Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

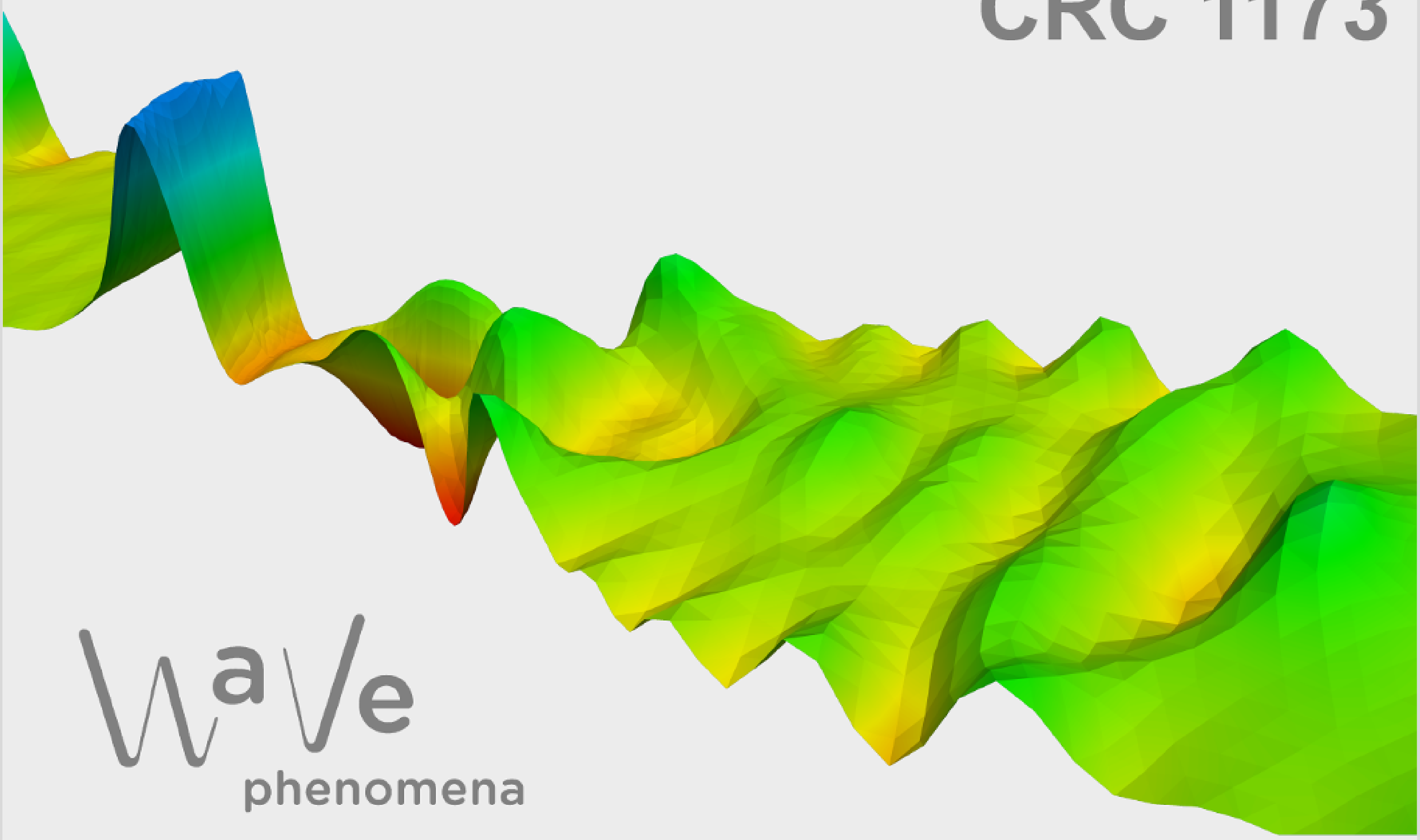
Far field operator splitting and completion in inverse medium scattering

Roland Griesmaier, Lisa Schätzle

CRC Preprint 2024/3, February 2024

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Wave phenomena

Participating universities

UNIVERSITÄT BONN

Universität Stuttgart

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

TU WIEN

Funded by

DFG

ISSN 2365-662X

Far field operator splitting and completion in inverse medium scattering

Roland Griesmaier and Lisa Schätzle*

January 10, 2024

Abstract

We study scattering of time-harmonic plane waves by compactly supported inhomogeneous objects in a homogeneous background medium. The far field operator associated to a fixed scatterer describes multi-static remote observations of scattered fields corresponding to arbitrary superpositions of plane wave incident fields at a single frequency. In this work we consider far field operators for systems of two well-separated scattering objects, and we discuss the nonlinear inverse problem to recover the far field operators associated to each of these two scatterers individually. This is closely related to the question whether the two components of the scatterer can be distinguished by means of inverse medium scattering in a stable way. We also study the restoration of missing or inaccurate components of an observed far field operator and comment on the benefits of far field operator splitting in this context. Both problems are ill-posed without further assumptions, but we give sufficient conditions on the diameter of the supports of the scatterers, the distance between them, and the size of the missing or corrupted data component to guarantee stable recovery whenever sufficient a priori information on the location of the unknown scatterers is available. We provide algorithms, error estimates, a stability analysis, and we demonstrate our theoretical predictions by numerical examples.

Mathematics subject classifications (MSC2010): 35R30 (65N21)
Keywords: Inverse medium scattering, Helmholtz equation, far field operator splitting, data completion
Short title: Far field operator splitting and completion

1 Introduction

The inverse medium scattering problem (see, e.g., [14, p. 439]) is to determine the shape and the refractive index of a compactly supported inhomogeneous object from a knowledge of the far field patterns of scattered waves corresponding to plane wave incident fields for all possible incident and observation directions on the unit sphere. These input data can be described by the far field operator, which maps densities of superpositions of plane wave incident fields to the far field patterns of the corresponding scattered fields. The far field operator can be viewed as an idealized measurement operator for the inverse medium scattering problem, and accordingly it plays a central role in several reconstruction methods (see, e.g., [1, 11, 12, 15, 26, 35] and the monographs [9, 10, 14, 38]). In this work we do not investigate the inverse scattering problem itself, but we focus on two inverse problems for the far field operator that are of immediate relevance to inverse medium scattering. We restrict the discussion to the two-dimensional case and consider scatterers that consist of two well-separated components. Assuming that a possibly

*Institut für Angewandte und Numerische Mathematik, Karlsruher Institut für Technologie, Englerstr. 2, 76131 Karlsruhe, Germany (roland.griesmaier@kit.edu, lisa.schaetzle@kit.edu).

noisy and incomplete version of the associated far field operator is available, we consider the following two questions:

(a) Is it possible to reconstruct the far field operators associated to the two components of the scatterer individually, if we have some a priori information on the location of these components?

(b) Can we recover missing components of the far field operator and filter possible data errors, and why can far field operator splitting be useful in this context?

Regarding problem (a), the uniqueness of solutions to the inverse medium scattering problem (see [8, 42, 44, 46]) tells us that the shape and the refractive index of the scattering object are uniquely determined by the far field operator. Accordingly, having some a priori information about the location of the two components of the scattering object, one could first reconstruct the scatterer, then split it into its two components, and finally calculate the far field operators corresponding to these two components to solve problem (a). However, solving the inverse medium scattering problem is time-consuming and severely ill-posed, and we will show that it is actually not required to solve problem (a). Similarly, it is well-known that the far field pattern depends analytically on both the incident and the observation direction (see, e.g., [14, 27]). Thus recovering missing components of the far field operator in problem (b) is usually possible, if the available data patch is not too small, but also severely ill-posed without further assumptions.

For both questions we will assume that we have access to an estimate of the location and possibly the size of the (two components of) the unknown scatterer, and our goal is to solve the inverse problems (a) and (b) without solving the inverse medium scattering problem itself. The a priori information on the location of the (two components of the) scatterer is used to develop sparse decompositions of the associated far field operators with respect to suitably modulated Fourier representations of the Hilbert Schmidt kernels of these operators. These sparse representations are the foundation of our reconstruction methods. Apart from that the main theoretical contributions of this work are error estimates and a rigorous stability analysis for our reconstruction algorithms. These results build on previous investigations of far field splitting and data completion for the inverse source problem in [25, 28, 29, 30, 31]. There are two main differences distinguishing the inverse source problem considered in these works and the inverse medium scattering problem considered here. The first difference is that, while for the inverse source problem one has access to just one far field pattern radiated by the unknown source, infinitely many but correlated far field patterns corresponding to plane wave incident fields for all possible incident directions are available for the inverse medium scattering problem. We will show that, by working with whole far field operators instead of individual far field patterns, correlations in this data set beyond simple reciprocity relations can be used to obtain better stability estimates for reconstruction methods for problems (a) and (b) for the inverse medium scattering problem, when compared to the corresponding stability estimates for the inverse source problem in [25, 29]. The second difference is that the far field splitting problem for inverse source problems is linear, while the far field operator splitting problem for the inverse medium scattering problem is nonlinear due to multiple scattering effects. To untangle multiple scattering we decompose far field operators corresponding to systems of two scatterers using the Born series and consider sparse representations of the various parts in this decomposition. These can then be distinguished by our splitting algorithm.

The (inverse) Born series has recently also been used directly in reconstruction methods for the inverse medium scattering problem in [17, 33, 34]. In [6] (see also [21, 22]) the authors apply a reduced order model to transform scattering data for a time-dependent scattering problem including multiple scattering effects to observations expected in the Born approximation, i.e., multiple scattering effects are removed. Both approaches are not directly related to the results in this work. We also note that alternate methods for far field splitting for inverse source problems have been proposed in [5, 45], and that splitting problems for time-dependent waves have more recently been considered in [3, 24, 32]. Data completion for far field operators as in problem (b) has recently also been discussed in [20, 41] (see also [2, 7] for related results for the Cauchy problem for the Helmholtz equation). In contrast to our work, these authors do not use sparse representations of far field patterns or far field operators with respect to modulated Fourier bases to stabilize their algorithms, which so far also lack a rigorous stability analysis.

Being able to split and complete far field operators has important implications for the inverse medium scattering problem. If one can stably split the far field operator corresponding to a system of two scatterers, then these scatterers can also be distinguished in the reconstruction, which tells something about resolution in the presence of noise. On the other hand incomplete data sets are common in applications, while reconstruction methods usually work more stably for complete data sets. Being able to reliably restore the full far field operator helps to reduce the effect of this additional source of ill-posedness.

The outline of this paper is as follows. After providing some theoretical background on inverse medium scattering, we develop sparse representations of far field operators based on the Born series and modulated Fourier expansions in Section 2. In Section 3 we pursue far field operator splitting, first using the Born approximation and neglecting multiple scattering, and later including multiple scattering effects. Far field operator completion is the topic of Section 4. In Section 5 we provide numerical examples to illustrate our theoretical findings, and we close with some concluding remarks.

2 Inhomogeneous medium scattering

We consider scattering of time-harmonic acoustic waves by compactly supported penetrable scattering objects in the plane. Let $D \subseteq \mathbb{R}^2$ be bounded such that the boundary ∂D of D is of Lipschitz class. Suppose that $n^2 = 1+q$ represents the *index of refraction*, where the real-valued contrast function $q \in L^\infty(D)$ satisfies $q > -1$ in D and is extended by $q = 0$ outside D.

Let $k > 0$ be the wave number, and let

$$u^i(x;d) := e^{\mathrm{i}kx\cdot d}, \qquad x \in \mathbb{R}^2, \tag{2.1a}$$

be an incident plane wave with illumination direction $d \in S^1$. The scattering problem is then to determine the *total field* $u_q \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ with

$$\Delta u_q + k^2 n^2 u_q = 0 \qquad \text{in } \mathbb{R}^2, \tag{2.1b}$$

such that the *scattered field* $u_q^s = u_q - u^i$ satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u_q^s}{\partial r}(x;d) - \mathrm{i}ku_q^s(x;d)\Big) = 0, \qquad r = |x|, \tag{2.1c}$$

uniformly with respect to all directions $x/|x| \in S^1$. As for the incident field, we indicate the dependence of the total and scattered fields on the incident direction by a second argument.

Remark 2.1. The Helmholtz equation (2.1b) has to be understood in the weak sense, i.e., $u_q \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ is a solution if and only if

$$\int_{\mathbb{R}^2} (\nabla u_q \cdot \nabla v - k^2 n^2 u_q v)\,\mathrm{d}x = 0 \qquad \text{for all } v \in H^1_0(\mathbb{R}^2).$$

Standard regularity results yield smoothness of u_q and u_q^s in $\mathbb{R}^2 \setminus \overline{B_R(0)}$, where $B_R(0)$ is a sufficiently large ball containing the scatterer D. In particular, the Sommerfeld radiation condition (2.1c) is well defined. $\diamondsuit$

The scattering problem (2.1) has a unique solution $u_q \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ (see, e.g., [37, Thm. 7.13]). This solution satisfies the *Lippmann-Schwinger integral equation*

$$u_q(x;d) \;=\; u^i(x;d) + k^2 \int_{\mathbb{R}^2} q(y)\Phi_k(x-y)u_q(y;d)\,\mathrm{d}y\,, \qquad x \in D\,, \tag{2.2}$$

where Φ_k denotes the *fundamental solution* to the Helmholtz equation (cf., e.g., [37, Thm. 7.12]). In fact, this integral equation is uniquely solvable in $L^2(D)$, and its solution can be extended by the right hand side of (2.2) to a solution $u_q \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ of the scattering problem (2.1) (see, e.g., [37, Thm. 7.12]).

The scattered field satisfies the *far field expansion*

$$u_q^s(x;d) \;=\; \frac{e^{\mathrm{i}\pi/4}}{\sqrt{8\pi}}\frac{e^{\mathrm{i}k|x|}}{\sqrt{k|x|}}u_q^\infty(\widehat{x};d) + O\big(|x|^{-\frac{3}{2}}\big)\,, \qquad |x| \to \infty\,,$$

uniformly in all directions $\widehat{x} := x/|x| \in S^1$. The *far field pattern* $u_q^\infty \in L^2(S^1)$ is given by

$$u_q^\infty(\widehat{x};d) \;=\; k^2 \int_D q(y)u_q(y;d)e^{-\mathrm{i}k\widehat{x}\cdot y}\,\mathrm{d}y\,, \qquad \widehat{x} \in S^1\,,$$

(see, e.g., [37, Thm. 7.15]). It satisfies the *reciprocity relation*

$$u_q^\infty(\widehat{x};d) \;=\; u_q^\infty(-d;-\widehat{x})\,, \qquad \widehat{x}, d \in S^1\,, \tag{2.3}$$

(see [14, Thm. 8.8]). The far field patterns $u_q^\infty(\widehat{x};d)$ for all $\widehat{x}, d \in S^1$ define our object of interest, the *far field operator*

$$F_q : L^2(S^1) \to L^2(S^1)\,, \quad (F_q g)(\widehat{x}) \;:=\; \int_{S^1} u_q^\infty(\widehat{x};d)g(d)\,\mathrm{d}s(d), \tag{2.4}$$

which maps densities of arbitrary superpositions of incident plane waves to the far field patterns of the corresponding scattered fields. The far field operator is compact and normal (see, e.g., [37, Thm. 7.20]), and it is a trace class operator (see [13]). In particular, $F_q \in \mathrm{HS}(L^2(S^1))$, where $\mathrm{HS}(L^2(S^1))$ denotes the space of Hilbert-Schmidt operators from $L^2(S^1)$ to $L^2(S^1)$.

2.1 Two inverse problems

In this work we are interested in the following two inverse problems.

(a) **Far field operator splitting:** Suppose that $D = D_1 \cup D_2$ for some well-separated bounded Lipschitz domains $D_1, D_2 \subseteq \mathbb{R}^2$, i.e., we assume that there exist balls $B_{R_j}(c_j)$, $j = 1,2$, with $D_j \subseteq B_{R_j}(c_j)$ and $|c_1 - c_2| > R_1 + R_2$. Accordingly, let $q_1 := q|_{D_1}$ and $q_2 := q|_{D_2}$ denote the contrast functions of the two components of the scatterer. The goal of *far field operator splitting* is then to recover the far field operators F_{q_1} and F_{q_2} corresponding to the two components of the scatterer from F_q. Since $q = q_1 + q_2$ is uniquely determined by F_q (see [8, 42, 44, 46]), this inverse problem is uniquely solvable, whenever sufficient a priori information on the locations of the scatterers D_1 and D_2 is available to determine q_1 and q_2 from q. In this work we will assume that balls $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$ as above containing the scatterers are known a priori, and we use this information to develop a stability estimate and a reconstruction algorithm.

(b) **Far field operator completion:** Suppose that the far field pattern $u_q^\infty(\widehat{x}; d)$ cannot be observed for all incident directions $d \in S^1$ and for all observation directions $\widehat{x} \in S^1$ but only observations of $u_q^\infty(\widehat{x}; d)$ for $(\widehat{x}, d) \in (S^1 \times S^1) \setminus \Omega$ for some $\Omega \subseteq S^1 \times S^1$ are available. Accordingly, we define the *restricted far field operator*

$$F_q|_{S^1\times S^1\setminus\Omega} : L^2(S^1) \to L^2(S^1)\,,$$
$$\big(F_q|_{S^1\times S^1\setminus\Omega} g\big)(\widehat{x}) \;:=\; \int_{S^1} \chi_{(S^1\times S^1)\setminus\Omega}(\widehat{x}, d) u_q^\infty(\widehat{x}; d) g(d)\,\mathrm{d}s(d)\,, \quad (2.5)$$

The goal of *far field operator completion* is then to recover F_q from $F_q|_{S^1\times S^1\setminus\Omega}$. Since the far field pattern is a real analytic function on $S^1 \times S^1$ (see, e.g., [14, 27]), this inverse problem has a unique solution as long as $(S^1 \times S^1) \setminus \Omega$ has an interior point. We will assume that a ball $B_R(c)$ with $D \subseteq B_R(c)$ is known a priori, and we use this information to develop a stability estimate and a reconstruction algorithm.

2.2 Far field translation

Translating the scatterer by $c \in \mathbb{R}^2$ changes the incident field at the location of the scatterer and thus also the scattered field as well as its far field pattern. To formalize this, we consider the shifted contrast function

$$q_c(x) \;:=\; q(x + c)\,, \qquad x \in \mathbb{R}^2\,. \quad (2.6)$$

We still consider an incident plane wave with illumination direction $d \in S^1$. Then, the Lippmann-Schwinger equation (2.2) for the associated solution $u_{q_c} \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ of the scattering problem (2.1) with q replaced by q_c reads

$$u_{q_c}(x - c; d) \;=\; e^{-\mathrm{i}kc\cdot d} u^i(x; d) + k^2 \int_{\mathbb{R}^2} q(y)\Phi_k(x - y) u_{q_c}(y - c; d)\,\mathrm{d}y\,, \qquad x \in D\,.$$

Accordingly,

$$u_{q_c}(x; d) \;=\; e^{-\mathrm{i}kc\cdot d} u_q(x + c; d)\,, \qquad x \in \mathbb{R}^2\,,$$

the associated scattered field satisfies

$$u^s_{q_c}(x; d) \;=\; e^{-\mathrm{i}kc\cdot d} u^s_q(x + c; d)\,, \qquad x \in \mathbb{R}^2\,,$$

and its far field pattern is given by

$$u^\infty_{q_c}(\widehat{x}; d) \;=\; k^2 \int_{\mathbb{R}^2} q_c(y) u_{q_c}(y; d) e^{-\mathrm{i}k\widehat{x}\cdot y}\,\mathrm{d}y \;=\; e^{-\mathrm{i}kc\cdot(d-\widehat{x})} u^\infty_q(\widehat{x}; d)\,, \qquad \widehat{x} \in S^1\,.$$

For the corresponding far field operator this means that

$$(F_{q_c} g)(\widehat{x}) \;=\; k^2 \int_{S^1} g(d) u^\infty_{q_c}(\widehat{x}; d)\,\mathrm{d}s(d) \;=\; e^{\mathrm{i}kc\cdot\widehat{x}} \int_{S^1} e^{-\mathrm{i}kc\cdot d} g(d) u^\infty_q(\widehat{x}; d)\,\mathrm{d}s(d)\,, \qquad \widehat{x} \in S^1\,.$$

Introducing the linear operators

$$T_c : L^2(S^1) \to L^2(S^1)\,, \quad (T_c g)(\widehat{x}) \;:=\; e^{\mathrm{i}kc\cdot\widehat{x}} g(\widehat{x})\,, \quad (2.7)$$

and

$$\mathcal{T}_c : \mathrm{HS}(L^2(S^1)) \to \mathrm{HS}(L^2(S^1))\,, \quad \mathcal{T}_c G \;:=\; T_c \circ G \circ T_{-c}\,, \quad (2.8)$$

we find that $F_{q_c} = \mathcal{T}_c F_q$. We call T_c and $\mathcal{T}_c$ *far field translation operators.*

Lemma 2.2. *Let $c \in \mathbb{R}^2$. The operator $\mathcal{T}_c \in \mathcal{L}(\mathrm{HS}(L^2(S^1)))$ is unitary with $\mathcal{T}_c^* = \mathcal{T}_{-c}$.*

Proof. We note that the adjoint of T_c from (2.7) is given by $T_c^* = T_{-c}$. Therewith, the definition (2.8) implies that, for any $G, H \in \mathrm{HS}(L^2(S^1))$,

$$\langle \mathcal{T}_c G, H \rangle_{\mathrm{HS}} \;=\; \mathrm{tr}\big((T_c G T_{-c})^* H\big) \;=\; \mathrm{tr}\big(T_{-c} H T_c G^*\big) \;=\; \mathrm{tr}\big(G^* T_{-c} H T_c\big) \;=\; \langle G, \mathcal{T}_{-c} H \rangle_{\mathrm{HS}}\,.$$

This shows that $\mathcal{T}_c^* = \mathcal{T}_{-c}$, which is the same as $\mathcal{T}_c^{-1}$. □

2.3 The Born series

For any $d \in S^1$ the Lippmann-Schwinger equation (2.2) can be understood as a fixpoint equation

$$u(\,\cdot\,; d) \;=\; u^i(\,\cdot\,; d) + L_{q,k} u(\,\cdot\,; d) \qquad \text{in } D$$

for the total field $u(\,\cdot\,; d)$, where the linear operator $L_{q,k} : L^2(D) \to L^2(D)$ is defined by

$$(L_{q,k} f)\,(x) \;:=\; k^2 \int_D q(y) f(y) \Phi_k(x-y)\,\mathrm{d}y\,, \qquad x \in D\,. \tag{2.9}$$

If the operator norm $\|L_{q,k}\|_{L^2(D)\to L^2(D)}$ is strictly less than one, then the solution to (2.2) can be written as

$$u_q(\,\cdot\,; d) \;=\; u^i(\,\cdot\,; d) + \sum_{l=1}^{\infty} L_{q,k}^l u^i(\,\cdot\,; d) \qquad \text{in } D\,. \tag{2.10}$$

This series is often called the *Born series*. It converges in $L^2(D)$ and uniformly with respect to the incident direction $d \in S^1$. Sufficient conditions on k and q for this to hold have, e.g., been discussed in [33, 36, 43, 49]. For instance, (2.9) immediately implies that

$$\begin{aligned}
\|L_{q,k} f\|_{L^2(D)}^2 \;&=\; \int_D \Big| k^2 \int_D q(y) f(y) \Phi_k(x-y)\,\mathrm{d}y \Big|^2 \,\mathrm{d}x \\
&\le\; k^4 \|f\|_{L^2(D)}^2 \int_D \int_D \big|q(y)\Phi_k(x-y)\big|^2 \,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

for all $f \in L^2(D)$, i.e.,

$$\|L_{q,k}\|_{L^2(D)\to L^2(D)}^2 \;\le\; k^4 \int_D \int_D \big|q(y)\Phi_k(x-y)\big|^2 \,\mathrm{d}y\,\mathrm{d}x \;=\; \|L_{q,k}\|_{\mathrm{HS}}^2 \tag{2.11}$$

(see [47, Thm. VI.23]). The asymptotic behavior of the fundamental solution $\Phi_k(x-y)$ as $|x-y| \to 0$ shows that the right hand side in (2.11) is finite. In particular, $L_{q,k}$ is a Hilbert-Schmidt operator, and the Born series converges when $\|L_{q,k}\|_{\mathrm{HS}} < 1$, which we assume henceforth.

The Born series describes the different levels of multiple scattering of the incident wave $u^i(\,\cdot\,; d)$ at the scatterer. Accordingly, we denote the lth summand of the Born series by

$$\begin{aligned}
u^{s,(l)}_{q,\dots,q}(\,\cdot\,; d) \;&:=\; L_{q,k}^l u^i(\,\cdot\,; d) \\
&=\; k^{2l} \int_D \cdots \int_D q(y_l) \cdots q(y_1) e^{\mathrm{i}k y_1 \cdot d} \Phi_k(\,\cdot - y_l) \Phi_k(y_l - y_{l-1}) \cdots \Phi_k(y_2 - y_1)\,\mathrm{d}y_1 \cdots \mathrm{d}y_l\,,
\end{aligned}$$

and refer to it as the scattered field component associated to scattering processes of order $l \ge 1$ on q.

Substituting (2.10) into (2.2) and extending the right hand side to all of $\mathbb{R}^2$, we find that the far field pattern of $u_q^s(\,\cdot\,;d)$ satisfies

$$u_q^\infty(\widehat{x};d) \;=\; k^2 \sum_{l=1}^{\infty} \int_D q(y) e^{-\mathrm{i}k\widehat{x}\cdot y} \big(L_{q,k}^{l-1} u^i(\,\cdot\,;d)\big)(y)\;\mathrm{d}y\,, \qquad \widehat{x}\in S^1\,. \tag{2.12}$$

This series converges in $L^2(S^1)$ and uniformly with respect to $d\in S^1$. We write

$$u_{q,\dots,q}^{\infty,(l)}(\widehat{x};d) \;:=\; k^{2l} \int_D \cdots \int_D q(y_l)\cdots q(y_1)\Phi_k(y_l-y_{l-1})\cdots\Phi_k(y_2-y_1) \\ e^{\mathrm{i}k(d\cdot y_1-\widehat{x}\cdot y_l)}\;\mathrm{d}y_1\cdots\mathrm{d}y_l\,, \qquad \widehat{x}\in S^1\,, \tag{2.13}$$

for the lth summand in (2.12) associated to scattering processes of order $l\geq 1$ on q. Then,

$$u_q^\infty(\widehat{x};d) \;=\; \sum_{l=1}^{\infty} u_{q,\dots,q}^{\infty,(l)}(\widehat{x};d)\,, \qquad \widehat{x}\in S^1\,, \tag{2.14}$$

and we define the *Born far field of order* $p\geq 1$ as the pth partial sum in (2.14), i.e., by

$$u_q^{\infty,(pB)}(\widehat{x};d) \;:=\; \sum_{l=1}^{p} u_{q,\dots,q}^{\infty,(l)}(\widehat{x};d)\,, \qquad \widehat{x}\in S^1\,.$$

For later reference, we note that (2.13) implies that each $u_{q,\dots,q}^{\infty,(l)}$, $l\geq 1$, satisfies the reciprocity relation

$$u_{q,\dots,q}^{\infty,(l)}(\widehat{x};d) \;=\; u_{q,\dots,q}^{\infty,(l)}(-d;-\widehat{x})\,, \qquad \widehat{x},d\in S^1\,. \tag{2.15}$$

Recalling (2.4) we introduce far field operator components associated to scattering processes of order $l\geq 1$,

$$F_{q,\dots,q}^{(l)} : L^2(S^1)\to L^2(S^1)\,, \quad \big(F_{q,\dots,q}^{(l)} g\big)(\widehat{x}) \;:=\; \int_{S^1} u_{q,\dots,q}^{\infty,(l)}(\widehat{x};d) g(d)\;\mathrm{d}s(d)\,, \tag{2.16}$$

and for any $p\geq 1$ we define the *Born far field operator of order* p by

$$F_q^{(pB)} : L^2(S^1)\to L^2(S^1), \quad \big(F_q^{(pB)} g\big)(\widehat{x}) \;:=\; \int_{S^1} u_q^{\infty,(pB)}(\widehat{x};d) g(d)\;\mathrm{d}s(d)\,. \tag{2.17}$$

Then (2.4) and (2.14) together with the realization of the Hilbert-Schmidt norm of an integral operator on $L^2(S^1)$ by the L^2-norm of its kernel (see [47, Thm. VI.23]) yields that $F_q = \sum_{l=1}^\infty F_{q,\dots,q}^{(l)} = \lim_{p\to\infty} F_q^{(pB)}$ in $\mathrm{HS}(L^2(S^1))$.

2.4 Sparse approximations of far field operators

Assuming that the support of the scatterer is contained in a ball $B_R(c)\subseteq\mathbb{R}^2$ for some $c\in\mathbb{R}^2$ and $R>0$, we show next that the components $F_{q,\dots,q}^{(l)}$ of the far field operator associated to scattering processes of order $l\geq 1$ in the Born series have sparse approximations with respect to some suitably modulated Fourier bases.

To begin with, we suppose that $D\subseteq B_R(0)$, i.e., we have $c=0$. We denote by $(\boldsymbol{e}_n)_n := (e^{\mathrm{i}n\arg(\,\cdot\,)}/\sqrt{2\pi})_n$ the Fourier basis of $L^2(S^1)$. Substituting the Jacobi-Anger expansion (see, e.g., [14, p. 75])

$$e^{\pm\mathrm{i}k\widehat{x}\cdot y} \;=\; \sum_{n\in\mathbb{Z}} (\pm\mathrm{i})^n e^{-\mathrm{i}n\arg y} J_n(k|y|) e^{\mathrm{i}n\arg\widehat{x}}\,, \qquad y\in\mathbb{R}^2\,,\ \widehat{x}\in S^1\,, \tag{2.18}$$

into (2.13) twice to replace the two plane wave terms shows that the far field component associated to scattering processes of order $l \geq 1$ satisfies

$$u^{\infty,(l)}_{q,\dots,q}(\widehat{x};d) \;=\; \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a^{(l)}_{m,n}\, \boldsymbol{e}_m(\widehat{x})\overline{\boldsymbol{e}_n(d)}\,, \qquad \widehat{x}, d \in S^1\,,$$

with

$$a^{(l)}_{m,n} \;=\; 2\pi k^{2l}(-\mathrm{i})^{m-n}\int_D\cdots\int_D q(y_l)\cdots q(y_1)\Phi_k(y_l-y_{l-1})\cdots\Phi_k(y_2-y_1) \\ e^{-\mathrm{i}(m\arg y_l - n\arg y_1)}J_m(k|y_l|)J_n(k|y_1|)\,\mathrm{d}y_1\cdots\mathrm{d}y_l\,. \tag{2.19}$$

Since $(\boldsymbol{e}_n)_n$ is a complete orthonormal system in $L^2(S^1)$, we obtain from (2.16), [47, Thm. VI.23], and Parseval's identity that

$$\|F^{(l)}_{q,\dots,q}\|_{\mathrm{HS}} \;=\; \|u^{\infty,(l)}_{q,\dots,q}\|_{L^2(S^1\times S^1)} \;=\; \|(a^{(l)}_{m,n})_{m,n}\|_{\ell^2\times\ell^2}\,.$$

Furthermore, applying the Cauchy-Schwarz inequality l times shows that

$$|a^{(l)}_{m,n}| \;\leq\; 2\pi k^2\|q\|_{L^\infty(D)}\|J_m(k|\cdot|)\|_{L^2(D)}\|J_n(k|\cdot|)\|_{L^2(D)}\|L_{q,k}\|^{l-1}_{\mathrm{HS}} \tag{2.20}$$

for all $l \geq 1$, and for all $m, n \in \mathbb{Z}$. We note that the last term on the right hand side of this equation already appeared in (2.11) and is assumed to be less than one throughout this work. Since $D \subseteq B_R(0)$, the estimate (2.20) implies that the coefficients $(a^{(l)}_{m,n})_{m,n}$ are essentially supported in the index range $|m|, |n| \lesssim kR$, and decay superlinearly as a function of $m, n \in \mathbb{Z}$ for $|m|, |n| \gtrsim kR$. In fact, we obtain from formula (3.8) in [29] that

$$\begin{aligned} k\|J_n(k|\cdot|)\|_{L^2(D)} &\leq k\|J_n(k|\cdot|)\|_{L^2(B_R(0))} \;=\; \|J_n(|\cdot|)\|_{L^2(B_{kR}(0))} \\ &\leq b_0\, n^{\frac13}\Bigl(1+\frac{1}{2n}\Bigr)^{\frac{n+1}{2}}\,\frac{kR}{n}\,\Bigl(\Bigl(\frac{kR}{n}\Bigr)^2 e^{1-\left(\frac{kR}{n}\right)^2}\Bigr)^{\frac{n}{2}}\,, \qquad \text{if } |n| \geq kR\,, \end{aligned} \tag{2.21}$$

where the constant b_0 satisfies $b_0 \approx 0.7928$. In addition, it has been shown in [29] that

$$\lim_{kR\to\infty}\frac{\|J_{\lceil \nu kR\rceil}(|\cdot|)\|^2_{L^2(B_{kR}(0))}}{2kR} \;=\; \begin{cases} \sqrt{1-\nu^2} & \text{if } \nu \leq 1\,,\\ 0 & \text{else}\,,\end{cases} \tag{2.22}$$

where $\lceil \nu kR\rceil$ denotes the smallest integer that is greater or equal to νkR. Numerical tests in [29] confirm that the values $\|J_n(|\cdot|)\|^2_{L^2(B_{kR}(0))}$ approach the asymptote $2\sqrt{(kR)^2-n^2}$ already for moderate values of kR and decay quickly for $n \gtrsim kR$ (see also Figure 2.1 below). Choosing $N \gtrsim kR$ this means that we may approximate

$$u^{\infty,(l)}_{q,\dots,q}(\widehat{x};d) \;\approx\; \sum_{|m|\leq N}\sum_{|n|\leq N} a^{(l)}_{m,n}\boldsymbol{e}_m(\widehat{x})\overline{\boldsymbol{e}_n(d)}\,, \qquad \widehat{x}, d \in S^1\,. \tag{2.23}$$

Substituting (2.23) into (2.16) yields that

$$(F^{(l)}_{q,\dots,q}g)(\widehat{x}) \;\approx\; \sum_{|m|\leq N}\sum_{|n|\leq N} a^{(l)}_{m,n}\,\boldsymbol{e}_m(\widehat{x})\int_{S^1} g(d)\overline{\boldsymbol{e}_n(d)}\,\mathrm{d}s(d)\,, \qquad \widehat{x} \in S^1\,, \tag{2.24}$$

is a reasonable approximation to the far field operator component associated to scattering processes of order $l \geq 1$.

Example 2.3. To further illustrate this sparse approximation, we consider the example $q = \chi_{B_R(0)}$ for the contrast function, i.e., the refractive index n^2 is piecewise constant with values $n^2 = 2$ in $D = B_R(0)$ and $n^2 = 1$ in $\mathbb{R}^2 \setminus \overline{B_R(0)}$. Then, the Fourier coefficients $(a^{(1)}_{m,n})_{m,n}$ from (2.19) with $l = 1$ satisfy

$$\begin{aligned}
a^{(1)}_{m,n} &= 2\pi k^2 (-\mathrm{i})^{m-n} \int_{B_R(0)} e^{-\mathrm{i}(m-n)\arg y} J_m(k|y|) J_n(k|y|) \,\mathrm{d}y \\
&= 2\pi k^2 (-\mathrm{i})^{m-n} \int_0^R J_m(kr) J_n(kr) r \,\mathrm{d}r \int_0^{2\pi} e^{-\mathrm{i}(m-n)\varphi} \,\mathrm{d}\varphi \\
&= \begin{cases} 2\pi \|J_n(|\cdot|)\|^2_{L^2(B_{kR}(0))} & \text{if } n = m\,, \\ 0 & \text{else}\,. \end{cases}
\end{aligned}$$

Recalling (2.22), this shows that the cut-off parameter N in (2.24) cannot be chosen smaller than kR. ♢

We define the finite dimensional subspace $\mathcal{V}_N \subseteq \mathrm{HS}(L^2(S^1))$ by

$$\mathcal{V}_N := \Bigl\{ G \in \mathrm{HS}(L^2(S^1)) \Bigm| Gg = \sum_{|m|\le N} \sum_{|n|\le N} a_{m,n}\, \boldsymbol{e}_m \langle g, \boldsymbol{e}_n \rangle_{L^2(S^1)} \Bigr\}\,. \tag{2.25}$$

Still assuming that $N \gtrsim kR$, we refer to $\mathcal{V}_N$ as the subspace of *non-evanescent far field operators* associated to scatterers supported in $B_R(0)$. Let $\mathcal{P}_N : \mathrm{HS}(L^2(S^1)) \to \mathrm{HS}(L^2(S^1))$ denote the orthonormal projection onto $\mathcal{V}_N$. Then the approximation error can be estimated using Parseval's identity and (2.20) by

$$\begin{aligned}
\|F^{(l)}_{q,\dots,q} - \mathcal{P}_N F^{(l)}_{q,\dots,q}\|^2_{\mathrm{HS}} &= \sum_{|m|>N} \sum_{|n|>N} |a^{(l)}_{m,n}|^2 \\
&\le \Bigl(2\pi \|q\|_{L^\infty(D)} \|L_{q,k}\|^{l-1}_{\mathrm{HS}} \sum_{|n|>N} \|J_n(|\cdot|)\|^2_{L^2(B_{kR}(0))} \Bigr)^2\,.
\end{aligned} \tag{2.26}$$

The last term on the right hand side can be further estimated by means of (2.21), or illustrated by numerical approximation. We note that [18, (10.22.5)] shows that

$$\|J_n(|\cdot|)\|^2_{L^2(B_{kR}(0))} = \pi (kR)^2 \bigl(J_n^2(kR) - J_{n-1}(kR) J_{n+1}(kR) \bigr)\,, \qquad n \in \mathbb{Z}\,.$$

In Figure 2.1 we show plots of $\sum_{|n|=N+1}^{1000} \|J_n(|\cdot|)\|^2_{L^2(B_{kR}(0))}$ as a function of $N \in \mathbb{N}$ for $kR = 10$ (left) and for $kR = 100$ (right). The exponential decay for $N \gtrsim kR$ is clearly visible.

Remark 2.4. Recalling (2.14), we find that the Fourier coefficients $(a_{m,n})_{m,n}$ of the full far field pattern

$$u^\infty_q(\widehat{x}; d) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a_{m,n}\, \boldsymbol{e}_m(\widehat{x}) \overline{\boldsymbol{e}_n(d)}\,, \qquad \widehat{x}, d \in S^1\,,$$

satisfy $a_{m,n} = \sum_{l=1}^\infty a^{(l)}_{m,n}$. The estimates for $|a^{(l)}_{m,n}|$ in (2.20) and (2.21) show that $|a_{m,n}|$ is also decaying superlinearly for $|m|, |n| \gtrsim kR$. Thus also the full far field operator F_q has a sparse approximation

$$(F_q g)(\widehat{x}) \approx \sum_{|m|\le N} \sum_{|n|\le N} a_{m,n}\, \boldsymbol{e}_m(\widehat{x}) \int_{S^1} g(d) \overline{\boldsymbol{e}_n(d)} \,\mathrm{d}s(d)\,, \qquad \widehat{x} \in S^1\,, \tag{2.27}$$

with $N \gtrsim kR$. ♢

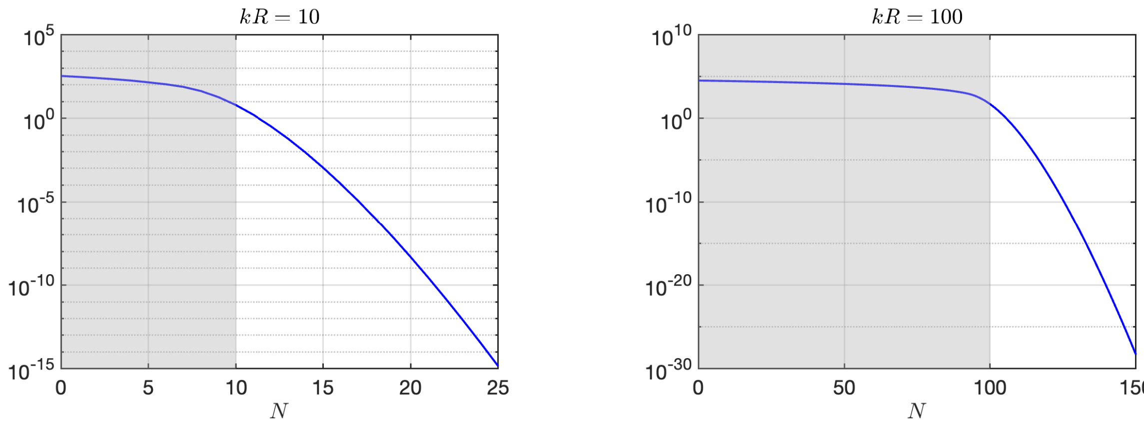


Figure 2.1: Plots of the factor $\sum_{|n|>N} \|J_n(|\cdot|)\|^2_{L^2(B_{kR}(0))}$ in (2.26) as a function of $N \in \mathbb{N}$ for $kR = 10$ (left) and for $kR = 100$ (right)

Remark 2.5. The reciprocity principle (2.15) gives that, for $l \geq 1$ and $\widehat{x}, d \in S^1$,

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a^{(l)}_{m,n}\, \boldsymbol{e}_m(\widehat{x})\overline{\boldsymbol{e}_n(d)} &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a^{(l)}_{m,n}\, \boldsymbol{e}_m(-d)\overline{\boldsymbol{e}_n(-\widehat{x})} \\ &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} (-1)^{m-n} a^{(l)}_{-n,-m}\, \boldsymbol{e}_m(\widehat{x})\overline{\boldsymbol{e}_n(d)}\,. \end{aligned} \tag{2.28}$$

In particular,

$$a^{(l)}_{m,n} = (-1)^{m-n} a^{(l)}_{-n,-m}\,, \qquad m, n \in \mathbb{Z}\,. \tag{2.29}$$

It would be possible to include (2.29) in the definition of the subspace $\mathcal{V}_N$ to obtain even sparser representations of non-evanescent far field operators associated to scatterers supported in $B_R(0)$. We will comment on this further in Remarks 3.10 and 3.16 below.

By (2.3) the formula (2.29) is also true for the Fourier coefficients $a_{m,n}$, $m, n \in \mathbb{Z}$, of the full far field pattern u^∞_q. $\diamondsuit$

If $D \subseteq B_R(c)$ for some $c \in \mathbb{R}^2$ with $c \neq 0$, then we can use the far field translation operator $\mathcal{T}_c$ from (2.8) to shift the scatterer into the origin. Recalling (2.13), the kernel of the integral representation of the shifted far field operator $\mathcal{T}_c F^{(l)}_{q,\dots,q}$ is given by

$$k^{2l} \int_D \cdots \int_D q(y_l) \cdots q(y_1) \Phi_k(y_l - y_{l-1}) \cdots \Phi_k(y_2 - y_1) e^{\mathrm{i}k(d\cdot(y_1-c) - \widehat{x}\cdot(y_l - c))}\, \mathrm{d}y_1 \cdots \mathrm{d}y_l\,,$$

which is the same as $u^{\infty,(l)}_{q_c,\dots,q_c}(\widehat{x}; d)$ (see also (2.6)). Now proceeding as in the special case $c = 0$, we find that $\mathcal{T}_c F^{(l)}_{q,\dots,q}$ has a sparse approximation

$$\bigl(\mathcal{T}_c F^{(l)}_{q,\dots,q} g\bigr)(\widehat{x}) \approx \sum_{|m|\leq N} \sum_{|n|\leq N} a^{(l)}_{m,n}\, \boldsymbol{e}_m \int_{S^1} g(d)\overline{\boldsymbol{e}_n(d)}\, \mathrm{d}s(d)\,, \qquad \widehat{x} \in S^1\,,$$

in $\mathcal{V}_N$ from (2.25) with $N \gtrsim kR$. Applying (2.8), we define the subspace $\mathcal{V}^c_N \subseteq \mathrm{HS}(L^2(S^1))$ of non-evanescent far field operators associated to scatterers supported in $B_R(c)$ by

$$\mathcal{V}^c_N := \Bigl\{ G \in \mathrm{HS}(L^2(S^1)) \Bigm| Gg = \sum_{|m|\leq N} \sum_{|n|\leq N} a_{m,n}\, e^{\mathrm{i}kc\cdot(\,\cdot\,)} \boldsymbol{e}_m \langle g, e^{\mathrm{i}kc\cdot(\,\cdot\,)} \boldsymbol{e}_n \rangle_{L^2(S^1)} \Bigr\}\,, \tag{2.30}$$

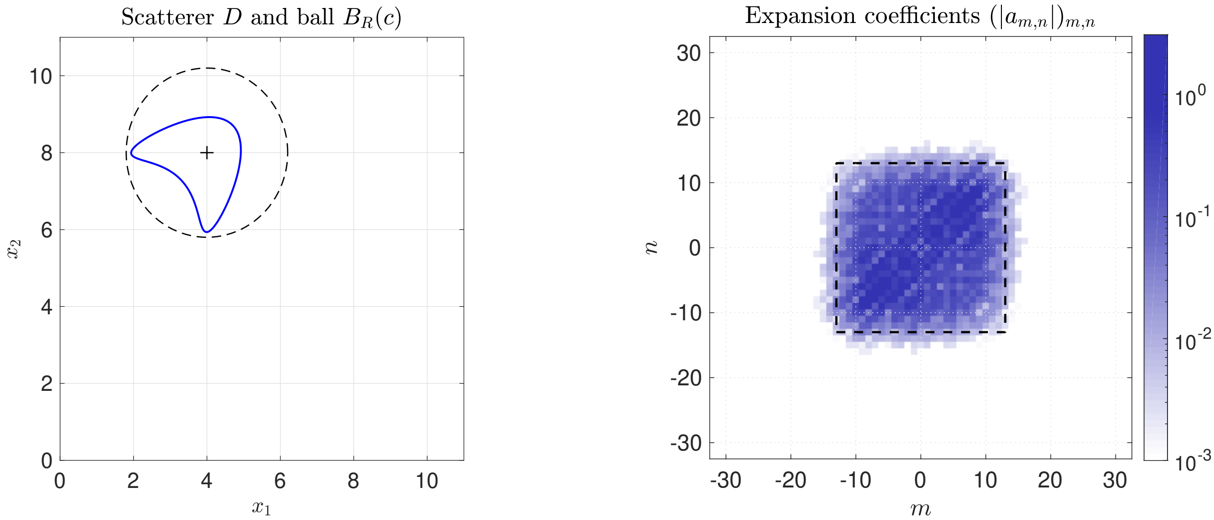


Figure 2.2: Left: Support of scatterer D (solid) and ball $B_R(c)$ (dashed) of radius $R = 2.2$ centered at $c = (4, 8)$ containing D. Right: Absolute values of modulated Fourier coefficients $(a_{m,n})_{m,n}$ of far field operator F_q at $k = 5$. Dashed square corresponds to expansion coefficients used by sparse approximation of F_q in $\mathcal{V}_N^c$ with $N = 13$.

i.e., $\mathcal{V}_N^c = \{G \in \mathrm{HS}(L^2(S^1)) \,|\, \mathcal{T}_c G \in \mathcal{V}_N\}$. We call $(e^{\mathrm{i}kc\cdot(\,\cdot\,)}\boldsymbol{e}_n)_n$ a modulated Fourier basis of $L^2(S^1)$, and accordingly the coefficients $(a_{m,n})_{m,n}$ in associated expansions as in (2.30) are called modulated Fourier coefficients. Let $\mathcal{P}_N^c : \mathrm{HS}(L^2(S^1)) \to \mathrm{HS}(L^2(S^1))$ denote the orthonormal projection onto $\mathcal{V}_N^c$. Then the error estimate (2.26) carries over, and we obtain that

$$\|F_{q,\dots,q}^{(l)} - \mathcal{P}_N^c F_{q,\dots,q}^{(l)}\|_{\mathrm{HS}} \;\le\; 2\pi \|q\|_{L^\infty(D)} \|L_{q,k}\|_{\mathrm{HS}}^{l-1} \sum_{|n|>N} \|J_n(|\,\cdot\,|)\|_{L^2(B_{kR}(0))}^2 \,. \tag{2.31}$$

The upper bound on the right hand side of (2.31) decays quickly for $N \gtrsim kR$ (see Figure 2.1).

The same reasoning that led to (2.27) shows that the Born far field operator F_q^{pB} of order $p \ge 1$ and also the full far field operator F_q have sparse approximations in $\mathcal{V}_N^c$.

Example 2.6. We illustrate our findings by a numerical example with contrast function $q = 2\chi_D$ for a kite-shaped scatterer D as shown in Figure 2.2 (left). For this example we use the wave number $k = 5$ and we simulate far field patterns $u^\infty(\widehat{x}_m; d_n)$ of solutions to the scattering problem (2.1) for $L = 256$ observation and incident directions

$$\widehat{x}_m, d_n \in \{(\cos\varphi_l, \sin\varphi_l) \in S^1 \mid \varphi_l = (l-1)2\pi/L\,,\ l = 1, \dots, L\}\,, \qquad 1 \le m, n \le L\,, \tag{2.32}$$

using a Nyström method for a boundary integral formulation of the scattering problem (see [14, pp. 91–96]). Accordingly, the matrix

$$\boldsymbol{F}_q \;=\; \frac{2\pi}{L}[u^\infty(\widehat{x}_m; d_n)]_{1\le m,n\le L} \in \mathbb{C}^{L\times L}\,,$$

approximates the far field operator F_q from (2.4).

The scatterer D is contained in the ball $B_R(c)$ of radius $R = 2.2$ centered at $c = (4, 8)$, which is shown as a dashed circle in Figure 2.2 (left). A two-dimensional fast Fourier transform of the shifted far field pattern $(e^{-\mathrm{i}kc\cdot(d_n - \widehat{x}_m)} u_q^\infty(\widehat{x}_m; d_n))_{m,n} \in \mathbb{C}^{L\times L}$ yields an approximation of the modulated Fourier coefficients $(a_{m,n})_{m,n}$ of F_q with respect to the modulated Fourier basis $(e^{\mathrm{i}kc\cdot(\,\cdot\,)}\boldsymbol{e}_n)_n$ of $L^2(S^1)$. In Figure 2.2 (right) the absolute values of these expansion coefficients are plotted for $-32 \le m, n \le 32$ in a logarithmic scale. It is nicely confirmed that the Fourier coefficients are essentially supported in a square $[-N, N]^2$ with $N \gtrsim kR = 11$. The dashed square in Figure 2.2 (right) corresponds to $N = 13$. $\Diamond$

3 Far field operator splitting

We consider the inverse problem (a) introduced in Section 2.1. Suppose that the scatterer consists of two well-separated components, i.e., $D = D_1 \cup D_2$ for some $D_1, D_2 \subseteq \mathbb{R}^2$ and $D_j \subseteq B_{R_j}(c_j)$, $j = 1, 2$, with $c_1, c_2 \in \mathbb{R}^2$ and $R_1, R_2 > 0$ such that $|c_1 - c_2| > R_1 + R_2$. Let $q_1 := q|_{D_1}$ and $q_2 := q|_{D_2}$ denote the contrast functions of these two components. Then, each summand in the Born series (2.14) for the far field pattern can be further decomposed. Recalling (2.13) we have for $l \ge 1$ that

$$u^{\infty,(l)}_{q,\dots,q}(\widehat{x}; d) \;=\; \sum_{j_l=1}^{2} \cdots \sum_{j_1=1}^{2} u^{\infty,(l)}_{q_{j_l},\dots,q_{j_1}}(\widehat{x}; d)\,, \qquad \widehat{x}, d \in S^1\,,$$

with

$$\begin{aligned} &u^{\infty,(l)}_{q_{j_l},\dots,q_{j_1}}(\widehat{x}; d) \\ &\quad := k^{2l} \int_{D_{j_l}} \cdots \int_{D_{j_1}} q_{j_l}(y_l) \cdots q_{j_1}(y_1) \Phi_k(y_l - y_{l-1}) \cdots \Phi_k(y_2 - y_1) e^{\mathrm{i}k(d\cdot y_1 - \widehat{x}\cdot y_l)} \,\mathrm{d}y_1 \cdots \mathrm{d}y_l\,, \\ &\qquad\qquad \widehat{x}, d \in S^1\,,\; j_1, \dots, j_l \in \{1, 2\}\,. \end{aligned} \tag{3.1}$$

As indicated by our notation, the term $u^{\infty,(l)}_{q_{j_l},\dots,q_{j_1}}$ describes the component of the far field pattern associated to the part of the scattered wave that results from l scattering processes starting at q_{j_1}, followed by q_{j_2} and so forth, until q_{j_l}. We note that these far field components satisfy the reciprocity relation

$$u^{\infty,(l)}_{q_{j_l},\dots,q_{j_1}}(\widehat{x}; d) \;=\; u^{\infty,(l)}_{q_{j_1},\dots,q_{j_l}}(-d; -\widehat{x})\,, \qquad \widehat{x}, d \in S^1\,. \tag{3.2}$$

Remark 3.1. Denoting the Fourier coefficients of $u^{\infty,(l)}_{q_{j_l},\dots,q_{j_1}}$ and $u^{\infty,(l)}_{q_{j_1},\dots,q_{j_l}}$ by $(a^{(l)}_{m,n})_{m,n}$ and $(b^{(l)}_{m,n})_{m,n}$, respectively, the same calculation as in (2.28), using (3.2) instead of (2.15), shows that

$$a^{(l)}_{m,n} \;=\; (-1)^{m-n} b^{(l)}_{-n,-m}\,, \qquad m, n \in \mathbb{Z}\,. \tag{3.3}$$

$\Diamond$

We define for $l \ge 1$ the far field operator components

$$F^{(l)}_{q_{j_l},\dots,q_{j_1}} : L^2(S^1) \to L^2(S^1)\,, \quad (F^{(l)}_{q_{j_l},\dots,q_{j_1}} g)(\widehat{x}) \;:=\; \int_{S^1} u^{\infty,(l)}_{q_{j_l},\dots,q_{j_1}}(\widehat{x}; d) g(d) \,\mathrm{d}s(d)\,.$$

Therewith, we can rewrite the Born far field operator of order p from (2.17) as

$$\begin{aligned} F^{(pB)}_q \;&=\; \sum_{l=1}^{p} F^{(l)}_{q,\dots,q} \;=\; \sum_{l=1}^{p} \Bigl(\sum_{j_l=1}^{2} \cdots \sum_{j_1=1}^{2} F^{(l)}_{q_{j_l},\dots,q_{j_1}} \Bigr) \\ &=\; F^{(pB)}_{q_1} + F^{(pB)}_{q_2} + \Bigl(\sum_{l=1}^{p} \sum_{(j_1,\dots,j_l)\in\{1,2\}^l \setminus (\{1\}^l \cup \{2\}^l)} F^{(l)}_{q_{j_l},\dots,q_{j_1}} \Bigr)\,. \end{aligned} \tag{3.4}$$

Remark 3.2. In the special case when $p = 1$ (i.e., Born approximation of order one, neglecting multiple scattering) the expansion (3.4) reduces to

$$F_q^{(1B)} = F_{q_1}^{(1B)} + F_{q_2}^{(1B)}, \tag{3.5}$$

while for $p = 2$ (i.e., Born approximation of order two, including multiple scattering of order one) we obtain that

$$F_q^{(2B)} = F_{q_1}^{(2B)} + F_{q_2}^{(2B)} + \left(F_{q_1,q_2}^{(2)} + F_{q_2,q_1}^{(2)}\right), \tag{3.6}$$

and for $p = 3$ (i.e., Born approximation of order three, including multiple scattering of order two) the expansion (3.4) reads

$$\begin{aligned} F_q^{(3B)} = F_{q_1}^{(3B)} + F_{q_2}^{(3B)} + \Big(&F_{q_1,q_2}^{(2)} + F_{q_2,q_1}^{(2)} \\ &+ F_{q_1,q_1,q_2}^{(3)} + F_{q_1,q_2,q_1}^{(3)} + F_{q_1,q_2,q_2}^{(3)} + F_{q_2,q_1,q_1}^{(3)} + F_{q_2,q_1,q_2}^{(3)} + F_{q_2,q_2,q_1}^{(3)}\Big). \end{aligned} \tag{3.7}$$

The Born far field operators $F_q^{(pB)}$, $p = 1, 2, 3$, on the left hand sides of (3.5)–(3.7) are increasingly accurate approximations of the far field operator F_q corresponding to the system of two scatterers, while the first two terms $F_{q_1}^{(pB)}$ and $F_{q_2}^{(pB)}$, $p = 1, 2, 3$, on the right hand sides of (3.5)–(3.7) approximate the far field operators F_{q_1} and F_{q_2} corresponding to the two individual scatterers in D_1 and D_2 increasingly well. On the other hand, the terms in brackets on the right hand side of (3.6)–(3.7) approximate multiple scattering effects. The goal of far field operator splitting is to recover approximations of F_{q_1} and F_{q_2} from F_q. To this end we will split F_q into three components, two corresponding to the first two terms on the right hand side of (3.4) (or (3.5)–(3.7)), and one corresponding to the terms in brackets on the right hand side of these equations. ♢

We have already seen at the end of Section 2.4 that the far field operator components $F_{q_1}^{(pB)}$ and $F_{q_2}^{(pB)}$ in (3.4) have sparse approximations in the subspaces $\mathcal{V}_{N_1}^{c_1}$ and $\mathcal{V}_{N_2}^{c_2}$ of non-evanescent far field operators associated to scatterers supported in $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$ with $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$, respectively. In fact the same reasoning shows that any term on the right hand side of (3.4) that is of the form $F_{q_1,q_{j_2},\dots,q_{j_{l-1}},q_1}^{(l)}$ or $F_{q_2,q_{j_2},\dots,q_{j_{l-1}},q_2}^{(l)}$, i.e., the first and the last interaction takes place at the same component of the scatterer, can be well approximated in $\mathcal{V}_{N_1}^{c_1}$ or $\mathcal{V}_{N_2}^{c_2}$, respectively. To obtain sparse approximations of the remaining terms on the right hand side of (3.4), which are all of the form $F_{q_1,q_{j_2},\dots,q_{j_{l-1}},q_2}^{(l)}$ or $F_{q_2,q_{j_2},\dots,q_{j_{l-1}},q_1}^{(l)}$, we introduce for $M, N \in \mathbb{N}$ the finite dimensional subspace $\mathcal{V}_{M,N} \subseteq \mathrm{HS}(L^2(S^1))$ by

$$\mathcal{V}_{M,N} := \left\{ G \in \mathrm{HS}(L^2(S^1)) \;\middle|\; Gg = \sum_{|m|\le M} \sum_{|n|\le N} a_{m,n}\, \boldsymbol{e}_m \langle g, \boldsymbol{e}_n \rangle_{L^2(S^1)} \right\}.$$

We note that $\mathcal{V}_{N,N} = \mathcal{V}_N$ from (2.25). Furthermore, we define for $b, c \in \mathbb{R}^2$ the *generalized far field translation operator*

$$\mathcal{T}_{b,c} : \mathrm{HS}(L^2(S^1)) \to \mathrm{HS}(L^2(S^1)), \quad \mathcal{T}_{b,c}G := T_b \circ G \circ T_{-c}, \tag{3.8}$$

where T_b and T_{-c} are defined as in (2.7). Then $\mathcal{T}_{c,c} = \mathcal{T}_c$ from (2.8).

Lemma 3.3. *Let $b, c \in \mathbb{R}^2$. The operator $\mathcal{T}_{b,c} \in \mathcal{L}(\mathrm{HS}(L^2(S^1)))$ is unitary with $\mathcal{T}_{b,c}^* = \mathcal{T}_{-b,-c}$.*

Proof. For any $G, H \in \mathrm{HS}(L^2(S^1))$ the definitions (3.8) and (2.7) imply that

$$\langle \mathcal{T}_{b,c}G, H\rangle_{\mathrm{HS}} = \mathrm{tr}\big((T_b G T_{-c})^* H\big) = \mathrm{tr}\big(T_{-b} H T_c G^*\big) = \mathrm{tr}\big(G^* T_{-b} H T_c\big) = \langle G, \mathcal{T}_{-b,-c}H\rangle_{\mathrm{HS}}.$$

This shows that $\mathcal{T}_{b,c}^* = \mathcal{T}_{-b,-c}$, which is the same as $\mathcal{T}_{b,c}^{-1}$. □

Substituting the Jacobi-Anger expansion (2.18) for the plane wave terms in (3.1) and proceeding as in Section 2.4, we find that terms of the form $\mathcal{T}_{c_1,c_2}F^{(l)}_{q_1,q_{j_2},\dots,q_{j_{l-1}},q_2}$ and $\mathcal{T}_{c_2,c_1}F^{(l)}_{q_2,q_{j_2},\dots,q_{j_{l-1}},q_1}$ have sparse approximations in $\mathcal{V}_{N_1,N_2}$ and $\mathcal{V}_{N_2,N_1}$ with $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$, respectively. Accordingly, we define

$$\mathcal{V}^{b,c}_{M,N} := \Big\{ G \in \mathrm{HS}(L^2(S^1)) \;\Big|\; Gg = \sum_{|m|\le M} \sum_{|n|\le N} a_{m,n} e^{\mathrm{i}kb\cdot(\,\cdot\,)} \boldsymbol{e}_m \langle g, e^{\mathrm{i}kc\cdot(\,\cdot\,)} \boldsymbol{e}_n \rangle_{L^2(S^1)} \,, \tag{3.9}$$

i.e., $\mathcal{V}^{b,c}_{M,N} = \{ G \in \mathrm{HS}(L^2(S^1)) \,|\, \mathcal{T}_{b,c}G \in \mathcal{V}_{M,N} \}$. We note that $\mathcal{V}^{c,c}_{N,N} = \mathcal{V}^c_N$ from (2.30). Then, $F^{(l)}_{q_1,q_{j_2},\dots,q_{j_{l-1}},q_2}$ and $F^{(l)}_{q_2,q_{j_2},\dots,q_{j_{l-1}},q_1}$ have sparse approximations in $\mathcal{V}^{c_1,c_2}_{N_1,N_2}$ and $\mathcal{V}^{c_2,c_1}_{N_2,N_1}$ with $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$, respectively. Denoting by $\mathcal{P}^{b,c}_{M,N} : \mathrm{HS}(L^2(S^1)) \to \mathrm{HS}(L^2(S^1))$ the orthonormal projection onto $\mathcal{V}^{b,c}_{M,N}$, the approximation error can be estimated similar to (2.26) and (2.31). We define $L_{q,k,D_j,D_l} : L^2(D_j) \to L^2(D_l)$ by

$$\left(L_{q,k,D_j,D_l} f \right)(x) := k^2 \int_{D_j} q(y) f(y) \Phi_k(x-y) \,\mathrm{d}y \,, \qquad x \in D_l \,.$$

Then,

$$\begin{aligned} &\| F^{(l)}_{q_{j_1},\dots,q_{j_l}} - \mathcal{P}^{c_{j_1},c_{j_l}}_{N_{j_1},N_{j_l}} F^{(l)}_{q_{j_1},\dots,q_{j_l}} \|^2_{\mathrm{HS}} \\ &\qquad\qquad \le 2\pi \|q_{j_l}\|_{L^\infty(D_{j_l})} \|L_{q,k,D_{j_1},D_{j_2}}\|_{\mathrm{HS}} \cdots \|L_{q,k,D_{j_{l-1}},D_{j_l}}\|_{\mathrm{HS}} \\ &\qquad\qquad\quad \times \Bigg(\sum_{|n|>N_{j_1}} \|J_n(|\cdot|)\|^2_{L^2(B_{kR_{j_1}}(0))} \sum_{|n|>N_{j_l}} \|J_n(|\cdot|)\|^2_{L^2(B_{kR_{j_l}}(0))} \Bigg)^{\frac12} . \end{aligned} \tag{3.10}$$

Recalling (2.21)–(2.22) and Figure 2.1, the right hand side of (3.10) is small as long as we choose $N_{j_1} \gtrsim kR_{j_1}$ and $N_{j_l} \gtrsim kR_{j_l}$.

Example 3.4. We consider a numerical example with contrast function $q = -0.5\chi_{D_1} + 2\chi_{D_2}$ for a kite-shaped scatterer D_1 and a nut-shaped scatterer D_2 as shown in Figure 3.1 (left). We choose $k = 5$ for the wave number and approximate the far field component $u^{\infty,(2)}_{q_1,q_2}(\widehat{x}_m; d_n)$ (cf. (3.1)) for $L = 256$ observation and incident directions as in (2.32) using trigonometric interpolation as described in [48, 50].

The scatterers D_1 and D_2 are contained in balls $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$ of radius $R_1 = 2.2$ and $R_2 = 1.1$ centered at $c_1 = (4,8)$ and $c_2 = (8,2)$, respectively. These are shown as dashed circles in Figure 3.1 (left). A two-dimensional fast Fourier transform of the shifted far field pattern $(e^{-\mathrm{i}k(c_2\cdot d_n - c_1\cdot\widehat{x}_m)} u^{\infty,(2)}_{q_1,q_2}(\widehat{x}_m; d_n))_{m,n} \in \mathbb{C}^{L\times L}$ yields an approximation of the Fourier coefficients $(a^{(2)}_{m,n})_{m,n}$ of $\mathcal{T}_{c_1,c_2}F^{\infty,(2)}_{q_1,q_2}$. In Figure 2.2 (right) the absolute values of these expansion coefficients are plotted for $-32 \le m,n \le 32$ in a logarithmic scale. It is nicely confirmed that the Fourier coefficients are essentially supported in a rectangle $[-N_1,N_1]\times[-N_2,N_2]$ with $N_1 \gtrsim kR_1 = 11$ and $N_2 \gtrsim kR_2 = 5.5$. The dashed rectangle in Figure 2.2 (right) corresponds to $N_1 = 13$ and $N_2 = 7$. $\Diamond$

We will use the sparse representations of far field operator components ensured by (2.31) and (3.10) to develop numerical algorithms for the inverse problem (a) introduced in Section 2.1 and to establish corresponding stability estimates. In Section 3.1 we first neglect multiple scattering and use (3.5) as ansatz for far field operator splitting. This leads to concise stability estimates that we compare with earlier results from [25, 29]. A more accurate method taking into account multiple scattering is the subject of Section 3.2.

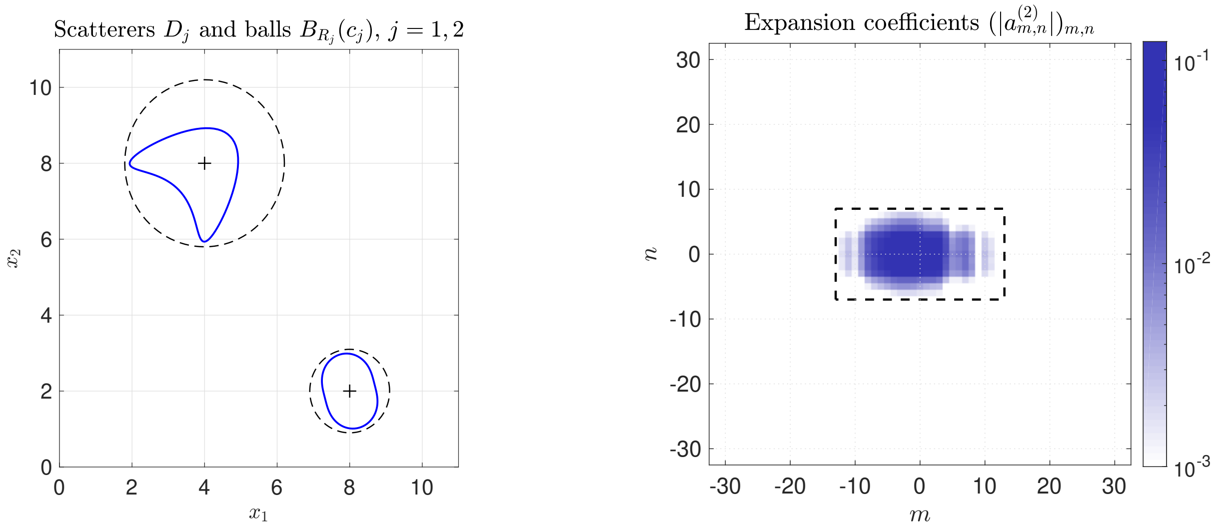


Figure 3.1: Left: Supports of kite-shaped scatterer D_1 and nut-shaped scatterer D_2 (solid) and balls $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$ (dashed) of radius $R_1 = 2.2$ and $R_2 = 1.1$ centered at $c_1 = (4, 8)$ and $c_2 = (8, 2)$ containing D_1 and D_2, respectively. Right: Absolute values of modulated Fourier coefficients $(a^{(2)}_{m,n})_{m,n}$ of far field operator component $F^{(2)}_{q_1,q_2}$ at $k = 5$. Dashed rectangle corresponds to expansion coefficients used by sparse approximation of $F^{(2)}_{q_1,q_2}$ in $\mathcal{V}^{c_1,c_2}_{N_1,N_2}$ with $N_1 = 13$ and $N_2 = 7$.

3.1 Far field operator splitting in Born approximation of order 1

Given the far field operator F_q associated to the two scattering objects with contrast functions q_1 and q_2 as defined at the beginning of Section 3, we seek approximations $\widetilde{F}_{q_1} \in \mathcal{V}^{c_1}_{N_1}$ and $\widetilde{F}_{q_2} \in \mathcal{V}^{c_2}_{N_2}$ of the far field operators F_{q_1} and F_{q_2}, corresponding to the two components of the scatterer, satisfying the least squares problem

$$F_q \overset{\mathrm{LS}}{=} \widetilde{F}_{q_1} + \widetilde{F}_{q_2} \qquad \text{in } \mathrm{HS}(L^2(S^1))\,, \tag{3.11}$$

or equivalently,

$$\langle F_q, \phi\rangle_{\mathrm{HS}} \;=\; \langle \widetilde{F}_{q_1}, \phi\rangle_{\mathrm{HS}} + \langle \widetilde{F}_{q_2}, \phi\rangle_{\mathrm{HS}} \qquad \text{for all } \phi \in \mathcal{V}^{c_1}_{N_1} \oplus \mathcal{V}^{c_2}_{N_2}\,.$$

Using the explicit representations of $\mathcal{V}^{c_1}_{N_1}$ and $\mathcal{V}^{c_2}_{N_2}$ in (2.30) this linear least squares problem can be solved straightforwardly using, e.g., conjugate gradients (see Section 5 below). In the following we discuss the conditioning of (3.11). To this end, we derive an upper bound on the cosine of the angle between $\mathcal{V}^{c_1}_{N_1}$ and $\mathcal{V}^{c_2}_{N_2}$, which immediately implies a bound the condition number of the splitting problem (3.11).[1]

Remark 3.5. In our stability analysis we will work with several different norms on $\mathrm{HS}(L^2(S^1))$. We recall that any Hilbert-Schmidt operator $G \in \mathrm{HS}(L^2(S^1))$ has an integral representation

$$(Gf)(\widehat{x}) \;=\; \int_{S^1} \kappa_G(\widehat{x}; d) f(d)\, \mathrm{d}s(d)\,, \qquad \widehat{x} \in S^1\,, \tag{3.12}$$

for some $\kappa_G \in L^2(S^1 \times S^1)$ (see [47, Thm. VI.23]). The kernel κ_G can be expanded into a Fourier series

$$\kappa_G(\widehat{x}; d) \;=\; \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a_{m,n}\, \boldsymbol{e}_m(\widehat{x}) \overline{\boldsymbol{e}_n(d)}\,, \qquad \widehat{x}, d \in S^1\,,$$

[1]The stability analysis presented here and in Section 4 below has been motivated by similar results for the discrete and continuous one-dimensional Fourier transform in [19].

and we use the notations

$$\|G\|_{L^p} := \|\kappa_G\|_{L^p(S^1\times S^1)} \qquad \text{and} \qquad \|G\|_{\ell^p\times\ell^p} := \|(a_{m,n})_{m,n}\|_{\ell^p\times\ell^p}\,, \qquad 1\le p\le\infty\,.$$

We also observe that, for any $G,H\in \mathrm{HS}(L^2(S^1))$,

$$\langle G,H\rangle_{\mathrm{HS}} = \langle G,H\rangle_{L^2} = \langle G,H\rangle_{\ell^2\times\ell^2} \tag{3.13}$$

(this can be seen as in the proof of [47, Thm. VI.23] and using Parseval's identity). Furthermore, we denote the area of the support of κ_G in $S^1\times S^1$ by $\|G\|_{L^0}$, and the number of nonzero Fourier coefficients of κ_G by $\|G\|_{\ell^0\times\ell^0}$. $\Diamond$

Lemma 3.6. *Let $c\in\mathbb{R}^2$, $c\neq 0$, and let $\mathcal{T}_c : \mathrm{HS}(L^2(S^1))\to\mathrm{HS}(L^2(S^1))$ be the far field translation operator in (2.8). Then, for all $1\le p\le\infty$,*

$$\|\mathcal{T}_c G\|_{L^p} = \|G\|_{L^p}\,, \qquad G\in\mathrm{HS}(L^2(S^1))\cap L^p(S^1\times S^1)\,, \tag{3.14}$$

and

$$\|\mathcal{T}_c G\|_{\ell^\infty\times\ell^\infty} \le (k|c|)^{-\frac{2}{3}}\|G\|_{\ell^1\times\ell^1}\,, \qquad G\in\mathrm{HS}(L^2(S^1))\cap\ell^1\times\ell^1\,. \tag{3.15}$$

If in addition $k|c|>2(M+N+1)$ for some $M,N\ge1$, then

$$\|\mathcal{T}_c G\|_{\ell^\infty[-M,M]\times\ell^\infty[-M,M]} \le (k|c|)^{-1}\|G\|_{\ell^1[-N,N]\times\ell^1[-N,N]}\,, \qquad G\in\mathrm{HS}(L^2(S^1))\cap\ell^1[-N,N]\times\ell^1[-N,N]\,. \tag{3.16}$$

We also have that

$$\|\mathcal{T}_c G\|_{L^\infty} \le \frac{1}{2\pi}\|\mathcal{T}_c G\|_{\ell^1\times\ell^1}\,, \qquad G\in\mathrm{HS}(L^2(S^1))\cap\ell^1\times\ell^1\,. \tag{3.17}$$

Proof. The isometry property (3.14) follows immediately from the definition of the far field translation operator $\mathcal{T}_c$ in (2.8) and (2.7).

To show (3.15), let $G\in\mathrm{HS}(L^2(S^1))\cap\ell^1\times\ell^1$ and denote the associated kernel in the integral representation (3.12) by $\kappa_G\in L^2(S^1\times S^1)$. The definitions (2.8) and (2.7) show that $\mathcal{T}_c$ acts on κ_G as a multiplication operator,

$$(\mathcal{T}_c\kappa_G)(\widehat{x};d) = e^{\mathrm{i}k(\widehat{x}-d)\cdot c}\kappa_G(\widehat{x};d)\,, \qquad \widehat{x},d\in S^1\,.$$

Accordingly, it operates on the Fourier coefficients $(a_{m,n})_{m,n}\in\ell^2\times\ell^2$ of κ_G as a convolution operator, and using (2.18) we obtain that the Fourier coefficients $(a^c_{m,n})_{m,n}$ of $\mathcal{T}_c\kappa_G$ satisfy

$$a^c_{m,n} = \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} a_{m-m',n-n'}\bigl(\mathrm{i}^{m'-n'}e^{-\mathrm{i}(m'-n')\arg c}J_{m'}(k|c|)J_{n'}(k|c|)\bigr)\,, \qquad m,n\in\mathbb{Z}\,.$$

Recalling that $|J_n(t)|<b/|t|^{1/3}$ for any $t\neq0$ with $b\approx0.6749$ (see [40, p. 199]) we find that

$$\|\mathcal{T}_c G\|_{\ell^\infty\times\ell^\infty} \le \|(J_n(k|c|))_n\|^2_{\ell^\infty}\|(a_{m,n})_{m,n}\|_{\ell^1\times\ell^1} \le (k|c|)^{-\frac{2}{3}}\|G\|_{\ell^1\times\ell^1}\,. \tag{3.18}$$

This shows (3.15).

Assuming that the support of $(a_{m,n})_{m,n}$ is contained in $[-N,N]\times[-N,N]$ we obtain similar to (3.18) that

$$\|\mathcal{T}_c G\|_{\ell^\infty[-M,M]\times\ell^\infty[-M,M]} \le \|(J_n(k|c|))_n\|^2_{\ell^\infty[-M-N,M+N]}\|(a_{m,n})_{m,n}\|_{\ell^1[-N,N]\times\ell^1[-N,N]}\,.$$

Using the assumption that $k|c| > 2(M+N+1)$ for some $M, N \geq 1$ and proceeding as in the proof of [31, Thm. 4.6] using [39, Thm. 2] gives

$$\sup_{|n|\leq M+N} |J_n(k|c|)| \leq b|c|^{-\frac{1}{2}} \qquad \text{with } b \approx 0.7595 .$$

Substituting this into (3.18) yields (3.16).

Finally, (3.17) follows from Hölder's inequality, which gives

$$\|\mathcal{T}_c G\|_{L^\infty} = \|\mathcal{T}_c \kappa_G\|_{L^\infty(S^1\times S^1)} = \Big\| \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a^c_{m,n}\, \boldsymbol{e}_m \overline{\boldsymbol{e}_n} \Big\|_{L^\infty(S^1\times S^1)} \leq \frac{1}{2\pi} \|\mathcal{T}_c G\|_{\ell^1\times\ell^1} .$$

□

Therewith we can now derive upper bounds on the cosine of the angle between $\mathcal{V}^{c_1}_{N_1}$ and $\mathcal{V}^{c_2}_{N_2}$. This extends a corresponding uncertainty principle for far field patterns from [29, Thms. 4.3 and 4.6] to far field operators. A related result on the correlation between the near fields radiated by due to two point sources located in c_1 and c_2, respectively, has recently been analyzed in the high-frequency limit in [23, Thm. 2.1].

Proposition 3.7. *Suppose that $G \in \mathcal{V}^{c_1}_{N_1}$ and $H \in \mathcal{V}^{c_2}_{N_2}$ for some $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$. Then,*

$$\frac{|\langle G, H\rangle_{\mathrm{HS}}|}{\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}}} \leq \frac{\sqrt{\|\mathcal{T}_{c_1}G\|_{\ell^0\times\ell^0}\|\mathcal{T}_{c_2}H\|_{\ell^0\times\ell^0}}}{(k|c_2-c_1|)^{\frac{2}{3}}} \leq \frac{(2N_1+1)(2N_2+1)}{(k|c_1-c_2|)^{\frac{2}{3}}} . \tag{3.19}$$

If in addition $k|c_1-c_2| > 2(N_1+N_2+1)$ and $N_1, N_2 \geq 1$, then

$$\frac{|\langle G, H\rangle_{\mathrm{HS}}|}{\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}}} \leq \frac{\sqrt{\|\mathcal{T}_{c_1}G\|_{\ell^0\times\ell^0}\|\mathcal{T}_{c_2}H\|_{\ell^0\times\ell^0}}}{k|c_2-c_1|} \leq \frac{(2N_1+1)(2N_2+1)}{k|c_1-c_2|} . \tag{3.20}$$

Proof. Using Lemma 2.2, Hölder's inequality, and (3.15) gives

$$\begin{aligned}
|\langle G, H\rangle_{\mathrm{HS}}| &= |\langle \mathcal{T}_{c_2}G, \mathcal{T}_{c_2}H\rangle_{\ell^2\times\ell^2}| \leq \|\mathcal{T}_{c_2}G\|_{\ell^\infty\times\ell^\infty}\|\mathcal{T}_{c_2}H\|_{\ell^1\times\ell^1} \\
&= \|\mathcal{T}_{c_2-c_1}\mathcal{T}_{c_1}G\|_{\ell^\infty\times\ell^\infty}\|\mathcal{T}_{c_2}H\|_{\ell^1\times\ell^1} \leq \frac{1}{(k|c_2-c_1|)^{\frac{2}{3}}}\|\mathcal{T}_{c_1}G\|_{\ell^1\times\ell^1}\|\mathcal{T}_{c_2}H\|_{\ell^1\times\ell^1} \\
&\leq \frac{\sqrt{\|\mathcal{T}_{c_1}G\|_{\ell^0\times\ell^0}\|\mathcal{T}_{c_2}H\|_{\ell^0\times\ell^0}}}{(k|c_2-c_1|)^{\frac{2}{3}}}\|\mathcal{T}_{c_1}G\|_{\ell^2\times\ell^2}\|\mathcal{T}_{c_2}H\|_{\ell^2\times\ell^2} \\
&\leq \frac{(2N_1+1)(2N_2+1)}{(k|c_2-c_1|)^{\frac{2}{3}}}\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}} .
\end{aligned}$$

If in addition $k|c_2-c_1| > 2(N_1+N_2+1)$ and $N_1, N_2 \geq 1$, then using (3.16) instead of (3.15) gives (3.20). □

In the following theorem F^δ_q denotes a noisy observation of a far field operator F_q corresponding to a scatterer $D = D_1 \cup D_2$ with two well-separated components $D_1 \subseteq B_{R_1}(c_1)$ and $D_2 \subseteq B_{R_2}(c_2)$. We assume that a priori information on the approximate location of these components is available, i.e., that the balls $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$ are known. Accordingly, we choose $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$ in the least squares problem (3.11) and compare the results for exact and noisy far field operators to establish a stability estimate. This generalizes the stability result for far field patterns from [29, Thm 5.3] to far field operators.

Theorem 3.8. *Suppose that $F_q, F_q^\delta \in \mathrm{HS}(L^2(S^1))$, and let $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$ such that*

$$C_{N_1,N_2}^{c_1,c_2} := \frac{(2N_1+1)(2N_2+1)}{(k|c_1-c_2|)^{\frac{2}{3}}} < 1\,.$$

Denote by $\widetilde{F}_{q_1}, \widetilde{F}_{q_2}$ and $\widetilde{F}_{q_1}^\delta, \widetilde{F}_{q_2}^\delta$ the solutions to the least squares problems

$$F_q \overset{\mathrm{LS}}{=} \widetilde{F}_{q_1} + \widetilde{F}_{q_2}\,, \qquad \widetilde{F}_{q_1} \in \mathcal{V}_{N_1}^{c_1}\,,\ \widetilde{F}_{q_2} \in \mathcal{V}_{N_2}^{c_2}\,, \tag{3.21a}$$

$$F_q^\delta \overset{\mathrm{LS}}{=} \widetilde{F}_{q_1}^\delta + \widetilde{F}_{q_2}^\delta\,, \qquad \widetilde{F}_{q_1}^\delta \in \mathcal{V}_{N_1}^{c_1}\,,\ \widetilde{F}_{q_2}^\delta \in \mathcal{V}_{N_2}^{c_2}\,, \tag{3.21b}$$

respectively. Then, for $j=1,2$,

$$\|\widetilde{F}_{q_j} - \widetilde{F}_{q_j}^\delta\|_{\mathrm{HS}}^2 \le \bigl(1-(C_{N_1,N_2}^{c_1,c_2})^2\bigr)^{-1}\|F_q - F_q^\delta\|_{\mathrm{HS}}^2\,. \tag{3.22}$$

Remark 3.9. If $N_1, N_2 \ge 1$, $k|c_1-c_2| > 2(N_1+N_2+1)$ in Theorem 3.8, and

$$\widetilde{C}_{N_1,N_2}^{c_1,c_2} := \frac{(2N_1+1)(2N_2+1)}{k|c_1-c_2|} < 1\,.$$

then (3.22) remains true with $C_{N_1,N_2}^{c_1,c_2}$ replaced by $\widetilde{C}_{N_1,N_2}^{c_1,c_2}$. ◊

Proof. Denoting $F := F_q - F_q^\delta \in \mathrm{HS}(L^2(S^1))$, $\widetilde{F}_1 := \widetilde{F}_{q_1} - \widetilde{F}_{q_1}^\delta \in \mathcal{V}_{N_1}^{c_1}$ and $\widetilde{F}_2 := \widetilde{F}_{q_2} - \widetilde{F}_{q_2}^\delta \in \mathcal{V}_{N_2}^{c_2}$, the least squares property (3.21) implies that

$$\|F\|_{\mathrm{HS}}^2 = \|\widetilde{F}_1 + \widetilde{F}_2\|_{\mathrm{HS}}^2 + \|F - (\widetilde{F}_1 + \widetilde{F}_2)\|_{\mathrm{HS}}^2 \ge \|\widetilde{F}_1 + \widetilde{F}_2\|_{\mathrm{HS}}^2\,.$$

Therefore, using (3.19) and the arithmetic-geometric mean inequality yields

$$\begin{aligned}
\|F\|_{\mathrm{HS}} &\ge \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 - 2|\langle \widetilde{F}_1, \widetilde{F}_2\rangle_{\mathrm{HS}}|\\
&\ge \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 - 2C_{N_1,N_2}^{c_1,c_2}\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_2\|_{\mathrm{HS}}\\
&\ge \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 - (C_{N_1,N_2}^{c_1,c_2})^2\|\widetilde{F}_1\|_{\mathrm{HS}}^2 - \|\widetilde{F}_2\|_{\mathrm{HS}}^2\,.
\end{aligned}$$

This shows (3.22).

To obtain the improved estimate stated in Remark 3.9, one has to replace (3.19) by (3.20) in this calculation. □

Remark 3.10. Including the reciprocity property (2.29), which is satisfied by the Fourier coefficients of actual far field patterns, into the definition (2.30) of the subspaces of non-evanescent far field operators allows to improve the constants $C_{N_1,N_2}^{c_1,c_2}$ and $\widetilde{C}_{N_1,N_2}^{c_1,c_2}$ in the stability estimates in Theorem 3.8. In fact, any $G \in \mathcal{V}_N$ with

$$Gg = \sum_{|m|\le N}\sum_{|n|\le N} a_{m,n}\,\boldsymbol{e}_m\langle g, \boldsymbol{e}_n\rangle_{L^2(S^1)}\,, \qquad g \in L^2(S^1)\,,$$

that satisfies (2.29) can be rewritten as

$$Gg = \sum_{|m|\le N}\sum_{n=-m}^{N} a_{m,n}\Bigl(1-\frac{1}{2}\delta_{n,-m}\Bigr)\bigl(\boldsymbol{e}_m\langle g, \boldsymbol{e}_n\rangle_{L^2(S^1)} + (-1)^{n+m}\overline{\boldsymbol{e}_n}\langle g, \overline{\boldsymbol{e}_m}\rangle_{L^2(S^1)}\bigr)\,.$$

Here, $\delta_{n,-m}$ denotes the Kronecker delta. Accordingly, we define for any $c \in \mathbb{R}^2$ and $N \in \mathbb{N}$ the finite dimensional subspace

$$\mathcal{W}_N^c := \bigl\{G \in \mathrm{HS}(L^2(S^1)) \bigm| \mathcal{T}_c G \in \mathcal{W}_N\bigr\}\,, \tag{3.23}$$

with

$$\mathcal{W}_N := \Big\{ G \in \mathrm{HS}(L^2(S^1)) \;\Big|\; Gg = \sum_{|m|\le N} \sum_{n=-m}^{N} a_{m,n} \Big(1 - \frac{1}{2}\delta_{n,-m}\Big) \big(\boldsymbol{e}_m \langle g, \boldsymbol{e}_n\rangle_{L^2(S^1)} + (-1)^{n+m} \overline{\boldsymbol{e}_n} \langle g, \overline{\boldsymbol{e}_m}\rangle_{L^2(S^1)} \big) \Big\} . \tag{3.24}$$

Then $\mathcal{W}_N^c \subseteq \mathcal{V}_N^c$ and thus the first inequality in (3.19) and (3.20), respectively, remains valid. However, since $\|G\|_{\ell^0\times\ell^0} \le (2N+1)(N+1)$ for any $G \in \mathcal{W}_N$, only about half as many coefficients are required when using this representation compared to the representation in (2.30). Accordingly, given $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$, and replacing $\mathcal{V}_{N_1}^{c_1}$ and $\mathcal{V}_{N_2}^{c_2}$ by $\mathcal{W}_{N_1}^{c_1}$ and $\mathcal{W}_{N_2}^{c_2}$ in the least squares problems (3.21) yields the improved constants

$$C_{N_1,N_2}^{c_1,c_2} := \frac{\sqrt{(2N_1+1)(N_1+1)(2N_2+1)(N_2+1)}}{(k|c_1-c_2|)^{\frac{2}{3}}} ,$$
$$\widetilde{C}_{N_1,N_2}^{c_1,c_2} := \frac{\sqrt{(2N_1+1)(N_1+1)(2N_2+1)(N_2+1)}}{k|c_1-c_2|}$$

in the stability estimates in Theorem 3.8. These constants are better by a factor of about 1/2. ♢

Remark 3.11. Recalling (2.4) it would be possible to split the far field operator F_q corresponding to a scatterer $D = D_1 \cup D_2$ with two components into two far field operators corresponding to each scatterer D_1 and D_2 by splitting the far field patterns $u_q^\infty(\,\cdot\,; d)$ for each incident direction $d \in S^1$ individually using the methods developed in [25, 28, 29], which also neglect multiple scattering. However, comparing the stability estimate for least squares splitting of single far field patterns $u_q^\infty(\,\cdot\,; d)$ established in [29, Thm. 5.3] with the stability estimate for least squares splitting of whole far field operators F_q in Theorem 3.8 we find that the latter is more stable. The stability estimates in [29, Thm. 5.3] and in Theorem 3.8 are of the same structure but the constants $C_{N_1,N_2}^{c_1,c_2}$ and $\widetilde{C}_{N_1,N_2}^{c_1,c_2}$ in Theorem 3.8 are the squares of the corresponding constants in [29, Thm. 5.3]. Taking into account the additional improvement that can be obtained by using the reciprocity principle as outlined in Remark 3.10, this shows a significant advantage of the algorithms developed in this work, when interested in splitting whole far field operators. Furthermore, the improved version of the splitting scheme, taking into account multiple scattering, that we discuss in the next section, is not applicable for splitting single far field patterns at all. ♢

3.2 Far field operator splitting in Born approximation of order 2

Next we include multiple scattering into the algorithm for far field operator splitting. The main idea is to replace the ansatz (3.5) in the least squares problem (3.11) by (3.6). We have already developed sparse representations of the additional components $F_{q_1,q_2}^{(2)}$ and $F_{q_2,q_1}^{(2)}$ at the beginning of Section 3 (see (3.9) and (3.10)). We note that using higher order Born approximations such as, e.g., (3.7) does not work with our approach, because terms like $F_{q_1,q_2,q_1}^{(3)}$ and $F_{q_2,q_1,q_2}^{(3)}$ cannot be distinguished from $F_{q_1}^{(3B)}$ and $F_{q_2}^{(3B)}$, respectively, using the techniques at hand. Essentially, we can only characterize the first and the last scatterer that interacts with the scattered field components described by far field operator components in (3.4).

Given the far field operator F_q associated to the two scattering objects with contrast functions q_1 and q_2 as defined at the beginning of Section 3, we seek approximations $\widetilde{F}_{q_1} \in \mathcal{V}_{N_1}^{c_1}$ and $\widetilde{F}_{q_2} \in \mathcal{V}_{N_2}^{c_2}$ of the far field operators F_{q_1} and F_{q_2}, corresponding to the two components of the scatterer, satisfying the least squares problem

$$F_q \overset{\text{LS}}{=} \widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1} \qquad \text{in } \mathrm{HS}(L^2(S^1)) \tag{3.25}$$

for some $\widetilde{F}_{q_1,q_2} \in \mathcal{V}^{c_1,c_2}_{N_1,N_2}$ and $\widetilde{F}_{q_2,q_1} \in \mathcal{V}^{c_2,c_1}_{N_2,N_1}$. This is equivalent to the Galerkin condition

$$\langle F_q, \phi\rangle_{\mathrm{HS}} \;=\; \langle \widetilde{F}_{q_1}, \phi\rangle_{\mathrm{HS}} + \langle \widetilde{F}_{q_2}, \phi\rangle_{\mathrm{HS}} + \langle \widetilde{F}_{q_1,q_2}, \phi\rangle_{\mathrm{HS}} + \langle \widetilde{F}_{q_2,q_1}, \phi\rangle_{\mathrm{HS}}$$
$$\text{for all } \phi \in \mathcal{V}^{c_1}_{N_1} \oplus \mathcal{V}^{c_2}_{N_2} \oplus \mathcal{V}^{c_1,c_2}_{N_1,N_2} \oplus \mathcal{V}^{c_2,c_1}_{N_2,N_1}\,.$$

In the following we discuss the conditioning of (3.25)

Lemma 3.12. *Let $c_1, c_2 \in \mathbb{R}^2$, and let $\mathcal{T}_{c_1,c_2} : \mathrm{HS}(L^2(S^1)) \to \mathrm{HS}(L^2(S^1))$ be the generalized far field translation operator in (3.8). Then, for any $G \in \mathrm{HS}(L^2(S^1)) \cap \ell^1 \times \ell^1$,*

$$\|\mathcal{T}_{c_1,c_2} G\|_{\ell^\infty\times\ell^\infty} \;\leq\; \begin{cases} (k^2|c_1||c_2|)^{-\frac13}\|G\|_{\ell^1\times\ell^1} & \text{if } c_1, c_2 \neq 0\,,\\ (k|c_1|)^{-\frac13}\|G\|_{\ell^1\times\ell^1} & \text{if } c_1 \neq 0\,,\ c_2 = 0\,,\\ (k|c_2|)^{-\frac13}\|G\|_{\ell^1\times\ell^1} & \text{if } c_1 = 0\,,\ c_2 \neq 0\,. \end{cases} \tag{3.26}$$

Proof. As in the proof of Lemma 3.6 we denote by $\kappa_G \in L^2(S^1\times S^1)$ the integral kernel of $G \in \mathrm{HS}(L^2(S^1)) \cap \ell^1 \times \ell^1$. According to (3.8), $\mathcal{T}_{c_1,c_2}$ acts as on κ_G as a multiplication operator,

$$(\mathcal{T}_{c_1,c_2}\kappa_G)(\widehat{x};d) \;=\; e^{\mathrm{i}k(\widehat{x}\cdot c_1 - d\cdot c_2)}\kappa_G(\widehat{x};d)\,, \qquad \widehat{x}, d \in S^1\,.$$

Therefore, it operates on the Fourier coefficients $(a_{m,n})_{m,n} \in \ell^2 \times \ell^2$ of κ_G as a convolution operator, and the Fourier coefficients $(a^{c_1,c_2}_{m,n})_{m,n}$ of $\mathcal{T}_{c_1,c_2}\kappa_G$ satisfy

$$a^{c_1,c_2}_{m,n} \;=\; \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} a_{m-m',n-n'}\bigl(\mathrm{i}^{m'-n'}e^{-\mathrm{i}m'\arg c_1}e^{\mathrm{i}n'\arg c_2}J_{m'}(k|c_1|)J_{n'}(k|c_2|)\bigr)\,, \qquad m,n\in\mathbb{Z}\,.$$

Using Hölder's inequality we find that

$$\|\mathcal{T}_{c_1,c_2}G\|_{\ell^\infty\times\ell^\infty} \;\leq\; \|(J_n(k|c_1|))_n\|_{\ell^\infty}\|(J_n(k|c_2|))_n\|_{\ell^\infty}\|G\|_{\ell^1\times\ell^1}\,.$$

Now recalling that $|J_n(t)| < \min\{1, 1/|t|^{1/3}\}$ (see, e.g., [25] and the proof of Lemma 3.12) shows (3.26). ☐

Proposition 3.13. *Suppose that $G \in \mathcal{V}^{c_1,c_2}_{N_1,N_2}$ and $H \in \mathcal{V}^{c_1',c_2'}_{N_1',N_2'}$ for some $c_1, c_1', c_2, c_2' \in \mathbb{R}^2$ and $N_1, N_1', N_2, N_2' \in \mathbb{N}$. Then,*

$$\frac{|\langle G,H\rangle_{\mathrm{HS}}|}{\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}}} \;\leq\; \begin{cases} \frac{\sqrt{(2N_1+1)(2N_2+1)(2N_1'+1)(2N_2'+1)}}{(k|c_1-c_1'|)^{\frac13}(k|c_2-c_2'|)^{\frac13}} & \text{if } c_1 \neq c_1' \text{ and } c_2 \neq c_2'\,,\\ \frac{\sqrt{(2N_1+1)(2N_2+1)(2N_1'+1)(2N_2'+1)}}{(k|c_1-c_1'|)^{\frac13}} & \text{if } c_1 = c_1' \text{ and } c_2 \neq c_2'\,,\\ \frac{\sqrt{(2N_1+1)(2N_2+1)(2N_1'+1)(2N_2'+1)}}{(k|c_2-c_2'|)^{\frac13}} & \text{if } c_1 \neq c_1' \text{ and } c_2 = c_2'\,. \end{cases} \tag{3.27}$$

Proof. Suppose that $c_1 \neq c_1'$ and $c_2 \neq c_2'$. Using Lemma 3.3, Hölder's inequality, and (3.26) gives

$$
\begin{aligned}
|\langle G,H\rangle_{\mathrm{HS}}| &= |\langle \mathcal{T}_{c_1',c_2'}G, \mathcal{T}_{c_1',c_2'}H\rangle_{\ell^2\times\ell^2}| \;\le\; \|\mathcal{T}_{c_1',c_2'}G\|_{\ell^\infty\times\ell^\infty}\|\mathcal{T}_{c_1',c_2'}H\|_{\ell^1\times\ell^1}\\
&= \|\mathcal{T}_{c_1'-c_1,c_2'-c_2}\mathcal{T}_{c_1,c_2}G\|_{\ell^\infty\times\ell^\infty}\|\mathcal{T}_{c_1',c_2'}H\|_{\ell^1\times\ell^1}\\
&\le \frac{1}{(k|c_1-c_1'|)^{\frac13}(k|c_2-c_2'|)^{\frac13}}\|\mathcal{T}_{c_1,c_2}G\|_{\ell^1\times\ell^1}\|\mathcal{T}_{c_1',c_2'}H\|_{\ell^1\times\ell^1}\\
&\le \frac{\sqrt{\|\mathcal{T}_{c_1,c_2}G\|_{\ell^0\times\ell^0}\|\mathcal{T}_{c_1',c_2'}H\|_{\ell^0\times\ell^0}}}{(k|c_1-c_1'|)^{\frac13}(k|c_2-c_2'|)^{\frac13}}\|\mathcal{T}_{c_1,c_2}G\|_{\ell^2\times\ell^2}\|\mathcal{T}_{c_1',c_2'}H\|_{\ell^2\times\ell^2}\\
&= \frac{\sqrt{(2N_1+1)(2N_2+1)(2N_1'+1)(2N_2'+1)}}{(k|c_1-c_1'|)^{\frac13}(k|c_2-c_2'|)^{\frac13}}\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}}\,.
\end{aligned}
$$

This shows the first inequality in (3.27), and the other two inequalities follow by using the corresponding estimates in (3.26) in this calculation. □

In the following theorem we establish a stability result for the least squares problem (3.25) that is similar to Theorem 3.8. Again F_q^δ denotes a noisy observation of a far field operator F_q corresponding to a scatterer $D = D_1\cup D_2$ with two well-separated components $D_1 \subseteq B_{R_1}(c_1)$ and $D_2 \subseteq B_{R_2}(c_2)$. Assuming that the a priori information $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$ on the approximate location of these components is available, we choose $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$ in (3.11) and compare the solutions of this least squares problem for exact and noisy far field operators.

Theorem 3.14. *Suppose that $F_q, F_q^\delta \in \mathrm{HS}(L^2(S^1))$, let $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$, and define*

$$
C_{N_1,N_2}^{c_1,c_2} := \frac{(2N_1+1)(2N_2+1)}{(k|c_1-c_2|)^{\frac23}}\,.
$$

We assume that, for all $(j,l) \in \{1,2\}^2$,

$$
M_{j,l} := \sqrt{C_{N_1,N_2}^{c_1,c_2}}\Big(\sqrt{C_{N_1,N_2}^{c_1,c_2}} + (2N_j+1) + (2N_l+1)\Big) < 1\,.
$$

Denote by $\widetilde F_{q_1}, \widetilde F_{q_2}, \widetilde F_{q_1,q_2}, \widetilde F_{q_2,q_1}$ and $\widetilde F_{q_1}^\delta, \widetilde F_{q_2}^\delta, \widetilde F_{q_1,q_2}^\delta, \widetilde F_{q_2,q_1}^\delta$ the solutions to the least squares problems

$$
\begin{aligned}
F_q &\overset{\mathrm{LS}}{=} \widetilde F_{q_1} + \widetilde F_{q_2} + \widetilde F_{q_1,q_2} + \widetilde F_{q_2,q_1}\,, &\qquad \widetilde F_{q_j} \in \mathcal{V}_{N_j}^{c_j}\,,\ \widetilde F_{q_j,q_l} \in \mathcal{V}_{N_j,N_l}^{c_j,c_l}\,, \qquad (3.28a)\\
F_q^\delta &\overset{\mathrm{LS}}{=} \widetilde F_{q_1}^\delta + \widetilde F_{q_2}^\delta + \widetilde F_{q_1,q_2}^\delta + \widetilde F_{q_2,q_1}^\delta\,, &\qquad \widetilde F_{q_j}^\delta \in \mathcal{V}_{N_j}^{c_j}\,,\ \widetilde F_{q_j,q_l}^\delta \in \mathcal{V}_{N_j,N_l}^{c_j,c_l}\,, \qquad (3.28b)
\end{aligned}
$$

respectively. Then,

$$
\|\widetilde F_{q_j} - \widetilde F_{q_j}^\delta\|_{\mathrm{HS}}^2 \le \big(1-M_{j,j}\big)^{-1}\|F_q - F_q^\delta\|_{\mathrm{HS}}^2\,, \qquad j=1,2\,. \tag{3.29}
$$

Proof. We define $F := F_q - F_q^\delta \in \mathrm{HS}(L^2(S^1))$, $\widetilde F_1 := \widetilde F_{q_1} - \widetilde F_{q_1}^\delta \in \mathcal{V}_{N_1}^{c_1}$, $\widetilde F_2 := \widetilde F_{q_2} - \widetilde F_{q_2}^\delta \in \mathcal{V}_{N_2}^{c_2}$, $\widetilde F_{1,2} := \widetilde F_{q_1,q_2} - \widetilde F_{q_1,q_2}^\delta \in \mathcal{V}_{N_1,N_2}^{c_1,c_2}$, and $\widetilde F_{2,1} := \widetilde F_{q_2,q_1} - \widetilde F_{q_2,q_1}^\delta \in \mathcal{V}_{N_2,N_1}^{c_2,c_1}$. Then the least squares property (4.6) implies that

$$
\begin{aligned}
\|F\|_{\mathrm{HS}}^2 &= \|\widetilde F_1 + \widetilde F_2 + \widetilde F_{1,2} + \widetilde F_{2,1}\|_{\mathrm{HS}}^2 + \|F - (\widetilde F_1 + \widetilde F_2 + \widetilde F_{1,2} + \widetilde F_{2,1})\|_{\mathrm{HS}}^2\\
&\ge \|\widetilde F_1 + \widetilde F_2 + \widetilde F_{1,2} + \widetilde F_{2,1}\|_{\mathrm{HS}}^2\,.
\end{aligned}
$$

Therefore, using the arithmetic-geometric mean inequality yields and (3.27) yields

$$\begin{aligned}
\|F\|_{\mathrm{HS}}^2 \geq\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \\
& - 2|\langle \widetilde{F}_1, \widetilde{F}_2\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_1, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_1, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}| \\
& - 2|\langle \widetilde{F}_2, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_2, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_{1,2}, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}| \\
\geq\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 \Bigl(1 - \frac{|\langle \widetilde{F}_1, \widetilde{F}_2\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_2\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_1, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_1, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_{2,1}\|_{\mathrm{HS}}}\Bigr) \\
& + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 \Bigl(1 - \frac{|\langle \widetilde{F}_1, \widetilde{F}_2\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_2\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_2, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_2\|_{\mathrm{HS}}\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_2, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_2\|_{\mathrm{HS}}\|\widetilde{F}_{2,1}\|_{\mathrm{HS}}}\Bigr) \\
& + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 \Bigl(1 - \frac{|\langle \widetilde{F}_1, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_2, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_2\|_{\mathrm{HS}}\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_{1,2}, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}\|\widetilde{F}_{2,1}\|_{\mathrm{HS}}}\Bigr) \\
& + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \Bigl(1 - \frac{|\langle \widetilde{F}_1, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_1\|_{\mathrm{HS}}\|\widetilde{F}_{2,1}\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_2, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_2\|_{\mathrm{HS}}\|\widetilde{F}_{2,1}\|_{\mathrm{HS}}} - \frac{|\langle \widetilde{F}_{1,2}, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}|}{\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}\|\widetilde{F}_{2,1}\|_{\mathrm{HS}}}\Bigr) \\
\geq\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2(1 - M_{1,1}) + \|\widetilde{F}_2\|_{\mathrm{HS}}^2(1 - M_{2,2}) + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2(1 - M_{1,2}) + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2(1 - M_{2,1})\,.
\end{aligned}$$

This shows (4.7). □

Remark 3.15. If $N_1, N_2 \geq 1$, $k|c_1 - c_2| > 2(N_1 + N_2 + 1)$ in Theorem 3.14, and

$$\widetilde{C}_{N_1,N_2}^{c_1,c_2} := \frac{(2N_1+1)(2N_2+1)}{k|c_1 - c_2|} < 1\,,$$

then (4.7) remains true with $C_{N_1,N_2}^{c_1,c_2}$ replaced by $\widetilde{C}_{N_1,N_2}^{c_1,c_2}$. This can be shown analogous to the improved estimate in Remark 3.9. ♢

Remark 3.16. Similar to Remark 3.10, including the reciprocity properties (2.29) and (3.3) into the ansatz spaces for the least squares problems (4.6), the constants $M_{j,l}$, $(j,l) \in \{1,2\}^2$, in Theorem 3.14 can be improved by a factor of about 1/2. Given $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$, one needs to replace the subspaces $\mathcal{V}_{N_1}^{c_1}$ and $\mathcal{V}_{N_2}^{c_2}$ in (4.6) by $\mathcal{W}_{N_1}^{c_1}$ and $\mathcal{W}_{N_2}^{c_2}$ as defined in (3.23). Moreover, the unknowns $\widetilde{F}_{q_1,q_2}$, $\widetilde{F}_{q_2,q_1}$ and $\widetilde{F}_{q_1,q_2}^\delta$, $\widetilde{F}_{q_2,q_1}^\delta$, respectively, have to be combined into a single unknown using (3.3). ♢

If a priori knowledge of the sizes R_j of the individual scatterers $D_j \subseteq B_{R_j}(c_j)$, $j = 1, 2$, which is required to determine the parameters $N_j \gtrsim kR_j$ of the ansatz spaces in the least squares formulation (3.25), is not available but at least the approximate positions c_1 and c_2 are known, then (3.25) can be replaced by an $\ell^1 \times \ell^1$ minimization problem. In Theorem 3.17 below we present this approach and establish an associated stability result. As before, F_q^δ represents a noisy observation of a far field operator F_q corresponding to a scatterer $D = D_1 \cup D_2$ with two well-separated components $D_1 \subseteq B_{R_1}(c_1)$ and $D_2 \subseteq B_{R_2}(c_2)$. Accordingly, let

$$F_q \approx \widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1} \tag{3.30}$$

be an approximate decomposition of the exact far field operator with $\widetilde{F}_{q_1} \in \mathcal{V}_{N_1}^{c_1}$, $\widetilde{F}_{q_2} \in \mathcal{V}_{N_2}^{c_2}$, $\widetilde{F}_{q_1,q_2} \in \mathcal{V}_{N_1,N_2}^{c_1,c_2}$, and $\widetilde{F}_{q_2,q_1} \in \mathcal{V}_{N_2,N_1}^{c_2,c_1}$ for some $N_1 \gtrsim kR_1$ and $N_2 \gtrsim kR_2$, which could be the least squares solution of (3.25) but does not have to be computed. The bound $\delta_0 > 0$ in (3.31) describes the accuracy of this approximate solution, which in case of the least squares solution corresponds to the error of the second order Born approximation and projection errors as in (2.31) and (3.10). Now the optimization problem (3.33) seeks an approximate decomposition of the given noisy far field operator F_q^δ in the spaces $\mathcal{V}_{N_1}^{c_1}$, $\mathcal{V}_{N_2}^{c_2}$, $\mathcal{V}_{N_1,N_2}^{c_1,c_2}$, and $\mathcal{V}_{N_2,N_1}^{c_2,c_1}$ but without specifying $N_1, N_2 > 0$ in advance. Here, the assumption (3.32) guarantees that the approximate split (3.30) is feasible. The theorem gives a stability estimate for the solution of this minimization problem.

Theorem 3.17. *Suppose that $F_q \in \mathrm{HS}(L^2(S^1))$, let $c_1, c_2 \in \mathbb{R}^2$ and $N_1, N_2 \in \mathbb{N}$ such that for all $(j,l) \in \{1,2\}^2$,*

$$C_{N_j,N_l}^{c_j,c_l} := \frac{12(2N_j+1)(2N_l+1)}{(k|c_1-c_2|)^{\frac{1}{3}}} < 1\,.$$

We assume that $\widetilde{F}_{q_1} \in \mathcal{V}_{N_1}^{c_1}$, $\widetilde{F}_{q_2} \in \mathcal{V}_{N_2}^{c_2}$, $\widetilde{F}_{q_1,q_2} \in \mathcal{V}_{N_1,N_2}^{c_1,c_2}$, and $\widetilde{F}_{q_2,q_1} \in \mathcal{V}_{N_2,N_1}^{c_2,c_1}$ are such that

$$\|F_q - (\widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1})\|_{\mathrm{HS}} < \delta_0 \tag{3.31}$$

for some $\delta_0 > 0$. Moreover, suppose that $F_q^\delta \in \mathrm{HS}(L^2(S^1))$ and $\delta \geq 0$ satisfy

$$\delta \geq \delta_0 + \|F_q - F_q^\delta\|_{\mathrm{HS}} \tag{3.32}$$

and let $(\widetilde{F}_{q_1}^\delta, \widetilde{F}_{q_2}^\delta, \widetilde{F}_{q_1,q_2}^\delta, \widetilde{F}_{q_2,q_1}^\delta) \in \mathrm{HS}(L^2(S^1))^4$ denote the solution to

$$\begin{aligned}\underset{F_{q_1},F_{q_2},F_{q_1,q_2},F_{q_2,q_1}}{\text{minimize}} \quad & \|\mathcal{T}_{c_1}F_{q_1}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2}F_{q_2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_1,c_2}F_{q_1,q_2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2,c_1}F_{q_2,q_1}\|_{\ell^1\times\ell^1}\\ & \text{subject to} \quad \|F_q^\delta - (F_{q_1} + F_{q_2} + F_{q_1,q_2} + F_{q_2,q_1})\|_{\mathrm{HS}} \leq \delta\,. \end{aligned}\tag{3.33}$$

Then,

$$\|\widetilde{F}_{q_j} - \widetilde{F}_{q_j}^\delta\|_{\mathrm{HS}}^2 \leq \big(1 - C_{N_j,N_j}^{c_j,c_j}\big)^{-1} 4\delta^2\,, \qquad j = 1,2\,.$$

Proof. We define $F := F_q - F_q^\delta$, $\widetilde{F}_1 := \widetilde{F}_{q_1} - \widetilde{F}_{q_1}^\delta$, $\widetilde{F}_2 := \widetilde{F}_{q_2} - \widetilde{F}_{q_2}^\delta$, $\widetilde{F}_{1,2} := \widetilde{F}_{q_1,q_2} - \widetilde{F}_{q_1,q_2}^\delta$, and $\widetilde{F}_{2,1} := \widetilde{F}_{q_2,q_1} - \widetilde{F}_{q_2,q_1}^\delta$. Moreover, we denote the $\ell^0 \times \ell^0$ support of $\mathcal{T}_{c_1}\widetilde{F}_{q_1}$, $\mathcal{T}_{c_2}\widetilde{F}_{q_2}$, $\mathcal{T}_{c_1,c_2}\widetilde{F}_{q_1,q_2}$, and $\mathcal{T}_{c_2,c_1}\widetilde{F}_{q_2,q_1}$ by W_1, W_2, $W_{1,2}$, and $W_{2,1}$, respectively, and their complements by W_1^c, W_2^c, $W_{1,2}^c$, and $W_{2,1}^c$. We estimate, for $j \in \{1,2\}$,

$$\begin{aligned}\|\mathcal{T}_{c_j}\widetilde{F}_{q_j}^\delta\|_{\ell^1\times\ell^1} &= \|\mathcal{T}_{c_j}(\widetilde{F}_{q_j} + \widetilde{F}_j)\|_{\ell^1\times\ell^1} = \|\mathcal{T}_{c_j}(\widetilde{F}_{q_j} + \widetilde{F}_j)\|_{(\ell^1\times\ell^1)(W_j)} + \|\mathcal{T}_{c_j}\widetilde{F}_j\|_{(\ell^1\times\ell^1)(W_j^c)}\\ &= \|\mathcal{T}_{c_j}(\widetilde{F}_{q_j} + \widetilde{F}_j)\|_{(\ell^1\times\ell^1)(W_j)} + \|\mathcal{T}_{c_j}\widetilde{F}_j\|_{\ell^1\times\ell^1} - \|\mathcal{T}_{c_j}\widetilde{F}_j\|_{(\ell^1\times\ell^1)(W_j)}\\ &\geq \|\mathcal{T}_{c_j}\widetilde{F}_{q_j}\|_{(\ell^1\times\ell^1)(W_j)} + \|\mathcal{T}_{c_j}\widetilde{F}_j\|_{\ell^1\times\ell^1} - 2\|\mathcal{T}_{c_j}\widetilde{F}_j\|_{(\ell^1\times\ell^1)(W_j)}\end{aligned}$$

and we can do the same for $\widetilde{F}_{q_j,q_l}^\delta$ with $j,l \in \{1,2\}$, $j \neq l$. Together with (3.33) this gives

$$\begin{aligned}&\|\mathcal{T}_{c_1}\widetilde{F}_1\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{\ell^1\times\ell^1}\\ &\qquad \leq 2\big(\|\mathcal{T}_{c_1}\widetilde{F}_1\|_{(\ell^1\times\ell^1)(W_1)} + \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{(\ell^1\times\ell^1)(W_2)}\\ &\qquad\qquad + \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{(\ell^1\times\ell^1)(W_{1,2})} + \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{(\ell^1\times\ell^1)(W_{2,1})}\big)\,.\end{aligned}\tag{3.34}$$

Using (3.31)–(3.33) and (3.26) we obtain

$$
\begin{aligned}
4\delta^2 \ge\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \\
&- 2|\langle \widetilde{F}_1, \widetilde{F}_2\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_1, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_1, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}| \\
&- 2|\langle \widetilde{F}_2, \widetilde{F}_{1,2}\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_2, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}| - 2|\langle \widetilde{F}_{1,2}, \widetilde{F}_{2,1}\rangle_{\mathrm{HS}}| \\
\ge\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \\
&- \frac{2}{(k|c_1-c_2|)^{\frac{2}{3}}} \big(\|\mathcal{T}_{c_1}\widetilde{F}_1\|_{\ell^1\times\ell^1} \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{\ell^1\times\ell^1} \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{\ell^1\times\ell^1} \big) \\
&- \frac{2}{(k|c_1-c_2|)^{\frac{1}{3}}} \big(\|\mathcal{T}_{c_1}\widetilde{F}_1\|_{\ell^1\times\ell^1} \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_1}\widetilde{F}_1\|_{\ell^1\times\ell^1} \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{\ell^1\times\ell^1} \\
&\qquad + \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{\ell^1\times\ell^1} \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{\ell^1\times\ell^1} \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{\ell^1\times\ell^1} \big) \,.
\end{aligned}
$$

For $a_1, \dots, a_4 \ge 0$ a simple calculation shows that $\sum_i \sum_{j\neq i} a_i a_j \le \frac{3}{4}\left(\sum_i a_i\right)^2 \le 3\sum_i a_i^2$. Together with (3.34) and the Cauchy Schwarz inequality this implies

$$
\begin{aligned}
4\delta^2 \ge\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \\
&- \frac{3}{4(k|c_1-c_2|)^{\frac{1}{3}}} \big(\|\mathcal{T}_{c_1}\widetilde{F}_1\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{\ell^1\times\ell^1} \big)^2 \\
\ge\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \\
&- \frac{3}{(k|c_1-c_2|)^{\frac{1}{3}}} \big(\|\mathcal{T}_{c_1}\widetilde{F}_1\|_{\ell^1\times\ell^1(W_1)} + \|\mathcal{T}_{c_2}\widetilde{F}_2\|_{\ell^1\times\ell^1(W_2)} \\
&\qquad + \|\mathcal{T}_{c_1,c_2}\widetilde{F}_{1,2}\|_{\ell^1\times\ell^1(W_{1,2})} + \|\mathcal{T}_{c_2,c_1}\widetilde{F}_{2,1}\|_{\ell^1\times\ell^1(W_{2,1})} \big)^2 \\
\ge\ & \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \\
&- \frac{3}{(k|c_1-c_2|)^{\frac{1}{3}}} \big(|W_1|^{\frac{1}{2}} \|\widetilde{F}_1\|_{\mathrm{HS}} + |W_2|^{\frac{1}{2}} \|\widetilde{F}_2\|_{\mathrm{HS}} + |W_{1,2}|^{\frac{1}{2}} \|\widetilde{F}_{1,2}\|_{\mathrm{HS}} + |W_{2,1}|^{\frac{1}{2}} \|\widetilde{F}_{2,1}\|_{\mathrm{HS}} \big)^2 \\
\ge\ & (1 - C_{N_1,N_1}^{c_1,c_1}) \|\widetilde{F}_1\|_{\mathrm{HS}}^2 + (1 - C_{N_2,N_2}^{c_2,c_2}) \|\widetilde{F}_2\|_{\mathrm{HS}}^2 + (1 - C_{N_1,N_2}^{c_1,c_2}) \big(\|\widetilde{F}_{1,2}\|_{\mathrm{HS}}^2 + \|\widetilde{F}_{2,1}\|_{\mathrm{HS}}^2 \big) \,.
\end{aligned}
$$

□

The assumptions of Theorem 3.17 are much more restrictive than the assumptions of Theorem 3.14 and the stability estimate is worse, but the $\ell^1 \times \ell^1$ minimization algorithm requires less a priori information and our numerical results in Section 5 suggest that both algorithms work comparably well.

Remark 3.18. Similar to Remark 3.16 the reciprocity principle can be used to improve the constants $C_{N_j,N_l}^{c_j,c_l}$, $(j,l) \in \{1,2\}^2$, in Theorem 3.17 by a factor of about 1/2. ♢

4 Far field operator completion

We consider the inverse problem (b) introduced in Section 2.1. Suppose that the scatterer $D \subseteq B_R(c)$ is contained in a ball of radius $R > 0$ centered at $c \in \mathbb{R}^2$, and that the far field pattern u_q^∞ cannot be observed on some subset $\Omega \subseteq S^1 \times S^1$. Without loss of generality we assume that Ω is symmetric in the sense of reciprocity, i.e., that $\Omega = \Omega^r$ with

$$\Omega^r := \{(-d, -\widehat{x}) \mid (\widehat{x}; d) \in \Omega\} \,.$$

Otherwise (2.3) can be used to extend the observed far field patterns from $S^1 \times S^1 \setminus \Omega$ to $S^1 \times S^1 \setminus (\Omega \cup \Omega^r)$. We define the infinite dimensional subspace

$$\mathcal{V}_\Omega := \Bigl\{ G \in \mathrm{HS}(L^2(S^1)) \;\Big|\; Gg = \int_{S^1} \chi_\Omega(\cdot, d)\alpha(\cdot; d)g(d) \,\mathrm{d}s(d)\,,\ \alpha \in L^2(S^1 \times S^1) \Bigr\}\,.$$

Given the observed far field operator $F_q|_{S^1\times S^1\setminus\Omega}$ from (2.5) we seek approximations $\widetilde{F_q} \in \mathcal{V}_N^c$ and $\widetilde{B} \in \mathcal{V}_\Omega$ of the far field operator F_q and of its non-observable part $B := F_q|_{S^1\times S^1\setminus\Omega} - F_q$ satisfying the least squares problem

$$F_q|_{S^1\times S^1\setminus\Omega} \overset{\mathrm{LS}}{=} \widetilde{F_q} + \widetilde{B} \qquad \text{in } \mathrm{HS}(L^2(S^1))\,, \tag{4.1}$$

or equivalently,

$$\langle F_q|_{S^1\times S^1\setminus\Omega}, \phi\rangle_{\mathrm{HS}} = \langle \widetilde{F_q}, \phi\rangle_{\mathrm{HS}} + \langle \widetilde{B}, \phi\rangle_{\mathrm{HS}} \qquad \text{for all } \phi \in \mathcal{V}_N^c \oplus \mathcal{V}_\Omega\,.$$

Again, the condition number of (4.1) is given by the cosecant of the angle between $\mathcal{V}_N^c$ and $\mathcal{V}_\Omega$, which can be estimated using upper bounds on the cosine of the angle between these two spaces.

Proposition 4.1. *Suppose that $G \in \mathcal{V}_N^c$ and $H \in \mathcal{V}_\Omega$ for some $c \in \mathbb{R}^2$, $N \in \mathbb{N}$, and $\Omega \subseteq S^1 \times S^1$. Then,*

$$\frac{|\langle G, H\rangle_{\mathrm{HS}}|}{\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}}} \leq \frac{(2N+1)\sqrt{|\Omega|}}{2\pi}\,. \tag{4.2}$$

Proof. Using (3.13), Hölder's inequality, (3.14) with $p = \infty$, and (3.17) yields

$$\begin{aligned}
|\langle G, H\rangle_{\mathrm{HS}}| &= |\langle G, H\rangle_{L^2}| \leq \|G\|_{L^\infty}\|H\|_{L^1} = \|\mathcal{T}_c G\|_{L^\infty}\|H\|_{L^1} \leq \frac{1}{2\pi}\|\mathcal{T}_c G\|_{\ell^1\times\ell^1}\|H\|_{L^1} \\
&\leq \frac{\sqrt{\|\mathcal{T}_c G\|_{\ell^0\times\ell^0}\|H\|_{L^0}}}{2\pi}\|\mathcal{T}_c G\|_{\ell^2\times\ell^2}\|H\|_{L^2} \leq \frac{(2N+1)\sqrt{|\Omega|}}{2\pi}\|G\|_{\mathrm{HS}}\|H\|_{\mathrm{HS}}\,.
\end{aligned}$$

□

In the following theorem $F_q^\delta|_{S^1\times S^1\setminus\Omega}$ denotes a noisy version of a restricted far field operator $F_q|_{S^1\times S^1\setminus\Omega}$ corresponding to a scatterer $D \subseteq B_R(c)$ that cannot be observed on $\Omega \subseteq S^1 \times S^1$. We assume that a priori information on the approximate location of the scatterer is available, i.e., that the ball $B_R(c)$ is known, and we establish a stability estimate for the least squares problem (4.1).

Theorem 4.2. *Suppose that $F_q, F_q^\delta \in \mathrm{HS}(L^2(S^1))$, $c \in \mathbb{R}^2$, $N \in \mathbb{N}$, and $\Omega \subseteq S^1 \times S^1$ such that*

$$C_N^\Omega := \frac{(2N+1)\sqrt{|\Omega|}}{2\pi} < 1\,.$$

Denote by $\widetilde{F}_q$ and $\widetilde{F}_q^\delta$ the solutions to the least squares problems

$$F_q|_{S^1\times S^1\setminus\Omega} \overset{\mathrm{LS}}{=} \widetilde{F}_q + \widetilde{B}_q\,, \qquad \widetilde{F}_q \in \mathcal{V}_N^c\,,\ \widetilde{B}_q \in \mathcal{V}_\Omega\,, \tag{4.3a}$$

$$F_q^\delta|_{S^1\times S^1\setminus\Omega} \overset{\mathrm{LS}}{=} \widetilde{F}_q^\delta + \widetilde{B}_q^\delta\,, \qquad \widetilde{F}_q^\delta \in \mathcal{V}_N^c\,,\ \widetilde{B}_q^\delta \in \mathcal{V}_\Omega\,, \tag{4.3b}$$

respectively. Then,

$$\|\widetilde{F}_q^\delta - \widetilde{F}_q\|_{\mathrm{HS}}^2 \leq \bigl(1 - (C_N^\Omega)^2\bigr)^{-1}\bigl\|F_q^\delta|_{S^1\times S^1\setminus\Omega} - F_q|_{S^1\times S^1\setminus\Omega}\bigr\|_{\mathrm{HS}}^2\,, \tag{4.4a}$$

$$\|\widetilde{B}_q^\delta - \widetilde{B}_q\|_{\mathrm{HS}}^2 \leq \bigl(1 - (C_N^\Omega)^2\bigr)^{-1}\bigl\|F_q^\delta|_{S^1\times S^1\setminus\Omega} - F_q|_{S^1\times S^1\setminus\Omega}\bigr\|_{\mathrm{HS}}^2\,. \tag{4.4b}$$

Proof. Denoting $F := F_q|_{S^1\times S^1\setminus\Omega} - F_q^\delta|_{S^1\times S^1\setminus\Omega}$, $\widetilde{F} := \widetilde{F}_q - \widetilde{F}_q^\delta \in \mathcal{V}_N^c$ and $\widetilde{B} := \widetilde{B}_q - \widetilde{B}_q^\delta \in \mathcal{V}_\Omega$, the least squares property (4.3) implies that

$$\|F\|_{\mathrm{HS}}^2 \;=\; \|\widetilde{F}+\widetilde{B}\|_{\mathrm{HS}}^2 + \|F-(\widetilde{F}+\widetilde{B})\|_{\mathrm{HS}}^2 \;\geq\; \|\widetilde{F}+\widetilde{B}\|_{\mathrm{HS}}^2 \,.$$

Therefore, using (4.2) and the arithmetic-geometric mean inequality yields

$$\begin{aligned}\|F\|_{\mathrm{HS}}^2 \;&\geq\; \|\widetilde{F}\|_{\mathrm{HS}}^2 + \|\widetilde{B}\|_{\mathrm{HS}}^2 - 2|\langle \widetilde{F},\widetilde{B}\rangle_{\mathrm{HS}}| \;\geq\; \|\widetilde{F}\|_{\mathrm{HS}}^2 + \|\widetilde{B}\|_{\mathrm{HS}}^2 - 2C_N^\Omega \|\widetilde{F}\|_{\mathrm{HS}}\|\widetilde{B}\|_{\mathrm{HS}}\\ &\geq\; \|\widetilde{F}\|_{\mathrm{HS}}^2 + \|\widetilde{B}\|_{\mathrm{HS}}^2 - (C_N^\Omega)^2\|\widetilde{F}\|_{\mathrm{HS}}^2 - \|\widetilde{B}\|_{\mathrm{HS}}^2 \,.\end{aligned} \tag{4.5}$$

This shows the first estimate in (4.4), and the second estimate is obtained by interchanging the roles of $\widetilde{F}$ and $\widetilde{B}$ in the last step of (4.5). □

Remark 4.3. Similar to Remark 3.10, including the reciprocity property (2.29) and replacing the space $\mathcal{V}_N^c$ in Theorem 4.2 by $\mathcal{W}_N^c$ from (3.23)–(3.24) allows to improve the constant C_N^Ω in the stability estimate (4.4) by a factor of about $1/\sqrt{2}$. ♢

Remark 4.4. As we already discussed in Remark 3.11 for far field operator splitting, it would be possible to complete the far field operator F_q by completing the far field patterns $u_q^\infty(\cdot;d)$ for each incident direction $d \in S^1$ individually using the methods developed in [29]. The stability estimate in [29, Thm. 5.5] and in Theorem 4.2 have the same structure, but again the constant C_N^Ω in Theorem 4.2 is roughly the square of the corresponding constant in [29, Thm. 5.5]. This means that one should use the data completion scheme developed in this work, when completing whole far field operators. ♢

Several variants of this data completion scheme have been discussed for single far field patterns in [29], and these can in principle also be generalized to the setting considered in this work. For instance one can avoid the required a priori knowledge of the size $R > 0$ of the scatterer by reformulating the data completion problem as an $\ell^1 \times \ell^1$ minimization problem similar to Theorem 3.17. Since the size of the support of the scatterer enters the stability estimate in Theorem 4.2 via the parameter $N \gtrsim kR$, it is often advantageous to combine far field operator completion with far field operator splitting to obtain better stability properties. In Theorem 4.5 below we provide a corresponding stability result. It can be shown by combining the proofs of the Theorems 3.14 and 4.2, and the proof is therefore omitted.

Theorem 4.5. *Suppose that $F_q, F_q^\delta \in \mathrm{HS}(L^2(S^1))$, and let $c_1, c_2 \in \mathbb{R}^2$, $N_1, N_2 \in \mathbb{N}$, and $\Omega \subseteq S^1 \times S^1$. We define*

$$C_{N_1,N_2}^{c_1,c_2} \;:=\; \frac{(2N_1+1)(2N_2+1)}{(k|c_1-c_2|)^{\frac{2}{3}}} \qquad \textit{and} \qquad C_{N_j,N_l}^\Omega \;:=\; \frac{\sqrt{(2N_j+1)(2N_l+1)}\sqrt{|\Omega|}}{2\pi}\,,$$

where $(j,l) \in \{1,2\}^2$. We assume that, for all $(j,l) \in \{1,2\}^2$,

$$\begin{aligned} M_{j,l} \;:=\; \sqrt{C_{N_1,N_2}^{c_1,c_2}}\Big(\sqrt{C_{N_1,N_2}^{c_1,c_2}} + (2N_j+1) + (2N_l+1)\Big) + C_{N_j,N_l}^\Omega \;&<\; 1\,,\\ C_{N_1,N_2}^\Omega + C_{N_1,N_2}^\Omega + C_{N_2,N_1}^\Omega + C_{N_2,N_2}^\Omega \;&<\; 1\,.\end{aligned}$$

Denote by $\widetilde{F}_{q_1}, \widetilde{F}_{q_2}, \widetilde{F}_{q_1,q_2}, \widetilde{F}_{q_2,q_1}, \widetilde{B}_q$ and $\widetilde{F}_{q_1}^\delta, \widetilde{F}_{q_2}^\delta, \widetilde{F}_{q_1,q_2}^\delta, \widetilde{F}_{q_2,q_1}^\delta, \widetilde{B}_q^\delta$ the solutions to the least squares problems

$$F_q|_{S^1\times S^1\setminus\Omega} \overset{\mathrm{LS}}{=} \widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1} + \widetilde{B}_q\,, \quad \widetilde{F}_{q_j} \in \mathcal{V}_{N_j}^{c_j}\,,\ \widetilde{F}_{q_j,q_l} \in \mathcal{V}_{N_j,N_l}^{c_j,c_l}\,,\ \widetilde{B}_q \in \mathcal{V}_\Omega\,, \tag{4.6a}$$

$$F_q^\delta|_{S^1\times S^1\setminus\Omega} \overset{\mathrm{LS}}{=} \widetilde{F}_{q_1}^\delta + \widetilde{F}_{q_2}^\delta + \widetilde{F}_{q_1,q_2}^\delta + \widetilde{F}_{q_2,q_1}^\delta + \widetilde{B}_q^\delta\,, \quad \widetilde{F}_{q_j}^\delta \in \mathcal{V}_{N_j}^{c_j}\,,\ \widetilde{F}_{q_j,q_l}^\delta \in \mathcal{V}_{N_j,N_l}^{c_j,c_l}\,,\ \widetilde{B}_q^\delta \in \mathcal{V}_\Omega\,, \tag{4.6b}$$

respectively. Then,

$$\|\widetilde{F}_{q_j} - \widetilde{F}^\delta_{q_j}\|^2_{\mathrm{HS}} \le \bigl(1 - (M_{j,j} + C^\Omega_{N_j,N_j})\bigr)^{-1} \|F_q - F^\delta_q\|^2_{\mathrm{HS}}\,, \qquad j = 1,2\,, \tag{4.7a}$$

$$\|\widetilde{B}_q - \widetilde{B}^\delta_q\|^2_{\mathrm{HS}} \le \bigl(1 - (C^\Omega_{N_1,N_1} + C^\Omega_{N_1,N_2} + C^\Omega_{N_2,N_1} + C^\Omega_{N_2,N_2})\bigr)^{-1} \|F_q - F^\delta_q\|^2_{\mathrm{HS}}\,. \tag{4.7b}$$

5 Numerical examples

We briefly comment on the numerical implementation and illustrate the performance of the far field operator splitting and completion methods discussed in Sections 3.2 and 4. As before, let $D = D_1 \cup D_2$ be a scatterer with two well separated components $D_j \subseteq B_{R_j}(c_j)$ for some $c_j \in \mathbb{R}^2$ and $R_j > 0$, $j = 1,2$. Assuming that far field patterns cannot be observed on a subdomain $\Omega \subseteq S^1 \times S^1$ satisfying $\Omega^r = \Omega$, i.e., given the noisy restricted far field operator $G := F^\delta_q|_{S^1\times S^1\setminus\Omega}$ we aim to recover the non-observable part $B := F_q|_{S^1\times S^1\setminus\Omega} - F_q$ and the far field operator components F_{q_1} and F_{q_2} associated to the two scatterer components individually.

The least squares approach from Theorem 4.5 requires to solve the linear least squares problem

$$\begin{aligned} G &\overset{\mathrm{LS}}{=} \widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1} + \widetilde{B}\,, \\ &\qquad \widetilde{F}_{q_1} \in \mathcal{V}^{c_1}_{N_1}\,,\ \widetilde{F}_{q_2} \in \mathcal{V}^{c_2}_{N_2}\,,\ \widetilde{F}_{q_1,q_2} \in \mathcal{V}^{c_1,c_2}_{N_1,N_2}\,,\ \widetilde{F}_{q_2,q_1} \in \mathcal{V}^{c_2,c_1}_{N_2,N_1}\,,\ \widetilde{B} \in \mathcal{V}_\Omega\,. \end{aligned}$$

Letting $\mathcal{P}_{j,l} := \mathcal{P}^{c_j,c_l}_{N_j,N_l}$ denote the orthogonal projection in $\mathrm{HS}(L^2(S^1))$ onto $\mathcal{V}^{c_j,c_l}_{N_j,N_l}$, $j,l = 1,2$, and $\mathcal{P}_\Omega$ the orthogonal projection onto $\mathcal{V}_\Omega$, this is equivalent to the linear system

$$\begin{array}{rcl} \widetilde{F}_{q_1} + \mathcal{P}_{1,1}\mathcal{P}_{2,2}\widetilde{F}_{q_2} + \mathcal{P}_{1,1}\mathcal{P}_{1,2}\widetilde{F}_{q_1,q_2} + \mathcal{P}_{1,1}\mathcal{P}_{2,1}\widetilde{F}_{q_2,q_1} + \mathcal{P}_{1,1}\mathcal{P}_\Omega\widetilde{B} &=& \mathcal{P}_{1,1}G\,, \\ \mathcal{P}_{2,2}\mathcal{P}_{1,1}\widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \mathcal{P}_{2,2}\mathcal{P}_{1,2}\widetilde{F}_{q_1,q_2} + \mathcal{P}_{1,1}\mathcal{P}_{2,1}\widetilde{F}_{q_2,q_1} + \mathcal{P}_{2,2}\mathcal{P}_\Omega\widetilde{B} &=& \mathcal{P}_{2,2}G\,, \\ \mathcal{P}_{1,2}\mathcal{P}_{1,1}\widetilde{F}_{q_1} + \mathcal{P}_{1,2}\mathcal{P}_{2,2}\widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \mathcal{P}_{1,2}\mathcal{P}_{2,1}\widetilde{F}_{q_2,q_1} + \mathcal{P}_{1,2}\mathcal{P}_\Omega\widetilde{B} &=& \mathcal{P}_{1,2}G\,, \\ \mathcal{P}_{2,1}\mathcal{P}_{1,1}\widetilde{F}_{q_1} + \mathcal{P}_{2,1}\mathcal{P}_{2,2}\widetilde{F}_{q_2} + \mathcal{P}_{2,1}\mathcal{P}_{1,2}\widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1} + \mathcal{P}_{2,1}\mathcal{P}_\Omega\widetilde{B} &=& \mathcal{P}_{2,1}G\,, \\ \mathcal{P}_\Omega\mathcal{P}_{1,1}\widetilde{F}_{q_1} + \mathcal{P}_\Omega\mathcal{P}_{2,2}\widetilde{F}_{q_2} + \mathcal{P}_\Omega\mathcal{P}_{1,2}\widetilde{F}_{q_1,q_2} + \mathcal{P}_\Omega\mathcal{P}_{2,1}\widetilde{F}_{q_2,q_1} + \widetilde{B} &=& 0\,. \end{array} \tag{5.1}$$

If the assumptions of Theorem 4.5 are fulfilled, then the self-adjoint block operator on the left side of (5.1) is strictly diagonally dominant and thus positive definit. Therefore, (5.1) can be solved efficiently using conjugate gradients.

On the other hand, the $\ell^1 \times \ell^1$ approach in Theorem 3.17 can also be extended to simultaneously splitting and completing far field operators. Introducing a Lagrange multiplier $1/\mu$ the associated unconstrained optimization problem consists in minimizing the functional

$$\begin{aligned} \Psi_\mu(F_{q_1}, F_{q_2}, F_{q_1,q_2}, F_{q_2,q_1}) &:= \Bigl\| G - \mathcal{P}_\Omega(F_{q_1} + F_{q_2} + F_{q_1,q_2} + F_{q_2,q_1}) \Bigr\|^2_{\mathrm{HS}} \\ &\quad + \mu\bigl(\|\mathcal{T}_{c_1}F_{q_1}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2}F_{q_2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_1,c_2}F_{q_1,q_2}\|_{\ell^1\times\ell^1} + \|\mathcal{T}_{c_2,c_1}F_{q_2,q_1}\|_{\ell^1\times\ell^1}\bigr) \end{aligned} \tag{5.2}$$

with respect to $(F_{q_1}, F_{q_2}, F_{q_1,q_2}, F_{q_2,q_1}) \in \mathrm{HS}(L^2(S^1))^4$ for some suitably chosen regularization parameter $\mu > 0$. The unique minimizer $(\widetilde{F}_{q_1}, \widetilde{F}_{q_2}, \widetilde{F}_{q_1,q_2}, \widetilde{F}_{q_2,q_1})$ of Ψ_μ can be approximated using iterative soft thresholding (see [4, 16]). Then $\widetilde{F}_q := \widetilde{F}_{q_1} + \widetilde{F}_{q_2} + \widetilde{F}_{q_1,q_2} + \widetilde{F}_{q_2,q_1}$ is an approximation to F_q, and $\widetilde{B} := -\mathcal{P}_\Omega\widetilde{F}_q$ approximates the non-observable part B.

We discuss three examples with a scattering object $D = D_1 \cup D_2$, where $D_1 \subseteq B_{R_1}(c_1)$ has the shape of a nut and $D_2 \subseteq B_{R_2}(c_2)$ the shape of a kite. Throughout, we use $q = -0.5\chi_{D_1} + \chi_{D_2}$ for the contrast function. Example 5.1 is concerned with far field operator splitting only, i.e., $\Omega = \emptyset$, while in Example 5.2 we consider far field operator completion but without splitting, i.e., we view D as a single object that is contained in a large ball $B_R(0)$. Finally, in Example 5.3 we

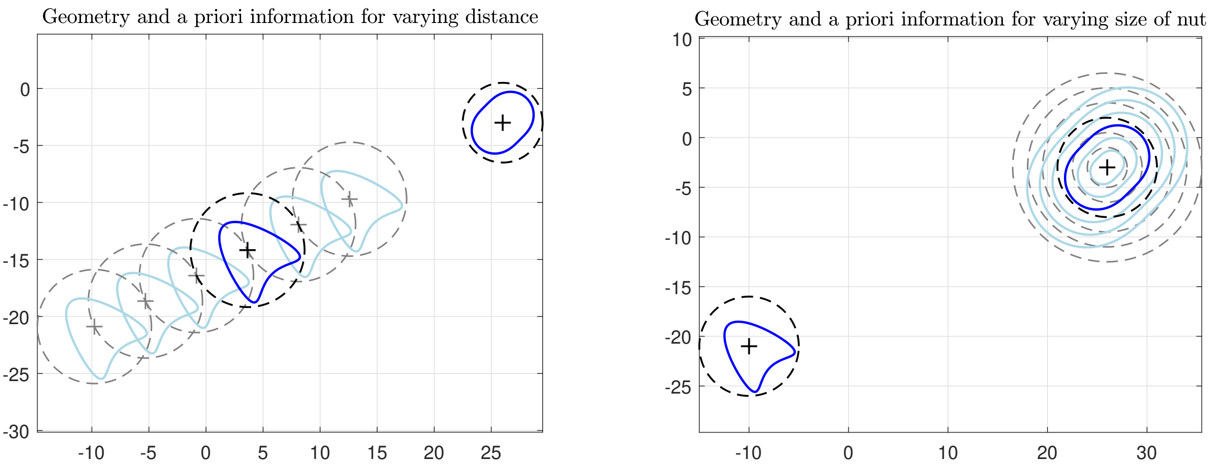


Figure 5.1: Left: Geometry of scatterer (solid) and a priori information on location and size of components (dashed) in Example 5.1 for varying distance ($|c_1 - c_2| = 25$ highlighted). Right: Geometry of two scatterer (solid) and a priori information on location and size of components (dashed) in Example 5.1 for varying size R_1 of nut-shaped component ($R_1 = 5$ highlighted).

combine far field operator completion with far field operator splitting and show that this yields better results for certain geometrical setups.

Example 5.1. We fix the wave number $k = 0.5$, i.e., the wave number is $\lambda = 2\pi/k \approx 12.57$. In our first test we study the accuracy of numerical reconstructions for far field operator splitting depending on the distance $|c_1 - c_2|$ between the centers of the two components of the scatterer. To this end we vary $|c_1 - c_2|$ as depicted in Figure 5.1 (left). We use $R_1 = 3.5$, $R_2 = 5$, $c_1 = (26, -3)$, and consider

$$c_2 = (26, -3) + \frac{|c_1 - c_2|}{\sqrt{5}}(-2, -1) \qquad \text{with} \qquad |c_1 - c_2| \in \{15, 20, 25, 30, 35, 40\}\,.$$

In our second test we fix the size of D_2 and the positions of both scatterers and vary the size R_1 of D_1, as shown in Figure 5.1 (right). We use $c_1 = (26, -3)$, $c_2 = (-10, -21)$, i.e., $|c_1 - c_2| \approx 40$, $R_2 = 5$, and consider

$$R_1 \in \{2, 3.5, 5, 6.5, 8, 9.5\}\,.$$

We apply a Nyström method to evaluate the far field patterns u_q^∞ for $L = 256$ incident and observation directions on an equidistant grid on the unit sphere for each configuration. This then gives an approximation of the associated far field operators F_q (see Example 2.6). We add 5% complex valued uniformly distributed additive relative error to F_q and denote the result by F_q^δ.

To solve the far field operator splitting problem using the least squares approach in Theorem 3.14, we assume the dashed circles in Figure 5.1 to be known a priori, i.e., we choose $N_1 = 5$ and $N_2 = 7$ for the first test, and $N_1 \in \{3, 5, 7, 9, 11, 13\}$ and $N_2 = 7$ for the second test. For the $l^1 \times l^1$ approach in Theorem 3.17, we use $\mu = 10^{-3}$ for the regularization parameter in (5.2). In contrast to the least squares method no a priori information on the approximate size of the components D_1 and D_2 but only the approximate positions c_1 and c_2 are required. We also simulate the far field operators F_{q_j}, $j = 1, 2$, corresponding to the two scatterers individually using the Nyström method and compare them to the results of our reconstruction method by evaluating relative reconstruction errors

$$\epsilon_{\mathrm{rel}}^j := \frac{\|F_{q_j} - \widetilde{F}_{q_j}\|_{\mathrm{HS}}}{\|F_{q_j}\|_{\mathrm{HS}}}\,, \qquad j = 1, 2\,.$$

In Table 5.1 these relative errors are shown. The left part of the table corresponds to our

	least squares		$\ell^1 \times \ell^1$-minimization	
$\lvert c_1 - c_2 \rvert$	ϵ^1_{rel}	ϵ^2_{rel}	ϵ^1_{rel}	ϵ^2_{rel}
15	0.171	0.340	0.099	0.077
20	0.023	0.068	0.043	0.054
25	0.019	0.020	0.027	0.023
30	0.017	0.018	0.029	0.022
35	0.015	0.015	0.030	0.023
40	0.015	0.013	0.030	0.021

	least squares		$\ell^1 \times \ell^1$-minimization	
R_1	ϵ^1_{rel}	ϵ^2_{rel}	ϵ^1_{rel}	ϵ^2_{rel}
2	0.025	0.012	0.056	0.019
3.5	0.014	0.012	0.026	0.020
5	0.016	0.013	0.021	0.022
6.5	0.029	0.017	0.021	0.026
8	0.044	0.020	0.028	0.037
9.5	0.079	0.022	0.054	0.050

Table 5.1: Left: Relative errors of far field operator splitting for varying distance $|c_1-c_2|$. Right: Relative errors of far field operator splitting for varying size R_1 of scatterer D_1.

first test with varying distances between the two components of the scatterers, and the right part of the table corresponds to our second test with varying size for the component D_1 of the scatterer. In both tables, the first two columns correspond to the reconstructions that have been obtained using the least squares approach, and the last two columns correspond to the reconstructions that have been obtained using the $\ell^1 \times \ell^1$ minimization. The relative errors decay with increasing distance $|c_1 - c_2|$ and also with decreasing size R_1 of the first components. This is to be expected because the accuracy of the second order Born approximation and also the stability of both reconstruction algorithms improve in this case (see Theorems 3.14 and 3.17). Both computational approaches yield satisfying results of comparable accuracy. $\diamondsuit$

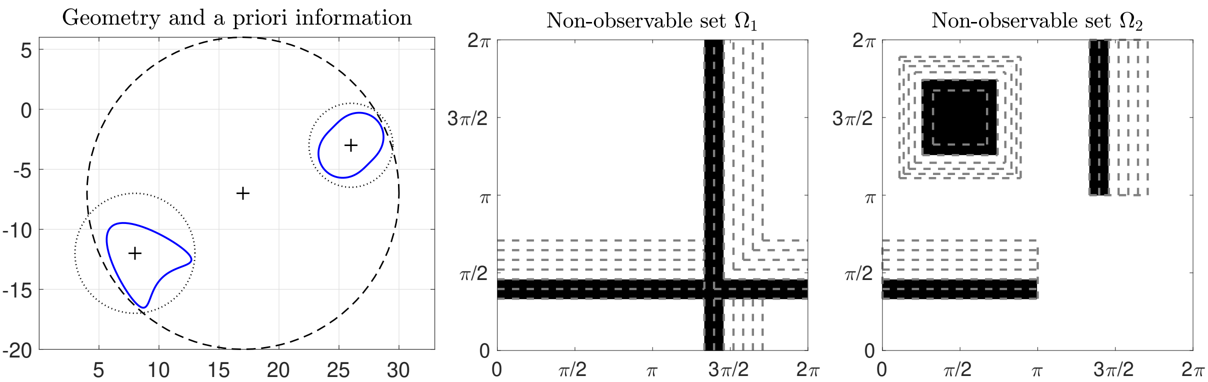


Figure 5.2: Left: Geometry of scatterer (solid), a priori information on location of scatterer (dashed) in Example 5.2, and a priori information on location of components of scatterer (dotted) in Example 5.3. Center: Support of missing data segment $\Omega_1(\alpha)$ in (5.3) for varying α ($\Omega_1(\pi/8)$ highlighted). Right: Support of missing data segment $\Omega_2(\alpha)$ in (5.4) for varying α ($\Omega_2(\pi/8)$ highlighted).

Example 5.2. Next we consider far field operator completion and study the accuracy of numerical reconstructions depending on the size $|\Omega|$ of the non-observable part Ω for two different geometrical setups for Ω. We use again the wave number $k = 0.5$, and consider the same contrast function $q = -0.5\chi_{D_1} + \chi_{D_2}$ as in Example 5.1. The geometry of the scatterer $D = D_1 \cup D_2$ is shown in Figure 5.2 (left), and we use the dashed circle as a priori information on the location of the scatterers, i.e., $c = (17, -7)$ and $R = 17$. In our first test, we assume that the missing data segment is supported on

$$\Omega_1(\alpha) := \left\{ \left(\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \right) \,\middle|\, (\varphi,\vartheta) \in [\pi/3, \pi/3+\alpha] \times [0,2\pi) \cup [0,2\pi) \times [4\pi/3, 4\pi/3+\alpha] \right\} \tag{5.3}$$

for $\alpha \in \{\pi/16, \pi/8, 3\pi/16, \pi/4, 5\pi/16, 3\pi/8\}$, as shown in the Figure 5.2 (center). In our second test, we consider a less structured non-observable part

$$\begin{aligned} \Omega_2(\alpha) := \Bigg\{ \left(\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \right) \,\Bigg|\, &(\varphi,\vartheta) \in [\pi/3, \pi/3+\alpha] \times [0,\pi] \cup [4\pi/3, 4\pi/3+\alpha] \times [\pi, 2\pi) \\ &\cup [(3\pi-\beta)/2, (3\pi+\beta)/2] \times [(\pi-\beta)/2, (\pi+\beta)/2] \Bigg\} \end{aligned} \tag{5.4}$$

for $\beta := \sqrt{\alpha(2\pi - \alpha)}$ and $\alpha \in \{\pi/16, \pi/8, 3\pi/16, \pi/4, 5\pi/16, 3\pi/8\}$, as shown in Figure 5.2 (right). Here, the parameter α controls the area of the non-observable sets, which coincide for same values of α,

$$\frac{|\Omega_1(\alpha)|}{4\pi^2} = \frac{|\Omega_2(\alpha)|}{4\pi^2} = \frac{\alpha(4\pi-\alpha)}{4\pi^2} \in \{6\%, 12\%, 18\%, 23\%, 29\%, 34\%\}\,.$$

We simulate the associated far field operators using a Nyström method as outlined in Example 2.6 with $L = 256$, and we add 5% complex valued uniformly distributed additive relative error.

To solve the far field operator completion problem using the least squares approach in Theorem 4.2, we assume that the dashed circle in Figure 5.2 to be known a priori, i.e., $N = 9$. For the $\ell^1 \times \ell^1$ approach (without splitting) we use $\mu = 10^{-3}$ for the Lagrange parameter. We evaluate relative reconstruction errors

$$\epsilon_{\mathrm{rel}} := \frac{\|F_q - \widetilde{F}_q\|_{\mathrm{HS}}}{\|F_q\|_{\mathrm{HS}}} \qquad \text{and} \qquad \epsilon_{\mathrm{rel}}^{\Omega} := \frac{\|B - \widetilde{B}\|_{\mathrm{HS}}}{\|B\|_{\mathrm{HS}}}$$

for the reconstructed far field operator $\widetilde{F}_q$ and for the reconstructed non-observable part $\widetilde{B}$. The

	least squares		$\ell^1 \times \ell^1$-minimization	
$\frac{\lvert\Omega\rvert}{4\pi^2}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$
6%	0.015	0.034	0.044	0.069
12%	0.022	0.049	0.051	0.106
18%	0.065	0.132	0.142	0.289
23%	0.166	0.307	0.250	0.462
29%	0.382	0.637	0.292	0.486
34%	0.621	0.946	0.372	0.566

	least squares		$\ell^1 \times \ell^1$-minimization	
$\frac{\lvert\Omega\rvert}{4\pi^2}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$
6%	0.059	0.200	0.072	0.225
12%	0.087	0.205	0.150	0.351
18%	0.168	0.332	0.187	0.365
23%	0.167	0.295	0.214	0.375
29%	0.247	0.393	0.283	0.447
34%	0.354	0.516	0.383	0.558

Table 5.2: Relative errors of far field operator completion. Left: Cross-shaped missing data segment $\Omega = \Omega_1$. Right: Disconnected missing data segment $\Omega = \Omega_2$.

results are shown in Table 5.2. The left part of this table corresponds to our first test with the missing data segment $\Omega = \Omega_1$, and the right part of this table corresponds to our second test with $\Omega = \Omega_2$. In both cases, the relative reconstruction errors are decaying with decaying area of Ω. The least squares approach yields slightly better reconstructions than the $\ell^1 \times \ell^1$ approach. The two examples show that the quality of the reconstruction not only depends on the area $|\Omega|$ of the missing data segment but also on the geometric structure of Ω. $\diamondsuit$

Example 5.3. We consider the same setting as in Example 5.2 but now we combine far field operator splitting and completion and show that this leads to better numerical reconstructions than those of Example 5.2. For the least squares approach in Theorem 4.5 we assume the dotted circles in Figure 5.2 (left) to be known a priori, i.e., $c_1 = (26, -3)$, $c_2 = (8, -12)$, $R_1 = 3.5$ and $R_2 = 5$. Accordingly, we use $N_1 = 3$ and $N_2 = 4$ in (5.1). For the associated $\ell^1 \times \ell^1$ minimization problem (5.2) we use $\mu = 10^{-3}$ for the Lagrange parameter. Again the data contain 5% complex valued uniformly distributed additive relative error. We evaluate relative reconstruction errors

$$\epsilon_{\mathrm{rel}} := \frac{\|F_q - \widetilde{F}_{q_1} - \widetilde{F}_{q_2} - \widetilde{F}_{q_1,q_2} - \widetilde{F}_{q_2,q_1}\|_{\mathrm{HS}}}{\|F_q\|_{\mathrm{HS}}} \qquad \text{and} \qquad \epsilon_{\mathrm{rel}}^{\Omega} := \frac{\|B - \widetilde{B}\|_{\mathrm{HS}}}{\|B\|_{\mathrm{HS}}}$$

for the reconstructed far field operator $\widetilde{F}_q$ and for the reconstructed non-observable part $\widetilde{B}$. The results are shown in Table 5.3. Again the left part of this table corresponds to our first test with the missing data segment $\Omega = \Omega_1$, and the right part of this table corresponds to our second test with $\Omega = \Omega_2$ (see Figure 5.2). When comparing the results Table 5.3 and Table 5.2, we find that combining far field operator completion with far field operator splitting yields more accurate results. This is due to the fact that $N_1 + N_2 = 7 < 9 = N$, where N denotes the parameter determining the dimension of the space $\mathcal{V}_N^c$ in Example 5.2. Accordingly, sparser representation are used when combining far field operator completion with far field operator splitting, which leads to increased stability (see also Theorem 4.5). $\lozenge$

	least squares		$\ell^1 \times \ell^1$-minimization	
$\frac{\lvert\Omega\rvert}{4\pi^2}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$
6%	0.014	0.027	0.045	0.074
12%	0.017	0.031	0.048	0.099
18%	0.044	0.088	0.068	0.132
23%	0.059	0.108	0.077	0.137
29%	0.127	0.211	0.089	0.144
34%	0.249	0.379	0.117	0.176

	least squares		$\ell^1 \times \ell^1$-minimization	
$\frac{\lvert\Omega\rvert}{4\pi^2}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$	ϵ_{rel}	$\epsilon_{\mathrm{rel}}^{\Omega}$
6%	0.019	0.058	0.045	0.066
12%	0.043	0.101	0.048	0.084
18%	0.090	0.177	0.059	0.101
23%	0.144	0.254	0.067	0.108
29%	0.248	0.394	0.071	0.104
34%	0.338	0.492	0.090	0.126

Table 5.3: Relative errors of simultaneous far field operator completion and splitting. Left: Cross-shaped missing data segment $\Omega = \Omega_1$. Right: Disconnected missing data segment $\Omega = \Omega_2$.

Conclusions

Untangling multiple scattering effects to approximate the scattering response of each scatterer in an ensemble of several scattering objects from scattering data for the whole ensemble is a basic question in inverse scattering theory. We have developed conditions on how far apart two scatterers have to be in order to be able to split their far field operator with a reasonable condition number. Closely related is the question of resolution in inverse scattering, i.e., how far apart do two scatterers have to be in order to be able to distinguish them in a reconstruction.

We have also discussed the data completion problem to recover missing or corrupted scattering data segments in remote observations. We have established conditions on how large the area covered by the affected incident or observation directions in the scattering data can be in order to guarantee that the associated far field operator can be reconstructed with a reasonable condition number.

For both inverse problems we have seen that these conditions for well-posedness are less restrictive for scattering problems and associated far field operators than for source problems and associated single far field patterns as considered in [29]. In fact, the correlation between scattered waves corresponding to different incident plane waves improves the stability.

Acknowledgments

The first author would like to thank John Sylvester for helpful conversations on the topic of this work. The research for this paper was supported by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

References

[1] L. Audibert and H. Haddar. A generalized formulation of the linear sampling method with exact characterization of targets in terms of farfield measurements. *Inverse Problems*, 30(3):035011, 20, 2014. doi:10.1088/0266-5611/30/3/035011.

[2] M. Aussal, Y. Boukari, and H. Haddar. Data completion method for the Helmholtz equation via surface potentials for partial Cauchy data. *Inverse Problems*, 36(5):055012, 17, 2020. doi:10.1088/1361-6420/ab730c.

[3] D. Baffet and M. J. Grote. On wave splitting, source separation and echo removal with absorbing boundary conditions. *J. Comput. Phys.*, 387:589–596, 2019. doi:10.1016/j.jcp.2019.03.004.

[4] A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.*, 2(1):183–202, 2009. doi:10.1137/080716542.

[5] F. ben Hassen, J. Liu, and R. Potthast. On source analysis by wave splitting with applications in inverse scattering of multiple obstacles. *J. Comput. Math.*, 25(3):266–281, 2007.

[6] L. Borcea, V. Druskin, A. V. Mamonov, and M. Zaslavsky. Untangling the nonlinearity in inverse scattering with data-driven reduced order models. *Inverse Problems*, 34(6):065008, 35, 2018. doi:10.1088/1361-6420/aabb16.

[7] Y. Boukari and H. Haddar. A convergent data completion algorithm using surface integral equations. *Inverse Problems*, 31(3):035011, 21, 2015. doi:10.1088/0266-5611/31/3/035011.

[8] A. L. Bukhgeim. Recovering a potential from Cauchy data in the two-dimensional case. *J. Inverse Ill-Posed Probl.*, 16(1):19–33, 2008. doi:10.1515/jiip.2008.002.

[9] F. Cakoni and D. Colton. *A qualitative approach to inverse scattering theory*, volume 188 of *Applied Mathematical Sciences*. Springer, New York, 2014. doi:10.1007/978-1-4614-8827-9.

[10] F. Cakoni, D. Colton, and H. Haddar. *Inverse scattering theory and transmission eigenvalues*, volume 98 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, [2023] ©2023. Second edition [of 3601119].

[11] S. Cogar, D. Colton, S. Meng, and P. Monk. Modified transmission eigenvalues in inverse scattering theory. *Inverse Problems*, 33(12):125002, 31, 2017. doi:10.1088/1361-6420/aa9418.

[12] D. Colton and A. Kirsch. A simple method for solving inverse scattering problems in the resonance region. *Inverse Problems*, 12(4):383–393, 1996. doi:10.1088/0266-5611/12/4/003.

[13] D. Colton and R. Kress. Eigenvalues of the far field operator for the Helmholtz equation in an absorbing medium. *SIAM J. Appl. Math.*, 55(6):1724–1735, 1995. doi:10.1137/S0036139993256114.

[14] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, Cham, fourth edition, 2019. doi:10.1007/978-3-030-30351-8.

[15] D. Colton and P. Monk. The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium. *Quart. J. Mech. Appl. Math.*, 41(1):97–125, 1988. doi:10.1093/qjmam/41.1.97.

[16] I. Daubechies, M. Defrise, and C. De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Comm. Pure Appl. Math.*, 57(11):1413–1457, 2004. doi:10.1002/cpa.20042.

[17] N. DeFilippis, S. Moskow, and J. C. Schotland. Born and inverse Born series for scattering problems with Kerr nonlinearities. *Inverse Problems*, 39(12):Paper No. 125015, 20, 2023. doi:10.1088/1361-6420/ad07a5.

[18] *NIST Digital Library of Mathematical Functions*. https://dlmf.nist.gov/, Release 1.1.11 of 2023-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: https://dlmf.nist.gov/.

[19] D. L. Donoho and P. B. Stark. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.*, 49(3):906–931, 1989. doi:10.1137/0149053.

[20] F. Dou, X. Liu, S. Meng, and B. Zhang. Data completion algorithms and their applications in inverse acoustic scattering with limited-aperture backscattering data. *J. Comput. Phys.*, 469:Paper No. 111550, 17, 2022. doi:10.1016/j.jcp.2022.111550.

[21] V. Druskin, A. V. Mamonov, A. E. Thaler, and M. Zaslavsky. Direct, nonlinear inversion algorithm for hyperbolic problems via projection-based model reduction. *SIAM J. Imaging Sci.*, 9(2):684–747, 2016. doi:10.1137/15M1039432.

[22] V. Druskin, A. V. Mamonov, and M. Zaslavsky. A nonlinear method for imaging with acoustic waves via reduced order model backprojection. *SIAM J. Imaging Sci.*, 11(1):164–196, 2018. doi:10.1137/17M1133580.

[23] B. Engquist and H. Zhao. Approximate separability of the Green's function of the Helmholtz equation in the high-frequency limit. *Comm. Pure Appl. Math.*, 71(11):2220–2274, 2018. doi:10.1002/cpa.21755.

[24] M. Graff, M. J. Grote, F. Nataf, and F. Assous. How to solve inverse scattering problems without knowing the source term: a three-step strategy. *Inverse Problems*, 35(10):104001, 20, 2019. doi:10.1088/1361-6420/ab2d5f.

[25] R. Griesmaier, M. Hanke, and J. Sylvester. Far field splitting for the Helmholtz equation. *SIAM J. Numer. Anal.*, 52(1):343–362, 2014. doi:10.1137/120891381.

[26] R. Griesmaier and B. Harrach. Monotonicity in inverse medium scattering on unbounded domains. *SIAM J. Appl. Math.*, 78(5):2533–2557, 2018. doi:10.1137/18M1171679.

[27] R. Griesmaier, N. Hyvönen, and O. Seiskari. A note on analyticity properties of far field patterns. *Inverse Probl. Imaging*, 7(2):491–498, 2013. doi:10.3934/ipi.2013.7.491.

[28] R. Griesmaier and J. Sylvester. Far field splitting by iteratively reweighted ℓ^1 minimization. *SIAM J. Appl. Math.*, 76(2):705–730, 2016. doi:10.1137/15M102839X.

[29] R. Griesmaier and J. Sylvester. Uncertainty principles for inverse source problems, far field splitting, and data completion. *SIAM J. Appl. Math.*, 77(1):154–180, 2017. doi:10.1137/16M1086157.

[30] R. Griesmaier and J. Sylvester. Uncertainty principles for three-dimensional inverse source problems. *SIAM J. Appl. Math.*, 77(6):2066–2092, 2017. doi:10.1137/17M111287X.

[31] R. Griesmaier and J. Sylvester. Uncertainty principles for inverse source problems for electromagnetic and elastic waves. *Inverse Problems*, 34(6):065003, 37, 2018. doi:10.1088/1361-6420/aab45c.

[32] M. J. Grote, M. Kray, F. Nataf, and F. Assous. Time-dependent wave splitting and source separation. *J. Comput. Phys.*, 330:981–996, 2017. doi:10.1016/j.jcp.2016.10.021.

[33] K. Kilgore, S. Moskow, and J. C. Schotland. Inverse Born series for scalar waves. *J. Comput. Math.*, 30(6):601–614, 2012. doi:10.4208/jcm.1205-m3935.

[34] K. Kilgore, S. Moskow, and J. C. Schotland. Convergence of the Born and inverse Born series for electromagnetic scattering. *Appl. Anal.*, 96(10):1737–1748, 2017. doi:10.1080/00036811.2017.1292349.

[35] A. Kirsch. Characterization of the shape of a scattering obstacle using the spectral data of the far field operator. *Inverse Problems*, 14(6):1489–1512, 1998. doi:10.1088/0266-5611/14/6/009.

[36] A. Kirsch. Remarks on the Born approximation and the factorization method. *Appl. Anal.*, 96(1):70–84, 2017. doi:10.1080/00036811.2016.1188286.

[37] A. Kirsch. *An introduction to the mathematical theory of inverse problems*, volume 120 of *Applied Mathematical Sciences*. Springer, third edition, 2021. doi:10.1007/978-3-030-63343-1.

[38] A. Kirsch and N. Grinberg. *The factorization method for inverse problems*, volume 36 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2008.

[39] I. Krasikov. Uniform bounds for Bessel functions. *J. Appl. Anal.*, 12(1):83–91, 2006. doi:10.1515/JAA.2006.83.

[40] L. J. Landau. Bessel functions: monotonicity and bounds. *J. London Math. Soc. (2)*, 61(1):197–215, 2000. doi:10.1112/S0024610799008352.

[41] X. Liu and J. Sun. Data recovery in inverse scattering: from limited-aperture to full-aperture. *J. Comput. Phys.*, 386:350–364, 2019. doi:10.1016/j.jcp.2018.10.036.

[42] A. I. Nachman. Reconstructions from boundary measurements. *Ann. of Math. (2)*, 128(3):531–576, 1988. doi:10.2307/1971435.

[43] F. Natterer. An error bound for the Born approximation. *Inverse Problems*, 20(2):447–452, 2004. doi:10.1088/0266-5611/20/2/009.

[44] R. G. Novikov. A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. *Funktsional. Anal. i Prilozhen.*, 22(4):11–22, 96, 1988. doi:10.1007/BF01077418.

[45] R. Potthast, F. M. Fazi, and P. A. Nelson. Source splitting via the point source method. *Inverse Problems*, 26(4):045002, 17, 2010. doi:10.1088/0266-5611/26/4/045002.

[46] A. G. Ramm. Recovery of the potential from fixed-energy scattering data. *Inverse Problems*, 4(3):877–886, 1988. URL: http://stacks.iop.org/0266-5611/4/877.

[47] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[48] J. Saranen and G. Vainikko. *Periodic integral and pseudodifferential equations with numerical approximation*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002. doi:10.1007/978-3-662-04796-5.

[49] J. Sylvester. An estimate for the free Helmholtz equation that scales. *Inverse Probl. Imaging*, 3(2):333–351, 2009. doi:10.3934/ipi.2009.3.333.

[50] G. Vainikko. Fast solvers of the Lippmann-Schwinger equation. In *Direct and inverse problems of mathematical physics (Newark, DE, 1997)*, volume 5 of *Int. Soc. Anal. Appl. Comput.*, pages 423–440. Kluwer Acad. Publ., Dordrecht, 2000. URL: https://doi.org/10.1007/978-1-4757-3214-6_25, doi:10.1007/978-1-4757-3214-6_25.