) - X(0)$ if the corresponding cumulative distribution function is strictly increasing and continuous on its support.

Basak and Shapiro (2001) chose the following approach to incorporate risk management concerns into the standard utility maximization problem. To keep the risk below a certain level, they bounded the value at risk from above by some endogenously given constant $K > 0$. Thus, they considered the constraint

$$\mathrm{VaR}_{1-\alpha}\big(X(T) - X(0)\big) \leq X(0) - K,$$

which entails

$$\alpha = \mathbb{P}\Big(X(0) - X(T) \leq \mathrm{VaR}_{1-\alpha}\big(X(T) - X(0)\big)\Big) \leq \mathbb{P}(X(T) \geq K).$$

Therefore, by including the additional constraint $\mathbb{P}(X(T) \geq K) \geq \alpha$ into the utility maximization problem, they bounded the risk arising from the portfolio while maximizing the expected utility of the terminal wealth. In support of the risk management feature of the problem, they explain that the constraint acts as a partial portfolio insurance which lies somewhere between the benchmark cases of the classical problem without the constraint ($\alpha = 0$) and the case of a portfolio insurer ($\alpha = 1$).

In order to adapt the model of Basak and Shapiro (2001) to include relative concerns, we replace the constant $K$ in the optimization problem of agent $i$ by a weighted arithmetic mean of the other $n-1$ agents' terminal wealth. Thus, we incorporate the constraint

$$\mathbb{P}\Big(X_i(T) \geq \sum_{j \neq i} \beta_j X_j(T)\Big) \geq \alpha_i \tag{8.1}$$

into the optimization problem of agent $i$, where $\alpha_i \in [0,1]$ and $\beta_j \in (0,1)$ for all $i, j \in \{1, \ldots, n\}$. By $X_j(T)$ we denote the terminal wealth of agent $j \in \{1, \ldots, n\}$. Under normal market conditions, i.e., in a fraction of $\alpha_i$ of the possible market scenarios, agent $i$ attains a terminal wealth which is at least as large as the weighted average of her competitors' terminal wealth. As opposed to the optimization problems considered in Chapters 3 – 7, the objective function of agent $i$ is given by the standard objective, i.e., the expected utility of her terminal wealth only.

The parameters $\beta_j$ in (8.1) are custom to each agent $j$, but not to agent $i$, meaning that the weight assigned to agent $j$ is the same in the optimization of agent $i$ for any $i \neq j$. A possible choice is $\beta_j = \frac{1}{n-1}$, but we allow for more generality at this point. It is, for example, possible to consider weights in terms of the initial capital invested by the agents so that a larger initial investment goes along with a larger weight assigned to the corresponding agent. In contrast to the parameters

[2]For a random variable $X$ and $\lambda \in (0,1)$, the lower and upper $\lambda$-quantiles are $q_X^-(\lambda) = \inf\{x \in \mathbb{R} \,|\, \mathbb{P}(X \leq x) \geq \lambda\}$ and $q_X^+(\lambda) = \inf\{x \in \mathbb{R} \,|\, \mathbb{P}(X \leq x) > \lambda\}$, respectively (see, e.g., Föllmer and Schied, 2016).

$\beta_j$, $j = 1, \ldots, n$, the level $\alpha_i \in [0, 1]$ is chosen by agent $i$ herself and has a similar interpretation as the competition weight $\theta_i$ used in earlier chapters. If $\alpha_i$ is chosen close to 1, agent $i$ wants to insure her terminal wealth against the other agents' wealth in almost all possible scenarios, while a value $\alpha_i$ close to 0 implies that she does not care as much about her performance with respect to the others. However, a large choice of $\alpha_i$ indicates risk aversion (since the agent bounds her risk in a larger percentage of cases) whereas a large choice of $\theta_i$ turns out to be more risk-seeking.

To conclude, let us give an overview of the chapter. In Section 8.1, we introduce the underlying financial market and state the optimization problem for $n$ agents using the VaR-based constraint (8.1). We present the optimal solution to the best response problem in terms of terminal wealth in Section 8.2. Due to the complicated structure of the optimal solution, we restrict the set of strategies to be able to find Nash equilibria. This gives rise to an $n$-dimensional fixed point problem. In Section 8.3, we solve the fixed point problem for two agents. Afterwards, the unique fixed point for a general number of agents is found and discussed in Section 8.4.

## 8.1. Problem formulation

In order to formalize the optimization problem described above, let us specify the underlying financial market first. The financial market is the same as the one used in Chapter 6. Nevertheless, we repeat the basics at this point. We base our analysis on the semimartingale financial market explained in Subsection 2.3.1. In summary, there are $d + 1$ assets in which $n$ agents can invest. The assets consist of one riskless bond with zero interest rate and $d$ risky stocks. The stock price processes $(S_k(t))_{t\in[0,T]}$, $k = 1, \ldots, d$, are $L^2(\mathbb{P})$–semimartingales with càdlàg paths (i.e., the paths are right continuous with left limits). To exclude arbitrage, we require the existence of an equivalent $\sigma$-martingale measure $\mathbb{Q}$. In contrast to Subsection 2.3.1, we make the additional assumption that $\mathbb{Q}$ is the *unique* equivalent $\sigma$-martingale measure. Under this assumption, the associated density process

$$Z_t^{\mathbb{Q}} = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|\mathcal{F}_t\right], \, t \in [0, T],$$

is unique as well. Thus, we write $Z_t := Z_t^{\mathbb{Q}}$, $t \in [0, T]$, throughout this chapter. The assumption that $\mathbb{Q}$ is the unique equivalent $\sigma$-martingale measure has the important consequence that each claim $X \in L^2(\mathbb{P})$ has a *unique* fair price given by

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}\left[Z_T X\right].$$

Let us now formalize the optimization problem motivated in the introduction to this chapter. Let $\beta_i \in (0, 1)$ and $\alpha_i \in [0, 1]$ for all $i \in \{1, \ldots, n\}$. The constant $\beta_j$ describes a weight assigned to the $j$-th agent's wealth in the stochastic constraint and $\alpha_i$ describes the probability assigned to the stochastic constraint in the optimization problem of agent $i$. Each agent $i$ uses an Inada utility function $U_i : (0, \infty) \to \mathbb{R}$ to measure her preferences (see Definition 2.11). Note that the requirements on the utility functions are a lot stricter than the ones considered in Chapter 3. First, $U_i$ is required to satisfy the Inada conditions and second, we only allow for utility functions defined on the strictly positive real numbers. However, the proofs presented in the current chapter rely heavily on both assumptions imposed on $U_1, \dots, U_n$. Fortunately, this requirement is not too restricting as there are commonly used utility functions such as the natural logarithm or power utility functions that satisfy the conditions. Finally, we assume that each agent $i$ is equipped with an initial capital $x_0^i > 0$, $i = 1, \dots, n$.

Following the motivation in the introduction to this chapter, agent $i \in \{1, \dots, n\}$ aims to solve the optimization problem

$$\begin{cases} & \max_{\varphi^i \in \mathcal{A}} \mathbb{E}\left[U_i(X_T^{i,\varphi^i})\right], \\ \text{s.t.} & X_T^{i,\varphi^i} = x_0^i + (\varphi^i \bullet S)_T,\ \mathbb{P}\Big(X_T^{i,\varphi^i} \geq \sum_{j \neq i} \beta_j X_T^{j,\varphi^j}\Big) \geq \alpha_i. \end{cases}$$

Similar to Chapter 6, we only consider the terminal wealth, not the replicating strategy. If the financial market is additionally assumed to be complete, replicating strategies do exist for every terminal wealth determined in this chapter. However, we do not consider the replicating strategies and leave this question open for future research. Thus, we state the optimization problem of agent $i$ directly in terms of her terminal wealth. Hence, the optimization problem of agent $i \in \{1, \dots, n\}$ is given as

$$\begin{cases} & \max_{X_i} \mathbb{E}\left[U_i(X_i)\right], \\ \text{s.t.} & X_i \text{ is } \mathcal{F}_T - \text{measurable},\ \mathbb{E}\left[Z_T X_i\right] \leq x_0^i,\ \mathbb{P}\left(X_i \geq \sum_{j \neq i} \beta_j X_j\right) \geq \alpha_i, \end{cases} \tag{8.2}$$

where $X_j$ denotes the terminal wealth of investor $j \in \{1, \dots, n\}$.

*Remark* 8.1. In Chapter 3, we displayed a method to solve portfolio optimization problems for relative investors who include their relative concerns into their expected utility function via the additive relative performance metric. Because of the arithmetic mean in the constraint (8.1), it would be possible to adjust (8.2) to make the method from Chapter 3 applicable. However, in order to be able to reduce the problem to an auxiliary single-agent problem, we would also have to adjust the argument in the objective function of (8.2) to match the constraint. Since this does not fit the motivation presented in the introduction to this chapter, we did not choose this approach. However, the resulting optimization problem seems more tractable when compared to (8.2). $\diamondsuit$

## 8.2. Discussion of terminal wealth

Let us consider the optimization problem (8.2) faced by agent $i \in \{1, \dots, n\}$. In the following, we state and prove the optimal solution to (8.2) for agent $i$ while the terminal wealth $X_j$, $j \neq i$, of the other agents is assumed to be fixed. To simplify notation, we introduce the abbreviation

$$X_\beta^{-i} \coloneqq \sum_{j \neq i} \beta_j X_j,\ i \in \{1, \dots, n\}.$$

Now the following proposition contains the optimal terminal wealth for agent $i$ in the optimization problem (8.2).

**Proposition 8.2.** *Let $\alpha_j \in [0,1]$, $\beta_j \in (0,1)$, and $x_0^j > 0$, $j = 1,\dots,n$. Further, let $i \in \{1,\dots,n\}$ and assume that $X_j$, $j \neq i$, are fixed and $\mathbb{P}$-almost surely positive. Moreover, assume that $U_i : (0,\infty) \to \mathbb{R}$ is an Inada utility function and let $I_i := (U_i')^{-1}$. Define the function $g_i : (0,\infty) \to \mathbb{R}$ by*

$$g_i(Z) := U_i\big(I_i(Z)\big) - ZI_i(Z) + ZX_\beta^{-i} \tag{8.3}$$

*and define*

$$X_i^* = \begin{cases} I_i\Big(\lambda_i^{(1)} Z_T\Big), & \lambda_i^{(1)} Z_T \le U_i'\big(X_\beta^{-i}\big), \\ X_\beta^{-i}, & U_i'\big(X_\beta^{-i}\big) < \lambda_i^{(1)} Z_T \le \bar{\xi}_i, \\ I_i\Big(\lambda_i^{(1)} Z_T\Big), & \lambda_i^{(1)} Z_T > \bar{\xi}_i, \end{cases} \tag{8.4}$$

*where $\bar{\xi}_i$ is chosen such that $g_i(\bar{\xi}_i) = \lambda_i^{(2)} + U_i\big(X_\beta^{-i}\big)$ and $\bar{\xi}_i > U_i'\big(X_\beta^{-i}\big)$ $\mathbb{P}$-almost surely, and $\lambda_i^{(1)}$, $\lambda_i^{(2)} > 0$ are chosen with respect to $\mathbb{E}[Z_T X_i^*] = x_0^i$ and $\mathbb{P}\big(X_i^* \ge X_\beta^{-i}\big) = \alpha_i$. We assume that $\bar{\xi}_i$, $\lambda_i^{(1)}$, and $\lambda_i^{(2)}$ exist. Then $X_i^*$ is the optimal solution to* (8.2).

*Proof.* Let $i \in \{1,\dots,n\}$ be arbitrary but fixed and define

$$L(X, z_1, z_2) := U_i(X) - z_1 X + z_2 \mathbb{1}\left\{X \ge X_\beta^{-i}\right\}.$$

For fixed $z_1, z_2 > 0$, the function $X \mapsto L(X, z_1, z_2)$ is increasing if, and only if, $X \le I_i(z_1)$. Moreover, $L(\cdot, z_1, z_2)$ is piecewise concave with a jump located at $X_\beta^{-i}$. Therefore, the maximizer is located either at $I_i(z_1)$ or at $X_\beta^{-i}$. It turns out that

$$X_i = I_i(z_1)\mathbb{1}\left\{z_1 \le U_i'\big(X_\beta^{-i}\big)\right\} + X_\beta^{-i}\mathbb{1}\left\{U_i'\big(X_\beta^{-i}\big) < z_1 \le \xi_z\right\} + I_i(z_1)\mathbb{1}\left\{z_1 > \xi_z\right\},$$

is the unique maximizer of $X \mapsto L(X, z_1, z_2)$, where $\xi_z$ is chosen such that $g_i(\xi_z) = U_i\big(X_\beta^{-i}\big) + z_2$ and $\xi_z > U_i'\big(X_\beta^{-i}\big)$ $\mathbb{P}$-almost surely. At this point, we assume that such a value $\xi_z$ exists. To prove this assertion, we use a case distinction with respect to $z_1$. First, let $z_1 \le U_i'\big(X_\beta^{-i}\big)$ and thus, $I_i(z_1) \ge X_\beta^{-i}$. Since the function $g_i$ from (8.3) is increasing if, and only if, $Z \ge U_i'\big(X_\beta^{-i}\big)$, we obtain

$$\begin{aligned} L(I_i(z_1), z_1, z_2) &= U_i\big(I_i(z_1)\big) - z_1 I(z_1) + z_2 \\ &= g_i(z_1) - z_1 X_\beta^{-i} + z_2 \\ &\ge g_i\big(U_i'\big(X_\beta^{-i}\big)\big) - z_1 X_\beta^{-i} + z_2 \\ &= U_i\big(X_\beta^{-i}\big) - z_1 X_\beta^{-i} + z_2 = L\big(X_\beta^{-i}, z_1, z_2\big). \end{aligned}$$

For $U_i'\big(X_\beta^{-i}\big) < z_1 \le \xi_z$, it follows

$$\begin{aligned} L\big(X_\beta^{-i}, z_1, z_2\big) &= U_i\big(X_\beta^{-i}\big) - z_1 X_\beta^{-i} + z_2 \\ &= g_i(\xi_z) - z_1 X_\beta^{-i} \\ &\ge g_i(z_1) - z_1 X_\beta^{-i} \\ &= U_i\big(I_i(z_1)\big) - z_1 I(z_1) = L(I(z_1), z_1, z_2), \end{aligned} \tag{8.5}$$

using the monotonicity of $g_i$ and the definition of $\xi_z$. Finally, if $z_1 > \xi_z$, the reverse of (8.5) holds. Thus, $X_i$ is the unique maximizer of $L(\cdot, z_1, z_2)$.

To prove that $X_i^*$ from (8.4) is optimal for (8.2), assume that $\widetilde{X}$ is another random variable satisfying the constraints of (8.2). Then it follows

$$\begin{aligned}
&\mathbb{E}\Big[U_i(X_i^*)\Big] - \mathbb{E}\Big[U_i(\widetilde{X})\Big] \\
=&\,\mathbb{E}\Big[U_i(X_i^*)\Big] - \mathbb{E}\Big[U_i(\widetilde{X})\Big] - \lambda_i^{(1)} x_0^i + \lambda_i^{(1)} x_0^i + \lambda_i^{(2)} \alpha_i - \lambda_i^{(2)} \alpha_i \\
\geq&\,\mathbb{E}\Big[U_i(X_i^*) - \lambda_i^{(1)} Z_T X_i^* + \lambda_i^{(2)} \mathbb{1}\left\{X_i^* \geq X_\beta^{-i}\right\}\Big] - \mathbb{E}\Big[U_i(\widetilde{X}) - \lambda_i^{(1)} Z_T \widetilde{X} + \lambda_i^{(2)} \mathbb{1}\left\{\widetilde{X} \geq X_\beta^{-i}\right\}\Big] \\
=&\,\mathbb{E}\Big[L\left(X_i^*, \lambda_i^{(1)} Z_T, \lambda_i^{(2)}\right) - L\left(\tilde{X}, \lambda_i^{(1)} Z_T, \lambda_i^{(2)}\right)\Big] \\
\geq&\, 0,
\end{aligned}$$

since, by construction and the previous observation, $X_i^*$ satisfies the constraints from (8.2) with equality and maximizes the function $L\Big(\cdot, \lambda_i^{(1)} Z_T, \lambda_i^{(2)}\Big)$ (pointwise for every $\omega \in \Omega$). This concludes our proof since $\bar{\xi}_i$, $\lambda_i^{(1)}$, and $\lambda_i^{(2)}$ exist by assumption. $\square$

The representation (8.4) of the best response for agent $i$ brings some serious difficulties. In general, existence and uniqueness of the expressions $\bar{\xi}_i$, $\lambda_i^{(1)}$, $\lambda_i^{(2)}$ are not clear. Moreover, we do not expect to be able to find analytical representations for these expressions, even in special cases like logarithmic utility. Thus, in order to find Nash equilibria for (8.2), we restrict the set of possible terminal wealth, although we might lose optimality compared to the more general set of admissible wealth profiles. Nevertheless, we only consider wealth profiles for agent $i \in \{1, \dots, n\}$ which are of the form

$$X_i = \begin{cases} I_i(\lambda_i Z_T), & I_i(\lambda_i Z_T) \geq X_\beta^{-i},\, Z_T \leq \chi_{\alpha_i}, \\ X_\beta^{-i}, & I_i(\lambda_i Z_T) < X_\beta^{-i},\, Z_T \leq \chi_{\alpha_i}, \\ I_i(\lambda_i Z_T), & Z_T > \chi_{\alpha_i}, \end{cases} \tag{8.6}$$

where $\chi_{\alpha_i}$ is the $\alpha_i$-quantile of $Z_T$ and $X_\beta^{-i} = \sum_{j \neq i} \beta_j X_j$, $i = 1, \dots, n$. The random variables $X_j$, $j \neq i$, describe the terminal wealth of the (other) agents $j \neq i$. The constant $\lambda_i > 0$ is chosen so that $\mathbb{E}[Z_T X_i^*] = x_0^i$. Thus, $\lambda_i$ is often called Lagrange multiplier. For now, we assume that the parameters of the model are chosen such that $\lambda_i$ exists and is unique. In general, this is the case if the initial capital is sufficiently large. In Section 8.3 as well as Section C.3 in the appendix, we give necessary and sufficient conditions for the existence and an explicit representation of the unique $\lambda_i$ in the special case of logarithmic utility and $n \in \{2, 3\}$. Moreover, it should be noted that, by construction, $X_i$ satisfies the constraints of (8.2).

*Remark* 8.3. a) The structure of (8.6) is motivated by the unique optimal terminal wealth found by Basak and Shapiro (2001). They consider the optimization problem

$$\begin{cases} & \max_X \mathbb{E}[U(X)], \\ \text{s.t.} & \mathbb{E}[Z_T X] \leq x_0,\, \mathbb{P}(X \geq K) \geq \alpha, \end{cases} \tag{8.7}$$

for constants $K > 0$, $x_0 > 0$, and $\alpha \in [0, 1]$, and an Inada utility function $U : (0, \infty) \to \mathbb{R}$. They prove that the optimal solution to (8.7) takes the form

$$X^* = \begin{cases} I(\lambda Z_T), & Z_T \leq \frac{U'(K)}{\lambda}, \\ K, & \frac{U'(K)}{\lambda} < Z_T \leq \chi_\alpha, \\ I(\lambda Z_T), & Z_T > \chi_\alpha, \end{cases} \tag{8.8}$$

where $\chi_\alpha$ denotes the $\alpha$-quantile of $Z_T$ and $I$ denotes the inverse of the first order derivative $U'$ of $U$. Moreover, $\lambda > 0$ is chosen with respect to the budget equation $\mathbb{E}[Z_T X^*] = x_0$.

b) Due to the structure of the optimal terminal wealth (8.8), which divides the solution into three cases depending on the value of the state price density $Z_T$, Basak and Shapiro (2001) call these three cases *good*, *intermediate*, and *bad states.* The good states are associated to smaller values of $Z_T$ while the bad states correspond to larger values of $Z_T$. However, this interpretation is not based on the optimal solution to the VaR-based problem (8.7) alone. It can also be justified in more generality. If, for example, we take a standard Black-Scholes market with one stock, one riskless bond with zero interest rate, and constant market parameters (see, e.g., Eberlein and Kallsen, 2019, Examples 9.1 and 9.17), the state price density at time $T$ is given by

$$Z_T = \exp\Big( -\frac{\mu}{\sigma} W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2} T\Big).$$

The stock value at time $T$ is given by

$$S_T = \exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)T + \sigma W_T\Big),$$

which can be rewritten as

$$S_T = \exp\Big( -\frac{\sigma^2}{\mu}\log(Z_T) + \frac{1}{2}\left(\mu - \sigma^2\right)T\Big).$$

Hence, large values of $Z_T$ result in small stock values which justifies the association of large values of $Z_T$ with bad market states.

◊

Since we aim to find Nash equilibria in the class of terminal wealth profiles of the form (8.6), we need to solve the emerging $n$-dimensional fixed point problem to find explicit representations of the components $X_1, \ldots, X_n$ in the Nash equilibrium. The search for and discussion of the $n$-dimensional fixed point is the focus of the subsequent Sections 8.3 and 8.4.

## 8.3. Solution of the fixed point problem for two agents

In this section, we set $n = 2$ and determine the unique two-dimensional Nash equilibrium in the class of wealth profiles of the form (8.6). Solving the fixed point problem for general $n$ is quite complicated, so for illustration we solve it for only two agents at first. Afterwards, Section 8.4 gives the general solution for $n$ agents.

The following theorem displays the unique solution to the fixed point problem (8.6) for two agents.

**Theorem 8.4.** *Let $U_i : (0, \infty) \to \mathbb{R}$ be Inada utility functions, $i = 1, 2$. Moreover, for $\alpha_i \in [0, 1]$, let $\chi_{\alpha_i}$ describe the $\alpha_i$–quantile of $Z_T$ and let $\beta_i \in (0, 1)$, $i = 1, 2$. Define*

$$X_1^*(\lambda_1, \lambda_2) = \begin{cases} I_1(\lambda_1 Z_T), & I_1(\lambda_1 Z_T) \geq \beta_2 I_2(\lambda_2 Z_T),\ Z_T \leq \chi_{\alpha_1}, \\ \beta_2 I_2(\lambda_2 Z_T), & I_1(\lambda_1 Z_T) < \beta_2 I_2(\lambda_2 Z_T),\ Z_T \leq \chi_{\alpha_1}, \\ I_1(\lambda_1 Z_T), & Z_T > \chi_{\alpha_1}, \end{cases}$$

$$X_2^*(\lambda_1, \lambda_2) = \begin{cases} I_2(\lambda_2 Z_T), & I_2(\lambda_2 Z_T) \geq \beta_1 I_1(\lambda_1 Z_T),\ Z_T \leq \chi_{\alpha_2}, \\ \beta_1 I_1(\lambda_1 Z_T), & I_2(\lambda_2 Z_T) < \beta_1 I_1(\lambda_1 Z_T),\ Z_T \leq \chi_{\alpha_2}, \\ I_2(\lambda_2 Z_T), & Z_T > \chi_{\alpha_2}, \end{cases}$$

*and assume that the system*

$$\mathbb{E}[Z_T X_i^*(\lambda_1, \lambda_2)] = x_0^i,\ i = 1, 2, \tag{8.9}$$

*has a unique solution $(\lambda_1^*, \lambda_2^*) \in (0, \infty)^2$. Then $X_i^*(\lambda_1^*, \lambda_2^*)$, $i = 1, 2$, is the unique Nash equilibrium for* (8.2) *in the class of wealth profiles of the form* (8.6).

*Remark* 8.5. a) Since $\beta_1, \beta_2 \in (0, 1)$ and $I_1$ and $I_2$ take only strictly positive values, it cannot happen that $I_1(\lambda_1 Z_T) < \beta_2 I_2(\lambda_2 Z_T)$ and $I_2(\lambda_2 Z_T) < \beta_1 I_1(\lambda_1 Z_T)$ hold at the same time. Thus, we notice that for any valid choice of parameters, utility functions, and for any realization of $Z_T$, at least one of the random variables $X_j^*$ is given by $I_j(\lambda_j Z_T)$. Note that, although the structure is the same, $I_j(\lambda_j Z_T)$ does not necessarily describe the solution of the classical problem without the VaR-constraint, since the Lagrange multiplier might be different from the one in the classical problem. It is also possible that both agents obtain the classical solution in the Nash equilibrium. This is the case if $I_1(\lambda_1 Z_T)$ and $I_2(\lambda_2 Z_T)$ are „close“ in the sense that $\beta_2 I_2(\lambda_2 Z_T) \leq I_1(\lambda_1 Z_T) \leq \beta_1^{-1} I_2(\lambda_2 Z_T)$. This observation becomes more tangible in the case where both investors use the natural logarithm as their utility function (see Remark 8.10).

b) The search for a solution of the system (8.9) of equations consists of two steps. First, we need to determine the expected value to receive a two-dimensional system of equations. Afterwards, we need to solve the emerging system of equations (which is most likely nonlinear) and make sure that there exists an admissible solution in the sense that $\lambda_i^* > 0$, $i = 1, 2$. If, for example, $x_0^j \geq \beta_{3-j} x_0^{3-j}$, $j = 1, 2$, we can ensure that for any fixed $\lambda_j > 0$ the equation $\mathbb{E}[Z_T X_{3-j}^*(\lambda_1, \lambda_2)] = x_0^{3-j}$ has a unique solution $\lambda_{3-j}^*$, depending on $\lambda_j$, using the monotonicity and continuity in terms of $\lambda_{3-j}$ and the intermediate value theorem. However, this does not ensure that the emerging two-dimensional system of equations is (uniquely) solvable. It should also be noted that this condition is sufficient, but in general not necessary. Lemma 8.8 gives necessary and sufficient conditions for the parameter choice in the case of logarithmic utility for both investors. ◊

Before we provide a proof of Theorem 8.4, we apply the result to the case that both agents use power utility functions. The example gives us more insight into the case distinction in the Nash equilibrium $(X_1^*, X_2^*)$.

*Example* 8.6. Suppose that both agents use power utility functions with parameters $\delta_i > 0$, $\delta_i \neq 1$, i.e.,

$$U_i(x) = \left(1 - \frac{1}{\delta_i}\right)^{-1} x^{1-\frac{1}{\delta_i}},\ I_i(x) = x^{-\delta_i},\ x > 0,\ i = 1, 2.$$

Assume without loss of generality that $\delta_2 > \delta_1$. Then

$$I_1(\lambda_1 Z_T) \geq \beta_2 I_2(\lambda_2 Z_T) \iff Z_T \geq \left(\beta_2 \frac{\lambda_1^{\delta_1}}{\lambda_2^{\delta_2}}\right)^{\frac{1}{\delta_2-\delta_1}} =: z_1,$$
$$I_2(\lambda_2 Z_T) \geq \beta_1 I_1(\lambda_1 Z_T) \iff Z_T \leq \left(\beta_1 \frac{\lambda_2^{\delta_2}}{\lambda_1^{\delta_1}}\right)^{\frac{1}{\delta_1-\delta_2}} =: z_2.$$

Since $\beta_1\beta_2 < 1$ and $\lambda_1, \lambda_2 > 0$, we obtain $z_1 < z_2$. Hence, we obtain the Nash equilibrium

$$\begin{aligned} X_1^* =& \beta_2 I_2(\lambda_2 Z_T) \mathbb{1}\left\{Z_T < z_1 \wedge \chi_{\alpha_1}\right\} + I_1(\lambda_1 Z_T) \mathbb{1}\left\{Z_T \geq z_1 \wedge \chi_{\alpha_1}\right\}, \\ X_2^* =& I_2(\lambda_2 Z_T) \mathbb{1}\left\{Z_T \leq z_2 \wedge \chi_{\alpha_2}\right\} + \beta_1 I_1(\lambda_1 Z_T) \mathbb{1}\left\{z_2 \wedge \chi_{\alpha_2} < Z_T \leq \chi_{\alpha_2}\right\} \\ &+ I_2(\lambda_2 Z_T) \mathbb{1}\left\{Z_T > \chi_{\alpha_2}\right\}. \end{aligned}$$

Note that $\delta_2 > \delta_1$ has the interpretation of investor 2 being more risk-taking than investor 1. If $\delta_1 = \delta_2 = \delta$, then

$$I_1(\lambda_1 Z_T) \geq \beta_2 I_2(\lambda_2 Z_T) \iff \lambda_2 \geq \beta_2^{1/\delta} \lambda_1,$$
$$I_2(\lambda_2 Z_T) \geq \beta_1 I_1(\lambda_1 Z_T) \iff \lambda_1 \geq \beta_1^{1/\delta} \lambda_2,$$

which shows that the case distinction does not depend on $Z_T$ but solely on the parameters chosen by the investors. Hence, we obtain the following case distinction for the Nash equilibrium $(X_1^*, X_2^*)$.

a) If $\lambda_1 \geq \beta_1^{1/\delta}\lambda_2$ and $\lambda_2 \geq \beta_2^{1/\delta}\lambda_1$, then

$$X_1^* = I_1(\lambda_1 Z_T),\ X_2^* = I_2(\lambda_2 Z_T).$$

b) If $\lambda_2 < \beta_2^{1/\delta}\lambda_1$, then $\lambda_1 > \beta_1^{1/\delta}\lambda_2$ and

$$X_1^* = \beta_2 I_2(\lambda_2 Z_T) \mathbb{1}\left\{Z_T \leq \chi_{\alpha_1}\right\} + I_1(\lambda_1 Z_T) \mathbb{1}\left\{Z_T > \chi_{\alpha_1}\right\},\ X_2^* = I_2(\lambda_2 Z_T).$$

c) If $\lambda_1 < \beta_1^{1/\delta}\lambda_2$, then $\lambda_2 > \beta_2^{1/\delta}\lambda_1$ and

$$X_1^* = I_1(\lambda_1 Z_T),\ X_2^* = \beta_1 I_1(\lambda_1 Z_T) \mathbb{1}\left\{Z_T \leq \chi_{\alpha_2}\right\} + I_2(\lambda_2 Z_T) \mathbb{1}\left\{Z_T > \chi_{\alpha_2}\right\}.$$

$\Diamond$

*Proof (Theorem 8.4).* To simplify notation throughout this proof, we abbreviate $I_i = I_i(\lambda_i Z_T)$ and $X_i^* = X_i^*(\lambda_1, \lambda_2)$, $i = 1, 2$. We define functions $f_i$ to rewrite $X_i$, $i = 1, 2$, from (8.6) as

$$X_1 = f_1(X_2) := I_1 \mathbb{1}\left\{I_1 \geq \beta_2 X_2,\ Z_T \leq \chi_{\alpha_1}\right\} + \beta_2 X_2 \mathbb{1}\left\{I_1 < \beta_2 X_2,\ Z_T \leq \chi_{\alpha_1}\right\} + I_1 \mathbb{1}\left\{Z_T > \chi_{\alpha_1}\right\}$$

for agent 1 and

$$X_2 = f_2(X_1) := I_2 \mathbb{1}\left\{I_2 \geq \beta_1 X_1,\, Z_T \leq \chi_{\alpha_2}\right\} + \beta_1 X_1 \mathbb{1}\left\{I_2 < \beta_1 X_1,\, Z_T \leq \chi_{\alpha_2}\right\} + I_2 \mathbb{1}\left\{Z_T > \chi_{\alpha_2}\right\}$$

for agent 2. To construct the two-dimensional fixed point for (8.6), we consider the equation

$$f_1(f_2(X_1)) = X_1 \tag{8.10}$$

and prove that the unique solution to (8.10) is given by $X_1^*$. The analysis of $X_2 = f_2(f_1(X_2))$ proceeds analogously.

We use a case distinction based on $Z_T$. If $Z_T > \chi_{\alpha_1} \vee \chi_{\alpha_2}$, (8.10) simplifies to $X_1 = I_1$ and thus, the unique solution is given by $X_1 = I_1$ in this case. For the case $\chi_{\alpha_1} \wedge \chi_{\alpha_2} < Z_T \leq \chi_{\alpha_1} \vee \chi_{\alpha_2}$, we consider two subcases. If $\chi_{\alpha_1} < \chi_{\alpha_2}$, then $Z_T > \chi_{\alpha_1}$ and (8.10) again simplifies to $X_1 = I_1$. If $\chi_{\alpha_1} \geq \chi_{\alpha_2}$, (8.10) reads as

$$X_1 = f_1(I_2) = I_1 \mathbb{1}\left\{I_1 \geq \beta_2 I_2\right\} + \beta_2 I_2 \mathbb{1}\left\{I_1 < \beta_2 I_2\right\}.$$

Since the right-hand side is constant in $X_1$, it is the unique solution to (8.10) in that case. Finally, assume that $Z_T \leq \chi_{\alpha_1} \wedge \chi_{\alpha_2}$. First, let us simplify the components of $f_1(f_2(X_1))$. We obtain

$$\begin{aligned}\mathbb{1}\left\{I_1 \geq \beta_2 f_2(X_1)\right\} =& \mathbb{1}\left\{I_1 \geq \beta_2 I_2,\, I_2 \geq \beta_1 X_1\right\} + \mathbb{1}\left\{I_1 \geq \beta_1\beta_2 X_1,\, I_2 < \beta_1 X_1\right\} \\ =& \mathbb{1}\left\{I_1 \geq \beta_2 I_2,\, X_1 \leq (\beta_1\beta_2)^{-1} I_1\right\}\end{aligned}$$

for the first indicator,

$$\begin{aligned}\mathbb{1}\left\{I_1 < \beta_2 f_2(X_1)\right\} =& \mathbb{1}\left\{I_1 < \beta_2 I_2,\, I_2 \geq \beta_1 X_1\right\} + \mathbb{1}\left\{I_1 < \beta_1\beta_2 X_1,\, I_2 < \beta_1 X_1\right\} \\ =& \mathbb{1}\left\{I_1 < \beta_2 I_2,\, X_1 \leq \beta_1^{-1} I_2\right\} + \mathbb{1}\left\{X_1 > \max\left\{\beta_1^{-1} I_2,\, (\beta_1\beta_2)^{-1} I_1\right\}\right\}\end{aligned}$$

for the second indicator, and thus,

$$\begin{aligned}&f_2(X_1)\mathbb{1}\left\{I_1 < \beta_2 f_2(X_1)\right\} \\ =& I_2 \mathbb{1}\left\{I_1 < \beta_2 I_2,\, X_1 \leq \beta_1^{-1} I_2\right\} + \beta_1 X_1 \mathbb{1}\left\{X_1 > \max\left\{\beta_1^{-1} I_2,\, (\beta_1\beta_2)^{-1} I_1\right\}\right\}.\end{aligned}$$

Therefore, (8.10) simplifies to

$$\begin{aligned}X_1 =& I_1 \mathbb{1}\left\{I_1 \geq \beta_2 I_2,\, X_1 \leq (\beta_1\beta_2)^{-1} I_1\right\} + \beta_2 I_2 \mathbb{1}\left\{I_1 < \beta_2 I_2,\, X_1 \leq \beta_1^{-1} I_2\right\} \\ &+ \beta_1\beta_2 X_1 \mathbb{1}\left\{X_1 > \max\left\{\beta_1^{-1} I_2,\, (\beta_1\beta_2)^{-1} I_1\right\}\right\}.\end{aligned} \tag{8.11}$$

If $I_1 \geq \beta_2 I_2$, the unique solution to (8.11) is given by $X_1 = I_1$, whereas for $I_1 < \beta_2 I_2$, the unique solution is given by $X_1 = \beta_2 I_2$. This is based on the fact that, in both cases, the right-hand side of (8.11) is constant for small values of $X_1$ and linear with slope between 0 and 1 for large values of $X_1$. In both cases, the unique point of intersection with the identity function is located in the interval where the right-hand side is constant.

To summarize, the unique solution to (8.10) is given by

$$X_1 = I_1 \mathbb{1}\{I_1 \geq \beta_2 I_2,\, Z_T \leq \chi_{\alpha_1}\} + \beta_2 I_2 \mathbb{1}\{I_1 < \beta_2 I_2,\, Z_T \leq \chi_{\alpha_1}\} + I_1 \mathbb{1}\{Z_T > \chi_{\alpha_1}\}.$$

Using similar arguments for $X_2$ yields that $(X_1^*, X_2^*)$ from the theorem is the unique fixed point for (8.6). □

*Remark* 8.7. In Section C.2 in the Appendix, the $n$-dimensional fixed point problem (8.6) is reduced to an $(n-1)$-dimensional fixed point problem. The reduced problem (C.7) can be used to determine the fixed point for $n = 2$ instead of the previously displayed direct calculation. It is also used to solve the fixed point problem explicitly for $n = 3$ (see Section C.3 in the appendix). Afterwards, the Lagrange multipliers $\lambda_1$, $\lambda_2$ are calculated explicitly if the agents use logarithmic utility functions and the parameters are chosen correctly. Finally, the explicit representations are used to generate some numerical results. ◊

## Example: Logarithmic Utility

In the following, we consider a special case of Theorem 8.4 where both agents use the natural logarithm as their utility function. In this setting, we explicitly determine the Lagrange multipliers mentioned in Theorem 8.4, which allows us to look at some numerical results.

Theorem 8.4 implies that, if both agents use logarithmic utility, the unique fixed point for (8.6) is given by

$$X_1^* = \begin{cases} \frac{1}{\lambda_1 Z_T}, & \lambda_1\beta_2 \leq \lambda_2,\, Z_T \leq \chi_{\alpha_1}, \\ \frac{\beta_2}{\lambda_2 Z_T}, & \lambda_1\beta_2 > \lambda_2,\, Z_T \leq \chi_{\alpha_1}, \\ \frac{1}{\lambda_1 Z_T}, & Z_T > \chi_{\alpha_1}, \end{cases} \qquad X_2^* = \begin{cases} \frac{1}{\lambda_2 Z_T}, & \lambda_2\beta_1 \leq \lambda_1,\, Z_T \leq \chi_{\alpha_2}, \\ \frac{\beta_1}{\lambda_1 Z_T}, & \lambda_2\beta_1 > \lambda_1,\, Z_T \leq \chi_{\alpha_2}, \\ \frac{1}{\lambda_2 Z_T}, & Z_T > \chi_{\alpha_2}. \end{cases} \tag{8.12}$$

Now we can explicitly determine the Lagrange multipliers $\lambda_1$ and $\lambda_2$ with respect to the conditions $\mathbb{E}[X_i^* Z_T] = x_0^i$, $i = 1, 2$, if the parameters are chosen correctly.

**Lemma 8.8.** *Let $x_0^1 > \alpha_1\beta_2 x_0^2$, $x_0^2 > \alpha_2\beta_1 x_0^1$, and let $X_i^* = X_i^*(\lambda_1, \lambda_2)$, $i = 1, 2$, be given by* (8.12). *Then the unique solution $\lambda_1$, $\lambda_2 > 0$ to the system*

$$\mathbb{E}[X_1^* Z_T] = x_0^1,\, \mathbb{E}[X_2^* Z_T] = x_0^2,$$

*is given by*

$$\begin{aligned} \lambda_1 &= \frac{1-\alpha_1}{x_0^1 - \alpha_1\beta_2 x_0^2} \mathbb{1}\left\{x_0^1 < \beta_2 x_0^2\right\} + \frac{1}{x_0^1} \mathbb{1}\left\{x_0^1 \geq \beta_2 x_0^2\right\}, \\ \lambda_2 &= \frac{1-\alpha_2}{x_0^2 - \alpha_2\beta_1 x_0^1} \mathbb{1}\left\{x_0^2 < \beta_1 x_0^1\right\} + \frac{1}{x_0^2} \mathbb{1}\left\{x_0^2 \geq \beta_1 x_0^1\right\}. \end{aligned}$$

A proof can be found in Section C.1 in the appendix. Using Lemma 8.8, the unique solution to the fixed point problem (8.6) can now be given explicitly.

**Proposition 8.9.** *Let* $x_0^1 > \alpha_1\beta_2 x_0^2$, $x_0^2 > \alpha_2\beta_1 x_0^1$, *and* $U_i(x) = \log(x)$, $x \in (0,\infty)$, $i = 1,2$. *Then the unique Nash equilibrium in the class of wealth profiles of the form* (8.6) *is given as follows.*

*a) If* $x_0^1 \geq \beta_2 x_0^2$ *and* $x_0^2 \geq \beta_1 x_0^1$, *then*

$$X_1^* = x_0^1 Z_T^{-1},\; X_2^* = x_0^2 Z_T^{-1}.$$

*b) If* $x_0^1 < \beta_2 x_0^2$, *then* $x_0^2 > \beta_1 x_0^1$ *and*

$$X_1^* = \beta_2 x_0^2 Z_T^{-1} \mathbb{1}\left\{Z_T \leq \chi_{\alpha_1}\right\} + \frac{x_0^1 - \alpha_1\beta_2 x_0^2}{1-\alpha_1} Z_T^{-1} \mathbb{1}\left\{Z_T > \chi_{\alpha_1}\right\}, \quad X_2^* = x_0^2 Z_T^{-1}.$$

*c) If* $x_0^2 < \beta_1 x_0^1$, *then* $x_0^1 > \beta_2 x_0^2$ *and*

$$X_1^* = x_0^1 Z_T^{-1}, \quad X_2^* = \beta_1 x_0^1 Z_T^{-1} \mathbb{1}\left\{Z_T \leq \chi_{\alpha_2}\right\} + \frac{x_0^2 - \alpha_2\beta_1 x_0^1}{1-\alpha_2} Z_T^{-1} \mathbb{1}\left\{Z_T > \chi_{\alpha_2}\right\}.$$

The assertion follows directly by inserting the Lagrange multipliers from Lemma 8.8 into (8.12). Nevertheless, we displayed the proof in Section C.1 in the appendix.

*Remark* 8.10. Proposition 8.9 reveals an important feature of the Nash equilibrium $(X_1^*, X_2^*)$. First, we notice that either $X_1^*$ or $X_2^*$ has a discontinuity while the other is given by the solution to the classical problem. The agent whose wealth in the Nash equilibrium is given by the classical solution $x_0^j Z_T^{-1}$ is always „the better one“ in the sense that her initial capital is significantly larger than the other investors' initial capital ($x_0^j > \beta_j^{-1} x_0^i > x_0^i$, $i \neq j$). If the initial investments $x_0^1$ and $x_0^2$ are close in the sense that $\beta_2 x_0^2 \leq x_0^1 \leq \frac{x_0^2}{\beta_1}$, then both investors obtain the terminal wealth $x_0^j Z_T^{-1}$ in the fixed point. ◊

## Numerical Results

Proposition 8.9 enables us to consider some numerical results. We use the special case of a classical Black-Scholes model (one stock, constant market parameters) with zero interest rate. Thus, the discounted state price density $Z_T$ is given by (see, e.g., Eberlein and Kallsen, 2019, Examples 9.1 and 9.17)

$$Z_T = \exp\left( -\frac{\mu}{\sigma} W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2} T \right).$$

Hence, $Z_T$ follows a lognormal distribution with parameters $\nu$ and $\tau^2$, where

$$\nu = -\frac{1}{2}\frac{\mu^2}{\sigma^2} T,\; \tau^2 = \frac{\mu^2}{\sigma^2} T.$$

The quantiles $\chi_\alpha$, $\alpha \in [0,1]$, of $Z_T$ are given by (see, e.g., Johnson et al., 1994, p. 213)

$$\chi_\alpha = \exp\left(\nu + \tau \cdot \Phi^{-1}(\alpha)\right),$$

where $\Phi^{-1}$ describes the quantile function of the standard normal distribution. Throughout our computations we set $\mu = 0.03$, $\sigma = 0.2$, and $T = 4$. The numerical results are shown below.

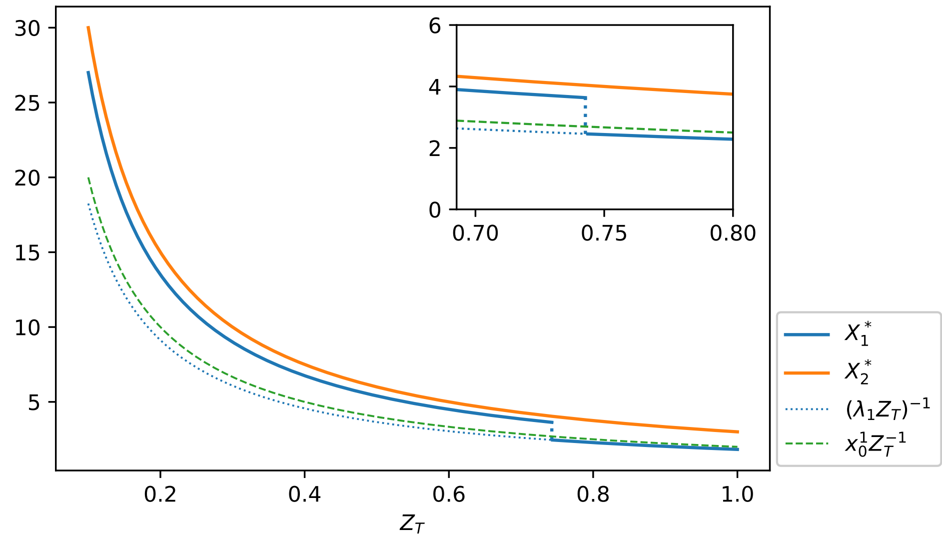
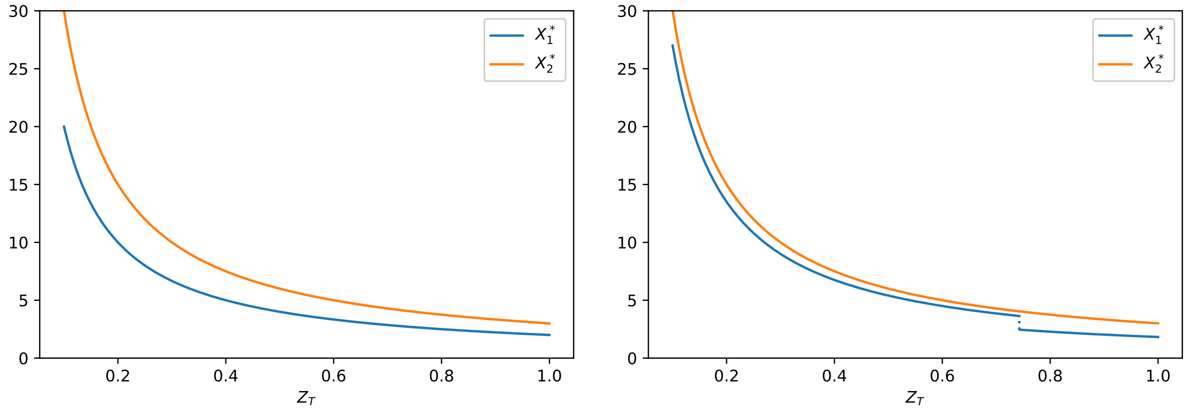



Figure 8.3.1.: $X_1^*$ and $X_2^*$ from Proposition 8.9 (solid) in terms of $Z_T$ for $\beta_1 = 0.5$, $\beta_2 = 0.9$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $x_0^1 = 2$, $x_0^2 = 3$, $\mu = 0.03$, $\sigma = 0.2$, $T = 4$. The dashed and dotted lines mark the classical solution $x_0^1 Z_T^{-1}$ and $(\lambda_1 Z_T)^{-1}$ for $\lambda_1$ from Lemma 8.8, respectively. The parameters are chosen such that case b) from Proposition 8.9 is present.




(a) $x_0^1 = 2$, $x_0^2 = 3$, $\beta_1 = 0.9$, $\beta_2 = 0.5$ (case a). (b) $x_0^1 = 2$, $x_0^2 = 3$, $\beta_1 = 0.5$, $\beta_2 = 0.9$ (case b).

Figure 8.3.2.: $X_1^*$ and $X_2^*$ from Proposition 8.9 in terms of $Z_T$ for $\mu = 0.03$, $\sigma = 0.2$, $T = 4$, $\alpha_1 = 0.2$ and $\alpha_2 = 0.4$.

Figure 8.3.1 gives a first glance at the behavior of the Nash equilibrium $(X_1^*, X_2^*)$ from Proposition 8.9. It shows $X_1^*$ and $X_2^*$ (solid) for the parameter choices $\beta_1 = 0.5$, $\beta_2 = 0.9$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $x_0^1 = 2$, and $x_0^2 = 3$. Note that, due to the choice of parameters, $X_1^*$ and $X_2^*$ are given in part b) of Proposition 8.9. For reference, the figure also displays $(\lambda_1 Z_T)^{-1}$ (dotted) in terms of $Z_T$. It is important to notice that $(\lambda_1 Z_T)^{-1}$ is not the solution to the classical problem, since the Lagrange multiplier $\lambda_1 = \frac{1-\alpha_1}{x_0^1 - \alpha_1 \beta_2 x_0^2} \approx 0.5479$ is different from the Lagrange multiplier $\lambda = (x_0^1)^{-1} = 0.5$ in the solution to the classical problem. For comparison, we also included the classical solution $x_0^1 Z_T^{-1}$ (dashed) for agent 1. While $X_2^*$ is simply given by the solution to the classical problem, $X_1^*$ shows a discontinuity at $\chi_{\alpha_1}$. For $Z_T \leq \chi_{\alpha_1}$ („good states“), $X_1^*$ is given by $\beta_2 (\lambda_2 Z_T)^{-1}$ and hence, strictly larger than $(\lambda_1 Z_T)^{-1}$. If $Z_T > \chi_{\alpha_1}$ („bad states“), $X_1^*$ is given by

$(\lambda_1 Z_T)^{-1}$. We observe that $X_1^*$ is slightly smaller than the classical solution $x_0^1 Z_T^{-1}$ if $Z_T > \chi_{\alpha_1}$, but significantly larger if $Z_T \leq \chi_{\alpha_1}$.

Figure 8.3.2 illustrates cases a) and b) from Proposition 8.9. Due to the symmetry between cases b) and c), only cases a) and b) are illustrated. For the first set of parameters, the conditions of case a) in Proposition 8.9 are satisfied so that $X_i^*$ is given by $(\lambda_i Z_T)^{-1}$, $i = 1, 2$, for any realization of $Z_T$. Hence, both $X_1^*$ and $X_2^*$ are continuous in $Z_T$. For the second set of parameters, the conditions of case b) are satisfied. In that case, $X_2^*$ is again given by $(\lambda_2 Z_T)^{-1}$ whereas $X_1^*$ is given by $\beta_2(\lambda_2 Z_T)^{-1}$ if $Z_T \leq \chi_{\alpha_1}$ and $(\lambda_1 Z_T)^{-1}$ if $Z_T > \chi_{\alpha_1}$, with a discontinuity located at $\chi_{\alpha_1}$.

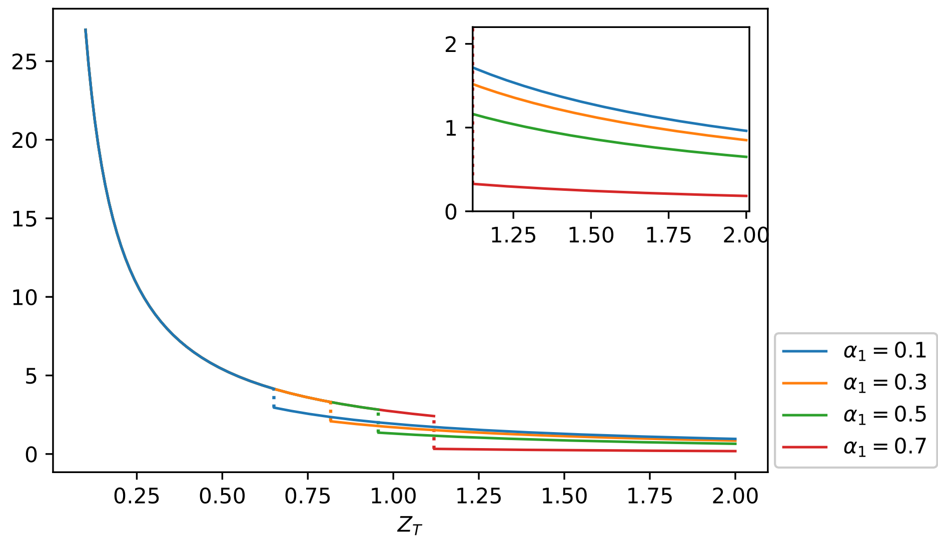



Figure 8.3.3.: $X_1^*$ from Proposition 8.9 in terms of $Z_T$ for $\beta_1 = 0.1$, $\beta_2 = 0.9$, $\alpha_2 = 0.4$ and different values of $\alpha_1$, $x_0^1 = 2$, $x_0^2 = 3$, $\mu = 0.03$, $\sigma = 0.2$, and $T = 4$. The parameters are chosen such that case b) from Proposition 8.9 is present.

Figures 8.3.3 and 8.3.4 illustrate how the choice of the parameters $\alpha_i$ and $\beta_i$, $i = 1, 2$, affects the component $X_1^*$ of the unique fixed point for two agents. All parameter choices used in the two figures satisfy the conditions of case b) in Proposition 8.9.

Figure 8.3.3 shows two different features of the influence of $\alpha_1$. We notice that the location of the discontinuity changes with $\alpha_1$, since the discontinuity is located at the quantile $\chi_{\alpha_1}$. We also notice that the value of $X_1^*$ is decreasing in terms of $\alpha_1$ if $Z_T > \chi_{\alpha_1}$. Hence, if a larger value of $\alpha_1$ is chosen, more states are insured with respect to agent 2, which outperforms agent 1 in this case. However, in the states in which there is no insurance with respect to agent 2, the terminal wealth is smaller for a larger choice of $\alpha_1$. The reason for this observation is the Lagrange multiplier $\lambda_1$ which is increasing in $\alpha_1$ in case b) of Proposition 8.9. Thus, we observe that a larger choice of $\alpha_1$ is related to agent 1 being more risk-averse. She wants to insure her wealth against the other agent's wealth in a larger fraction of the possible market scenarios. However, she has to „pay the price“ for this choice with an even smaller terminal wealth in the „bad“ states. This reveals a general criticism of the value at risk – it only limits the probability of losses above a certain threshold; the magnitude of such losses is not taken into account (see, e.g., Föllmer and Schied, 2016, p. 231).

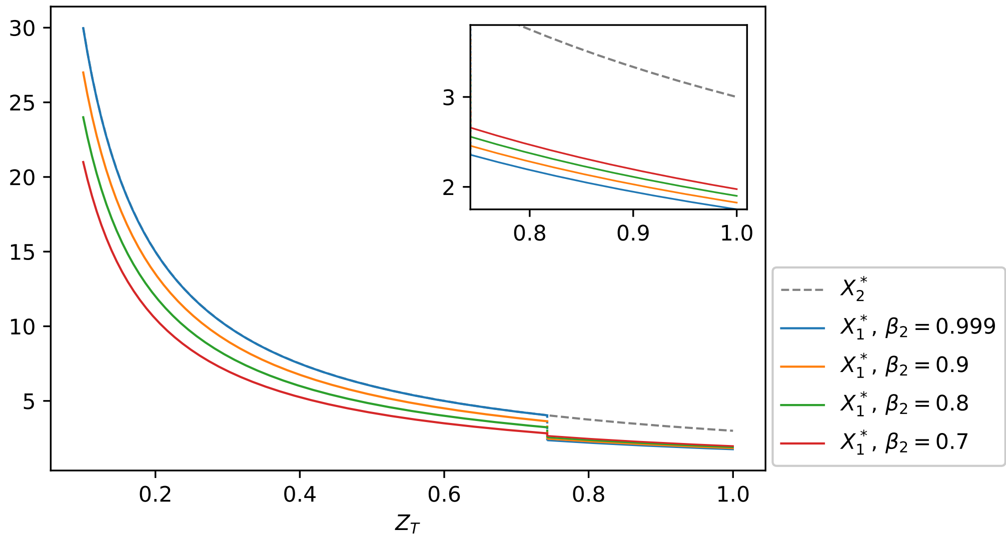



Figure 8.3.4.: $X_1^*$ (solid) and $X_2^*$ (dashed) from Proposition 8.9 in terms of $Z_T$ for different values of $\beta_2$, $\beta_1 = 1 - \beta_2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $x_0^1 = 2$, $x_0^2 = 3$, $\mu = 0.03$, $\sigma = 0.2$, and $T = 4$. The parameters are chosen such that case b) from Proposition 8.9 is present.

Figure 8.3.4 illustrates the influence of the parameter $\beta_2$ on $X_1^*$. Here we chose $x_0^1 = 2$ and $x_0^2 = 3$, so that, as long as $\beta_2 > 2/3$, case b) of Proposition 8.9 is present. Note that the parameter $\beta_1$ has no effect in this case, since $X_1^*$ does not depend on $\beta_1$, nor does the condition which ensures that case b) is present. Figure 8.3.4 shows that a larger value of $\beta_2$ results in a larger value of $X_1^*$ in the „good states“ ($Z_T \leq \chi_{\alpha_1}$) and in a smaller value of $X_1^*$ in the „bad states“.

## 8.4. Solution of the fixed point problem for $n$ agents

In this section, we show that, if $\beta_j \leq \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$, a fixed point for (8.6) is either unique or all components are equal. Moreover, we prove an important property of the fixed point, which is similar to the case distinction for three investors given in Table C.1 in the appendix. It shows that any possible value of the triple $(X_1^*, X_2^*, X_3^*)$ contains some (at least one, at most three) entries $X_j^*$ that are given by $I_j(\lambda_j Z_T)$ while the other entries are linear combinations of these $I_j$'s. We provide a similar result for a general number of investors. Finally, if we consider the case $\beta_j = \frac{1}{n-1}$ and $\alpha_j = \alpha \in [0, 1]$ for all $j \in \{1, \dots, n\}$, we can give a closed-form representation of the unique Nash equilibrium in the class of strategies of the form (8.6) (under an additional assumption which is conjectured to hold for general $n$ and is proven to hold for $n \in \{2, 3\}$). The solution for investor $i$ is given as a maximum of $2^{n-1}$ terms if $Z_T \leq \chi_\alpha$, and by $I_i(\lambda_i Z_T)$ if $Z_T > \chi_\alpha$. We also state and justify a conjecture on a statement allowing for more general choices of $\beta_j$, $j = 1, \dots, n$.

Using (8.6), $X_i$ equals $I_i(\lambda_i Z_T)$ if $Z_T > \chi_{\alpha_i}$, or if $Z_T \leq \chi_{\alpha_i}$ as well as $I_i \geq \sum_{j \neq i} \beta_j X_j$, and $\sum_{j \neq i} \beta_j X_j$ for other values of $Z_T$. Hence, to solve this $n$-dimensional fixed point problem, we can distinguish how many of the $n$ investors use the „classical“ solution $I_i(\lambda_i Z_T)$. Note that we need to be careful with the expression „classical solution“ since the Lagrange multiplier $\lambda_i$ is not necessarily given as the solution to $\mathbb{E}[Z_T I_i(\lambda_i Z_T)] = x_0^i$ since there might be values of $Z_T$ for which $X_i^*$ is not equal to $I_i(\lambda_i Z_T)$. Hence, although the structure is the same as in the classical solution for some values of $Z_T$, the value might not be equal to the value of the classical solution.

We can use this idea in the following to solve the $n$-dimensional fixed point problem emerging from (8.6). But first, we need to introduce some notation. For a finite set $S \subset \mathbb{R}$, we denote the set of $k$-combinations of pairwise distinct elements from $S$ by $\binom{S}{k}$, i.e.,

$$\binom{S}{k} := \{(i_1, \ldots, i_k) \,|\, i_j \in S,\, j = 1, \ldots, k,\, i_1 < i_2 < \ldots < i_k\},\, k \in \{0, 1, \ldots, |S|\},$$
$$\binom{S}{k} := \emptyset,\, k \notin \{0, 1, \ldots, |S|\}.$$

If $k = 0$, the set $\binom{S}{0}$ contains only the empty tuple $()$. A standard combinatorial result then states that the number of elements in $\binom{S}{k}$ equals $\binom{|S|}{k}$ if $k \in \{0, \ldots, |S|\}$, where

$$\binom{|S|}{k} = \frac{|S|!}{k!(|S| - k)!}$$

denotes the binomial coefficient (see, e.g., Harris et al., 2008, p. 133). We also use the abbreviation

$$[L] := \{1, 2, \ldots, L\},\, L \in \mathbb{N}.$$

We can then define

$$d_L(j_1, \ldots, j_L) := 1 - \sum_{k=2}^{L} (k-1) \sum_{(i_1, \ldots, i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} \tag{8.13}$$

and

$$A_L(j_1, \ldots, j_L) := \begin{pmatrix} 1 & -\beta_{j_2} & -\beta_{j_3} & \ldots & -\beta_{j_L} \\ -\beta_{j_1} & 1 & -\beta_{j_3} & \ddots & -\beta_{j_L} \\ \vdots & \ddots & \ddots & \ldots & \vdots \\ -\beta_{j_1} & \ldots & -\beta_{j_{L-2}} & 1 & -\beta_{j_L} \\ -\beta_{j_1} & -\beta_{j_2} & \ldots & -\beta_{j_{L-1}} & 1 \end{pmatrix}$$

for some $L \in \{1, \ldots, n\}$, where $A_1(j_1) := (1)$. Hence, the entries of $A_L(j_1, \ldots, j_L)$ are given by

$$A_L(j_1, \ldots, j_L)[i, k] = \begin{cases} 1, & i = k, \\ -\beta_{j_k}, & i \neq k. \end{cases}$$

The first goal of this section is to prove the following theorem which presents an $n$-dimensional analogue of Table C.1.

**Theorem 8.11.** *Let $\beta_j \leq \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$ and let the value of $Z_T$ be arbitrary but fixed. If there exists a solution $(X_1, \dots, X_n)$ to the fixed point problem (8.6), one of the following two cases is present. Either the fixed point satisfies $X_i = X_j$ for all $i, j \in \{1, \dots, n\}$ or it is unique with the following property: If, for some $L \in \{0, \dots, n-1\}$ and an $L$-combination $(j_1, \dots, j_L) \in \binom{[n]}{L}$,*

$$X_{j_\ell} \neq I_{j_\ell}(\lambda_{j_\ell} Z_T),\ \ell \in \{1, \dots, L\}, \quad X_k = I_k(\lambda_k Z_T),\ k \notin \{j_1, \dots, j_L\},$$

*then*

$$X_{j_\ell} = \frac{\prod\limits_{i=1, i \neq \ell}^{L} (1 + \beta_{j_i})}{d_L(j_1, \dots, j_L)} \sum_{\substack{k=1 \\ k \notin \{j_1, \dots, j_L\}}}^{n} \beta_k I_k(\lambda_k Z_T)$$

*holds for all $\ell \in \{1, \dots, L\}$.*

The proof uses Lemmas 8.12 - 8.14 stated below. Their proofs can be found in Section C.4 in the appendix.

**Lemma 8.12.** *Let $L \in \{1, \dots, n\}$ and $(j_1, \dots, j_L) \in \binom{[n]}{L}$. If $\beta_j \leq \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$, then*

$$d_L(j_1, \dots, j_L) \geq \left(\frac{n}{n-1}\right)^{L-1} \frac{n-L}{n-1}.$$

*Further, $d_L(j_1, \dots, j_L) > 0$ holds if, and only if, either $L \in \{1, \dots, n-1\}$ or $L = n$ and $\prod_{j=1}^{n} \beta_j < \left(\frac{1}{n-1}\right)^n$.*

**Lemma 8.13.** *Assume that $\beta_j \leq \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$. Then, if either $L \in \{1, \dots, n-1\}$, or $L = n$ and $\prod_{j=1}^{n} \beta_j < \left(\frac{1}{n-1}\right)^n$, the matrix $A_L(j_1, \dots, j_L)$ is regular with inverse $A_L(j_1, \dots, j_L)^{-1}$ given by*

$$A_L(j_1, \dots, j_L)^{-1}[i,k] = \begin{cases} \dfrac{1 - \sum\limits_{m=2}^{L-1} (m-1) \sum\limits_{(i_1, \dots, i_m) \in \binom{\{j_1, \dots, j_L\} \setminus \{j_i\}}{m}} \prod\limits_{\ell=1}^{m} \beta_{i_\ell}}{d_L(j_1, \dots, j_L)}, & i = k, \\[2ex] \dfrac{\beta_{j_k} \prod\limits_{\ell=1, \ell \neq i,k}^{L} (1 + \beta_{j_\ell})}{d_L(j_1, \dots, j_L)}, & i \neq k, \end{cases}$$

*for any $(j_1, \dots, j_L) \in \binom{[n]}{L}$. If $L = n$ and $\beta_j = \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$, then $A_n(1, \dots, n)$ is not regular.*

**Lemma 8.14.** *Let $\beta_j \leq \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$. Further, let either $L \in \{1, \dots, n-1\}$ or $L = n$ and $\prod_{j=1}^{n} \beta_j < \left(\frac{1}{n-1}\right)^n$. Finally, let $(j_1, \dots, j_L) \in \binom{[n]}{L}$. Then the sum of the $i$-th row of $A_L(j_1, \dots, j_L)^{-1}$, $i \in \{1, \dots, L\}$, is given by*

$$\sum_{k=1}^{L} A_L(j_1, \dots, j_L)^{-1}[i,k] = \frac{1}{d_L(j_1, \dots, j_L)} \prod_{\ell=1, \ell \neq i}^{L} (1 + \beta_{j_\ell}).$$

Using these three lemmas, we can now prove Theorem 8.11.

*Proof (Theorem 8.11).* Considering the $n$-dimensional fixed point problem (8.6), any fixed point $(X_1, \ldots, X_n)$ has the property that, for any value of $Z_T$, $X_i$ is given by either $I_i(\lambda_i Z_T)$ or $\sum_{j \neq i} \beta_j X_j$. To prove the assertion in the theorem, we use a case distinction based on the number $M$ of indices $j \in \{1, \ldots, n\}$ for which $X_j = I_j(\lambda_j Z_T)$ holds.

First, we need to consider the case that $X_i = \sum_{j \neq i} \beta_j X_j$ for all $i \in \{1, \ldots, n\}$, i.e., $M = 0$. This yields a system of linear equations which can be written as

$$A_n(1, \ldots, n)(X_1, \ldots, X_n)^\top = \mathbf{0}_n, \tag{8.14}$$

where $\mathbf{0}_n$ denotes the $n$-dimensional vector of zeros. If $\prod_{j=1}^n \beta_j < \left(\frac{1}{n-1}\right)^n$, Lemma 8.13 implies that $A_n(1, \ldots, n)$ is regular and thus, the unique solution to this system is given by $\mathbf{0}_n$. However, $X_i = \sum_{j \neq i} \beta_j X_j$ implies that $0 = \sum_{j \neq i} \beta_j X_j > I_i(\lambda_i Z_T)$, which is a contradiction since $I_i$ takes only strictly positive values. Now let $\beta_j = \frac{1}{n-1}$ for all $j \in \{1, \ldots, n\}$. Then (8.14) reads as

$$X_i = \frac{1}{n-1} \sum_{j \neq i} X_j \text{ or, equivalently, } X_i = \frac{1}{n} \sum_{j=1}^n X_j,$$

for all $i \in \{1, \ldots, n\}$. Thus, all components of the fixed point are equal.

Now let $M \in \{1, \ldots, n\}$ describe the number of indices $j \in \{1, \ldots, n\}$ with $X_j = I_j(\lambda_j Z_T)$. Further, let $L \coloneqq n - M \in \{0, \ldots, n-1\}$ and denote the $L$-tuple of indices $\ell$ for which[3] $X_\ell \neq I_\ell(\lambda_\ell Z_T)$ by $(j_1, \ldots, j_L)$. For $L = 0$, $X_j = I_j(\lambda_j Z_T)$ holds for all $j \in \{1, \ldots, n\}$. Thus, we consider $L \in \{1, \ldots, n-1\}$. Hence, for all $\ell \in \{1, \ldots, L\}$,

$$X_{j_\ell} = \sum_{\substack{i=1 \\ i \neq j_\ell}}^n \beta_i X_i = \sum_{\substack{p=1 \\ p \neq \ell}}^L \beta_{j_p} X_{j_p} + \sum_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_L\}}}^n \beta_k I_k(\lambda_k Z_T)$$

and we obtain the system of linear equations

$$X_{j_\ell} - \sum_{\substack{p=1 \\ p \neq \ell}}^L \beta_{j_p} X_{j_p} = \sum_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_L\}}}^n \beta_k I_k(\lambda_k Z_T), \quad \ell = 1, \ldots, L,$$

which we can also write as

$$A_L(j_1, \ldots, j_L)(X_{j_1}, \ldots, X_{j_L})^\top = \sum_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_L\}}}^n \beta_k I_k(\lambda_k Z_T) \cdot \mathbf{1}_L, \tag{8.15}$$

where $\mathbf{1}_L$ denotes the $L$-dimensional vector of ones. Since, for $L \in \{1, \ldots, n-1\}$, $A_L(j_1, \ldots, j_L)$ is regular with known inverse $A_L(j_1, \ldots, j_L)^{-1}$ (see Lemma 8.13), the unique solution to (8.15) is given by

$$(X_{j_1}, \ldots, X_{j_L})^\top = \sum_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_L\}}}^n \beta_k I_k(\lambda_k Z_T) \cdot A_L(j_1, \ldots, j_L)^{-1} \mathbf{1}_L.$$

[3]Note that, although not relevant for our proof, this does imply that $Z_T \leq \chi_{\alpha_{j_\ell}}$ for all $\ell \in \{1, \ldots, L\}$.

Therefore, the $\ell$-th entry of the solution is given by

$$X_{j_\ell} = \left( \sum_{\substack{k=1 \\ k \notin \{j_1, \dots, j_L\}}}^{n} \beta_k I_k(\lambda_k Z_T) \right) \cdot \sum_{p=1}^{L} A_L(j_1, \dots, j_L)^{-1}[\ell, p].$$

Combining Lemma 8.13 and Lemma 8.14 yields

$$X_{j_\ell} = \frac{\prod_{p=1, p \neq \ell}^{L} (1 + \beta_{j_p})}{d_L(j_1, \dots, j_L)} \cdot \sum_{\substack{k=1 \\ k \notin \{j_1, \dots, j_L\}}}^{n} \beta_k I_k(\lambda_k Z_T).$$

Since we can use this argumentation for any solution of the fixed point problem, it follows that, if there exists a fixed point for (8.6), it is unique or all components are equal. Moreover, it satisfies the asserted property from the theorem. □

*Remark* 8.15. Although the fixed point problem (8.6) contains a case distinction on whether $Z_T \leq \chi_\alpha$ or $Z_T > \chi_\alpha$, we did not need to separate these two cases in the proof of Theorem 8.11. We just needed to consider how many and which of the indices $j \in \{1, \dots, n\}$ satisfy $X_j = I_j(\lambda_j Z_T)$, which could be due to $Z_T > \chi_{\alpha_j}$ or to $I_j(\lambda_j Z_T) > \sum_{k \neq j} \beta_k X_k$. This enables us to state the theorem without the assumption $\alpha_i = \alpha$ for all $i \in \{1, \dots, n\}$. ♢

Theorem 8.11 shows that if there exists a fixed point for (8.6), it is either unique or all components of the fixed point are equal. Moreover, in the case where not all components are equal, it can be expressed implicitly via a case distinction. However, it does not prove the existence of a fixed point or give an explicit representation. In the following, we provide a closed-form representation of the unique fixed point (in a slightly more restrictive setting) and thus, the unique Nash equilibrium in the case that $\alpha_i = \alpha \in [0, 1]$ and $\beta_i = \frac{1}{n-1}$ for all $i \in \{1, \dots, n\}$. We also state and justify two conjectures about a more general theorem (see Remarks 8.17 and 8.19). The following theorem displays the mentioned closed-form representation of the unique fixed point.

**Theorem 8.16.** *Let $\alpha_j = \alpha \in [0, 1]$ and $\beta_j = \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$. Then the $\mathbb{P}$-almost surely unique Nash equilibrium for* (8.2) *in the class of wealth profiles $X_j$, $j = 1, \dots, n$, of the form* (8.6) *for which*

$$\mathbb{P}\Big( X_i = X_j \text{ for all } i, j \in \{1, \dots, n\} \Big) = 0 \tag{8.16}$$

*holds, is given by $(X_1^*, \dots, X_n^*)$ for*

$$\begin{aligned} X_i^* = \max & \left\{ \{I_i(\lambda_i Z_T)\} \cup \bigcup_{L=0}^{n-2} \bigcup_{(j_1, \dots, j_L) \in \binom{[n] \setminus \{i\}}{L}} \left\{ \frac{\prod_{\ell=1}^{L} (1 + \beta_{j_\ell})}{d_{L+1}(i, j_1, \dots, j_L)} \sum_{\substack{k=1 \\ k \notin \{i, j_1, \dots, j_L\}}}^{n} \beta_k I_k(\lambda_k Z_T) \right\} \right\} \\ & \cdot \mathbb{1}\{Z_T \leq \chi_\alpha\} + I_i(\lambda_i Z_T) \mathbb{1}\{Z_T > \chi_\alpha\}, \end{aligned} \tag{8.17}$$

*$i = 1, \dots, n$, where we assume that the $n$-dimensional system $\mathbb{E}[Z_T X_i^*] = x_0^i$, $i = 1, \dots, n$, of equations has a unique solution $(\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$.*

*Remark* 8.17. We conjecture that (8.16) holds for any Nash equilibrium for (8.2) in the class of wealth profiles of the form (8.6). Thus, we believe that Theorem 8.16 holds without the restriction

(8.16) to tuples $(X_1, \dots, X_n)$ with components that are almost surely not identical. For $n \in \{2, 3\}$, we can prove that (8.17) is the unique Nash equilibrium in the class of wealth profiles of the form (8.6) without the additional restriction (8.16) (see Theorem 8.4 and Theorem C.1 in the appendix). $\Diamond$

The proof of Theorem 8.16 uses the following lemma.

**Lemma 8.18.** *Let $L \in \{0, \dots, n-1\}$ and $(j_1, \dots, j_{L+1}) \in \binom{[n]}{L+1}$. Moreover, let $\beta_j \le \frac{1}{n-1}$, $j = 1, \dots, n$. Then*

$$d_{L+1}(j_1, \dots, j_L, j_{L+1}) = (1 + \beta_{j_{L+1}}) d_L(j_1, \dots, j_L) - \beta_{j_{L+1}} \prod_{\ell=1}^{L} (1 + \beta_{j_\ell}).$$

The proof can be found in Section C.4 in the appendix.

*Proof (Theorem 8.16).* The conditions of Theorem 8.16 allow us to use the statement of Theorem 8.11. We only need to consider the case $Z_T \le \chi_\alpha$, because otherwise we already know that $X_i^*$ is given by $I_i(\lambda_i Z_T)$ for all $i \in \{1, \dots, n\}$. Thus, we assume that $Z_T \le \chi_\alpha$ throughout this proof.

To prove the claimed representation of $X_i^*$, $i = 1, \dots, n$, we have to show that the $n$-tuple $(X_1^*, \dots, X_n^*)$ solves the fixed point problem (8.6). To keep notation simple, we consider $i = 1$ and assume that $X_j^*$, $j = 2, \dots, n$, are given by (8.17). Then we need to show that $X_1^*$ can also be written in the form of (8.17). For the remainder of this proof, we abbreviate $I_j = I_j(\lambda_j Z_T)$.

Analogously to the proof of Theorem 8.11, we use a case distinction based on the number $N \in \{0, \dots, n-1\}$ of indices $j \in \{2, \dots, n\}$ for which $X_j^* = I_j$ holds. If $N = 0$, i.e., if $X_j^* \ne I_j(\lambda_j Z_T)$ for all $j \in \{2, \dots, n\}$, it follows immediately from assumption (8.16) that $X_1^* = I_1(\lambda_1 Z_T)$ $\mathbb{P}$-almost surely. In fact, the proof of Theorem 8.11 implies that, if $X_j^* \ne I_j(\lambda_j Z_T)$ for all $j \in \{1, \dots, n\}$, it follows that $X_i^* = X_j^*$ for all $i, j \in \{1, \dots, n\}$, which is $\mathbb{P}$-almost surely not the case using (8.16). Thus, we can assume that there are $N \in \{1, \dots, n-1\}$ indices $j \in \{2, \dots, n\}$ for which $X_j^* = I_j$ holds. Further, let $L = n-1-N \in \{0, \dots, n-2\}$ and $j_1, \dots, j_L \in \{2, \dots, n\}$ describe the pairwise distinct indices for which $X_{j_\ell}^* \ne I_{j_\ell}$, $\ell = 1, \dots, L$. Then it follows analogously to the proof of Theorem 8.11 that $X_1^*$ is given by either $I_1$ or

$$\frac{\prod_{\ell=1}^{L}(1 + \beta_{j_\ell})}{d_{L+1}(1, j_1, \dots, j_L)} \sum_{\substack{k=2 \\ k \notin \{j_1, \dots, j_L\}}}^{n} \beta_k I_k.$$

Hence, it remains to show that

$$\begin{aligned}
&\max\left\{\{I_1\} \cup \bigcup_{M=0}^{n-2} \bigcup_{(m_1,\dots,m_M) \in \binom{[n]\setminus\{1\}}{M}} \left\{\frac{\prod_{p=1}^{M}(1+\beta_{m_p})}{d_{M+1}(1, m_1, \dots, m_M)} \sum_{\substack{r=2 \\ r \notin \{m_1,\dots,m_M\}}}^{n} \beta_r I_r\right\}\right\} \\
&= \max\left\{I_1, \frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1, j_1, \dots, j_L)} \sum_{\substack{k=2 \\ k \notin \{j_1,\dots,j_L\}}}^{n} \beta_k I_k\right\}.
\end{aligned}$$

In order to do this, we prove the inequality

$$\frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1,j_1,\dots,j_L)} \sum_{\substack{k=2\\ k\notin\{j_1,\dots,j_L\}}}^{n} \beta_k I_k \geq \frac{\prod_{p=1}^{M}(1+\beta_{m_p})}{d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{r=2\\ r\notin\{m_1,\dots,m_M\}}}^{n} \beta_r I_r \tag{8.18}$$

for all $M \in \{0,\dots,n-2\}$ and $(m_1,\dots,m_M) \in \binom{\{2,\dots,n\}}{M}$ with $(m_1,\dots,m_M) \neq (j_1,\dots,j_L)$. Hence, let $M \in \{0,\dots,n-2\}$ and $(m_1,\dots,m_M) \in \binom{\{2,\dots,n\}}{M}$ with $(m_1,\dots,m_M) \neq (j_1,\dots,j_L)$ be arbitrary but fixed.

Using that $X_j^*$, $j=2,\dots,n$, are given by (8.17) and that $X_k^* = I_k$ for $k \in \{2,\dots,n\}\backslash\{j_1,\dots,j_L\}$ yields

$$I_k \geq \frac{\prod_{q=1}^{K}(1+\beta_{i_q})}{d_{K+1}(k,i_1,\dots,i_K)} \sum_{\substack{s=1\\ s\notin\{k,i_1,\dots,i_K\}}}^{n} \beta_s I_s \tag{8.19}$$

for all $K \in \{0,\dots,n-2\}$, $(i_1,\dots,i_K) \in \binom{[n]\backslash\{k\}}{K}$, and $k \in \{2,\dots,n\}\backslash\{j_1,\dots,j_L\}$. We can use (8.19) to show (8.18) by choosing $K$ and $(i_1,\dots,i_K)$ accordingly. The choice of $K$ and $(i_1,\dots,i_K)$ is divided into two cases. The reason is that the lower bound (8.19) only holds if the sum on the right-hand side does not contain the summand $\beta_k I_k$. Hence, we need to distinguish the cases $k \in \{m_1,\dots,m_M\}$ and $k \notin \{m_1,\dots,m_M\}$.

First, we consider the indices $k$ contained in $\{m_1,\dots,m_M\}\backslash\{j_1,\dots,j_L\}$. Note that the index set $\{m_1,\dots,m_M\}\backslash\{j_1,\dots,j_L\}$ is non-empty by assumption. For these indices, we can choose $K=M$ and $\{i_1,\dots,i_K\} = \{1,m_1,\dots,m_M\}\backslash\{k\}$, and obtain

$$I_k \geq \frac{\prod_{p\in\{1,m_1,\dots,m_M\}\backslash\{k\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{r=1\\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r. \tag{8.20}$$

Now we consider the case $k \in \{2,\dots,n\}\backslash\{j_1,\dots,j_L,m_1,\dots,m_M\}$. Then we choose $K = M+1$ and $\{i_1,\dots,i_K\} = \{1,m_1,\dots,m_M\}$ and obtain

$$I_k \geq \frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+2}(1,m_1,\dots,m_M,k)} \sum_{\substack{r=1\\ r\notin\{1,m_1,\dots,m_M,k\}}}^{n} \beta_r I_r,$$

which we can equivalently rewrite as

$$\left(1+\beta_k\frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+2}(1,m_1,\dots,m_M,k)}\right) I_k \geq \frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+2}(1,m_1,\dots,m_K,k)} \sum_{\substack{r=1\\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r.$$

We can rewrite the constant factor on the left-hand side using Lemma 8.18. It follows

$$\begin{aligned} 1+\beta_k\frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+2}(1,m_1,\dots,m_M,k)} &= \frac{d_{M+2}(1,m_1,\dots,m_M,k)+\beta_k\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+2}(1,m_1,\dots,m_M,k)} \\ &= \frac{(1+\beta_k)d_{M+1}(1,m_1,\dots,m_M)}{d_{M+2}(1,m_1,\dots,m_M,k)}, \end{aligned}$$

which is strictly positive using Lemma 8.12. Hence, we obtain

$$I_k \geq \frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{(1+\beta_k)d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{r=1 \\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r. \tag{8.21}$$

Now we combine (8.20) and (8.21) to obtain

$$\begin{aligned}
\sum_{\substack{k=2 \\ k\notin\{j_1,\dots,j_L\}}}^{n} \beta_k I_k &= \sum_{\substack{k\in\{m_1,\dots,m_M\} \\ k\notin\{j_1,\dots,j_L\}}} \beta_k I_k + \sum_{\substack{k=2 \\ k\notin\{m_1,\dots,m_M\} \\ k\notin\{j_1,\dots,j_L\}}}^{n} \beta_k I_k \\
&\geq \sum_{\substack{k\in\{m_1,\dots,m_M\} \\ k\notin\{j_1,\dots,j_L\}}} \beta_k \frac{\prod_{p\in\{1,m_1,\dots,m_M\}\setminus\{k\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{r=1 \\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r \\
&\quad + \sum_{\substack{k=2 \\ k\notin\{m_1,\dots,m_M\} \\ k\notin\{j_1,\dots,j_L\}}}^{n} \beta_k \frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{(1+\beta_k)d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{r=1 \\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r \\
&= \frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{k=2 \\ k\notin\{j_1,\dots,j_L\}}}^{n} \frac{\beta_k}{1+\beta_k} \sum_{\substack{r=1 \\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1,j_1,\dots,j_L)} \sum_{\substack{k=2 \\ k\notin\{j_1,\dots,j_L\}}}^{n} \beta_k I_k \\
\geq& \frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1,j_1,\dots,j_L)} \frac{\prod_{p\in\{1,m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{k=2 \\ k\notin\{j_1,\dots,j_L\}}}^{n} \frac{\beta_k}{1+\beta_k} \sum_{\substack{r=1 \\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r \\
=& \underbrace{\frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1,j_1,\dots,j_L)}(1+\beta_1) \sum_{\substack{k=2 \\ k\notin\{j_1,\dots,j_L\}}}^{n} \frac{\beta_k}{1+\beta_k}}_{=:\gamma_{L,1}} \cdot \frac{\prod_{p\in\{m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)} \sum_{\substack{r=1 \\ r\notin\{1,m_1,\dots,m_M\}}}^{n} \beta_r I_r. \quad (8.22)
\end{aligned}$$

Finally, we need to check if $\gamma_{L,1} \geq 1$ which then concludes our proof. If we use the assumption $\beta_j \leq \frac{1}{n-1}$, $j = 1,\dots,n$, used throughout the current section, we obtain

$$\begin{aligned}
\gamma_{L,1} &= \frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1,j_1,\dots,j_L)}(1+\beta_1) \sum_{\substack{k=2 \\ k\notin\{j_1,\dots,j_L\}}}^{n} \frac{\beta_k}{1+\beta_k} \\
&\leq \frac{\left(1+\frac{1}{n-1}\right)^L}{\frac{n-1-L}{n-1}\left(1+\frac{1}{n-1}\right)^L} \cdot \left(1+\frac{1}{n-1}\right)(n-1-L) \cdot \frac{\frac{1}{n-1}}{1+\frac{1}{n-1}} = 1.
\end{aligned}$$

The lower bound on $d_{L+1}(1,j_1,\dots,j_L)$ is given in Lemma 8.12. Hence, we can only achieve $\gamma_{L,1} \geq 1$ (or, to be more precise, $\gamma_{L,1} = 1$) if we choose $\beta_j = \frac{1}{n-1}$. Then we can make the final step in (8.22)

$$\frac{\prod_{\ell=1}^{L}(1+\beta_{j_\ell})}{d_{L+1}(1,j_1,\dots,j_L)}\sum_{\substack{k=2\\k\notin\{j_1,\dots,j_L\}}}^{n}\beta_k I_k \geq \gamma_{L,1}\cdot\frac{\prod_{p\in\{m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)}\sum_{\substack{r=1\\r\notin\{1,m_1,\dots,m_M\}}}^{n}\beta_r I_r$$
$$=\frac{\prod_{p\in\{m_1,\dots,m_M\}}(1+\beta_p)}{d_{M+1}(1,m_1,\dots,m_M)}\sum_{\substack{r=1\\r\notin\{1,m_1,\dots,m_M\}}}^{n}\beta_r I_r,$$

which concludes our proof. □

*Remark* 8.19. Note that, although we assumed that $\beta_j = \frac{1}{n-1}$ for all $j \in \{1,\dots,n\}$, we displayed $X_i^*$ in terms of $\beta_1,\dots,\beta_n$. The reason is that we conjecture that (8.17) holds for general $\beta_j \leq \frac{1}{n-1}$, not just for $\beta_j = \frac{1}{n-1}$. We expect that the lower bounds (8.20) and (8.21) in the proof of Theorem 8.16 are too small. The lower bound (8.19) holds for *any* $K \in \{0,\dots,n-2\}$ and *any* $(i_1,\dots,i_K) \in \binom{[n]\setminus\{k\}}{K}$. Although the specific choice in the proof enables us to prove (8.17) for general $n$, different choices for $K \in \{0,\dots,n-2\}$ and $(i_1,\dots,i_K) \in \binom{[n]\setminus\{k\}}{K}$ might result in better (larger) lower bounds for $I_k$, $k \in \{2,\dots,n\}\setminus\{j_1,\dots,j_L\}$, and hence, also a better lower bound for the sum $\sum_k \beta_k I_k$. This might yield the desired lower bound (8.18) under the weaker assumption $\beta_j \leq \frac{1}{n-1}$ for all $j \in \{1,\dots,n\}$.

In the case $n = 3$, we are able to prove the assertion for general $\beta_j \leq \frac{1}{2}$. In the proof of Table C.1, we needed to use all $3 = 2^{n-1}-1$ lower bounds resulting from the representation of $X_i^*$ as a maximum of four arguments. For general $n \in \mathbb{N}$, it is simply not possible to consider $2^{n-1}-1$ different lower bounds in order to find the best one. However, we strongly believe that it would give us the desired result for general $\beta_j \leq \frac{1}{n-1}$. ◊

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# Appendix A

## Additional material for Chapter 6

The following chapter contains supplementary material for Chapter 6. First, we explain the application of the dominated convergence theorem in the proof of Lemma 6.6 in more depth. Afterwards, we prove two results regarding the expected value of the product of a normal and a corresponding lognormal random variable (appearing in Remark 6.7 and Example 6.11).

### A.1. Application of the dominated convergence theorem in Lemma 6.6

In Lemma 6.6, we determined the optimal collective competitive terminal wealth using the collective competitive utility function $\widetilde{U}_\beta$ from Lemma 6.2. In order to argue that there exists a unique Lagrange multiplier, we needed to apply the dominated convergence theorem. In what follows, we explain the application of the dominated convergence theorem in more depth. To be more specific, we justify the following interchange of convergence and expectation

$$\lim_{k\to\infty} \mathbb{E}\left[Z_T \cdot I_i\left(\frac{n\lambda_k Z_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right] = \mathbb{E}\left[Z_T \cdot I_i\left(\frac{n\lambda Z_T}{\beta_i\,(n+\theta_i)\,(1-\hat{\theta})}\right)\right],$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of real numbers with $\lim_{k\to\infty}\lambda_k = \lambda$.

For simplicity, let $\tilde{I}_i(x) := I_i\left(\frac{nx}{\beta_i(n+\theta_i)(1-\hat{\theta})}\right)$, $x \in (0,\infty)$. We consider two different cases: First, assume that $\mathcal{D}_i = (0,\infty)$. Let $0 < \varepsilon < \lambda$. Since $\lambda_k \to \lambda,\ k\to\infty$, there exists some $k_0 = k_0(\varepsilon)$ such that $\lambda_k \in (\lambda-\varepsilon, \lambda+\varepsilon)$ for all $k \geq k_0$. Hence, since $\tilde{I}_i$ is strictly decreasing, it follows

$$Z_T\tilde{I}_i\left(\lambda_k Z_T\right) \leq Z_T\tilde{I}_i\left((\lambda-\varepsilon)Z_T\right) \quad \forall\, k \geq k_0.$$

Moreover, let $\lambda_{\min} := \min_{1\leq k<k_0} \lambda_k$. Then

$$Z_T\tilde{I}_i\left(\lambda_k Z_T\right) \leq Z_T\tilde{I}_i\left(\lambda_{\min} Z_T\right) \quad \forall\, k < k_0.$$

In summary, we obtain

$$\begin{aligned} Z_T\tilde{I}_i\left(\lambda_k Z_T\right) &\leq \max\left\{Z_T\tilde{I}_i\left((\lambda-\varepsilon)Z_T\right), Z_T\tilde{I}_i\left(\lambda_{\min} Z_T\right)\right\} \\ &\leq Z_T\tilde{I}_i\left((\lambda-\varepsilon)Z_T\right) + Z_T\tilde{I}_i\left(\lambda_{\min} Z_T\right) \end{aligned}$$

for all $k \in \mathbb{N}$. By assumption, the right-hand side is integrable and hence, we can apply the dominated convergence theorem.

Now assume that $\mathcal{D}_i = \mathbb{R}$. Again, let $0 < \varepsilon < \lambda$. Then there exists some $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $\lambda_k \in (\lambda-\varepsilon, \lambda+\varepsilon)$. Hence, using the monotonicity of $\tilde{I}_i$, we have

$$\left|Z_T\tilde{I}_i(\lambda_k Z_T)\right| \leq \max\left\{\left|Z_T\tilde{I}_i((\lambda-\varepsilon)Z_T)\right|, \left|Z_T\tilde{I}_i((\lambda+\varepsilon)Z_T)\right|\right\}$$

for all $k \geq k_0$. Moreover, let

$$\lambda_{\max} = \underset{\lambda\in\{\lambda_1,\ldots,\lambda_{k_0-1}\}}{\arg\max} \left|Z_T\tilde{I}_i(\lambda_k Z_T)\right|.$$

Then it follows

$$\left|Z_T\tilde{I}_i(\lambda_k Z_T)\right| \leq \left|Z_T\tilde{I}_i(\lambda_{\max} Z_T)\right| \;\; \forall k < k_0.$$

In summary, we obtain

$$\begin{aligned} &\left|Z_T\tilde{I}_i(\lambda_k Z_T)\right| \\ &\leq \max\left\{\left|Z_T\tilde{I}_i((\lambda-\varepsilon)Z_T)\right|, \left|Z_T\tilde{I}_i((\lambda+\varepsilon)Z_T)\right|, \left|Z_T\tilde{I}_i(\lambda_{\max} Z_T)\right|\right\} \\ &\leq \left|Z_T\tilde{I}_i((\lambda-\varepsilon)Z_T)\right| + \left|Z_T\tilde{I}_i((\lambda+\varepsilon)Z_T)\right| + \left|Z_T\tilde{I}_i(\lambda_{\max} Z_T)\right| \end{aligned}$$

for all $k \in \mathbb{N}$. By assumption, the right-hand side is integrable and hence, we can apply the dominated convergence theorem.

## A.2. Expectation of a normal and a lognormal random variable

The following lemma treats the expectation of the product of a normally distributed random variable and the corresponding lognormally distributed random variable. The results are used in Remark 6.7 and Example 6.11.

**Lemma A.1.** *Let $X \sim \mathcal{N}(\mu, \sigma^2)$ for $\mu \in \mathbb{R}$, $\sigma > 0$. Then*

$$\mathbb{E}\left[|X|\exp(X)\right] < \infty, \quad \mathbb{E}\left[X\exp(X)\right] = \left(\mu+\sigma^2\right)\exp\left(\frac{\sigma^2}{2}+\mu\right).$$

*Proof.* A straightforward calculation yields

$$\begin{aligned}
\mathbb{E}[|X|\exp(X)] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}|x|\exp(x)\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\mathrm{d}x \\
&= \exp\left(\mu+\frac{\sigma^2}{2}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}|x|\exp\Big(-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}\Big)\mathrm{d}x \\
&= \exp\left(\mu+\frac{\sigma^2}{2}\right)\mathbb{E}[|\widetilde{X}|] < \infty,
\end{aligned}$$

where $\widetilde{X} \sim \mathcal{N}(\mu+\sigma^2, \sigma^2)$. The expectation in the last line is finite since $|x| \leq \frac{1}{2}(x^2+1)$ holds for all $x \in \mathbb{R}$ and the second moment of a normally distributed random variable is finite.

We can reuse the previous calculation for the second assertion of the lemma. Thus,

$$\begin{aligned}
\mathbb{E}[X\exp(X)] &= \exp\left(\mu+\frac{\sigma^2}{2}\right)\mathbb{E}[\widetilde{X}] \\
&= \left(\mu+\sigma^2\right)\exp\left(\mu+\frac{\sigma^2}{2}\right). \qquad \square
\end{aligned}$$

# Appendix B

## Additional material for Chapter 7

In the following, we prove two verification results used in the proofs of Theorems 7.2 and 7.12.

### B.1. Supplements for the proof of Theorem 7.2

Throughout this section, we work in the setting of Section 7.3 and use the notation defined there. Lemma B.1 verifies that the optimal strategy determined in the proof of Theorem 7.2 is, in fact, the unique optimal solution to the best response problem.

**Lemma B.1.** *Suppose that the assumptions of Theorem 7.2 are satisfied. For some arbitrary but fixed $i \in \{1, \dots, n\}$, let*

$$G(t, y) = -\exp\left(-\rho(T-t) - \frac{1}{\delta_i} y\right), \; t \in [0, T], \; y \in \mathbb{R}, \tag{B.1}$$

*where $\rho$ is taken from (7.21), and let $u^* = u^*(t, y)$ be the unique maximizer of*

$$h : \mathbb{R}^d \to \mathbb{R}, \; \varphi \mapsto \frac{G(t, y)}{\delta_i^2} \left(\frac{1}{2}\varphi^\top \left(\sigma\sigma^\top - \frac{2\delta_i}{n} A\right)\varphi - \delta_i \varphi^\top \widetilde{\mu}^{-i}\right)$$

*for $(t, y) \in [0, T] \times \mathbb{R}$.*

*Then $J(t, y) = G(t, y)$ for all $(t, y) \in [0, T] \times \mathbb{R}$ and $\varphi_t^* = u^*(t, Y_t^*)$ is the unique optimal solution to (7.11), where $Y^*$ solves the stochastic differential equation*

$$\mathrm{d}Y_t = u^*(t, Y_t)^\top \left(\left(\widetilde{\mu}^{-i} + \frac{1}{n} A u^*(t, Y_t)\right)\mathrm{d}t + \sigma \mathrm{d}W(t)\right), \; t \in [0, T], \; Y_0 = \widetilde{x}_0^i. \tag{B.2}$$

*Proof.* The proof follows arguments of standard verification theorems (see, e.g., Björk, 2004, pp. 280–282; Pham, 2009, pp. 47–49). First, we define an operator $\mathcal{T}$ to rewrite the HJB equation

(7.19) as follows

$$0=\sup_{\varphi\in\mathbb{R}^d}\left\{F_t+F_y\varphi^\top\left(\widetilde{\mu}^{-i}+\frac{1}{n}A\varphi\right)+\frac{1}{2}F_{yy}\varphi^\top\sigma\sigma^\top\varphi\right\}=:\sup_{\varphi\in\mathbb{R}^d}\mathcal{T}F(t,y,\varphi),\tag{B.3}$$

where $F\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, $t\in[0,T]$, and $y\in\mathbb{R}$. Now we apply the Itô-Doeblin formula (Theorem 2.1) to the function $G$ from (B.1) and the process $Y^{i,\varphi}$ from (7.16), where $\varphi\in\mathcal{A}$, over the time interval $[t,T]$ for some $t\in[0,T]$. It follows

$$\begin{aligned}
&G(T,Y_T^{i,\varphi})\\
&=G(t,Y_t^{i,\varphi})+\int_t^T G_t(s,Y_s^{i,\varphi})\mathrm{d}s+\int_t^T G_y(s,Y_s^{i,\varphi})\mathrm{d}Y_s^{i,\varphi}+\frac{1}{2}\int_t^T G_{yy}(s,Y_s^{i,\varphi})\mathrm{d}\langle Y^{i,\varphi}\rangle_s\\
&=G(t,Y_t^{i,\varphi})+\int_t^T G_t(s,Y_s^{i,\varphi})\mathrm{d}s+\int_t^T G_y(s,Y_s^{i,\varphi})\varphi(s)^\top\left(\widetilde{\mu}^{-i}+\frac{1}{n}A\varphi(s)\right)\mathrm{d}s\\
&\quad+\int_t^T G_y(s,Y_s^{i,\varphi})\varphi(s)^\top\sigma\mathrm{d}W(s)+\frac{1}{2}\int_t^T G_{yy}(s,Y_s^{i,\varphi})\varphi(s)^\top\sigma\sigma^\top\varphi(s)\mathrm{d}s\\
&=G(t,Y_t^{i,\varphi})+\int_t^T \mathcal{T}G(s,Y_s^{i,\varphi},\varphi(s))\mathrm{d}s+\int_t^T G_y(s,Y_s^{i,\varphi})\varphi(s)^\top\sigma\mathrm{d}W(s).
\end{aligned}\tag{B.4}$$

By $G_t$, $G_y$, and $G_{yy}$, we denote the respective partial derivatives of $G$ with respect to $t$ and $y$. Further, we applied Lemma 2.2 to determine the quadratic variation of $Y^{i,\varphi}$. Let us now assume that the process

$$\int_0^t G_y(s,Y_s^{i,\varphi})\varphi(s)^\top\sigma\mathrm{d}W(s),\ t\in[0,T],\tag{B.5}$$

is a martingale for any admissible $\varphi$. Then we can apply the conditional expectation $\mathbb{E}^{t,y}$ for some $y\in\mathbb{R}$ to both sides of (B.4), using that $G(T,y)=-\mathrm{e}^{-\frac{1}{\delta_i}y}$, to obtain

$$\begin{aligned}
&\mathbb{E}^{t,y}\left[-\exp\left(-\frac{1}{\delta_i}Y_T^{i,\varphi}\right)\right]\\
&=G(t,y)+\mathbb{E}^{t,y}\left[\int_t^T\mathcal{T}G(s,Y_s^{i,\varphi},\varphi(s))\mathrm{d}s+\int_t^T G_y(s,Y_s^{i,\varphi})\varphi(s)^\top\sigma\mathrm{d}W(s)\right]\\
&\leq G(t,y)+\mathbb{E}^{t,y}\left[\int_t^T G_y(s,Y_s^{i,\varphi})\varphi(s)^\top\sigma\mathrm{d}W(s)\right] &\text{(B.6)}\\
&=G(t,y). &\text{(B.7)}
\end{aligned}$$

The last step (B.7) holds due to the assumption that (B.5) is a martingale. Moreover, (B.6) follows since $G$ is a solution to the HJB equation and hence, $\mathcal{T}G(s,Y_s^{i,\varphi},\varphi(s))\leq 0$ for all $s\in[0,T]$ and any admissible $\varphi$. Then it follows by the definition of the value function $J$

$$J(t,y)=\sup_{\varphi\in\mathcal{A}}\mathbb{E}^{t,y}\left[-\exp\left(-\frac{1}{\delta_i}Y_T^{i,\varphi}\right)\right]\leq G(t,y)\tag{B.8}$$

for all $(t,y)\in[0,T]\times\mathbb{R}$. Moreover, since $u^*=u^*(t,y)$ maximizes $h$ and thus, $\mathcal{T}G(t,y,\cdot)$, we obtain equality in (B.6) for $\varphi^*(s)=u^*(s,Y_s^*)$, where $Y^*$ solves the stochastic differential equation (B.2). Thus, we also obtain equality in (B.8). Hence, $\varphi^*(s)=u^*(s,Y_s^*)$ is an optimal solution to (7.11). Moreover, the solution $G$ to the HJB equation is equal to the almost surely uniquely defined value function $J$ and thus, $G$ is the unique solution to the HJB equation. To see that $\varphi^*$

is the unique (up to modifications) optimal solution to (7.13), we note that any optimal strategy satisfies the Bellman optimality principle. Since we already found the (unique) value function, this implies that optimal strategies are precisely given by the extremal points in (B.3). Thus, the optimal strategy is unique due to the strict concavity of the expression inside the supremum.

Hence, it only remains to show that the process (B.5) is a martingale. For now, we assume that $\varphi$ is bounded, i.e., there exists a constant $K > 0$ such that $\|\varphi(t)\| \leq K$ $\mathbb{P}$-almost surely for all $t \in [0, T]$. In order to show that (B.5) is a martingale, it suffices to prove that the integrand is square integrable, i.e., that

$$\mathbb{E}\left[\int_0^T \left\| G_y(t, Y_t^{i,\varphi})\varphi(t)^\top \sigma \right\|^2 \mathrm{d}t\right] < \infty.$$

This is a general result arising in the construction of the Itô integral with respect to a Brownian motion. It can, for example, be found in Theorem 4.3.1 in Shreve (2004). Moreover, since we assumed that $\varphi$ is bounded and $\sigma$ is deterministic and constant, it is enough to show that

$$\mathbb{E}\left[\int_0^T G_y(t, Y_t^{i,\varphi})^2 \mathrm{d}t\right] = \int_0^T \mathbb{E}\left[G_y(t, Y_t^{i,\varphi})^2\right] \mathrm{d}t < \infty, \tag{B.9}$$

where we used Fubini's theorem in the first equality. Using the representation

$$Y_t^{i,\varphi} = \widetilde{x}_0^i + \int_0^t \varphi(s)^\top \left(\widetilde{\mu}^{-i} + \frac{1}{n} A\varphi(s)\right) \mathrm{d}s + \int_0^t \varphi(s)^\top \sigma \mathrm{d}W(s),$$

we can calculate the expectation in (B.9)

$$\begin{aligned}
&\mathbb{E}\left[G_y(t, Y_t^{i,\varphi})^2\right] \\
=&\mathbb{E}\left[\frac{1}{\delta_i^2} \exp\left(-2\rho(T-t) - \frac{2}{\delta_i} Y_t^{i,\varphi}\right)\right] \\
=&\frac{1}{\delta_i^2} \mathrm{e}^{-2\rho(T-t) - \frac{2}{\delta_i}\widetilde{x}_0^i} \mathbb{E}\left[\exp\left(-\frac{2}{\delta_i}\int_0^t \varphi(s)^\top \left(\widetilde{\mu}^{-i} + \frac{1}{n}A\varphi(s)\right)\mathrm{d}s - \frac{2}{\delta_i}\int_0^t \varphi(s)^\top \sigma \mathrm{d}W(s)\right)\right] \\
=:&\frac{1}{\delta_i^2} \mathrm{e}^{-2\rho(T-t) - \frac{2}{\delta_i}\widetilde{x}_0^i} \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\exp\left(-\frac{2}{\delta_i}\int_0^t \varphi(s)^\top \left(\widetilde{\mu}^{-i} + \frac{1}{n}A\varphi(s) - \frac{1}{\delta_i}\sigma\sigma^\top\varphi(s)\right)\mathrm{d}s\right)\right] &\text{(B.10)} \\
\leq&\frac{1}{\delta_i^2} \mathrm{e}^{-2\rho(T-t) - \frac{2}{\delta_i}\widetilde{x}_0^i} \cdot \exp(C \cdot t). &\text{(B.11)}
\end{aligned}$$

In (B.10), we introduced an equivalent probability measure $\widetilde{\mathbb{Q}} \sim \mathbb{P}$ with density

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{2}{\delta_i}\int_0^t \varphi(s)^\top \sigma \mathrm{d}W(s) - \frac{2}{\delta_i^2}\int_0^t \varphi(s)^\top \sigma\sigma^\top \varphi(s)\mathrm{d}s\right).$$

Since $\varphi$ was assumed to be bounded, this is in fact a density. In the last step (B.11), we used the assumption that $\varphi$ is bounded again and thus, the integrand is bounded by some constant $C > 0$. Now we can deduce further that the integral in (B.9) is finite and thus, (B.5) is in fact a martingale.

Since the strategy we derived in the proof of Theorem 7.2 is constant and not at the boundary of the interval the strategy $\varphi$ was restricted to in the beginning, the dominated convergence theorem for stochastic processes (Theorem 32 in Protter, 2005, p.176) yields that the assumption that $\varphi$ is bounded is not a restriction (see also Korn and Desmettre, 2014, p.294; Pham, 2009, pp.47–48). This concludes our proof. □

## B.2. Supplements for the proof of Theorem 7.12

We use notation taken from Section 7.4 throughout this section. The lemma below verifies that the optimal strategy determined in the proof of Theorem 7.12 is, in fact, the unique optimal solution to the best response problem (7.37) for constant Nash equilibria.

**Lemma B.2.** *Assume that the assumptions of Theorem 7.12 are satisfied. For some arbitrary but fixed $i \in \{1, \dots, n\}$, let $\pi^j$, $j \neq i$, be deterministic and constant, and*

$$G(t,x,y) = \frac{\delta_i}{\delta_i - 1} \mathrm{e}^{\rho(T-t)} \left( x y^{-\frac{\theta_i}{n}} \right)^{\frac{\delta_i - 1}{\delta_i}}, \; t \in [0,T],\; x, y \in (0,\infty), \tag{B.12}$$

*where $\rho \in \mathbb{R}$ is taken from (7.46), and let $u^* = u^*(t,x,y)$ be the unique maximizer of $h : \mathbb{R}^d \to \mathbb{R}$,*

$$h(\pi) = \frac{\delta_i - 1}{\delta_i^2} G(t,x,y) \cdot \pi^\top \left\{ \delta_i \mu + \left( \frac{\delta_i}{n}\Big(1 - \frac{\theta_i}{n}\Big) A - \frac{\theta_i}{n}(\delta_i - 1)\sigma\sigma^\top \right) \sum_{j \neq i} \pi^j - \frac{1}{2}\left( \sigma\sigma^\top - \frac{2\delta_i}{n} A \right) \pi \right\}$$

*for $(t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty)$.*

*Then $J(t,x,y) = G(t,x,y)$ for all $(t,x,y) \in [0,T] \times (0,\infty) \times (0,\infty)$ and $\pi_t^* = u^*\left(t, \widetilde{X}_t^*, \widetilde{Y}_t^{-i}\right)$ is the unique optimal solution to (7.37), where $X^*$ solves the stochastic differential equation*

$$\mathrm{d}X_t = X_t u^*(t, X_t, \widetilde{Y}_t^{-i})^\top \left( \left( \mu + \frac{1}{n} A u^*(t, X_t, \widetilde{Y}_t^{-i}) + \frac{1}{n}\Big(1 - \frac{\theta_i}{n}\Big) A \sum_{j \neq i} \pi^j \right) \mathrm{d}t + \sigma \mathrm{d}W(t) \right) \tag{B.13}$$

*for $t \in [0,T]$ and $X_0 = x_0^i$.*

*Proof.* The proof follows arguments of standard verification theorems (see, e.g., Björk, 2004, pp. 280–282; Pham, 2009, pp. 47–49). First, we define an operator $\mathcal{T}$ by

$$\begin{aligned}
&\mathcal{T}F(t,x,y,\pi) \\
&= F_t + yF_y \left( \sum_{j \neq i} (\pi^j)^\top \left( \mu + \frac{1}{n} A \sum_{j \neq i} \pi^j \right) + \frac{1}{2} \sum_{\substack{h \neq j \\ h,j \neq i}} (\pi^h)^\top \sigma\sigma^\top \pi^j \right) + \frac{1}{2} y^2 F_{yy} \sum_{h,j \neq i} (\pi^h)^\top \sigma\sigma^\top \pi^j \\
&\qquad + xF_x \pi^\top \mu + \pi^\top \left( \frac{1}{n} xF_x A + \frac{1}{2} x^2 F_{xx} \sigma\sigma^\top \right) \pi + \pi^\top \left( \frac{1}{n}\Big(1 - \frac{\theta_i}{n}\Big) xF_x A + xyF_{xy} \sigma\sigma^\top \right) \sum_{j \neq i} \pi^j,
\end{aligned}$$

where $\pi \in \mathbb{R}^d$, $t \in [0,T]$, $x, y \in (0,\infty)$, and $F \in \mathcal{C}^{1,2,2}([0,T] \times (0,\infty)^2)$. Note that the arguments of $F$ were omitted to simplify notation. Then the HJB equation (7.44) can be written as

$$0 = \sup_{\pi \in \mathbb{R}^d} \mathcal{T}F(t,x,y,\pi) \tag{B.14}$$

with terminal condition $F(T,x,y) = \widetilde{U}(x,y) := \frac{\delta_i}{\delta_i - 1}\left(xy^{-\frac{\theta_i}{n}}\right)^{\frac{\delta_i - 1}{\delta_i}}$.

Now we consider the function $G$ given in (B.12) and apply the Itô-Doeblin formula (Theorem 2.1) to $G$, $\widetilde{X}^{i,\pi}$, and $\widetilde{Y}^{-i}$, where $\pi \in \mathcal{A}$, over the interval $[t,T]$ for some $t \in [0,T]$. It follows

$$\begin{aligned}
&G(T, \widetilde{X}_T^{i,\pi}, \widetilde{Y}_T^{-i}) \\
&= G(t, \widetilde{X}_t^{i,\pi}, \widetilde{Y}_t^{-i}) + \int_t^T G_t \mathrm{d}s + \int_t^T G_x \mathrm{d}\widetilde{X}_s^{i,\pi} + \int_t^T G_y \mathrm{d}\widetilde{Y}_s^{-i} \\
&\quad + \frac{1}{2}\int_t^T G_{xx}\mathrm{d}\langle \widetilde{X}^{i,\pi}\rangle_s + \int_t^T G_{xy}\mathrm{d}\langle \widetilde{X}^{i,\pi}, \widetilde{Y}^{-i}\rangle_s + \frac{1}{2}\int_t^T G_{yy}\mathrm{d}\langle \widetilde{Y}^{-i}\rangle_s \\
&= G(t, \widetilde{X}_t^{i,\pi}, \widetilde{Y}_t^{-i}) + \int_t^T \mathcal{T}G(s, \widetilde{X}_s^{i,\pi}, \widetilde{Y}_s^{-i}, \pi(s))\mathrm{d}s + \int_t^T \Big(G_x \widetilde{X}_s^{i,\pi}\pi(s)^\top \sigma + G_y \widetilde{Y}_s^{-i}\sum_{j\neq i}(\pi^j)^\top\sigma\Big)\mathrm{d}W(s).
\end{aligned}$$

By $G_t$, we denote the first order partial derivative of $G$ with respect to $t$. The other partial derivatives with respect to $t$, $x$, and $y$ are denoted similarly. Let us now assume that the process

$$\left(\int_0^t \Big(G_x(s, \widetilde{X}_s^{i,\pi}, \widetilde{Y}_s^{-i})\widetilde{X}_s^{i,\pi}\pi(s)^\top\sigma + G_y(s, \widetilde{X}_s^{i,\pi}, \widetilde{Y}_s^{-i})\widetilde{Y}_s^{-i}\sum_{j\neq i}(\pi^j)^\top\sigma\Big)\mathrm{d}W(s)\right)_{t\in[0,T]} \tag{B.15}$$

is a martingale for any admissible $\pi$. Then we can apply the conditional expectation $\mathbb{E}^{t,x,y}$ on both sides of the above representation of $G(T, \widetilde{X}_T^{i,\pi}, \widetilde{Y}_T^{-i})$ to obtain

$$\begin{aligned}
\mathbb{E}^{t,x,y}\left[\frac{\delta_i}{\delta_i - 1}\Big(\widetilde{X}_T^{i,\pi}(\widetilde{Y}_T^{-i})^{-\frac{\theta_i}{n}}\Big)^{\frac{\delta_i-1}{\delta_i}}\right] &= \mathbb{E}^{t,x,y}\left[G(t, \widetilde{X}_t^{i,\pi}, \widetilde{Y}_t^{-i}) + \int_t^T \mathcal{T}G(s, \widetilde{X}_s^{i,\pi}, \widetilde{Y}_s^{-i}, \pi(s))\mathrm{d}s\right] \\
&\leq G(t,x,y).
\end{aligned} \tag{B.16}$$

On the left-hand side, we used that $G(T,x,y) = \frac{\delta_i}{\delta_i-1}(xy^{-\frac{\theta_i}{n}})^{\frac{\delta_i-1}{\delta_i}}$ for all $x, y \in (0,\infty)$. In the first equality, we used the assumption that (B.15) is a martingale which implies that the conditional expectation of the stochastic integral vanishes. Further, the inequality in the second step follows since $G$ solves the HJB equation and thus, the supremum of $\mathcal{T}G(t,x,y,\pi)$ over all $\pi \in \mathbb{R}^d$ is equal to 0. Therefore, it follows by the definition of the value function $J$ that

$$J(t,x,y) = \sup_{\pi\in\mathcal{A}} \mathbb{E}^{t,x,y}\left[\frac{\delta_i}{\delta_i - 1}\Big(\widetilde{X}_T^{i,\pi}(\widetilde{Y}_T^{-i})^{-\frac{\theta_i}{n}}\Big)^{\frac{\delta_i-1}{\delta_i}}\right] \leq G(t,x,y), \tag{B.17}$$

i.e., the first assertion of the lemma holds.

Now let $u^* = u^*(t,x,y)$ be the maximizer of the function $h$ given in the lemma. Noticing that the operator $\mathcal{T}$, applied to $G$ and evaluated at $t, x, y$, and $\pi$, can be written as a sum of $h(\pi)$ and some expression independent of $\pi$ (that takes the form $C \cdot G(t,x,y)$ for some constant $C \in \mathbb{R}$), we can deduce that using $\pi = u^*$ gives equality in (B.16) and thus, equality in (B.17). Hence, $\pi_t^* = u^*(t, X_t^*, \widetilde{Y}_t^{-i})$ is an optimal control for the best response problem for (7.37) and $G$ is equal to the value function $J$. By $X^*$, we denote the solution to the stochastic differential equation (B.13) given in the lemma. To verify the uniqueness of $\pi^*$, note that any optimal control satisfies the Bellman optimality principle. Thus, the optimal strategies are precisely the extremal points of the supremum in (B.14). Since the function inside the supremum has a unique maximizer (which we have shown in the proof of Theorem 7.12), $\pi^*$ is the unique (up to modifications) optimal solution to the best response problem associated to (7.37).

The last step remaining in this proof is to show that the process (B.15) is in fact a martingale. For the moment, assume that $\pi$ is bounded, i.e., that there exists a constant $K > 0$ such that $\|\pi(t)\| \leq K$ $\mathbb{P}$-almost surely for all $t \in [0,T]$. First, we notice that it suffices to prove that both summands in (B.15) are martingales to prove that the sum itself is a martingale. According to Theorem 4.3.1 in Shreve (2004), it suffices to show that the integrand is square integrable, i.e., that

$$\mathbb{E}\bigg[\int_0^T \Big\|G_x(t,\widetilde{X}_t^{i,\pi},\widetilde{Y}_t^{-i})\widetilde{X}_t^{i,\pi}\pi(t)^\top\sigma\Big\|^2 \mathrm{d}t\bigg] < \infty,$$

to prove that the first summand defines a martingale. Further, since $\sigma$ is deterministic and constant and $\pi$ is assumed to be bounded, it only remains to show that

$$\mathbb{E}\bigg[\int_0^T \Big(G_x(t,\widetilde{X}_t^{i,\pi},\widetilde{Y}_t^{-i})\widetilde{X}_t^{i,\pi}\Big)^2 \mathrm{d}t\bigg] = \int_0^T \mathbb{E}\Big[\Big(G_x(t,\widetilde{X}_t^{i,\pi},\widetilde{Y}_t^{-i})\widetilde{X}_t^{i,\pi}\Big)^2\Big]\mathrm{d}t < \infty. \tag{B.18}$$

We used Fubini's theorem to interchange integral and expectation in the previous line. Using $x \cdot G_x(t,x,y) = \mathrm{e}^{\rho(T-t)}(xy^{-\frac{\theta_i}{n}})^{\frac{\delta_i-1}{\delta_i}}$ for any $t \in [0,T]$, $x,y \in (0,\infty)$, and recalling that

$$\begin{aligned}
\widetilde{X}_t^{i,\pi} =& x_0^i \exp\bigg(\int_0^t \pi(s)^\top\Big(\mu + \Big(\frac{1}{n}A - \frac{1}{2}\sigma\sigma^\top\Big)\pi(s) + \frac{1}{n}\Big(1-\frac{\theta_i}{n}\Big)A\sum_{j\neq i}\pi^j\Big)\mathrm{d}s + \int_0^t \pi(s)^\top\sigma\mathrm{d}W(s)\bigg),\\
\widetilde{Y}_t^{-i} =& y_0^i \exp\bigg(\int_0^t \Big(\sum_{j\neq i}\pi^j\Big)^\top\Big(\mu + \frac{1}{n}A\sum_{j\neq i}\pi^j\Big)\mathrm{d}s + \int_0^t \Big(\sum_{j\neq i}\pi^j\Big)^\top\sigma\mathrm{d}W(s) - \frac{1}{2}\int_0^t \sum_{j\neq i}(\pi^j)^\top\sigma\sigma^\top\pi^j\mathrm{d}s\bigg),
\end{aligned}$$

we obtain the following bound for the expectation in (B.18)

$$\begin{aligned}
&\mathrm{e}^{-2\rho(T-t)}\Big(x_0^i(y_0^i)^{-\frac{\theta_i}{n}}\Big)^{-\frac{2(\delta_i-1)}{\delta_i}}\mathbb{E}\Big[\Big(G_x(t,\widetilde{X}_t^{i,\pi},\widetilde{Y}_t^{-i})\widetilde{X}_t^{i,\pi}\Big)^2\Big]\\
=&\mathbb{E}\bigg[\exp\bigg(\frac{2(\delta_i-1)}{\delta_i}\bigg\{\int_0^t\pi(s)^\top\Big(\mu+\Big(\frac{1}{n}A-\frac{1}{2}\sigma\sigma^\top\Big)\pi(s)+\frac{1}{n}\Big(1-\frac{\theta_i}{n}\Big)A\sum_{j\neq i}\pi^j\Big)\mathrm{d}s+\int_0^t\pi(s)^\top\sigma\mathrm{d}W(s)\bigg\}\\
&-\frac{2\theta_i(\delta_i-1)}{n\delta_i}\bigg\{\int_0^t\bigg(\Big(\sum_{j\neq i}\pi^j\Big)^\top\Big(\mu+\frac{1}{n}A\sum_{j\neq i}\pi^j\Big)-\frac{1}{2}\sum_{j\neq i}(\pi^j)^\top\sigma\sigma^\top\pi^j\bigg)\mathrm{d}s+\int_0^t\Big(\sum_{j\neq i}\pi^j\Big)^\top\sigma\mathrm{d}W(s)\bigg\}\bigg)\bigg]\\
=&\exp\bigg(-\frac{2\theta_i(\delta_i-1)}{n\delta_i}\bigg(\Big(\sum_{j\neq i}\pi^j\Big)^\top\Big(\mu+\frac{1}{n}A\sum_{j\neq i}\pi^j\Big)-\frac{1}{2}\sum_{j\neq i}(\pi^j)^\top\sigma\sigma^\top\pi^j\bigg)t\bigg)\\
&\cdot\mathbb{E}\bigg[\exp\bigg(\frac{2(\delta_i-1)}{\delta_i}\bigg\{\int_0^t\pi(s)^\top\Big(\mu+\Big(\frac{1}{n}A-\frac{1}{2}\sigma\sigma^\top\Big)\pi(s)+\frac{1}{n}\Big(1-\frac{\theta_i}{n}\Big)A\sum_{j\neq i}\pi^j\Big)\mathrm{d}s+\int_0^t\pi(s)^\top\sigma\mathrm{d}W(s)\bigg\}\\
&-\frac{2\theta_i(\delta_i-1)}{n\delta_i}\int_0^t\Big(\sum_{j\neq i}\pi^j\Big)^\top\sigma\mathrm{d}W(s)\bigg)\bigg]\\
=:&\exp(C\cdot t)\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\bigg[\exp\bigg(\frac{2(\delta_i-1)}{\delta_i}\int_0^t\pi(s)^\top\bigg(\mu+\Big(\frac{1}{n}A-\frac{1}{2}\sigma\sigma^\top\Big)\pi(s)+\frac{1}{n}\Big(1-\frac{\theta_i}{n}\Big)A\sum_{j\neq i}\pi^j\bigg)\mathrm{d}s\bigg)
\end{aligned}$$

$$\cdot \exp\Bigg(\frac{2(\delta_i-1)^2}{\delta_i^2}\int_0^t \Big(\pi(s)-\frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^\top \sigma\sigma^\top \Big(\pi(s)-\frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)\mathrm{d}s\Bigg)\Bigg] \tag{B.19}$$

$$\leq \exp(\widetilde{C}\cdot t). \tag{B.20}$$

In (B.19), we abbreviated the constant in the first exponential factor by $C$ and introduced an equivalent probability measure $\widetilde{\mathbb{Q}} \sim \mathbb{P}$ with density

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \exp\Bigg(\frac{2(\delta_i-1)}{\delta_i}\int_0^t \Big(\pi(s)-\frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^\top \sigma \mathrm{d}W(s) - \frac{2(\delta_i-1)^2}{\delta_i^2}\int_0^t \Big(\pi(s)-\frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)^\top \sigma\sigma^\top \Big(\pi(s)-\frac{\theta_i}{n}\sum_{j\neq i}\pi^j\Big)\mathrm{d}s\Bigg). \tag{B.21}$$

Note that, since $\pi$ was assumed to be bounded, (B.21) is indeed a density. Moreover, in (B.20) we used that $\pi$ is bounded and that the other expressions appearing in the integral are constant. Thus, the integrand can be bounded by a constant. In combination with the constant $C$ introduced in (B.19), we obtain the bound (B.20). Thus, the integral in (B.18) is finite and the first summand of (B.15) is a martingale. Proving that the second summand in (B.15) is also a martingale proceeds analogously since

$$y\cdot G_y(t,x,y) = -\frac{\theta_i}{n}\mathrm{e}^{\rho(T-t)}\Big(xy^{-\frac{\theta_i}{n}}\Big)^{\frac{\delta_i-1}{\delta_i}}$$

for all $t\in[0,T]$ and $x,y\in(0,\infty)$. Thus, up to a constant, the expectation

$$\mathbb{E}\Bigg[\int_0^T \Big(G_y(t,\widetilde{X}_t^{i,\pi},\widetilde{Y}_t^{-i})\widetilde{Y}_t^{-i}\Big)^2\mathrm{d}t\Bigg]$$

coincides with (B.18) and is, thus, finite. Finally, let us recall that the candidate optimal strategy determined in the proof of Theorem 7.12 is constant and not at the boundary of the restriction interval. Hence, the dominated convergence theorem for stochastic integrals (Theorem 32 in Protter, 2005, p.176) yields that the assumption that $\pi$ is bounded is not a restriction (see also Korn and Desmettre (2014), p.294; Pham, 2009, pp.47–48). This concludes the proof. □

# APPENDIX C

## Additional material for Chapter 8

The following chapter contains additional material for Chapter 8. Section C.1 comprises proofs of Lemma 8.8 and Proposition 8.9, which provide an explicit representation of the Nash equilibrium in terms of terminal wealth of two agents using logarithmic utility. In Section C.2, we explain how the $n$–dimensional fixed point problem (8.6) can be reduced to an $(n-1)$–dimensional fixed point problem. This technique is used in Section C.3 to solve the fixed point problem for three agents. Afterwards, we apply the result for three agents to the special case of logarithmic utility. At the end of Section C.3, we illustrate the Nash equilibrium for three agents under logarithmic utility for varying values of the state price density $Z_T$ and for different parameter choices. The fourth and last section of this chapter contains the proofs of four lemmas used to prove Theorem 8.11 and Theorem 8.16.

### C.1. Proofs of Lemma 8.8 and Proposition 8.9

First, we display the proof of Lemma 8.8, in which the Lagrange multipliers from (8.12) are determined explicitly if both agents use the natural logarithm as their utility function.

*Proof (Lemma 8.8).* The proof consists of two steps. In the first step, we compute the expected values $\mathbb{E}[X_i^* Z_T]$, $i = 1, 2$. Afterwards, we solve a two-dimensional system of nonlinear equations to determine the Lagrange multipliers $\lambda_1$ and $\lambda_2$.

Let us determine the expected value $\mathbb{E}[Z_T X_1^*]$. It follows

$$X_1^* Z_T = \left(\frac{1}{\lambda_1}\mathbb{1}\left\{\lambda_1\beta_2 \leq \lambda_2\right\} + \frac{\beta_2}{\lambda_2}\mathbb{1}\left\{\lambda_1\beta_2 > \lambda_2\right\}\right)\mathbb{1}\left\{Z_T \leq \chi_{\alpha_1}\right\} + \frac{1}{\lambda_1}\mathbb{1}\left\{Z_T > \chi_{\alpha_1}\right\}.$$

Hence, since $\chi_{\alpha_1}$ denotes the $\alpha_1$-quantile of $Z_T$,

$$\begin{aligned}\mathbb{E}[X_1^* Z_T] &= \alpha_1\left(\frac{1}{\lambda_1}\mathbb{1}\left\{\lambda_1\beta_2\le\lambda_2\right\}+\frac{\beta_2}{\lambda_2}\mathbb{1}\left\{\lambda_1\beta_2>\lambda_2\right\}\right)+\frac{1}{\lambda_1}(1-\alpha_1)\\ &=: \alpha_1\left(\gamma_1\mathbb{1}\left\{\gamma_1\ge\beta_2\gamma_2\right\}+\beta_2\gamma_2\mathbb{1}\left\{\gamma_1<\beta_2\gamma_2\right\}\right)+(1-\alpha_1)\gamma_1\\ &=: f_1(\gamma_1,\gamma_2),\end{aligned}$$

where we defined $\gamma_i := \lambda_i^{-1}$, $i=1,2$, to simplify notation. Analogously, we obtain

$$\begin{aligned}\mathbb{E}[X_2^* Z_T] &= \alpha_2\left(\gamma_2\mathbb{1}\left\{\gamma_2\ge\beta_1\gamma_1\right\}+\beta_1\gamma_1\mathbb{1}\left\{\gamma_2<\beta_1\gamma_1\right\}\right)+(1-\alpha_2)\gamma_2\\ &=: f_2(\gamma_1,\gamma_2).\end{aligned}$$

Now it remains to solve the system of equations

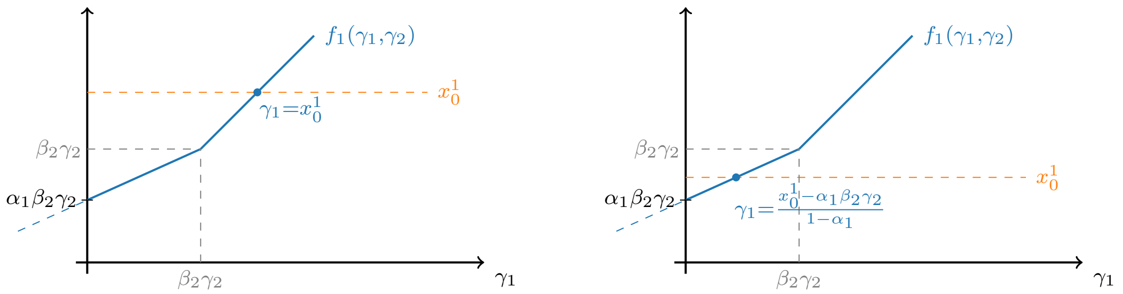
$$f_1(\gamma_1,\gamma_2)=x_0^1,\ f_2(\gamma_1,\gamma_2)=x_0^2.$$

Clearly, $f_1(\gamma_1,\gamma_2)$ can be rewritten as

$$f_1(\gamma_1,\gamma_2)=\gamma_1\mathbb{1}\left\{\gamma_1\ge\beta_2\gamma_2\right\}+(\alpha_1\beta_2\gamma_2+(1-\alpha_1)\gamma_1)\mathbb{1}\left\{\gamma_1<\beta_2\gamma_2\right\}\overset{!}{=}x_0^1.$$

For fixed $\gamma_2$, the unique solution $\gamma_1$ to this equation is given by (using a case distinction based on whether or not $x_0^1\ge\beta_2\gamma_2$ and the sketch of $f_1$ below)

$$\gamma_1 = x_0^1\mathbb{1}\left\{x_0^1\ge\beta_2\gamma_2\right\}+\frac{x_0^1-\alpha_1\beta_2\gamma_2}{1-\alpha_1}\mathbb{1}\left\{x_0^1<\beta_2\gamma_2\right\}. \tag{C.1}$$




(a) $x_0^1\ge\beta_2\gamma_2$ (b) $x_0^1<\beta_2\gamma_2$

Since the representation (C.1) of $\gamma_1$ depends on $\gamma_2$, we insert $\gamma_1$ into the equation $f_2(\gamma_1,\gamma_2)=x_0^2$ to find an explicit solution for $\gamma_2$, which can then be inserted into (C.1) to determine an explicit solution for $\gamma_1$ as well. First, we rewrite $f_2(\gamma_1,\gamma_2)$ as

$$f_2(\gamma_1,\gamma_2)=\gamma_2\mathbb{1}\left\{\gamma_2\ge\beta_1\gamma_1\right\}+(\alpha_2\beta_1\gamma_1+(1-\alpha_2)\gamma_2)\mathbb{1}\left\{\gamma_2<\beta_1\gamma_1\right\}.$$

Now it remains to find an explicit representation for $\gamma_1$. Using (C.1), it follows

$$\left\{\gamma_2\ge\beta_1\gamma_1\right\}=\left(\left\{\gamma_2\ge\beta_1x_0^1\right\}\cap\left\{x_0^1\ge\beta_2\gamma_2\right\}\right)\cup\left(\left\{\gamma_2\ge\beta_1\frac{x_0^1-\alpha_1\beta_2\gamma_2}{1-\alpha_1}\right\}\cap\left\{x_0^1<\beta_2\gamma_2\right\}\right)$$

$$
\begin{aligned}
&= \left\{\beta_1\beta_2 x_0^1 \le \beta_2\gamma_2 \le x_0^1\right\} \cup \left\{\beta_2\gamma_2 > x_0^1\right\} && \text{(C.2)}\\
&= \left\{\gamma_2 \ge \beta_1 x_0^1\right\}.
\end{aligned}
$$

(C.2) holds since

$$
\gamma_2 \ge \beta_1 \frac{x_0^1 - \alpha_1\beta_2\gamma_2}{1-\alpha_1} \iff \gamma_2 \ge \frac{\beta_1}{1-\alpha_1(1-\beta_1\beta_2)} x_0^1
$$

and

$$
\begin{aligned}
\beta_1\beta_2 < 1 &\iff (1-\alpha_1)\beta_1\beta_2 < 1-\alpha_1\\
&\iff \beta_1\beta_2 < 1-\alpha_1(1-\beta_1\beta_2)\\
&\iff \frac{\beta_1}{1-\alpha_1(1-\beta_1\beta_2)} < \frac{1}{\beta_2}.
\end{aligned}
$$

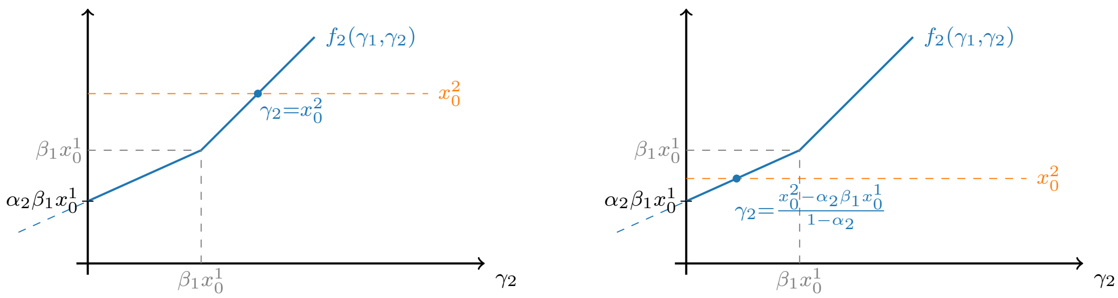
Since $\frac{x_0^1}{\beta_2} > x_0^1 > \beta_1 x_0^1$, $\gamma_1$ given in (C.1) takes the value $x_0^1$ if $\gamma_2 < \beta_1 x_0^1$. Therefore,

$$
f_2(\gamma_1,\gamma_2) = \gamma_2 \mathbb{1}\left\{\gamma_2 \ge \beta_1 x_0^1\right\} + (\alpha_2\beta_1 x_0^1 + (1-\alpha_2)\gamma_2)\mathbb{1}\left\{\gamma_2 < \beta_1 x_0^1\right\} \stackrel{!}{=} x_0^2.
$$

The previous equation has the following unique solution (using a case distinction on whether or not $x_0^2 \ge \beta_1 x_0^1$ and the sketch of $f_2$ below)

$$
\gamma_2 = x_0^2 \mathbb{1}\left\{x_0^2 \ge \beta_1 x_0^1\right\} + \frac{x_0^2 - \alpha_2\beta_1 x_0^1}{1-\alpha_2}\mathbb{1}\left\{x_0^2 < \beta_1 x_0^1\right\}, \tag{C.3}
$$

which is positive due to the assumption $x_0^2 > \alpha_2\beta_1 x_0^1$ of the lemma.




(a) $x_0^2 \ge \beta_1 x_0^1$ (b) $x_0^2 < \beta_1 x_0^1$

Using (C.3), it follows

$$
\begin{aligned}
\{x_0^1 \ge \beta_2\gamma_2\} &= \left(\left\{x_0^1 \ge \beta_2 x_0^2\right\} \cap \left\{x_0^2 \ge \beta_1 x_0^1\right\}\right) \cup \left(\left\{x_0^1 \ge \beta_2 \frac{x_0^2 - \alpha_2\beta_1 x_0^1}{1-\alpha_2}\right\} \cap \left\{x_0^2 < \beta_1 x_0^1\right\}\right)\\
&= \left\{\beta_1\beta_2 x_0^2 \le \beta_1 x_0^1 \le x_0^2\right\} \cup \left\{x_0^2 < \beta_1 x_0^1\right\} && \text{(C.4)}\\
&= \left\{x_0^1 \ge \beta_2 x_0^2\right\}. && \text{(C.5)}
\end{aligned}
$$

(C.4) follows analogously to (C.2). Moreover, since $\frac{x_0^2}{\beta_1} > x_0^2 > \beta_2 x_0^2$, $\gamma_2$ takes the value $x_0^2$ if $x_0^1 < \beta_2 x_0^2$. Now we can insert (C.3) into (C.1) and apply (C.5) to find an explicit representation of $\gamma_1$. It follows

$$\gamma_1 = x_0^1 \mathbb{1}\left\{x_0^1 \geq \beta_2 x_0^2\right\} + \frac{x_0^1 - \alpha_1\beta_2 x_0^2}{1-\alpha_1} \mathbb{1}\left\{x_0^1 < \beta_2 x_0^2\right\}.$$

Note that the assumption $x_0^1 > \alpha_1\beta_2 x_0^2$ of the lemma implies $\gamma_1 > 0$. Using the definition of $\gamma_i = \lambda_i^{-1}$, $i = 1, 2$, concludes our proof. □

With the Lagrange multipliers from the previous proof, we can now give an explicit representation of the fixed point for two agents using the natural logarithm as their utility functions. Thus, the proof of Proposition 8.9 proceeds as follows.

*Proof (Proposition 8.9).* a) If $x_0^1 \geq \beta_2 x_0^2$ and $x_0^2 \geq \beta_1 x_0^1$, using Lemma 8.8, the Lagrange multipliers are given by

$$\lambda_1 = \frac{1}{x_0^1},\ \lambda_2 = \frac{1}{x_0^2}.$$

Hence, it follows that

$$\lambda_1\beta_2 \leq \lambda_2,\ \lambda_2\beta_1 \leq \lambda_1.$$

Therefore, (8.12) implies

$$X_1^* = \frac{1}{\lambda_1 Z_T} = x_0^1 Z_T^{-1},\ X_2^* = \frac{1}{\lambda_2 Z_T} = x_0^2 Z_T^{-1}.$$

b) If $x_0^1 < \beta_2 x_0^2$, it follows

$$x_0^2 > \frac{x_0^1}{\beta_2} > x_0^1 > \beta_1 x_0^1$$

and thus,

$$\lambda_1 = \frac{1-\alpha_1}{x_0^1 - \alpha_1\beta_2 x_0^2},\ \lambda_2 = \frac{1}{x_0^2}.$$

This implies

$$\beta_2\lambda_1 > \lambda_2,\ \beta_1\lambda_2 \leq \lambda_1$$

and therefore, using (8.12),

$$\begin{aligned}
X_1^* &= \frac{\beta_2}{\lambda_2 Z_T}\mathbb{1}\{Z_T \leq \chi_{\alpha_1}\} + \frac{1}{\lambda_1 Z_T}\mathbb{1}\{Z_T > \chi_{\alpha_1}\} \\
&= \beta_2 x_0^2 Z_T^{-1}\mathbb{1}\{Z_T \leq \chi_{\alpha_1}\} + \frac{x_0^1 - \alpha_1\beta_2 x_0^2}{1-\alpha_1} Z_T^{-1}\mathbb{1}\{Z_T > \chi_{\alpha_1}\}, \\
X_2^* &= \frac{1}{\lambda_2 Z_T} = x_0^2 Z_T^{-1}.
\end{aligned}$$

c) The proof in case c) follows, due to symmetry, analogously to case b). □

## C.2. Dimensional reduction of the fixed point problem

In the following, we demonstrate how to reduce the $n$-dimensional fixed point problem (8.6) to an $(n-1)$-dimensional fixed point problem with a similar structure. We use the abbreviation $I_i = I_i(\lambda_i Z_T)$, $i = 1, \ldots, n$, throughout this section.

We can write the fixed point problem (8.6) as

$$\begin{aligned} X_i &= \left(I_i \mathbb{1}\left\{I_i \geq X_\beta^{-i}\right\} + X_\beta^{-i} \mathbb{1}\left\{I_i < X_\beta^{-i}\right\}\right) \mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} + I_i \mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\} \\ &= \max\left\{I_i, X_\beta^{-i}\right\} \mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} + I_i \mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}, \end{aligned} \tag{C.6}$$

$i = 1, \ldots, n$, where we defined

$$X_\beta^{-j_1,\ldots,j_k} := \sum_{\substack{j=1 \\ j \notin \{j_1,\ldots,j_k\}}}^{n} \beta_j X_j, \;\; 0 \leq k \leq n, \;\; j_\ell \in \{1, \ldots, n\}, \;\; 1 \leq \ell \leq k.$$

In the following, we insert $X_n$ into $X_i$ for all $1 \leq i \leq n-1$ to reduce the $n$-dimensional fixed point problem given by (C.6) to an $(n-1)$-dimensional fixed point problem. If we are able to solve the reduced problem, we can easily determine $X_n$ by inserting the solution $X_1, \ldots, X_{n-1}$ to the reduced problem.

Without loss of generality, we assume that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. Moreover, let $1 \leq i \leq n-1$. First, we insert $X_n$ into the first indicator function in (C.6) to obtain

$$\begin{aligned} &\mathbb{1}\left\{I_i \geq X_\beta^{-i}\right\} \\ =&\mathbb{1}\left\{I_i \geq X_\beta^{-i,n} + \beta_n \left(I_n \mathbb{1}\left\{I_n \geq X_\beta^{-n}\right\} + X_\beta^{-n} \mathbb{1}\left\{I_n < X_\beta^{-n}\right\}\right) \mathbb{1}\left\{Z_T \leq \chi_{\alpha_n}\right\} + \beta_n I_n \mathbb{1}\left\{Z_T > \chi_{\alpha_n}\right\}\right\} \\ =&\mathbb{1}\left\{Z_T \leq \chi_{\alpha_n}\right\} \left(\mathbb{1}\left\{I_i \geq X_\beta^{-i,n} + \beta_n I_n\right\} \mathbb{1}\left\{I_n \geq X_\beta^{-n}\right\} + \mathbb{1}\left\{I_i \geq X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right\} \mathbb{1}\left\{I_n < X_\beta^{-n}\right\}\right) \\ &+ \mathbb{1}\left\{Z_T > \chi_{\alpha_n}\right\} \mathbb{1}\left\{I_i \geq X_\beta^{-i,n} + \beta_n I_n\right\}. \end{aligned}$$

The second indicator function can be simplified analogously. Since we assumed that $\alpha_1, \ldots, \alpha_n$ are in ascending order, it follows

$$\begin{aligned} &\mathbb{1}\left\{I_i \geq X_\beta^{-i}\right\} \cdot \mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} \\ =&\mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} \left(\mathbb{1}\left\{I_i \geq X_\beta^{-i,n} + \beta_n I_n\right\} \mathbb{1}\left\{I_n \geq X_\beta^{-n}\right\} + \mathbb{1}\left\{I_i \geq X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right\} \mathbb{1}\left\{I_n < X_\beta^{-n}\right\}\right). \end{aligned}$$

Now we insert $X_n$ into the summand $X_\beta^{-i} \mathbb{1}\left\{I_i < X_\beta^{-i}\right\} \mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\}$ of (C.6) to obtain

$$\begin{aligned} &X_\beta^{-i} \mathbb{1}\left\{I_i < X_\beta^{-i}\right\} \mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} \\ =&\mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} \left(X_\beta^{-i,n} + \left(\beta_n I_n \mathbb{1}\left\{I_n \geq X_\beta^{-n}\right\} + \beta_n X_\beta^{-n} \mathbb{1}\left\{I_n < X_\beta^{-n}\right\}\right)\right) \\ &\cdot \left(\mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n I_n\right\} \mathbb{1}\left\{I_n \geq X_\beta^{-n}\right\} + \mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right\} \mathbb{1}\left\{I_n < X_\beta^{-n}\right\}\right) \\ =&\mathbb{1}\left\{Z_T \leq \chi_{\alpha_i}\right\} \left(\left(X_\beta^{-i,n} + \beta_n I_n\right) \mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n I_n\right\} \mathbb{1}\left\{I_n \geq X_\beta^{-n}\right\}\right. \\ &\left. + \left(X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right) \mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right\} \mathbb{1}\left\{I_n < X_\beta^{-n}\right\}\right). \end{aligned}$$

Finally, we obtain the following implicit representation of $X_i$ in terms of only $X_\ell,\ 1 \le \ell \le n-1$:

$$X_i = \Bigg( \mathbb{1}\left\{I_n \ge X_\beta^{-n}\right\} \left(I_i \mathbb{1}\left\{I_i \ge X_\beta^{-i,n} + \beta_n I_n\right\} + \left(X_\beta^{-i,n} + \beta_n I_n\right)\mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n I_n\right\}\right)$$
$$+ \mathbb{1}\left\{I_n < X_\beta^{-n}\right\} \left(I_i \mathbb{1}\left\{I_i \ge X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right\} + \left(X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right)\mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n X_\beta^{-n}\right\}\right)\Bigg)$$
$$\cdot \mathbb{1}\left\{Z_T \le \chi_{\alpha_i}\right\} + I_i \cdot \mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}.$$

In order to find an explicit representation of $X_i$, we define the right-hand side of the previous representation as $f_i(X_i)$ and solve the one-dimensional fixed point problem $f_i(X_i) = X_i$. First, we rewrite the indicator functions in the definition of $f_i$ in terms of $X_i$ to receive

$$\begin{aligned} f_i(X_i) =& \mathbb{1}\left\{Z_T \le \chi_{\alpha_i}\right\} \cdot \Bigg( \mathbb{1}\left\{X_i \le \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)\right\} \left(I_i \mathbb{1}\left\{I_i \ge X_\beta^{-i,n} + \beta_n I_n\right\}\right. \\ &\left. + \left(X_\beta^{-i,n} + \beta_n I_n\right)\mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n I_n\right\}\right) \\ &+ \mathbb{1}\left\{X_i > \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)\right\} \left(I_i \mathbb{1}\left\{X_i \le \frac{1}{\beta_i \beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right)\right\}\right. \\ &\left. + \left(\beta_i\beta_n X_i + (1+\beta_n)X_\beta^{-i,n}\right)\mathbb{1}\left\{X_i > \frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right)\right\}\right)\Bigg) \\ &+ I_i \cdot \mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}. \end{aligned}$$

In the following, the fixed point, which turns out to be unique, is determined via case distinction. If $Z_T > \chi_{\alpha_i}$, the unique fixed point is apparently given by $X_i = I_i$. Hence, we assume $Z_T \le \chi_{\alpha_i}$ for the remainder of the fixed point search.

**Case 1:** $I_i \ge X_\beta^{-i,n} + \beta_n I_n$

In this case we have

$$\frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right) \ge \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)$$

and therefore, we need to determine $X_i$ such that

$$\begin{aligned} X_i =& I_i \mathbb{1}\left\{X_i \le \frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right)\right\} \\ &+ \left(\beta_i\beta_n X_i + (1+\beta_n)X_\beta^{-i,n}\right)\mathbb{1}\left\{X_i > \frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right)\right\} \\ =& \max\left\{I_i, \beta_i\beta_n X_i + (1+\beta_n)X_\beta^{-i,n}\right\}. \end{aligned}$$

Since the right-hand side is continuous in $X_i$, constant for $X_i \le \frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right)$ and linear with positive slope smaller than 1 otherwise, the unique fixed point is given by

$$X_i = \begin{cases} I_i, & I_i \le \frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right), \\ \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}, & I_i > \frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right). \end{cases}$$

**Case 2:** $I_i < X_\beta^{-i,n} + \beta_n I_n$

Then it follows that
$$\frac{1}{\beta_i\beta_n}\left(I_i - (1+\beta_n)X_\beta^{-i,n}\right) < \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)$$
and the fixed point is given as the solution to the following equation

$$\begin{aligned}X_i = &\left(X_\beta^{-i,n} + \beta_n I_n\right)\mathbb{1}\left\{X_i \le \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)\right\}\\ &+\left(\beta_i\beta_n X_i + (1+\beta_n)X_\beta^{-i,n}\right)\mathbb{1}\left\{X_i > \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)\right\}.\end{aligned}$$

The right-hand side is continuous in $X_i$, constant for $X_i \le \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right)$ and linear with positive slope smaller than 1 otherwise. Hence, the unique fixed point is given by

$$X_i = \begin{cases} X_\beta^{-i,n} + \beta_n I_n, & X_\beta^{-i,n} + \beta_n I_n \le \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right),\\ \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}, & X_\beta^{-i,n} + \beta_n I_n > \frac{1}{\beta_i}\left(I_n - X_\beta^{-i,n}\right).\end{cases}$$

In summary, the unique fixed point is given by

$$\begin{aligned}X_i =&\mathbb{1}\left\{Z_T \le \chi_{\alpha_i}\right\}\Bigg(\mathbb{1}\Big\{I_i \ge X_\beta^{-i,n} + \beta_n I_n\Big\}\left(I_i\mathbb{1}\left\{I_i \ge \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}\right\}\right.\\ &\qquad\left.+\frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}\mathbb{1}\left\{I_i < \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}\right\}\right)\\ &\qquad+\mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n I_n\right\}\left(\left(X_\beta^{-i,n} + \beta_n I_n\right)\mathbb{1}\left\{I_n \ge \frac{1+\beta_i}{1-\beta_i\beta_n}X_\beta^{-i,n}\right\}\right.\\ &\qquad\left.+\frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}\mathbb{1}\left\{I_n < \frac{1+\beta_i}{1-\beta_i\beta_n}X_\beta^{-i,n}\right\}\right)\Bigg)\\ &+I_i\mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}\\ =&\Bigg(\max\left\{I_i, \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}\right\}\mathbb{1}\left\{I_i \ge X_\beta^{-i,n} + \beta_n I_n\right\}\\ &\quad+\max\left\{X_\beta^{-i,n} + \beta_n I_n, \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}\right\}\mathbb{1}\left\{I_i < X_\beta^{-i,n} + \beta_n I_n\right\}\Bigg)\mathbb{1}\left\{Z_T \le \chi_{\alpha_i}\right\}\\ &+I_i\mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}\\ =&\max\left\{I_i, \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}, X_\beta^{-i,n} + \beta_n I_n\right\}\mathbb{1}\left\{Z_T \le \chi_{\alpha_i}\right\} + I_i\mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}.\end{aligned}$$

Finally, the original $n$-dimensional fixed point problem can now be solved using the $(n-1)$-dimensional fixed point problem

$$X_i = \max\left\{I_i, \frac{1+\beta_n}{1-\beta_i\beta_n}X_\beta^{-i,n}, X_\beta^{-i,n} + \beta_n I_n\right\}\mathbb{1}\left\{Z_T \le \chi_{\alpha_i}\right\} + I_i\mathbb{1}\left\{Z_T > \chi_{\alpha_i}\right\}, \tag{C.7}$$

$i = 1, \dots, n-1$.

## C.3. Solution of the fixed point problem for three agents

The following theorem gives the unique solution to the fixed point problem (8.6) for three agents.

**Theorem C.1.** *Let $n = 3$, $\alpha_i = \alpha \in [0,1]$, and $\beta_i \in \big(0, \frac{1}{2}\big]$ for all $i \in \{1,2,3\}$. Moreover, let $U_i : (0,\infty) \to \mathbb{R}$ be Inada utility functions and define $I_i := I_i(\lambda_i Z_T)$ for notational convenience. Finally, define*

$$X_i^*(\lambda_1, \lambda_2, \lambda_3) = \max\left\{I_i, \frac{1+\beta_k}{1-\beta_i\beta_k}\beta_j I_j, \frac{1+\beta_j}{1-\beta_i\beta_j}\beta_k I_k, \beta_j I_j + \beta_k I_k\right\} \mathbb{1}\{Z_T \leq \chi_\alpha\} + I_i \mathbb{1}\{Z_T > \chi_\alpha\}$$

*for pairwise distinct $i, j, k \in \{1,2,3\}$, and assume that the system*

$$\mathbb{E}[Z_T X_i^*(\lambda_1, \lambda_2, \lambda_3)] = x_0^i,\ i = 1,2,3,$$

*has a unique solution $(\lambda_1^*, \lambda_2^*, \lambda_3^*) \in (0,\infty)^3$. Then the unique Nash equilibrium in the class of wealth profiles of the form (8.6) is given by $X_i^*(\lambda_1^*, \lambda_2^*, \lambda_3^*)$, $i = 1,2,3$.*

*Remark* C.2. Similar to Remark 8.5, we can ensure that, for fixed $\lambda_j > 0$, $j \neq i$, the equation $\mathbb{E}[Z_T X_i^*] = x_0^i$ has a unique solution $\lambda_i^* > 0$ in terms of $\lambda_j$, $j \neq i$, if we assume that $x_0^i \geq \sum_{j\neq i} \beta_j x_0^j$ for all $i = 1,2,3$. Again, this assumption is sufficient, but not necessary, and does also not ensure that the emerging three-dimensional system of equations is (uniquely) solvable. ♢

*Proof.* First, we rewrite the wealth profile from (8.6) of agent $i$ as

$$X_i = \max\left\{I_i, X_\beta^{-i}\right\} \mathbb{1}\{Z_T \leq \chi_\alpha\} + I_i \mathbb{1}\{Z_T > \chi_\alpha\},\ i = 1,2,3.$$

The three-dimensional fixed point problem from (8.6) can be reduced to a two-dimensional problem using the reduced problem (C.7). Hence, we need to solve the following two-dimensional fixed point problem

$$\begin{aligned} X_1 &= \max\left\{I_1, \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 X_2,\ \beta_2 X_2 + \beta_3 I_3\right\} \mathbb{1}\{Z_T \leq \chi_\alpha\} + I_1 \mathbb{1}\{Z_T > \chi_\alpha\}, \\ X_2 &= \max\left\{I_2, \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 X_1,\ \beta_1 X_1 + \beta_3 I_3\right\} \mathbb{1}\{Z_T \leq \chi_\alpha\} + I_2 \mathbb{1}\{Z_T > \chi_\alpha\}. \end{aligned}$$

Inserting $X_2$ into $X_1$ yields for $Z_T \leq \chi_\alpha$

$$\begin{aligned} X_1 = \max\Bigg\{ & I_1,\ \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2,\ \frac{(1+\beta_3)^2}{(1-\beta_1\beta_3)(1-\beta_2\beta_3)}\beta_1\beta_2 X_1, \\ & \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_1\beta_2 X_1 + \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2\beta_3 I_3,\ \beta_2 I_2 + \beta_3 I_3,\ \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1\beta_2 X_1 + \beta_3 I_3, \\ & \beta_1\beta_2 X_1 + \beta_3(1+\beta_2) I_3 \Bigg\} \\ =: \max\Big\{ & m_1,\ g_1(X_1),\ g_2(X_1),\ g_3(X_1),\ g_4(X_1)\Big\}, \end{aligned} \tag{C.8}$$

where we defined

$$
\begin{aligned}
m_1 &:= \max\left\{I_1, \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2, \beta_2 I_2 + \beta_3 I_3\right\}, \\
g_1(X_1) &:= \frac{(1+\beta_3)^2}{(1-\beta_1\beta_3)(1-\beta_2\beta_3)}\beta_1\beta_2 X_1, \qquad g_3(X_1) := \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1\beta_2 X_1 + \beta_3 I_3, \\
g_2(X_1) &:= \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_1\beta_2 X_1 + \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2\beta_3 I_3, \quad g_4(X_1) := \beta_1\beta_2 X_1 + \beta_3(1+\beta_2) I_3.
\end{aligned}
$$

Now we compare $g_2(X_1)$ and $g_3(X_1)$. The constant term of $g_2(X_1)$ is smaller than the constant term of $g_3(X_1)$, since

$$
\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 \leq 1
$$

using the assumption $\beta_i \in (0, 1/2]$ for all $i \in \{1,2,3\}$. Moreover, if we assume that $\beta_1 \leq \beta_2$, $g_2(X_1)$ is smaller than $g_3(X_1)$ (pointwise for any $X_1 > 0$) and can therefore be neglected inside the maximum. Note that we could make this conclusion because $X_1$ is strictly positive due to the above representation as the maximum of $I_1$ (which is strictly positive by assumption) and some other expressions. Hence, we now need to solve the following one-dimensional fixed point problem

$$
\begin{aligned}
X_1 &= \max\left\{m_1, \frac{(1+\beta_3)^2}{(1-\beta_1\beta_3)(1-\beta_2\beta_3)}\beta_1\beta_2 X_1,\ \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1\beta_2 X_1 + \beta_3 I_3,\ \beta_1\beta_2 X_1 + \beta_3(1+\beta_2) I_3\right\} \\
&= \max\left\{m_1, g_1(X_1), g_3(X_1), g_4(X_1)\right\}. \qquad \text{(C.9)}
\end{aligned}
$$

The second argument $g_1(X_1)$ inside the maximum cannot yield the fixed point because the intersection with the identity is located at 0. Thus, we consider $g_3(X_1)$ and $g_4(X_1)$. The value of $g_4$ at zero is strictly larger than the value of $g_3$, while the slope of $g_3$ is larger than the slope of $g_4$. Thus, we need to analyze whether the point of intersection of $g_3$ and $g_4$, denoted by $z_{3,4}$, is smaller or larger than the intersection $z_{4,\mathrm{id}}$ of $g_4$ and the identity function.

Simple calculations imply

$$
z_{3,4} = \frac{1-\beta_2\beta_3}{\beta_1(1+\beta_2)} I_3,\ z_{4,\mathrm{id}} = \frac{\beta_3(1+\beta_2)}{1-\beta_1\beta_2} I_3.
$$

Hence, $z_{4,\mathrm{id}} \leq z_{3,4}$ (using the assumption $0 < \beta_i \leq 1/2$ for all $i \in \{1,2,3\}$) and therefore, the fixed point is given by the maximum of $m_1$ and $z_{4,\mathrm{id}}$, i.e.,

$$
X_1 = \max\left\{m_1, \frac{\beta_3(1+\beta_2)}{1-\beta_1\beta_2} I_3\right\} \mathbb{1}\{Z_T \leq \chi_\alpha\} + I_1 \mathbb{1}\{Z_T > \chi_\alpha\}. \qquad \text{(C.10)}
$$

Now assume that $\beta_1 > \beta_2$. Hence, we cannot omit $g_2(X_1)$ in the maximum in (C.8). Let us now argue that adding $g_2(X_1)$ to the maximum (C.9) does not change the fixed point. In order to do this, we compare the intersection of $g_2$ and the identity, which is denoted by $z_{2,\mathrm{id}}$, with $z_{4,\mathrm{id}}$. The intersection of $g_2$ and the identity is given by

$$
z_{2,\mathrm{id}} := \frac{\beta_2\beta_3(1+\beta_3)}{1-\beta_1\beta_2-\beta_1\beta_3-\beta_1\beta_2\beta_3} I_3.
$$

A straightforward calculation using $\beta_i \leq 1/2$ for all $i \in \{1,2,3\}$ yields $z_{2,\text{id}} \leq z_{4,\text{id}}$. Hence, the term $g_2(X_1)$ does not influence the fixed point and $X_1$ from (C.10) is the unique solution to the fixed point problem.

To summarize, the first component of the fixed point for (8.6) takes the form

$$X_1 = \max\left\{I_1, \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2, \frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 I_3, \beta_2 I_2 + \beta_3 I_3\right\}.$$

Now the assertion follows due to the symmetry of the original fixed point problem. □

Theorem C.1 shows a very compact representation of the unique Nash equilibrium in the class of wealth profiles of the form (8.6) for three agents. However, it does hide some structural properties of the triple $(X_1^*, X_2^*, X_3^*)$. Table C.1 shows all possible combinations of the wealth profiles for agents 1, 2, and 3, given that $Z_T \leq \chi_\alpha$.

| | $X_1^*$ | $X_2^*$ | $X_3^*$ |
|---|---|---|---|
| a) | $I_1(\lambda_1 Z_T)$ | $I_2(\lambda_2 Z_T)$ | $I_3(\lambda_3 Z_T)$ |
| b) | $\beta_2 I_2(\lambda_2 Z_T) + \beta_3 I_3(\lambda_3 Z_T)$ | $I_2(\lambda_2 Z_T)$ | $I_3(\lambda_3 Z_T)$ |
| c) | $I_1(\lambda_1 Z_T)$ | $\beta_1 I_1(\lambda_1 Z_T) + \beta_3 I_3(\lambda_3 Z_T)$ | $I_3(\lambda_3 Z_T)$ |
| d) | $I_1(\lambda_1 Z_T)$ | $I_2(\lambda_2 Z_T)$ | $\beta_1 I_1(\lambda_1 Z_T) + \beta_2 I_2(\lambda_2 Z_T)$ |
| e) | $I_1(\lambda_1 Z_T)$ | $\frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 I_1(\lambda_1 Z_T)$ | $\frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 I_1(\lambda_1 Z_T)$ |
| f) | $\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2(\lambda_2 Z_T)$ | $I_2(\lambda_2 Z_T)$ | $\frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2(\lambda_2 Z_T)$ |
| g) | $\frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 I_3(\lambda_3 Z_T)$ | $\frac{1+\beta_1}{1-\beta_1\beta_2}\beta_3 I_3(\lambda_3 Z_T)$ | $I_3(\lambda_3 Z_T)$ |

Table C.1.: Possible values of the triple $(X_1^*, X_2^*, X_3^*)$ from Theorem C.1 for $Z_T \leq \chi_\alpha$.

We can divide the behavior of $(X_1^*, X_2^*, X_3^*)$ into three different cases. First, it is possible that all three investors perform equally well in the sense that their terminal wealth in the Nash equilibrium is given by the solution $I_i(\lambda_i Z_T)$, $i = 1,2,3$, to the standard utility maximization problem (case a)). Moreover, it is also possible that two of the three investors perform equally well while the third investor performs worse, so that her wealth in the Nash equilibrium consists of a linear combination of the other two agent's wealth (cases b)-d)). Finally, if one investor outperforms the other two agents, their terminal wealth is a multiple of the classical solution for the „winning“ investor (cases e)-g)).

The previous observations become more clearly visible at the end of this section where we consider the special case of logarithmic utility functions. The numerical results presented there also show that the initial capital of the investors mainly determines which of the three cases is present.

*Proof (Table C.1).* In the following, we prove that Table C.1 contains every possible combination of values of $X_1^*$, $X_2^*$, and $X_3^*$ from Theorem C.1. Again, we abbreviate $I_i = I_i(\lambda_i Z_T)$, $i = 1,2,3$.

First, it is obviously possible that $X_1^* = I_1$, $X_2^* = I_2$, and $X_3^* = I_3$ occur simultaneously (case a)). Next, assume that $X_1^* = \beta_2 I_2 + \beta_3 I_3$. Then

$$I_3 \geq \frac{1}{\beta_3}\left(\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2 - \beta_2 I_2\right) = \frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2, \tag{C.11}$$

$$I_2 \geq \frac{1}{\beta_2}\left(\frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 I_3 - \beta_3 I_3\right) = \frac{1+\beta_1}{1-\beta_1\beta_2}\beta_3 I_3, \tag{C.12}$$

since $\beta_2 I_2 + \beta_3 I_3 \geq \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2 \vee \frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 I_3$. Further, we obtain

$$\begin{aligned}\beta_1 I_1 + \beta_2 I_2 &\leq \beta_1(\beta_2 I_2 + \beta_3 I_3) + \beta_2 I_2 = \beta_2(1+\beta_1)I_2 + \beta_1\beta_3 I_3 \\ &\leq \beta_2(1+\beta_1)\frac{1-\beta_1\beta_3}{1+\beta_1}\frac{1}{\beta_2}I_3 + \beta_1\beta_3 I_3 = I_3,\end{aligned} \tag{C.13}$$

where we used $\beta_2 I_2 + \beta_3 I_3 \geq I_1$ in the first and (C.11) in the second inequality. Analogously, we can deduce

$$\beta_1 I_1 + \beta_3 I_3 \leq I_2 \tag{C.14}$$

using $\beta_2 I_2 + \beta_3 I_3 \geq I_1$ and (C.12). Finally, it follows,

$$I_3 \geq \beta_1 I_1 + \beta_2 I_2 \geq \beta_1 I_1 + \beta_2(\beta_1 I_1 + \beta_3 I_3) = \beta_1(1+\beta_2)I_1 + \beta_2\beta_3 I_3,$$

using (C.13) in the first and (C.14) in the second inequality. This implies further that

$$I_3 \geq \frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 I_1.$$

Analogously,

$$I_2 \geq \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 I_1.$$

In summary, $X_1^* = \beta_2 I_2 + \beta_3 I_3$ implies $X_2^* = I_2$ and $X_3^* = I_3$.

Using very similar arguments, we can deduce the following implications

$$\begin{aligned}X_2^* = \beta_1 I_1 + \beta_3 I_3 &\implies X_1^* = I_1,\ X_3^* = I_3, \\ X_3^* = \beta_1 I_1 + \beta_2 I_2 &\implies X_1^* = I_1,\ X_2^* = I_2.\end{aligned}$$

So far, we justified lines a)-d) of Table C.1. Now assume that $X_1^* = \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2$. Then it follows

$$\beta_1 I_1 + \beta_2 I_2 \leq \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_1\beta_2 I_2 + \beta_2 I_2 = \frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2,$$

since $I_1 \leq \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2$, using the representation of $X_1^*$ in Theorem C.1 and the assumption $X_1^* = \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2$. Moreover, we obtain

$$\frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 I_1 \leq \frac{1+\beta_2}{1-\beta_2\beta_3}\cdot\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_1\beta_2 I_2 \leq \frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2.$$

Again, we used Theorem C.1 and the assumption on $X_1^*$ for the first inequality. The second inequality is valid since

$$\frac{(1+\beta_2)(1+\beta_3)}{1-\beta_2\beta_3}\beta_1 \le 3\beta_1 \le 1+\beta_1,$$

which holds due to the assumption $\beta_j \le \frac{1}{2}$, $j = 1,2,3$. Finally, $\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2 \ge \beta_2 I_2 + \beta_3 I_3$ implies

$$I_3 \le \frac{1}{\beta_3}\left(\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2 - \beta_2 I_2\right) = \frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2. \tag{C.15}$$

Thus, $X_3^* = \frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2$ holds. For $X_2^*$, we obtain

$$I_2 \ge \underbrace{\frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2}_{\le 1} I_2 \ge I_3 \ge \underbrace{\frac{1+\beta_1}{1-\beta_1\beta_2}\beta_3}_{\le 1} I_3$$

from the assumption $\beta_j \le 1/2$ for all $j \in \{1,2,3\}$, and the previous representation of $X_3^*$. Analogously,

$$I_2 \ge \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 I_1.$$

Finally, using (C.15) and $I_1 \le \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2 = X_1^*$,

$$I_2 \ge \beta_1 I_2 + \beta_3 I_2 \ge \beta_1 \cdot \underbrace{\frac{1-\beta_1\beta_3}{1+\beta_3}\frac{1}{\beta_2}}_{\ge 1} I_1 + \beta_3 \cdot \underbrace{\frac{1-\beta_1\beta_3}{1+\beta_1}\frac{1}{\beta_2}}_{\ge 1} I_3 \ge \beta_1 I_1 + \beta_3 I_3.$$

Therefore, it follows that $X_2^* = I_2$.

Using similar arguments, we can also prove the following implications

$$\begin{aligned}
X_1^* = \frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 I_3 &\implies X_2^* = \frac{1+\beta_1}{1-\beta_1\beta_2}\beta_3 I_3,\ X_3^* = I_3,\\
X_2^* = \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 I_1 &\implies X_1^* = I_1,\ X_3^* = \frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 I_1,\\
X_2^* = \frac{1+\beta_1}{1-\beta_1\beta_2}\beta_3 I_3 &\implies X_1^* = \frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 I_3,\ X_3^* = I_3,\\
X_3^* = \frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 I_1 &\implies X_1^* = I_1,\ X_2^* = \frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 I_1,\\
X_3^* = \frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 I_2 &\implies X_1^* = \frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 I_2,\ X_2^* = I_2.
\end{aligned}$$

Hence, lines e)-g) in Table C.1 are valid as well. The line of arguments also implies that lines a)-g) of Table C.1 contain all possible combinations of $X_1^*$, $X_2^*$, and $X_3^*$. □

### Example: Logarithmic utility

If all investors use the natural logarithm as their utility function, we can determine the unique Lagrange multipliers $\lambda_i$, $i = 1,2,3$, mentioned in Theorem C.1 explicitly. Using the cases a)-g) introduced in Table C.1, we obtain the following values of $X_i^*$, $i = 1,2,3$.

**Proposition C.3.** *Let $n = 3$ and assume that $U_i(x) = \log(x)$, $x > 0$, $i = 1,2,3$. Further, let $\alpha_i = \alpha \in [0,1]$ and $\beta_i \in (0,1/2]$, $i = 1,2,3$, and let*

$$x_0^i > \alpha \max\left\{\beta_j x_0^j + \beta_k x_0^k,\ \frac{1+\beta_j}{1-\beta_i\beta_j}x_0^k,\ \frac{1+\beta_k}{1-\beta_i\beta_k}\beta_j x_0^j\right\}$$

*for all pairwise distinct $i,j,k \in \{1,2,3\}$. Then the triples $(X_1^*, X_2^*, X_3^*)$ displayed in Table C.1 take the following form*

*a)* $X_i^* = \frac{x_0^i}{Z_T}$, $i = 1,2,3$,

*b)-d)* $X_i^* = \frac{\beta_j x_0^j + \beta_k x_0^k}{Z_T}\mathbb{1}\{Z_T \le \chi_\alpha\} + \frac{x_0^i - \alpha(\beta_j x_0^j + \beta_k x_0^k)}{(1-\alpha)Z_T}\mathbb{1}\{Z_T > \chi_\alpha\}$, $X_j^* = \frac{x_0^j}{Z_T}$, $X_k^* = \frac{x_0^k}{Z_T}$ *for pairwise distinct $i,j,k \in \{1,2,3\}$,*

*e)-g)* $X_i^* = \frac{1+\beta_j}{1-\beta_i\beta_j}\beta_k x_0^k Z_T^{-1}\mathbb{1}\{Z_T \le \chi_\alpha\} + \frac{1}{1-\alpha}\left(x_0^i - \alpha\frac{1+\beta_j}{1-\beta_i\beta_j}\beta_k x_0^k\right)Z_T^{-1}\mathbb{1}\{Z_T > \chi_\alpha\}$,
$X_j^* = \frac{1+\beta_i}{1-\beta_i\beta_j}\beta_k x_0^k Z_T^{-1}\mathbb{1}\{Z_T \le \chi_\alpha\} + \frac{1}{1-\alpha}\left(x_0^j - \alpha\frac{1+\beta_i}{1-\beta_i\beta_j}\beta_k x_0^k\right)Z_T^{-1}\mathbb{1}\{Z_T > \chi_\alpha\}$,
$X_k = \frac{x_0^k}{Z_T}$ *for pairwise distinct $i,j,k \in \{1,2,3\}$.*

*Proof.* Since $U_i(x) = \log(x)$, $I_i(x) = x^{-1}$ holds for all $i \in \{1,2,3\}$ and $x > 0$. In the following, we determine the Lagrange multipliers $\lambda_i$, $i = 1,2,3$, in the cases a)-g) displayed in Table C.1. As already stated in Theorem C.1, $\lambda_i$ is determined via the budget constraint $\mathbb{E}[X_i^* Z_T] = x_0^i$, $i = 1,2,3$.

a) In case a), the wealth profile of agent $i$ is given by $X_i^* = (\lambda_i Z_T)^{-1}$ for all $i \in \{1,2,3\}$, i.e., by the solution to the classical problem without the VaR-constraint. Hence,

$$\mathbb{E}[X_i^* Z_T] = \frac{1}{\lambda_i} \overset{!}{=} x_0^i$$

and thus, $\lambda_i = (x_0^i)^{-1}$, $i = 1,2,3$.

b) In case b), we obtain $X_1^* = (\frac{\beta_2}{\lambda_2 Z_T} + \frac{\beta_3}{\lambda_3 Z_T})\mathbb{1}\{Z_T \le \chi_\alpha\} + (\lambda_1 Z_T)^{-1}\mathbb{1}\{Z_T > \chi_\alpha\}$, $X_2^* = (\lambda_2 Z_T)^{-1}$, $X_3^* = (\lambda_3 Z_T)^{-1}$. Analogously to a), we obtain $\lambda_2 = (x_0^2)^{-1}$, $\lambda_3 = (x_0^3)^{-1}$. Hence, it follows

$$\begin{aligned}\mathbb{E}[X_1^* Z_T] &= (\beta_2 x_0^2 + \beta_3 x_0^3)\mathbb{E}[\mathbb{1}\{Z_T \le \chi_\alpha\}] + \frac{1}{\lambda_1}\mathbb{E}[\mathbb{1}\{Z_T > \chi_\alpha\}] \\ &= \alpha(\beta_2 x_0^2 + \beta_3 x_0^3) + \frac{1-\alpha}{\lambda_1} = x_0^1,\end{aligned}$$

which implies $\lambda_1 = \frac{1-\alpha}{x_0^1 - \alpha(\beta_2 x_0^2 + \beta_3 x_0^3)}$. Note that we used that $\chi_\alpha$ denotes the $\alpha$-quantile of $Z_T$.

c) Analogously to case b), we obtain $\lambda_1 = (x_0^1)^{-1}$, $\lambda_2 = \frac{1-\alpha}{x_0^2 - \alpha(\beta_1 x_0^1 + \beta_3 x_0^3)}$, and $\lambda_3 = (x_0^3)^{-1}$.

d) Analogously to case b), we obtain $\lambda_1 = (x_0^1)^{-1}$, $\lambda_2 = (x_0^2)^{-1}$, and $\lambda_3 = \frac{1-\alpha}{x_0^3 - \alpha(\beta_1 x_0^1 + \beta_2 x_0^2)}$.

e) In case e), we have $X_1^* = (\lambda_1 Z_T)^{-1}$, so that $\lambda_1 = (x_0^1)^{-1}$ follows. Moreover, $X_2^*$ and $X_3^*$ are given by

$$X_2^* = \frac{1+\beta_3}{1-\beta_2\beta_3}\frac{\beta_1 x_0^1}{Z_T}\mathbb{1}\{Z_T \le \chi_\alpha\} + \frac{1}{\lambda_2 Z_T}\mathbb{1}\{Z_T > \chi_\alpha\},$$

$$X_3^* = \frac{1+\beta_2}{1-\beta_2\beta_3}\frac{\beta_1 x_0^1}{Z_T}\mathbb{1}\left\{Z_T \leq \chi_\alpha\right\} + \frac{1}{\lambda_3 Z_T}\mathbb{1}\left\{Z_T > \chi_\alpha\right\},$$

and therefore

$$\mathbb{E}[X_2^* Z_T] = \alpha\frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 x_0^1 + \frac{1-\alpha}{\lambda_2} \overset{!}{=} x_0^2,$$
$$\mathbb{E}[X_3^* Z_T] = \alpha\frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 x_0^1 + \frac{1-\alpha}{\lambda_3} \overset{!}{=} x_0^3,$$

which imply that

$$\lambda_2 = \frac{1-\alpha}{x_0^2 - \alpha\frac{1+\beta_3}{1-\beta_2\beta_3}\beta_1 x_0^1}, \lambda_3 = \frac{1-\alpha}{x_0^3 - \alpha\frac{1+\beta_2}{1-\beta_2\beta_3}\beta_1 x_0^1}.$$

f) Analogously to case e), we obtain $\lambda_2 = (x_0^2)^{-1}$ and

$$\lambda_1 = \frac{1-\alpha}{x_0^1 - \alpha\frac{1+\beta_3}{1-\beta_1\beta_3}\beta_2 x_0^2}, \lambda_3 = \frac{1-\alpha}{x_0^3 - \alpha\frac{1+\beta_1}{1-\beta_1\beta_3}\beta_2 x_0^2}.$$

g) Analogously to case e), we obtain $\lambda_3 = (x_0^3)^{-1}$ and

$$\lambda_1 = \frac{1-\alpha}{x_0^1 - \alpha\frac{1+\beta_2}{1-\beta_1\beta_2}\beta_3 x_0^3}, \lambda_2 = \frac{1-\alpha}{x_0^2 - \alpha\frac{1+\beta_1}{1-\beta_1\beta_2}\beta_3 x_0^3}.$$

Finally, inserting $\lambda_i^*$ into $X_i^*$ from Table C.1 for all $i \in \{1, 2, 3\}$ yields the asserted representation of the triple $(X_1^*, X_2^*, X_3^*)$. □

## ILLUSTRATION OF PROPOSITION C.3

The following figures show an application of Proposition C.3. The displayed results are based on a classical Black-Scholes market with one stock, drift $\mu = 0.03$, and volatility $\sigma = 0.2$, and one riskless bond with zero interest rate. Moreover, we chose $\alpha = 0.5$ and $T = 4$ (similar to the setting used in Section 8.3).

Figures C.3.1 - C.3.3 display $X_i^*$, $i = 1, 2, 3$, for varying values of the state price density $Z_T$. Figure C.3.1 shows case a) of Table C.1 in which the parameters of the problem are chosen so that the terminal wealth in the Nash equilibrium is the classical solution for all three agents. In Figure C.3.2, the parameters are chosen so that case b) is present. The figure displays, as expected, that agents 2 and 3 use the classical solution while agent 1 uses a linear combination of $X_2^*$ and $X_3^*$ for small values of $Z_T$ (i.e., $Z_T \leq \chi_\alpha$) and the „classical solution" $I_1(\lambda_1 Z_T)$ for large values of $Z_T$. Again, we have to be careful with the term „classical solution" since the Lagrange multiplier $\lambda_1$ is not the same as in the classical problem due to the discontinuity at $\chi_\alpha$. The last of the three figures, Figure C.3.3, shows case e) in which the parameters of the problem are chosen so that agent 1 uses the classical solution while the other two agents use a multiple of the first agents optimal wealth for small values of $Z_T$ and their „classical solution" $I_i(\lambda_i Z_T)$, $i = 2, 3$, for large values of $Z_T$.

Figure C.3.4 shows the behavior of $X_1^*$ in terms of $Z_T$ for different choices of $\beta_2$, while $\beta_1$ and $\beta_3$ are fixed. For the smallest value of $\beta_2$, i.e., $\beta_2 = 0.2$, case a) is present and we can observe that $X_1^*$ is simply given by the solution to the classical problem. For the other three values of $\beta_2$, case f) of Table C.1 is present, which explains the discontinuity of $X_1^*$ at $Z_T = \chi_\alpha$. As the value of $\beta_2$ increases, we can observe that the value of $X_1^*$ increases for small values of $Z_T$ ($Z_T \leq \chi_\alpha$) and decreases for large values of $Z_T$ ($Z_T > \chi_\alpha$).

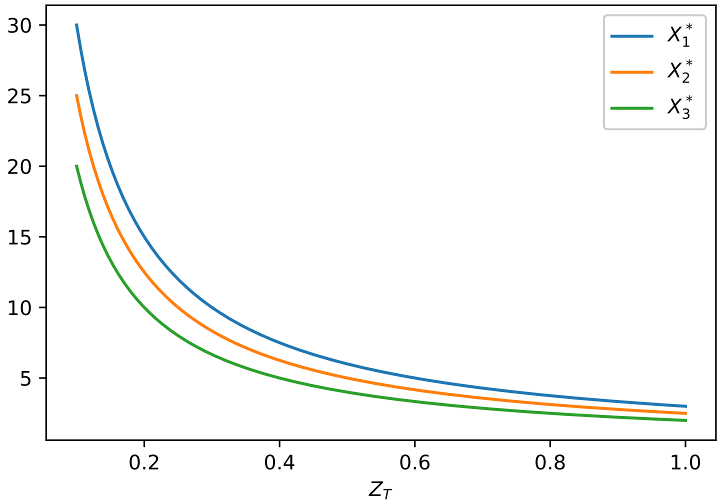



Figure C.3.1.: $X_i^*$, $i = 1, 2, 3$, from Proposition C.3 in terms of $Z_T$ for $\alpha = 0.5$, $\beta_1 = 0.4$, $\beta_2 = 0.3$, $\beta_3 = 0.4$, $x_0^1 = 3$, $x_0^2 = 2.5$, $x_0^3 = 3$. The parameter choice implies that case a) of Table C.1 is present. The market parameters are $d = 1$, $r = 0$, $\mu = 0.03$, $\sigma = 0.2$, $T = 4$.

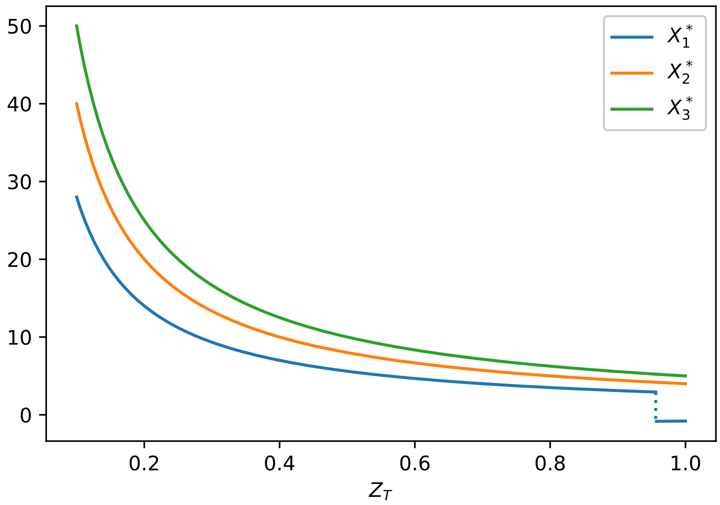



Figure C.3.2.: $X_i^*$, $i = 1, 2, 3$, from Proposition C.3 in terms of $Z_T$ for $\alpha = 0.5$, $\beta_1 = 0.2$, $\beta_2 = 0.2$, $\beta_3 = 0.4$, $x_0^1 = 1$, $x_0^2 = 4$, $x_0^3 = 5$. The parameter choice implies that case b) of Table C.1 is present. The market parameters are $d = 1$, $r = 0$, $\mu = 0.03$, $\sigma = 0.2$, $T = 4$.

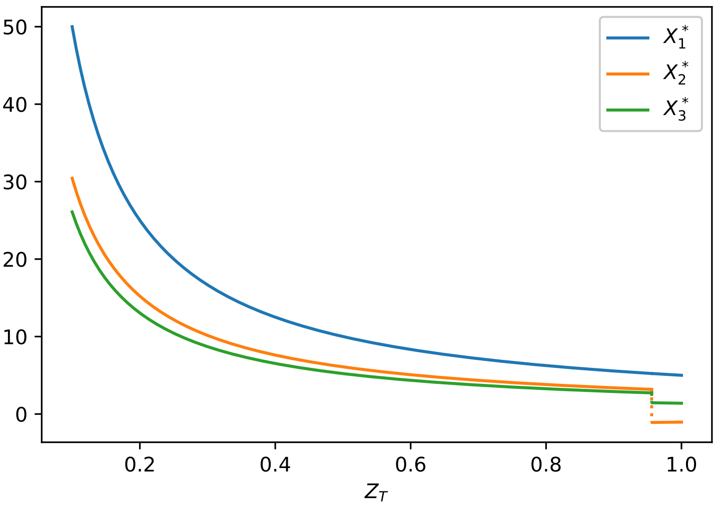



Figure C.3.3.: $X_i^*$, $i = 1, 2, 3$, from Proposition C.3 in terms of $Z_T$ for $\alpha = 0.5$, $\beta_1 = 0.4$, $\beta_2 = 0.2$, $\beta_3 = 0.4$, $x_0^1 = 5$, $x_0^2 = 1$, $x_0^3 = 2$. The parameter choice implies that case e) of Table C.1 is present. The market parameters are $d = 1$, $r = 0$, $\mu = 0.03$, $\sigma = 0.2$, $T = 4$.

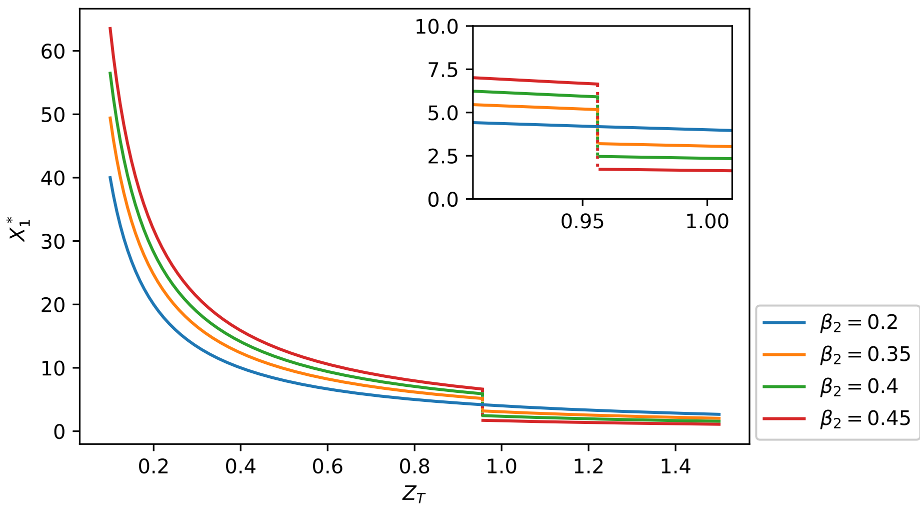



Figure C.3.4.: $X_1^*$ from Proposition C.3 in terms of $Z_T$, for different choices of $\beta_2$, while $\alpha = 0.5$, $\beta_1 = 0.3$, $\beta_3 = 0.5$, $x_0^1 = 4$, $x_0^2 = 8$, $x_0^3 = 3$. For $\beta_2 = 0.2$, case a) from Table C.1 is present; the other parameters imply that case f) is present. The market parameters are $d = 1$, $r = 0$, $\mu = 0.03$, $\sigma = 0.2$, $T = 4$.

## C.4. Proofs of lemmas from Section 8.4

In the following, we provide the proofs of Lemmas 8.12 - 8.14, as well as Lemma 8.18, that were left out of Section 8.4. The lemmas are used to find and characterize the unique fixed point for (8.6) for general $n \in \mathbb{N}$.

*Proof (Lemma 8.12).* Let $L \in \{1, \dots, n\}$ and $(j_1, \dots, j_L) \in \binom{[n]}{L}$. Using $d_L(j_1, \dots, j_L)$ defined in (8.13) and the Binomial theorem (Harris et al., 2008, p. 139) yields

$$\begin{aligned}
d_L(j_1, \dots, j_L) &= 1 - \sum_{k=2}^{L} (k-1) \sum_{(i_1,\dots,i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} \\
&\geq 1 - \sum_{k=2}^{L} (k-1) \sum_{(i_1,\dots,i_k) \in \binom{[L]}{k}} \left(\frac{1}{n-1}\right)^k \qquad \text{(C.16)} \\
&= 1 - \sum_{k=2}^{L} k \binom{L}{k} \left(\frac{1}{n-1}\right)^k + \sum_{k=2}^{L} \binom{L}{k} \left(\frac{1}{n-1}\right)^k \\
&= 1 - \frac{L}{n-1} \sum_{k=1}^{L-1} \binom{L-1}{k} \left(\frac{1}{n-1}\right)^k + \sum_{k=2}^{L} \binom{L}{k} \left(\frac{1}{n-1}\right)^k \\
&= 1 - \frac{L}{n-1} \left(1 + \frac{1}{n-1}\right)^{L-1} + \frac{L}{n-1} + \left(1 + \frac{1}{n-1}\right)^{L} - 1 - \frac{L}{n-1} \\
&= \left(\frac{n}{n-1}\right)^{L-1} \frac{n-L}{n-1}. \qquad \text{(C.17)}
\end{aligned}$$

If $L = n$ and $\beta_j = \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$, we obtain equality in (C.16) and the expression in the last line (C.17) is equal to 0. Otherwise, we obtain either a strict inequality in (C.16) or a strictly positive lower bound in (C.17). This concludes our proof. $\square$

*Proof (Lemma 8.13).* First, let $L = n$ and $\beta_j = \frac{1}{n-1}$ for all $j \in \{1, \dots, n\}$. Then the entries on the diagonal of $A_n(1, \dots, n)$ are equal to 1 while all other entries are equal to $-\frac{1}{n-1}$. Thus, the rows of $A_n(1, \dots, n)$ sum up to the $n$-dimensional vector $\mathbf{0}_n$ of zeros and $A_n(1, \dots, n)$ is not regular.

Now assume that either $L \in \{1, \dots, n-1\}$, or $L = n$ and $\prod_{j=1}^{n} \beta_j < \left(\frac{1}{n-1}\right)^n$. Let $(j_1, \dots, j_L) \in \binom{[n]}{L}$ be arbitrary but fixed. Throughout this proof, we denote $A_L(j_1, \dots, j_L)$ by $A$ and the claimed inverse by $A^{-1}$. Moreover, we set $B := A^{-1}A$ and $d_L := d_L(j_1, \dots, j_L)$. Note that Lemma 8.12 implies $d_L > 0$. We show that $A^{-1}$ is the left inverse of $A$. The proof that it is also the right inverse of $A$ proceeds analogously. First, we consider the elements on the diagonal of $B$. Let $i \in \{1, \dots, L\}$. Then

$$\begin{aligned}
B[i,i] &= \sum_{k=1}^{L} A^{-1}[i,k] A[k,i] \\
&= \frac{1}{d_L} \left( 1 - \sum_{k=2}^{L-1} (k-1) \sum_{(i_1,\dots,i_k) \in \binom{\{j_1,\dots,j_L\} \setminus \{j_i\}}{k}} \prod_{\ell=1}^{k} \beta_{i_\ell} + \sum_{\substack{k=1 \\ k \neq i}}^{L} \beta_{j_k} \prod_{\substack{\ell=1 \\ \ell \neq i,k}}^{L} (1 + \beta_{j_\ell}) \cdot (-\beta_{j_i}) \right) \\
&= \frac{1}{d_L} \left( 1 - \sum_{k=2}^{L-1} (k-1) \sum_{(i_1,\dots,i_k) \in \binom{\{j_1,\dots,j_L\} \setminus \{j_i\}}{k}} \prod_{\ell=1}^{k} \beta_{i_\ell} \right. \\
&\qquad \left. - \beta_{j_i} \sum_{\substack{k=1 \\ k \neq i}}^{L} \beta_{j_k} \sum_{m=0}^{L-2} \sum_{(i_1,\dots,i_m) \in \binom{\{j_1,\dots,j_L\} \setminus \{j_i, j_k\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d_L}\Bigg(1 - \sum_{k=2}^{L-1}(k-1) \sum_{(i_1,\dots,i_k)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{k}} \prod_{\ell=1}^{k} \beta_{i_\ell} \\
&\qquad - \beta_{j_i}\sum_{m=2}^{L}(m-1) \sum_{(i_1,\dots,i_{m-1})\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m-1}} \prod_{\ell=1}^{m-1} \beta_{i_\ell}\Bigg) \\
&= \frac{1}{d_L}\Bigg(1 - \sum_{k=2}^{L}(k-1) \sum_{\substack{(i_1,\dots,i_k)\in\binom{\{j_1,\dots,j_L\}}{k} \\ j_i\notin(i_1,\dots,i_k)}} \prod_{\ell=1}^{k} \beta_{i_\ell} - \sum_{m=2}^{L}(m-1) \sum_{\substack{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}}{m} \\ j_i\in(i_1,\dots,i_m)}} \prod_{\ell=1}^{m-1} \beta_{i_\ell}\Bigg) \\
&= \frac{1}{d_L}\Bigg(1 - \sum_{k=2}^{L}(k-1) \sum_{(i_1,\dots,i_k)\in\binom{\{j_1,\dots,j_L\}}{k}} \prod_{\ell=1}^{k} \beta_{i_\ell}\Bigg) = 1.
\end{aligned}
$$

Now let $i,k\in\{1,\dots,L\}$, $i\neq k$. Then

$$
\begin{aligned}
&B[i,k] = \sum_{p=1}^{L} A^{-1}[i,p]\cdot A[p,k] \\
&= \frac{1}{d_L}\Bigg(\Bigg(1 - \sum_{m=2}^{L-1}(m-1) \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell}\Bigg)\cdot(-\beta_{j_k}) + \beta_{j_k}\prod_{\substack{\ell=1 \\ \ell\neq i,k}}^{L}(1+\beta_{j_\ell})\cdot 1 \\
&\qquad + \sum_{\substack{p=1 \\ p\neq i,k}}^{L} \beta_{j_p}\prod_{\substack{\ell=1 \\ \ell\neq i,p}}^{L}(1+\beta_{j_p})\cdot(-\beta_{j_k})\Bigg) \\
&= -\frac{\beta_{j_k}}{d_L}\Bigg(1 - \sum_{m=2}^{L-1}(m-1) \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} - \prod_{\substack{\ell=1 \\ \ell\neq i,k}}^{L}(1+\beta_{j_\ell}) + \sum_{\substack{p=1 \\ p\neq i,k}}^{L}\beta_{j_p}\prod_{\substack{\ell=1 \\ \ell\neq i,p}}^{L}(1+\beta_{j_p})\Bigg) \\
&= -\frac{\beta_{j_k}}{d_L}\Bigg(\sum_{\substack{p=1 \\ p\neq i,k}}^{L}\beta_{j_p}\sum_{m=0}^{L-2} \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i,j_p\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} - \sum_{m=2}^{L-1}(m-1) \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} \\
&\qquad - \sum_{m=1}^{L-2} \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i,j_k\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell}\Bigg) \\
&= -\frac{\beta_{j_k}}{d_L}\Bigg(\sum_{\substack{p=1 \\ p\neq i,k}}^{L}\beta_{j_p}\sum_{m=0}^{L-2} \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i,j_p\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} - \sum_{m=2}^{L-1}(m-1) \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} \\
&\qquad - \sum_{m=1}^{L-1} \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} + \sum_{m=0}^{L-2}\beta_{j_k} \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i,j_k\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell}\Bigg) \\
&= -\frac{\beta_{j_k}}{d_L}\Bigg(\sum_{\substack{p=1 \\ p\neq i}}^{L}\beta_{j_p}\sum_{m=0}^{L-2} \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i,j_p\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} - \sum_{m=1}^{L-1} m \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell}\Bigg) \\
&= -\frac{\beta_{j_k}}{d_L}\Bigg(\sum_{m=1}^{L-1} m \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} - \sum_{m=1}^{L-1} m \sum_{(i_1,\dots,i_m)\in\binom{\{j_1,\dots,j_L\}\setminus\{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell}\Bigg) = 0.
\end{aligned}
$$

Hence, $B$ equals the identity matrix in $\mathbb{R}^{L\times L}$. □

*Proof (Lemma 8.14).* Let $L \in \{1, \ldots, n-1\}$ or $L = n$ and $\prod_{j=1}^{n} \beta_j < \left(\frac{1}{n-1}\right)^n$. Further, let $(j_1, \ldots, j_L) \in \binom{[n]}{L}$ be arbitrary but fixed. Throughout this proof, we denote $A_L(j_1, \ldots, j_L)^{-1}$ by $A^{-1}$ and $d_L(j_1, \ldots, j_L)$ by $d_L$. Let $i \in \{1, \ldots, L\}$. Then, using Lemma 8.13,

$$
\begin{aligned}
d_L \cdot \sum_{k=1}^{L} A^{-1}[i,k] &= 1 - \sum_{m=2}^{L-1} (m-1) \sum_{(i_1,\ldots,i_m) \in \binom{\{j_1,\ldots,j_L\} \setminus \{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} + \sum_{\substack{k=1\\k \neq i}}^{L} \beta_{j_k} \prod_{\substack{\ell=1\\\ell \neq i,k}}^{L} (1 + \beta_{j_\ell}) \\
&= 1 - \sum_{m=2}^{L-1} (m-1) \sum_{(i_1,\ldots,i_m) \in \binom{\{j_1,\ldots,j_L\} \setminus \{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} + \sum_{\substack{k=1\\k \neq i}}^{L} \beta_{j_k} \sum_{m=0}^{L-2} \sum_{(i_1,\ldots,i_m) \in \binom{\{j_1,\ldots,j_L\} \setminus \{j_i, j_k\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} \\
&= 1 - \sum_{m=2}^{L-1} (m-1) \sum_{(i_1,\ldots,i_m) \in \binom{\{j_1,\ldots,j_L\} \setminus \{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} + \sum_{m=1}^{L-1} m \sum_{(i_1,\ldots,i_m) \in \binom{\{j_1,\ldots,j_L\} \setminus \{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} \\
&= 1 + \sum_{m=1}^{L-1} \sum_{(i_1,\ldots,i_m) \in \binom{\{j_1,\ldots,j_L\} \setminus \{j_i\}}{m}} \prod_{\ell=1}^{m} \beta_{i_\ell} \\
&= \prod_{\substack{\ell=1\\\ell \neq i}}^{L} (1 + \beta_{j_\ell}).
\end{aligned}
$$

□

*Proof (Lemma 8.18).* Let $L \in \{1, \ldots, n-1\}$ and $(j_1, \ldots, j_{L+1}) \in \binom{[n]}{L+1}$. Then, by definition of $d_{L+1}(j_1, \ldots, j_{L+1})$,

$$
\begin{aligned}
d_{L+1}(j_1, \ldots, j_{L+1}) &= 1 - \sum_{k=2}^{L+1} (k-1) \sum_{(i_1,\ldots,i_k) \in \binom{[L+1]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} \\
&= 1 - \beta_{j_{L+1}} \sum_{k=2}^{L+1} (k-1) \sum_{(i_1,\ldots,i_{k-1}) \in \binom{[L]}{k-1}} \prod_{\ell=1}^{k-1} \beta_{j_{i_\ell}} - \sum_{k=2}^{L} (k-1) \sum_{(i_1,\ldots,i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} \\
&= d_L(j_1, \ldots, j_L) - \beta_{j_{L+1}} \sum_{k=2}^{L+1} (k-1) \sum_{(i_1,\ldots,i_{k-1}) \in \binom{[L]}{k-1}} \prod_{\ell=1}^{k-1} \beta_{j_{i_\ell}} \\
&= d_L(j_1, \ldots, j_L) - \beta_{j_{L+1}} \sum_{k=1}^{L} k \sum_{(i_1,\ldots,i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} \\
&= d_L(j_1, \ldots, j_L) - \beta_{j_{L+1}} \sum_{k=2}^{L} (k-1) \sum_{(i_1,\ldots,i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} - \beta_{j_{L+1}} \sum_{k=1}^{L} \sum_{(i_1,\ldots,i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} \\
&= d_L(j_1, \ldots, j_L) - \beta_{j_{L+1}} \sum_{k=2}^{L} (k-1) \sum_{(i_1,\ldots,i_k) \in \binom{[L]}{k}} \prod_{\ell=1}^{k} \beta_{j_{i_\ell}} + \beta_{j_{L+1}} - \beta_{j_{L+1}} \prod_{\ell=1}^{L} (1 + \beta_{j_k}) \\
&= (1 + \beta_{j_{L+1}}) d_L(j_1, \ldots, j_L) - \beta_{j_{L+1}} \prod_{\ell=1}^{L} (1 + \beta_{j_k}).
\end{aligned}
$$

□