Published for SISSA by Springer

Received: *August 24, 2023*
Revised: *November 20, 2023*
Accepted: *December 28, 2023*
Published: *January 15, 2024*

# Linear power corrections to top quark pair production in hadron collisions

**Sergei Makarov**,[a] **Kirill Melnikov**,[a] **Paolo Nason**[b] **and Melih A. Ozcelik**[c]

[a]*Institute for Theoretical Particle Physics, KIT, 76128 Karlsruhe, Germany*
[b]*INFN, Sezione di Milano-Bicocca, and Università di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy*
[c]*Université Paris-Saclay, CNRS, IJCLab, 91405 Orsay, France*

*E-mail:* sergei.makarov@kit.edu, kirill.melnikov@kit.edu, paolo.nason@mib.infn.it, melih.ozcelik@ijclab.in2p3.fr

Abstract: We compute, in the framework of renormalon calculus, the $\mathcal{O}(\Lambda_{\text{QCD}})$ corrections to the production of $t\bar{t}$ pairs in hadron collisions under the assumption that $q\bar{q} \to t\bar{t}$ is the dominant partonic channel. This assumption is not applicable to top quark pair production at the LHC but it is valid for the Tevatron where collisions of protons and anti-protons were studied. We show that the linear power correction to the total $t\bar{t}$ production cross section vanishes provided one uses a short-distance scheme for the top quark mass. We also derive relatively simple formulas for the power corrections to top quark kinematic distributions. Although small numerically, these power corrections exhibit interesting dependencies on top quark kinematics.

Keywords: Nonperturbative Effects, Top Quark, Scattering Amplitudes, Specific QCD Phenomenology

ArXiv ePrint: 2308.05526



https://doi.org/10.1007/JHEP01(2024)074

JHEP01(2024)074

## Contents



## 1 Introduction

Top quark studies play a central role in the current exploration of the Standard Model of particle physics and in the quest to discover physics beyond it. Of particular interest are top quark couplings to electroweak gauge bosons and the Higgs boson, as well as its mass and width [1, 2]. It is well-known that the lifetime of the top quark is so short that the hadronisation mechanism has no time to set in. As a result, many properties of "free" top quarks, such as their polarisations and spin correlations, can be accessed by studying kinematic distributions of their decay products.

Given the very rich research program that can be pursued by studying top quarks, experimental and theoretical exploration of top quark pair production progressed rapidly in recent years and reached a very advanced stage [3–12]. In fact, progress in theory and experiment allows us to study various properties of top quarks with very high precision making a better understanding of subtle effects desirable and even mandatory in certain cases.

One important class of such effects are the non-perturbative power corrections. In the case of top quark pair production, these power corrections are especially important for the extraction of the top quark mass from the total cross section and from kinematic distributions [13–23]. Since a sound theoretical understanding of power corrections to top quark pair production is lacking,[1] it cannot be excluded that such extractions are biased.

A valuable approach to the study of power corrections is the renormalon calculus in the large-$b_0$ approximation. It can be applied to processes that, at the Born level, are described by Feynman diagrams without gluons. The method consists in adding one soft gluon (virtual or real), dressed with an arbitrary number of quark-anti-quark vacuum polarisation insertions, to the Born process. The underlying abelianised model corresponds to QCD in the limit of a large, negative number of quark flavours. In this limit, the theory remains asymptotically free, and exhibits infrared renormalons. It turns out that the results in the large-$b_0$ approximation can be easily obtained from calculations in QCD where the gluon carries a small mass $\lambda$. In particular, $\mathcal{O}(\Lambda_{\rm QCD})$ corrections are associated with corrections of order $\lambda$ to the considered observables. This procedure is well known; for example, it is reviewed in ref. [24], where many applications are illustrated. Furthermore, a comprehensive account of the method is given in appendix B of ref. [25].

In two recent papers [26, 27], some of us have used the renormalon calculus to discuss linear power corrections to some collider observables.[2] In particular, in ref. [27], the case of $t$-channel single top production was considered. This process does not have any gluon at the Born level and is thus amenable to renormalon calculus. It was found that no linear power corrections are present in the total inclusive cross section of the $t$-channel single top production, provided that the cross section is expressed in terms of a short-distance mass of the top quark. We note in passing that, in general, certain input parameters, such as e.g. masses of heavy quarks in the on-shell renormalisation scheme, may receive linear power corrections, and it is better to avoid them by switching to input parameters defined at short distances. However, employing the short-distance quark masses is not sufficient to get rid of linear power corrections in general since, as it was shown in ref. [27], they exist in top-quark kinematic distributions. Such corrections are easily calculable within the same renormalon framework that we use to study total inclusive cross sections.

The goal of this paper is to go one step further in the computation of the linear power corrections to top quark production in hadron collisions, and to use the renormalon calculus to study such corrections in the $q\bar{q} \to t\bar{t}$ partonic channel. We note that this process is mediated by a gluon exchange at leading order and that such a gluon is highly virtual. As we explain below, the large virtuality of the gluon allows us to use the Low-Burnett-Kroll (LBK) theorem [32, 33],[3] to uniquely reconstruct the first subleading term in the soft expansion of

[1]We note that power corrections to processes and observables amenable to the operator product expansion can be characterised in terms of expectation values of higher-dimensional operators. Unfortunately, understanding collider processes in such a framework is an open problem.

[2]Other approaches to understanding power corrections to hard processes at lepton and hadron colliders are discussed in refs. [28–31].

[3]For recent literature on the LBK theorem see ref. [34] and references therein.

the $q\bar{q} \to t\bar{t}$ amplitude, and in this way compute the linear power correction. On the contrary, the $gg \to t\bar{t}$ channel, which is dominant at the LHC, has on-shell gluons as external lines, and we will not deal with it in the present work.

The remainder of the paper is organised as follows. In section 2 we describe the generalisation of the Low-Burnett-Kroll theorem to processes with arbitrary number of quarks and anti-quarks as external particles. In section 3 we discuss aspects of the large-$N_f$ limit of QCD which concern the presence of a virtual gluon in the Born amplitude. In section 4 we explore the structure of cancellations of various $\mathcal{O}(\lambda)$ terms and show that they occur *independently* for different colour dipoles responsible for soft QCD radiation and for contributions where the same parton emits and absorbs soft radiation. In section 5 we continue with the discussion of $\mathcal{O}(\lambda)$ corrections to top-quark kinematic distributions. We apply these general results to the partonic process $q\bar{q} \to t\bar{t}$ in section 6, where we also compute non-perturbative corrections to the top quark pair production in $p\bar{p}$ collisions. We conclude in section 7. The appendices contain useful technical information and results for non-perturbative corrections to observables in a general $q\bar{q} \to t\bar{t} + X$ process.

## 2 The Low-Burnett-Kroll theorem

The goal of this section is to discuss the Low-Burnett-Kroll theorem for processes with an arbitrary number of external quarks and anti-quarks that carry non-Abelian charges, and an arbitrary number of colour-neutral particles. We consider the process

$$\varnothing \to \sum_{i=1}^{N} f_i(p_i) + X(p_X), \tag{2.1}$$

where $f_i(p_i)$ is a quark or an anti-quark of flavour $i$ with momentum $p_i$ and mass $m_i$, and $X(p_X)$ denotes, collectively, a system of colour-neutral particles, and the symbol $\varnothing$ in the initial state symbolises the vacuum. We only consider final-state particles, since amplitudes with initial-state particles can be obtained from our results by crossing. The total number of colour-charged particles is $N$ and we will use $N_q$ and $N_{\bar{q}}$ to refer to the total number of quarks and anti-quarks, respectively.[4] With a slight abuse of notation we will also indicate with $N$, $N_q$ and $N_{\bar{q}}$ the sets of all quark and anti-quark indices. We stress that gluons cannot appear as external on-shell particles but virtual gluons can be present as internal lines in the Born amplitude.

### 2.1 Real emission contribution

We consider the emission of a gluon with momentum $k$ in the process shown in eq. (2.1),

$$\varnothing \to \sum_{i=1}^{N} f_i(p_i) + X(p_X) + g(k). \tag{2.2}$$

This gluon is considered to be massive, with a tiny mass $\lambda$. We are interested in a situation when the emitted gluon is soft and has an energy comparable to its mass.

[4]Obviously, colour conservation requires $N_q = N_{\bar{q}}$.

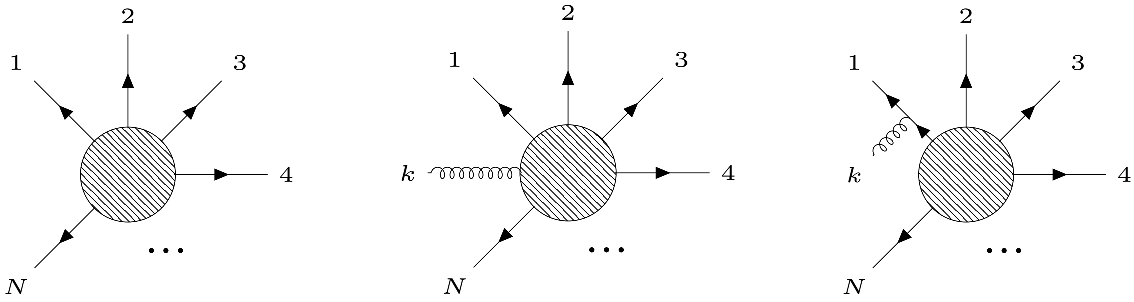


**Figure 1.** Leading order and the relevant real-emission contributions to the process $\varnothing \to \sum_i f_i(p_i) + X(p_X) + g(k)$. Arrows show directions of outgoing momenta for both quarks and anti-quarks.

We extract the gluon polarisation vector $\epsilon$ and write the amplitude of the process in eq. (2.2) as

$$\mathcal{A}_{\text{real}} = \epsilon_\mu \langle c|\mathcal{M}^{a,\mu}\rangle, \tag{2.3}$$

where $|c\rangle$ indicates colour quantum numbers of all particles in eq. (2.1) and $a$ is the gluon colour index. Thus $\mathcal{M}^{a,\mu}$ is a vector in colour space, in a representation that is the direct product of fundamental or anti-fundamental representations, each one associated to an outgoing particle or anti-particle. In this context we also define the colour space operator $T_i^a$, corresponding to a Gell-Mann matrix acting on the vector subspace of the outgoing particle $i$. For an outgoing anti-particle it is minus the transpose of the Gell-Mann matrix that acts by left multiplication. It is more convenient to us to drop the minus sign, and have instead the Gell-Mann matrix $T_i^a$ act by right multiplication upon the colour index of the outgoing anti-particle.

Separating the emissions from the external legs and the "structure-dependent" radiation from the internal lines as shown in figure 1, we write the reduced amplitude $\mathcal{M}^{a,\mu}$ as[5]

$$\begin{aligned}\mathcal{M}^{a,\mu} = {}& g_s \sum_{i\in N_q} \bar{u}(p_i)\gamma^\mu \frac{\not{p}_i + \not{k} + m_i}{d_i}\, T_i^a\, \mathbf{N}_i(p_1,p_2,\ldots,p_i+k,\ldots) \\ &+ g_s \sum_{i\in N_{\bar{q}}} \mathbf{N}_i(p_1,p_2,\ldots,p_i+k,\ldots)\, T_i^a \frac{-\not{p}_i - \not{k} + m_i}{d_i} \gamma^\mu v(p_i) + \mathcal{M}^{a,\mu}_{\text{reg}}(p_1,\ldots,p_N;k).\end{aligned} \tag{2.4}$$

Here, $d_i = (p_i+k)^2 - m_i^2$, $T_i^a$ refers to the colour charge of parton $i$ and $\mathbf{N}_i(p_1,p_2,\ldots,p_i+k,\ldots)$ is the Green's function of the process in eq. (2.1) with an amputated off-shell leg $i$. The amplitude $\mathcal{M}^{a,\mu}_{\text{reg}}$ describes the structure-dependent radiation; it is regular in the $k \sim \lambda \to 0$ limit.

The LBK theorem stems from the observation that one can determine $\mathcal{M}^{a,\mu}_{\text{reg}}$ at $k=0$ by requiring that $k_\mu \mathcal{M}^{a,\mu} = 0$.[6] We will apply this observation to eq. (2.4). Before we do

[5]We will write $\mathcal{M}^{a,\mu}$ rather than $|\mathcal{M}^{a,\mu}\rangle$ in what follows to simplify the notations.

[6]We note that recently the validity of LBK theorem has been questioned in refs. [35–37]. Although some of the criticism in these references might be justified, we believe that our derivation of the theorem is consistent and leads to correct results, see section 4.3 for further discussion of this point.

that, it is convenient to rewrite the parts of the amplitude that describe the gluon emissions from the external legs. Using the Dirac equation, we obtain

$$\bar{u}_i \gamma^\mu \frac{\not{p}_i + \not{k} + m_i}{d_i} = \bar{u}_i \left( J_i^\mu + \mathbf{S}_i^\mu \right), \tag{2.5}$$

where

$$J_i^\mu = \frac{2p_i^\mu + k^\mu}{d_i}, \qquad \mathbf{S}_i^\mu = \frac{\sigma^{\mu\nu} k_\nu}{d_i}, \tag{2.6}$$

with $\sigma^{\mu\nu} = 1/2\, [\gamma^\mu, \gamma^\nu]$. We note that these quantities have the following properties

$$k_\mu J_i^\mu = 1, \qquad k_\mu \mathbf{S}_i^\mu = 0. \tag{2.7}$$

For an anti-quark, we find a similar equation

$$\frac{-\not{p}_i - \not{k} + m_i}{d_i} \gamma^\mu v_i = \left( -J_i^\mu + \mathbf{S}_i^\mu \right) v_i. \tag{2.8}$$

We contract eq. (2.4) with $k_\mu$, use eq. (2.7) and obtain

$$0 = g_s \sum_{i=1}^{N} \eta_i \mathcal{N}_i^a(p_1, \ldots, p_i + k, \ldots) + k_\mu \mathcal{M}_{\text{reg}}^{a,\mu}(p_1, \ldots, p_N; k), \tag{2.9}$$

where we introduced

$$\mathcal{N}_i^a = \bar{u}_i T_i^a \mathbf{N}_i \quad \text{or} \quad \mathcal{N}_i^a = \mathbf{N}_i T_i^a v_i, \tag{2.10}$$

as appropriate for a quark or an anti-quark, and $\eta_i = 1$ or $-1$ if $i$ is a quark or an anti-quark.

Expanding equation (2.9) in Taylor series through linear terms in $k$, we obtain[7]

$$\begin{aligned} 0 &= \sum_{i=1}^{N} \eta_i\, \mathcal{N}_i^a(p_1, \ldots, p_i, \ldots, p_N), \\ 0 &= k^\mu \left( g_s \sum_{i=1}^{N} \eta_i \bar{D}_{i,\mu} \mathcal{N}_i^a(p_1, \ldots, p_i, \ldots, p_N) + \mathcal{M}_{\text{reg},\mu}^a(p_1, \ldots, p_N; 0) \right), \end{aligned} \tag{2.11}$$

where $D_{i,\mu} = \partial/\partial p_i^\mu$ and $\bar{D}$ indicates that the differential operator does not act on the external spinors that appear in $\mathcal{N}_i^a$ defined in eq. (2.10).

We also note that at $k = 0$ the functions $\mathcal{N}_i^a$ can be written as

$$\mathcal{N}_i^a = T_i^a |\mathcal{M}_0(p_1, p_2, \ldots, p_N)\rangle, \tag{2.12}$$

where $\mathcal{M}_0(p_1, p_2, \ldots, p_N)$ is the amplitude of the process in eq. (2.1) and we have written it as a vector in colour space.

[7]Note that our conventions for the colour generators differ from the ones commonly used in the literature. In this paper, we use $(T_i)^a_{\alpha\beta} = t^a_{\alpha\beta}$ both for outgoing particles and anti-particles, except that in the latter case $(T_i)^a_{\alpha\beta}$ acts on the right, corresponding to the transposed matrix acting on the left. Besides transposing, the anti-fundamental representation requires a minus sign, that we absorb into the factors $\eta_i$.



JHEP01(2024)074

The first equation in eq. (2.11) is the colour conservation condition. The second equation has to be satisfied for arbitrary $k$ so that

$$|\mathcal{M}^{a,\mu}_{\rm reg}(p_1,\ldots,p_N;0)\rangle = -g_s \sum_{i=1}^{N} \eta_i \bar{D}^{\mu}_i \mathcal{N}^a_i = -g_s \sum_{i=1}^{N} \eta_i \bar{D}^{\mu}_i T^a_i |\mathcal{M}_0\rangle. \tag{2.13}$$

Having determined the structure-dependent part of the real-emission amplitude, we can now write the full amplitude as an expansion in the gluon momentum with $\mathcal{O}(k^0)$ accuracy. We obtain

$$\mathcal{M}^{\mu} = g_s \sum_{i\in N} \eta_i (J^{\mu}_i + \bar{L}^{\mu}_i) T^a_i\, |\mathcal{M}_0\rangle + g_s \sum_{i\in N_q} \bar{u}_i \mathbf{S}^{\mu}_i \mathbf{N}^a_i + g_s \sum_{i\in N_{\bar{q}}} \mathbf{N}^a_i \mathbf{S}^{\mu}_i v_i + \mathcal{O}(k), \tag{2.14}$$

where

$$\mathbf{N}^a_i = T^a_i \mathbf{N}_i \quad \text{or} \quad \mathbf{N}_i T^a_i, \tag{2.15}$$

depending on whether parton $i$ is a quark or an anti-quark, and

$$\bar{L}^{\mu}_i = J^{\mu}_i k^{\nu} \bar{D}_{i,\nu} - \bar{D}^{\mu}_i. \tag{2.16}$$

As the next step, we need to compute the square of the gluon emission amplitude summed over polarisations and colours of external particles. Working through the first subleading term in the expansion of the gluon momentum, we find

$$\begin{aligned} \mathcal{M}^{\dagger}_{\mu}\mathcal{M}^{\mu} = g_s^2 &\sum_{i,j\in N} \eta_i\eta_j \langle \mathcal{M}_0 | J^{\mu}_i J_{j,\mu} T^a_i T^a_j + \overleftarrow{\bar{L}}_{j,\mu} J^{\mu}_i T^a_j T^a_i + J^{\mu}_j T^a_j T^a_i \bar{L}_{i,\mu} | \mathcal{M}_0\rangle \\ &+ g_s^2 \sum_{i,j\in N} \eta_i J_{i,\mu} \left( \langle \mathcal{M}^{s,\mu}_{0,j} | T^a_j T^a_i | \mathcal{M}_0\rangle + \langle \mathcal{M}_0 | T^a_i T^a_j | \mathcal{M}^{s,\mu}_{0,j}\rangle \right), \end{aligned} \tag{2.17}$$

where

$$|\mathcal{M}^{s,\mu}_{0,j}\rangle = \begin{cases} \bar{u}_j \mathbf{S}^{\mu}_j \mathbf{N}_j, & j\in N_q, \\ \mathbf{N}_j \mathbf{S}^{\mu}_j v_j, & j \in N_{\bar{q}}. \end{cases} \tag{2.18}$$

We note that

$$\langle \mathcal{M}^{s,\mu}_{0,j}| = (-1) \begin{cases} \bar{\mathbf{N}}_j \mathbf{S}^{\mu}_j u_j, & j\in N_q, \\ \bar{v}_j \mathbf{S}^{\mu}_j \bar{\mathbf{N}}_j, & j \in N_{\bar{q}}, \end{cases} \tag{2.19}$$

because the spin operators defined in eq. (2.6) are anti-hermitian.

We now discuss the various terms that appear in eq. (2.17). The terms in the first line can be easily simplified if we use the fact that $T^a_j T^a_i = T^a_i T^a_j$. Then we find

$$\begin{aligned} &\sum_{i,j\in N} \eta_i\eta_j \langle \mathcal{M}_0 | J^{\mu}_i J_{j,\mu} T^a_i T^a_j + \overleftarrow{\bar{L}}_{j,\mu} J^{\mu}_i T^a_j T^a_i + J^{\mu}_j T^a_j T^a_i \bar{L}_{i,\mu} | \mathcal{M}_0\rangle \\ &= \sum_{i,j\in N} \eta_i\eta_j \left( J^{\mu}_i J_{j,\mu} + J^{\mu}_i \bar{L}_{j,\mu} \right) F^{ij}_{\rm LO}, \end{aligned} \tag{2.20}$$

where

$$F^{ij}_{\rm LO} = \langle \mathcal{M}_0 | T^a_i T^a_j | \mathcal{M}_0 \rangle, \tag{2.21}$$

is the colour-correlated matrix element squared of the process in eq. (2.1).



JHEP01(2024)074

Next, we need to consider the various contributions that depend on the spin operators $\mathbf{S}_i^\mu$. As we will see, in this case one should be careful with the relative signs between the quark and the anti-quark cases. Consider the expression

$$\sum_{i\in N, j\in N_q} \eta_i J_i^\mu \left( \langle \mathcal{M}_{0,j}^{s,\mu} | T_j^a T_i^a | \mathcal{M}_0 \rangle + \langle \mathcal{M}_0 | T_i^a T_j^a | \mathcal{M}_{0,j}^{s,\mu} \rangle \right). \tag{2.22}$$

To simplify it, we note that we can write a tree-level amplitude as

$$|\mathcal{M}_0\rangle = \bar{u}_j \mathbf{N}_j, \tag{2.23}$$

"factoring out" a spinor $u_j$. Then we find

$$\begin{aligned} &\sum_{i\in N, j\in N_q} \eta_i J_i^\mu \left( \langle \mathcal{M}_{0,j}^{s,\mu} | T_j^a T_i^a | \mathcal{M}_0 \rangle + \langle \mathcal{M}_0 | T_i^a T_j^a | \mathcal{M}_{0,j}^{s,\mu} \rangle \right) \\ &= \sum_{i\in N, j\in N_q} \eta_i J_i^\mu \left( -\bar{u}_j \mathbf{N}_j T_i^a T_j^a \bar{\mathbf{N}}_j \mathbf{S}_{j,\mu} u_j + \bar{u}_j \mathbf{S}_{j,\mu} \mathbf{N}_j T_i^a T_j^a \bar{\mathbf{N}}_j u_j \right) \\ &= \sum_{i\in N, j\in N_q} \eta_i J_i^\mu \mathrm{Tr} \left( [\hat{\rho}_{q,j}, \mathbf{S}_{j,\mu}] \mathbf{N}_j T_i^a T_j^a \bar{\mathbf{N}}_j \right), \end{aligned} \tag{2.24}$$

where $\hat{\rho}_{q,j} = \not{p}_j + m_j$ is the density matrix associated with the quark $j$. A similar calculation for an anti-quark gives

$$\begin{aligned} &\sum_{i\in N, j\in N_{\bar{q}}} \eta_i J_i^\mu \left( \langle \mathcal{M}_{0,j}^{s,\mu} | T_j^a T_i^a | \mathcal{M}_0 \rangle + \langle \mathcal{M}_0 | T_i^a T_j^a | \mathcal{M}_{0,j}^{s,\mu} \rangle \right) \\ &= \sum_{i\in N, j\in N_{\bar{q}}} \eta_i J_i^\mu \left( -\bar{v}_j \mathbf{S}_{j,\mu} \bar{\mathbf{N}}_j T_i^a T_j^a \mathbf{N}_j v_j + \bar{v}_j \bar{\mathbf{N}}_j T_i^a T_j^a \mathbf{N}_j \mathbf{S}_{j,\mu} v_j \right) \\ &= - \sum_{i\in N, j\in N_{\bar{q}}} \eta_i J_i^\mu \mathrm{Tr} \left( [\hat{\rho}_{\bar{q},j}, \mathbf{S}_{j,\mu}] \bar{\mathbf{N}}_j T_i^a T_j^a \mathbf{N}_j \right), \end{aligned} \tag{2.25}$$

where $\hat{\rho}_{\bar{q},j} = \not{p}_j - m_j$. Since

$$[\hat{\rho}_j, \mathbf{S}_j^\mu] = L_j^\mu \hat{\rho}_j, \tag{2.26}$$

regardless of whether the density matrix refers to a quark or an anti-quark, we observe that eqs. (2.24)–(2.25) combine with the last term of eq. (2.20) into

$$\sum_{i,j\in N} \eta_i \eta_j J_i^\mu L_{j,\mu} F_{\mathrm{LO}}^{ij}. \tag{2.27}$$

We emphasise that the differential operator $L_j^\mu$ in the above equation *does* act on all $p_j$-dependent terms in $F_{\mathrm{LO}}^{ij}$ without any restrictions.

We conclude that the amplitude squared that describes the emission of soft gluons in the process $\varnothing \to \sum_{i\in N} f_i(p_i) + X(p_X)$ with subleading accuracy in $k$ can be written as follows

$$|\mathcal{A}_{\mathrm{real}}|^2 = -g_s^2 \sum_{i,j\in N} \eta_i \eta_j W_i^\mu W_{j,\mu} F_{\mathrm{LO}}^{ij} + \mathcal{O}(k^0). \tag{2.28}$$



JHEP01(2024)074

In eq. (2.28) we introduced the generalised current $W_i^\mu$ which reads

$$W_i^\mu = J_i^\mu + \frac{1}{2} L_i^\mu, \tag{2.29}$$

and it is understood that the differential operator $L_i^\mu$ does not act on the eikonal currents $J_i^\mu$ but only on the colour-correlated matrix element $F_{\rm LO}^{ij}$.

Finally, we note that, for processes with some particles in the initial state, $|\mathcal{A}_{\rm real}|^2$ can be obtained from our result by crossing. To this end, to describe an initial-state (anti)-particle $i$, one starts with eq. (2.28) and inverts the corresponding momentum $p_i \to -p_i$ in the definitions of $J_i$, $D_i$, $L_i$ and $d_i$. In addition, one needs to set $\eta_i = -1$ if $i$ is an initial-state quark and $\eta_i = 1$ if $i$ is an initial-state anti-quark. This completes the discussion of the real-emission part.

### 2.2 Virtual corrections

We need to analyse the virtual corrections in a similar way. The one-loop diagrams that contribute to the process of eq. (2.1) can be divided into three distinct groups. The first group includes diagrams where the virtual gluon is not connected to any of the external lines. The second group comprises all diagrams where the virtual gluon is attached to one and only one external line. The third group includes diagrams where the virtual gluon connects two external on-shell lines. These different contributions are shown in figure 2. We will analyse each one of them in turn. We note that we will also have to include the wave function renormalisation contribution that corresponds to self-energy insertions on the external lines and account for mass counterterms.

It is straightforward to convince oneself that diagrams of the first group cannot contain $\mathcal{O}(\lambda)$ terms. However, this is not the case for the diagrams in the second and third groups. To analyse the diagrams that belong to the second group, we note that their sum can be written in the following way

$$|\mathcal{A}_{V_2}\rangle = \int \frac{{\rm d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} \, |\mathcal{M}_{V_2}\rangle, \tag{2.30}$$

where

$$\begin{aligned} |\mathcal{M}_{V_2}\rangle = {}& g_s^2 \sum_{i \in N_q} \bar{u}_i \gamma_\mu \frac{\not{p}_i + \not{k} + m_i}{d_i} T_i^a \mathbf{N}_i^{a,\mu}(p_1, p_2, \dots, p_i + k, \dots | -k) \\ & + g_s^2 \sum_{i \in N_{\bar{q}}} \mathbf{N}_i^{a,\mu}(p_1, p_2, \dots, p_i + k, \dots | -k) T_i^a \frac{-\not{p}_i - \not{k} + m_i}{d_i} \gamma_\mu v_i. \end{aligned} \tag{2.31}$$

In the above equation $d_i = (p_i + k)^2 - m_i^2$ and $\mathbf{N}_i^{a,\mu}(p_1, p_2, \dots, p_i + k, \dots | -k)$ is the Green's function that describes the structure-dependent radiation in the process of eq. (2.2) with the off-shell leg $i$. A simple power counting shows that, for the purpose of computing $\mathcal{O}(\lambda)$ contributions to the virtual amplitude, the above expression can be simplified to

$$\begin{aligned} |\mathcal{M}_{V_2}\rangle = {}& g_s^2 \sum_{i \in N_q} J_{i,\mu} \, T_i^a \bar{u}_i \mathbf{N}_i^{a,\mu}(p_1, p_2, \dots, p_i, \dots | 0) \\ & - g_s^2 \sum_{i \in N_{\bar{q}}} J_{i,\mu} \mathbf{N}_i^{a,\mu}(p_1, p_2, \dots, p_i, \dots | 0) \, v_i T_i^a + \mathcal{O}(k^0), \end{aligned} \tag{2.32}$$

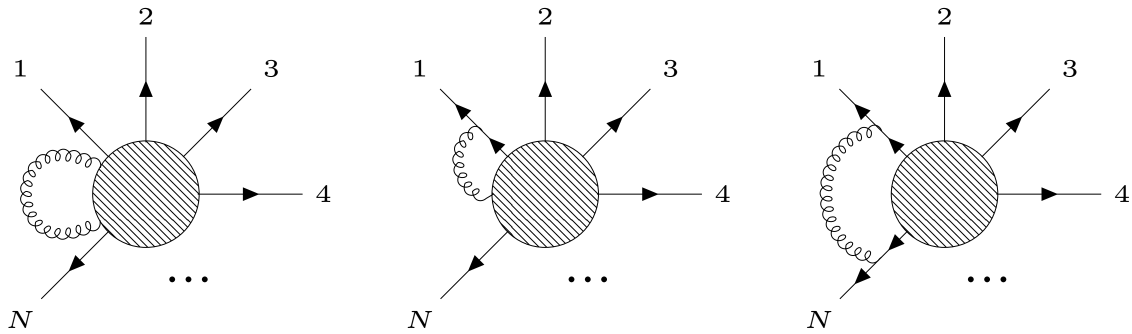


**Figure 2.** Loop contributions that need to be considered. The first diagram belongs to group $V_1$, while the second and third diagrams belong to groups $V_2$ and $V_3$ respectively (see text for details).

where $J_i^\mu = 2p_i^\mu/d_i$ is the eikonal current and it is indicated that we need the function $\mathbf{N}_i^{a,\mu}$ at $k=0$. To compute it, we proceed similarly to what was done in the previous section and write

$$g_s \mathbf{N}_{i_{\bar q}}^{a,\mu}(p_1,p_2,\ldots,p_i,\ldots|0)\, v_{i_{\bar q}} = g_s \bar u_{i_q} \mathbf{N}_{i_q}^{a,\mu}(p_1,p_2,\ldots,p_i,\ldots|0) = \mathcal{M}_{\mathrm{reg}}^{a,\mu}(p_1,p_2,\ldots,p_N|0), \tag{2.33}$$

for all $i_q$ and $i_{\bar q}$, and with $\mathcal{M}_{\mathrm{reg}}^{a,\mu}$ given in eq. (2.13). Using eq. (2.13), we find

$$|\mathcal{M}_{V_2}\rangle = -g_s^2 \sum_{i,j\in N} \eta_i\eta_j\; J_i^\mu \bar D_{j,\mu}\; T_i^a T_j^a\; |\mathcal{M}_0\rangle. \tag{2.34}$$

We will need to compute the interference of the one-loop amplitude with the leading order amplitude. Using eq. (2.34), we find

$$\langle\mathcal{M}_0|\mathcal{A}_{V_2}\rangle + \langle\mathcal{A}_{V_2}|\mathcal{M}_0\rangle = -g_s^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-i}{k^2-\lambda^2} \sum_{i,j\in N} \eta_i\eta_j\; J_i^\mu \bar D_{j,\mu} F_{\mathrm{LO}}^{ij}. \tag{2.35}$$

Next, we consider gluon exchanges between two external lines. There are three similar but distinct cases that need to be studied, namely the exchanges between two quarks, two anti-quarks and a quark and an anti-quark. We write

$$|\mathcal{A}_{V_3}\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-i}{k^2-\lambda^2}\,|\mathcal{M}_{V_3}\rangle, \tag{2.36}$$

where

$$\begin{aligned}|\mathcal{M}_{V_3}\rangle &= g_s^2 \sum_{i_q<j_q} \bar u_i\,(J_i^\mu + \mathbf{S}_i^\mu)\,\bar u_j\,(I_{j,\mu} - \mathbf{\Sigma}_{j,\mu})\,T_i^aT_j^a\mathbf{N}_{ij}(\ldots,p_i+k,\ldots,p_j-k,\ldots)\\ &+ g_s^2 \sum_{i_q,j_{\bar q}} \bar u_i\,(J_i^\mu + \mathbf{S}_i^\mu)\,T_i^aT_j^a\mathbf{N}_{ij}(\ldots,p_i+k,\ldots,p_j-k,\ldots)\,(-I_{j,\mu} - \mathbf{\Sigma}_{j,\mu})\,v_j\\ &+ g_s^2 \sum_{i_{\bar q}<j_{\bar q}} T_i^aT_j^a\mathbf{N}_{ij}(\ldots,p_i+k,\ldots,p_j-k,\ldots)\,(-I_{j,\mu} - \mathbf{\Sigma}_{j,\mu})\,v_j\,(-J_i^\mu + \mathbf{S}_i^\mu)\,v_i.\end{aligned} \tag{2.37}$$

Analogous to the definition of $\mathbf{N}_i$, the quantity $\mathbf{N}_{ij}(\ldots,p_i+k,\ldots,p_j-k,\ldots)$ is the Green's function of the process in eq. (2.1) with amputated off-shell legs $i$ and $j$. In the above



JHEP01(2024)074

equation, $J_i^\mu$ and $\mathbf{S}_i^\mu$ have already been defined and

$$I_j^\mu = \frac{2p_j^\mu - k^\mu}{d_j^-}, \qquad \mathbf{\Sigma}_j^\mu = \frac{\sigma^{\mu\nu}k_\nu}{d_j^-}, \tag{2.38}$$

with $d_j^- = (p_j - k)^2 - m_j^2$. We expand $|\mathcal{M}_{V_3}\rangle$ through relevant order in $k$ and find

$$\begin{aligned} |\mathcal{M}_{V_3}\rangle = &\frac{g_s^2}{2}\sum_{i\neq j\in N}\eta_i\eta_j\left(J_i^\mu I_{j,\mu} + J_i^\mu I_{j,\mu}\, k_\nu\, \bar{\Delta}_{ij}^\nu\right)T_i^a T_j^a|\mathcal{M}_0\rangle \\ &+\frac{g_s^2}{2}\sum_{i\neq j\in N}T_i^a T_j^a\left(\eta_j I_{j,\mu}|\mathcal{M}_{0,i}^{s,\mu}\rangle - \eta_i J_i^\mu|\mathcal{M}_{0,j}^{\sigma,\mu}\rangle\right), \end{aligned} \tag{2.39}$$

where $\bar{\Delta}_{ij}^\nu = \bar{D}_i^\nu - \bar{D}_j^\nu$ and we used the fact that $I_i^\mu(k) = J_i^\mu(-k)$ and $\mathbf{\Sigma}_j^\mu(-k) = -\mathbf{S}_j^\mu(k)$. $|\mathcal{M}_{0,j}^{\sigma,\mu}\rangle$ is defined similarly to eq. (2.18) with $\mathbf{S}_i^\mu$ being replaced by $\mathbf{\Sigma}_i^\mu$. Next, we need to compute

$$\langle\mathcal{M}_0|\mathcal{M}_{V_3}\rangle + \langle\mathcal{M}_{V_3}|\mathcal{M}_0\rangle. \tag{2.40}$$

The computation is very similar to what has been done for the real-emission contribution. We recall that the key point is to rewrite the commutators of the spin operators with the quark and anti-quark density matrices as the derivatives of the density matrices with respect to the external momenta, and then combine these derivatives with $\bar{\Delta}_{ij}^\nu$ acting on $F_{\rm LO}^{ij}$. We find

$$\begin{aligned} &\langle\mathcal{M}_0|\mathcal{M}_{V_3}\rangle + \langle\mathcal{M}_{V_3}|\mathcal{M}_0\rangle \\ &= \frac{g_s^2}{2}\sum_{i\neq j\in N}\eta_i\eta_j\left[2J_i^\mu I_{j,\mu} + J_i^\mu I_{j,\mu}k_\nu\bar{\Delta}_{ij}^\nu + I_j^\mu\tilde{L}_{i,\mu} - J_i^\mu\tilde{K}_{j,\mu}\right]F_{\rm LO}^{ij}, \end{aligned} \tag{2.41}$$

where

$$L_i^\mu = J_i^\mu k^\nu D_{i,\nu} - D_i^\mu, \qquad K_i^\mu = I_i^\mu k^\nu D_{i,\nu} + D_i^\mu, \tag{2.42}$$

so that

$$[\hat{\rho}_i, \mathbf{S}_i^\mu] = L_i^\mu\hat{\rho}_i, \qquad [\hat{\rho}_j, \mathbf{\Sigma}_{j,\mu}] = K_{j,\mu}\hat{\rho}_j, \tag{2.43}$$

and tilde indicates that these differential operators *only* act on the density matrices in $F_{\rm LO}^{ij}$.

To obtain the final result, we combine eq. (2.41) and eq. (2.34), where in the latter equation we separate the contributions with $i = j$ from those with $i \neq j$. We find

$$\begin{aligned} &\langle\mathcal{M}_0|\mathcal{M}_{V_2} + \mathcal{M}_{V_3}\rangle + \langle\mathcal{M}_{V_2} + \mathcal{M}_{V_3}|\mathcal{M}_0\rangle \\ &= \frac{g_s^2}{2}\sum_{i\neq j\in N}\eta_i\eta_j\left[2J_i^\mu I_{j,\mu} + J_i^\mu I_{j,\mu}k_\nu\bar{\Delta}_{ij}^\nu + I_j^\mu\tilde{L}_{i,\mu} - J_i^\mu\tilde{K}_{j,\mu}\right]F_{\rm LO}^{ij} \\ &\quad - g_s^2\sum_{i\neq j\in N}\eta_i\eta_j\, J_i^\mu\bar{D}_{j,\mu}F_{\rm LO}^{ij} - g_s^2 C_F\sum_{i=1}^{N}J_i^\mu\bar{D}_{i,\mu}F_{\rm LO}, \end{aligned} \tag{2.44}$$

where in the last term we used $\sum_a T_i^a T_i^a = C_F$ for $i \in N$.

It is convenient to rewrite certain terms that appear in the above expression. First we note that

$$J_i^\mu I_{j,\mu} k_\nu \bar{\Delta}_{ij}^\nu = I_{j,\mu} \bar{L}_i^\mu + I_{j,\mu} \bar{D}_i^\mu - J_i^\mu \bar{K}_{j,\mu} + J_i^\mu \bar{D}_{j,\mu}. \tag{2.45}$$

Since $\bar{K}_{i,\mu} + \tilde{K}_{i,\mu} = K_{i,\mu}$ and $\bar{L}_{i,\mu} + \tilde{L}_{i,\mu} = L_{i,\mu}$, we find

$$J_i^\mu I_{j,\mu} k_\nu \bar{\Delta}_{ij}^\nu + I_j^\mu \tilde{L}_{i,\mu} - J_i^\mu \tilde{K}_{j,\mu} = I_{j,\mu} L_i^\mu - J_i^\mu K_{j,\mu} + I_{j,\mu} \bar{D}_i^\mu + J_i^\mu \bar{D}_{j,\mu}. \tag{2.46}$$

Next, we combine the last two terms from the above equation with the next-to-last term in eq. (2.44). We find

$$\begin{aligned} &\frac{g_s^2}{2} \sum \eta_i \eta_j \left( I_{j,\mu} \bar{D}_i^\mu + J_i^\mu \bar{D}_{j,\mu} - 2 J_i^\mu \bar{D}_{j,\mu} \right) F_{\mathrm{LO}}^{ij} \\ &= \frac{g_s^2}{2} \sum \eta_i \eta_j \left( I_{j,\mu} \bar{D}_i^\mu - J_i^\mu \bar{D}_{j,\mu} \right) F_{\mathrm{LO}}^{ij} \to 0. \end{aligned} \tag{2.47}$$

The last step follows from the fact that, to compute the one-loop amplitude, we will have to integrate eq. (2.47) over $k$ with the weight $1/(k^2 - \lambda^2)$, and from the observation that through leading order in $k$, $I_j^\mu(-k) = J_j^\mu(k)$. Hence, we obtain

$$\begin{aligned} &\langle \mathcal{M}_0 | \mathcal{M}_{V_2} + \mathcal{M}_{V_3} \rangle + \langle \mathcal{M}_{V_2} + \mathcal{M}_{V_3} | \mathcal{M}_0 \rangle \\ &= \frac{g_s^2}{2} \sum_{i \neq j \in N} \eta_i \eta_j \left[ 2 J_i^\mu I_{j,\mu} + I_{j,\mu} L_i^\mu - J_i^\mu K_{j,\mu} \right] F_{\mathrm{LO}}^{ij} - g_s^2 C_F \sum_{i=1}^N J_i^\mu \bar{D}_{i,\mu} F_{\mathrm{LO}}. \end{aligned} \tag{2.48}$$

It is further convenient to rewrite the last term as follows

$$\sum_i J_i^\mu \bar{D}_{i,\mu} F_{\mathrm{LO}} = \sum_i J_i^\mu \left( D_{i,\mu} - \tilde{D}_{i,\mu} \right) F_{\mathrm{LO}}. \tag{2.49}$$

Then

$$\sum_i J_i^\mu \tilde{D}_{i,\mu} \, F_{\mathrm{LO}} = \sum_i \frac{2}{d_i} F_{\mathrm{LO}}|_{\hat{\rho}_i = \not{p}_i} = \sum_i \frac{2}{d_i} F_{\mathrm{LO}} - \sum_i \frac{2\eta_i}{d_i} F_{\mathrm{LO}}|_{\hat{\rho}_i = m_i \mathbf{1}}. \tag{2.50}$$

In this equation, the subscripts indicate that the density matrix of a fermion $i$ should be replaced either with $\not{p}_i$ or with $m_i$ times the identity matrix $\mathbf{1}$.

Putting everything together and using $I_j^\mu(k) = J_j^\mu(-k)$ and $K_j^\mu(k) = -L_j^\mu(-k)$, we obtain

$$\begin{aligned} &\langle \mathcal{M}_0 | \mathcal{M}_{V_2} + \mathcal{M}_{V_3} \rangle + \langle \mathcal{M}_{V_2} + \mathcal{M}_{V_3} | \mathcal{M}_0 \rangle \\ &= \frac{g_s^2}{2} \sum_{i \neq j \in N} 2 \eta_i \eta_j W_i^\mu(k) W_{j,\mu}(-k) F_{\mathrm{LO}}^{ij} - g_s^2 C_F \sum_{i=1}^N \left( J_i^\mu D_{i,\mu} F_{\mathrm{LO}} - \frac{2}{d_i} F_{\mathrm{LO}} + \frac{2\eta_i}{d_i} F_{\mathrm{LO}}|_{\hat{\rho}_i = m_i \mathbf{1}} \right), \end{aligned} \tag{2.51}$$

where $W^\mu(k)$ is defined in eq. (2.29). The final result for the $\mathcal{O}(\lambda)$ contribution that originates from the one-loop amplitude reads

$$\begin{aligned} |\mathcal{M}_V|^2 = g_s^2 \mathcal{T}_\lambda \Bigg[ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} \Bigg\{ &\sum_{i \neq j \in N} \eta_i \eta_j \; W_i^\mu(k) W_{j,\mu}(-k) F_{\mathrm{LO}}^{ij} \\ &- C_F \sum_{i=1}^N \left( J_i^\mu D_{i,\mu} F_{\mathrm{LO}} - \frac{2}{d_i} F_{\mathrm{LO}} + \frac{2\bar{\eta}_i}{d_i} F_{\mathrm{LO}}|_{\hat{\rho}_i \to m_i \mathbf{1}} \right) \Bigg\} \Bigg], \end{aligned} \tag{2.52}$$

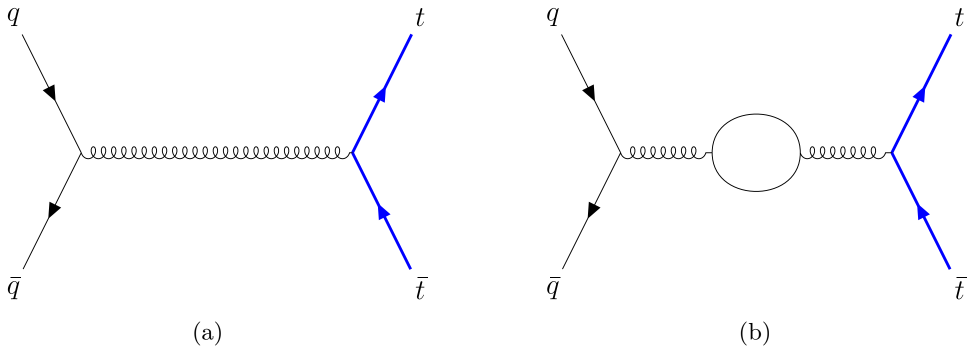


**Figure 3.** (a) Leading order diagram and (b) $N_f$-dependent vacuum polarisation contribution to $q\bar{q} \to t\bar{t}$ process.

where $\mathcal{T}_\lambda$ is an operator that extracts the contribution linear in $\lambda$ from the expression it is applied to. We have introduced the quantity $\bar{\eta}_i$ in the above equation to enable crossing to the initial state. We define $\bar{\eta}_i$ to be equal to $\eta_i$ $(-\eta_i)$ if $i$ is in the final (initial) state. Furthermore, for dipoles involving initial-state particles, we can use the same expression, eq. (2.52), and we need to apply the same changes as in the real-emission part. This means that the corresponding momenta should be inverted $p_i \to -p_i$ in the definitions of $J_i$, $D_i$, $L_i$ and $d_i$ and the correct $\eta_i$-values for initial-state quarks and anti-quarks have to be assigned.

## 3 Connection to the large-$N_f$ limit of QCD

There is an important difference between the calculation that we just described and traditional applications of the renormalon calculus. This difference is related to the fact that, in the current case, virtual gluons appear already in the tree-level diagrams. This leads to the appearance of perturbative corrections that scale as $\mathcal{O}(\alpha_s(Q)N_f)$ where $Q$ is the hard scale of the process, and $N_f$ is the number of massless fermions. These corrections are peculiar because the relation between renormalon calculus and calculations where the gluon is assigned a small but non-vanishing mass is derived by considering the $N_f \to -\infty$ limit; in this limit the behaviour of the $\mathcal{O}(\alpha_s(Q)N_f)$ corrections needs to be clarified.

To explain the origin of these corrections, we focus on an example where a top quark pair is produced in the collision of a massless quark and an anti-quark. At leading order, there is just one diagram that contributes to this process, it is shown in figure 3a. The one-loop corrections include the vacuum polarisation diagram shown in figure 3b which, together with $\mathcal{O}(N_f)$ contribution to the strong coupling renormalisation constant, combine to fix the scale of $\alpha_s$ in the leading order amplitude to the hard scale $Q^2 = (p_q + p_{\bar{q}})^2$. Moreover, if we choose the so-called $V$-scheme to renormalise the strong coupling constant, we can absorb the entire $N_f$-dependent vacuum polarisation contribution into the strong coupling constant [38, 39]. The leading order matrix element then reads

$$|\mathcal{M}_0\rangle = \frac{i4\pi\alpha_{s,V}(Q)}{Q^2}\,[\bar{v}(p_{\bar{q}})\gamma^\mu T^a u(p_q)]\;\;[\bar{u}_t(p_t)T^a\gamma^\mu v_{\bar{t}}(p_{\bar{t}})]\,. \tag{3.1}$$

We now discuss what happens at NNLO and, as an example, we consider an emission of a soft gluon in diagrams with the vacuum polarisation insertions. These diagrams are shown in

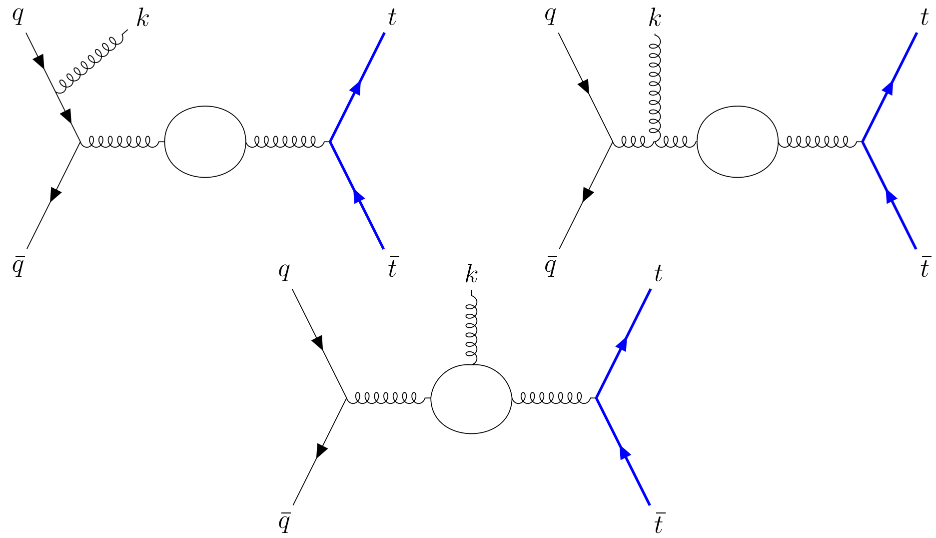


**Figure 4.** Examples of contributions to $q\bar{q} \to t\bar{t}$ which are proportional to $\mathcal{O}(g_s \alpha_s(Q) N_f)$.

figure 4. The important feature of all these diagrams is that they are *hard*, in the sense that the $k \to 0$ limit does not induce additional singularities in the $N_f$-dependent parts of the diagrams. Therefore, these diagrams can then be studied following the proof of the LBK theorem.

It is then straightforward to show that all diagrams similar to the ones displayed in figure 4 can indeed be obtained from the LBK theorem provided that $|\mathcal{M}_0\rangle$ is chosen as in eq. (3.1). Interestingly, this implies that when the structure-dependent radiation is computed by differentiating the tree-level amplitude as in eq. (2.13), the derivative of the strong coupling constant in eq. (3.1) also needs to be calculated. However, beyond that, there seem to be no additional implications for the computation of linear power corrections related to the presence of off-shell gluons in the leading order matrix elements. It is very plausible that this result generalises also to higher orders both within and beyond the large-$N_f$ resummation framework; we will, however, leave an all-orders investigation of this factorisation to future work.

## 4 Cancellation of $\mathcal{O}(\lambda)$ contributions to the total cross section

In the previous section we computed the next-to-leading soft terms in the radiative corrections to a process involving an arbitrary number of external quarks, anti-quarks and colour-neutral particles, caused by the production or exchange of a soft massive gluon. This allows us to calculate the expansion of the cross section in the gluon mass $\lambda$ including $\mathcal{O}(\lambda)$ terms. The question that we would like to answer in this section is whether such terms are present in the total cross section of the $q\bar{q} \to t\bar{t}$ process.

According to the analysis in the previous section, two types of terms appear in the sum of the real and virtual contributions. First, there are terms that depend on a particular colour-correlated amplitude squared $F_{\rm LO}^{ij}$. We will refer to such contributions as "dipole". Second, there are terms which depend on the leading order amplitude squared $F_{\rm LO}$ multiplied by the Casimir operator $C_F$. We will refer to such terms as "monopole". We will show that the cancellation of the $\mathcal{O}(\lambda)$ terms takes place individually for each of the dipole and monopole terms, and therefore it is convenient to study them separately.

In this respect, we note that, for processes with massive quarks, the cancellation of the $\mathcal{O}(\lambda)$ terms requires us to introduce the renormalisation of the quark mass parameter and the wave function renormalisation. In addition, it is to be expected that the cancellation requires us to express the cross section in terms of a mass parameter that is free of $\mathcal{O}(\lambda)$ terms (see e.g. ref. [27] for a related analysis). Thus, if the calculation is performed in the on-shell mass scheme, one has to rewrite the leading order cross section in terms of the new mass parameter. Since all these renormalisation factors and mass shifts are proportional to the Casimir factor $C_F$, these contributions will have to be added to the monopole terms to ensure the cancellation.

In what follows, we will explicitly study the $q\bar{q} \to t\bar{t} + X$ process. There are six dipoles ($q\bar{q}$, $qt$, $q\bar{t}$, $\bar{q}t$, $\bar{q}\bar{t}$ and $t\bar{t}$), and four monopoles ($qq$, $\bar{q}\bar{q}$, $tt$ and $\bar{t}\bar{t}$) to consider. Since $q$ and $\bar{q}$ are massless, according to ref. [26], the corresponding dipole $q\bar{q}$ and the two monopoles $qq$ and $\bar{q}\bar{q}$ do not produce $\mathcal{O}(\lambda)$ terms and can be discarded. We will also make use of the fact that masses of $t$ and $\bar{t}$ are identical which allows us to combine the $tt$ and $\bar{t}\bar{t}$ monopoles into a single contribution. Hence, we write

$$
\begin{aligned}
\mathcal{T}_\lambda \left[\mathrm{d}\sigma(q\bar{q} \to t\bar{t} + X)\right] = \mathcal{T}_\lambda \left[\mathrm{d}\sigma_R^{t\bar{t}} + \mathrm{d}\sigma_V^{t\bar{t}}\right] + \sum_{f_1=q,\bar{q}} \sum_{f_2=t,\bar{t}} \mathcal{T}_\lambda \left[\mathrm{d}\sigma_R^{f_1 f_2} + \mathrm{d}\sigma_V^{f_1 f_2}\right] \\
+ \sum_{f=t,\bar{t}} \mathcal{T}_\lambda \left[\mathrm{d}\sigma_R^{ff} + \mathrm{d}\sigma_V^{ff} + \mathrm{d}\sigma_{\mathrm{ren}} + \mathrm{d}\sigma_{\mathrm{mass}}\right],
\end{aligned}
\tag{4.1}
$$

where in the first line we collected the various dipole contributions and in the second line the two monopole contributions together with terms generated by the renormalisation and mass redefinition. We will now proceed with the analysis of the various terms in the above equation. However, before we dive into this discussion, we will have to describe the momenta mappings required to enable the integration over the gluon momentum in the real-emission contributions.

### 4.1 Momenta mappings

We consider the process $q(p_q) + \bar{q}(p_{\bar{q}}) \to t(q_t) + \bar{t}(q_{\bar{t}}) + X(p_X) + g(k)$. In order to integrate out the gluon momentum $k$ and express the result in terms of the LO cross section in a process-independent manner, we will need to factorise out the gluon momentum. This will allow us to combine real emission and virtual contributions in a convenient manner. To remove the momentum of the gluon from the delta-function that enforces the energy-momentum conservation, we change the momenta of top quark and anti-quark. Specifically, we write

$$
\begin{aligned}
q_t &= p_t - \alpha k + A(k) p_t + B(k) p_{\bar{t}}, \\
q_{\bar{t}} &= p_{\bar{t}} - \beta k - A(k) p_t - B(k) p_{\bar{t}},
\end{aligned}
\tag{4.2}
$$

where two parameters $\alpha$ and $\beta$ are $k$-independent, and $A$ and $B$ are two $\mathcal{O}(k)$ functions. The mapping must satisfy the condition

$$
q_t + q_{\bar{t}} + k = p_t + p_{\bar{t}}, \tag{4.3}
$$

which implies

$$
1 = \alpha + \beta. \tag{4.4}
$$

Furthermore, imposing the conditions

$$p_t^2 = p_{\bar{t}}^2 = q_t^2 = q_{\bar{t}}^2 = m_t^2, \tag{4.5}$$

we find the following results

$$\begin{aligned} A &= -\frac{\alpha\, m_t^2\, (p_t k) + \beta\, (p_t p_{\bar{t}})\, (p_{\bar{t}} k)}{(p_t p_{\bar{t}})^2 - m_t^4}, \\ B &= \frac{\alpha\, (p_t p_{\bar{t}})\, (p_t k) + \beta\, m_t^2\, (p_{\bar{t}} k)}{(p_t p_{\bar{t}})^2 - m_t^4}. \end{aligned} \tag{4.6}$$

Using the mapping in eq. (4.2), it is straightforward to find the phase space transformation. Keeping $\mathcal{O}(k)$ contributions and neglecting higher order terms, we obtain

$$\begin{aligned} \mathrm{dLips}(p_q, p_{\bar{q}}; q_t, q_{\bar{t}}, p_X, k) = \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X) \frac{\mathrm{d}^4 k}{(2\pi)^4} \delta(k^2 - \lambda^2) \times \\ \times \left(1 + \frac{\left[(p_t p_{\bar{t}})\, ((p_{\bar{t}} k) - (p_t k))\, (\alpha - \beta) - 2m_t^2\, (\alpha\, (p_t k) + \beta\, (p_{\bar{t}} k))\right]}{(p_t p_{\bar{t}})^2 - m_t^4}\right). \end{aligned} \tag{4.7}$$

The momenta mappings shown in eq. (4.2) will have to be applied to the leading order matrix element squared $F_{\mathrm{LO}}$ that appears in both the dipole and monopole terms. Working through the first order in $k$, we find

$$F_{\mathrm{LO}}(q_t, q_{\bar{t}}) = \left[1 + (A p_t^\nu + B p_{\bar{t}}^\nu - \alpha k^\nu) D_{t,\nu} - (A p_t^\nu + B p_{\bar{t}}^\nu + \beta k^\nu) D_{\bar{t},\nu}\right] F_{\mathrm{LO}}(p_t, p_{\bar{t}}). \tag{4.8}$$

The mappings shown in eq. (4.2) also affect the $t$ and $\bar{t}$ propagators that appear explicitly in the eikonal currents. The expansion of these propagators through linear terms in $k$ is straightforward and we do not present it here.

### 4.2 Individual dipole and monopole contributions to the $t\bar{t}$ cross section

In the following, we will discuss the individual dipole and monopole contributions to the $q\bar{q} \to t\bar{t} + X$ processes.

#### 4.2.1 The case of the $t\bar{t}$ dipole

We start by considering the contribution of the $t\bar{t}$ dipole to the cross section. In this case, we only need to combine the real-emission contribution and the contribution of the virtual corrections. As explained earlier, no renormalisation contributions need to be added in this case.

We use eq. (2.28) to write the real-emission contribution in the following way

$$\mathcal{T}_\lambda \left[\mathrm{d}\sigma_R^{t\bar{t}}\right] = \mathcal{T}_\lambda \left[g_s^2 \int \mathrm{dLips}(p_q, p_{\bar{q}}; q_t, q_{\bar{t}}, p_X, k) \left(2 J_t^\mu J_{\bar{t},\mu} + J_t^\mu L_{\bar{t},\mu} + J_{\bar{t}}^\mu L_{t,\mu}\right) F_{\mathrm{LO}}^{t\bar{t}}(q_t, q_{\bar{t}})\right]. \tag{4.9}$$



JHEP01(2024)074

We then perform the momentum transformation using the formulas in the previous section, integrate over $k$ with the help of the phase-space integrals collected in appendix A.1, and find

$$\mathcal{T}_\lambda\left[\mathrm{d}\sigma_R^{t\bar{t}}\right] = -\frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X) \times \frac{1}{(p_t p_{\bar{t}} - m_t^2)}\times \\ \times\left[2(m_t^2 - 2p_t p_{\bar{t}}) + m_t^2(p_{\bar{t},\nu}D_t^\nu + p_{t,\nu}D_{\bar{t}}^\nu) - (p_t p_{\bar{t}})(p_{t,\nu}D_t^\nu + p_{\bar{t},\nu}D_{\bar{t}}^\nu)\right] F_{\mathrm{LO}}^{t\bar{t}}(p_t, p_{\bar{t}}). \tag{4.10}$$

We note that this result is obtained for arbitrary $\alpha$ and $\beta$, subject to the constraint $\alpha + \beta = 1$. We observe that the dependence on these parameters has disappeared from the final result.

The contribution from the virtual corrections can be extracted from the general formula in eq. (2.52). In this case, no momentum mapping is involved and one can integrate the relevant expression over the four-momentum of the virtual gluon. The relevant integrals are collected in appendix A.2. We find

$$\begin{aligned}\mathcal{T}_\lambda\left[\mathrm{d}\sigma_V^{t\bar{t}}\right] &= \mathcal{T}_\lambda\Big[-g_s^2\int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X)\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2-\lambda^2}\times \\ &\quad\times\Big(2J_{\bar{t}}^\mu(k)J_t^\mu(-k) + J_{\bar{t}}^\mu(k)L_{t,\mu}(-k) + J_t^\mu(-k)L_{\bar{t},\mu}(k)\Big)\, F_{\mathrm{LO}}^{t\bar{t}}(p_t, p_{\bar{t}})\Big] \\ &= -\frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X)\times\frac{1}{(p_t p_{\bar{t}} - m_t^2)}\times \\ &\quad\times\Big[2(2p_t p_{\bar{t}} - m_t^2) - m_t^2(p_{\bar{t},\nu}D_t^\nu + p_{t,\nu}D_{\bar{t}}^\nu) \\ &\quad + (p_t p_{\bar{t}})(p_{t,\nu}D_t^\nu + p_{\bar{t},\nu}D_{\bar{t}}^\nu)\Big] F_{\mathrm{LO}}^{t\bar{t}}(p_t, p_{\bar{t}}).\end{aligned} \tag{4.11}$$

Combining the above results for the real and virtual corrections, we obtain

$$\mathcal{T}_\lambda\left[\mathrm{d}\sigma_R^{t\bar{t}}\right] + \mathcal{T}_\lambda\left[\mathrm{d}\sigma_V^{t\bar{t}}\right] = 0. \tag{4.12}$$

#### 4.2.2 The case of the $tq$ dipole

We continue with the discussion of the $tq$ dipole. In principle, the calculation is very similar to the one for the $t\bar{t}$ dipole but there is a subtlety related to the fact that the momentum mapping, eq. (4.2), involves the momentum of $\bar{t}$, that does not belong to the $tq$ dipole. The consequence of this is the appearance of the derivative with respect to the $\bar{t}$ momentum in the real-emission contribution. However, such a derivative does not appear in the virtual correction to the $tq$ dipole because no momentum mapping is required there. Hence, the minimal requirement for the cancellation of the $\mathcal{O}(\lambda)$ corrections to occur in the sum of the real and virtual contributions to the $tq$ dipole (independently of other dipoles and monopoles) is the disappearance of the $\partial F_{\mathrm{LO}}^{tq}/\partial p_{\bar{t}}^\mu$ term after the integration over $k$ in the real-emission contribution.

To understand how this can be arranged, we consider eq. (4.8), which is the only source of derivatives w.r.t. $p_{\bar{t}}^\mu$. Since the coefficient of this derivative is already $O(k)$, we conclude that the potentially offending term reads

$$\mathcal{T}_\lambda\left[\int\frac{\mathrm{d}^4k}{(2\pi)^4}\delta(k^2-\lambda^2)J_t^\mu J_{q,\mu}(Ap_t^\nu + Bp_{\bar{t}}^\nu + \beta k^\nu)D_{\bar{t},\nu}\right] F_{\mathrm{LO}}^{tq}, \tag{4.13}$$

where the two eikonal currents should be taken at leading power. We would like the above expression to vanish after the integration over $k$. To see how this can occur, we note that $A$ and $B$ are linear combinations of $\alpha(p_t k)$ and $\beta(p_{\bar{t}} k)$. Since

$$\mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \delta(k^2 - \lambda^2) J_t^\mu J_{q,\mu} \, (p_t k) \right] = 0, \tag{4.14}$$

we conclude that the non-vanishing contribution in eq. (4.13) is proportional to $\beta$

$$\mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \delta(k^2 - \lambda^2) J_t^\mu J_{q,\mu} (A p_t^\nu + B p_{\bar{t}}^\nu + \beta k^\nu) D_{\bar{t},\nu} F_{\mathrm{LO}}^{tq} \right] \sim \beta \, V^\nu \, D_{\bar{t},\nu} \, F_{\mathrm{LO}}^{tq}, \tag{4.15}$$

where $V^\nu$ is a non-vanishing vector that arises as the result of the integration over $k$. Hence, the only way to remove this term from the real emission contribution to the $tq$ dipole is to choose a mapping with $\beta = 0$. It is important to stress that, although choosing $\beta = 0$ is a necessary condition, it is not obvious that it is a sufficient one to ensure a cancellation of the $\mathcal{O}(\lambda)$ corrections within the $tq$ dipole independently of all other contributions. However, an explicit calculation shows that this is the case.

To illustrate this point, we choose $\beta = 0$ and compute the real-emission contribution to the $tq$ dipole. Since the $J_q$ and $L_q$ are defined for the outgoing momenta, we will need to invert the momentum of the initial-state quark in their definitions. In addition, we need to set $\eta_q = -1$. We then find that

$$\begin{aligned}
\mathcal{T}_\lambda \left[ \mathrm{d}\sigma_R^{tq} \right] &= \mathcal{T}_\lambda \Big[ g_s^2 \int \mathrm{dLips}(p_q, p_{\bar{q}}; q_t, q_{\bar{t}}, p_X, k) \frac{d^4 k}{(2\pi)^3} \delta_+(k^2 - \lambda^2) \\
&\quad \times \left( 2 J_t^\mu J_{q,\mu} + J_t^\mu L_{q,\mu} + J_q^\mu L_{t,\mu} \right) F_{\mathrm{LO}}^{tq}(q_t, q_{\bar{t}}) \Big] \\
&= \frac{\alpha_s}{2\pi} \frac{\pi\lambda}{m_t} \int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X) \bigg( 2 - \frac{m_t^2}{p_t p_q} + p_{t,\nu} D_t^\nu \\
&\quad - \frac{m_t^2}{p_t p_q} p_{q,\nu} (D_q^\nu + D_t^\nu) \bigg) F_{\mathrm{LO}}^{tq}(p_t, p_{\bar{t}}).
\end{aligned} \tag{4.16}$$

A straightforward computation of the virtual corrections, using the integrals presented in appendix A.2, gives

$$\begin{aligned}
\mathcal{T}_\lambda \left[ \mathrm{d}\sigma_V^{tq} \right] &= \mathcal{T}_\lambda \Big[ - g_s^2 \int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X) \frac{d^4 k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2} \\
&\quad \times \left( 2 J_q^\mu(k) J_{t,\mu}(-k) + J_{q,\mu}(k) L_{t,\mu}(-k) + J_{t,\mu}(-k) L_{q,\mu}(k) \right) F_{\mathrm{LO}}^{tq}(p_t, p_{\bar{t}}) \Big] \\
&= \frac{\alpha_s}{2\pi} \frac{\pi\lambda}{m_t} \int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X) \bigg( - 2 + \frac{m_t^2}{p_t p_q} - p_{t,\nu} D_t^\nu \\
&\quad + \frac{m_t^2}{p_t p_q} p_{q,\nu} (D_q^\nu + D_t^\nu) \bigg) F_{\mathrm{LO}}^{tq}(p_t, p_{\bar{t}}).
\end{aligned} \tag{4.17}$$

Combining the above results, we find

$$\mathcal{T}_\lambda \left[ \mathrm{d}\sigma_V^{tq} \right] + \mathcal{T}_\lambda \left[ \mathrm{d}\sigma_R^{tq} \right] = 0. \tag{4.18}$$

### 4.2.3 Remaining dipoles

The remaining dipoles $t\bar{q}$, $\bar{t}\bar{q}$ and $\bar{t}q$ can be analysed in the same way as the $tq$ dipole. For all of them we use the momentum mapping of eq. (4.2). The cancellations of the $\mathcal{O}(\lambda)$ terms occur independently for each of these dipoles if we choose $\beta = 0$ for the $\bar{q}t$ and $\alpha = 0$ for the $q\bar{t}$ and $\bar{q}\bar{t}$ dipoles. This completes the discussion of the cancellation of $\mathcal{O}(\lambda)$ terms for all of the dipoles that potentially contribute to the $q\bar{q} \to t\bar{t} + X$ partonic process.

### 4.2.4 The monopole $tt + \bar{t}\bar{t}$ contributions

The last contributions that we need to consider are the monopole contributions, related to the $t$ and $\bar{t}$ quarks in the final state. In principle, one can design a procedure that deals with each of them separately but, for simplicity, we will consider both of them at once. The main difference with respect to the dipole contributions is the need to account for the renormalisation and to redefine the top quark mass. Hence, the pattern of cancellations becomes more involved.

The real-emission contribution reads

$$\begin{aligned}\mathcal{T}_\lambda\left[\mathrm{d}\sigma_R^{tt} + \mathrm{d}\sigma_R^{\bar{t}\bar{t}}\right] &= \mathcal{T}_\lambda\Big[ - C_F g_s^2 \int \mathrm{dLips}(p_q, p_{\bar{q}}; q_t, q_{\bar{t}}, p_X, k)(J_t^\mu J_{t,\mu} + J_{\bar{t}}^\mu J_{\bar{t},\mu} \\ &\qquad + J_t^\mu L_{t,\mu} + J_{\bar{t}}^\mu L_{\bar{t},\mu}) F_{\mathrm{LO}}(q_t, q_{\bar{t}})\Big] \\ &= \frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}\int \mathrm{dLips}_{\mathrm{LO}}(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X) \times \frac{1}{(p_t p_{\bar{t}} - m_t^2)} \times \\ &\qquad \times \left[m_t^2(-1 + p_{\bar{t},\nu}D_t^\nu + p_{t,\nu}D_{\bar{t}}^\nu) - (p_t p_{\bar{t}})(1 + p_{t,\nu}D_t^\nu + p_{\bar{t},\nu}D_{\bar{t}}^\nu)\right] F_{\mathrm{LO}}(p_t, p_{\bar{t}}),\end{aligned} \tag{4.19}$$

where the dependence on $\alpha$ and $\beta$ has cancelled out.

The virtual corrections evaluate to

$$\begin{aligned}\mathcal{T}_\lambda\left[\mathrm{d}\sigma_V^{tt} + \mathrm{d}\sigma_V^{\bar{t}\bar{t}}\right] &= \mathcal{T}_\lambda\Big[ - C_F g_s^2 \int \mathrm{dLips}_{\mathrm{LO}} \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2 - \lambda^2} \times \\ &\qquad \times \left(\left(J_t^\mu D_{t,\mu} + J_{\bar{t}}^\mu D_{\bar{t},\mu} - \frac{2}{d_t} - \frac{2}{d_{\bar{t}}}\right) F_{\mathrm{LO}} + \frac{2}{d_t} F_{\mathrm{LO}}|_{\hat{\rho}_t = m_t \mathbf{1}} - \frac{2}{d_{\bar{t}}} F_{\mathrm{LO}}|_{\hat{\rho}_{\bar{t}} = m_t \mathbf{1}}\right)\Big] \\ &= \frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}\int \mathrm{dLips}_{\mathrm{LO}}\Big[(-2 + p_{t,\nu}D_t^\nu + p_{\bar{t},\nu}D_{\bar{t}}^\nu) F_{\mathrm{LO}} \\ &\qquad + F_{\mathrm{LO}}|_{\hat{\rho}_t = m_t \mathbf{1}} - F_{\mathrm{LO}}|_{\hat{\rho}_{\bar{t}} = m_t \mathbf{1}}\Big].\end{aligned} \tag{4.20}$$

The above results for the real and virtual corrections have to be supplemented with the renormalisation contributions since they are proportional to the Casimir invariant $C_F$ and the leading order amplitude squared $F_{\mathrm{LO}}$. The computation is analogous to the single top production case discussed in ref. [27]. We obtain

$$\mathcal{T}_\lambda\left[\mathrm{d}\sigma_{\mathrm{ren}}\right] = \frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}\int \mathrm{dLips}_{\mathrm{LO}}\left[3F_{\mathrm{LO}} + m_t \mathrm{Tr}\left[\hat{\rho}_t \frac{\partial \mathbf{N}}{\partial m_t}\hat{\rho}_{\bar{t}}\bar{\mathbf{N}}\right] + m_t \mathrm{Tr}\left[\hat{\rho}_t \mathbf{N}\hat{\rho}_{\bar{t}}\frac{\partial \bar{\mathbf{N}}}{\partial m_t}\right]\right], \tag{4.21}$$

where derivatives w.r.t. the mass parameter $m_t$ in the last two terms arise because of the on-shell counterterm mass insertions on the internal lines.



JHEP01(2024)074

Finally, for the cancellation of the $\mathcal{O}(\lambda)$ terms, it is necessary to express the cross section through a short-distance mass parameter. To do this, it is important to recognise that the dependence of the cross section on the top quark masses arises in two distinct ways: 1) through the *explicit* appearance of $m_t$ in the matrix elements and 2) through the *implicit* dependence of the momenta of the final state particles on $m_t$.

The explicit dependence is accounted for by writing $m_t = \tilde{m}_t + \delta m_t$ in the function $F_{\mathrm{LO}}$ and then expanding in $\delta m_t$ to first order. The corresponding change in the leading order cross section reads

$$\begin{aligned}\delta\sigma_{\mathrm{mass}}^{\mathrm{expl}} &= \delta m_t \int \mathrm{dLips}_{\mathrm{LO}} \frac{\partial F_{\mathrm{LO}}}{\partial m_t} \\ &= \delta m_t \int \mathrm{dLips}_{\mathrm{LO}} \left( \mathrm{Tr}\Big[\mathbf{1}\mathbf{N}\hat{\rho}_{\bar{t}}\bar{\mathbf{N}}\Big] + \mathrm{Tr}\Big[\hat{\rho}_t\mathbf{N}(-\mathbf{1})\bar{\mathbf{N}}\Big] + \mathrm{Tr}\left[\hat{\rho}_t\left(\frac{\partial \mathbf{N}}{\partial m_t}\hat{\rho}_{\bar{t}}\bar{\mathbf{N}} + \mathbf{N}\hat{\rho}_{\bar{t}}\frac{\partial \bar{\mathbf{N}}}{\partial m_t}\right)\right]\right).\end{aligned} \tag{4.22}$$

The change in the cross section due to the dependence of the momenta of the final-state particles on $m_t$ can be computed by redefining the momenta of the top quark and the anti-top quark as follows

$$p_t = (1-\kappa)\tilde{p}_t + \kappa\tilde{p}_{\bar{t}}, \quad p_{\bar{t}} = (1-\kappa)\tilde{p}_{\bar{t}} + \kappa\tilde{p}_t, \tag{4.23}$$

where $\kappa$ is $\mathcal{O}(\lambda)$. From this, it follows that

$$p_t^2 = m_t^2 = \tilde{p}_t^2 + 2\kappa\left(\tilde{p}_t\tilde{p}_{\bar{t}} - \tilde{m}_t^2\right) + \mathcal{O}(\kappa^2). \tag{4.24}$$

Thus, by choosing

$$\kappa = \frac{\delta m_t^2}{2\left(\tilde{p}_t\tilde{p}_{\bar{t}} - \tilde{m}_t^2\right)}, \tag{4.25}$$

the mass-shell condition for $\tilde{p}_t$ becomes

$$\tilde{p}_t^2 = \tilde{m}_t^2 = m_t^2 - \delta m_t^2. \tag{4.26}$$

Following the discussion of the momenta mapping of the real-emission contribution in section 4.1 and slightly modifying it where necessary, we obtain

$$\mathrm{dLips}\left(p_q, p_{\bar{q}}; p_t, p_{\bar{t}}, p_X; m_t^2\right) = \mathrm{dLips}\left(p_q, p_{\bar{q}}; \tilde{p}_t, \tilde{p}_{\bar{t}}, p_X; \tilde{m}_t^2\right)\left(1 - 2\kappa + \mathcal{O}\Big(\lambda^2\Big)\right). \tag{4.27}$$

Finally, expanding the leading order amplitude squared, we determine the change of the cross section due to the implicit mass change

$$\begin{aligned}\delta\sigma_{\mathrm{mass}}^{\mathrm{impl}} &= \int \mathrm{dLips}\left(\ldots, \tilde{p}_t, \tilde{p}_{\bar{t}}, \ldots\right)\left[-2\kappa - \kappa\left(\tilde{p}_t^\mu - \tilde{p}_{\bar{t}}^\mu\right)\left(\frac{\partial}{\partial\tilde{p}_t^\mu} - \frac{\partial}{\partial\tilde{p}_{\bar{t}}^\mu}\right)\right] F_{\mathrm{LO}}(\tilde{p}_t, \tilde{p}_{\bar{t}}) \\ &= \int \mathrm{dLips}\left(\ldots, p_t, p_{\bar{t}}, \ldots\right)\frac{\delta m_t^2}{2\left(\tilde{m}_t^2 - p_t p_{\bar{t}}\right)}\left[2 + \left(p_t^\mu - p_{\bar{t}}^\mu\right)\left(\frac{\partial}{\partial p_t^\mu} - \frac{\partial}{\partial p_{\bar{t}}^\mu}\right)\right] F_{\mathrm{LO}}(p_t, p_{\bar{t}}),\end{aligned} \tag{4.28}$$

where in the last step we relabelled the momenta $\tilde{p}_t$ and $\tilde{p}_{\bar{t}}$ back to $p_t$ and $p_{\bar{t}}$. While the short-distance masses can be defined in many different ways [40–45], the guiding principle is that they should not contain linear $\mathcal{O}(\Lambda_{\rm QCD})$ terms. Therefore, for our purposes, it is sufficient to write

$$m_t = \tilde{m}_t \left(1 - \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t}\right), \tag{4.29}$$

so that

$$\delta m_t = -m_t \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t}, \qquad \delta m_t^2 = -2m_t^2 \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t}. \tag{4.30}$$

Combining the different terms, we obtain the change of the cross section due to the mass shift

$$\begin{aligned} {\rm d}\sigma_{\rm LO}(m_t) - {\rm d}\sigma_{\rm LO}(\tilde{m}_t) &= \delta\sigma_{\rm mass}^{\rm expl} + \delta\sigma_{\rm mass}^{\rm impl} \\ &= \frac{\alpha_s C_F}{2\pi} \frac{\pi\lambda}{m_t} \int {\rm dLips}_{\rm LO} \Bigg[ \frac{m_t^2}{(p_t p_{\bar{t}} - m_t^2)} \left[2 + \left(p_t^\mu - p_{\bar{t}}^\mu\right)\left(D_{t,\mu} - D_{\bar{t},\mu}\right)\right] F_{\rm LO} \\ &\quad - \left[F_{\rm LO}|_{\hat{\rho}_t = m_t \mathbf{1}} - F_{\rm LO}|_{\hat{\rho}_{\bar{t}} = m_t \mathbf{1}}\right] - m_t {\rm Tr}\left[\hat{\rho}_t \left(\frac{\partial \mathbf{N}}{\partial m_t} \hat{\rho}_{\bar{t}} \bar{\mathbf{N}} + \mathbf{N} \hat{\rho}_{\bar{t}} \frac{\partial \bar{\mathbf{N}}}{\partial m_t}\right)\right]\Bigg]. \end{aligned} \tag{4.31}$$

Finally, we use eqs. (4.19), (4.20), (4.21), (4.31) to compute the various $\mathcal{O}(\lambda)$ contributions to the sum of the $tt$ and $\bar{t}\bar{t}$ monopoles and find that the result vanishes

$$\delta\sigma_{\rm mass}^{\rm expl} + \delta\sigma_{\rm mass}^{\rm impl} + \mathcal{T}_\lambda \left[{\rm d}\sigma_{\rm ren}\right] + \mathcal{T}_\lambda \left[{\rm d}\sigma_V^{tt} + {\rm d}\sigma_V^{\bar{t}\bar{t}}\right] + \mathcal{T}_\lambda \left[{\rm d}\sigma_R^{tt} + {\rm d}\sigma_R^{\bar{t}\bar{t}}\right] = 0. \tag{4.32}$$

As we explained earlier, the $\mathcal{O}(\lambda)$ contribution to $q\bar{q} \to t\bar{t} + X$ cross section can be calculated as a sum of various dipole and monopole terms. In this section we have shown that, for each of these terms, the $\mathcal{O}(\lambda)$ contribution vanishes. Hence, we conclude that within the renormalon model, there are no $\mathcal{O}(\Lambda_{\rm QCD})$ corrections to top quark pair production in hadron collisions provided that the leading partonic process is the $q\bar{q}$ annihilation channel.

### 4.3 On the validity of the LBK theorem

Recently, in refs. [35–37] objections were raised about the validity of the LBK theorem, and one may wonder whether these objections have implications for the results reported in this and earlier (e.g. [27]) papers. As discussed in [35–37], potential problems with the derivation of the LBK result stem from the need to consider the off-shell extensions of the Born amplitude, or its extensions to external momenta that do not satisfy momentum conservation. It is argued in [35–37] that such extensions may lead to ambiguities because they cannot uniquely follow from amplitudes computed for external on-shell momenta that satisfy momentum conservation. To illustrate this point, we note that if we replace $(p_t+p_{\bar{t}})^2$ either with $(p_q+p_{\bar{q}})^2$ or with $(2m_t^2 + 2p_t \cdot p_{\bar{t}})$ in the leading order amplitude $F_{\rm LO}^{ij}$, computations of derivatives of $F_{\rm LO}^{ij}$ w.r.t. $t$ or $\bar{t}$ momentum will yield different results. It is therefore important to clarify if and how the uniqueness of the result for the radiative amplitude is restored.

To understand this, it is useful to realise that the off-shell continuation problem and the momentum conservation problem have different origins and resolutions. Let us first focus on the off-shell continuation. In this case, the *final* result eq. (2.28) is *independent* of any particular off-shell extension of the amplitude squared. To see this, we note that in eq. (2.28) terms of the form $J_i^\mu J_{j,\mu} F_{\rm LO}^{ij}$ (the $JJ$ terms from now on) as well as terms of the form $J_i^\mu L_{i,\mu} F_{\rm LO}^{ij}$ (the $JL$ terms) appear. The momenta appearing in the $JJ$ terms are on shell, so they do not depend upon the off-shell continuation. The $JL$ terms could in principle be affected by the off-shell continuation, but this is not the case, since the operators $L$ yield zero when applied to the square of the external momenta,

$$\bar{L}_i^\mu p_i^2 = (J_i^\mu k^\nu \bar{D}_{i,\nu} - \bar{D}_i^\mu) p_i^2 \sim J_i^\mu k \cdot p_i - p_i^\mu = 0. \tag{4.33}$$

Thus the $L$ derivatives treat the invariants associated with the off-shell extensions of external legs as constants, so that, as far as the derivatives are concerned, working with the on-shell $F_{\rm LO}$ functions does not affect the result. Therefore, the off-shell continuation of the truncated Born amplitude is not needed.[8]

On the contrary, the leading order amplitude with external momenta that do not satisfy momentum conservation is only introduced for bookkeeping purposes at intermediate steps in our construction. In fact, we note that in addition to writing the expansion of the amplitude squared in the small gluon momentum, our computation involves a second step where we redefine momenta ($q_t \to p_t$ etc.) to ensure the momentum conservation without the need to account for the gluon momentum and then reexpand the amplitude around these conserved momenta values. We find that both the $JJ$ and $JL$ terms are affected by the momentum non-conservation issue but, once $F_{\rm LO}$ is rewritten in terms of the conserved momenta, the ambiguities must cancel out. Hence, we conclude that our final result for the real-emission contribution to the coefficient of the $\lambda$ term, given by the sum of eqs. (4.10), (4.31) and eq. (4.16) with all its variants for $\bar{t}q$, $\bar{t}\bar{q}$ and $t\bar{q}$ dipoles, is not affected by the issues with the LBK theorem pointed out in refs. [35–37].

In more detail, any possible contribution of an off-shell extension to the on-shell amplitude disappears separately in each dipole/monopole. However, the momentum non-conservation extension is more subtle since it requires adding together various dipole and monopole contributions. In a particularly simple case of $e^+e^- \to t\bar{t}$, the cancellation is quite evident, since only the $t\bar{t}$ dipole and the $tt + \bar{t}\bar{t}$ monopoles contribute, they have the same colour factor, and the derivative terms are equal and opposite, so that they cancel in the sum. The case of $q\bar{q} \to t\bar{t}$ is more involved. There we verified that the derivative terms, when acting on the combination $(p_q + p_{\bar{q}})^2 - (p_t + p_{\bar{t}})^2$, sum up to zero, thereby yielding a further check of the correctness of our procedure.

[8]This also follows from the fact that when expanding the amplitude in the off-shellness of external legs, one removes denominators of the eikonal currents and generates terms that are indistinguishable from the structure-dependent radiation amplitude. Then, the current conservation requirement expresses both the structure-dependent amplitude *and* the off-shell terms through derivatives of the *on-shell* amplitude.

## 5 Kinematic distributions

We will now study the kinematic distributions in top quark pair production processes. We consider an observable $X$ that depends exclusively on the momentum of the top quark

$$O_X = \int \mathrm{d}\sigma \; X(p_t). \tag{5.1}$$

To compute the $\mathcal{O}(\lambda)$ contribution to $O_X$, we follow the approach described in the previous sections and write

$$O_X = \int \mathrm{d}\sigma_{\mathrm{LO}} \; X(p_t) + \int \mathrm{d}\sigma_{\mathrm{NLO}} \; X(p_t). \tag{5.2}$$

We can write the NLO contribution to the cross section as the sum of dipoles and monopoles

$$\int \mathrm{d}\sigma_{\mathrm{NLO}} \; X(p_t) = \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_{\mathrm{NLO}}^{(\mathfrak{a})} \; X(p_t), \tag{5.3}$$

where $\mathfrak{a}$ denotes a particular dipole or the combination of the $tt$ and $\overline{t}\overline{t}$ monopoles. In the real-emission contribution of each dipole or monopole, we apply the appropriate momentum mapping defined in section 4.1 in order to factorise the $k$ integration in the phase space. The difference with respect to the case of the inclusive cross section is the appearance of the observable $X$ in the integrand in eq. (5.1). For the real-emission part, we therefore have that

$$\int \mathrm{d}\sigma_R(q_t,\ldots) \; X(q_t) = \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_R^{(\mathfrak{a})}(q_t,\ldots) X(p_t) + \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_R^{(\mathfrak{a})}(q_t,\ldots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(\mathfrak{a})} p_t^\mu, \tag{5.4}$$

where $\delta^{(\mathfrak{a})} p_t^\mu$ is the shift in the top quark momentum given in eq. (4.2). Since $\delta^{(\mathfrak{a})} p_t^\mu \sim \mathcal{O}(k)$, one needs, in the second term, $\mathrm{d}\sigma_R^{(\mathfrak{a})}(q_t,\ldots)$ in the leading soft approximation only. Hence, we obtain

$$\int \mathrm{d}\sigma_{\mathrm{NLO}} \; X(p_t) = \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_{\mathrm{NLO}}^{(\mathfrak{a})}(p_t,\ldots) X(p_t) + \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_R^{(\mathfrak{a})}(p_t,\ldots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(\mathfrak{a})} p_t^\mu, \tag{5.5}$$

where the first term on the right-hand side includes all the terms that contribute to the calculation of the inclusive cross section for a particular dipole and monopole except for terms that originate from the mass redefinition.

The mass redefinition terms affect both the leading order cross section as well as the observable function $X(q_t)$ that multiplies it. Redefining the mass, we obtain

$$\begin{aligned} O_X = \int \mathrm{d}\sigma_{\mathrm{LO}} \; X(p_t)|_{m_t \to \bar{m}_t} &+ \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_{\mathrm{NLO}}^{(\mathfrak{a})}(p_t,\ldots) X(p_t) + \int \mathrm{d}\sigma_{\mathrm{NLO}}^{\mathrm{mass}} \; X(p_t) \\ &+ \int \mathrm{d}\sigma_{\mathrm{LO}} \; \frac{\partial X(p_t)}{\partial p_t^\mu} \; \delta^{\mathrm{mass}} p_t^\mu + \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_R^{(\mathfrak{a})}(p_t,\ldots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(\mathfrak{a})} p_t^\mu, \end{aligned} \tag{5.6}$$

where $\mathrm{d}\sigma_{\mathrm{NLO}}^{\mathrm{mass}}$ is the change in the cross section due to the mass redefinition and $\delta^{\mathrm{mass}} p_t$ is the related shift in the top quark momentum. As was shown in the previous sections,

$$\mathcal{T}_\lambda \left[ \sum_{\mathfrak{a}} \int \mathrm{d}\sigma_{\mathrm{NLO}}^{(\mathfrak{a})}(p_t,\ldots) X(p_t) + \int \mathrm{d}\sigma_{\mathrm{NLO}}^{\mathrm{mass}} \; X(p_t) \right] = 0, \tag{5.7}$$

we conclude that

$$O_X = \bar{O}_X^{\rm LO} + \int {\rm d}\sigma_{\rm LO}\, \frac{\partial X(p_t)}{\partial p_t^\mu}\, \delta^{\rm mass} p_t^\mu + \sum_{\mathfrak{a}} \int {\rm d}\sigma_R^{(\mathfrak{a})}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(\mathfrak{a})} p_t^\mu, \tag{5.8}$$

where $\bar{O}_X^{\rm LO}$ is the observable $X$ computed at leading order with the short-distance mass.

In what follows, we will discuss the different contributions to the above equation. We combine the term that originates from the mass shift with the $tt$ and $\bar{t}\bar{t}$ monopole contributions. The general expression for $\delta^{(\mathfrak{a})} p_t^\mu$ in eq. (4.2) involves the parameters $\alpha$ and $\beta$, and we will specify our choices for them when we discuss the individual dipole and monopole contributions.

**Monopoles $tt + \bar{t}\bar{t}$.** In this case, we do not need to choose particular values for $\alpha$ and $\beta$, and we use the phase-space integrals in appendix A.1 in order to integrate over the gluon momentum $k$. We also combine the mass-redefinition contribution with those of the $tt + \bar{t}\bar{t}$ monopoles.[9] We find

$$\begin{aligned} &\mathcal{T}_\lambda\Big[\int {\rm d}\sigma_{\rm LO}\, \frac{\partial X(p_t)}{\partial p_t^\mu}\, \delta^{\rm mass} p_t^\mu + \int {\rm d}\sigma_R^{(tt+\bar{t}\bar{t})}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(tt+\bar{t}\bar{t})} p_t^\mu\Big] \\ &= -\frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t} \int {\rm d}\sigma_{\rm LO}\, \frac{\partial X(p_t)}{\partial p_t^\mu}\, p_t^\mu. \end{aligned} \tag{5.9}$$

**Dipole $t\bar{t}$.** For this dipole, we also do not need to specify the $\alpha$ and $\beta$ values. We find

$$\begin{aligned} &\mathcal{T}_\lambda\Big[\int {\rm d}\sigma_R^{(t\bar{t})}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(t\bar{t})} p_t^\mu\Big] \\ &= \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \int {\rm d}\sigma_{\rm LO}^{t\bar{t}}\, \frac{\partial X(p_t)}{\partial p_t^\mu} \left(\frac{2(p_t p_{\bar{t}})\left((p_t p_{\bar{t}})\, p_t^\mu - m_t^2\, p_{\bar{t}}^\mu\right)}{(p_t p_{\bar{t}})^2 - m_t^4}\right). \end{aligned} \tag{5.10}$$

**Dipole $tq$.** To compute the contribution of this dipole, we take a mapping with $\alpha = 1$ and $\beta = 0$. We then find

$$\mathcal{T}_\lambda\Big[\int {\rm d}\sigma_R^{(tq)}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(tq)} p_t^\mu\Big] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \int {\rm d}\sigma_{\rm LO}^{tq}\, \frac{\partial X(p_t)}{\partial p_t^\mu} \left(2p_t^\mu - \frac{2m_t^2}{(p_t p_q)} p_q^\mu\right). \tag{5.11}$$

**Remaining dipoles.** In a similar fashion, by choosing corresponding values for $\alpha$ and $\beta$, one can easily derive similar expressions for the other dipoles,

$$\mathcal{T}_\lambda\Big[\int {\rm d}\sigma_R^{(t\bar{q})}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(t\bar{q})} p_t^\mu\Big] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \int {\rm d}\sigma_{\rm LO}^{t\bar{q}}\, \frac{\partial X(p_t)}{\partial p_t^\mu} \left(-2p_t^\mu + \frac{2m_t^2}{(p_t p_{\bar{q}})} p_{\bar{q}}^\mu\right), \tag{5.12}$$

$$\mathcal{T}_\lambda\Big[\int {\rm d}\sigma_R^{(\bar{t}q)}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(\bar{t}q)} p_t^\mu\Big] = 0, \tag{5.13}$$

$$\mathcal{T}_\lambda\Big[\int {\rm d}\sigma_R^{(\bar{t}\bar{q})}(p_t,\dots) \frac{\partial X(p_t)}{\partial p_t^\mu} \delta^{(\bar{t}\bar{q})} p_t^\mu\Big] = 0. \tag{5.14}$$

[9]We remark that for an observable which depends only on $p_t$, the inclusion of $\bar{t}\bar{t}$ dipole can in principle be avoided. If one uses the alternative treatment of the self-energy contributions to $\bar{t}\bar{t}$ monopole (see section 7 in ref. [27]), this monopole does not contribute.



JHEP01(2024)074

**Linear shift in the observable distributions.** We now combine the results derived for the individual dipoles and monopoles. It is easy to see that for processes that have the same colour structure as $q\bar{q} \to t\bar{t}$, we can express the colour-correlated cross section through combinations of Casimir invariants and the leading order cross section. We then find

$$\mathrm{d}\sigma_{\mathrm{LO}}^{\mathfrak{a}} = C^{\mathfrak{a}}\, \mathrm{d}\sigma_{\mathrm{LO}}, \tag{5.15}$$

where coefficients $C^{\mathfrak{a}}$ are dipole-specific colour factors. They read

$$\begin{aligned} C^{tt} &= C^{\bar{t}\bar{t}} = C_F, & C^{t\bar{t}} &= C_F - C_A/2, \\ C^{tq} &= C^{\bar{t}\bar{q}} = 2C_F - C_A/2, & C^{\bar{t}q} &= C^{t\bar{q}} = 2C_F - C_A, \end{aligned} \tag{5.16}$$

Using these colour factors, we write the expression for the observable in the following way

$$O_X = \int \mathrm{d}\sigma_{\mathrm{LO}} \left[ X(p_t)|_{m_t \to \bar{m}_t} + \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \left( \sum_{\mathfrak{a}} C^{\mathfrak{a}}\, l_{\mathfrak{a}}^{\mu} \right) \frac{\partial X(p_t)}{\partial p_t^{\mu}} \right], \tag{5.17}$$

where the momenta $l_{\mathfrak{a}}$ can be extracted from the results derived in the previous subsection,

$$l_{\mathfrak{a}}^{\mu} = \begin{cases} -p_t^{\mu}, & \text{for } (\mathfrak{a}) = (tt + \bar{t}\bar{t}), \\ 2(p_t p_{\bar{t}}) \left( (p_t p_{\bar{t}})\, p_t^{\mu} - m_t^2\, p_{\bar{t}}^{\mu} \right) / \left( (p_t p_{\bar{t}})^2 - m_t^4 \right), & \text{for } (\mathfrak{a}) = (t\bar{t}), \\ 2p_t^{\mu} - 2m_t^2\, p_q^{\mu}/(p_t p_q), & \text{for } (\mathfrak{a}) = (tq), \\ -2p_t^{\mu} + 2m_t^2\, p_{\bar{q}}^{\mu}/(p_t p_{\bar{q}}), & \text{for } (\mathfrak{a}) = (t\bar{q}), \\ 0, & \text{for } (\mathfrak{a}) = (\bar{t}q) \text{ and } (\bar{t}\bar{q}). \end{cases} \tag{5.18}$$

It is clear that the above result can be written as the shift in the argument of the function $X$. We find

$$\mathcal{O}_X = \int \mathrm{d}\sigma_{\mathrm{LO}}\; X\!\left( p_t + \frac{\alpha_s}{2\pi} \sum_{\mathfrak{a}} C^{\mathfrak{a}}\, \delta p_{t,\mathfrak{a}} \right), \tag{5.19}$$

where

$$\delta p_{t,\mathfrak{a}} = \frac{\pi\lambda}{m_t}\, l_{\mathfrak{a}}. \tag{5.20}$$

In a similar fashion, one can derive the corresponding expressions for observables that depend on the momentum of the anti-top. We provide the complete expressions in appendix B.

## 6 Applications to simple kinematic distributions

In this section we compute the linear power corrections to three simple observables — the top quark transverse momentum, the top quark rapidity and the $t\bar{t}$ invariant mass — focusing on the process $q\bar{q} \to t\bar{t}$ with no additional colour-neutral particles in the final state. Complete formulas for other processes e.g. $q\bar{q} \to t\bar{t} + X$ and $e^+e^- \to t\bar{t} + X$ are given in appendix B.

Before we begin, we recall that, in the large $n_f$ framework, the term linear in $\lambda$ in the cross sections is related to the leading factorial growth of the coefficients of the perturbative expansion in $\alpha_s$, according to the formula reported, for example, in appendix A of ref. [46]. From that reference (following the same notation) it is also clear that the renormalon ambiguity is obtained by replacing $\alpha_s\lambda$ with a non-perturbative scale of order $\Lambda_{\mathrm{QCD}}$. For the following estimates, we will only need to know that we should replace $\alpha_s\lambda$ with a scale parameter, that we will fix according to current estimates of the renormalon ambiguity in the on-shell top mass.

JHEP01(2024)074

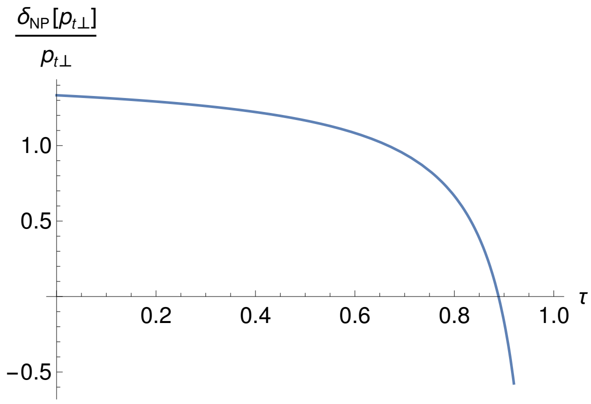


**Figure 5.** Plot of $\delta_{\rm NP}\,[p_{t\perp}]\,/p_{t\perp}$ as function of $\tau$. The global factor of $\alpha_s/(2\pi)\,\pi\lambda/m_t$ has been set to one.

The well-known expressions for the top quark transverse momentum, its rapidity in the partonic center-of-mass frame and the $t\bar{t}$ invariant mass read

$$p_{t\perp} = \sqrt{p_t^\mu g_{\perp,\mu\nu} p_t^\nu}, \qquad y_t = \frac{1}{2}\ln\frac{p_{\bar{q}}p_t}{p_q p_t}, \qquad s_{t\bar{t}} = (p_t + p_{\bar{t}})^2, \tag{6.1}$$

where

$$g_\perp^{\mu\nu} = \frac{p_q^\mu p_{\bar{q}}^\nu + p_{\bar{q}}^\mu p_q^\nu}{p_q p_{\bar{q}}} - g^{\mu\nu}. \tag{6.2}$$

Applying the formalism of section 5 and defining $\tau = 4m_t^2/s_{t\bar{t}}$, we find

$$\frac{\delta_{\rm NP}\,[p_{t\perp}]}{p_{t\perp}} = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\,\frac{(2C_F - C_A\tau)}{2(1-\tau)}, \tag{6.3}$$

$$\delta_{\rm NP}\,[y_t] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\,\left[(3C_A - 8C_F)\,\tau\cosh^2 y_t - (C_A - 2C_F)\,\frac{\tau(2-\tau)}{4(1-\tau)}\sinh{(2y_t)}\right], \tag{6.4}$$

$$\frac{\delta_{\rm NP}\,[s_{t\bar{t}}]}{s_{t\bar{t}}} = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\,\Big[2C_F\,(1-\tau) - C_A\,\tau\cosh{(2y_t)} + (3C_A - 8C_F)\,\tau\sinh{(2y_t)}\Big]. \tag{6.5}$$

Interestingly, these shifts exhibit non-trivial dependencies on the QCD colour factors and on the kinematics of the underlying $q\bar{q}\to t\bar{t}$ process.[10] To visualise them, we display the shifts in figures 5–7. We observe that the transverse momentum shift is large and negative around the partonic threshold and that the sign is driven by the non-Abelian Casimir $C_A$. The transverse momentum shift changes the sign at

$$\sqrt{s} = 2m_t\sqrt{\frac{C_A}{2C_F}}, \tag{6.6}$$

which, numerically, is $\mathcal{O}(20)$ GeV above the $t\bar{t}$ threshold. At larger invariant masses, the non-perturbative shift is dominated by the "Abelian" contribution proportional to $C_F$.

[10] We note that very close to the threshold, fixed-order perturbative computations break down. This means that our results in this region should be interpreted with care.

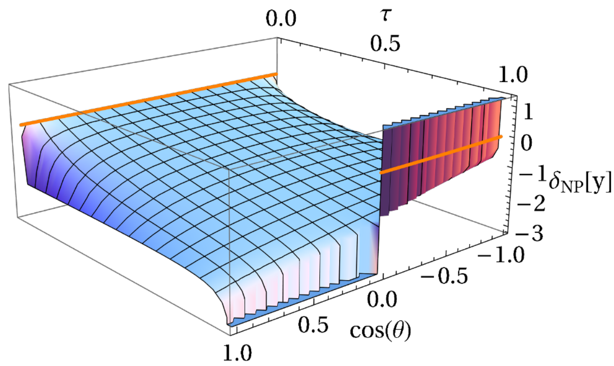


**Figure 6.** Plot of $\delta_{\rm NP}\left[y_t\right]$ as function of $\tau$ and $\cos\theta$ (see text for details). The global factor of $\alpha_s/(2\pi)\,\pi\lambda/m_t$ has been set to one. The orange lines indicate the intersection with the plane of vanishing shift.

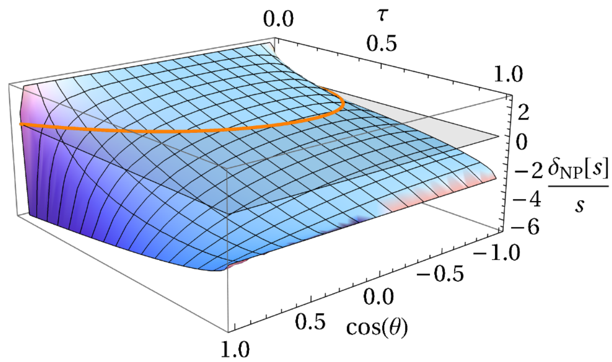


**Figure 7.** Plot of $\delta_{\rm NP}\left[s_{t\bar{t}}\right]/s_{t\bar{t}}$ shift as function of $\tau$ and $\cos\theta$ (see text for details). The global factor of $\alpha_s/(2\pi)\,\pi\lambda/m_t$ has been set to one. In the plot, we have added a transparent plane of vanishing shift. The orange lines indicate the intersection with the plane of vanishing shift.

The shifts in $y_t$ and $s_{t\bar{t}}$ depend on both the invariant mass of the $t\bar{t}$ pair and the rapidity of the top quark. Since we work in the partonic center-of-mass frame, it is convenient to express the rapidity of the top quark through the scattering angle $\theta$ of $t$ relative to $q$ using

$$y_t = \frac{1}{2}\log\left(\frac{1+\sqrt{1-\tau}\cos\theta}{1-\sqrt{1-\tau}\cos\theta}\right). \tag{6.7}$$

Hence, to visualise the shifts in $y_t$ and $s_{t\bar{t}}$, we use two-dimensional plots in $\tau$ and $\cos\theta$, see figures 6–7.

A peculiar feature of these shifts is that they induce forward-backward asymmetry in $t\bar{t}$ production. This is obvious from the presence of $\sinh(2y_t)$ terms in eqs. (6.4), (6.5). Moreover, these $y_t$-odd shifts are again enhanced in the threshold region. To see this, we expand eq. (6.4) around threshold, $\tau = 1$, and find

$$\lim_{\tau\to 1}\delta_{\rm NP}[y_t] = -\frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t}\frac{(C_A - 2C_F)}{2(1-\tau)}\,y_t. \tag{6.8}$$

Comparing this shift with the shift of $p_{t\perp}$ in the threshold region, we observe that the relative shifts are, in fact, identical and determined by the same colour factors involving



JHEP01(2024)074

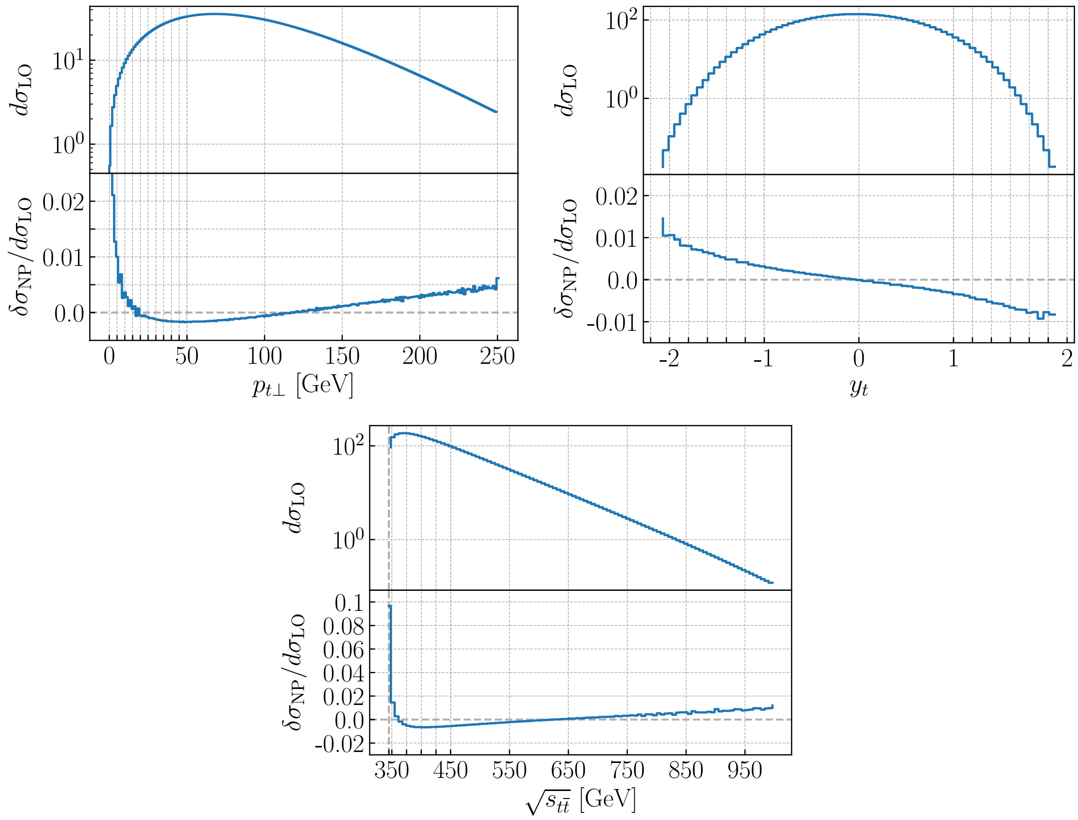


**Figure 8.** Non-perturbative shifts in top quark transverse momentum, lab-frame rapidity and $t\bar{t}$ invariant mass distributions at the Tevatron for the $q\bar{q} \to t\bar{t}$ process. The center-of-mass energy is set to $\sqrt{s} = 1.8$ TeV. The upper pane shows the leading order distribution. The lower pane shows the ratio $\delta\sigma_{\rm NP}/{\rm d}\sigma_{\rm LO} = [{\rm d}\sigma_{\rm LO}(v+\delta v_{\rm NP}) - {\rm d}\sigma_{\rm LO}(v)]/{\rm d}\sigma_{\rm LO}(v)$ for an observable $v$ affected by a non-perturbative shift $\delta v_{\rm NP}$. See text for details.

both $C_F$ and $C_A$,

$$\lim_{\tau\to 1} \frac{\delta_{\rm NP}[y_t]}{y_t} = \lim_{\tau\to 1} \frac{\delta_{\rm NP}\left[p_{t\perp}\right]}{p_{t\perp}}. \tag{6.9}$$

In contrast to this, the relative shift for the $t\bar{t}$ invariant mass in the threshold region is constant and involves only the non-Abelian colour factor,

$$\lim_{\tau\to 1} \frac{\delta_{\rm NP}\left[s_{t\bar{t}}\right]}{s_{t\bar{t}}} = -\frac{\alpha_s C_A}{2\pi}\frac{\pi\lambda}{m_t}. \tag{6.10}$$

In the opposite $\tau = 0$ limit which correspond to the high-energy regime, we note that, while the shift in $y_t$ vanishes, the relative shifts of $p_{t\perp}$ and $s_{t\bar{t}}$ are purely "Abelian" and can be related to the shift in the mass redefinition as follows

$$\frac{\delta_{\rm NP}\left[m_t\right]}{m_t} = \lim_{\tau\to 0} \frac{\delta_{\rm NP}\left[p_{t\perp}\right]}{p_{t\perp}} = \frac{1}{2}\lim_{\tau\to 0} \frac{\delta_{\rm NP}\left[s_{t\bar{t}}\right]}{s_{t\bar{t}}} = \frac{\alpha_s C_F}{2\pi}\frac{\pi\lambda}{m_t}. \tag{6.11}$$

We have also computed the non-perturbative shifts for basic top-quark kinematic distributions in the $p\bar{p} \to t\bar{t}$ process at the Tevatron; the results are shown in figure 8. To



JHEP01(2024)074

assign a numerical value to the product of $\alpha_s$ and the gluon mass $\lambda$, we assume that the non-perturbative shift in the value of the top quark pole mass is 200 MeV [47–49]. Then, using eq. (4.29) we obtain

$$\alpha_s \lambda = \frac{0.4 \text{ GeV}}{C_F} = 0.3 \text{ GeV}. \tag{6.12}$$

Furthermore, we employ the central value of the NNPDF31_lo_as_0118 parton distribution function [50], take $m_t = 172.5$ GeV and set the factorisation and the renormalisation scales to $\mu_F = \mu_R = m_t$.[11]

We observe (cf. figure 8) that non-perturbative corrections in $p_{t\perp}$ and $s_{t\bar{t}}$ distributions can be significant in the corresponding threshold regions. Although in $p_{t\perp}$ distribution large effects are confined to a region which ends about 5 GeV above the $p_{t\perp}$-threshold, for the $t\bar{t}$ invariant mass distribution $\mathcal{O}(1\%)$ effects appear in a broader interval of the invariant masses that extends to about 450 GeV. Non-perturbative corrections to the rapidity distribution are small at central rapidities but become larger at $|y_t| > 1.5$ where the leading order rapidity distribution starts to decrease rapidly.

## 7 Conclusions

In this paper we computed linear non-perturbative $\mathcal{O}(\Lambda_{\text{QCD}})$ corrections to top quark pair production in hadron collisions under the assumption that $q\bar{q} \to t\bar{t}$ is the dominant partonic channel. Our starting point is the renormalon model. Traditionally, the renormalon calculus is used to compute linear power corrections to processes without gluons at the tree level, which is clearly not the case for the $t\bar{t}$ production in hadron collisions. However, we have argued that, for *quark initiated* partonic processes, i.e. for $q\bar{q} \to t\bar{t}$, the renormalon calculus is still applicable, because of the large virtuality of the gluon in the Born diagram.

We have shown how to compute the linear power corrections efficiently using a generalisation of the Low-Burnett-Kroll theorem to processes with colour charges. In this case, the first subleading soft corrections can be written in terms of colour-correlated matrix elements, in a form that exhibits the dipole structure typical of soft radiation. We have further shown that, for inclusive total cross sections expressed through a short-distance top quark mass, the $\mathcal{O}(\Lambda_{\text{QCD}})$ contributions vanish and, if a proper mapping of momenta is chosen, this occurs separately for each of the colour dipoles.

Finally, we studied the non-perturbative corrections to kinematic distributions that depend on the momenta of the top and anti-top quarks. Our formalism allows us to compute them in a straightforward manner. Although these are not particularly large numerically, they exhibit interesting dependencies on the kinematics of the Born process and on the QCD colour factors. For example, the relative correction to the transverse momentum distribution of the top, $p_{t\perp}$, is large and negative close to the $t\bar{t}$ threshold, where the sign is driven by the non-Abelian colour factor $C_A$. However, the sign changes at $\sqrt{s_{t\bar{t}}} = 2m_t\sqrt{C_A/(2C_F)}$ which is about 20 GeV above the $t\bar{t}$ threshold. Furthermore, the $\mathcal{O}(\Lambda_{\text{QCD}})$ corrections to the top quark

[11]The numerical value of the top quark mass is chosen for the illustration purposes only. In principle, as we mentioned several times in the text, we must use a short-distance top quark mass to ensure that $\mathcal{O}(\Lambda_{\text{QCD}})$ corrections to the total cross section vanish.

rapidity distribution induce forward-backward asymmetry, which is particularly enhanced in the threshold region. Hence, it appears from our analysis that even for a relatively simple $2 \to 2$ process that we consider here, the renormalon model predicts interesting kinematic dependencies of non-perturbative power suppressed effects that relate to such fundamental properties of QCD as gluon self-interactions.

As the last observation, we notice that both in the single top production case discussed in ref. [27] and in the present case, no linear power corrections are present in the inclusive total cross section if one uses a short-distance mass scheme. Although the full analysis of hadronic $t\bar{t}$ production that incorporates the $gg$ partonic channel remains an outstanding task, these persistent cancellations hint at the possibility that this property holds in general. Assuming that this is the case, this would imply that short-distance mass schemes (e.g. the $\overline{\text{MS}}$ scheme [40], or schemes of refs. [41–45]) are preferable for computing the total cross section and, if a heavy quark mass parameter is extracted from the cross-section measurement, the quoted result should be in a chosen short-distance mass scheme.


## Acknowledgments

The research of K.M. and S.M. was supported by the German Research Foundation (DFG, Deutsche Forschungsgemeinschaft) under grant 396021762-TRR 257. P. N. acknowledges the support of the Humboldt foundation.


## A Loop and real-emission integrals required for computing linear power corrections

In this appendix we give the results for the phase-space and loop integrals that occur in the real emission and virtual contributions respectively. In order to present the results in a compact form, we make use of the variable

$$\delta = \frac{1}{(2\pi)^2} \frac{\lambda \pi}{m_t}. \tag{A.1}$$

### A.1 Real emission integrals

The phase-space integrals required for computing the real-emission contribution to top quark pair production read[12]

$$I_1 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta_+(k^2 - \lambda^2) \, \frac{1}{(2p_t k)} \right] = -\frac{\delta}{4}, \tag{A.2}$$

$$I_2 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta_+(k^2 - \lambda^2) \, \frac{k^\mu}{(2p_t k)^2} \right] = -\frac{\delta}{8} \frac{1}{m_t^2} \, p_t^\mu, \tag{A.3}$$

$$I_3 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta_+(k^2 - \lambda^2) \, \frac{k^\mu k^\nu}{(2p_t k)^3} \right] = \frac{\delta}{32} \frac{1}{m_t^2} \left( g^{\mu\nu} - \frac{3}{m_t^2} \, p_t^\mu p_t^\nu \right), \tag{A.4}$$

$$I_4 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta_+(k^2 - \lambda^2) \, \frac{k^\mu}{(2p_t k)(2p_{\bar{t}} k)} \right] = -\frac{\delta}{8} \, \frac{1}{(p_t p_{\bar{t}}) + m_t^2} \left( p_t^\mu + p_{\bar{t}}^\mu \right), \tag{A.5}$$

[12]We only display $\mathcal{O}(\lambda)$ contributions to these integrals.

$$I_5 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4k}{(2\pi)^3} \delta_+(k^2-\lambda^2) \, \frac{\lambda^2}{(2p_t k)^2 (2p_{\bar{t}} k)} \right] = \frac{\delta}{16} \, \frac{1}{(p_t p_{\bar{t}}) + m_t^2}. \tag{A.6}$$

$$I_6 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4k}{(2\pi)^3} \delta_+(k^2-\lambda^2) \, \frac{k^\mu}{(2p_t k)(-2p_q k)} \right] = \frac{\delta}{8} \frac{1}{(p_t p_q)} \left( p_t^\mu - \frac{m_t^2}{(p_t p_q)} \, p_q^\mu \right). \tag{A.7}$$

$$I_7 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4k}{(2\pi)^3} \delta_+(k^2-\lambda^2) \, \frac{\lambda^2}{(2p_t k)^2 (-2p_q k)} \right] = -\frac{\delta}{16} \frac{1}{(p_t p_q)}, \tag{A.8}$$

$$I_8 = \mathcal{T}_\lambda \left[ \int \frac{\mathrm{d}^4k}{(2\pi)^3} \delta_+(k^2-\lambda^2) \, \frac{\lambda^2}{(2p_t k)(-2p_q k)^2} \right] = -\frac{\delta}{16} \frac{m_t^2}{(p_t p_q)^2}. \tag{A.9}$$

### A.2 Loop integrals

The required loop integrals read

$$V_1 = \mathcal{T}_\lambda \left[ -i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)} \frac{1}{(2p_t k)} \right] = -\frac{\delta}{4}, \tag{A.10}$$

$$V_2 = \mathcal{T}_\lambda \left[ -i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)} \frac{k^\mu}{(2p_t k)\,(-2p_{\bar{t}} k)} \right] = \frac{\delta}{8} \frac{1}{(p_t p_{\bar{t}}) - m_t^2} \left( p_t^\mu - p_{\bar{t}}^\mu \right), \tag{A.11}$$

$$V_3 = \mathcal{T}_\lambda \left[ -i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)} \frac{\lambda^2}{(2p_t k)\,(-2p_{\bar{t}} k)^2} \right] = -\frac{\delta}{16} \frac{1}{(p_t p_{\bar{t}}) - m_t^2}. \tag{A.12}$$

$$V_4 = \mathcal{T}_\lambda \left[ -i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)} \frac{k^\mu}{(2p_t k)(2p_q k)} \right] = -\frac{\delta}{8} \frac{1}{(p_t p_q)} \left( p_t^\mu - \frac{m_t^2}{(p_t p_q)} \, p_q^\mu \right). \tag{A.13}$$

$$V_5 = \mathcal{T}_\lambda \left[ -i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)} \frac{\lambda^2}{(2p_t k)^2 (2p_q k)} \right] = \frac{\delta}{16} \frac{1}{(p_t p_q)}, \tag{A.14}$$

$$V_6 = \mathcal{T}_\lambda \left[ -i \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)} \frac{\lambda^2}{(2p_t k)(2p_q k)^2} \right] = -\frac{\delta}{16} \frac{m_t^2}{(p_t p_q)^2}. \tag{A.15}$$

## B Observables for a more general process of $q\bar{q} \to t\bar{t} + X$

In this appendix we give the relevant expressions for observables in a more general setting $q\bar{q} \to t\bar{t} + X$ and also briefly discuss the case for $e^+e^- \to t\bar{t} + X$. In the presence of $X$, the symmetry between $t$ and $\bar{t}$ breaks down and, hence, we need to consider this case explicitly.

For observables depending on the $\bar{t}$ momentum, we acquire the shifts

$$\bar{l}_{\mathfrak{a}}^\mu = \begin{cases} -p_{\bar{t}}^\mu, & \text{for } (\mathfrak{a}) = (tt + \overline{tt}), \\ 2(p_t p_{\bar{t}}) \left( (p_t p_{\bar{t}})\, p_{\bar{t}}^\mu - m_t^2 \, p_t^\mu \right) / \left( (p_t p_{\bar{t}})^2 - m_t^4 \right), & \text{for } (\mathfrak{a}) = (t\bar{t}), \\ -2p_{\bar{t}}^\mu + 2m_t^2 \, p_q^\mu / (p_{\bar{t}} p_q), & \text{for } (\mathfrak{a}) = (\bar{t}q), \\ 2p_{\bar{t}}^\mu - 2m_t^2 \, p_{\bar{q}}^\mu / (p_{\bar{t}} p_{\bar{q}}), & \text{for } (\mathfrak{a}) = (\bar{t}\bar{q}), \\ 0, & \text{for } (\mathfrak{a}) = (tq) \text{ and } (t\bar{q}). \end{cases} \tag{B.1}$$

Using this, for general observables that depend on both momenta $p_t$, $p_{\bar{t}}$ and the top mass

$m_t$, the linear shift reads

$$\mathcal{T}_\lambda[O_X] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \int \mathrm{d}\sigma_{\mathrm{LO}} \left[ \left(\sum_{\mathfrak{a}} C^{\mathfrak{a}}\, l^\mu_{\mathfrak{a}}\right) \frac{\partial X(p_t,p_{\bar{t}},m_t^2)}{\partial p_t^\mu} + \left(\sum_{\mathfrak{a}} C^{\mathfrak{a}}\, \bar{l}^\mu_{\mathfrak{a}}\right) \frac{\partial X(p_t,p_{\bar{t}},m_t^2)}{\partial p_{\bar{t}}^\mu} - m_t \frac{\partial X(p_t,p_{\bar{t}},m_t^2)}{\partial m_t}\right]. \tag{B.2}$$

In the following, we consider the same observables as in section 6 for $q\bar{q} \to t\bar{t} + X$, where $s = (p_q + p_{\bar{q}})^2 \neq s_{t\bar{t}} = (p_t + p_{\bar{t}})^2$. We will give the split-down for the different monopole and dipole contributions and only explicitly insert the colour coefficients for the monopoles, the remaining coefficients can be extracted from eq. (5.16). The definitions for $p_{\bar{t}\perp}$ and $y_{\bar{t}}$ follow the equivalent definitions for $p_{t\perp}$ and $y_t$ in eq. (6.1) respectively, but with $p_t$ replaced by $p_{\bar{t}}$ instead.

The shift in the transverse momenta for the top and anti-top read,

$$\frac{\delta_{\mathrm{NP}}\left[p_{t\perp}\right]}{p_{t\perp}} = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \Bigg[ C_F\,(-1) + C^{tq}\,(2) + C^{t\bar{q}}\,(-2) + C^{t\bar{t}} \left( \frac{8(p_tp_{\bar{t}})\left(m_t^2(p_qp_{\bar{t}})(p_{\bar{q}}p_t) + m_t^2(p_{\bar{q}}p_{\bar{t}})(p_qp_t) - 2(p_qp_t)(p_{\bar{q}}p_t)(p_tp_{\bar{t}})\right)}{\left(s_{t\bar{t}} - 4m_t^2\right)s_{t\bar{t}}\left(m_t^2(p_qp_{\bar{q}}) - 2(p_qp_t)(p_{\bar{q}}p_t)\right)} \right) \Bigg], \tag{B.3}$$

$$\frac{\delta_{\mathrm{NP}}\left[p_{\bar{t}\perp}\right]}{p_{\bar{t}\perp}} = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \Bigg[ C_F\,(-1) + C^{\bar{t}\bar{q}}\,(2) + C^{\bar{t}q}\,(-2) + C^{t\bar{t}} \left( \frac{8(p_tp_{\bar{t}})\left(m_t^2(p_qp_{\bar{t}})(p_{\bar{q}}p_t) + m_t^2(p_{\bar{q}}p_{\bar{t}})(p_qp_t) - 2(p_qp_{\bar{t}})(p_{\bar{q}}p_{\bar{t}})(p_tp_{\bar{t}})\right)}{\left(s_{t\bar{t}} - 4m_t^2\right)s_{t\bar{t}}\left(m_t^2(p_qp_{\bar{q}}) - 2(p_qp_{\bar{t}})(p_{\bar{q}}p_{\bar{t}})\right)} \right) \Bigg]. \tag{B.4}$$

For the rapidity of the top and anti-top, we have that

$$\delta_{\mathrm{NP}}\left[y_t\right] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \Bigg[ C^{t\bar{t}} \left( \frac{4m_t^2(p_tp_{\bar{t}})}{\left(s_{t\bar{t}} - 4m_t^2\right)s_{t\bar{t}}} \left( \frac{p_qp_{\bar{t}}}{p_qp_t} - \frac{p_{\bar{q}}p_{\bar{t}}}{p_{\bar{q}}p_t} \right) \right) + C^{tq} \left( -\frac{m_t^2\,(p_qp_{\bar{q}})}{(p_qp_t)(p_{\bar{q}}p_t)} \right) + C^{t\bar{q}} \left( -\frac{m_t^2\,(p_qp_{\bar{q}})}{(p_qp_t)(p_{\bar{q}}p_t)} \right) \Bigg], \tag{B.5}$$

$$\delta_{\mathrm{NP}}\left[y_{\bar{t}}\right] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \Bigg[ C^{t\bar{t}} \left( \frac{4m_t^2(p_tp_{\bar{t}})}{\left(s_{t\bar{t}} - 4m_t^2\right)s_{t\bar{t}}} \left( \frac{p_qp_t}{p_qp_{\bar{t}}} - \frac{p_{\bar{q}}p_t}{p_{\bar{q}}p_{\bar{t}}} \right) \right) + C^{\bar{t}q} \left( \frac{m_t^2\,(p_qp_{\bar{q}})}{(p_qp_{\bar{t}})(p_{\bar{q}}p_{\bar{t}})} \right) + C^{\bar{t}\bar{q}} \left( \frac{m_t^2\,(p_qp_{\bar{q}})}{(p_qp_{\bar{t}})(p_{\bar{q}}p_{\bar{t}})} \right) \Bigg]. \tag{B.6}$$

The shift in the invariant mass of the $t\bar{t}$ pair reads,

$$\delta_{\mathrm{NP}}\left[s_{t\bar{t}}\right] = \frac{\alpha_s}{2\pi}\frac{\pi\lambda}{m_t} \Bigg[ C_F\left(-2s_{t\bar{t}}\right) + C^{t\bar{t}}\left(4s_{t\bar{t}} - 8m_t^2\right) + C^{tq}\left(4p_tp_{\bar{t}} - 4m_t^2\frac{p_{\bar{t}}p_q}{p_tp_q}\right) + C^{t\bar{q}}\left(-4p_tp_{\bar{t}} + 4m_t^2\frac{p_{\bar{t}}p_{\bar{q}}}{p_tp_{\bar{q}}}\right) + C^{\bar{t}q}\left(-4p_tp_{\bar{t}} + 4m_t^2\frac{p_tp_q}{p_{\bar{t}}p_q}\right) + C^{\bar{t}\bar{q}}\left(4p_tp_{\bar{t}} - 4m_t^2\frac{p_tp_{\bar{q}}}{p_{\bar{t}}p_{\bar{q}}}\right) \Bigg]. \tag{B.7}$$

For processes of type $e^+e^- \to t\bar{t} + X$ and the same observables, we can make use of the same expressions above but now need to adjust only for the colour coefficients as

$$C^{tt} = C^{\bar{t}\bar{t}} = C^{t\bar{t}} = C_F, \qquad C^{tq} = C^{\bar{t}\bar{q}} = C^{\bar{t}q} = C^{t\bar{q}} = 0. \tag{B.8}$$



## References


[1] ATLAS collaboration, *Observation of Higgs boson production in association with a top quark pair at the LHC with the ATLAS detector*, *Phys. Lett. B* **784** (2018) 173 [arXiv:1806.00425] [INSPIRE].

[2] ATLAS collaboration, *CP Properties of Higgs Boson Interactions with Top Quarks in the $t\bar{t}H$ and $tH$ Processes Using $H \to \gamma\gamma$ with the ATLAS Detector*, *Phys. Rev. Lett.* **125** (2020) 061802 [arXiv:2004.04545] [INSPIRE].

[3] J. Mazzitelli et al., *Top-pair production at the LHC with MINNLO$_{PS}$*, *JHEP* **04** (2022) 079 [arXiv:2112.12135] [INSPIRE].

[4] ATLAS collaboration, *Measurement of the $t\bar{t}$ production cross-section in the lepton+jets channel at $\sqrt{s} = 13$ TeV with the ATLAS experiment*, *Phys. Lett. B* **810** (2020) 135797 [arXiv:2006.13076] [INSPIRE].

[5] CMS collaboration, *Measurements of $\mathrm{t\bar{t}}H$ Production and the CP Structure of the Yukawa Interaction between the Higgs Boson and Top Quark in the Diphoton Decay Channel*, *Phys. Rev. Lett.* **125** (2020) 061801 [arXiv:2003.10866] [INSPIRE].

[6] M. Czakon, P. Fiedler and A. Mitov, *Total Top-Quark Pair-Production Cross Section at Hadron Colliders Through $O(\alpha_S^4)$*, *Phys. Rev. Lett.* **110** (2013) 252004 [arXiv:1303.6254] [INSPIRE].

[7] M. Czakon, P. Fiedler, D. Heymes and A. Mitov, *NNLO QCD predictions for fully-differential top-quark pair production at the Tevatron*, *JHEP* **05** (2016) 034 [arXiv:1601.05375] [INSPIRE].

[8] S. Catani et al., *Top-quark pair hadroproduction at NNLO: differential predictions with the $\overline{MS}$ mass*, *JHEP* **08** (2020) 027 [arXiv:2005.00557] [INSPIRE].

[9] A. Kulesza et al., *Associated top quark pair production with a heavy boson: differential cross sections at NLO+NNLL accuracy*, *Eur. Phys. J. C* **80** (2020) 428 [arXiv:2001.03031] [INSPIRE].

[10] N. Kidonakis and A. Tonero, *Higher-order corrections in $t\bar{t}\gamma$ cross sections*, *Phys. Rev. D* **107** (2023) 034013 [arXiv:2212.00096] [INSPIRE].

[11] G. Bevilacqua, M. Lupattelli, D. Stremmer and M. Worek, *Study of additional jet activity in top quark pair production and decay at the LHC*, *Phys. Rev. D* **107** (2023) 114027 [arXiv:2212.04722] [INSPIRE].

[12] A. Denner, D. Lombardi and G. Pelliccioli, *Complete NLO corrections to off-shell $t\bar{t}Z$ production at the LHC*, *JHEP* **09** (2023) 072 [arXiv:2306.13535] [INSPIRE].

[13] ATLAS collaboration, *Measurement of the $t\bar{t}$ production cross-section using $e\mu$ events with b-tagged jets in pp collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS detector*, *Eur. Phys. J. C* **74** (2014) 3109 [*Addendum ibid.* **76** (2016) 642] [arXiv:1406.5375] [INSPIRE].

[14] CMS collaboration, *Measurement of the t-tbar production cross section in the e-mu channel in proton-proton collisions at $\sqrt{s} = 7$ and 8 TeV*, *JHEP* **08** (2016) 029 [arXiv:1603.02303] [INSPIRE].

[15] CMS collaboration, *Measurement of the* $\mathrm{t\bar{t}}$ *production cross section, the top quark mass, and the strong coupling constant using dilepton events in pp collisions at* $\sqrt{s} = 13$ *TeV*, *Eur. Phys. J. C* **79** (2019) 368 [arXiv:1812.10505] [INSPIRE].

[16] ATLAS collaboration, *Measurement of the* $t\bar{t}$ *production cross-section and lepton differential distributions in* $e\mu$ *dilepton events from pp collisions at* $\sqrt{s} = 13$ *TeV with the ATLAS detector*, *Eur. Phys. J. C* **80** (2020) 528 [arXiv:1910.08819] [INSPIRE].

[17] ATLAS collaboration, *Measurement of lepton differential distributions and the top quark mass in* $t\bar{t}$ *production in pp collisions at* $\sqrt{s} = 8$ *TeV with the ATLAS detector*, *Eur. Phys. J. C* **77** (2017) 804 [arXiv:1709.09407] [INSPIRE].

[18] CMS collaboration, *Measurement of* $\mathrm{t\bar{t}}$ *normalised multi-differential cross sections in pp collisions at* $\sqrt{s} = 13$ *TeV, and simultaneous determination of the strong coupling strength, top quark pole mass, and parton distribution functions*, *Eur. Phys. J. C* **80** (2020) 658 [arXiv:1904.05237] [INSPIRE].

[19] CMS collaboration, *Determination of the Top-Quark Pole Mass and Strong Coupling Constant from the* $t\bar{t}$ *Production Cross Section in pp Collisions at* $\sqrt{s} = 7$ *TeV*, *Phys. Lett. B* **728** (2014) 496 [arXiv:1307.1907] [INSPIRE].

[20] A. Juste et al., *Determination of the top quark mass circa 2013: methods, subtleties, perspectives*, *Eur. Phys. J. C* **74** (2014) 3119 [arXiv:1310.0799] [INSPIRE].

[21] A.H. Hoang, *The Top Mass: Interpretation and Theoretical Uncertainties*, in the proceedings of the *7th International Workshop on Top Quark Physics*, Cannes, France (2014) [arXiv:1412.3649] [INSPIRE].

[22] P. Nason, *The Top Mass in Hadronic Collisions*, in *From My Vast Repertoire …: Guido Altarelli's Legacy*, A. Levy, S. Forte and G. Ridolfi eds., World Scientific (2019), pg. 123 [DOI:10.1142/9789813238053_0008] [arXiv:1712.02796] [INSPIRE].

[23] P. Azzi et al., *Report from Working Group 1: Standard Model Physics at the HL-LHC and HE-LHC*, *CERN Yellow Rep. Monogr.* **7** (2019) 1 [arXiv:1902.04070] [INSPIRE].

[24] M. Beneke, *Renormalons*, *Phys. Rept.* **317** (1999) 1 [hep-ph/9807443] [INSPIRE].

[25] S. Ferrario Ravasio, P. Nason and C. Oleari, *All-orders behaviour and renormalons in top-mass observables*, *JHEP* **01** (2019) 203 [arXiv:1810.10931] [INSPIRE].

[26] F. Caola et al., *On linear power corrections in certain collider observables*, *JHEP* **01** (2022) 093 [arXiv:2108.08897] [INSPIRE].

[27] S. Makarov, K. Melnikov, P. Nason and M.A. Ozcelik, *Linear power corrections to single top production processes at the LHC*, *JHEP* **05** (2023) 153 [arXiv:2302.02729] [INSPIRE].

[28] B. Dehnadi, A.H. Hoang, O.L. Jin and V. Mateu, *Top quark mass calibration for Monte Carlo event generators — an update*, *JHEP* **12** (2023) 065 [arXiv:2309.00547] [INSPIRE].

[29] B. Bachu et al., *Boosted top quarks in the peak region with NL3L resummation*, *Phys. Rev. D* **104** (2021) 014026 [arXiv:2012.12304] [INSPIRE].

[30] A.H. Hoang, S. Mantry, A. Pathak and I.W. Stewart, *Extracting a Short Distance Top Mass with Light Grooming*, *Phys. Rev. D* **100** (2019) 074021 [arXiv:1708.02586] [INSPIRE].


JHEP01(2024)074

[31] A.H. Hoang, S. Mantry, A. Pathak and I.W. Stewart, *Nonperturbative Corrections to Soft Drop Jet Mass*, *JHEP* **12** (2019) 002 [arXiv:1906.11843] [INSPIRE].

[32] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974 [INSPIRE].

[33] T.H. Burnett and N.M. Kroll, *Extension of the low soft photon theorem*, *Phys. Rev. Lett.* **20** (1968) 86 [INSPIRE].

[34] T. Engel, A. Signer and Y. Ulrich, *Universal structure of radiative QED amplitudes at one loop*, *JHEP* **04** (2022) 097 [arXiv:2112.07570] [INSPIRE].

[35] P. Lebiedowicz, O. Nachtmann and A. Szczurek, *High-energy $\pi\pi$ scattering without and with photon radiation*, *Phys. Rev. D* **105** (2022) 014022 [arXiv:2107.10829] [INSPIRE].

[36] P. Lebiedowicz, O. Nachtmann and A. Szczurek, *Soft-photon theorem for pion-proton elastic scattering revisited*, arXiv:2307.12673 [INSPIRE].

[37] P. Lebiedowicz, O. Nachtmann and A. Szczurek, *Soft-Photon Theorem for Pion-Proton Scattering: Next to Leading Term*, arXiv:2307.13291 [INSPIRE].

[38] M. Peter, *The Static quark-anti-quark potential in QCD to three loops*, *Phys. Rev. Lett.* **78** (1997) 602 [hep-ph/9610209] [INSPIRE].

[39] Y. Schroder, *The Static potential in QCD to two loops*, *Phys. Lett. B* **447** (1999) 321 [hep-ph/9812205] [INSPIRE].

[40] G. 't Hooft, *Dimensional regularization and the renormalization group*, *Nucl. Phys. B* **61** (1973) 455 [INSPIRE].

[41] A. Czarnecki, K. Melnikov and N. Uraltsev, *NonAbelian dipole radiation and the heavy quark expansion*, *Phys. Rev. Lett.* **80** (1998) 3189 [hep-ph/9708372] [INSPIRE].

[42] M. Beneke, *A Quark mass definition adequate for threshold problems*, *Phys. Lett. B* **434** (1998) 115 [hep-ph/9804241] [INSPIRE].

[43] A.H. Hoang, *1S and MS-bar bottom quark masses from Upsilon sum rules*, *Phys. Rev. D* **61** (2000) 034005 [hep-ph/9905550] [INSPIRE].

[44] A. Pineda, *Determination of the bottom quark mass from the Upsilon(1S) system*, *JHEP* **06** (2001) 022 [hep-ph/0105008] [INSPIRE].

[45] A.H. Hoang, A. Jain, I. Scimemi and I.W. Stewart, *Infrared Renormalization Group Flow for Heavy Quark Masses*, *Phys. Rev. Lett.* **101** (2008) 151602 [arXiv:0803.4214] [INSPIRE].

[46] S. Ferrario Ravasio, G. Limatola and P. Nason, *Infrared renormalons in kinematic distributions for hadron collider processes*, *JHEP* **06** (2021) 018 [arXiv:2011.14114] [INSPIRE].

[47] M. Beneke, P. Marquard, P. Nason and M. Steinhauser, *On the ultimate uncertainty of the top quark pole mass*, *Phys. Lett. B* **775** (2017) 63 [arXiv:1605.03609] [INSPIRE].

[48] A.H. Hoang, C. Lepenik and M. Preisser, *On the Light Massive Flavor Dependence of the Large Order Asymptotic Behavior and the Ambiguity of the Pole Mass*, *JHEP* **09** (2017) 099 [arXiv:1706.08526] [INSPIRE].

[49] K. Agashe et al., *Report of the Topical Group on Top quark physics and heavy flavor production for Snowmass 2021*, arXiv:2209.11267 [INSPIRE].

[50] NNPDF collaboration, *Parton distributions from high-precision collider data*, *Eur. Phys. J. C* **77** (2017) 663 [arXiv:1706.00428] [INSPIRE].


JHEP01(2024)074