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# Wellposedness for a $(1+1)$ -dimensional wave equation with quasilinear boundary condition

Sebastian Ohrem, Wolfgang Reichel\*  
and Roland Schnaubelt

Institute for Analysis, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany

E-mail: [wolfgang.reichel@kit.edu](mailto:wolfgang.reichel@kit.edu)

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## Abstract

We consider the linear wave equation  $V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0$  on  $[0, \infty) \times [0, \infty)$  with initial conditions and a nonlinear Neumann boundary condition  $u_x(0,t) = (f(u_t(0,t)))_t$  at  $x=0$ . This problem is an exact reduction of a nonlinear Maxwell problem in electrodynamics. In the case where  $f: \mathbb{R} \rightarrow \mathbb{R}$  is an increasing homeomorphism we study global existence, uniqueness and wellposedness of the initial value problem by the method of characteristics and fixed point methods. We also prove conservation of energy and momentum and discuss why there is no wellposedness in the case where  $f$  is a decreasing homeomorphism. Finally we show that previously known time-periodic, spatially localized solutions (breathers) of the wave equation with the nonlinear Neumann boundary condition at  $x=0$  have enough regularity to solve the initial value problem with their own initial data.

\* Author to whom any correspondence should be addressed.



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(Some figures may appear in colour only in the online journal)

### 1. Introduction and main results

In this paper we study the initial value problem for the following 1+1-dimensional wave equation with quasilinear boundary condition:

$$\begin{cases} V(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in [0, \infty), t \in [0, \infty), \\ u_x(0,t) = (f(u_t(0,t)))_t, & x = 0, t \in [0, \infty), \\ u(x,t_0) = u_0(x), u_t(x,t_0) = u_1(x), & x \in [0, \infty), t = 0. \end{cases} \tag{1}$$

This initial value problem has two main features: the wave equation on the half-axis  $[0, \infty)$  is linear with a space-dependent speed of propagation and the boundary condition at  $x = 0$  is a rather singular, quasilinear, 2nd-order in time Neumann-condition. We show wellposedness on all time intervals  $[0, T]$  with  $T > 0$ , and preservation of energy and momentum.

Our interest in (1) stems from the fact that it appears in the context of electromagnetics as an exact reduction of a nonlinear Maxwell system. We recall the Maxwell equations in the absence of charges and currents

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, & \mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}), \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= \partial_t \mathbf{D}, & \mathbf{B} &= \mu_0 \mathbf{H} \end{aligned}$$

with the electric field  $\mathbf{E}$ , the electric displacement field  $\mathbf{D}$ , the polarization field  $\mathbf{P}$ , the magnetic field  $\mathbf{B}$ , and the magnetic induction field  $\mathbf{H}$ . Particular properties of the underlying material are modelled by the specification of the relations between  $\mathbf{E}, \mathbf{D}, \mathbf{P}$  on one hand, and  $\mathbf{B}, \mathbf{H}$  on the other hand. Here, we assume a magnetically inactive material, i.e.  $\mathbf{B} = \mu_0 \mathbf{H}$ , but on the electric side we assume a material with a Kerr-type nonlinear behaviour, see [1], section 2.3, given through

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi_1(\mathbf{x}) \mathbf{E} + \varepsilon_0 \chi_{NL}(\mathbf{x}) g(|\mathbf{E}|^2) \mathbf{E}$$

with  $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$  and  $|\cdot|$  the Euclidean norm on  $\mathbb{R}^3$ . For simplicity we assume that  $\chi_1, \chi_{NL}$  are given scalar valued functions instead of the more general situation where they are matrix valued. The scalar constants  $\varepsilon_0, \mu_0$  are such that  $c = (\varepsilon_0 \mu_0)^{-1/2}$  is the speed of light in vacuum. Local existence, wellposedness and regularity results for the general nonlinear Maxwell system have been shown on  $\mathbb{R}^3$  by Kato [2] and on domains by Spitz [3, 4].

In its second order formulation the Maxwell system becomes

$$0 = \nabla \times \nabla \times \mathbf{E} + \partial_t^2 \left( \mu_0 \varepsilon_0 (1 + \chi_1(\mathbf{x})) \mathbf{E} + \mu_0 \varepsilon_0 \chi_{NL}(\mathbf{x}) g(|\mathbf{E}|^2) \mathbf{E} \right). \tag{2}$$

We assume additionally that  $\chi_1(\mathbf{x}) = \chi_1(x), \chi_{NL}(\mathbf{x}) = \chi_{NL}(x)$  and that  $\mathbf{E}$  takes the form of a polarized traveling wave

$$\mathbf{E}(\mathbf{x}, t) = (0, 0, U(x, \kappa^{-1}y - t))^T. \tag{3}$$

Then the quasilinear vectorial wave-type equation (2) turns into the scalar equation

$$V(x) U_{tt} - U_{xx} + \Gamma(x) (g(U^2) U)_{tt} = 0 \tag{4}$$

for  $U = U(x, t)$ , where  $V(x) = \mu_0 \varepsilon_0 (1 + \chi_1(x)) - \kappa^{-2}$  and  $\Gamma(x) = \mu_0 \varepsilon_0 \chi_{NL}(x)$ . Note that (4) is an exact reduction of the Maxwell problem, from which all fields can be reconstructed. E.g. the magnetic induction  $\mathbf{B}$  can be retrieved from  $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$  by time-integration and it will satisfy  $\nabla \cdot \mathbf{B} = 0$  provided it does so at time  $t = 0$ . By assumption the magnetic field is given by  $\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}$  and it satisfies  $\nabla \times \mathbf{H} = \partial_t \mathbf{D}$ . It remains to check that the displacement field  $\mathbf{D}$  satisfies the Gauss law  $\nabla \cdot \mathbf{D} = 0$  in the absence of external charges. This follows directly from the constitutive equation  $\mathbf{D} = \varepsilon_0 (1 + \chi_1(\mathbf{x})) \mathbf{E} + \varepsilon_0 \chi_{NL}(\mathbf{x}) g(|\mathbf{E}|^2) \mathbf{E}$  and the assumption of the polarized form of the electric field in (3).

An extreme case for the potential  $\Gamma$  in front of the nonlinearity arises when  $\Gamma(x) = 2\delta_0(x)$  is a multiple of the  $\delta$ -distribution at 0, see [5, 6]. If additionally  $V(x)$  is even and  $U(x, t) = u_t(x, t)$  for an even function  $u(x, t) = u(-x, t)$ , by removing one time derivative (4) becomes

$$\begin{cases} V(x)u_{tt}(x, t) - u_{xx}(x, t) = 0, & x \in [0, \infty), t \in [0, \infty), \\ u_x(0, t) = (f(u_t(0, t)))_t, & x = 0, t \in [0, \infty) \end{cases} \tag{5}$$

with  $f(s) := g(s^2)s$ . Clearly (1) is the initial value problem for (5). This extreme model describes the concentration of the entire nonlinear behaviour in a waveguide-like structure where the width of the waveguide has been shrunk to zero and the strength of the nonlinearity has been sent to infinity. A similar model, where the potential  $V$  in front of the linear term is taken as a multiple of a  $\delta$ -distribution, has been considered in [7]. Clearly,  $\delta$ -distributions are inserted purely for mathematical simplicity, and may be considered as a step towards physically more realistic models with  $L^\infty$ -potentials.

From the point of view of time-periodic solutions, problem (5) with  $f(s) = \pm s^3$  has been considered in [6]. Under specific assumptions on the linear potential  $V$  the existence of infinitely many breathers, i.e. real-valued, time-periodic, spatially localized solutions of (5), was shown. Typical examples of  $V$  were given in classes of piecewise continuous functions having jump discontinuities. Under different assumptions on  $V$  and  $\Gamma$ , but still including  $\delta$ -distributions, problem (5) was considered in [5] and real-valued breathers were constructed. A series of works considering linear and nonlinear wave equations with Neumann boundary conditions emerged from [8, 9]. Attention was given to global existence and well-posedness as well as to blow-up phenomena arising from nonlinear terms in either the boundary condition or the equation. In [8, 10] decay and global attractors were obtained. We point out that in contrast to our work these papers consider nonlinear terms at the boundary which are only of first order in time and have a damping character. This may also be the reason why even higher-dimensional cases are by now well-understood in this first-order case. Perhaps closest to our set-up is the paper [11], where a linear wave equation in the domain is coupled to a linear wave equation at the boundary and the well-posedness is shown to be true exactly in dimension one—in complete accordance with our set-up.

Our goal is to study the initial value problem (1) from the point of view of wellposedness, to derive the conservation of momentum and energy, and to verify that known time-periodic solutions from [6] satisfy (1) with their own initial values. Note that the boundary condition in (1) becomes  $u_x(0, t) = \pm 3u_t(0, t)^2 u_{tt}(0, t)$  in the model case  $f(s) = \pm s^3$ . Hence, (1) is a singular initial value problem which is not covered by typical theories like, e.g. energy methods or monotone operators. Instead, our approach will be to prove existence by making use of the

method of characteristics. Uniqueness, wellposedness, global existence, and the conservation of energy and momentum will build upon this.

Our basic assumptions on the initial data  $u_0, u_1$  are:

$$u_0 \in C^1([0, \infty)), \quad u_1 \in C([0, \infty)). \tag{A0}$$

Here  $C^k([0, \infty)) = C^k([0, \infty), \mathbb{R})$ , and in general all function spaces consist of real-valued functions unless the codomain is explicitly mentioned. Motivated by the results from [6] we are interested in the case where the coefficient  $V$  may have discontinuities. In particular, we consider piecewise  $C^1$  functions  $V$ . We have chosen the setting  $(u_0, u_1) \in C^1([0, \infty)) \times C([0, \infty))$  since it perfectly fits to our method which is inspired by the method of characteristics. We bridge the gap to Sobolev space-based weak solutions (introduced in definition 1.6) by proposition 5.2 in section 5.

Let  $I \subseteq \mathbb{R}$  be a closed interval. We call a function  $\phi : I \rightarrow \mathbb{R}$  piecewise  $C^k$  if there exists a discrete set  $D \subseteq I$  such that  $\phi \in C^k(I \setminus D)$  and the limits  $\phi^{(j)}(x-)$  and  $\phi^{(j)}(x+)$  exist for all  $x \in D(\phi)$  and  $0 \leq j \leq k$ , although they do not need to coincide. If  $I$  is bounded from below (or above), in addition we require  $\phi^{(j)}(\min I+)$  (or  $\phi^{(j)}(\max I-)$ ) to exist for all  $0 \leq j \leq k$ . Let  $PC^k(I)$  denote the set of piecewise  $C^k$  functions on  $I$ , and for  $\phi \in PC(I) := PC^0(I)$  let us denote by  $D(\phi)$  the set of discontinuities of  $\phi$ .

For the coefficient  $V$  and the nonlinear function  $f$  we assume

$$V \in PC^1([0, \infty)), V, V' \in L^\infty, \inf V > 0, \tag{A1}$$

$$\inf \{|d_1 - d_2| \text{ with } d_1, d_2 \in D(V) \cup \{0\}, d_1 \neq d_2\} > 0, \tag{A2}$$

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ is an increasing homeomorphism.} \tag{A3}$$

The main theorem of this paper is given next.

**Theorem 1.1.** *Assume (A0)–(A3). Then (1) admits a unique and global  $C^1$ -solution. Moreover, (1) is wellposed on every finite time interval  $[0, T]$  with  $T > 0$ .*

In proposition 6.1 our concept of continuous dependence on data is stated precisely. In the above result the assumption (A3) is crucial. For a decreasing homeomorphism  $f$  the result of theorem 1.1 does not hold, see remark 1.7. Since we have already used the notion of a  $C^1$ -solution, we are going to explain it in detail next. As the notion of a  $C^1$ -solution will also be used for subdomains of  $[0, \infty) \times [0, \infty)$  we first define the notion of an admissible domain.

**Definition 1.2 (admissible domain).** We call a set  $\Omega \subseteq [0, \infty) \times [0, \infty)$  an *admissible domain* if it is of the form

$$\Omega = \{(x, t) \in [0, \infty) \times [0, \infty) \mid t \leq h(x)\}$$

where  $h \equiv +\infty$  or  $h : [0, \infty) \rightarrow \mathbb{R}$  is Lipschitz with  $|h_x(x)| \leq \sqrt{V(x)}$  for almost all  $x$ . We denote the relative interior of  $\Omega$  by

$$\Omega^\circ := \{(x, t) \in [0, \infty) \times [0, \infty) \mid t < h(x)\}.$$

In order to explain the notion of a  $C^1$ -solution let us first mention that we cannot expect that a solution of (1) has everywhere second derivatives  $u_{tt}$  or  $u_{xx}$ . This is essentially due to the nonlinear boundary condition and the discontinuities of second derivatives which propagate

away from  $x = 0$ . However, if we denote by  $c(x) := \frac{1}{\sqrt{V(x)}}$  the inverse of the  $x$ -dependent wave speed, then we can factorize the wave operator as

$$\partial_t^2 - c(x)^2 \partial_x^2 = (\partial_t - c(x) \partial_x) (\partial_t + c(x) \partial_x) + c(x) c_x(x) \partial_x.$$

It is then reasonable for a  $C^1$ -solution to have almost everywhere a mixed second directional derivative  $\partial_{\nu,\mu}^2$  with directions  $\nu = (1, -c(x))$  and  $\mu = (1, c(x))$ . This is the basis for the following definition.

**Definition 1.3 (solution).** A function  $u \in C^1(\Omega)$  on an admissible domain  $\Omega$  is called a  $C^1$ -solution to (1) if the following hold:

- (i) For all  $(x, t) \in \Omega \setminus (D(c) \cup D(c_x) \times \mathbb{R})$  we have  $(\partial_t - c(x)\partial_x)(u_t + c(x)u_x)(x, t) = -c(x)c_x(x)u_x(x, t)$ .
- (ii)  $(f(u_t(0, t)))_t = u_x(0, t)$  for all  $(0, t) \in \Omega^\circ$ .
- (iii)  $u(x, 0) = u_0(x)$  for all  $(x, 0) \in \Omega$ ,  $u_t(x, 0) = u_1(x)$  for all  $(x, 0) \in \Omega^\circ$ .

Problem (1) has a momentum given by

$$M(u, t) := \int_0^\infty V(x) u_t dx + f(u_t(0, t)) \tag{6}$$

and an energy given by

$$E(u, t) := \frac{1}{2} \int_0^\infty (V(x) u_t(x, t)^2 + u_x(x, t)^2) dx + F(u_t(0, t)) \tag{7}$$

where  $F(s) := sf(s) - \int_0^s f(\sigma) d\sigma$ . If, e.g.  $f$  is continuously differentiable, then  $F(s)$  is a primitive of  $sf'(s)$ . The conservation of momentum and energy is stated next.

**Theorem 1.4.** Assume (A0)–(A3) and that  $u$  is a  $C^1$ -solution of (1) with  $u_0'(x), u_1(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Then the momentum given by (6) and the energy given by (7) are time-invariant.

**Remark 1.5.** Note that  $F(s) = \int_0^s (f(s) - f(\sigma)) d\sigma$  tends to  $+\infty$  as  $s \rightarrow \pm\infty$ , since by assumption (A3) we have  $f(s) \rightarrow \pm\infty$  as  $s \rightarrow \pm\infty$ . Therefore, due to theorem 1.4,  $u_x(\cdot, t)$  and  $u_t(\cdot, t)$  are bounded in  $L^2([0, \infty))$  and  $u_t(0, t)$  is bounded as well.

Another common notion of solution for (1) is the notion of a weak solution, which we only give for  $\Omega = [0, \infty)^2$ . The fact that a  $C^1$ -solution to (1) is also a weak solution to (1) holds true and will be proven in proposition 5.2 in section 5.

**Definition 1.6 (weak solution).** A function  $u \in W_{loc}^{1,1}([0, \infty) \times [0, \infty))$  is called a weak solution to (1) if  $f(u_t(0, \cdot)) \in L_{loc}^1([0, \infty))$ ,  $u(\cdot, 0) = u_0$ , and  $u$  satisfies

$$\begin{aligned} 0 = & \int_0^\infty \int_0^\infty (V(x) u_t \varphi_t - u_x \varphi_x) dx dt + \int_0^\infty f(u_t(0, t)) \varphi_t(0, t) dt \\ & + \int_0^\infty V(x) u_1(x) \varphi(x, 0) dx + f(u_1(0)) \varphi(0, 0) \end{aligned}$$

for all  $\varphi \in C_c^\infty([0, \infty) \times [0, \infty))$ .

**Remark 1.7.** Due to assumption (A3) we have only considered increasing functions  $f$ . If we instead allow  $f: \mathbb{R} \rightarrow \mathbb{R}$  to be a decreasing homeomorphism, then (1) will not be wellposed in general and can have multiple solutions. Consider for example the cubic term  $f(y) = -y^3$  with constant potential  $V = 1$  and homogeneous initial data:

$$\begin{cases} u_{tt}(x, t) - u_{xx}(x, t) = 0, & x \in [0, \infty), t \in [0, \infty), \\ u_x(0, t) = -(u_t(0, t))^3, & x = 0, t \in [0, \infty), \\ u(x, t_0) = 0, u_t(x, t_0) = 0, & x \in [0, \infty), t = 0. \end{cases} \quad (8)$$

By direct calculation one can show that the right-traveling wave

$$u_p(x, t) = \begin{cases} \left(\frac{2}{3}(t-x)\right)^{\frac{3}{2}}, & x < t, \\ 0, & x \geq t \end{cases}$$

is a nontrivial solution to (8). In fact,  $u_p$  is a  $C^1$ -solution of  $(\partial_x + \partial_t)u = 0$ . But (8) also has the trivial solution  $u = 0$ , or  $u(x, t) = \pm u_p(x, t - \tau)$  for any  $\tau \geq 0$ . However, due to the continuity of  $f^{-1}$ , one can still show existence of solutions to (1) in the case where  $f$  grows at least linearly, cf (A4). This follows from the arguments in sections 3 and 4. Theorem 1.4 also holds when  $f$  is decreasing, but now the quantity  $F(y)$  tends to  $-\infty$  as  $y \rightarrow \pm\infty$ , so that (7) does not give rise to estimates on  $u$ . Lastly, also in this case  $C^1$ -solutions to (1) are weak solutions.

In addition to the problem being posed on the positive real half-line  $x \in [0, \infty)$ , we can also consider the same quasilinear problem posed on a bounded domain  $x \in [0, L]$  where we impose a homogeneous Dirichlet condition at  $x = L$ :

$$\begin{cases} V(x)u_{tt}(x, t) - u_{xx}(x, t) = 0, & x \in [0, L], t \in [0, \infty), \\ u_x(0, t) = (f(u_t(0, t)))_t, & t \in [0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in [0, L], \\ u(L, t) = 0, & t \in [0, \infty). \end{cases} \quad (9)$$

Both theorems 1.1 and 1.4 remain valid when making the obvious adaptations to this setting.

**Theorem 1.8.** Assume (A0)–(A3). Then (9) admits a unique and global  $C^1$ -solution  $u$ . Moreover, the energy given by

$$E(u, t) := \frac{1}{2} \int_0^L \left( V(x)u_t(x, t)^2 + u_x(x, t)^2 \right) dx + F(u_t(0, t)).$$

is time-invariant.

**Remark 1.9.** For Dirichlet boundary data, momentum is in general not conserved.

The paper is structured as follows. In section 2 we provide a change of variables which turns the wave operator with variable wave speed in (1) into a constant coefficient operator with an additional first order term. The well-known constant coefficient operator is useful since it provides implicit solution formulas which are analysed in section 3. In section 4 we prove the existence and uniqueness part of theorem 1.1 under an extra assumption on the nonlinearity  $f$ . We use that the wave equation has finite speed of propagation to argue locally. Difficulties arise at the boundary  $x = 0$  where the nonlinearity comes in to play, and near jumps of  $V$  where wave breaking occurs. We use the method of characteristics and section 3 to obtain a reduced

problem that is a retarded ordinary differential equation, which can be treated using fixed-point arguments. Since the ODE at the boundary is nonlinear we use the extra assumption on  $f$  to close the fixed-point argument. In section 5 we prove energy and momentum conservation as stated in theorem 1.4, and the fact that  $C^1$ -solutions of (1) in the sense of definition 1.3 are also weak solutions, cf proposition 5.2. Using the conservation laws, we also obtain *a priori* bounds that allow us to remove the extra assumption on  $f$  from section 4. The wellposedness part of theorem 1.1 is shown in section 6 using similar methods as in the existence and uniqueness parts. Finally, in section 7 we verify that the breather solutions obtained in [6] satisfy (1) with their own initial values. This is mainly a problem of regularity, as we have to show that these breathers are of class  $C^1$ . To achieve this, we follow ideas from [6] and improve upon their bootstrapping argument. The appendices A and B contain some technical results used in the proofs of the main results.

## 2. A change of variables

It will be convenient to normalize the wave speed to 1. To achieve this, we introduce a new variable  $z = \kappa(x) = \int_0^x \frac{1}{c(s)} ds$ , and thus a new coordinate system  $(z, t)$ . Avoiding new notation we denote the functions  $V, c, u, u_0, u_1$  transformed into this new coordinate system again by  $V, c, u, u_0, u_1$ . The relation between the two coordinate systems is given by

$$\frac{\partial z}{\partial x} = \frac{1}{c(x)} \quad \text{or} \quad c(x) \partial_x = \partial_z \quad \text{or} \quad dx = c(x) dz.$$

From now on until the end of section 5, we will exclusively work with the coordinate system  $(z, t)$ . As before we denote the points where  $c$  is discontinuous by  $D(c)$  and the points where  $c_z$  is discontinuous by  $D(c_z)$ .

Formally the initial value problem (1) transforms into

$$\begin{cases} u_{tt}(z, t) - u_{zz}(z, t) = -\frac{c_z(z)}{c(z)} u_z(z, t), & z \in [0, \infty), t \in [0, \infty), \\ \frac{1}{c(0)} u_z(0, t) = (f(u_t(0, t)))_t, & t \in [0, \infty), \\ u(z, 0) = u_0(z), u_t(z, 0) = u_1(z), & z \in [0, \infty) \end{cases} \quad (10)$$

where we need to take into account that  $u_x = \frac{1}{c} u_z$  is continuous (and not  $u_z$  itself) and that the differential equation does not hold at the discontinuities of  $c$  and  $c_z$ . A detailed definition of the solution concept is given below in definition 2.3.

We begin by rephrasing definitions 1.2 and 1.3 for the new coordinate system.

**Definition 2.1 (admissible domain).** We call a set  $\Omega \subseteq [0, \infty) \times [0, \infty)$  an *admissible domain* if it is of the form

$$\Omega = \{(z, t) \in [0, \infty) \times [0, \infty) \mid t \leq h(z)\}$$

where  $h \equiv +\infty$  or  $h: [0, \infty) \rightarrow \mathbb{R}$  is Lipschitz continuous with Lipschitz constant 1. We denote its relative interior by

$$\Omega^\circ := \{(z, t) \in [0, \infty) \times [0, \infty) \mid t < h(z)\}.$$

Next we introduce function spaces that capture the condition of the continuity of  $\frac{1}{c} u_z$ .



**Definition 2.2 (x-dependent function spaces).** Let the transformation between  $(x, t)$  and  $(z, t)$ -coordinates be given by  $\tilde{\kappa}(x, t) := (\kappa(x), t) = (z, t)$ . For  $\Omega \subseteq [0, \infty) \times [0, \infty)$  we write

$$C^1_{(x,t)}(\Omega) := \{u: \Omega \rightarrow \mathbb{R} \mid u \circ \tilde{\kappa} \in C^1(\tilde{\kappa}^{-1}(\Omega))\}$$

where we understand  $u$  to be a function of  $(z, t)$  variables, and  $\tilde{u} := u \circ \tilde{\kappa}$  is the  $(x, t)$ -dependent version of  $u$ , i.e.  $\tilde{u}(x, t) = u(z, t)$  holds. Note that  $u \in C^1_{(x,t)}(\Omega)$  if and only if  $u, u_t, \frac{1}{c}u_z \in C(\Omega)$ .

Similarly, for an interval  $I \subseteq [0, \infty)$  we define

$$C^1_x(I) := \{v: I \rightarrow \mathbb{R} \mid v \circ \kappa \in C^1(\kappa^{-1}(I))\}.$$

where again we understand  $v$  to be a function of  $z$ .

**Definition 2.3 (solution).** A function  $u \in C^1_{(x,t)}(\Omega)$  on an admissible domain  $\Omega$  is called a  $C^1$ -solution to (10) if the following hold:

- (i) For all  $(z, t) \in \Omega \setminus (D(c) \cup D(c_z) \times \mathbb{R})$  we have  $(\partial_t - \partial_z)(u_t + u_z)(z, t) = -\frac{c_z(z)}{c(z)}u_z(z, t)$ .
- (ii)  $f(u_t(0, t))_t = \frac{1}{c(0)}u_z(0, t)$  for all  $(0, t) \in \Omega^\circ$ .
- (iii)  $u(z, 0) = u_0(z)$  for all  $(z, 0) \in \Omega$ ,  $u_t(z, 0) = u_1(z)$  for all  $(z, 0) \in \Omega^\circ$ .

**Remark 2.4.** Note that  $u: \Omega \rightarrow \mathbb{R}$  is a  $C^1$ -solution to (1) in the  $(x, t)$ -coordinates if and only if it is a  $C^1$ -solution to (10) in the  $(z, t)$ -coordinates.

### 3. Auxiliary results on the linear part

In this section we gather some auxiliary results and estimates on the linear wave equation. These will prove useful for the study of the nonlinear initial boundary value problem (10). All results of this section hold under the assumptions (A0)–(A3).

We first note that the wave equation has finite speed of propagation; if we know its behaviour at time  $t_0$  on an interval  $[z_0 - r, z_0 + r]$ , then we can defer its accurate behaviour on the space-time triangle with corners  $(z_0 - r, t_0)$ ,  $(z_0 + r, t_0)$  and  $(z_0, t_0 + r)$ .

**Definition 3.1.** For  $(z_0, t_0) \in \mathbb{R}^2$  and  $r > 0$  we denote the triangle with corners  $(z_0 - r, t_0)$ ,  $(z_0 + r, t_0)$  and  $(z_0, t_0 + r)$  by

$$\Delta(z_0, t_0, r) := \{(z, t) \in \mathbb{R}^2 \mid t \geq t_0, |z - z_0| + |t - t_0| \leq r\},$$

its base projected onto the  $z$ -axis is given by  $P_z\Delta(z_0, t_0, r) = [z_0 - r, z_0 + r]$  with projection  $P_z(z, t) := z$ . Similarly, we define left and right half triangles

$$\Delta_-(z_0, t_0, r) := \Delta(z_0, t_0, r) \cap \{z \leq z_0\}, \quad \Delta_+(z_0, t_0, r) := \Delta(z_0, t_0, r) \cap \{z \geq z_0\}$$

whose bases are given by

$$P_z\Delta_-(z_0, t_0, r) = [z_0 - r, z_0], \quad P_z\Delta_+(z_0, t_0, r) = [z_0, z_0 + r].$$

Recall the solution formula for the one-dimensional wave equation:

**Theorem 3.2.** Let  $(z_0, t_0) \in \mathbb{R}^2$ ,  $r > 0$ ,  $\Delta := \Delta(z_0, t_0, r)$  and  $B := P_z \Delta$ . Assume that  $u_0 \in C^1(B)$ ,  $u_1 \in C(B)$ , and  $g \in L^\infty(\Delta)$  is continuous outside a set  $L$  consisting of finitely many lines of the form  $\{z = \text{const}\}$ . Then the function

$$u(z, t) = \frac{1}{2} (u_0(z + t - t_0) + u_0(z - t + t_0)) + \frac{1}{2} \int_{z-t+t_0}^{z+t-t_0} u_1(y) dy + \frac{1}{2} \int_{\Delta(z, t_0, t-t_0)} g(y, \tau) d(y, \tau)$$

belongs to  $C^1(\Delta)$  and is the unique  $C^1$ -solution of the problem

$$\begin{cases} (\partial_t - \partial_z)(u_t + u_z) = g, & (z, t) \in \Delta, \\ u(z, t_0) = u_0(z), \quad u_t(z, t_0) = u_1(z), & z \in B \end{cases}$$

in the following sense:  $u(\cdot, t_0) = u_0(\cdot)$ ,  $u_t(\cdot, t_0) = u_1(\cdot)$  on  $B$  and the directional derivative  $(\partial_t - \partial_z)(u_t + u_z)$  exists and equals  $g$  on  $\Delta^\circ \setminus L$ .

**Remark 3.3.** For every  $C^1$ -solution  $u$  of  $(\partial_t - \partial_z)(u_t + u_z) = g$  on a domain we have that  $(\partial_t + \partial_z)(u_t - u_z) = (\partial_t - \partial_z)(u_t + u_z)$  wherever  $g$  is continuous, cf Schwarz’s theorem in [12, theorem 9.41]. As a consequence, any of the two factorizations of the wave operator  $(\partial_t - \partial_z)(\partial_t + \partial_z)$  or  $(\partial_t + \partial_z)(\partial_t - \partial_z)$  can be used and yields the same solution.

By combining the above theorem 3.2 with a fixed point argument, we can treat the initial value problem for  $(\partial_t - \partial_z)(u_t + u_z) = -\frac{c_z(z)}{c(z)}u_z$  on sufficiently small triangles  $\Delta$ . In order to have a slightly more general situation available we work with a piecewise continuous function  $\lambda$  instead of  $\frac{c_z}{c}$ .

**Corollary 3.4.** Let  $(z_0, t_0) \in \mathbb{R}^2$  and  $\Delta := \Delta(z_0, t_0, r)$ ,  $B := P_z \Delta$  for  $r > 0$ . Assume  $u_0 \in C^1(B)$ ,  $u_1 \in C(B)$  and  $\lambda \in PC(B)$  such that  $r \|\lambda\|_\infty < 1$ . Then

$$\begin{cases} (\partial_t - \partial_z)(u_t + u_z) = -\lambda(z)u_z, & (z, t) \in \Delta, \\ u(z, t_0) = u_0(z), u_t(z, t_0) = u_1(z), & z \in B \end{cases} \tag{11}$$

has a unique solution  $u \in C^1(\Delta)$  in the sense of theorem 3.2 with  $g = -\lambda u_z$  and  $L = D(\lambda) \times \mathbb{R}$ . We denote this solution by  $\Phi(u_0, u_1) := u$ .

**Remark 3.5.** If additionally  $u_0, u_1$  are odd around  $z = z_0$  and  $\lambda$  is odd around  $z = z_0$ , then the solution of (11) is odd around  $z = z_0$ . To see this, notice that under these assumptions the odd reflection of the solution  $u$  of (11) again solves (11) – but with the opposite factorization of the wave operator. Hence, by remark 3.3 and uniqueness of solutions,  $u$  coincides with its odd reflection.

**Proof of corollary 3.4.** W.l.o.g. we assume  $(z_0, t_0) = (0, 0)$ . Let  $u \in C^1(\Delta)$ . Then by theorem 3.2  $u$  is a solution if and only if

$$u(z, t) = \frac{1}{2} (u_0(z + t) + u_0(z - t)) + \frac{1}{2} \int_{z-t}^{z+t} u_1(y) dy - \frac{1}{2} \int_{\Delta(z, 0, t)} \lambda(y) u_z(y, \tau) d(y, \tau) \tag{12}$$

holds for  $(z, t) \in \Delta$ . Taking the derivative w.r.t.  $z$  we obtain

$$u_z(z, t) = \frac{1}{2} (u'_0(z+t) + u'_0(z-t)) + \frac{1}{2} (u_1(z+t) - u_1(z-t)) - \frac{1}{2} \int_0^t \lambda(z+t-s) u_z(z+t-s, s) ds + \frac{1}{2} \int_0^t \lambda(z-t+s) u_z(z-t+s, s) ds. \quad (13)$$

We consider (13) as a fixed point problem for  $u_z \in C(\Delta)$ . If we denote the right-hand side of (13) by  $T(u_z)(z, t)$ , then clearly  $T$  maps  $C(\Delta)$  into itself. Furthermore, one has

$$\begin{aligned} & \|T(u_z) - T(w_z)\|_\infty \\ &= \frac{1}{2} \sup_{(z,t) \in \Delta} \left| - \int_0^t \lambda(z+s) \cdot [u_z - w_z](z+s, t-s) ds \right. \\ &\quad \left. + \int_0^t \lambda(z-s) \cdot [u_z - w_z](z-s, t-s) ds \right| \\ &\leq \|\lambda\|_\infty r \cdot \|u_z - w_z\|_\infty \end{aligned}$$

so that by Banach’s fixed-point theorem there exists a unique solution  $u_z$  of (13). With the help of  $u_z$  we define  $u$  as in (12) and thus get the claimed result.  $\square$

In the setting of the above proof, we can obtain estimates on the solution  $u$ . First, if we set  $q := r\|\lambda\|_\infty$ , then by Banach’s fixed-point theorem we have

$$\|u_z - 0\|_\infty \leq \frac{1}{1-q} \|T(0) - 0\|_\infty.$$

Using  $\|T(0)\|_\infty \leq \|u'_0\|_\infty + \|u_1\|_\infty$ , we obtain

$$\|u_z\|_\infty \leq \frac{1}{1-q} (\|u'_0\|_\infty + \|u_1\|_\infty).$$

From

$$\begin{aligned} u(z, t) &= \frac{1}{2} (u_0(z+t) + u_0(z-t)) + \frac{1}{2} \int_{z-t}^{z+t} u_1(y) dy - \frac{1}{2} \int_0^t \int_{z-(t-\tau)}^{z+(t-\tau)} \lambda(y) u_z(y, \tau) dy d\tau, \\ u_t(z, t) &= \frac{1}{2} (u'_0(z+t) - u'_0(z-t)) + \frac{1}{2} (u_1(z+t) + u_1(z-t)) \\ &\quad - \frac{1}{2} \int_0^t \lambda(z+s) u_z(z+s, t-s) ds - \frac{1}{2} \int_0^t \lambda(z-s) u_z(z-s, t-s) ds \end{aligned}$$

we also obtain

$$\begin{aligned} \|u\|_\infty &\leq \|u_0\|_\infty + r\|u_1\|_\infty + \frac{1}{2} r^2 \|\lambda\|_\infty \|u_z\|_\infty, \\ \|u_t\|_\infty &\leq \|u'_0\|_\infty + \|u_1\|_\infty + r\|\lambda\|_\infty \|u_z\|_\infty. \end{aligned}$$

Combining these estimates, we get the following result.

**Corollary 3.6.** *In the setting of corollary 3.4, the following estimates hold with  $q := r\|\lambda\|_\infty$ :*

$$\begin{aligned} \|u\|_\infty &\leq \|u_0\|_\infty + \frac{rq}{2(1-q)} \|u'_0\|_\infty + \frac{r(1-\frac{1}{2}q)}{1-q} \|u_1\|_\infty, \\ \|u_z\|_\infty &\leq \frac{1}{1-q} (\|u'_0\|_\infty + \|u_1\|_\infty), \\ \|u_t\|_\infty &\leq \frac{1}{1-q} (\|u'_0\|_\infty + \|u_1\|_\infty). \end{aligned}$$

In particular, there exists a constant  $C = C(r, \|\lambda\|_\infty)$  such that the operator-norm of the linear solution operator  $\Phi : C^1(B) \times C(B) \rightarrow C^1(\Delta)$ , which maps the data  $(u_0, u_1) \in C^1(B) \times C(B)$  to the solution of (11), satisfies

$$\|\Phi\| \leq C.$$

Recall that in definition 2.3 we required  $\frac{u_z}{c}$  to be continuous. Since  $c$  may have jumps, e.g. at  $z_0$ , we also need to treat the jump condition

$$\frac{u_z(z_0+, t)}{c(z_0+)} = \frac{u_z(z_0-, t)}{c(z_0-)}.$$

We prepare this in the following lemma by adding to (11) the inhomogeneous Dirichlet condition  $u(z_0, t) \stackrel{!}{=} b(t)$  at the spatial boundary  $z = z_0$ .

**Lemma 3.7.** Let  $(z_0, t_0) \in \mathbb{R}^2$  and  $\Delta_+ := \Delta_+(z_0, t_0, r)$ ,  $B_+ := P_z \Delta_+$  for  $r > 0$ . Assume  $u_0 \in C^1(B_+)$ ,  $u_1 \in C(B_+)$ ,  $b \in C^1([t_0, t_0 + r])$  with  $b(t_0) = u_0(z_0)$ ,  $b'(t_0) = u_1(z_0)$  and  $\lambda \in PC(B_+)$  such that  $r\|\lambda\|_\infty < 1$ . Then the problem

$$\begin{cases} (\partial_t - \partial_z)(u_t + u_z) = -\lambda(z)u_z, & (z, t) \in \Delta_+^\circ, \\ u(z_0, t) = b(t), & t \in [t_0, t_0 + r], \\ u(z, t_0) = u_0(z), u_t(z, t_0) = u_1(z), & z \in B_+, \end{cases} \tag{14}$$

has a unique  $C^1$ -solution  $u : \Delta_+ \rightarrow \mathbb{R}$  in the sense of theorem 3.2 with  $g = -\lambda u_z$  and  $L = D(\lambda) \times \mathbb{R}$ . We denote this solution by  $\Phi_+(b, u_0, u_1) := u$ . The assertion also holds for the left half triangle  $\Delta_- := \Delta_-(z_0, t_0, r)$  with corresponding solution operator  $\Phi_-$ .

**Proof.** Note that the function  $G^b$  defined on  $\Delta_+$  by

$$G^b(z, t) = \begin{cases} b(t_0) + (t - t_0)b'(t_0), & z - z_0 > t - t_0 \geq 0, \\ b(t + z_0 - z) + (z - z_0)b'(t_0), & t - t_0 \geq z - z_0 \geq 0 \end{cases} \tag{15}$$

belongs to  $C^1(\Delta_+)$ , solves the homogenous wave equation  $(\partial_t - \partial_z)(\partial_t + \partial_z)G^b = 0$  on  $\Delta_+$ , and satisfies  $G^b(z_0, t) = b(t)$ . Setting  $v := u - G^b$ , problem (14) can be rewritten as

$$\begin{cases} (\partial_t - \partial_z)(v_t + v_z) = -\lambda(z)(v_z + G_z^b), & (z, t) \in \Delta_+^\circ, \\ v(z_0, t) = 0, & t \in [t_0, t_0 + r], \\ v(z, t_0) = u_0(z) - b(t_0) =: v_0(z), & z \in B_+, \\ v_t(z, t_0) = u_1(z) - b'(t_0) =: v_1(z), & z \in B_+. \end{cases} \tag{16}$$

Note that  $v_0(z_0) = v_1(z_0) = 0$  by assumption. If we extend the functions  $v_0, v_1,$  and  $\lambda$  in an odd way and  $G^b$  in an even way around  $z = z_0,$  we can consider the problem

$$\begin{cases} (\partial_t - \partial_z)(\tilde{v}_t + \tilde{v}_z) = -\lambda_{\text{odd}}(z) \cdot (\tilde{v}_z + G_{\text{even},z}^b) & (z, t) \in \Delta^\circ, \\ \tilde{v}(z, t_0) = v_{0,\text{odd}}(z), & z \in B, \\ \tilde{v}_t(z, t_0) = v_{1,\text{odd}}(z), & z \in B, \end{cases} \tag{17}$$

where  $\Delta := \Delta(z_0, t_0, r)$  and  $B := P_z \Delta.$  Arguing as in the proof of corollary 3.4, we see that due to the Banach fixed-point theorem, (17) has a unique solution, which must be odd, cf remark 3.5. Now, on one hand the solution of (17) solves (after restriction to  $\Delta_+$ ) (16) and, on the other hand, after odd extension around  $z = z_0$  every solution of (16) solves (17). This shows existence and uniqueness for (16) and hence for (14).  $\square$

**Remark 3.8.** One can show that there exists a constant  $C = C(r, \|\lambda\|_\infty)$  such that

$$\Phi_\pm : \mathcal{D}(\Phi_\pm) \subseteq C^1([t_0, t_0 + r]) \times C^1(B_\pm) \times C(B_\pm) \rightarrow C^1(\Delta_\pm)$$

satisfy  $\|\Phi_\pm\| \leq C,$  where the domain  $\mathcal{D}(\Phi_\pm)$  consists of those  $(b, u_0, u_1)$  that satisfy  $b(t_0) = u_0(z_0)$  and  $b'(t_0) = u_1(z_0).$

When treating the nonlinear problem (1), the operators  $\Phi_\pm$  play an important role and the estimate in remark 3.8 will be used. However, we need to investigate the dependency of  $\Phi_\pm$  on the datum  $b$  more precisely. This will be achieved next in the case where  $u_0 = u_1 = 0.$

**Lemma 3.9 (Estimate on  $\Phi_\pm$  in the case  $u_0 = u_1 = 0$ ).** *Let  $\Delta_\pm$  and  $\lambda$  be as in lemma 3.7 with  $q := r\|\lambda\|_\infty < 1.$  Assume  $b \in C^1([t_0, t_0 + r])$  and  $b(t_0) = b'(t_0) = 0.$  Then for  $u := \Phi_\pm(b, 0, 0)$  one has*

$$|u_z(z, t) \pm b'(m)| \leq \alpha |z - z_0| |b'(m)| + \beta \int_{t_0}^m |b'(\tau)| \, d\tau,$$

where  $m := \max\{t_0, t - |z - z_0|\}, \alpha := \frac{2}{4-q} \|\lambda\|_\infty,$  and  $\beta := \frac{4}{(2-q)(4-q)} \|\lambda\|_\infty.$

**Proof.** We only give the proof in the ‘+’-case and for  $(z_0, t_0) = (0, 0).$  We revisit the proof of lemma 3.7 where  $\Phi_+$  is defined. From (13) we know that  $v_z$  satisfies

$$\begin{aligned} v_z(z, t) = & -\frac{1}{2} \int_0^t \lambda_{\text{odd}}(z + s) \cdot (G_{\text{even},z}^b(z + s, t - s) + v_z(z + s, t - s)) \, ds \\ & + \frac{1}{2} \int_0^t \lambda_{\text{odd}}(z - s) \cdot (G_{\text{even},z}^b(z - s, t - s) + v_z(z - s, t - s)) \, ds. \end{aligned}$$

We denote the term on the right-hand side by  $T(v_z)(z, t)$  and already know that  $T$  is Lipschitz continuous with constant  $q < 1.$  Therefore we may write the solution as  $v_z := \lim_{n \rightarrow \infty} T^n(0)$  and thus have to study  $v_z^{(n)} := T^n(0).$  The claimed inequality for  $u_z$  will follow once we have shown that

$$|v_z(z, t)| \leq \alpha |z - z_0| |b'(m)| + \beta \int_{t_0}^m |b'(\tau)| \, d\tau.$$

Due to  $v_z = \lim_{n \rightarrow \infty} T^n(0)$  it is sufficient to show that this estimate holds for all  $v_z^{(n)}$ . Since  $v_z^{(0)} = 0$ , there is nothing left to show for  $n = 0$ . Now assume that the estimate has been shown for some fixed  $n$ . Recalling the definition of  $G^b$  from (15), we have

$$G_{\text{even},z}^b(z,t) = -\text{sign}(z) b'(\max\{t - |z|, 0\}).$$

Notice that  $G_{\text{even},z}^b(z,t)$  vanishes for  $|z| \geq t$ . Therefore, if  $v_z^{(n)}$  vanishes for  $|z| \geq t$  then also  $v_z^{(n+1)} = T(v_z^{(n)})$  vanishes on this set. So in the following we may assume  $|z| < t$ . We will only consider  $z \geq 0$  as  $z < 0$  can be treated similarly. For  $z \geq 0$  and  $t > z$  the expression  $m = \max\{t - |z|, 0\}$  simplifies to  $m = t - z$ . We begin by estimating the terms which are independent of  $v_z^{(n)}$ :

$$\begin{aligned} & \left| \int_0^t \lambda_{\text{odd}}(z+s) G_{\text{even},z}^b(z+s,t-s) \, ds \right| \\ &= \left| - \int_0^t \lambda_{\text{odd}}(z+s) b'(\max\{t-z-2s, 0\}) \, ds \right| \\ &\leq \frac{1}{2} \|\lambda\|_\infty \int_0^{t-z} |b'(\tau)| \, d\tau = \frac{1}{2} \|\lambda\|_\infty \int_0^m |b'(\tau)| \, d\tau, \\ & \left| \int_0^t \lambda_{\text{odd}}(z-s) G_{\text{even},z}^b(z-s,t-s) \, ds \right| \\ &= \left| - \int_0^z \lambda_{\text{odd}}(z-s) b'(t-z) \, ds + \int_z^t \lambda_{\text{odd}}(z-s) b'(\max\{t+z-2s, 0\}) \, ds \right| \\ &\leq \|\lambda\|_\infty z |b'(t-z)| + \frac{1}{2} \|\lambda\|_\infty \int_0^{t-z} |b'(\tau)| \, d\tau = \|\lambda\|_\infty z |b'(m)| + \frac{1}{2} \|\lambda\|_\infty \int_0^m |b'(\tau)| \, d\tau. \end{aligned}$$

The remaining two summands are treated by

$$\begin{aligned} & \left| \int_0^t \lambda_{\text{odd}}(z+s) v_z^{(n)}(z+s,t-s) \, ds \right| \\ &\leq \|\lambda\|_\infty \int_0^t \left( \alpha(z+s) |b'(\max\{t-z-2s, 0\})| + \beta \int_0^{\max\{t-z-2s, 0\}} |b'(\tau)| \, d\tau \right) \, ds \\ &= \|\lambda\|_\infty \int_0^{\frac{t-z}{2}} \left( \alpha(z+s) |b'(t-z-2s)| + \beta \int_0^{t-z-2s} |b'(\tau)| \, d\tau \right) \, ds \\ &\leq \|\lambda\|_\infty \int_0^{\frac{t-z}{2}} \left( \alpha \frac{t+z}{2} |b'(t-z-2s)| + \beta \int_0^{t-z} |b'(\tau)| \, d\tau \right) \, ds \\ &= \|\lambda\|_\infty \left( \alpha \frac{t+z}{4} + \beta \frac{t-z}{2} \right) \int_0^m |b'(\tau)| \, d\tau, \\ & \left| \int_0^t \lambda_{\text{odd}}(z-s) v_z^{(n)}(z-s,t-s) \, ds \right| \\ &\leq \|\lambda\|_\infty \int_0^t \left( \alpha |z-s| |b'(\max\{t-s-|z-s|, 0\})| + \beta \int_0^{\max\{t-s-|z-s|, 0\}} |b'(\tau)| \, d\tau \right) \, ds \end{aligned}$$

$$\begin{aligned}
 &= \|\lambda\|_\infty \int_0^z \left( \alpha(z-s)|b'(t-z)| + \beta \int_0^{t-z} |b'(\tau)| \, d\tau \right) \, ds \\
 &\quad + \|\lambda\|_\infty \int_z^{\frac{t+z}{2}} \left( \alpha(s-z)|b'(t+z-2s)| + \beta \int_0^{t+z-2s} |b'(\tau)| \, d\tau \right) \, ds \\
 &\leq \|\lambda\|_\infty \left( \alpha \frac{z^2}{2} |b'(m)| + \beta z \int_0^m |b'(\tau)| \, d\tau \right) \\
 &\quad + \|\lambda\|_\infty \left( \alpha \frac{t-z}{4} + \beta \frac{t-z}{2} \right) \int_0^m |b'(\tau)| \, d\tau.
 \end{aligned}$$

Summing up all four estimates, we obtain

$$\begin{aligned}
 &2 \left| v_z^{(n+1)}(z,t) \right| \\
 &\leq \frac{1}{2} \|\lambda\|_\infty \int_0^m |b'(\tau)| \, d\tau \\
 &\quad + \|\lambda\|_\infty z |b'(m)| + \frac{1}{2} \|\lambda\|_\infty \int_0^m |b'(\tau)| \, d\tau \\
 &\quad + \|\lambda\|_\infty \left( \alpha \frac{t+z}{4} + \beta \frac{t-z}{2} \right) \int_0^m |b'(\tau)| \, d\tau \\
 &\quad + \|\lambda\|_\infty \left( \alpha \frac{z^2}{2} |b'(m)| + \beta z \int_0^m |b'(\tau)| \, d\tau \right) \\
 &\quad + \|\lambda\|_\infty \left( \alpha \frac{t-z}{4} + \beta \frac{t-z}{2} \right) \int_0^m |b'(\tau)| \, d\tau \\
 &= \|\lambda\|_\infty \left( 1 + \alpha \frac{z}{2} \right) z |b'(m)| \\
 &\quad + \|\lambda\|_\infty \left( \frac{1}{2} + \frac{1}{2} + \alpha \frac{t+z}{4} + \beta \frac{t-z}{2} + \beta z + \alpha \frac{t-z}{4} + \beta \frac{t-z}{2} \right) \int_0^m |b'(\tau)| \, d\tau \\
 &=: 2C_1 z |b'(m)| + 2C_2 \int_0^m |b'(\tau)| \, d\tau.
 \end{aligned}$$

It remains to verify  $C_1 \leq \alpha$  and  $C_2 \leq \beta$ . In fact, using  $t, z \leq r$ , we obtain

$$\begin{aligned}
 2C_1 &\leq \|\lambda\|_\infty + \frac{q}{2} \alpha = 2\alpha, \\
 2C_2 &\leq \|\lambda\|_\infty + \frac{q}{2} \alpha + q\beta = 2\alpha + q\beta = 2\beta,
 \end{aligned}$$

where the equalities hold by definition of  $\alpha$  and  $\beta$ , respectively. □

#### 4. Main part of proof of theorem 1.1

In this section, we will prove the existence and uniqueness part of the main theorem 1.1 under the additional assumption that  $f$  grows at least linearly, i.e. for some  $a, A > 0$  we have

$$|f(x)| \geq a|x| - A \quad \text{for } x \in \mathbb{R}. \tag{A4}$$

The assumption (A4) will be used in lemma 4.3 below as an upper bound on  $f^{-1}$  which helps in the construction of solutions to (10). In section 5 we show that the argument of  $f$ , that is  $u_t(0, t)$ ,

is uniformly bounded on finite time intervals, and thereby eliminate the growth assumptions on  $f$ . The wellposedness part of theorem 1.1 will be completed in section 6.

We will again use that the wave equation has finite speed of propagation so that we may argue locally. To be more specific, we will work on the following types of triangular domains:

- A *jump triangle* is a triangle  $\Delta = \Delta(z_0, 0, r)$  with base  $B = P_z\Delta \subseteq (0, \infty)$ , where  $z_0 \in D(c)$  and  $B$  intersects  $D(c)$  in no other point. These are useful for the study of the jump condition  $\frac{u_z(z+,t)}{c(z+)} = \frac{u_z(z-,t)}{c(z-)}$ .
- A *boundary triangle* is a half-triangle  $\Delta_+ = \Delta_+(0, 0, r)$  with base  $B_+ = P_z\Delta_+ = [0, r]$  where  $B_+$  does not intersect  $D(c)$ . These are used to study the nonlinear Neumann condition  $\frac{u_z}{c(0)} = (f(u_t))_t$ .
- A *plain triangle* is a triangle  $\Delta = \Delta(z_0, 0, r)$  with base  $B = P_z\Delta \subseteq (0, \infty)$  not intersecting  $D(c)$ . These are used to cover the remaining space.

In the next three lemmata, we show that (10) is wellposed on all three types of domains.

**Lemma 4.1.** *Let  $\Delta$  be a plain triangle with base  $B$ . Assume  $r \|\frac{c_z}{c}\|_\infty < 1$ . Then (10) has a unique  $C^1$ -solution  $u$  on  $\Delta$  and there exists a constant  $C = C(r, \|\frac{c_z}{c}\|_\infty)$  such that the solution operator  $\Phi : C^1(B) \times C(B) \rightarrow C^1(\Delta), (u_0, u_1) \mapsto u$  satisfies  $\|\Phi\| \leq C$ .*

**Proof.** Since  $\Delta$  is disjoint from the spatial boundary  $z=0$ , the boundary condition (ii) in definition 2.3 is trivially satisfied on  $\Delta$ . By corollary 3.4 we have uniqueness of solutions, and the estimate holds by corollary 3.6.  $\square$

**Lemma 4.2.** *Let  $\Delta$  be a jump triangle with base  $B$ . Assume  $r \|\frac{c_z}{c}\|_\infty < 1$ . Then (10) has a unique  $C^1$ -solution  $u$  on  $\Delta$  and there exists a constant  $C = C(r, \|\frac{c_z}{c}\|_\infty)$  such that the solution operator  $\Phi : C^1_x(B) \times C(B) \rightarrow C^1_{(x,t)}(\Delta), (u_0, u_1) \mapsto u$  satisfies  $\|\Phi\| \leq C$ .*

**Proof.** As in lemma 4.1, the boundary condition at  $z=0$  trivially holds. Now let  $\Delta = \Delta(z_0, 0, r)$ . If  $u : \Delta \rightarrow \mathbb{R}$  is a solution of (10), then by defining  $b : [0, r] \rightarrow \mathbb{R}, b(t) = u(z_0, t)$  and using lemma 3.7 we have

$$u(z, t) = \begin{cases} \Phi_+(b, u_0, u_1)(z, t), & z \geq z_0, \\ \Phi_-(b, u_0, u_1)(z, t), & z \leq z_0. \end{cases} \tag{18}$$

On the other hand, if  $b \in C^1([0, r])$  with  $b(0) = u_0(z_0)$  and  $b'(0) = u_1(z_0)$  is given, then the function  $u$  defined by (18) satisfies  $u, u_t \in C(\Delta)$  as  $\Phi_\pm(b, u_0, u_1)$  and  $\Phi_\pm(b, u_0, u_1)_t$  coincide with  $b$  resp.  $b'$  at the boundary  $z = z_0$ . Hence,  $u$  solves (10) if and only if  $u_x$  is continuous, i.e.

$$\frac{u_z(z_0+, t)}{c(z_0+)} = \frac{u_z(z_0-, t)}{c(z_0-)} \tag{19}$$

holds for all  $t \in [0, r]$ . Using (18), we can write (19) as

$$\frac{1}{c(z_0-)} \Phi_-(b, u_0, u_1)_z(z_0, t) = \frac{1}{c(z_0+)} \Phi_+(b, u_0, u_1)_z(z_0, t)$$

or as

$$b'(t) = \gamma \left( \frac{1}{c(z_0-)} (b'(t) - \Phi_-(b, u_0, u_1)_z(z_0, t)) + \frac{1}{c(z_0+)} (b'(t) + \Phi_+(b, u_0, u_1)_z(z_0, t)) \right) \tag{20}$$



with

$$\gamma := \left( \frac{1}{c(z_{0-})} + \frac{1}{c(z_{0+})} \right)^{-1}.$$

We denote the right-hand side of (20) by  $T(b)(t)$  and show now that  $\Psi : b \mapsto u_0(z_0) + \int_0^{(\cdot)} T(b)(\tau) d\tau$  is a strict contraction in the space  $X := \{b \in C^1([0, r]) \mid b(0) = u_0(z_0)\}$  with norm  $\|b\|_X = \sup \{e^{-\mu t} |b'(t)| : t \in [0, r]\}$ , where  $\mu > 0$  will be chosen later. So let  $b, \tilde{b} \in X$  and write  $\hat{b} := b - \tilde{b}$ . Next we estimate

$$\begin{aligned} & \left| \Psi(b)'(t) - \Psi(\tilde{b})'(t) \right| \\ &= \gamma \left| \frac{1}{c(z_{0-})} \left( \hat{b}'(t) - \Phi_-(\hat{b}, 0, 0)_z(z_0, t) \right) + \frac{1}{c(z_{0+})} \left( \hat{b}'(t) + \Phi_+(\hat{b}, 0, 0)_z(z_0, t) \right) \right| \\ &\leq \gamma \left( \frac{1}{c(z_{0-})} \beta \int_0^t |\hat{b}'(\tau)| d\tau + \frac{1}{c(z_{0+})} \beta \int_0^t |\hat{b}'(\tau)| d\tau \right) \\ &= \beta \int_0^t |\hat{b}'(\tau)| d\tau \leq \beta \| \hat{b} \|_X \int_0^t e^{\mu\tau} d\tau \leq \frac{\beta}{\mu} e^{\mu t} \| \hat{b} \|_X, \end{aligned}$$

where  $\beta$  is the constant from lemma 3.9. If we choose  $\mu > \beta$ , then  $\Psi$  is a strict contraction so that  $b = \Psi(b)$  has a unique solution by Banach's fixed-point theorem. Using remark 3.8, the fixed-point theorem also shows that  $b$  linearly and continuously depends on  $u_0$  and  $u_1$ . Moreover, boundedness of the linear solution operator  $\Phi$  then follows from (18). □

Next we discuss wellposedness on boundary triangles. Unlike for the other types of triangles, now the nonlinear boundary condition of (10) appears, and becomes the main object of our study.

**Lemma 4.3.** *Let  $\Delta_+$  be a boundary triangle with base  $B_+$ . Assume  $r \|\frac{c_z}{c}\|_\infty < 1$ . Then (10) has a unique  $C^1$ -solution on  $\Delta_+$ .*

Let us give a motivation of this result. As in lemma 4.2 it will be convenient to rephrase the problem as an ordinary differential equation. Again we write  $b(t) = u(0, t)$  so that  $u$  is a solution on  $\Delta_+$  if and only if

$$u = \Phi_+(b, u_0, u_1) \quad \text{and} \quad \frac{df(u_t(0, t))}{dt} = \frac{u_z(0, t)}{c(0)}$$

hold. We may rewrite the latter equation as

$$\frac{df(b'(t))}{dt} = \frac{1}{c(0)} \Phi_+(b, u_0, u_1)_z(0, t),$$

eliminating  $u$ . We rewrite this as an equation in  $d(t) := f(b'(t))$ , where  $b$  can be reconstructed from  $d$  via  $b_d(t) := u_0(0) + \int_0^t f^{-1}(d(\tau)) d\tau$ . We are left with solving

$$d'(t) = \frac{1}{c(0)} \Phi_+(b_d, u_0, u_1)_z(0, t), \quad d(0) = f(u_1(0)). \tag{21}$$

We have  $\Phi_+(b, u_0, u_1)_z(0, t) = -b'(t) + g(t)$  where  $g$  depends (up to a small error) only on the initial data  $u_0, u_1$ , hence

$$d'(t) = \frac{1}{c(0)} [g(t) - b'_d(t)] = \frac{1}{c(0)} [g(t) - f^{-1}(d(t))]. \tag{22}$$

Ignoring the error, (22) would be an ODE with monotone decreasing right-hand side (in  $d(t)$ ), which is known to be uniquely solvable. Lemma 3.9 gives us an estimate on this small error and is the main ingredient in the uniqueness proof, and we use the estimate (A4) and a fixed-point argument to show existence.

**Proof of lemma 4.3.** It suffices to show that (21) has a unique solution.

**Uniqueness:** Assume that  $d, \tilde{d}$  are solutions to (21) that coincide up to time  $t_* \geq 0$ , but not at time  $t_n$  for some  $t_n \geq 0$  with  $t_n \downarrow t_*$  as  $n \rightarrow \infty$ . Define  $\delta(t) := |f^{-1}(d(t)) - f^{-1}(\tilde{d}(t))|$ . For  $\varepsilon > 0$  consider the function

$$h_\varepsilon(t) := \varepsilon(1 + t - t_*) + \frac{1}{c(0)} \int_{t_*}^t \left( -\delta(s) + \beta \int_{t_*}^s \delta(\tau) \, d\tau \right) ds,$$

where  $\beta$  is the constant from lemma 3.9.

*Claim:* The inequality  $|d(t) - \tilde{d}(t)| < h_\varepsilon(t)$  holds for all  $t \geq t_*$ .

Clearly, the claim holds true for  $t = t_*$ , and thus by continuity for  $t$  close to  $t_*$ . Assume the claim is false. Then there exists some minimal  $t_i > t_*$  such that  $|d(t_i) - \tilde{d}(t_i)| = h_\varepsilon(t_i)$ . W.l.o.g. assume that  $d(t_i) \geq \tilde{d}(t_i)$ . Since  $d(t) - \tilde{d}(t) < h_\varepsilon(t)$  for  $t_* \leq t < t_i$ , we get  $d'(t_i) - \tilde{d}'(t_i) \geq h'_\varepsilon(t_i)$  which implies

$$\frac{1}{c(0)} \Phi_+(b_d, 0, 0)_z(0, t_i) - \frac{1}{c(0)} \Phi_+(b_{\tilde{d}}, 0, 0)_z(0, t_i) \geq \varepsilon + \frac{1}{c(0)} \left( -\delta(t_i) + \beta \int_{t_*}^{t_i} \delta(\tau) \, d\tau \right)$$

and hence

$$\Phi_+(b_d - b_{\tilde{d}}, 0, 0)_z(0, t_i) + \delta(t_i) > \beta \int_{t_*}^{t_i} \delta(\tau) \, d\tau \geq 0. \tag{23}$$

On the other hand, setting  $b := b_d - b_{\tilde{d}}$  we have

$$|\Phi_+(b, 0, 0)_z(0, t_i) + b'(t_i)| \leq \beta \int_{t_*}^{t_i} |b'(\tau)| \, d\tau$$

due to lemma 3.9. Since  $b'(t_i) = f^{-1}(d(t_i)) - f^{-1}(\tilde{d}(t_i))$  and since  $f^{-1}$  is increasing, we see that  $b'(t_i) = \delta(t_i)$ . Combining these facts, we find

$$|\Phi_+(b, 0, 0)_z(0, t_i) + \delta(t_i)| \leq \beta \int_{t_*}^{t_i} \delta(\tau) \, d\tau$$

which contradicts (23). So the claim holds.

Letting  $\varepsilon$  go to 0, we obtain

$$|d(t) - \tilde{d}(t)| \leq \frac{1}{c(0)} \int_{t_*}^t \left( -\delta(s) + \beta \int_{t_*}^s \delta(\tau) \, d\tau \right) ds$$

for any  $t \geq t_*$ . Fubini implies that the term on the right-hand side is negative for  $t \in (t_*, t_* + \frac{1}{\beta})$ , a contradiction.

**Existence:** Let  $D, \mu > 0$ . Consider the set

$$K := \{d \in W^{1,\infty}([0, r]) : d(t_0) = f^{-1}(u_1(0)), |d(t)| \leq De^{\mu t}, |d'(t)| \leq D\mu e^{\mu t} \text{ for } t \in [0, r]\},$$

which is a convex and compact subset of  $C([0, r])$ , as well as the operator

$$T: K \rightarrow C([0, r]), \quad T(d)(t) = f^{-1}(u_1(0)) + \frac{1}{c(0)} \int_0^t \Phi_+(b_d, u_0, u_1)_z(0, \tau) \, d\tau.$$

We choose  $D := \max\{|f^{-1}(u_1(0))|, 1\}$ , so that  $K$  is nonempty as it contains the constant function  $d \equiv f^{-1}(u_1(0))$ . To see that  $T$  is continuous, let  $d_n \in K$  with  $d_n \rightarrow d$  in  $C([0, r])$  as  $n \rightarrow \infty$ . As  $f^{-1}$  is uniformly continuous on  $[-De^{\mu r}, De^{\mu r}]$ , we have  $f^{-1} \circ d_n \rightarrow f^{-1} \circ d$  in  $C([0, r])$ , from which it follows that

$$b_{d_n} = u_0(0) + \int_0^{(\cdot)} f^{-1}(d_n(\tau)) \, d\tau$$

converges to

$$b_d = u_0(0) + \int_0^{(\cdot)} f^{-1}(d(\tau)) \, d\tau.$$

in  $C^1([0, r])$ . Due to remark 3.8, the operator  $\Phi_+(\cdot, u_0, u_1): C^1([0, r]) \rightarrow C^1(\Delta_+)$  is continuous. Hence  $T(d_n) \rightarrow T(d)$  in  $C([0, r])$  as  $n \rightarrow \infty$ .

To check that  $T$  maps into  $K$ , we need to verify that for any  $d \in K$  one has

$$|T(d)'(t)| \leq D\mu e^{\mu t}. \tag{24}$$

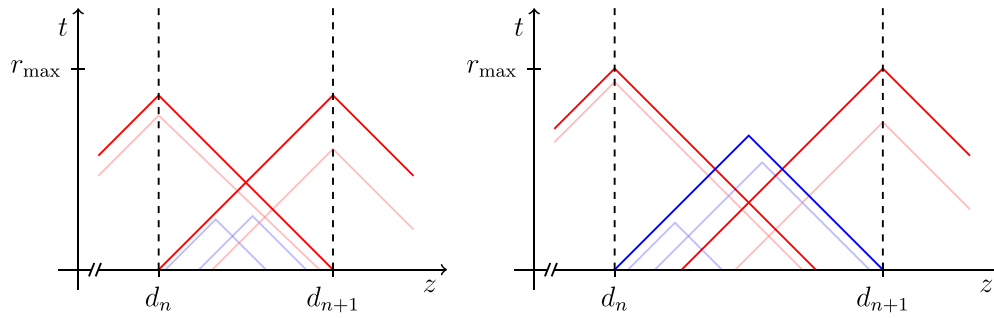
Notice that  $|d(t)| \leq De^{\mu t}$  follows from (24) by integration. By assumption (A4) on the growth on  $f$  we have  $|f^{-1}(y)| \leq \frac{|y|+A}{a}$ , and in particular  $|b'_d(t)| = |f^{-1}(d(t))| \leq \frac{De^{\mu t}+A}{a}$ . We use this inequality,  $|b_d(t)| \leq |u_0(0)| + t\|b'_d\|_\infty$  as well as remark 3.8 to estimate

$$\begin{aligned} |T(d)'(t)| &= \frac{1}{c(0)} |\Phi_+(b_d, u_0, u_1)_z(0, t)| \\ &\leq \frac{C}{c(0)} (\|b_d\|_{[0,t],C^1} + \|u_0\|_{C^1} + \|u_1\|_\infty) \\ &\leq \frac{C}{c(0)} ((1+t)\|b'_d\|_{[0,t],\infty} + 2\|u_0\|_{C^1} + \|u_1\|_\infty) \\ &\leq \frac{C}{c(0)} \left( (1+t) \frac{De^{\mu t} + A}{a} + 2\|u_0\|_{C^1} + \|u_1\|_\infty \right) \\ &\leq \frac{C}{c(0)} \left( (1+r) \frac{D+A}{a} + 2\|u_0\|_{C^1} + \|u_1\|_\infty \right) e^{\mu t}. \end{aligned}$$

Therefore  $T$  maps  $K$  into itself if we choose

$$\mu := \frac{C}{c(0)D} \left( (1+r) \frac{D+A}{a} + 2\|u_0\|_{C^1} + \|u_1\|_\infty \right).$$

Hence existence follows by applying Schauder’s fixed-point theorem. □



**Figure 1.** Sketch: Covering of  $[0, \infty) \times [0, \infty)$  by jump triangles (red) and plain triangles (blue), with  $r_{\max} := \left\| \frac{c_z}{c} \right\|_{\infty}^{-1}$  being the maximum height of triangles. Black dashed lines indicate jumps of  $c$ . Left:  $|d_{n+1} - d_n| < r_{\max}$  where covering has height  $\frac{1}{2} |d_{n+1} - d_n|$ , right:  $|d_{n+1} - d_n| > r_{\max}$  where covering has height  $\frac{1}{2} r_{\max}$ .

Using the existence and uniqueness results on plain, jump, and boundary triangles shown above, next we prove existence and uniqueness on the whole space  $[0, \infty) \times [0, \infty)$  by covering it with these specific triangles.

**Proof of theorem 1.1 with additional assumption (A4).** We show existence and uniqueness of the solution to (1) under the assumption (A4). Wellposedness will be discussed in section 6.

**Existence:** Denote by  $\mathcal{C}$  the set containing all jump, boundary and plain triangles where the heights  $r$  have to satisfy  $r \left\| \frac{c_z}{c} \right\|_{\infty} < 1$ . As we have just shown in the previous three lemmata, (10) admits a unique solution on each  $\Delta \in \mathcal{C}$ . Since  $\mathcal{C}$  is closed with respect to finite intersection, we obtain a solution  $u$  of (10) on  $\cup_{\Delta \in \mathcal{C}} \Delta$ . With

$$h := \frac{1}{2} \min \left\{ \left\| \frac{c_z}{c} \right\|_{\infty}^{-1}, |z_1 - z_2| : z_1, z_2 \in D(c) \cup \{0\}, z_1 \neq z_2 \right\}$$

we have  $[0, \infty) \times [0, h) \subseteq \cup_{\Delta \in \mathcal{C}} \Delta$ , see figure 1 for an illustration of this covering property. By restriction, we therefore obtain a solution  $u^{(1)}$  of (1) on  $[0, \infty) \times [0, \tilde{h}]$  for any  $0 < \tilde{h} < h$ . The same argument, used with initial data  $u_0^{(2)}(z) := u^{(1)}(z, \tilde{h})$  and  $u_1^{(2)}(z) := u_t^{(1)}(z, \tilde{h})$  instead of  $u_0, u_1$ , yields another solution  $u^{(2)}$  on  $[0, \infty) \times [0, \tilde{h}]$ . Repeating this, we construct solutions  $u^{(k)}$  on  $[0, \infty) \times [0, \tilde{h}]$  with  $u^{(k+1)}(z, 0) = u^{(k)}(z, \tilde{h})$  and  $u_t^{(k+1)}(z, 0) = u_t^{(k)}(z, \tilde{h})$  for  $k \in \mathbb{N}$ . Finally, we define the map  $u : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  by  $u(z, (k-1)\tilde{h} + \tau) = u^{(k)}(z, \tau)$  for  $\tau \in [0, \tilde{h}]$ , which solves (1).

**Uniqueness:** Assume that  $u, \tilde{u} : \Omega \rightarrow \mathbb{R}$  are two different solutions to (10), where  $\Omega = \{(z, t) \mid t \leq h(z)\}$  is an admissible domain. So there exists  $(z_0, t_0) \in \Omega$  with  $u(z_0, t_0) \neq \tilde{u}(z_0, t_0)$ . Consider the (possibly cut-off) triangle  $\Delta := \Delta(z_0, 0, t_0) \cap \{z \geq 0\}$  and define the set  $N := \{(z, t) \in \Delta \mid u(z, t) \neq \tilde{u}(z, t)\}$  and  $t_{\inf} := \inf P_t(N)$ , where  $P_t$  denotes the projection onto the second variable. Choose some sequence  $(z_n, t_n) \in N$  with  $t_n \rightarrow t_{\inf}$  and  $z_n \rightarrow z_{\infty} \in [0, \infty)$ .

For  $\varepsilon > 0$  consider the (possibly cut-off) triangle  $\Delta_{\varepsilon} := \Delta \cap \Delta(z_{\infty}, t_{\inf}, \varepsilon) \cap \{z \geq 0\}$  with base  $B_{\varepsilon}$ .

*Claim:*  $u(z, t_{\inf}) = \tilde{u}(z, t_{\inf})$  and  $u_t(z, t_{\inf}) = \tilde{u}_t(z, t_{\inf})$  hold for all  $z \in B_{\varepsilon}$ .

If  $t_{\inf} = 0$ , this holds because both  $u$  and  $\tilde{u}$  satisfy the same initial conditions. If  $t_{\inf} > 0$ , by assumption we have  $u(z, t) = \tilde{u}(z, t)$  for  $z \in B_{\varepsilon}$  and  $t < t_{\inf}$  as  $(z, t) \in \Delta$  and therefore also  $u_t(z, t) = \tilde{u}_t(z, t)$ , so that the claim is obtained by taking the limit  $t \rightarrow t_{\inf}$ .

If we choose  $\varepsilon$  small enough, then  $\Delta_\varepsilon$  is a jump (if  $z_\infty \in D(c)$ ), boundary (if  $z_\infty = 0$ ) or plain triangle (otherwise). By the previously established uniqueness results on these triangles,  $u$  and  $\tilde{u}$  must coincide on  $\Delta_\varepsilon$ . But since  $t_n \geq t_{\text{inf}}$  for all  $n$ , we have  $(z_n, t_n) \in \Delta_\varepsilon$  for  $n$  sufficiently large, so that  $u(z_n, t_n) = \tilde{u}(z_n, t_n)$ . This cannot be since  $(z_n, t_n) \in N$ .  $\square$

**Remark 4.4 (Modifications for the bounded domain version).** In order to capture the homogeneous Dirichlet boundary condition for the bounded domain version of the theorem, we also need to consider ‘Dirichlet’ triangles  $\Delta_-$  with centre  $z_0 = L$ . Problem (1) is well-defined on the domain  $\Delta_-$  assuming  $r \|\frac{c_z}{c}\|_\infty < 1$ . In fact the solution on ‘Dirichlet’ triangles is simply given by  $u = \Phi_-(0, u_0, u_1)$ . We can then proceed as in the above proof to show existence and uniqueness of solutions, i.e. theorem 1.8. Conservation of energy can be shown as in section 5.

### 5. Energy, momentum, and completion of theorem 1.1

Using  $V(x) = \frac{1}{c(x)^2}$ , the energy (7) can be written as

$$\begin{aligned} E(u, t) &:= \frac{1}{2} \int_0^\infty \left( V(x) u_t(x, t)^2 + u_x(x, t)^2 \right) dx + F(u_t(0, t)) \\ &= \frac{1}{2} \int_0^\infty \left( \frac{1}{c(z)^2} u_t(z, t)^2 + \left( \frac{u_z(z, t)}{c(z)} \right)^2 \right) \cdot c(z) dz + F(u_t(0, t)) \\ &= \frac{1}{2} \int_0^\infty \frac{1}{c(z)} \left( u_t(z, t)^2 + u_z(z, t)^2 \right) dz + F(u_t(0, t)) \end{aligned}$$

where  $F(y) = yf(y) - \int_0^y f(v) dv$ . In  $(z, t)$ -coordinates the momentum reads

$$M(u, t) = \int_0^\infty \frac{1}{c(z)} u_t(z, t) dz + f(u_t(0, t)).$$

We now show that both quantities are time-invariant.

**Proof of theorem 1.4.** Let  $\Omega \subseteq [0, \infty) \times [0, \infty)$  be a Lipschitz domain such that  $c$  is  $C^1$  on  $\Omega$ . Recall that  $(\partial_t \mp \partial_z)(u_t \pm u_z) + \frac{c_z}{c} u_z = 0$ . In the following, for a term  $a(\pm, \mp)$  which may have  $\pm$  or  $\mp$  signs, we write  $\sum^\pm a(\pm, \mp) := a(+, -) + a(-, +)$ .

**Part 1: Energy.** With  $\nu$  being the outer normal at  $\partial\Omega$  we calculate

$$\begin{aligned} 0 &= \sum^\pm \int_\Omega \left[ (\partial_t \mp \partial_z)(u_t \pm u_z) + \frac{c_z}{c} u_z \right] \cdot \frac{1}{c} (u_t \pm u_z) d(z, t) \\ &= \sum^\pm \int_{\partial\Omega} (\nu_2 \mp \nu_1) \frac{1}{c} (u_t \pm u_z)^2 d\sigma \\ &\quad + \sum^\pm \int_\Omega \left( \frac{c_z}{c^2} u_z (u_t \pm u_z) - \frac{1}{c} (u_t \pm u_z) \cdot (\partial_t \mp \partial_z)(u_t \pm u_z) \mp \frac{c_z}{c^2} (u_t \pm u_z)^2 \right) d(z, t). \end{aligned}$$

The sum  $\sum^\pm$  over the boundary integrals can be simplified to

$$\sum^\pm \int_{\partial\Omega} (\nu_2 \mp \nu_1) \frac{1}{c} (u_t \pm u_z)^2 d\sigma = \int_{\partial\Omega} \left( \frac{2}{c} \nu_2 (u_t^2 + u_z^2) - \frac{4}{c} \nu_1 u_t u_z \right) d\sigma.$$

The sum  $\sum^\pm$  of the integrands in the integral over  $\Omega$  vanishes as can be seen by the following calculation using once more the differential equation  $(\partial_t \mp \partial_z)(u_t \pm u_z) + \frac{c_z}{c} u_z = 0$  :

$$\begin{aligned} & \sum^\pm \left( \frac{c_z}{c^2} u_z (u_t \pm u_z) - \frac{1}{c} (u_t \pm u_z) \cdot (\partial_t \mp \partial_z)(u_t \pm u_z) \mp \frac{c_z}{c^2} (u_t \pm u_z)^2 \right) \\ &= \sum^\pm \left( \frac{c_z}{c^2} u_z (u_t \pm u_z) + \frac{1}{c} (u_t \pm u_z) \frac{c_z}{c} u_z \mp \frac{c_z}{c^2} (u_t \pm u_z)^2 \right) \\ &= \frac{c_z}{c^2} \sum^\pm \left( 2u_z (u_t \pm u_z) \mp (u_t \pm u_z)^2 \right) = 0. \end{aligned}$$

Hence

$$\int_{\partial\Omega} \left( \frac{2}{c} \nu_2 (u_t^2 + u_z^2) - \frac{4}{c} \nu_1 u_t u_z \right) d\sigma = 0. \tag{25}$$

Since  $D(c)$  and  $D(c_z)$  are discrete sets, we find an increasing sequence  $0 = a_1 < a_2 < a_3 < \dots$  with  $a_k \rightarrow \infty$  as  $k \rightarrow \infty$  such that  $D(c) \cup D(c_z) \subseteq \{a_k : k \in \mathbb{N}\}$ .

Now let  $t_1 < t_2 \in \mathbb{R}$  and  $K \in \mathbb{N}$ . We choose  $\Omega = [a_k, a_{k+1}] \times [t_1, t_2]$  and sum (25) from  $k = 1$  to  $K$ . As terms along common boundaries cancel, we obtain

$$0 = \int_{\partial([0, a_{K+1}] \times [t_1, t_2])} \left( \frac{2}{c} \nu_2 (u_t^2 + u_z^2) - \frac{4}{c} \nu_1 u_t u_z \right) d\sigma$$

or equivalently

$$\begin{aligned} & \frac{1}{2} \int_0^{a_{K+1}} \left( \frac{1}{c} u_t^2 + \frac{1}{c} u_z^2 \right) dz \Big|_{t=t_2} \\ &= \frac{1}{2} \int_0^{a_{K+1}} \left( \frac{1}{c} u_t^2 + \frac{1}{c} u_z^2 \right) dz \Big|_{t=t_1} - \int_{t_1}^{t_2} \frac{1}{c} u_t u_z dt \Big|_{z=a_{K+1}} + \int_{t_1}^{t_2} \frac{1}{c} u_t u_z dt \Big|_{z=0}. \end{aligned}$$

The estimates established in corollary 3.6 and the assumptions on the initial conditions  $u_0, u_1$  show that  $u_t(z, t)$  and  $u_z(z, t)$  converge to 0 as  $z \rightarrow \infty$  uniformly on  $[t_1, t_2]$ . In the limit  $K \rightarrow \infty$ , we thus obtain

$$\frac{1}{2} \int_0^\infty \left( \frac{1}{c} u_t^2 + \frac{1}{c} u_z^2 \right) dz \Big|_{t=t_2} = \frac{1}{2} \int_0^\infty \left( \frac{1}{c} u_t^2 + \frac{1}{c} u_z^2 \right) dz \Big|_{t=t_1} + \int_{t_1}^{t_2} \frac{1}{c} u_t u_z dt \Big|_{z=0}.$$

Switching back to  $(x, t)$ -coordinates, we infer

$$\begin{aligned} \int_{t_1}^{t_2} u_t u_x dt \Big|_{x=0} &= \int_{t_1}^{t_2} u_t(0, t) u_x(0, t) dt \\ &= \int_{t_1}^{t_2} u_t(0, t) f(u_t(0, t))_t dt = F(u_t(0, t_2)) - F(u_t(0, t_1)) \end{aligned}$$

where the last equality is due to lemma A.1. This shows the claimed energy conservation:

$$\frac{1}{2} \int_0^\infty (V(x) u_t^2 + u_x^2) dx + F(u_t(0, t)) \Big|_{t=t_2} = \frac{1}{2} \int_0^\infty (V(x) u_t^2 + u_x^2) dx + F(u_t(0, t)) \Big|_{t=t_1}.$$

**Part 2: Momentum.** We calculate

$$\begin{aligned}
 0 &= \sum^\pm \int_\Omega \frac{1}{c} \left[ (\partial_t \pm \partial_z)(u_t \mp u_z) + \frac{c_z}{c} u_z \right] d(z, t) \\
 &= \sum^\pm \int_{\partial\Omega} (\nu_2 \pm \nu_1) \frac{1}{c} (u_t \mp u_z) d\sigma \\
 &\quad + \sum^\pm \int_\Omega \left( \pm \frac{c_z}{c^2} (u_t \mp u_z) + \frac{c_z}{c^2} u_z \right) d(z, t) \\
 &= 2 \int_{\partial\Omega} \left( \nu_2 \frac{1}{c} u_t - \nu_1 \frac{1}{c} u_z \right) d\sigma. \tag{26}
 \end{aligned}$$

Again we choose  $\Omega = [a_k, a_{k+1}] \times [t_1, t_2]$ , and sum (26) from  $k = 1$  to  $K$ . As before all terms along common boundaries cancel, whence we obtain

$$\int_0^{a_{K+1}} \frac{1}{c} u_t dz \Big|_{t=t_2} = \int_0^{a_{K+1}} \frac{1}{c} u_t dz \Big|_{t=t_1} + \int_{t_1}^{t_2} \frac{1}{c} u_z dt \Big|_{z=a_{K+1}} - \int_{t_1}^{t_2} \frac{1}{c} u_z dt \Big|_{z=0}.$$

Since

$$\int_{t_1}^{t_2} \frac{1}{c} u_z dt \Big|_{z=0} = \int_{t_1}^{t_2} f(u_t(0, t))_t dt = f(u_t(0, t_2)) - f(u_t(0, t_1)),$$

in the limit  $K \rightarrow \infty$  we find the claimed momentum conservation:

$$\int_{t_1}^\infty \frac{1}{c^2} u_t dx + f(u_t(0, t)) \Big|_{t=t_2} = \int_0^\infty \frac{1}{c^2} u_t dx + f(u_t(0, t)) \Big|_{t=t_1}.$$

□

In section 4, we required an extra growth condition (A4) on  $f$  in order to prove a first version of theorem 1.1. We now discuss how to exploit the energy conservation to eliminate this extra growth assumption and prove theorem 1.1 in full generality.

**Lemma 5.1.** For  $t > 0$  the estimate

$$F(u_t(0, t)) \leq F(u_1(0)) + \frac{1}{2} \int_0^{\kappa^{-1}(t)} \left( V(x) u_1(x)^2 + u_{0,x}(x)^2 \right) dx$$

holds, where  $\kappa(x) = \int_0^x \frac{1}{c(s)} ds = \int_0^x \sqrt{V(s)} ds$ .

**Proof.** Fix  $t_1 > 0$ , let  $\varepsilon > 0$  and define modified initial data  $\tilde{u}_0, \tilde{u}_1 : [0, \infty) \rightarrow \mathbb{R}$  by setting

$$\tilde{u}'_0(z) = \begin{cases} u'_0(z), & z \leq t_1, \\ \frac{t_1 + \varepsilon - z}{\varepsilon} u'_0(t_1), & t_1 \leq z \leq t_1 + \varepsilon, \\ 0, & z \geq t_1 + \varepsilon, \end{cases} \quad \tilde{u}_1(z) = \begin{cases} u_1(z), & z \leq t_1, \\ \frac{t_1 + \varepsilon - z}{\varepsilon} u_1(t_1), & t_1 \leq z \leq t_1 + \varepsilon, \\ 0, & z \geq t_1 + \varepsilon, \end{cases}$$

and  $\tilde{u}_0(0) = u_0(0)$ . Denote the solution to (10) corresponding to these initial data by  $\tilde{u}$ . By uniqueness of the solution,  $u(z, t) = \tilde{u}(z, t)$  for  $|z| + |t| \leq t_1$ . In particular,  $\tilde{u}_t(0, t_1) = u_t(0, t_1)$ . This yields

$$\begin{aligned}
 &F(u_t(0, t_1)) \\
 &= F(\tilde{u}_t(0, t_1)) \leq E(\tilde{u}, t_1) = E(\tilde{u}, 0) \\
 &= F(\tilde{u}_t(0, 0)) + \frac{1}{2} \int_0^\infty (V(x)\tilde{u}_1(x)^2 + \tilde{u}'_0(x)^2) dx \\
 &= F(u_1(0)) + \frac{1}{2} \int_0^{\kappa^{-1}(t_1)} (V(x)u_1(x)^2 + u'_0(x)^2) dx \\
 &\quad + \frac{1}{2} \int_{\kappa^{-1}(t_1)}^{\kappa^{-1}(t_1+\varepsilon)} (V(x)\tilde{u}_1(x)^2 + \tilde{u}'_0(x)^2) dx.
 \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , the last term goes to 0. □

**Proof of theorem 1.1 without additional assumption (A4).** We show existence and uniqueness of the solution to (1). Wellposedness will be discussed in section 6. Fix  $T > 0$  and let

$$C := F(u_1(0)) + \frac{1}{2} \int_0^{\kappa^{-1}(T)} (V(x)u_1(x)^2 + u_{0,x}(x)^2) dx.$$

Recall from remark 1.5 that  $F(y) \rightarrow \infty$  as  $y \rightarrow \pm\infty$ . Therefore the set  $\{y: F(y) \leq C\}$  is contained in the interval  $[-K, K]$  for some  $K > 0$ . Now consider the cut-off version of  $f$  given by

$$f_K(y) = \begin{cases} y - K + f(K), & y \geq K, \\ f(y), & -K \leq y \leq K, \\ y + K + f(-K), & y \leq -K, \end{cases}$$

which satisfies the growth condition (A4). As we have shown in section 4, there exists a unique solution  $u_K$  of (1) with  $f$  replaced by  $f_K$ . Then, using  $F_K(y) = yf_K(y) - \int_0^y f_K(s) ds$ , lemma 5.1 gives  $F_K(u_{K,t}(0, t)) \leq C$  for  $t \leq T$ , so that  $u_{K,t}(0, t)$  takes values in  $[-K, K]$  where the functions  $f, F$  and  $f_k, F_k$  coincide. Hence  $u_K$  is the unique solution of the original problem (1) up to time  $T$ . □

Next, we verify that  $C^1$ -solutions to (1) are indeed weak solutions in the sense of definition 1.6.

**Proposition 5.2.** *A  $C^1$ -solution to (1) is also a weak solution to (1).*

**Proof.** Let  $u$  be a  $C^1$ -solution to (1). We have to show that

$$\begin{aligned}
 0 &= \int_0^\infty \int_0^\infty (V(x)u_t\varphi_t - u_x\varphi_x) dx dt + \int_0^\infty f(u_t(0, t))\varphi_t(0, t) dt \\
 &\quad + \int_0^\infty V(x)u_1(x)\varphi(x, 0) dx + f(u_1(0))\varphi(0, 0)
 \end{aligned}$$

holds for all  $\varphi \in C_c^\infty([0, \infty) \times [0, \infty))$ .

Let  $\Omega \subseteq [0, \infty) \times [0, \infty)$  be a Lipschitz domain such that  $c$  is  $C^1$  on  $\Omega$ . Denoting the outer normal at  $\partial\Omega$  by  $\nu$ , we obtain



$$\begin{aligned}
 0 &= \int_{\Omega} \left[ (\partial_t - \partial_z)(u_t + u_z) + \frac{c_z}{c} u_z \right] \cdot \frac{1}{c} \varphi \, d(z, t) \\
 &= \int_{\partial\Omega} \frac{1}{c} (u_t + u_z) \varphi \cdot (\nu_2 - \nu_1) \, d\sigma + \int_{\Omega} \left( \frac{c_z}{c^2} u_z \varphi - (u_t + u_z) (\partial_t - \partial_z) \left[ \frac{1}{c} \varphi \right] \right) d(z, t) \\
 &= \int_{\partial\Omega} \left( \frac{1}{c} u_t \varphi \nu_2 - \frac{1}{c} u_z \varphi \nu_1 \right) d\sigma + \int_{\Omega} \left( \frac{1}{c} u_z \varphi_z - \frac{1}{c} u_t \varphi_t \right) d(z, t) \\
 &\quad + \int_{\partial\Omega} \left( \frac{1}{c} u_z \varphi \nu_2 - \frac{1}{c} u_t \varphi \nu_1 \right) d\sigma + \int_{\Omega} \left( u_t \partial_z \left[ \frac{1}{c} \varphi \right] - u_z \partial_t \left[ \frac{1}{c} \varphi \right] \right) d(z, t).
 \end{aligned}$$

We next show that the sum of the last two integrals equals zero. First, we calculate

$$\begin{aligned}
 &\int_{\partial\Omega} \left( \frac{1}{c} u_z \varphi \nu_2 - \frac{1}{c} u_t \varphi \nu_1 \right) d\sigma + \int_{\Omega} \left( u_t \partial_z \left[ \frac{1}{c} \varphi \right] - u_z \partial_t \left[ \frac{1}{c} \varphi \right] \right) d(z, t) \\
 &= \int_{\partial\Omega} \left( \frac{1}{c} u_z \varphi \nu_2 - \frac{1}{c} u_t \varphi \nu_1 + u \partial_z \left[ \frac{1}{c} \varphi \right] \nu_2 - u \partial_t \left[ \frac{1}{c} \varphi \right] \nu_1 \right) d\sigma \\
 &= \int_{\partial\Omega} (\nu_2 \partial_z - \nu_1 \partial_t) \left[ \frac{1}{c} u \varphi \right] d\sigma.
 \end{aligned}$$

Let  $\gamma: [0, l] \rightarrow \mathbb{R}^2$  be a positively oriented parametrization of  $\partial\Omega$  by arc length. As  $\nu$  is the outer normal at  $\partial\Omega$ , the identity  $\gamma' = (\nu_2, -\nu_1)^\top$  holds. Hence,

$$\begin{aligned}
 \int_{\partial\Omega} (\nu_2 \partial_z - \nu_1 \partial_t) \left[ \frac{1}{c} u \varphi \right] d\sigma &= \int_{\partial\Omega} \begin{pmatrix} \nu_2 \\ -\nu_1 \end{pmatrix} \cdot \nabla \left[ \frac{1}{c} u \varphi \right] d\sigma \\
 &= \int_0^l \gamma'(s) \cdot \nabla \left[ \frac{1}{c} u \varphi \right] (\gamma(s)) \, ds = 0
 \end{aligned}$$

as  $\gamma$  is closed. Thus we have shown

$$0 = \int_{\partial\Omega} \left( \frac{1}{c} u_t \varphi \nu_2 - \frac{1}{c} u_z \varphi \nu_1 \right) d\sigma + \int_{\Omega} \left( \frac{1}{c} u_z \varphi_z - \frac{1}{c} u_t \varphi_t \right) d(z, t). \tag{27}$$

As in the proof of theorem 1.4 we choose an increasing sequence  $0 = a_1 < a_2 < a_3 < \dots$  with  $a_k \rightarrow \infty$  as  $k \rightarrow \infty$  such that  $D(c) \cup D(c_z) \subseteq \{a_k : k \in \mathbb{N}\}$ . We take  $\Omega = [a_k, a_{k+1}] \times [n, n + 1]$  in (27) and sum over  $k \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ . Using that boundary terms along common boundaries cancel out, the fact that  $\varphi$  has compact support, and (1), we obtain

$$\begin{aligned}
 0 &= \int_{\partial[0, \infty)^2} \left( \frac{1}{c} u_t \varphi \nu_2 - \frac{1}{c} u_z \varphi \nu_1 \right) d\sigma + \int_{[0, \infty)^2} \left( \frac{1}{c} u_z \varphi_z - \frac{1}{c} u_t \varphi_t \right) d(z, t) \\
 &= - \int_0^\infty \left[ \frac{1}{c} u_t \varphi \right] (z, 0) \, dz + \int_0^\infty \left[ \frac{1}{c} u_z \varphi \right] (0, t) \, dt + \int_0^\infty \int_0^\infty \left( \frac{1}{c} u_z \varphi_z - \frac{1}{c} u_t \varphi_t \right) \, dz \, dt \\
 &= - \int_0^\infty V(x) u_t(x, 0) \varphi(x, 0) \, dx + \int_0^\infty u_x(0, t) \varphi(0, t) \, dt + \int_0^\infty \int_0^\infty (u_x \varphi_x - V(x) u_t \varphi_t) \, dx \, dt \\
 &= - \int_0^\infty V(x) u_1(x) \varphi(x, 0) \, dx + \int_0^\infty (f(u_t(0, t)))_t \varphi(0, t) \, dt \\
 &\quad + \int_0^\infty \int_0^\infty (u_x \varphi_x - V(x) u_t \varphi_t) \, dx \, dt
 \end{aligned}$$

$$\begin{aligned}
 &= - \int_0^\infty V(x)u_1(x)\varphi(x,0) \, dx - \int_0^\infty f(u_t(0,t))\varphi_t(0,t) \, dt - f(u_1(0))\varphi(0,0) \\
 &\quad + \int_0^\infty \int_0^\infty (u_x\varphi_x - V(x)u_t\varphi_t) \, dx \, dt
 \end{aligned}$$

which finishes the proof. □

### 6. Wellposedness

The section completes the proof of the wellposedness claim stated in theorem 1.1. To be precise, (1) is wellposed in the following sense. The spaces  $C^1_{(x,t)}([0, \infty) \times [0, T])$ ,  $C^1_x([0, \infty))$ , and  $C([0, \infty))$  are endowed with uniform convergence on compact sets.

**Proposition 6.1.** *Assume that  $u_0^{(n)}, u_1^{(n)}$  are initial data with  $u_0^{(n)} \rightarrow u_0$  in  $C^1_x([0, \infty))$  and  $u_1^{(n)} \rightarrow u_1$  in  $C([0, \infty))$ , and denote by  $u^{(n)}$  and  $u$  the solutions of (10) corresponding to these initial data. Then for any  $T > 0$ , we have  $u^{(n)} \rightarrow u$  in  $C^1_{(x,t)}([0, \infty) \times [0, T])$ .*

**Sketch of proof.** We proceed similar to the proof of theorem 1.1. Choose some

$$0 < \bar{r} < \min \left\{ \left(5 - \sqrt{17}\right) \left\| \frac{c_z}{c} \right\|_\infty^{-1}, |z_1 - z_2| : z_1, z_2 \in D(c) \cup \{0\}, z_1 \neq z_2 \right\}$$

and let  $\beta$  be as in lemma 3.9 with  $r = \bar{r}$  and  $\lambda = \frac{c_z}{c}$ . The choice of  $\bar{r}$  implies  $\beta\bar{r} < \frac{4(5-\sqrt{17})}{(-3+\sqrt{17})(-1+\sqrt{17})} = 1$  as well as  $q := \bar{r} \left\| \frac{c_z}{c} \right\|_\infty < 1$ .

Denote by  $\mathcal{C}$  the set containing all triangles  $\Delta$  that are of jump-type or plain-type and such that their base-radii  $r$  are at most  $\bar{r}$ . Then by lemma 4.1, 4.2, there exists a constant  $C > 0$  such that

$$\left\| u^{(n)} - u \right\|_{C^1_{(x,t)}(\Delta)} \leq C \max \left\{ \left\| u_0^{(n)} - u_0 \right\|_{C^1_x([0, \infty))}, \left\| u_1^{(n)} - u_1 \right\|_{C([0, \infty))} \right\}$$

holds for each  $\Delta \in \mathcal{C}$ .

We also consider a single boundary-type triangle  $\Delta_+$  with centre  $z_0 = 0$  and height  $\bar{r}$ . Writing  $b(t) := u(0,t)$ ,  $b^{(n)}(t) := u^{(n)}(0,t)$ ,  $d(t) := f(u_t(0,t))$  as well as  $d^{(n)}(t) := f(u_t^{(n)}(0,t))$ , as in the proof of lemma 4.3 we obtain

$$d'(t) = \frac{1}{c(0)} \Phi_+(b, u_0, u_1)_z(0,t), \quad \left( d^{(n)} \right)'(t) = \frac{1}{c(0)} \Phi_+(b^{(n)}, u_0^{(n)}, u_1^{(n)})_z(0,t).$$

Setting  $\hat{b}(t) := u_0^{(n)}(0) - u_0(0) + t(u_1^{(n)}(0) - u_1(0))$ , we find

$$\begin{aligned}
 &c(0) \left( \left( d^{(n)} \right)'(t) - d'(t) \right) \\
 &= \Phi_+ \left( b^{(n)} - b, u_0^{(n)} - u_0, u_1^{(n)} - u_1 \right)_z(0,t) \\
 &= \Phi_+ \left( \hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1 \right)_z(0,t) + \Phi_+ \left( b^{(n)} - b - \hat{b}, 0, 0 \right)_z(0,t) \\
 &= \Phi_+ \left( \hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1 \right)_z(0,t) \\
 &\quad - \left[ f^{-1} \left( d^{(n)}(t) \right) - f^{-1} \left( d(t) \right) - \left( u_1^{(n)}(0) - u_1(0) \right) \right] + \rho(n,t)
 \end{aligned}$$

where lemma 3.9 gives

$$|\rho(n, t)| \leq \beta \int_0^t \left| f^{-1}(d^{(n)}(\tau)) - f^{-1}(d(\tau)) - u_1^{(n)}(0) + u_1(0) \right| d\tau.$$

Multiplying with sign  $(d^{(n)}(t) - d(t))$  and integrating, we obtain

$$\begin{aligned} & c(0) \left| d^{(n)}(t) - d(t) \right| \\ & \leq c(0) \left| d^{(n)}(0) - d(0) \right| \\ & \quad + \int_0^t \left( \left| \Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0, s) \right| - \left| f^{-1}(d^{(n)}(s)) - f^{-1}(d(s)) \right| \right. \\ & \quad \left. + \left| u_1^{(n)}(0) - u_1(0) \right| \right) ds \\ & \quad + \beta \int_0^t \int_0^s \left| f^{-1}(d^{(n)}(\tau)) - f^{-1}(d(\tau)) - u_1^{(n)}(0) + u_1(0) \right| d\tau ds \\ & \leq \int_0^t \left( \left| \Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0, s) \right| - \left| f^{-1}(d^{(n)}(s)) - f^{-1}(d(s)) \right| \right. \\ & \quad \left. + \left| u_1^{(n)}(0) - u_1(0) \right| \right) ds \\ & \quad + \beta \int_0^{\bar{r}} \int_0^t \left( \left| f^{-1}(d^{(n)}(\tau)) - f^{-1}(d(\tau)) \right| + \left| u_1^{(n)}(0) - u_1(0) \right| \right) d\tau ds \\ & = \int_0^t \left| \Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0, s) \right| ds + (1 + \bar{r}\beta)t \left| u_1^{(n)}(0) - u_1(0) \right| \\ & \quad - (1 - \bar{r}\beta) \int_0^t \left| f^{-1}(d^{(n)}(s)) - f^{-1}(d(s)) \right| ds \\ & \leq \int_0^t \left| \Phi_+(\hat{b}, u_0^{(n)} - u_0, u_1^{(n)} - u_1)_z(0, s) \right| ds + (1 + \bar{r}\beta)t \left| u_1^{(n)}(0) - u_1(0) \right| \\ & \leq \tilde{C} \left( \bar{r}, \left\| \frac{c_z}{c} \right\|_\infty \right) \max \left\{ \left\| u_0^{(n)} - u_0 \right\|_{C^1([0, \infty))}, \left\| u_1^{(n)} - u_1 \right\|_{C([0, \infty))} \right\}. \end{aligned}$$

This shows the uniform convergence of  $d^{(n)}$  to  $d$  on  $[0, \bar{r}]$  as  $n \rightarrow \infty$ . Since

$$b(t) = u_0(0) + \int_0^t f^{-1}(d(\tau)) d\tau, \quad b^{(n)}(t) = u_0(0) + \int_0^t f^{-1}(d^{(n)}(\tau)) d\tau$$

for  $t \in [0, \bar{r}]$ , it follows that  $b^{(n)} \rightarrow b$  in  $C^1([0, \bar{r}])$  as  $n \rightarrow \infty$ , and therefore we see that  $u^{(n)} = \Phi_+(b^{(n)}, u_0^{(n)}, u_1^{(n)}) \rightarrow \Phi_+(b, u_0, u_1) = u$  in  $C^1(\Delta_+)$ .

Combined, we find that  $u^{(n)} \rightarrow u$  in  $C^1_{(x,t)}(\mathcal{D})$  where  $\mathcal{D} := \cup_{\Delta \in \mathcal{C}} \Delta$ . Note that  $[0, \infty) \times [0, \frac{\bar{r}}{2}] \subseteq \mathcal{D}$ , so in particular  $u^{(n)} \rightarrow u$  in  $C^1_{(x,t)}([0, \infty) \times [0, \frac{\bar{r}}{2}])$ . Applying this result repeatedly  $k$  times, we see that  $u^{(n)} \rightarrow u$  in  $C^1_{(x,t)}([0, \infty) \times [0, k\frac{\bar{r}}{2}])$  where  $k \in \mathbb{N}$  is chosen such that  $k\frac{\bar{r}}{2} \geq T$ .  $\square$

### 7. Breather solutions and their regularity

One can also consider (1) in the context of breather solutions, where a *breather* is a time-periodic and spatially localized function. With time-period denoted by  $T$ , the time domain becomes the torus  $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$  and after dropping the initial data, (1) reads

$$\begin{cases} V(x)u_{tt}(x, t) - u_{xx}(x, t) = 0, & x \in [0, \infty), t \in \mathbb{T}, \\ u_x(0, t) = (f(u_t(0, t)))_t, & t \in \mathbb{T}. \end{cases} \tag{28}$$

In [6] the case of a cubic boundary term  $f(y) = \frac{1}{2}\gamma y^3$  ( $\gamma \in \mathbb{R} \setminus \{0\}$ ) and a  $2\pi$ -periodic step potential  $V: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$V(x) = \begin{cases} a, & |x| < \pi\theta, \\ b, & \theta\pi < |x| < \pi, \end{cases} \tag{A5}$$

where  $b > a > 0$  and  $\theta \in (0, 1)$  was discussed. It was shown that if  $V$  satisfies

$$4\sqrt{a}\theta\omega \in 2\mathbb{N}_0 + 1 \quad \text{and} \quad 4\sqrt{b}(1 - \theta)\omega \in 2\mathbb{N}_0 + 1, \tag{A6}$$

where  $\omega := \frac{2\pi}{T}$  is the frequency, then there exist infinitely many weak breather solutions  $u$  of (28) with time-period  $T$ . A weak solution of (28) is defined next.

**Definition 7.1.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be an increasing, odd homeomorphism. A *weak solution* of (28) is a function  $u \in H^1([0, \infty) \times \mathbb{T})$  with  $u(0, \cdot) \in W^{1,1}(\mathbb{T})$  and  $f(u_t(0, \cdot)) \in L^1(\mathbb{T})$  which satisfies

$$\int_{[0, \infty) \times \mathbb{T}} -V(x)u_t\varphi_t + u_x\varphi_x \, d(x, t) - \int_{\mathbb{T}} f(u_t(0, t))\varphi_t(0, t) \, dt = 0$$

for all test functions  $\varphi \in C_c^\infty([0, \infty) \times \mathbb{T})$ .

**Remark 7.2.** We require that the trace  $u(0, \cdot)$  of  $u$  at  $x = 0$  has an integrable weak first-order time derivative in order to give a pointwise meaning to  $u_t(0, t)$  and, in particular, to define  $f(u_t(0, t))$  pointwise almost everywhere.

In the setting of [6] where  $f(y) = \frac{1}{2}\gamma y^3$ , one requires  $u_t(0, t) \in L^3(\mathbb{T})$  and

$$2 \int_{[0, \infty) \times \mathbb{T}} -V(x)u_t\varphi_t + u_x\varphi_x \, d(x, t) - \gamma \int_{\mathbb{T}} u_t(0, t)^3\varphi_t(0, t) \, dt = 0.$$

In [6, theorem 4] it was furthermore shown that weak solutions to (28) constructed in [6] lie in  $H^{\frac{5}{4}-\varepsilon}(\mathbb{T}, L^2(0, \infty)) \cap H^{\frac{1}{4}-\varepsilon}(\mathbb{T}, H^1(0, \infty))$  for  $\varepsilon > 0$ . Here, the Bochner spaces  $H^s(\mathbb{T}, X)$  are defined by

$$\|u\|_{H^s(\mathbb{T}, X)}^2 := \sum_{k \in \mathbb{Z}} (1 + k^2)^s \|\hat{u}_k\|_X^2 < \infty.$$

In this section, we show the following improved regularity result for breather solutions of (28):

**Theorem 7.3.** Assume (A3), (A5), (A6) that  $f^{-1}$  is  $r$ -Hölder continuous with  $r \in (0, 1)$  and that  $u$  is a weak solution to (28). Then  $u$  is  $\frac{T}{2}$ -antiperiodic, lies in  $C^{1,r}([0, \infty) \times \mathbb{T})$  and is a  $C^1$ -solution to (1) with its own initial data, i.e.  $u_0(x) = u(x, 0)$  and  $u_1(x) = u_t(x, 0)$ . In addition, there exists  $C > 0$  such that  $|u(x, t)| \leq Ce^{-\rho x}$  where  $\rho := \frac{\log(b) - \log(a)}{4\pi}$ .

Note that in the setting of [6], the assumptions of theorem 7.3 are satisfied with  $r = \frac{1}{3}$ . In the following, we are going to prove theorem 7.3 and we will always assume the assumptions of theorem 7.3. We begin with a discussion of the linear operator  $V(x)\partial_t^2 - \partial_x^2$  appearing in (28).

7.1. Fourier decomposition of  $V(x)\partial_t^2 - \partial_x^2$

We denote by  $e_k(t) := \frac{1}{\sqrt{T}}e^{ik\omega t}$  the orthonormal Fourier base of  $L^2(\mathbb{T})$  and decompose  $u$  in its Fourier series with respect to  $t$ :

$$u(x, t) = \sum_{k \in \mathbb{Z}} \hat{u}_k(x) e_k(t) =: \mathcal{F}^{-1}(\hat{u})$$

with

$$\hat{u}_k(x) := \mathcal{F}_k(u) := \int_{\mathbb{T}} u(x, t) \overline{e_k(t)} dt.$$

Writing  $L := V(x)\partial_t^2 - \partial_x^2$  and  $L_k := -\partial_x^2 - k^2\omega^2V(x)$ , we see that any solution  $u$  of (28) satisfies

$$0 = Lu$$

and therefore also

$$0 = \mathcal{F}_k Lu = L_k \mathcal{F}_k u = L_k \hat{u}_k \tag{29}$$

for all  $k \in \mathbb{Z}$ . Since

$$\|u\|_{L^2([0, \infty) \times \mathbb{T})}^2 + \|u_x\|_{L^2([0, \infty) \times \mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} \|\hat{u}_k\|_{L^2(0, \infty)}^2 + \|(\hat{u}_k)_x\|_{L^2(0, \infty)}^2,$$

each  $\hat{u}_k$  is an  $H^1((0, \infty), \mathbb{C})$ -solution of (29). As  $V$  (and therefore also  $L_k$ ) is given explicitly, we can characterize the space of solutions of (29) as follows.

**Proposition 7.4.** *If  $k \in \mathbb{Z}$  is even, then the only solution  $\hat{u}_k \in H^1((0, \infty), \mathbb{C})$  to (29) is  $\hat{u}_k = 0$ . If  $k$  is odd, there exists  $\phi_k \in H^2((0, \infty), \mathbb{R})$  such that a function  $\hat{u}_k \in H^1((0, \infty), \mathbb{C})$  solves (29) if and only if  $\hat{u}_k = \lambda \phi_k$  for some  $\lambda \in \mathbb{C}$ . Furthermore,  $\phi_k$  satisfies*

$$\phi_k(0) = 1, \quad \phi_k'(0) = Ck(-1)^{(k-1)/2}, \quad \phi_k(x + 4\pi) = \frac{a}{b} \phi_k(x)$$

for  $x > 0$ , where  $C = C(T, a) \in \mathbb{R}$  is a constant independent of  $k$ . The function  $\phi_k$  is called fundamental Bloch mode of (29).

A proof of proposition 7.4 for  $k$  odd can be found in [6, appendix A2]. The nonexistence result for even  $k$  can be obtained using similar arguments: For  $k \neq 0$  the monodromy matrix for  $L_k$  is the identity matrix so that (29) only has spatially periodic solutions. For  $k = 0$ , the solutions of (29) are affine.

Next we establish a bootstrapping argument that will be used to obtain the  $C^{1,r}$  regularity in theorem 7.3.

7.2. Bootstrapping argument

Assume that  $u$  is a weak solution to (28) in the sense of definition 7.1. By proposition 7.4, all even Fourier modes of  $u$  vanish and there exists a complex sequence  $\hat{\alpha}_k$  such that

$$u(x, t) = \sum_{k \in \mathbb{Z}_{\text{odd}}} \hat{\alpha}_k \phi_k(x) e_k(t) \tag{30}$$

where  $\mathbb{Z}_{\text{odd}} := 2\mathbb{Z} + 1$ . In particular,  $u$  is  $\frac{T}{2}$ -antiperiodic. Since  $\phi_k(0) = 1$ , we have  $\alpha(t) := \sum_{k \in \mathbb{Z}_{\text{odd}}} \hat{\alpha}_k e_k(t) = u(0, t)$ . In addition to  $\alpha$ , we also consider the quantity  $\beta(t) := f(u_t(0, t))$ . Thus we have

$$\alpha = \partial_t^{-1} f^{-1}(\beta). \tag{31}$$

Let us explain the idea of the proof on a formal level. First, by (30) we can express  $u$ , and terms derived from  $u$ , as a function of  $\alpha$ . Setting  $\Psi(\alpha)(t) := u_x(0, t)$ , the boundary condition of (28) can be written as  $\Psi(\alpha) = f(u_t(0, \cdot))_t = \beta'$  or as

$$\beta = \partial_t^{-1} \Psi(\alpha). \tag{32}$$

As shown below, the maps  $\beta \mapsto \partial_t^{-1} f^{-1}(\beta) = \alpha$  and  $\alpha \mapsto \partial_t^{-1} \Psi(\alpha) = \beta$  are regularity improving. Bootstrapping this regularity improvement, we show in lemma 7.7 that  $\alpha$  (which by definition lies in  $W^{1,1}(\mathbb{T})$ ) and  $\beta$  (which by definition lies in  $L^1(\mathbb{T})$ ), both are in  $C^{1,r}(\mathbb{T})$ . From there, using  $u(0, \cdot) = \alpha \in C^{1,r}(\mathbb{T})$ ,  $u_x(0, \cdot) = \beta' \in C^r(\mathbb{T})$  and that the wave equation is regularity preserving, we show  $u \in C^{1,r}([0, \infty) \times \mathbb{T})$ .

Note that  $u$  satisfies the boundary condition of (28) in a weak sense, so that  $u_x(0, \cdot) = \beta'$  is not clear *a priori* and will be shown as part of the proof of theorem 7.3. Moreover, since  $u$  lies in  $H^1([0, \infty) \times \mathbb{T})$ , its derivative  $u_x$  does not admit traces in general so that  $\Psi(\alpha)$  need not be defined. This is not an issue, because next we establish a (rigorous) identity that we replace (32) with.

Using definition 7.1 with  $\varphi(x, t) = \psi(x) \overline{e_k(t)}$  for  $k \in \mathbb{Z}_{\text{odd}}$ , where  $\psi \in C_c^\infty([0, \infty))$  and  $\psi(0) = 1$ , we obtain

$$\begin{aligned} 0 &= \int_{[0, \infty) \times \mathbb{T}} \left[ -V(x) u_t \psi(x) \overline{e_k'(t)} + u_x \psi'(x) \overline{e_k(t)} \right] dx dt - \int_{\mathbb{T}} f(u_t(0, t)) \psi(0) \overline{e_k'(t)} dt \\ &= \int_0^\infty \left[ -V(x) i k \omega \hat{\alpha}_k \phi_k(x) \overline{i k \omega} \psi(x) + \hat{\alpha}_k \phi_k'(x) \psi'(x) \right] dx + i k \omega \hat{\beta}_k \\ &= \int_0^\infty \left[ -\hat{\alpha}_k k^2 \omega^2 V(x) \phi_k(x) \psi(x) - \hat{\alpha}_k \phi_k''(x) \psi(x) \right] dx - \hat{\alpha}_k \phi_k'(0) \psi(0) + i k \omega \hat{\beta}_k \\ &= -\phi_k'(0) \hat{\alpha}_k + i k \omega \hat{\beta}_k, \end{aligned}$$

or

$$\hat{\beta}_k = \frac{\phi_k'(0)}{i k \omega} \hat{\alpha}_k. \tag{33}$$

Since  $u(0, \cdot)$  is  $\frac{T}{2}$ -antiperiodic, the even Fourier coefficients of  $\alpha = u(0, \cdot)$  vanish, and since  $f$  is odd the even Fourier coefficients of  $\beta = f(u_t(0, \cdot))$  also vanish. Hence from (33) we obtain

$$\beta = \mathcal{F}^{-1} \left( \left( \frac{\phi_k'(0)}{i k \omega} \hat{\alpha}_k \right)_{k \in \mathbb{Z}_{\text{odd}}} \right), \tag{34}$$

assuming the sequence on the right-hand side lies in  $\mathcal{F}(L^1(\mathbb{T}))$ . In the following, we use (34) instead of the formal equation (32).

We next investigate the properties of the maps defined by (31) and (34), which we consider as maps between the fractional Sobolev–Slobodeckij spaces  $W^{s,p}(\mathbb{T})$  or between Hölder spaces  $C^s(\mathbb{T})$ . The definition and all employed properties of the spaces  $W^{s,p}(\mathbb{T})$  can be found in appendix B. In the following we use the suffix ‘anti’ to denote that the space consists of functions which are  $\frac{T}{2}$ -antiperiodic in time.

**Lemma 7.5.** *The map*

$$\beta \mapsto \partial_t^{-1} f^{-1}(\beta)$$

is well-defined from  $W_{\text{anti}}^{s,p}(\mathbb{T})$  to  $W_{\text{anti}}^{1+rs,p/r}(\mathbb{T})$  for any  $s \in [0, 1)$  and  $p \in [1, \infty)$  as well as from  $C_{\text{anti}}^{0,s}(\mathbb{T})$  to  $C_{\text{anti}}^{1,rs}(\mathbb{T})$  for any  $s \in [0, 1]$ .

**Proof.** If  $\beta \in C_{\text{anti}}^{0,s}(\mathbb{T})$ , then  $f^{-1}(\beta) \in C_{\text{anti}}^{0,rs}(\mathbb{T})$  since  $f^{-1}$  is  $r$ -Hölder regular, and thus  $\partial_t^{-1} f^{-1}(\beta) \in C_{\text{anti}}^{1,rs}(\mathbb{T})$ . If  $\beta \in W_{\text{anti}}^{s,p}(\mathbb{T})$ , then  $f^{-1}(\beta) \in W_{\text{anti}}^{rs,p/r}(\mathbb{T})$  by lemma B.2 and thus  $\partial_t^{-1} f^{-1}(\beta) \in W_{\text{anti}}^{1+rs,p/r}(\mathbb{T})$ . □

**Lemma 7.6.** *The map*

$$\alpha \mapsto \mathcal{F}^{-1} \left( \left( \frac{\phi'_k(0)}{ik\omega} \hat{\alpha}_k \right)_{k \in \mathbb{Z}_{\text{odd}}} \right)$$

is well-defined from  $W_{\text{anti}}^{s,p}(\mathbb{T})$  to  $W_{\text{anti}}^{s,p}(\mathbb{T})$  for all  $s \in (0, \infty)$  and  $p \in [1, \infty)$  as well as from  $C_{\text{anti}}^{k,s}(\mathbb{T})$  to  $C_{\text{anti}}^{k,s}(\mathbb{T})$  for all  $k \in \mathbb{N}_0$  and  $s \in [0, 1]$ .

**Proof.** We begin by taking a closer look at the Fourier multiplier  $\hat{M}_k := \frac{\phi'_k(0)}{ik\omega}$  which is defined for  $k \in \mathbb{Z}_{\text{odd}}$  and extended by 0 to the whole of  $\mathbb{Z}$ . By proposition 7.4 we have  $\phi'_k(0) = Ck(-1)^{(k-1)/2}$  for a real constant  $C$  depending only on  $T$  and  $a$ . From this we obtain

$$\hat{M}_k = -\frac{iC}{\omega} \text{Im}(i^k)$$

for all  $k \in \mathbb{Z}$ . Now,  $\hat{M}_k$  is the Fourier series of

$$M(t) := \frac{\sqrt{TC}}{2\omega} (\delta_{T/4}(t) - \delta_{-T/4}(t))$$

where  $\delta_x$  denotes the Dirac measure at  $x$ . In particular,  $M$  is a finite measure. For  $\alpha \in L^1_{\text{anti}}(\mathbb{T})$  we calculate

$$\begin{aligned} \mathcal{F}_k \left( \frac{1}{\sqrt{T}} M * \alpha \right) &= \frac{1}{\sqrt{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \alpha(t-s) dM(s) \overline{e_k(t)} dt \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \alpha(t-s) \overline{e_k(t-s)} dt \overline{e_k(s)} dM(s) = \hat{M}_k \hat{\alpha}_k. \end{aligned}$$

so that  $\mathcal{F}^{-1} \left( k \mapsto \hat{M}_k \hat{\alpha}_k \right)$  exists and equals  $\frac{1}{\sqrt{T}} M * \alpha$ . To see that  $\frac{1}{\sqrt{T}} M * (\cdot)$  maps  $W_{\text{anti}}^{s,p}(\mathbb{T})$  into  $W_{\text{anti}}^{s,p}(\mathbb{T})$  and  $C_{\text{anti}}^{k,s}(\mathbb{T})$  into  $C_{\text{anti}}^{k,s}(\mathbb{T})$ , let  $\|\cdot\|$  be  $\|\cdot\|_{W^{s,p}}$  or  $\|\cdot\|_{C^{k,s}}$  (or any translation invariant norm). Then

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left( \left( \hat{M}_k \hat{\alpha}_k \right)_{k \in \mathbb{Z}_{\text{odd}}} \right) \right\| &= \left\| \frac{1}{\sqrt{T}} M * \alpha \right\| = \frac{1}{\sqrt{T}} \left\| \int_{\mathbb{T}} \alpha(\cdot - s) \, dM(s) \right\| \\ &\leq \frac{1}{\sqrt{T}} \int_{\mathbb{T}} \|\alpha(\cdot - s)\| \, d|M|(s) = \frac{|M|(\mathbb{T})}{\sqrt{T}} \|\alpha\|. \end{aligned}$$

□

With the previous two lemmata, we can complete the bootstrapping argument stated next.

**Lemma 7.7.** *If the pair  $(\alpha, \beta)$  satisfies (31) and (34) with  $\alpha, \beta \in L^1_{\text{anti}}(\mathbb{T})$ , then  $\alpha, \beta \in C^{1,r}_{\text{anti}}(\mathbb{T})$ .*

**Proof.** By lemma 7.5 we have  $\alpha \in W^{1,1/r}_{\text{anti}}(\mathbb{T})$ , and therefore  $\beta \in W^{1,1/r}_{\text{anti}}(\mathbb{T})$  by lemma 7.6. Applying lemmas 7.5 and 7.6 again, we get  $\alpha, \beta \in W^{1+r-\varepsilon, 1/r^2}_{\text{anti}}(\mathbb{T})$  for any  $\varepsilon > 0$ . Repeating this  $n$  times, we obtain  $\alpha, \beta \in W^{1+r-\varepsilon, 1/r^{2+n}}_{\text{anti}}(\mathbb{T})$ . If  $n \in \mathbb{N}$  is large enough, then  $W^{1+r-\varepsilon, 1/r^{2+n}}_{\text{anti}}(\mathbb{T})$  embeds continuously into  $C^1_{\text{anti}}(\mathbb{T})$  by lemma B.3, so in particular we have  $\alpha, \beta \in C^1_{\text{anti}}(\mathbb{T})$ . Now, applying lemmas 7.5 and 7.6 one last time yields  $\alpha, \beta \in C^{1,r}_{\text{anti}}(\mathbb{T})$ . □

Next we prove the main theorem of this section, theorem 7.3.

**Proof of theorem 7.3.** Note that  $\alpha \in W^{1,1}(\mathbb{T}), \beta \in L^1(\mathbb{T})$  hold by definition 7.1, and both are  $\frac{T}{2}$ -antiperiodic as we have seen above. So lemma 7.7 is applicable and yields  $\alpha, \beta \in C^{1,r}_{\text{anti}}(\mathbb{T})$ .

By  $d_1 := \theta\pi, d_2 := (2 - \theta)\pi, d_3 := (2 + \theta)\pi, \dots$  we label the discontinuities of  $V$ . We start by showing that  $u \in C^{1,r}_{\text{anti}}([0, d_1] \times \mathbb{T})$ . To do this, consider

$$w(x, t) := \frac{1}{2} (\alpha(t + \sqrt{ax}) + \alpha(t - \sqrt{ax})) + \frac{1}{2\sqrt{a}} (\beta(t + \sqrt{ax}) - \beta(t - \sqrt{ax})). \tag{35}$$

Note that  $w$  is  $\frac{T}{2}$ -antiperiodic in time. The  $k$ th Fourier coefficient of  $w$  is given by

$$\begin{aligned} \hat{w}_k(x) &= \frac{\hat{\alpha}_k}{2} \left( e^{ik\omega\sqrt{ax}} + e^{-ik\omega\sqrt{ax}} \right) + \frac{\hat{\beta}_k}{2\sqrt{a}} \left( e^{ik\omega\sqrt{ax}} - e^{-ik\omega\sqrt{ax}} \right) \\ &= \hat{\alpha}_k \cos(k\omega\sqrt{ax}) + \frac{\hat{\beta}_k i}{\sqrt{a}} \sin(k\omega\sqrt{ax}). \end{aligned}$$

We see that  $\hat{w}_k$  solves  $L_k \hat{w}_k = 0$  on  $[0, d_1]$  and at  $x = 0$  it satisfies

$$\hat{w}_k(0) = \hat{\alpha}_k = \hat{\alpha}_k \phi_k(0) \quad \text{and} \quad \hat{w}'_k(0) = \frac{\hat{\beta}_k i}{\sqrt{a}} k\omega\sqrt{a} = \hat{\alpha}_k \phi'_k(0),$$

where we have used (34). So  $\hat{w}_k(x) = \hat{\alpha}_k \phi_k(x)$  must hold, and from this we obtain

$$w(x, t) = \sum_{k \in \mathbb{Z}_{\text{odd}}} \hat{w}_k(x) e_k(t) = \sum_{k \in \mathbb{Z}_{\text{odd}}} \hat{\alpha}_k \phi_k(x) e_k(t) = u(x, t).$$

As  $w$  is given by (35),  $u = w \in C^{1,r}_{\text{anti}}([0, d_1] \times \mathbb{T})$  follows immediately.



Now assume that  $u \in C_{\text{anti}}^{1,r}([0, d_n] \times \mathbb{T})$  holds for some  $n \in \mathbb{N}$ . We aim to show  $u \in C_{\text{anti}}^{1,r}([0, d_{n+1}])$ . Denote by  $v \in \{a, b\}$  the value of  $V$  on  $(d_n, d_{n+1})$  and define a function  $w$  by

$$w(x, t) = \frac{1}{2} (u(d_n, t + \sqrt{v}(x - d_n)) + u(d_n, t - \sqrt{v}(x - d_n))) + \frac{1}{2\sqrt{v}} \int_{t - \sqrt{v}(x - d_n)}^{t + \sqrt{v}(x - d_n)} u_x(d_n, \tau) \, d\tau \tag{36}$$

for  $x \in [d_n, d_{n+1}]$  and  $t \in \mathbb{T}$ . Then  $w \in C_{\text{anti}}^{1,r}([d_n, d_{n+1}] \times \mathbb{T})$  follows immediately from (36). Arguing as above, one can show  $L_k \hat{w}_k(x) = 0$  for all  $k \in \mathbb{Z}$ . Since  $\hat{w}_k(d_n) = \hat{u}_k(d_n) = \hat{\alpha}_k \phi_k(d_n)$  and  $\hat{w}'_k(d_n) = \hat{\alpha}_k \phi'_k(d_n)$ , we again get  $\hat{w}_k(x) = \hat{\alpha}_k \phi_k(x)$  and thus  $w = u$  on  $[d_n, d_{n+1}] \times \mathbb{T}$ .

Next we show the uniform bound  $|u(x, t)| \leq Ce^{-\rho x}$  with  $\rho = \frac{\log(b) - \log(a)}{4\pi}$ . By proposition 7.4,  $u$  satisfies  $u(x + 4\pi, t) = \frac{a}{b}u(x, t)$  for all  $x \in [0, \infty)$  and  $t \in \mathbb{T}$ . Hence we can choose

$$C := \max_{x \in [0, 4\pi], t \in \mathbb{T}} e^{\rho x} |u(x, t)|.$$

To show that  $u$  is a  $C^1$ -solution to (1), first from (35) it follows that the directional derivative

$$(\partial_t - c(x)\partial_x)(u_t + c(x)u_x)$$

exists and equals 0 for  $x \in (0, d_1)$  as  $c(x) = \frac{1}{\sqrt{a}}$  here. Similarly, using (36) we obtain

$$(\partial_t - c(x)\partial_x)(u_t + c(x)u_x) = 0$$

for  $x \in (d_n, d_{n+1})$  as  $c(x) = \frac{1}{\sqrt{v}}$ . Lastly, due to (35) and the definitions of  $\alpha, \beta$  we have

$$u_x(0, t) = w_x(0, t) = \beta'(t) = (f(\alpha'(t)))_t = (f(u_t(0, t)))_t.$$

for all  $t \in \mathbb{T}$ . This shows that  $u$  is a  $C^1$ -solution to (1) with its own initial data. □

**Data availability statement**

No new data were created or analysed in this study.

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### Appendix A

Recall that  $F'(s) = sf'(s)$  formally holds, so that  $(F \circ g)'(s) = g(s)(f \circ g)'(s)$ . Integrating the second equality from  $t_0$  to  $t_1$ , the resulting identity holds pointwise as we show next.

**Lemma A.1.** For  $t_0, t_1 \in \mathbb{R}$  with  $t_0 < t_1$  and  $g \in C([t_0, t_1], \mathbb{R})$  with  $f \circ g \in C^1([t_0, t_1], \mathbb{R})$ , the equation

$$F(g(t_1)) - F(g(t_0)) = \int_{t_0}^{t_1} g(t) \frac{df(g(t))}{dt} dt$$

holds.

**Proof.** Assume first that  $f$  and  $g$  are both  $C^1$  in which case the definition  $F(y) = yf(y) - \int_0^y f(s) ds$  and integration by parts yield the result

$$\begin{aligned} \int_{t_0}^{t_1} g(t) \frac{df(g(t))}{dt} dt &= [g(t)f(g(t))]_{t=t_0}^{t_1} - \int_{t_0}^{t_1} g'(t)f(g(t)) dt \\ &= [g(t)f(g(t))]_{t=t_0}^{t_1} - \int_{g(t_0)}^{g(t_1)} f(v) dv = F(g(t_1)) - F(g(t_0)). \end{aligned} \tag{37}$$

For the general case, choose a sequence of non-negative smooth mollifiers  $\phi_n : \mathbb{R} \rightarrow [0, \infty)$  converging to  $\delta_0$ , each with support in  $[-\frac{1}{n}, \frac{1}{n}]$  and with average  $\int_{\mathbb{R}} \phi_n(x) dx = 1$ . Since  $f$  is strictly increasing, so is  $f_n := \phi_n * f$ . In particular,  $f_n$  is bijective and we may define  $g_n := (f_n)^{-1} \circ f \circ g$  so that  $f_n \circ g_n = f \circ g$ .

Clearly,  $f_n \rightarrow f$  uniformly on compacts. To see that  $g_n \rightarrow g$  uniformly on compacts, it suffices to show  $\| (f_n)^{-1} - f^{-1} \|_{\infty} \leq \frac{1}{n}$  for  $n \in \mathbb{N}$ . Note that

$$f_n(x - \frac{1}{n}) = \int_{x - \frac{2}{n}}^x f(y) \phi_n(x - \frac{1}{n} - y) dy \leq \int_{x - \frac{2}{n}}^x f(x) \phi_n(x - \frac{1}{n} - y) dy = f(x).$$

If we choose  $x := f^{-1}(y)$  for arbitrary  $y \in \mathbb{R}$  and apply  $(f_n)^{-1}$  to both sides of the above inequality, we get  $f^{-1}(y) - \frac{1}{n} \leq (f_n)^{-1}(y)$ . Similarly,  $f^{-1}(y) + \frac{1}{n} \geq (f_n)^{-1}(y)$  holds so that the estimate  $\| (f_n)^{-1} - f^{-1} \|_{\infty} \leq \frac{1}{n}$  is shown. Letting  $F_n(s) := sf_n(s) - \int_0^s f_n(\sigma) d\sigma$ , by (37) we have

$$F_n(g_n(t_1)) - F_n(g_n(t_0)) = \int_{t_0}^{t_1} g_n(t) \frac{df_n(g_n(t))}{dt} dt = \int_{t_0}^{t_1} g_n(t) \frac{df(g(t))}{dt} dt.$$

For  $n \rightarrow \infty$ , the desired result follows. □

### Appendix B. Sobolev–Slobodeckij space

We give a definition of the fractional Sobolev–Slobodeckij space  $W^{s,p}(\mathbb{T})$  on the torus, and present two results on it.

**Definition B.1.** Denote the distance on the torus  $\mathbb{T}$  by  $d$ . Then, for  $s \in (0, 1)$  and  $p \in [1, \infty)$  define the Sobolev–Slobodeckij space  $W^{s,p}(\mathbb{T}) := \left\{ u \in L^p(\mathbb{T}) : [u]_{W^{s,p}(\mathbb{T})} < \infty \right\}$  with

$$[u]_{W^{s,p}(\mathbb{T})}^p = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|u(t_1) - u(t_2)|^p}{d(t_1, t_2)^{1+sp}} dt_1 dt_2.$$

Also let  $W^{0,p}(\mathbb{T}) := L^p(\mathbb{T})$  and  $W^{k+s,p}(\mathbb{T}) := \{ u \in W^{k,p}(\mathbb{T}) : u^{(k)} \in W^{s,p}(\mathbb{T}) \}$  for  $k \in \mathbb{N}$ ,  $s \in [0, 1)$  and  $p \in [1, \infty)$ .

**Lemma B.2.** If  $g : \mathbb{R} \rightarrow \mathbb{R}$  is  $r$ -Hölder continuous, then the map

$$W^{s,p}(\mathbb{T}) \rightarrow W^{rs,p/r}(\mathbb{T}), u \mapsto g \circ u$$

is well-defined for  $s \in [0, 1)$  and  $p \in [1, \infty)$ .

**Proof.** By assumption, there exists  $C > 0$  such that  $|g(x) - g(y)| \leq C|x - y|^r$  holds for all  $x, y \in \mathbb{R}$ . First, let  $u \in L^p(\mathbb{T})$ . Then

$$\begin{aligned} \|g(u)\|_{L^{p/r}(\mathbb{T})}^{p/r} &= \int_{\mathbb{T}} |g(u(t))|^{p/r} dt \leq 2^{p/r-1} \int_{\mathbb{T}} (|g(u(t)) - g(0)|^{p/r} + |g(0)|^{p/r}) dt \\ &\leq 2^{p/r-1} \int_{\mathbb{T}} (C^{p/r} |u(t)|^p + |g(0)|^{p/r}) dt = 2^{p/r-1} (C^{p/r} \|u\|_{L^p(\mathbb{T})}^p + T|g(0)|^{p/r}), \end{aligned}$$

so  $g(u) \in L^{p/r}(\mathbb{T})$ . Now let  $u \in W^{s,p}(\mathbb{T})$  with  $s \in (0, 1)$ . Then

$$\begin{aligned} [g(u)]_{W^{rs,p/r}(\mathbb{T})}^{p/r} &= \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|g(u(t_1)) - g(u(t_2))|^{p/r}}{d(t_1, t_2)^{1+sp}} dt_1 dt_2 \\ &\leq \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{C^{p/r} |u(t_1) - u(t_2)|^p}{d(t_1, t_2)^{1+sp}} dt_1 dt_2 = C^{p/r} [u]_{W^{s,p}(\mathbb{T})}^p. \end{aligned}$$

□

**Lemma B.3.**  $W^{1+s,p}(\mathbb{T}) \hookrightarrow C^{1, s-\frac{1}{p}}(\mathbb{T})$  for  $s \in (0, 1), p \in (1, \infty)$  with  $sp > 1$ .

**Proof.** Consider the fractional Sobolev–Slobodeckij space  $W^{s,p}([0, T])$  which is similarly defined using the seminorm

$$[v]_{W^{s,p}([0, T])}^p = \int_0^T \int_0^T \frac{|v(t_1) - v(t_2)|^p}{|t_1 - t_2|^{1+sp}} dt_1 dt_2.$$

We have  $[u']_{W^{s,p}([0, T])}^p \leq [u']_{W^{s,p}(\mathbb{T})}^p < \infty$ , so that  $u' \in W^{s,p}([0, T])$  and from [13, theorem 8.2] it follows that  $u' \in C^{(sp-1)/p}([0, T])$ . □

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