Bounding the first excursion probability of stochastic oscillators under randomness and imprecision

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Abstract
In this paper, we propose a method to bound the first excursion probability of uncertain linear systems subjected to Gaussian loading. Specifically, a case is considered, where the structural behaviour is affected by both interval and random variables. Further, the structure is subjected to a Gaussian stochastic process load, and it is of interest to estimate the bounds on the corresponding first excursion probability. To solve this kind of problem, an extension to the recently developed operator norm framework is proposed. This approach becomes applicable to oscillators that are described by random quantities. Specifically, this is obtained by representing the parametric dependency of the oscillator's system matrices to the random quantities by means of a first-order Taylor series expansion. A case study of a three-story concrete frame subjected to a stochastic wind load is included in the paper to illustrate the functionality and efficiency of the approach.

1 Introduction

Structural reliability theory offers a sound framework for quantifying the effects of aleatory uncertainty over the behavior of structural systems [1]. In this way, it is possible to synthesize the level of safety of a structural systems in terms of a failure probability [2]. The practical implementation of methods of structural reliability demands defining a probability model which, in turn, demands selecting crisp values for parameters such as mean value or standard deviation. In many cases, selecting such crisp values may be challenging due to issues such as incompleteness (lack of knowledge) and imprecision of the data. Here, there is an additional source of uncertainty of the epistemic type [3]. A possible means for coping with both sources of uncertainty is resorting to the concept of imprecise probabilities [4], where aleatory and epistemic sources of uncertainty are explicitly acknowledged (but not mixed). The concept is also known as polymorphic or hybrid uncertainty modeling [5] and is already used in various engineering disciplines, e.g. in non-linear concrete damage analysis [6], design of wooden structures [7], real-time risk assessment of tunneling-induced building damage [8] and shell buckling [9, 10]. The advantage of such formulation is that it allows to consider a set of possible probabilistic models instead of a single one, as usually considered in classical reliability analysis.

The framework provided by imprecise probabilities [4, 11] offers a suitable approach for coping with aleatory and epistemic uncertainty. Nonetheless, its practical application entails considerable challenges, as usually one must perform uncertainty propagation in a double-loop approach. That is, for each realization of the epistemic parameters, one has to perform a full reliability analysis to propagate aleatory uncertainty. In view of this challenge, this work proposes a framework for bounding the probability of failure of a structural system under the presence of aleatory and epistemic uncertainty. It is assumed that the structural model behaves linear and that some of its parameters can be characterized either as intervals or random variables. Furthermore, it is assumed that the load acting on the structure follows a Gaussian model, which can also depend on interval parameters. The proposed framework is based on the operator norm approach as presented in [12, 13], which is extended herein to account for aleatory uncertainty affecting the structural behavior. In particular, by linearizing the stochastic finite element structural model around the mean of the aleatory uncertain parameter, it is possible to extend the application of the operator norm approach. This allows determining those values of the epistemic uncertain parameters that yield an extremum in the failure probability without solving the associated reliability problem. Hence, the double loop that is typically associated to this type of problems is effectively broken. A case study illustrates the robustness and efficiency of the proposed method.

2 Formulation of the problem

Consider a Gaussian process, which models the load acting on a structural system. It is assumed that this process can be indexed with respect to position or time. Such process can be represented as follows:

$$\boldsymbol{f}(\boldsymbol{\theta}_f, \boldsymbol{z}) = \boldsymbol{\mu}(\boldsymbol{\theta}_f) + \boldsymbol{B}(\boldsymbol{\theta}_f)\boldsymbol{z} \quad . \tag{1}$$

In Eq. (1), $\boldsymbol{f}$ is a vector of dimension n_f representing a realization of the loading; $\boldsymbol{\mu}$ denotes the mean of the loading; $\boldsymbol{B}$ is a matrix of dimension $n_r \times n_z$ related with the covariance of the Gaussian process; $\boldsymbol{z}$ is a realization of a standard Gaussian probability distribution $p_{\boldsymbol{Z}}(\boldsymbol{z})$ of dimension n_z; and $\boldsymbol{\theta}_f$ is a vector, which contains parameters that are related with the characteristics of the Gaussian process (e.g. correlation length). It is assumed that there may be uncertainty regarding the precise value of this vector and hence, it is modeled as interval-valued, which implies that $\boldsymbol{\theta}_f \in [\underline{\boldsymbol{\theta}}_f, \overline{\boldsymbol{\theta}}_f]$, where $\underline{(\cdot)}$ and $\overline{(\cdot)}$ denote lower and upper bound.
The Gaussian process loading described above acts over a structural system, which behaves within the linear range. Some parameters of this structural model are uncertain and are modeled either as:

- interval-valued. This set of parameters is represented by $\boldsymbol{\theta}_s \in [\underline{\boldsymbol{\theta}}_s, \overline{\boldsymbol{\theta}}_s]$.
- random variables. A realization of this set of parameters is denoted by $\boldsymbol{y}$ while its probability distribution is $p_{\boldsymbol{Y}}(\boldsymbol{y})$.

For practical purposes, it is of interest monitoring n_r responses of the structural system. None of these responses (in absolute value) should exceed a prescribed threshold value $\boldsymbol{b}$.
Taking into account the previous definitions, it is possible to express the normalized structural response of the structure as:

$$\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{z}) = (\mathrm{diag}(\boldsymbol{b}))^{-1} \boldsymbol{A}(\boldsymbol{\theta}_s, \boldsymbol{y})\boldsymbol{f}(\boldsymbol{\theta}_f, \boldsymbol{z}) \quad , \tag{2}$$

where vector $\boldsymbol{\theta}$ groups $\boldsymbol{\theta}_f$ and $\boldsymbol{\theta}_s$; and $\boldsymbol{A}$ is a matrix representing structural behavior (e.g. inverse of stiffness matrix in static analysis). As Eq. (2) includes threshold levels, it has the advantage of producing dimensionless responses.
The probability that an undesirable behavior occurs is calculated by the following integral:

$$p_F(\boldsymbol{\theta}) = \int_{\boldsymbol{z}\in\mathbb{R}^{n_z}} \int_{\boldsymbol{y}\in\Omega_y} I_F(\boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{z}) p_{\boldsymbol{Y}}(\boldsymbol{y}) p_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{y} d\boldsymbol{z} \quad , \tag{3}$$

where p_F denotes the failure probability; and $I_F(\cdot,\cdot,\cdot)$ is the indicator function, which is equal to one in case $\|\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{y}, \boldsymbol{z})\|_\infty \geq 1$ and zero, otherwise (note that $\|\cdot\|_\infty$ denotes infinity norm). As the failure probability is dependent on the interval-valued vector $\boldsymbol{\theta}$, the failure probability itself becomes interval-valued. Naturally, it is of interest determining its lower and upper bounds (denoted as $\underline{p}_F$ and $\overline{p}_F$, respectively), that is:

$$\underline{p}_F = \min_{\boldsymbol{\theta}\in[\underline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}}]} \left(p_F(\boldsymbol{\theta})\right) \tag{4}$$

$$\overline{p}_F = \max_{\boldsymbol{\theta}\in[\underline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}}]} \left(p_F(\boldsymbol{\theta})\right) \quad . \tag{5}$$

The solution of these two optimization problems can be very cost-intensive from a numerical point of view. In fact, this corresponds to a double-loop problem, as in the outer loop the epistemic uncertainties have to be propagated, while in the inner loop the failure probabilities have to be evaluated .

3 Standard Deviation as a Proxy of the Failure Probability

The calculation of bounds of the failure probability can be quite challenging due to the necessity of coping with epistemic and aleatory uncertainty. A possible means for overcoming such issue is applying the operator norm framework, as proposed in [12, 13]. A detailed review of those contributions is beyond the scope of the present paper. Nonetheless, the results obtained in [12, 13] indicate that the minimum/maximum value of the maximum standard deviation associated with the normalized response $\boldsymbol{\eta}$ for a given $\boldsymbol{\theta}$ offers an excellent proxy for identifying the minimum/maximum value that the failure probability may attain. Hence, it is proposed to identify the set of values of the interval parameters that leads to a minimum/maximum value of the failure probability by solving the following two optimization problems.

$$\underline{\boldsymbol{\theta}}^* = \underset{\boldsymbol{\theta}\in[\underline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}}]}{\text{argmin}} \left(\sigma_{\max}(\boldsymbol{\theta})\right) \tag{6}$$

$$\overline{\boldsymbol{\theta}}^* = \underset{\boldsymbol{\theta}\in[\underline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}}]}{\text{argmax}} \left(\sigma_{\max}(\boldsymbol{\theta})\right) \quad , \tag{7}$$

where $\underline{\boldsymbol{\theta}}^*$ and $\overline{\boldsymbol{\theta}}^*$ denote the set of values of the uncertain interval parameters, which yield the minimum/maximum value of $\sigma_{\max}$. Here, $\sigma_{\max}$ denotes the maximum standard deviation of the response $\boldsymbol{\eta}$. Thus, the bounds of the failure probability are approximated as:

$$\underline{p}_F \approx p_F\left(\underline{\boldsymbol{\theta}}^*\right) \tag{8}$$

$$\overline{p}_F \approx p_F\left(\overline{\boldsymbol{\theta}}^*\right) \quad . \tag{9}$$

As noted from Eqs. (6) and (7), the problem of minimization/maximization of the failure probability is replaced by minimization/maximization of the maximum standard deviation. Upon examination of Eq. (2), it is noted that the calculation of the maximum standard deviation of the response $\boldsymbol{\eta}$ cannot be solved in closed-form for general cases. Nonetheless, if matrix $\boldsymbol{A}$ related with the structural response is approximated by means of the linear expansion

$$\boldsymbol{A}(\boldsymbol{\theta}_s, \boldsymbol{y}) = \boldsymbol{A}(\boldsymbol{\theta}_s, \boldsymbol{y}^0) + \sum_{i=1}^{n_y} \left.\frac{\partial \boldsymbol{A}(\boldsymbol{\theta}_s, \boldsymbol{y})}{\partial y_i}\right|_{\boldsymbol{y}=\boldsymbol{y}^0} (y_i - y_i^0) \quad , \tag{10}$$

it is possible to determine the maximum standard deviation in closed-form, following classical perturbation theory [14]. Note that $\boldsymbol{y}^0$ in Eq.(10) is the expansion point for linearization, which is taken as the expected value of the random variable vector $\boldsymbol{Y}$.

4 Example: Three-story concrete frame subjected to a stochastic wind load

A three-story concrete frame taken from [15] is modeled as a three-mass oscillator. The presented approach based on the maximum standard deviation is intended to reduce computing times in such simulations significantly. Moreover, the objective of this example is to show that the bounds can also be identified for more complex behaviors of the first excursion probability. The simplified model of the three-story concrete frame with a stochastic wind load $F(t)$ is depicted in Figure 1. The concrete floors are modeled as rigid

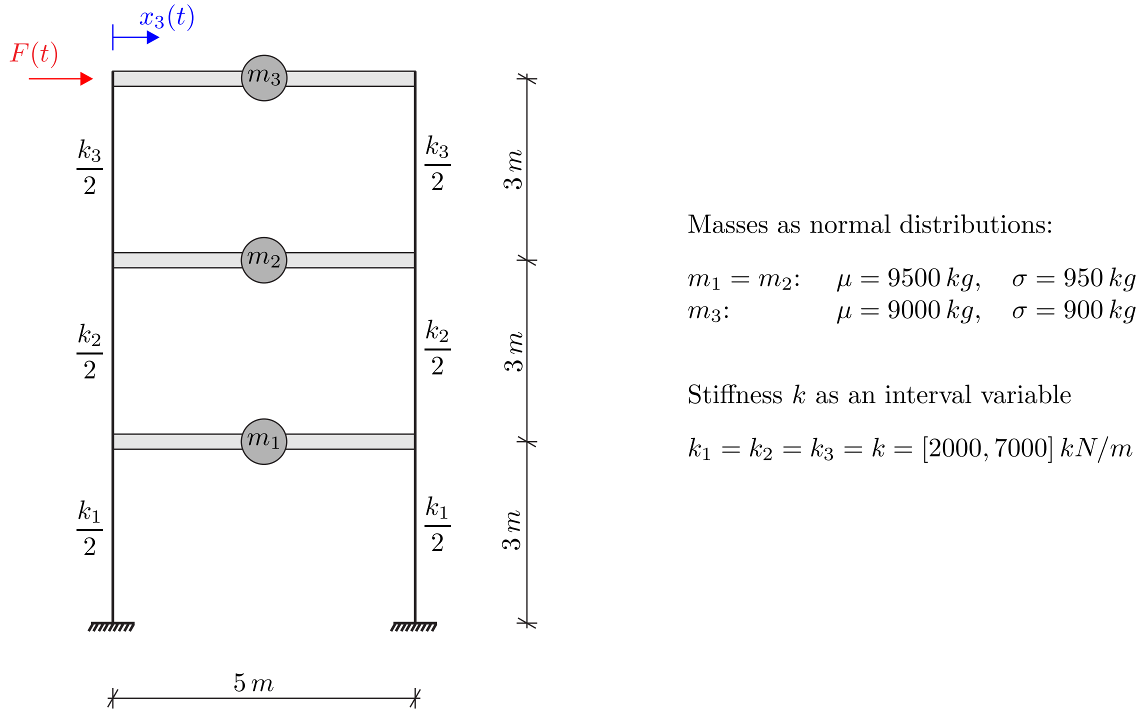

Figure 1: Three-degree-of-freedom model of the concrete frame

bars. Dead and traffic loads are fully considered in the point masses which are characterized as (truncated) Gaussian random variables. The expected value and standard deviation of the masses $y_1 = m_1 = m_2$ hold $\mu = 9500\,kg$ and $\sigma = 950\,kg$. The mass of the top floor $y_2 = m_3$ is slightly lower than the others with $\mu = 9000\,kg$ and $\sigma = 900\,kg$. For bending stiffness of the concrete columns an interval-valued variable $k = \theta_1 = [1000, 7000]\,kN/m$ is defined. The damping of the model is neglected. On the top of the frame a stochastic wind load is simulated with the homogeneous correlation function defined in Eq. (11). The variance is specified with $\sigma^2 = 400\,kN^2$ and the autocorrelation function to simulate a realistic wind loading is defined as follows

$$\rho(\tau) = \cos(A\sqrt{1 - B^2}\,\tau)\exp(-C\,\tau) \quad \text{with} \quad A = 30, B^2 = 0.05 \text{ and } C = 0.3 \quad , \tag{11}$$

where τ represents the time lag. The autocorrelation function given by Eq. (11) is shown in Figure 2. The stochastic process of the wind loading with a total duration $T = 10\,s$ is represented considering a time step discretization $\tau = 0.01\,s$ by means of the Karhunen-Loève expansion. The number of responses to be controlled is equal to $n_R = (10/0.01 + 1) = 1001$. A truncation of the series is performed when the sum of eigenvalues exceeds 99% of the total amount. This means $n_{KL} = 111$ terms have to be used. The response of interest within the duration of the stochastic wind process is the top displacement $x_3(t)$. In the duration, the threshold value $x_3(t) = 0.02\,m$ should not be exceeded. The results of the failure probability $p_F(k)$ and the maximum standard deviation $\sigma_{\text{max}}(k)$ are shown in Figure 3.
For the proposed and the direct optimization approach based on the MCS to calculate $p_F(k)$, the epistemic parameter k is discretized on $N_e = 200$ points. To calculate $p_F(k)$, a MCS is performed with 10^4 samples. This means, the model is evaluated 2×10^6 times. It is interesting to note that two maxima of the failure

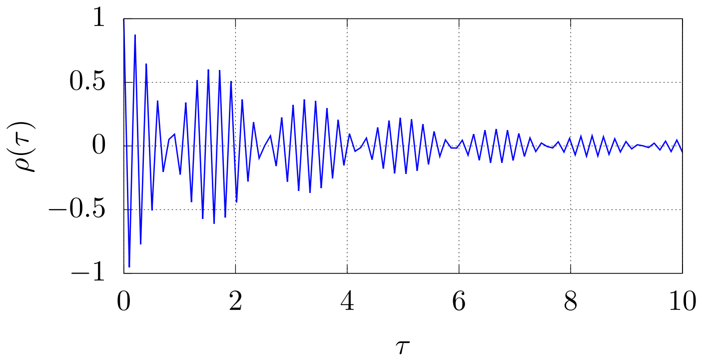

Figure 2: Autocorrelation function to simulate a stochastic wind load.

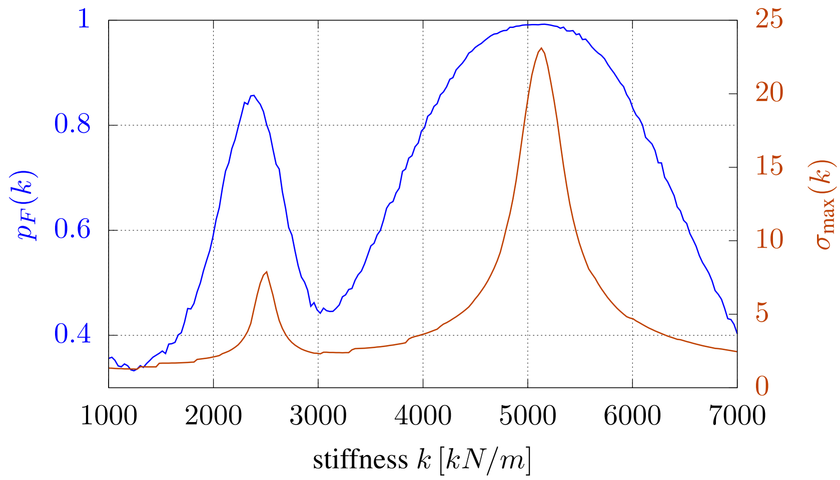

Figure 3: Maximum standard deviation $\sigma_{\max}(k)$ and failure probability $p_F(k)$ versus stiffness k.

probability can be observed with increasing stiffness. The extreme values of $\sigma_{\max}(k)$ correlate reasonably well with those of the function $p_F(k)$, see Table 1.

Table 1: Bounds of failure probability for three-story concrete frame subjected to a stochastic wind load

	Proposed approach		Direct optimization (MCS)	
	lower bound	upper bound	lower bound	upper bound
p_F	0.3358	0.9921	0.3324	0.9923
k [N/m]	1271.4	5130.7	1241.2	5160.8
Relative computation time	1		1463	

In this example the approach based on the maximum standard deviation is 1463 times faster than the double-loop Monte Carlo simulation. Also if a more efficient optimization-based interval analysis would be applied, where maybe less than the 200 MCS would be required, the proposed approach is still faster compared to the double-loop approach.

5 Conclusion

This contribution addressed the problem of estimation of probability bounds for structural systems subject to epistemic and aleatory uncertainty. The results presented in this contribution suggest that the maximum standard deviation of the response of a model serves as an excellent proxy for determining the bounds of the failure probability. By considering such proxy, it is possible to replace a classical double-loop approach for computing the probability bounds with a decoupled approach. The numerical example indicates that such replacement may lead to substantial numerical gains without sacrificing accuracy in the estimates of the bounds. While the results presented are promising, only linear systems can be approximated. An idea for further research is to integrate the presented approach in the concept of polymorphic/hybrid uncertainties, where some input parameters are described as fuzzy variables. Then, a computationally expensive $\alpha-$level optimization has to be performed. The aim is to reduce the computational effort in such concepts significantly.

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