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Local Message Passing on Frustrated Systems

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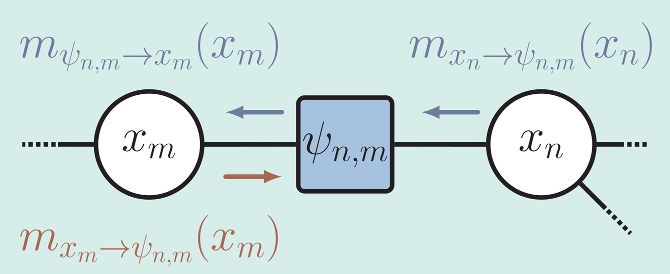
1. Motivation: Marginal Inference

- **Task**: Compute marginal distribution $p(x_n|\boldsymbol{y}) = \sum_{\sim\{x_n\}} p(x_1,\ldots,x_N|\boldsymbol{y})$
  - Marginalization $\sum_{\sim\{x_n\}}$: Sum over all variables except x_n
  - $\boldsymbol{y}$: Noisy observation of the system $x_1,\ldots,x_N$
  - Many practical problems require marginalization
- Efficient approach: **Message passing on graphical models**
  - Widely used: **Sum-product algorithm (SPA)** / belief propagation (BP)
- **Problem**: SPA is derived for graphical models with tree structure
  → **Suboptimal** for graphical models with many short cycles, which often occur in practical applications

2. Message Passing on Graphs with Cycles

- **Idea**: Instead of fixing the mismatched SPA, **find a new message update rule** that is especially taylored to graphs with (many) cycles!
- Message update rule:

  A mapping from one or multiple incident messages to one outgoing message, which is applied **locally** at a (factor) node.



- Sum-product algorithm (SPA):

$$m_{\psi\to x}(x) = \sum_{\sim\{x\}}\left(\psi(\mathcal{X})\prod_{x'\in\mathcal{X}\setminus\{x\}} m_{x'\to\psi}(x')\right),\quad \mathcal{X}\text{: Neighbors of } \psi$$

- General update rule for pairwise factors $\psi_{n,m}(x_m,x_n)$:

  Extrinsic: $\mathrm{FN}_{\mathrm{e}}(\psi_{n,m}) : m_{x_n\to\psi_{n,m}}(x_n) \mapsto m_{\psi_{n,m}\to x_m}(x_m)$

  novel: **Non-extrinsic:** $\mathrm{FN}(\psi_{n,m}) : \begin{pmatrix} m_{x_n\to\psi_{n,m}}(x_n) \\ m_{x_m\to\psi_{n,m}}(x_m)\end{pmatrix} \mapsto m_{\psi_{n,m}\to x_m}(x_m)$

- **cycBP/cycBP$_\mathrm{e}$**: **Represent** (non-)extrinsic **mapping** by compact multilayer perceptron (1 hidden layer à 7 neurons)
  - Optimize local mapping towards good **end-to-end inference performance**

3. Objective Functions for Marginal Inference

- Supervised Training: **Kullback-Leibler (KL) divergence**

$$\mathcal{L}_{\mathrm{KL}} := D_{\mathrm{KL}}\left(b_n(x_n)\|p(x_n)\right)$$

  - $b_n(x_n), b_{n,m}(x_n,x_m)$: single, pairwise beliefs of message passing
  - $p(x_n) = \sum_{\sim\{x_n\}} p(x_1,\ldots,x_N)$: Exact marginal distributions
- Unsupervised Training: **Regularized Bethe free energy**

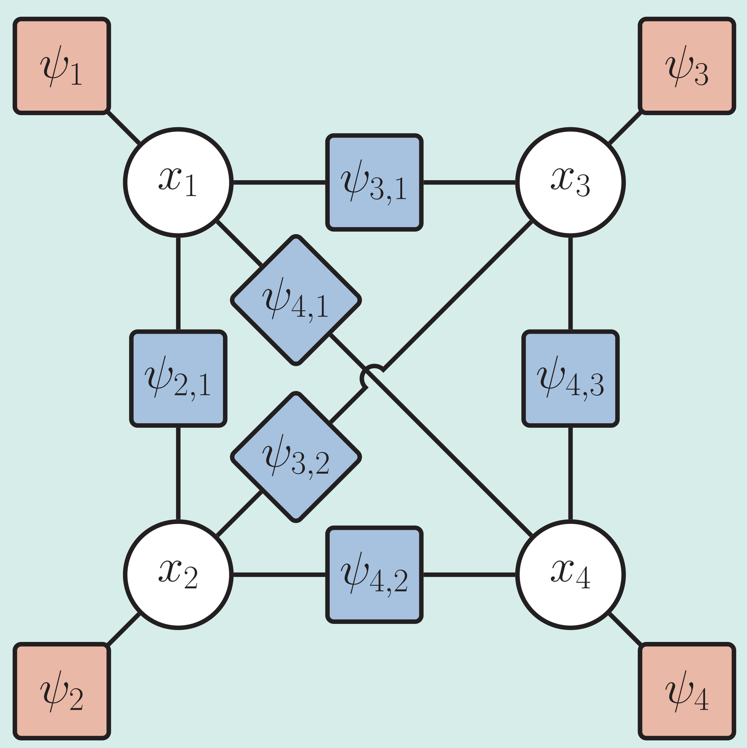
$$\mathcal{L}_{\mathrm{Bethe}} := F_{\mathrm{Bethe}} + \alpha\mathcal{L}_{\mathbb{L}},\quad \alpha\in\mathbb{R}^+$$

  - F_{Bethe}: Bethe free energy
  - **Bethe consistency distance** (*proposed*):

$$\mathcal{L}_{\mathbb{L}} := D_{\mathrm{KL}}\left(\sum_{x_m} b_{n,m}(x_n,x_m)\Big\| b_n(x_n)\right) + D_{\mathrm{KL}}\left(\sum_{x_n} b_{n,m}(x_n,x_m)\Big\| b_m(x_m)\right)$$

4. Example: 2×2 Ising Model

- System: $p(x_1,\ldots,x_N|\boldsymbol{y}) = \frac{1}{Z}\prod_{n=1}^{N}\psi_n(x_n)\prod_{n>m}\psi_{nm}(x_m,x_n)$
  - $\psi_n(x_n) = \exp(\theta_n x_n)$ with **local fields** θ_n, $n=1,\ldots,N$
  - $\psi_{n,m}(x_n,x_m) = \exp(E_{n,m}x_n x_m)$ with **local couplings** $E_{n,m}$, $n>m$
- $N=4$ binary variables $x_n\in\{+1,-1\}$
  - Use log-likelihood ratios $L_{\psi_{nm}\to x_n} := \log\left(\frac{m_{\psi_{nm}\to x_n}(x_n=+1)}{m_{\psi_{nm}\to x_n}(x_n=-1)}\right)$

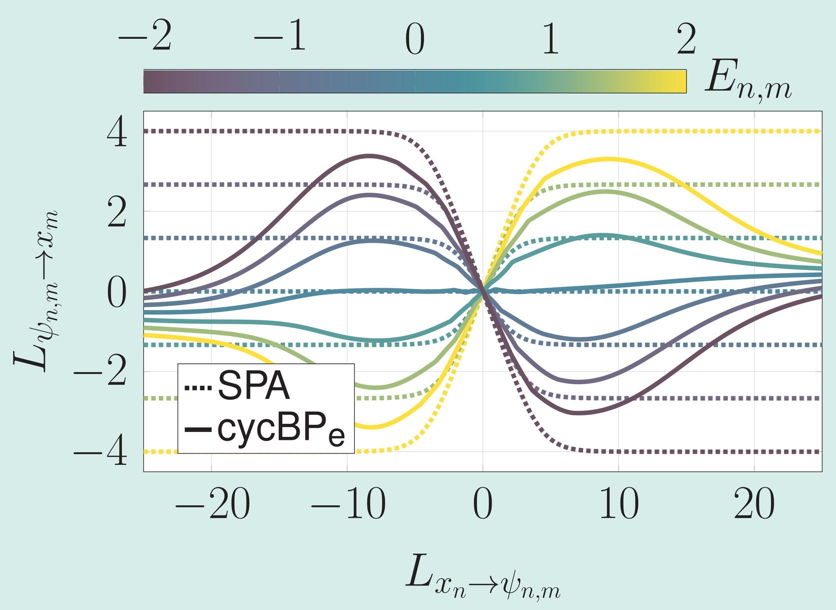


- **Spin glass**: θ_n and $E_{n,m}$ independently sampled from $\mathcal{U}[-2,+2]$
  - **Frustrated system**, e.g., for $E_{n,m}\ll 0,\ \forall n,m$

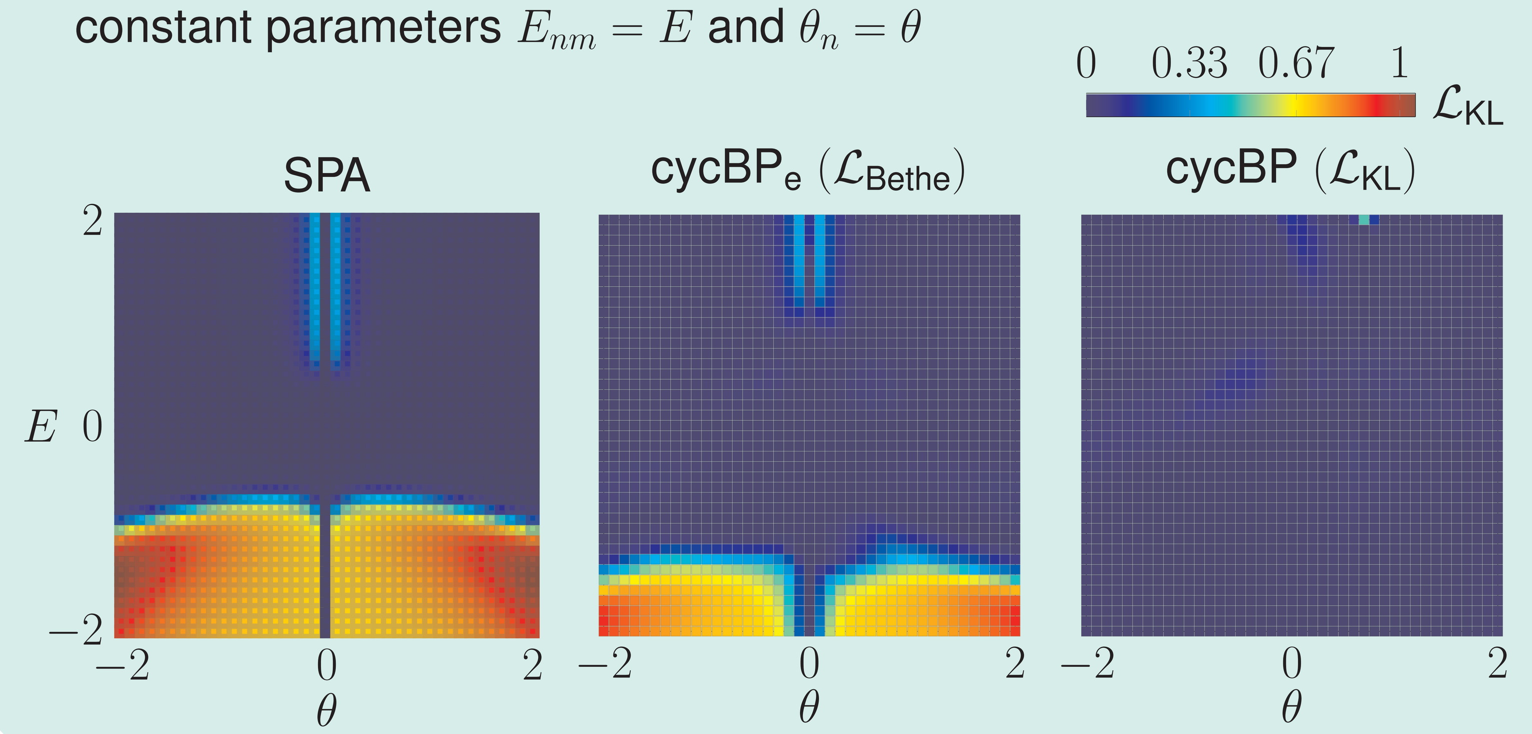
5. Experiment 1: 2×2 Ising Spin Glasses

- **Training** and **evaluation** on samples of the Ising spin glass model

Algo.	Loss	$\mathcal{L}_{\mathrm{KL}}$	$\mathcal{L}_{\mathbb{L}}$
SPA	-	0.087	0.30
cycBP$_\mathrm{e}$	$\mathcal{L}_{\mathrm{KL}}$	0.040	0.17
cycBP	$\mathcal{L}_{\mathrm{KL}}$	0.014	0.48
cycBP$_\mathrm{e}$	$\mathcal{L}_{\mathrm{Bethe}}$	0.030	0.11
cycBP	$\mathcal{L}_{\mathrm{Bethe}}$	0.027	0.027

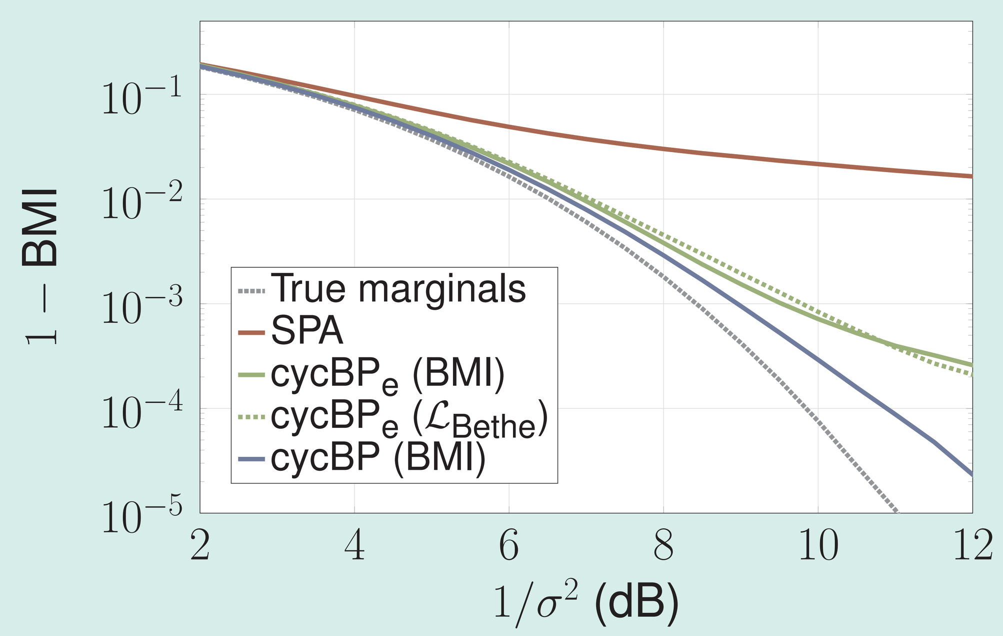


- **Generalizability**: Evaluate pretrained mappings on graphs with constant parameters $E_{nm}=E$ and $\theta_n=\theta$



7. Example 2: Symbol Detection

- **Digital transmission** of binary symbols $x_n\in\{\pm1\}$ over additive white Gaussian noise (AWGN) channel with linear inter-symbol interference
- Efficient symbol detection by message passing on factor graphs



- **BMI**: bitwise mutual information
- σ^2: Variance of the AWGN