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Calculus of Variations

Block-radial symmetry breaking for ground states of biharmonic NLS

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Abstract

We prove that the biharmonic NLS equation

$$\Delta^2 u + 2\Delta u + (1+\varepsilon)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^d$$

has at least $k+1$ geometrically distinct solutions if $\varepsilon > 0$ is small enough and $2 < p < 2_\star^k$, where $2_\star^k$ is an explicit critical exponent arising from the Fourier restriction theory of $O(d-k) \times O(k)$-symmetric functions. This extends the recent symmetry breaking result of Lenzmann–Weth (Symmetry breaking for ground states of biharmonic NLS via Fourier extension estimates, 2023) and relies on a chain of strict inequalities for the corresponding Rayleigh quotients associated with distinct values of k. We further prove that, as $\varepsilon \to 0^+$, the Fourier transform of each ground state concentrates near the unit sphere and becomes rough in the scale of Sobolev spaces.

Mathematics Subject Classification 42B10

1 Introduction

A *ground state* for the biharmonic nonlinear Schrödinger equation,

$$\Delta^2 u + 2\Delta u + (1+\varepsilon)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^d, \quad (p > 2, \varepsilon > 0) \tag{1.1}$$

is a nonzero solution $u \in H^2(\mathbb{R}^d)$ of (1.1), at which the infimum

$$R_\varepsilon(p) := \inf_{0 \neq u \in H^2(\mathbb{R}^d)} \frac{q_\varepsilon(u)}{\|u\|_p^2}, \quad \text{where} \quad q_\varepsilon(u) := \int_{\mathbb{R}^d} \left(|\Delta u|^2 - 2|\nabla u|^2 + (1+\varepsilon)|u|^2\right) \mathrm{d}x,$$

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is attained. Lenzmann and Weth recently established the existence of *nonradial* ground states $u \in H^2(\mathbb{R}^d)$ of (1.1). More precisely, [7, Theorem 1.2] guarantees the existence of a threshold $\varepsilon_0 > 0$ such that every ground state $u \in H^2(\mathbb{R}^d)$ of (1.1) is a nonradial function if $0 < \varepsilon < \varepsilon_0$, as long as $d \geq 2$ and $2 < p < 2_\star$. Here, $2_\star := 2\frac{d+1}{d-1}$ is the endpoint Stein–Tomas exponent [14, 15] from Fourier restriction theory. This symmetry breaking result is especially interesting in view of the recent work by Lenzmann and Sok [6] on sufficient conditions for radial symmetry of ground states in the general framework of elliptic pseudodifferential equations. Our goal is to shed further light on the interplay between symmetries and ground states of elliptic PDEs by generalizing the symmetry breaking result from [7] to a broader class of (pseudo-)differential equations and by including the symmetry groups

$$G_k := O(d-k) \times O(k), \qquad k \in \{1, \ldots, d-1\}$$

in the analysis. A function $f : \mathbb{R}^d \to \mathbb{C}$ is G_k*-symmetric* if $f \circ A = f$ holds for every $A \in G_k$. Our main result implies that, for suitable exponents $p > 2$ and sufficiently small $\varepsilon > 0$, *there exist radial and block-radial solutions of* (1.1) *whose energy is larger than that of the ground states*. By our construction, this immediately implies that *not only do ground states of* (1.1) *fail to be radial functions, but they also fail to be block-radial*. The sharp Stein–Tomas exponent $2_\star^k$ for G_k-symmetric functions from [12] will play a central role. It satisfies $2_\star^k = 2_\star^{d-k}$ and is explicitly given by

$$2_\star^k := 2\frac{d + (d-k) \wedge k}{d - 2 + (d-k) \wedge k} \qquad (x \wedge y := \min\{x, y\}). \tag{1.2}$$

We shall consider nonlinear elliptic (pseudo-)differential equations

$$g_\varepsilon(|D|)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^d, \tag{1.3}$$

for a certain class of so-called (s, γ)*-admissible* symbols g_ε. Here, $g_\varepsilon(|D|)u = \mathcal{F}^{-1}(g_\varepsilon(|\cdot|)\hat{u})$. In the case $s = \gamma = 2$, the symbol $g_\varepsilon(|\xi|) = (|\xi|^2 - 1)^2 + \varepsilon$ of the biharmonic NLS is a prototypical example.

Definition 1.1 Let $s > \frac{d}{d+1}$ and $\gamma > 1$. The symbol $g_\varepsilon : \mathbb{R} \to (0, \infty)$ is (s, γ)*-admissible* if it is measurable and satisfies the two-sided estimates

$$c(1 + |\xi|^{2s}) \leq g_\varepsilon(|\xi|) \leq C(1 + |\xi|^{2s}), \qquad \text{for all } \ ||\xi| - 1| \geq \frac{1}{2}, \tag{1.4}$$

$$c(\varepsilon + ||\xi| - 1|^\gamma) \leq g_\varepsilon(|\xi|) \leq C(\varepsilon + ||\xi| - 1|^\gamma), \qquad \text{for all } \ ||\xi| - 1| \leq \frac{1}{2}, \tag{1.5}$$

for some constants $0 < c < C < \infty$ independent of $\varepsilon \in (0, 1)$.

Some comments are in order. We assume $\gamma > 1$ in order to present a complete and unified analysis. Optimal results for $\gamma \in (0, 1]$ require substantial modifications, which will not be investigated here. The assumption $s > d/(d+1)$ is equivalent to $2_\star < 2d/(d-2s)_+$, and thus ensures that the endpoint Stein–Tomas exponent is Sobolev-subcritical for the problem under investigation. In light of [2], this ensures the existence of ground states for (1.3), i.e., nonzero solutions which attain the infimum

$$\mathrm{R}_\varepsilon^\circ(p) := \inf_{\mathbf{0} \neq u \in H^s(\mathbb{R}^d)} \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2}, \quad \text{where } \mathrm{q}_\varepsilon(u) := \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 g_\varepsilon(|\xi|)\, \mathrm{d}\xi. \tag{1.6}$$

For any fixed $\varepsilon > 0$, the functional q_ε is positive and continuous on $H^s(\mathbb{R}^d)$ as long as the symbol g_ε is (s, γ)-admissible. More precisely, $u \mapsto \sqrt{\mathrm{q}_\varepsilon(u)}$ is then an equivalent norm on

$H^s(\mathbb{R}^d)$. Being interested in symmetric solutions of (1.3), we define the following radial and G_k-symmetric versions of the Rayleigh quotient in (1.6):

$$\mathrm{R}_\varepsilon^{\mathrm{rad}}(p) := \inf_{\mathbf{0}\neq u \in H^s_{\mathrm{rad}}} \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2}; \qquad \mathrm{R}_\varepsilon^{k}(p) := \inf_{\mathbf{0}\neq u \in H^s_{k}} \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2},$$

where the infima are taken over the Hilbert spaces $H^s_{\mathrm{rad}} := \{f \in H^s(\mathbb{R}^d) : f \text{ is radial}\}$ and $H^s_k := \{f \in H^s(\mathbb{R}^d) : f \text{ is } G_k\text{-symmetric}\}$, respectively. We have $\mathrm{R}^k_\varepsilon(p) = \mathrm{R}^{d-k}_\varepsilon(p)$ with identical minimizers up to a trivial change of coordinates. Hence in our main result we focus on $k \in \{1, \dots, \lfloor d/2 \rfloor\}$.

Theorem 1.2 *Assume $d \geq 2$, $k \in \{1, \dots, \lfloor d/2 \rfloor\}$, $2 < p < 2^k_\star$. Let g_ε be (s, γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Then there exists $\varepsilon_0 = \varepsilon_0(p, d, k, s, \gamma)$ such that, for $0 < \varepsilon < \varepsilon_0$, the chain of strict inequalities*

$$\mathrm{R}^\circ_\varepsilon(p) \vee \mathrm{R}^1_\varepsilon(p) < \mathrm{R}^2_\varepsilon(p) < \cdots < \mathrm{R}^k_\varepsilon(p) < \mathrm{R}^{k+1}_\varepsilon(p) \wedge \ldots \wedge \mathrm{R}^{\lfloor d/2 \rfloor}_\varepsilon(p) \leq \mathrm{R}^{\mathrm{rad}}_\varepsilon(p) < \infty \tag{1.7}$$

holds and each of these Rayleigh quotients, except possibly $\mathrm{R}^1_\varepsilon(p)$, is attained at some nontrivial solution of (1.3). In particular, there exist $k + 1$ geometrically distinct nontrivial solutions and the ground state is neither radial nor block-radial.

Specializing to $g_\varepsilon(|\xi|) = (|\xi|^2 - 1)^2 + \varepsilon$ and $(s, \gamma, k) = (2, 2, 1)$, we recover the symmetry breaking result from [7, Theorem 1.2] since $2^1_\star = 2\frac{d+1}{d-1} = 2_\star$. Recall that geometrically distinct solutions have distinguishable group orbits with respect to the symmetry group of the action functional (here: rotations and translations). This is implied by different energy levels and hence follows from (1.7). The strict inequalities in (1.7) rely on lower and upper bounds for the Rayleigh quotients that allow us to determine the asymptotic behaviour of $\mathrm{R}^*_\varepsilon(p)$ as $\varepsilon \to 0^+$, for $* \in \{\circ, k, \mathrm{rad}\}$. This is the content of our next result, Theorem 1.3. We introduce the interpolation parameter

$$\alpha_k := \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{2^k_\star}} = \begin{cases} (d+k)(\frac{1}{2} - \frac{1}{p}), & \text{if } k \leq \frac{d}{2}, \\ (2d-k)(\frac{1}{2} - \frac{1}{p}), & \text{if } k > \frac{d}{2}, \end{cases} \tag{1.8}$$

which satisfies $\alpha_k \in (0, 1)$ if and only if $2 < p < 2^k_\star$. We further define

$$\alpha_{\mathrm{rad}} := \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{2^{\mathrm{rad}}_\star}} = 2d\left(\frac{1}{2} - \frac{1}{p}\right), \text{ where } 2^{\mathrm{rad}}_\star := \frac{2d}{d-1}, \tag{1.9}$$

which satisfies $\alpha_{\mathrm{rad}} \in (0, 1)$ if and only if $2 < p < 2^{\mathrm{rad}}_\star$. We also need the Sobolev exponent[1]

$$2^\star_s := \begin{cases} \frac{2d}{d-2s}, & \text{if } 0 \leq s < \frac{d}{2}, \\ \infty, & \text{if } s \geq \frac{d}{2}, \end{cases} \tag{1.10}$$

which ensures the largest possible range of validity in the next result.

[1] We take this opportunity to record that the exponents defined in (1.2), (1.9) and (1.10) satisfy

$$2 < 2^{\mathrm{rad}}_\star < 2^k_\star \leq 2^1_\star = 2_\star < 2^\star_s \leq \infty.$$

Theorem 1.3 *(*Lower & upper bounds) Assume $d \geq 2$, $k \in \{1, \dots, d-1\}$, $2 < p < 2_s^\star$. Let g_ε be (s, γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. As $\varepsilon \to 0^+$, it holds that

$$\mathrm{R}_\varepsilon^\circ(p) \simeq \varepsilon^{1-\frac{1\wedge\alpha_1}{\gamma}}, \tag{1.11}$$

$$\mathrm{R}_\varepsilon^k(p) \simeq \varepsilon^{1-\frac{1\wedge\alpha_k}{\gamma}}, \tag{1.12}$$

$$\mathrm{R}_\varepsilon^{\mathrm{rad}}(p) \simeq \begin{cases} \varepsilon^{1-\frac{1\wedge\alpha_{\mathrm{rad}}}{\gamma}}, & \text{if } p \neq 2_\star^{\mathrm{rad}}, \\ \varepsilon^{1-\frac{1}{\gamma}}|\log(\varepsilon)|^{\frac{1-d}{d}}, & \text{if } p = 2_\star^{\mathrm{rad}}. \end{cases} \tag{1.13}$$

Here, $A_\varepsilon \simeq B_\varepsilon$ means that there exist constants $0 < c < C < \infty$, independent of $\varepsilon > 0$, such that $cB_\varepsilon \leq A_\varepsilon \leq CB_\varepsilon$ holds for all sufficiently small $\varepsilon > 0$. We remark that the logarithmic term appearing when $p = 2_\star^{\mathrm{rad}}$ is new even in the special case of the biharmonic NLS, and thus refines the estimates from [7]. This particular bound relies on a simple but nonstandard interpolation result that the interested reader may find in Appendix 1.

The proof of Theorem 1.3 naturally splits into two parts: lower and upper bounds for each of the Rayleigh quotients. The lower bounds hinge on the following recent G_k-symmetric refinement of the Stein–Tomas inequality. If $d \geq 2$ and $k \in \{1, \dots, d-1\}$, then [12, Theorem 1.1] ensures the validity of the adjoint restriction inequality

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \leq C(k, d, p)\|f\|_{L^2(\mathbb{S}^{d-1})} \text{ for all } p \geq 2_\star^k, \tag{1.14}$$

for every G_k-symmetric function $f : \mathbb{R}^d \to \mathbb{C}$, where the adjoint restriction (or *extension*) operator to the unit sphere $\mathbb{S}^{d-1} := \{\omega \in \mathbb{R}^d : |\omega| = 1\}$ is given by

$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^{d-1}} e^{i\omega\cdot x} f(\omega)\, \mathrm{d}\sigma(\omega). \quad (x \in \mathbb{R}^d) \tag{1.15}$$

More precisely, estimate (1.14) was shown in [12] for $k \in \{2, \dots, d-2\}$, whereas the case $k \in \{1, d-1\}$ is a direct consequence of the classical Stein–Tomas inequality since $2_\star^1 = 2_\star^{d-1} = 2_\star$. That the range of exponents in (1.14) is optimal for L^2-densities follows from [12, Theorem 1.3] and is based on a careful analysis of a G_k-symmetric version of Knapp's well-known construction. Inequality (1.14) and these G_k-symmetric Knapp-type constructions together pave the way towards the precise asymptotics given by Theorem 1.3. The asymptotics from Theorem 1.3 imply the chain of inequalities (1.7) stated in Theorem 1.2. To finish the proof of the latter, one still has to establish the existence of minimizers within the relevant class of functions. In turn, this amounts to a straightforward argument relying on certain compact embeddings of H_k^s, $k \in \{2, \dots, d-2\}$, that we recall in Appendix 1. If $k \in \{1, d-1\}$, then the existence of a minimizer remains an open question, which we discuss in Appendix 1.

Once the existence of minimizers has been established, their qualitative properties become of interest. To explore this matter, define the interval

$$I_{\varepsilon,\delta} := [\delta\varepsilon^{1/\gamma}, \delta^{-1}\varepsilon^{1/\gamma}], \quad (\varepsilon, \delta > 0) \tag{1.16}$$

and the associated spherical shell

$$A_{\varepsilon,\delta} := \{\xi \in \mathbb{R}^d : ||\xi| - 1| \in I_{\varepsilon,\delta}\}. \tag{1.17}$$

Given a minimizer u_ε for $\mathrm{R}_\varepsilon^*(p)$ with $* \in \{\circ, k, \mathrm{rad}\}$, we split

$$u_\varepsilon = v_\varepsilon + w_\varepsilon, \text{ where } \widehat{v}_\varepsilon = \mathbf{1}_{A_{\varepsilon,\delta}}\widehat{u}_\varepsilon. \tag{1.18}$$

This decomposition preserves radiality and G_k-symmetry. Our next result reveals that u_ε *concentrates on the unit sphere in Fourier space*, as $\varepsilon \to 0^+$.

Theorem 1.4 *Assume $d \geq 2$, $k \in \{1, \ldots, d-1\}$, $2 < p < 2_s^\star$. Let g_ε be (s,γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Let $* \in \{\circ, k, \mathrm{rad}\}$. If u_ε is a minimizer for $\mathrm{R}_\varepsilon^*(p)$, decomposed as in (1.18) with $\delta = \delta_\varepsilon$ for a given positive null sequence $(\delta_\varepsilon)_{\varepsilon>0}$, then*

$$\lim_{\varepsilon\to0^+}\frac{\|w_\varepsilon\|_p}{\|v_\varepsilon\|_p} = \lim_{\varepsilon\to0^+}\frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\mathrm{q}_\varepsilon(v_\varepsilon)} = 0.$$

We emphasize that this result also applies to the case $* = k \in \{1, d-1\}$ where the existence of a minimizer is still open.

In the smaller G_k-symmetry breaking range $2 < p < 2_\star^k$, minimizers *become rough in Fourier space*, as $\varepsilon \to 0^+$. The intuition comes from the fact that the upper bounds in Theorem 1.3 were proved via test functions that resemble Knapp counterexamples. The Fourier transform of such functions vary sharply along spheres. Our next result indicates that such behaviour is somewhat necessary in order to be energetically efficient. The regularity is measured in the Sobolev space $H^t(\mathbb{S}^{d-1})$ of functions having $t \geq 0$ derivatives in $L^2(\mathbb{S}^{d-1})$. This space is defined via spherical harmonic expansions, e.g. as in [9, §1.7.3, Remark 7.6], or equivalently by considering a smooth finite partition of unity and diffeomorphisms onto the unit ball of $\mathbb{R}^{d-1}$ together with the usual Sobolev norm on $\mathbb{R}^{d-1}$.

Theorem 1.5 *Assume $d \geq 2$, $k \in \{1, \ldots, d-1\}$, $2 < p < 2_\star^k$. Let g_ε be (s,γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Let $* \in \{\circ, k\}$. If u_ε is a minimizer for $\mathrm{R}_\varepsilon^*(p)$, then for every $t > 0$ and every positive null sequence $(\delta_\varepsilon)_{\varepsilon>0}$, it holds that*

$$\lim_{\varepsilon\to0^+}\sup_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}}\frac{\|\widehat{u}_\varepsilon(r\,\cdot)\|_{H^t(\mathbb{S}^{d-1})}}{\|\widehat{u}_\varepsilon(r\,\cdot)\|_{L^2(\mathbb{S}^{d-1})}} = \infty. \tag{1.19}$$

Given that radially symmetric functions are constant on spheres around the origin, this phenomenon cannot occur in the radial case.

1.1 Structure of the paper

We prove Theorem 1.3 in Sect. 2, and this implies the first part of Theorem 1.2. We prove Theorem 1.4 in Sect. 3, and Theorem 1.5 in Sect. 4. We recall the classical proof of the second part of Theorem 1.2 in Appendix 1 and establish a technical interpolation result that is needed for the proof of Theorem 1.3 in Appendix 1. Finally, Appendix 1 contains some further considerations regarding the exceptional cases $k \in \{1, d-1\}$.

1.2 Notation

We use $X \lesssim Y$ or $Y \gtrsim X$ to denote the estimate $|X| \leq CY$ for an absolute positive constant C, $X \simeq Y$ to denote the estimates $X \lesssim Y \lesssim X$, and $X \cong Y$ to denote the identity $X = CY$. The indicator function of a set E is denoted by $\mathbf{1}_E$.

2 Lower and upper bounds

In this section, we prove Theorem 1.3. The following simple result will be useful for both lower and upper bounds.

Lemma 2.1 *Let $\varepsilon \in (0,1)$ and let g_ε be (s,γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Then*

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \simeq \varepsilon^{\frac{1}{\gamma}-1}.$$

Proof This follows from a simple change of variables and $\gamma > 1$:

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \overset{(1.5)}{\simeq} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{\varepsilon + |r-1|^\gamma} = 2\varepsilon^{\frac{1}{\gamma}-1} \int_0^{\frac{1}{2\varepsilon^{1/\gamma}}} \frac{\mathrm{d}t}{1+t^\gamma} \simeq \varepsilon^{\frac{1}{\gamma}-1}.$$

□

2.1 Lower bounds

We start by addressing the lower bound in (1.12). We decompose an arbitrary G_k-symmetric function $u \in H^s_k$ as

$$u = v + w, \text{ where } \widehat{v}(\xi) = \mathbf{1}_{||\xi|-1| \le \frac{1}{2}} \widehat{u}(\xi) \text{ and } \widehat{w}(\xi) = \mathbf{1}_{||\xi|-1| > \frac{1}{2}} \widehat{u}(\xi), \tag{2.1}$$

and observe that v, w are still G_k-symmetric. We then have $\|u\|_p \le \|v\|_p + \|w\|_p$ and $\mathrm{q}_\varepsilon(u) = \mathrm{q}_\varepsilon(v) + \mathrm{q}_\varepsilon(w)$. It follows that

$$\frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2} \ge \frac{\mathrm{q}_\varepsilon(v) + \mathrm{q}_\varepsilon(w)}{(\|v\|_p + \|w\|_p)^2} \ge \frac{1}{2} \frac{\mathrm{q}_\varepsilon(v) + \mathrm{q}_\varepsilon(w)}{\|v\|_p^2 + \|w\|_p^2} \ge \frac{1}{2}\left(\frac{\mathrm{q}_\varepsilon(v)}{\|v\|_p^2} \wedge \frac{\mathrm{q}_\varepsilon(w)}{\|w\|_p^2} \right).$$

Hence it suffices to prove lower bounds for the quotients corresponding to v and w separately. Since $2 < p < 2_s^\star$, Sobolev embedding ensures $H^s \subset L^2 \cap L^{2_s^\star} \subset L^p$, and thus

$$\|w\|_p^2 \lesssim \|w\|_{H^s}^2 = \int_{\mathbb{R}^d} |\widehat{w}(\xi)|^2 (1+|\xi|^2)^s \,\mathrm{d}\xi \simeq \int_{\mathbb{R}^d} |\widehat{w}(\xi)|^2 g_\varepsilon(|\xi|)\,\mathrm{d}\xi = \mathrm{q}_\varepsilon(w), \tag{2.2}$$

where the $\simeq$-estimate holds uniformly with respect to small $\varepsilon > 0$ thanks to (1.4). The lower bound for $\|w\|_p^{-2}\mathrm{q}_\varepsilon(w)$ follows at once, and so we focus on v.

As in [7] the proof of lower bounds for $\|v\|_p^{-2}\mathrm{q}_\varepsilon(v)$ splits into two cases, according to whether or not p is larger than $2^k_\star$. If $p \ge 2^k_\star$, then we may use Minkowski's inequality in integral form and the G_k-symmetric Stein–Tomas inequality (1.14) as follows:

$$\begin{aligned}
\|v\|_p^2 &\simeq \left\| \int_{\mathbb{R}^d} e^{i\langle \xi, \cdot\rangle} \widehat{v}(\xi)\,\mathrm{d}\xi \right\|_p^2 = \left\| \int_{\frac{1}{2}}^{\frac{3}{2}} r^{d-1} \left(\int_{\mathbb{S}^{d-1}} e^{i\langle \omega, r\cdot\rangle} \widehat{v}(r\cdot)\,\mathrm{d}\sigma(\omega) \right) \mathrm{d}r \right\|_p^2 \\
&\lesssim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle \omega, r\cdot\rangle} \widehat{v}(r\omega)\,\mathrm{d}\sigma(\omega) \right\|_p \mathrm{d}r \right)^2 \lesssim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|\widehat{v}(r\cdot)\|_{L^2(\mathbb{S}^{d-1})}\,\mathrm{d}r \right)^2 \\
&\le \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|\widehat{v}(r\cdot)\|_{L^2(\mathbb{S}^{d-1})}^2 g_\varepsilon(r)\,\mathrm{d}r \right) \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \right) \\
&\lesssim \left(\int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 g_\varepsilon(|\xi|)\,\mathrm{d}\xi \right) \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \right).
\end{aligned}$$

Here, we first used Fourier inversion and then passed to polar coordinates in Fourier space, namely $\xi = r\omega$ with $0 \le r < \infty$ and $\omega \in \mathbb{S}^{d-1}$, so that $\mathrm{d}\xi = r^{d-1}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}r$. By Lemma 2.1, we then conclude

$$\|v\|_p^2 \lesssim \mathrm{q}_\varepsilon(v) \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \simeq \varepsilon^{\frac{1}{\gamma}-1} \mathrm{q}_\varepsilon(v), \tag{2.3}$$

which implies the claimed lower bound. If $2 < p < 2_\star^k$, then the interpolation parameter from (1.8) satisfies $\alpha_k \in (0,1)$, and $\|v\|_p \le \|v\|_2^{1-\alpha_k} \|v\|_{2_\star^k}^{\alpha_k}$. Plancherel's identity, the Fourier support assumption on v and the two-sided estimate (1.5) lead to

$$\|v\|_2^2 \simeq \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2\,\mathrm{d}\xi \lesssim \varepsilon^{-1} \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 g_\varepsilon(|\xi|)\,\mathrm{d}\xi = \varepsilon^{-1}\mathrm{q}_\varepsilon(v). \tag{2.4}$$

From (2.3) with $p = 2_\star^k$ and (2.4), it then follows that

$$\|v\|_p^2 \le (\|v\|_2^2)^{1-\alpha_k} (\|v\|_{2_\star^k}^2)^{\alpha_k} \lesssim (\varepsilon^{-1}\mathrm{q}_\varepsilon(v))^{1-\alpha_k} (\varepsilon^{\frac{1}{\gamma}-1}\mathrm{q}_\varepsilon(v))^{\alpha_k} = \varepsilon^{\frac{\alpha_k}{\gamma}-1}\mathrm{q}_\varepsilon(v). \tag{2.5}$$

This concludes the proof of (1.12). The lower bounds in (1.11) are proved analogously given that it suffices to replace the exponent $2_\star^k$ originating from the G_k-symmetric Stein–Tomas inequality by the exponent $2_\star = 2_\star^1$ coming from the classical Stein–Tomas inequality. We omit the details. (See [7, §4] for a proof in the special case $s = \gamma = 2$.)

We now verify the lower bound in (1.13). Using the decomposition (2.1) and reasoning as in (2.2), it suffices to consider a radial function $v \in H^s_{\mathrm{rad}}$ whose Fourier support is contained in the spherical shell $\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le \frac{3}{2}\}$. By Fourier inversion,

$$v(x) \cong |x|^{\frac{2-d}{2}} \int_{\frac{1}{2}}^{\frac{3}{2}} \widehat{v}(r) J_{\frac{d-2}{2}}(r|x|) r^{\frac{d}{2}}\,\mathrm{d}r,$$

where $\widehat{v}(r) = \widehat{v}(r\omega)$ for $r > 0$ and almost every $\omega \in \mathbb{S}^{d-1}$. The appearance of the Bessel function is due to

$$\widehat{\sigma}(\xi) = (2\pi)^{\frac{d}{2}} |\xi|^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(|\xi|). \tag{2.6}$$

Standard asymptotics for Bessel functions at zero and at infinity lead to the pointwise estimate

$$\begin{aligned}
|v(x)| &\lesssim (1+|x|)^{\frac{1-d}{2}} \int_{\frac{1}{2}}^{\frac{3}{2}} |\widehat{v}(r)|\,\mathrm{d}r \\
&\le (1+|x|)^{\frac{1-d}{2}} \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{\varepsilon + |r-1|^\gamma} \right)^{\frac{1}{2}} \left(\int_{\frac{1}{2}}^{\frac{3}{2}} (\varepsilon + |r-1|^\gamma) |\widehat{v}(r)|^2\,\mathrm{d}r \right)^{\frac{1}{2}} \\
&\simeq (1+|x|)^{\frac{1-d}{2}} \varepsilon^{\frac{1}{2}(\frac{1}{\gamma}-1)} \sqrt{\mathrm{q}_\varepsilon(v)},
\end{aligned}$$

where we used the Cauchy–Schwarz inequality, Lemma 2.1, and assumption (1.5) on g_ε. Recall that $2_\star^{\mathrm{rad}} = \frac{2d}{d-1}$. We thus infer

$$\|v\|_p + \|v\|_{2_\star^{\mathrm{rad}},\infty} \lesssim \varepsilon^{\frac{1}{2}(\frac{1}{\gamma}-1)} \sqrt{\mathrm{q}_\varepsilon(v)}, \text{ for every } p > 2_\star^{\mathrm{rad}}, \tag{2.7}$$

which implies the lower bound (1.13) in the range $p > 2_\star^{\mathrm{rad}}$. If $2 < p < 2_\star^{\mathrm{rad}}$, then real interpolation between (2.7) and the simple L^2-bound $\|v\|_2^2 \lesssim \varepsilon^{-1}\mathrm{q}_\varepsilon(v)$ from (2.4) yields

$$\|v\|_p^2 \lesssim (\|v\|_2^2)^{1-\alpha_{\mathrm{rad}}} (\|v\|_{2_\star^{\mathrm{rad}},\infty}^2)^{\alpha_{\mathrm{rad}}} \lesssim \varepsilon^{\frac{\alpha_{\mathrm{rad}}}{\gamma}-1}\mathrm{q}_\varepsilon(v);$$

see [5, Prop. 1.1.14]. Finally, the critical case $p = 2_\star^{\mathrm{rad}}$ is a consequence of Proposition 2 with parameters

$$(r, q, C_1, C_2) = \left(2, 2_\star^{\mathrm{rad}}, (\varepsilon^{-1}\mathrm{q}_\varepsilon(v))^{\frac{1}{2}}, (\varepsilon^{\frac{1}{\gamma}-1}\mathrm{q}_\varepsilon(v))^{\frac{1}{2}}\right),$$

as follows: if $\varepsilon > 0$ is sufficiently small, then

$$\|v\|_{2_\star^{\mathrm{rad}}}^2 \lesssim C_2^2(1+\log_+(C_1/C_2))^{\frac{2}{2_\star^{\mathrm{rad}}}} \simeq \varepsilon^{\frac{1}{\gamma}-1}|\log(\varepsilon)|^{\frac{d-1}{d}}\mathrm{q}_\varepsilon(v).$$

This concludes the verification of the lower bounds in Theorem 1.3.

2.2 Upper bounds

We first consider the upper bound in (1.12) for G_k-symmetric functions. To this end, we use a G_k-symmetric trial function v, defined via its Fourier transform as follows:

$$\widehat{v}(\xi) := \mathbf{1}_{||\xi|-1|\le m} \cdot \mathbf{1}_{\mathcal{C}_\delta^k}\left(|\xi|^{-1}\xi\right) \cdot a(||\xi|-1|). \tag{2.8}$$

Here, $\mathcal{C}_\delta^k \subset \mathbb{S}^{d-1}$ is the following union of two spherical caps of radius $\delta \in (0,1)$:

$$\mathcal{C}_\delta^k := \{(\eta, \zeta) \in \mathbb{R}^{d-k} \times \mathbb{R}^k : |\eta|^2 + |\zeta|^2 = 1, |\eta| < \delta\}.$$

The set $\mathcal{C}_\delta^k$ is G_k-symmetric and measurable, with surface area $\sigma(\mathcal{C}_\delta^k) \cong \delta^{d-k}$. We shall let the parameter δ tend to zero. The precise choice of $m \in (0,1)$ and of the profile function a will be given below. Our estimates for v make use of the following uniform bounds for small $m, \delta > 0$.

Proposition 2.2 *Let $d \ge 2$ and $k \in \{1, \dots, d-1\}$. There exist $c_0, c_1, c_2 > 0$ with the following property: for every $(m, \delta) \in (0, \frac{1}{2})^2$, there exist disjoint measurable sets $E_j \subset \mathbb{R}^d$, $j \in \{1, \dots, \lfloor \frac{c_0}{\delta^2+m} \rfloor\}$, satisfying $|E_j| \ge c_1\delta^{k-d}j^{k-1}$ and*

$$\int_{\mathbb{S}^{d-1}} e^{irx\cdot\omega}\mathbf{1}_{\mathcal{C}_\delta^k}(\omega)\,\mathrm{d}\sigma(\omega) \ge c_2\delta^{d-k}j^{\frac{1-k}{2}},$$

whenever $|r-1| \le m$ and $x \in E_j$.

Proof If $k \in \{2, \dots, d-2\}$, then this result is a variant of the computations from the proof of [12, Theorem 1.3], which we recall for the reader's convenience. The starting point is the formula

$$\begin{aligned}\widehat{\mathbf{1}_{\mathcal{C}_\delta^k}\sigma}(rx) \simeq \int_0^\delta \rho^{d-k-1}(1-\rho^2)^{\frac{k-2}{2}} \\ \times (\rho r|y|)^{\frac{2-d+k}{2}} J_{\frac{d-k-2}{2}}(\rho r|y|) \cdot (\sqrt{1-\rho^2}r|z|)^{\frac{2-k}{2}} J_{\frac{k-2}{2}}(\sqrt{1-\rho^2}r|z|)\,\mathrm{d}\rho\end{aligned} \tag{2.9}$$

where $x = (y, z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$ and $|r-1| \le m$; see [12, Eq. (6.3)]. Define the set

$$\tilde{E}_j(r) := \left\{(y, z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k : 0 \le r|y| \le c\delta^{-1}, \frac{z_j - c}{\sqrt{1-\delta^2}} \le r|z| \le z_j + c\right\}$$

for some sufficiently small absolute constant $c > 0$ satisfying $c < \frac{1}{4}\inf\{z_{j+1} - z_j : j \in \mathbb{N}\}$, where $\{z_j\}_{j\ge1}$ denotes the increasing sequence of local maxima of the Bessel function

$J_{(k-2)/2}$. It is well-known that

$$z_j = 2\pi j + O(1) \quad \text{and} \quad J_{\frac{k-2}{2}}(z_j) \sim j^{-1/2} \quad \text{as } j \to \infty.$$

In [12], the expression from (2.9) is estimated from below by $c_1 \delta^{d-k} j^{\frac{1-k}{2}}$, provided that $x = (y, z)$ belongs to $\tilde{E}_j(r)$. Indeed, from (2.9) we infer that

$$\widehat{\mathbf{1}_{C_\delta^k}\sigma}(rx) \gtrsim \int_0^\delta \rho^{d-k-1}(1-\rho^2)^{\frac{k-2}{2}} \cdot 1 \cdot j^{\frac{2-k}{2}} j^{-\frac{1}{2}} \, \mathrm{d}\rho \gtrsim \delta^{d-k} j^{\frac{1-k}{2}}.$$

For every $r \in [1-m, 1+m]$ and $m \in (0, \frac{1}{2})$, we have $E_j \subset \tilde{E}_j(r)$, where

$$E_j := \left\{ (y,z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k : 0 \le |y| \le \frac{c}{\delta(1+m)}, \frac{z_j - c}{\sqrt{1-\delta^2(1-m)}} \le |z| \le \frac{z_j + c}{1+m} \right\}.$$

To bound the measure of E_j from below, choose $\alpha, \beta > 0$ such that the estimates

$$\left(\frac{z_j - c}{z_j + c}\right)^k \le 1 - \frac{\alpha}{j}, \qquad \left(\frac{1+m}{\sqrt{1-\delta^2(1-m)}}\right)^k \le 1 + \beta(\delta^2 + m)$$

hold for $(\delta, m) \in (0, \frac{1}{2})^2$ and all $j \in \mathbb{N}$. We then find, for $j = 1, \ldots, \lfloor \frac{c_0}{\delta^2+m} \rfloor$ with $c_0 := \frac{\alpha}{2\beta}$,

$$\begin{aligned}
|E_j| &\gtrsim \left(\frac{c}{\delta(1+m)}\right)^{d-k} \left[\left(\frac{z_j + c}{1+m}\right)^k - \left(\frac{z_j - c}{\sqrt{1-\delta^2(1-m)}}\right)^k \right] \\
&\gtrsim \delta^{k-d} j^k \left[1 - \left(\frac{z_j - c}{z_j + c}\right)^k \left(\frac{1+m}{\sqrt{1-\delta^2(1-m)}}\right)^k \right] \\
&\gtrsim \delta^{k-d} j^k \left[1 - (1 - \frac{\alpha}{j}) \cdot (1 + \beta(\delta^2 + m)) \right] \\
&\gtrsim \delta^{k-d} j^k \left[\frac{\alpha}{j} - \beta(\delta^2 + m) \right] \\
&\gtrsim \delta^{k-d} j^{k-1}.
\end{aligned}$$

This finishes the proof for $k \in \{2, \ldots, d-2\}$. Formula (2.9) continues to hold in the case $k \in \{1, d-1\}$ by taking into account the fact that

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z).$$

The same proof, with z_j replaced by $2\pi j$, then yields the result. □

Lemma 2.3 *Assume $2 < p < \frac{2k}{k-1}$. For $\delta, m \in (0, \frac{1}{2})$, nonnegative functions $a \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ and v as defined in (2.8), we have*

$$\mathrm{q}_\varepsilon(v) \simeq \delta^{d-k} \int_0^m a^2(s)(\varepsilon + s^\gamma) \, \mathrm{d}s, \tag{2.10}$$

$$\|v\|_p \gtrsim \delta^{\frac{d-k}{p'}} (\delta^2 + m)^{\frac{k-1}{2} - \frac{k}{p}} \int_0^m a(s) \, \mathrm{d}s. \tag{2.11}$$

These estimates are uniform with respect to $\varepsilon, \delta, m, a$.

Proof Estimate (2.10) is a simple consequence of first passing to polar coordinates in Fourier space, and then applying the two-sided inequality (1.5) together with the change of variables $s = r - 1$:

$$\begin{aligned} \mathrm{q}_\varepsilon(v) &\simeq \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 g_\varepsilon(|\xi|)\, \mathrm{d}\xi \\ &= \sigma(\mathcal{C}^k_\delta) \int_{1-m}^{1+m} a^2(|r-1|) g_\varepsilon(r)\, \mathrm{d}r \simeq \delta^{d-k} \int_0^m a^2(s)(\varepsilon + s^\gamma)\, \mathrm{d}s. \end{aligned}$$

Note that this estimate is uniform with respect to $\varepsilon, \delta, m, a$. Estimate (2.11) stems from the following identity, obtained via Fourier inversion:

$$v(x) \cong \int_{1-m}^{1+m} a(|r-1|) \left(\int_{\mathbb{S}^{d-1}} e^{irx\cdot\omega} \mathbf{1}_{\mathcal{C}^k_\delta}(\omega)\, \mathrm{d}\sigma(\omega) \right) r^{d-1}\, \mathrm{d}r. \tag{2.12}$$

Choosing $c_0 > 0$ and $E_j \subset \mathbb{R}^d$ as in Proposition 2.2 we obtain, for $j \in \{1, \ldots, \lfloor \frac{c_0}{\delta^2+m} \rfloor\}$,

$$|v(x)| \gtrsim \delta^{d-k} j^{\frac{1-k}{2}} \int_0^m a(s)\, \mathrm{d}s \quad \text{for every } x \in E_j. \tag{2.13}$$

Using that $|E_j| \gtrsim \delta^{k-d} j^{k-1}$, we thus obtain for $2 < p < \frac{2k}{k-1}$:

$$\begin{aligned} \|v\|_p &\overset{(2.13)}{\gtrsim} \delta^{d-k} \left(\sum_{j=1}^{\lfloor \frac{c_0}{\delta^2+m} \rfloor} (j^{\frac{1-k}{2}})^p |E_j| \right)^{\frac{1}{p}} \int_0^m a(s)\, \mathrm{d}s \\ &\gtrsim \delta^{d-k} \left(\sum_{j=1}^{\lfloor \frac{c_0}{\delta^2+m} \rfloor} j^{\frac{1-k}{2}(p-2)} \delta^{k-d} \right)^{\frac{1}{p}} \int_0^m a(s)\, \mathrm{d}s \\ &\simeq \delta^{\frac{d-k}{p'}} (\delta^2+m)^{\frac{k-1}{2}-\frac{k}{p}} \int_0^m a(s)\, \mathrm{d}s, \end{aligned}$$

which is all we had to show. Note that we used $p < \frac{2k}{k-1}$ in the last estimate. Given that these estimates are again uniform with respect to $\varepsilon, \delta, m, a$, the result follows. □

In the radial case, the following substitute holds for simpler trial functions of the form

$$\widehat{v}(\xi) := \mathbf{1}_{||\xi|-1| \le m} \cdot a(||\xi| - 1|). \tag{2.14}$$

Lemma 2.4 *For all $m \in (0, \frac{1}{2})$ and nonnegative functions $a \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ and v as defined in (2.14),*

$$q_\varepsilon(v) \simeq \int_0^m a^2(s)(\varepsilon + s^\gamma)\, \mathrm{d}s, \tag{2.15}$$

$$\|v\|_p \gtrsim \int_0^m a(s)\, \mathrm{d}s \cdot \begin{cases} m^{\frac{d-1}{2}-\frac{d}{p}}, & \text{if } p \in (2, 2^{\mathrm{rad}}_\star), \\ |\log(m)|^{\frac{1}{p}}, & \text{if } p = 2^{\mathrm{rad}}_\star, \\ 1, & \text{if } p \in (2^{\mathrm{rad}}_\star, \infty). \end{cases} \tag{2.16}$$

Proof The proof is analogous to that of Lemma 2.3, so we just highlight the differences. The lower bound (2.16) relies on (2.6) for the representation

$$v(x) = c_d |x|^{\frac{2-d}{2}} \int_{1-m}^{1+m} a(|r-1|) J_{\frac{d-2}{2}}(r|x|) r^{\frac{d}{2}}\, \mathrm{d}r.$$

Let again $z_j = 2\pi j + O(1)$, as $j \to \infty$, denote the sequence of local maxima of $J_{\frac{d-2}{2}}$. With $c > 0$ as in the proof of Proposition 2.2, it holds that

$$|J_{\frac{d-2}{2}}(r|x|)| \gtrsim j^{-1/2}, \text{ if } |r-1| \le m \text{ and } \frac{z_j - c}{1-m} \le |x| \le \frac{z_j + c}{1+m}.$$

For such x, we get the pointwise lower bound

$$|v(x)| \gtrsim j^{\frac{1-d}{2}} \int_0^m a(s)\, \mathrm{d}s.$$

Arguing as above, we find that this lower bound holds for x belonging to annular regions $E_j \subset \mathbb{R}^d$ satisfying

$$|E_j| \gtrsim \left(\frac{z_j + c}{1+m}\right)^d - \left(\frac{z_j - c}{1-m}\right)^d \gtrsim j^{d-1}$$

provided that $j \in \{1, 2, \ldots, \lfloor \frac{c_0}{m} \rfloor\}$ for some suitably small $c_0 > 0$. Integrating these estimates and summing up the resulting bounds yields

$$\|v\|_p \gtrsim \left(\sum_{j=1}^{\lfloor \frac{c_0}{m} \rfloor} j^{\frac{p(1-d)}{2}} |E_j|\right)^{\frac{1}{p}} \int_0^m a_\varepsilon(s)\, \mathrm{d}s \gtrsim \left(\sum_{j=1}^{\lfloor \frac{c_0}{m} \rfloor} j^{\frac{(p-2)(1-d)}{2}}\right)^{\frac{1}{p}} \int_0^m a(s)\, \mathrm{d}s.$$

The three cases in (2.16) correspond to the exponent $\frac{(p-2)(1-d)}{2}$ being larger than, equal to, or smaller than -1. $\square$

Proof of Theorem 1.3 (Upper bounds) **Radial case** (1.13). By Lemma 2.4, for all $m \in (0, \frac{1}{2})$ and nonnegative functions $a \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ it holds that

$$\mathrm{R}_\varepsilon^{\mathrm{rad}}(p) \lesssim \frac{\int_0^m a^2(s)(\varepsilon + s^\gamma)\, \mathrm{d}s}{\left(\int_0^m a(s)\, \mathrm{d}s\right)^2} \times \begin{cases} m^{1-d+\frac{2d}{p}}, & \text{if} p \in (2, 2_\star^{\mathrm{rad}}), \\ |\log(m)|^{-\frac{2}{p}}, & \text{if } p = 2_\star^{\mathrm{rad}}, \\ 1, & \text{if } p \in (2_\star^{\mathrm{rad}}, \infty). \end{cases}$$

If $a(s) = (\varepsilon + s^\gamma)^{-1}$, then the first factor simplifies to $\left(\int_0^m (\varepsilon + s^\gamma)^{-1}\, \mathrm{d}s\right)^{-1}$. Choosing $m := \varepsilon^{\frac{1}{\gamma}}$, we find that $m \in (0, \frac{1}{2})$ for small $\varepsilon > 0$, as well as

$$\mathrm{R}_\varepsilon^{\mathrm{rad}}(p) \lesssim \begin{cases} \varepsilon^{1-\frac{1}{\gamma}} \cdot \varepsilon^{\frac{1}{\gamma}(1-d+\frac{2d}{p})}, & \text{if} p \in (2, 2_\star^{\mathrm{rad}}), \\ \varepsilon^{1-\frac{1}{\gamma}} \cdot |\log(\varepsilon)|^{-\frac{2}{p}}, & \text{if } p = 2_\star^{\mathrm{rad}}, \\ \varepsilon^{1-\frac{1}{\gamma}}, & \text{if } p \in (2_\star^{\mathrm{rad}}, \infty) \end{cases}$$
$$\simeq \begin{cases} \varepsilon^{1-\frac{\alpha_{\mathrm{rad}}}{\gamma}}, & \text{if } p \in (2, 2_\star^{\mathrm{rad}}), \\ \varepsilon^{1-\frac{1}{\gamma}} |\log(\varepsilon)|^{-\frac{d-1}{d}}, & \text{if } p = 2_\star^{\mathrm{rad}}, \\ \varepsilon^{1-\frac{1}{\gamma}}, & \text{if } p \in (2_\star^{\mathrm{rad}}, \infty). \end{cases}$$

G_k**-symmetric case** (1.12). Since $\mathrm{R}_\varepsilon^k(p) = \mathrm{R}_\varepsilon^{d-k}(p)$ it suffices to prove the claimed inequality for $k \le \lfloor d/2 \rfloor$, so (1.8) yields $\alpha_k = (d+k)(\frac{1}{2} - \frac{1}{p})$. If $p > 2_\star^k$, then $p > 2_\star^{\mathrm{rad}}$ and thus

$$\mathrm{R}_\varepsilon^k(p) \le \mathrm{R}_\varepsilon^{\mathrm{rad}}(p) \lesssim \varepsilon^{1-\frac{1}{\gamma}},$$

as claimed. Thus we may assume $2 < p \leq 2_\star^k$, which in particular forces $p < \frac{2k}{k-1}$. So Lemma 2.3 applies and the choices $a(s) = (\varepsilon + s^\gamma)^{-1}$ and $m = \delta^2$ lead to

$$\mathrm{R}_\varepsilon^k(p) \lesssim \frac{\int_0^{\delta^2} a^2(s)(\varepsilon + s^\gamma)\,\mathrm{d}s \cdot \delta^{d-k}}{\left(\delta^{\frac{d-k}{p'}}(\delta^2)^{\frac{k-1}{2}-\frac{k}{p}} \int_0^{\delta^2} a(s)\,\mathrm{d}s\right)^2} \simeq \delta^{2-2\alpha_k}\left(\int_0^{\delta^2} \frac{1}{\varepsilon + s^\gamma}\,\mathrm{d}s\right)^{-1}.$$

Setting $\delta = \varepsilon^{\frac{1}{2\gamma}}$ yields $\mathrm{R}_\varepsilon^k(p) \lesssim \varepsilon^{1-\frac{\alpha_k}{\gamma}}$, which proves the claim.

Non-symmetric case (1.11). In this case, we have

$$\mathrm{R}_\varepsilon^\circ(p) \lesssim \mathrm{R}_\varepsilon^1(p) \simeq \varepsilon^{1-\frac{1\wedge\alpha_1}{\gamma}}.$$

This concludes the proof of Theorem 1.3. □

3 Fourier concentration

In this section, we prove Theorem 1.4.

Lemma 3.1 *Assume $d \geq 2$, $k \in \{1, \ldots, d-1\}$, $2 < p < 2_s^\star$. Let g_ε be (s, γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Let $* \in \{\circ, k, \mathrm{rad}\}$. Assume that u_ε is a minimizer for $\mathrm{R}_\varepsilon^*(p)$ and that there exist nonzero functions $v_\varepsilon, w_\varepsilon$, such that $u_\varepsilon = v_\varepsilon + w_\varepsilon$ and $\widehat{v}_\varepsilon \cdot \widehat{w}_\varepsilon = 0$ almost everywhere. Moreover, suppose that there exists $M_\varepsilon > 1$, such that*

$$\frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\|w_\varepsilon\|_p^2} \geq M_\varepsilon \mathrm{R}_\varepsilon^*(p). \tag{3.1}$$

Then the following estimates hold:

$$\frac{\|w_\varepsilon\|_p}{\|v_\varepsilon\|_p} \leq \frac{2}{M_\varepsilon - 1}, \quad \text{and} \quad \frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\mathrm{q}_\varepsilon(v_\varepsilon)} \leq \frac{4}{M_\varepsilon(1 - M_\varepsilon^{-1})^2}. \tag{3.2}$$

Proof The assumption $\widehat{v}_\varepsilon \cdot \widehat{w}_\varepsilon = 0$ implies $\mathrm{q}_\varepsilon(u_\varepsilon) = \mathrm{q}_\varepsilon(v_\varepsilon) + \mathrm{q}_\varepsilon(w_\varepsilon)$. Minkowski's inequality and assumption (3.1) then imply

$$\mathrm{R}_\varepsilon^*(p) = \frac{\mathrm{q}_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_p^2} \geq \frac{\mathrm{q}_\varepsilon(v_\varepsilon) + \mathrm{q}_\varepsilon(w_\varepsilon)}{(\|v_\varepsilon\|_p + \|w_\varepsilon\|_p)^2} \geq \mathrm{R}_\varepsilon^*(p) \frac{\|v_\varepsilon\|_p^2 + M_\varepsilon \|w_\varepsilon\|_p^2}{(\|v_\varepsilon\|_p + \|w_\varepsilon\|_p)^2}.$$

This can be rewritten as

$$(\|v_\varepsilon\|_p + \|w_\varepsilon\|_p)^2 \geq \|v_\varepsilon\|_p^2 + M_\varepsilon \|w_\varepsilon\|_p^2,$$

which leads to the first estimate in (3.2) after elementary manipulations. The second estimate in (3.2) follows similarly:

$$\mathrm{R}_\varepsilon^*(p) \geq \frac{\mathrm{q}_\varepsilon(v_\varepsilon) + \mathrm{q}_\varepsilon(w_\varepsilon)}{(\|v_\varepsilon\|_p + \|w_\varepsilon\|_p)^2} \geq \frac{\mathrm{q}_\varepsilon(v_\varepsilon) + \mathrm{q}_\varepsilon(w_\varepsilon)}{\left(\sqrt{\frac{\mathrm{q}_\varepsilon(v_\varepsilon)}{\mathrm{R}_\varepsilon^*(p)}} + \sqrt{\frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{M_\varepsilon \mathrm{R}_\varepsilon^*(p)}}\right)^2}$$

can be rewritten as

$$\left(\sqrt{\mathrm{q}_\varepsilon(v_\varepsilon)} + M_\varepsilon^{-\frac{1}{2}} \sqrt{\mathrm{q}_\varepsilon(w_\varepsilon)}\right)^2 \geq \mathrm{q}_\varepsilon(v_\varepsilon) + \mathrm{q}_\varepsilon(w_\varepsilon),$$

which leads to the second estimate in (3.2). □

Theorem 1.4 follows by verifying the lower bound (3.1) for some M_ε such that $M_\varepsilon \to \infty$, as $\varepsilon \to 0^+$. The latter condition is the content of our next result.

Lemma 3.2 *Assume $d \geq 2$, $k \in \{1, \dots, d-1\}$, $2 < p < 2_s^\star$. Let g_ε be (s, γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Let $* \in \{\circ, k, \mathrm{rad}\}$. Let $u_\varepsilon, v_\varepsilon, w_\varepsilon$ be as in Theorem 1.4, such that $\widehat{w}_\varepsilon$ vanishes identically on the spherical shell $A_{\varepsilon,\delta_\varepsilon}$ from (1.16)–(1.17) for a given positive null sequence $\{\delta_\varepsilon\}$. Then*

$$\frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\|w_\varepsilon\|_p^2} \gtrsim M_\varepsilon \mathrm{R}_\varepsilon^*(p) \tag{3.3}$$

holds for some $M_\varepsilon = M_\varepsilon(p, d, k, s, \gamma) > 0$ satisfying $M_\varepsilon \to \infty$, as $\varepsilon \to 0^+$.

Proof We only consider the case $* = k$ and split the analysis into two cases. If $2_\star^k \leq p < 2_s^\star$, then reasoning as in (2.3) we obtain

$$\begin{aligned}
\|w_\varepsilon\|_p^2 &\simeq \left\| \int_{\mathbb{R}^d} e^{i\langle \xi, \cdot \rangle} \widehat{w}_\varepsilon(\xi)\, \mathrm{d}\xi \right\|_p^2 \\
&= \left\| \int_{r \notin I_{\varepsilon,\delta}} r^{d-1} \left(\int_{\mathbb{S}^{d-1}} e^{i\langle \omega, r\cdot \rangle} \widehat{w}_\varepsilon(r\cdot)\, \mathrm{d}\sigma(\omega) \right) \mathrm{d}r \right\|_p^2 \\
&\lesssim \left(\int_{r \notin I_{\varepsilon,\delta}} \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle \omega, r\cdot \rangle} \widehat{w}_\varepsilon(r\omega)\, \mathrm{d}\sigma(\omega) \right\|_p \mathrm{d}r \right)^2 \lesssim \left(\int_{r \notin I_{\varepsilon,\delta}} \|\widehat{w}_\varepsilon(r\cdot)\|_{L^2(\mathbb{S}^{d-1})}\, \mathrm{d}r \right)^2 \\
&\lesssim \left(\int_{\mathbb{R}^d} |\widehat{w}_\varepsilon(\xi)|^2 g_\varepsilon(|\xi|)\, \mathrm{d}\xi \right) \left(\int_{r \notin I_{\varepsilon,\delta}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \right) \\
&\lesssim \mathrm{q}_\varepsilon(w_\varepsilon) \left(\int_0^{\delta_\varepsilon \varepsilon^{1/\gamma}} \varepsilon^{-1}\, \mathrm{d}s + \int_{\delta_\varepsilon^{-1}\varepsilon^{1/\gamma}}^{\infty} s^{-\gamma}\, \mathrm{d}s \right) \\
&\simeq \mathrm{q}_\varepsilon(w_\varepsilon)(\delta_\varepsilon + \delta_\varepsilon^{\gamma-1})\varepsilon^{\frac{1}{\gamma}-1}.
\end{aligned}$$

Here, we used $I_{\varepsilon,\delta} = [\delta\varepsilon^{1/\gamma}, \delta^{-1}\varepsilon^{1/\gamma}]$ and that g_ε satisfies the lower bound in (1.5). In light of (1.12), it follows that (3.3) holds with $M_\varepsilon \simeq (\delta_\varepsilon + \delta_\varepsilon^{\gamma-1})^{-1}$. If $2 < p < 2_\star^k$, then reasoning as in (2.4)–(2.5) yields

$$\|w_\varepsilon\|_p^2 \leq (\|w_\varepsilon\|_2^2)^{1-\alpha_k} (\|w_\varepsilon\|_{2_\star^k}^2)^{\alpha_k} \lesssim (\varepsilon^{-1}\mathrm{q}_\varepsilon(w_\varepsilon))^{1-\alpha_k} \left((\delta_\varepsilon + \delta_\varepsilon^{\gamma-1})\varepsilon^{\frac{1}{\gamma}-1}\mathrm{q}_\varepsilon(w_\varepsilon) \right)^{\alpha_k},$$

so (1.12) yields (3.3) with $M_\varepsilon \simeq (\delta_\varepsilon + \delta_\varepsilon^{\gamma-1})^{-\alpha_k}$. Since $2 < p < 2_\star^k$ implies $\alpha_k > 0$, this proves the claim. □

4 Roughness

In this section, we prove Theorem 1.5, which should be regarded as a quantified version of the fact that non-radial minimizers of the Rayleigh quotient asymptotically exhibit some Knapp-type behaviour. In particular, their Fourier transforms must develop certain singularities along spheres close to $\mathbb{S}^{d-1}$. To prove this, we use Sobolev estimates for the adjoint restriction operator (1.15). In the general (non-symmetric) case, optimal results were established by Cho–Guo–Lee [3] but curiously their methods do not seem to allow for the required improved estimates in the G_k-symmetric setting, which are key to our analysis.

Proposition 4.1 *Let $d \geq 2$ and $k \in \{1, \dots, d-1\}$. There exists $t^\star = t^\star(d) > 0$ such that, for all $t \in [0, t^\star)$ and G_k-symmetric functions $f \in H^t(\mathbb{S}^{d-1})$, the following estimate holds:*

$$\|\widehat{f\sigma}\|_{2_\star^k(t)} \lesssim_t \|f\|_{H^t(\mathbb{S}^{d-1})}, \quad \textit{where } \frac{1}{2_\star^k(t)} := \frac{1 - t/t^\star}{2_\star^k} + \frac{t/t^\star}{2_\star^{\mathrm{rad}}}. \tag{4.1}$$

Moreover, estimate (4.1) for $k \in \{1, d-1\}$ holds for every function $f \in H^t(\mathbb{S}^{d-1})$.

Proof In view of (1.14), the estimate

$$\|\widehat{f\sigma}\|_{2_\star^k} \lesssim \|f\|_{L^2(\mathbb{S}^{d-1})} \tag{4.2}$$

holds for every G_k-symmetric function $f : \mathbb{S}^{d-1} \to \mathbb{C}$. On the other hand, it was proved in [11, Prop. 1] that the pointwise decay estimate

$$|\widehat{f\sigma}(x)| \lesssim \|f\|_{C^m(\mathbb{S}^{d-1})}(1 + |x|)^{\frac{1-d}{2}}$$

holds for every $f \in C^m(\mathbb{S}^{d-1})$ with $m = \lfloor \frac{d-1}{2} \rfloor + 1$. By Sobolev embedding, we may choose $t^\star = t^\star(d) > 0$ such that $H^{t^\star}(\mathbb{S}^{d-1})$ embeds into $C^m(\mathbb{S}^{d-1})$. Then

$$\|\widehat{f\sigma}\|_{2_\star^{\mathrm{rad}},\infty} \lesssim \|f\|_{C^m(\mathbb{S}^{d-1})} \|(1 + |x|)^{\frac{1-d}{2}}\|_{2_\star^{\mathrm{rad}},\infty} \lesssim \|f\|_{H^{t^\star}(\mathbb{S}^{d-1})}, \tag{4.3}$$

for every $f \in H^{t^\star}(\mathbb{S}^{d-1})$. The desired conclusion follows from interpolating (4.2) and (4.3). Indeed, given $t \in (0, t^\star)$, define $\theta_t := t/t^\star$, so that real interpolation [1, Ch. 3] implies

$$\|\widehat{f\sigma}\|_{2_\star^k(t)} \lesssim \|\widehat{f\sigma}\|_{2_\star^k(t),2} \simeq \|\widehat{f\sigma}\|_{(L^{2_\star^k}, L^{2_\star^{\mathrm{rad}},\infty})_{\theta_t,2}} \lesssim \|f\|_{(L^2(\mathbb{S}^{d-1}), H^{t^\star}(\mathbb{S}^{d-1}))_{\theta_t,2}} \simeq \|f\|_{H^t(\mathbb{S}^{d-1})}.$$

In the last equality, we used the identity

$$(L^2(\mathbb{S}^{d-1}), H^{t^\star}(\mathbb{S}^{d-1}))_{\theta_t,2} = H^t(\mathbb{S}^{d-1}),$$

which holds by definition of θ_t according to [1, Theorem 6.2.4 and Theorem 6.4.4]. □

Proof of Theorem 1.5 We focus on the proof for G_k-symmetric functions. In view of Proposition 4.1, the non-symmetric case $* = \circ$ is treated as the case $k \in \{1, d-1\}$.

Assume $2 < p < 2_\star^k$. Lowering $t > 0$ if necessary, we lose no generality in further assuming $2 < p < 2_\star^k(t)$ since $2_\star^k(t) \nearrow 2_\star^k$, as $t \to 0^+$. Aiming at a contradiction, suppose that (1.19) does not hold, i.e., there exists $C < \infty$, such that

$$\sup_{|r-1| \in I_{\varepsilon,\delta_\varepsilon}} \frac{\|\widehat{u}_\varepsilon(r\cdot)\|_{H^t(\mathbb{S}^{d-1})}}{\|\widehat{u}_\varepsilon(r\cdot)\|_{L^2(\mathbb{S}^{d-1})}} \leq C, \text{ as } \varepsilon \to 0^+. \tag{4.4}$$

Our aim is to use (4.4) in order to verify the hypothesis (3.1) in Lemma 3.1 with $M_\varepsilon \to \infty$, as $\varepsilon \to 0^+$. To this end, split $u_\varepsilon = v_\varepsilon + w_\varepsilon$ as in (1.18), so that $\widehat{w}_\varepsilon$ is supported outside the spherical shell $A_{\varepsilon,\delta_\varepsilon}$ where $\delta_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$. In Lemma 3.2, we have already verified that this part is asymptotically negligible. More precisely,

$$\frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\|w_\varepsilon\|_p^2} \gtrsim M_\varepsilon \mathrm{R}_\varepsilon^k(p), \tag{4.5}$$

with $M_\varepsilon \to \infty$, as $\varepsilon \to 0^+$. Next we use (4.4) in order to prove lower bounds for the corresponding Rayleigh quotient involving v_ε. To this end, we invoke Fourier inversion, the

Sobolev estimate (4.1), assumption (4.4) and Lemma 2.1, yielding

$$\begin{aligned}
\|v_\varepsilon\|^2_{2^k_\star(t)} &\lesssim \left\| \int_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}} \left(\int_{\mathbb{S}^{d-1}} e^{ir\omega\cdot} \widehat{v}_\varepsilon(r\omega)\, \mathrm{d}\sigma(\omega) \right) \mathrm{d}r \right\|^2_{2^k_\star(t)} \\
&\lesssim \left(\int_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}} \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle\omega,\cdot\rangle} \widehat{v}_\varepsilon(r\omega)\, \mathrm{d}\sigma(\omega) \right\|_{2^k_\star(t)} \mathrm{d}r \right)^2 \\
&\lesssim \left(\int_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}} \|\widehat{v}_\varepsilon(r\cdot)\|_{H^t(\mathbb{S}^{d-1})}\, \mathrm{d}r \right)^2 \le C^2 \left(\int_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}} \|\widehat{v}_\varepsilon(r\cdot)\|_{L^2(\mathbb{S}^{d-1})}\, \mathrm{d}r \right)^2 \\
&\lesssim \mathrm{q}_\varepsilon(v_\varepsilon) \left(\int_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \right) \lesssim \varepsilon^{\frac{1}{\gamma}-1} \mathrm{q}_\varepsilon(v_\varepsilon).
\end{aligned}$$

The key observation is that the exponent $2^k_\star(t)$ is strictly smaller than $2^k_\star$, and therefore the corresponding interpolation parameter

$$\alpha_k(t) := \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{2^k_\star(t)}} \in (0, 1)$$

satisfies $\alpha_k(t) > \alpha_k$; recall (1.8). For $2 < p < 2^k_\star(t)$, we then have

$$\begin{aligned}
\|v_\varepsilon\|^2_p &\le (\|v_\varepsilon\|^2_2)^{1-\alpha_k(t)} (\|u_\varepsilon\|^2_{2^k_\star(t)})^{\alpha_k(t)} \\
&\lesssim (\varepsilon^{-1}\mathrm{q}_\varepsilon(v_\varepsilon))^{1-\alpha_k(t)} (\varepsilon^{\frac{1}{\gamma}-1}\mathrm{q}_\varepsilon(v_\varepsilon))^{\alpha_k(t)} = \varepsilon^{\frac{\alpha_k(t)}{\gamma}-1}\mathrm{q}_\varepsilon(v_\varepsilon).
\end{aligned}$$

In view of (1.12), it follows that

$$\frac{\mathrm{q}_\varepsilon(v_\varepsilon)}{\|v_\varepsilon\|^2_p} \gtrsim \varepsilon^{1-\frac{\alpha_k(t)}{\gamma}} \simeq \varepsilon^{\frac{\alpha_k-\alpha_k(t)}{\gamma}} \mathrm{R}^k_\varepsilon(p). \tag{4.6}$$

From (4.5) and (4.6), we then conclude

$$\mathrm{R}^k_\varepsilon(p) = \frac{\mathrm{q}_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|^2_p} \ge \frac{1}{2}\left(\frac{\mathrm{q}_\varepsilon(v_\varepsilon)}{\|v_\varepsilon\|^2_p} \wedge \frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\|w_\varepsilon\|^2_p} \right) \gtrsim \left(\varepsilon^{\frac{\alpha_k-\alpha_k(t)}{\gamma}} \wedge M_\varepsilon \right) \mathrm{R}^k_\varepsilon(p),$$

which is absurd since $\alpha_k - \alpha_k(t) < 0$ and $M_\varepsilon \to \infty$, as $\varepsilon \to 0^+$. So the assumed uniform bound (4.4) cannot hold, and this completes the proof of the theorem. □

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Appendix A: Existence of ground states

Proposition 1 *Assume $d \geq 2$ and $2 < p < 2_s^\star$. Given $\varepsilon > 0$, let g_ε be (s, γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Then the infimum defining*

$$\mathrm{R}_\varepsilon^*(p) \textit{ for } * \in \{\circ, \mathrm{rad}\} \quad \textit{or} \quad \mathrm{R}_\varepsilon^k(p) \textit{ for } k \in \{2, \dots, d-2\}$$

is attained. Any minimizer is a non-trivial weak solution of (1.3) after multiplication by a nonzero constant.

Proof The non-symmetric case of $\mathrm{R}_\varepsilon^\circ(p)$ is covered by [2, Theorem 1]. The discussion of the radially symmetric case $* = \mathrm{rad}$ is almost identical to the G_k-symmetric case for $k \in \{2, \dots, d-2\}$, so we only discuss the latter.

Let $\{u_n\} \subset H_k^s$ be a minimizing sequence such that $\|u_n\|_p = 1$ for each n, so that

$$\mathrm{q}_\varepsilon(u_n) \searrow \mathrm{R}_\varepsilon^k(p), \text{ as } n \to \infty.$$

Then the sequence $\{\mathrm{q}_\varepsilon(u_n)\}$ is bounded in $\mathbb{R}$, and so $\{u_n\}$ is likewise bounded in H_k^s. The latter statement is a consequence of the definition of q_ε and the inequalities (1.4)–(1.5) for the (s, γ)-admissible function g_ε. By Alaoglu's theorem, there exists a subsequence $\{u_{n_\ell}\} \subset H_k^s$ satisfying $u_{n_\ell} \rightharpoonup u$ in H_k^s, as $\ell \to \infty$. Since $k \in \{2, \dots, d-2\}$, H_k^s compactly embeds into $L^p(\mathbb{R}^d)$; see [10, Théorème III.3]. As a consequence, $u_{n_\ell} \to u$ in $L^p(\mathbb{R}^d)$, as $\ell \to \infty$, and in particular $\|u\|_p = 1$. Being continuous and convex, the functional q_ε is weakly lower semicontinuous on H_k^s. Hence,

$$\mathrm{R}_\varepsilon^k(p) = \lim_{\ell \to \infty} \mathrm{q}_\varepsilon(u_{n_\ell}) \geq \mathrm{q}_\varepsilon(u) = \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2},$$

and it follows that u is a minimizer for $\mathrm{R}_\varepsilon^k(p)$. Lagrange's multiplier rule implies that any minimizer u is a weak solution after multiplication by a nonzero constant. □

Appendix B: A real interpolation bound

Proposition 2 *Let $0 < r < q < \infty$ and $u : \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that $\|u\|_r \leq C_1$ and $|u(x)| \leq C_2(1+|x|)^{-\frac{d}{q}}$ for almost every $x \in \mathbb{R}^d$. Then*

$$\|u\|_q \lesssim_{r,q,d} C_2 \left(1 + \log_+ \left(\frac{C_1}{C_2}\right)\right)^{\frac{1}{q}},$$

where $\log_+ := \log \vee \mathbf{0}$.

Proof We use the layer cake representation [8, § 1.13] for $|u|$, and set $\lambda(t) := |\{x \in \mathbb{R}^d : |u(x)| \geq t\}|$. The pointwise assumption implies

$$\lambda(t) \lesssim_d (C_2 t^{-1})^q \text{ for every } t > 0, \qquad \lambda(t) = 0 \text{ if } t \geq C_2.$$

For any $\delta \in [0, C_2]$, we thus have that

$$\|u\|_q^q = q \int_0^{C_2} t^{q-1}\lambda(t)\,\mathrm{d}t \lesssim_d \delta^{q-r} \int_0^{\delta} t^{r-1}\lambda(t)\,\mathrm{d}t + \int_\delta^{C_2} t^{q-1}(C_2 t^{-1})^q\,\mathrm{d}t$$
$$\lesssim \delta^{q-r}\|u\|_r^r + C_2^q \int_\delta^{C_2} t^{-1}\,\mathrm{d}t \le C_1^r \delta^{q-r} + C_2^q \log(C_2\delta^{-1}).$$

Minimizing the latter quantity with respect to $\delta \in [0, C_2]$ leads to the choice

$$\delta = \begin{cases} ((q-r)C_1^r C_2^{-q})^{\frac{1}{r-q}}, & \text{if } C_1 C_2^{-1} \ge (q-r)^{-\frac{1}{r}}, \\ C_2, & \text{if } C_1 C_2^{-1} < (q-r)^{-\frac{1}{r}}. \end{cases}$$

In the first case this yields

$$\|u\|_q^q \lesssim_d \frac{C_2^q}{q-r} + C_2^q\left(\log(C_2) + \frac{\log(q-r)}{q-r} + \frac{r}{q-r}\log(C_1) - \frac{q}{q-r}\log(C_2)\right)$$
$$= C_2^q\left(\frac{1+\log(q-r)}{q-r} - \frac{r}{q-r}\log(C_2) + \frac{r}{q-r}\log(C_1)\right)$$
$$\lesssim_{r,q} C_2^q\left(1 + \log_+\left(\frac{C_1}{C_2}\right)\right),$$

and in the second case we even obtain the better bound $\|u\|_q \lesssim C_2$. □

Appendix C: The case $k \in \{1, d-1\}$

We now explain in more detail why the symmetry groups $G_k = O(d-k) \times O(k)$ must be treated differently when $k \in \{1, d-1\}$. We focus on $k = 1$. The first observation is that the optimal Stein–Tomas exponent for G_1-symmetric functions is identical to the optimal exponent for general functions. In other words,

$$2_\star^1 = 2_\star = 2\frac{d+1}{d-1},$$

as observed in [12, Remark 6.1]. As a consequence, our methods do not allow to prove the strict inequality $\mathrm{R}_\varepsilon^\circ(p) < \mathrm{R}_\varepsilon^1(p)$ for sufficiently small $\varepsilon > 0$ and any $p > 2$, and it is an open question whether this holds or not. A related open question is whether some or even all ground states (i.e., minimizers for $\mathrm{R}_\varepsilon^\circ$) are G_1-symmetric up to a change of coordinates or whether G_1-symmetric solutions exist at all. In fact, our existence proof of Proposition 1 does not carry over to $k = 1$ due to the non-compactness of the embedding of H_1^s into $L^p(\mathbb{R}^d)$. Indeed, take any nonzero test function $\varphi \in C_0^\infty(\mathbb{R}^2)$ and define $u_n(x) := \varphi(|x'|^2, |x^d - n|^2) + \varphi(|x'|^2, |x^d + n|^2)$ where $x = (x', x^d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Then $\{u_n\}$ is bounded in H_1^s, but does not admit any $L^p(\mathbb{R}^d)$-convergent subsequence, so a compact embedding cannot exist. More generally, it is unclear whether minimizing sequences for the Rayleigh quotient $\mathrm{R}_\varepsilon^1(p)$ behave according to the “dichotomy” or the “compactness” alternative in Lions’ concentration-compactness principle. In view of [13, Lemma 2.4], “vanishing” does not occur in the Sobolev-subcritical regime $2 < p < 2_s^\star$.

Nevertheless, when $d \ge 3$ we can prove the existence of solutions with the slightly smaller symmetry group $G := O(d-1) \times \{1\} \subset O(d)$, where the evenness requirement with respect to the last coordinate is dropped. Denote the corresponding Sobolev space by H_G^s. We first establish the following uniform decay estimate.

Proposition 3 *Let $d \geq 2$, $q > 2$, and $t > 0$. Then there exists $\tau = \tau(d,q,t) > 0$ such that, for all radially symmetric functions $u \in H^t(\mathbb{R}^d)$, the following holds for all $R > 0$:*

$$\|u\|_{L^q(\mathbb{R}^d \setminus B_R(0))} \lesssim R^{-\tau} \|u\|_{H^t(\mathbb{R}^d)}.$$

Proof Fix $\varepsilon > 0$. By [4, Theorem 3.1], there exists $\vartheta \in (0,1)$ such that[2]

$$|u(x)| \lesssim |x|^{\frac{1-d}{2}} [u]^{\vartheta}_{H^{\frac{1}{2}+\varepsilon}} \|u\|_2^{1-\vartheta} \lesssim |x|^{\frac{1-d}{2}} \|u\|_{H^{\frac{1}{2}+\varepsilon}}.$$

This implies, for $\frac{2d}{d-1} < \tilde{q} \leq \infty$,

$$\|u\|_{L^{\tilde{q}}(\mathbb{R}^d \setminus B_R(0))} \lesssim R^{\frac{d}{\tilde{q}} + \frac{1-d}{2}} \|u\|_{H^{\frac{1}{2}+\varepsilon}}.$$

We interpolate this estimate with the trivial bound $\|u\|_{L^2(\mathbb{R}^d \setminus B_R(0))} \leq \|u\|_2 = \|u\|_{H^0}$, and obtain

$$\|u\|_{L^q(\mathbb{R}^d \setminus B_R(0))} \lesssim R^{\theta(\frac{d}{\tilde{q}} + \frac{1-d}{2})} \|u\|_{H^{\theta(\frac{1}{2}+\varepsilon)}}, \quad \text{if } \frac{\theta}{\tilde{q}} + \frac{1-\theta}{2} = \frac{1}{q}, \ \theta \in [0,1].$$

The freedom to arbitrarily choose $\theta \in (0,1]$, $\tilde{q} > \frac{2d}{d-1}$ and $\varepsilon > 0$ allows us to conclude. □

Theorem 4 *Assume $d \geq 3$, $2 < p < 2^s_\star$, and $\varepsilon > 0$. Let g_ε be (s,γ)-admissible for some $s > \frac{d}{d+1}$ and $\gamma > 1$. Then the infimum defining the Rayleigh quotient*

$$\inf_{\mathbf{0} \neq u \in H^s_G} \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2} \tag{4.7}$$

is attained. Any minimizer is a G-symmetric non-trivial weak solution of (1.3) after multiplication by a nonzero constant.

Proof The argument mimicks the proof of [2, Theorem 1], so we concentrate on the novel aspects. As in [2], one shows that there exist a minimizing sequence $\{u_n\} \subset H^s_G$, a sequence $\{x_n\} \subset \mathbb{R}^d$ and a nonzero function $u \in H^s(\mathbb{R}^d)$, such that $\|u_n\|_p = 1$ and $u_n(\cdot + x_n) \rightharpoonup u$, as $n \to \infty$; see [2, Eq. (2.2)]. Writing $x_n = (x_n', x_n^d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, we shall prove below that the sequence $\{x_n'\}$ is bounded in $\mathbb{R}^{d-1}$ and hence convergent to some limit $x_\infty' \in \mathbb{R}^{d-1}$, possibly after extraction of a subsequence. We then obtain that $\tilde{u}_n(x) := u_n(x', x^d + x_n^d)$ is a minimizing sequence in H^s_G that weakly converges to the nonzero function $\tilde{u} \in H^s_G$ given by $\tilde{u}(x) := u(x' - x_\infty', x^d)$. Here we used that H^s_G is weakly closed in $H^s(\mathbb{R}^d)$. The argument can then be completed as in [2], yielding $\tilde{u}$ as a minimizer for the Rayleigh quotient (4.7) and establishing the claim.

It remains to prove that the sequence $\{x_n'\} \subset \mathbb{R}^{d-1}$ is bounded. Suppose not. Choose a test function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with support in $K := \overline{B_R(0)} \times [-R, R] \subset \mathbb{R}^d$ such that

[2] Here, $[u]^2_{H^s} := \int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 |\xi|^{2s} \,\mathrm{d}\xi$.

$\mu := \int_K u\varphi \neq 0$; here, $B_R(0)$ denotes the ball in $\mathbb{R}^{d-1}$ with radius R. Then

$$\begin{aligned}
\mu &= \int_K u(x)\varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_K u_n(x+x_n)\varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{K+(x_n',0)} u_n(x', x^d + x_n^d)\varphi(x'-x_n', x^d)\,\mathrm{d}x'\,\mathrm{d}x^d \\
&= \lim_{n\to\infty} \int_{-R}^{R} \left(\int_{B_R(0)+x_n'} u_n(x', x^d + x_n^d)\varphi(x'-x_n', x^d)\,\mathrm{d}x' \right) \mathrm{d}x^d.
\end{aligned}$$

We now apply Proposition 3 to the radially symmetric functions

$$\mathbb{R}^{d-1} \ni x' \mapsto u_n(x', x^d + x_n^d) \in \mathbb{R}$$

with $t \in (0, s - \frac{1}{2})$ and spatial dimension $d - 1 \geq 2$. This and the fractional Trace Theorem, i.e., the uniform boundedness of the trace operator $H^s(\mathbb{R}^d) \ni u \mapsto u|_{\mathbb{R}^{d-1}\times\{z\}} \in H^t(\mathbb{R}^{d-1})$ with respect to $z \in \mathbb{R}$, implies, for some $q > 2$ and $\tau > 0$,

$$\begin{aligned}
\mu &\leq \liminf_{n\to\infty} \int_{-R}^{R} \left(\|u_n(\cdot, x^d + x_n^d)\|_{L^q(B_R(0)+x_n')} \|\varphi(\cdot, x^d)\|_{L^{q'}(B_R(0)+x_n')} \right) \mathrm{d}x^d \\
&\lesssim_R \liminf_{n\to\infty} \|\varphi\|_\infty \int_{-R}^{R} \|u_n(\cdot, x^d + x_n^d)\|_{L^q(\mathbb{R}^{d-1}\setminus B_{|x_n'|-R}(0))}\,\mathrm{d}x^d \\
&\lesssim_R \liminf_{n\to\infty} \|\varphi\|_\infty \int_{-R}^{R} (|x_n'| - R)^{-\tau} \|u_n(\cdot, x^d + x_n^d)\|_{H^t(\mathbb{R}^{d-1})}\,\mathrm{d}x^d \\
&\lesssim_R \liminf_{n\to\infty} \|\varphi\|_\infty (|x_n'| - R)^{-\tau} \|u_n\|_{H^s}.
\end{aligned}$$

The last term is zero because $\{u_n\}$ is bounded in H^s_G and $|x_n'| \to \infty$ by assumption. This contradicts $\mu \neq 0$, so $\{x_n'\} \subset \mathbb{R}^{d-1}$ must be bounded. □

The preceding proof does not work for G_1-symmetric functions since *a priori* the functions $\tilde{u}_n(x) = u_n(x', x + x_n^d)$ are only G-symmetric. In other words, the $\tilde{u}_n$ might not be even in the last coordinate. But if u denotes a G-symmetric minimizer/solution, then its reflection with respect to x^d is another minimizer/solution. As a consequence, we find either one G_1-symmetric minimizer or two distinct G-symmetric minimizers which are related to each other by a reflection. Given Theorems 4 and 1.2, one may ask whether ground states are G-symmetric or not. Furthermore, as far as the asymptotic regime is concerned, one may check that the Fourier transform of any G-symmetric minimizer concentrates on the unit sphere and becomes rough as $\varepsilon \to 0^+$; the proof is identical to the G_k-symmetric case.

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