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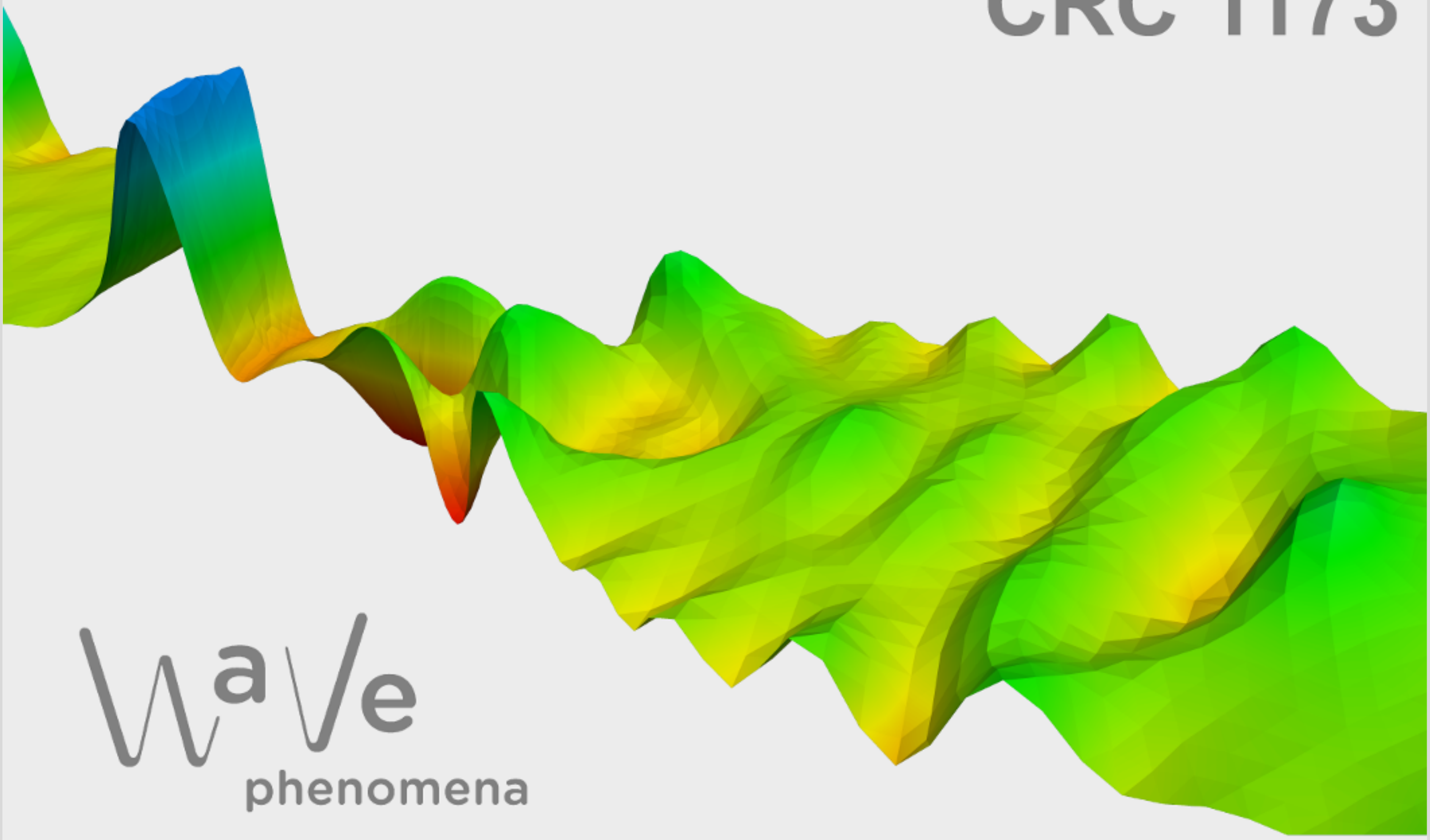

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ABSENCE OF THE EFIMOV EFFECT FOR A SYSTEM OF CONFINED PARTICLES

MARVIN R. SCHULZ, SYLVAIN ZALCZER

Abstract. We consider a system a three particles interacting via short-range potentials, such that two of the particles are on parallel lines in a plane and the third one is on a line perpendicular to this plane. We prove that the corresponding Schrödinger operator only has a finite number of discrete eigenvalues which disproves a recent prediction made in physics literature [11].

1. Introduction

The Efimov effect can be described as follows: The three-body Schrödinger operator of a system of three three-dimensional particles that interact via short-range potentials has an infinite number of negative eigenvalues, if the Hamiltonians of the two-body subsystems have no negative eigenvalues and at least two of them have a zero-energy resonance. Moreover, the eigenvalues form a geometric sequence whose common ratio is independent from the nature of the potentials. Such an curious phenomenon was first predicted by the physicist Vitaly Efimov in 1970 [5]. In 1974, Yafaev gave the first rigorous mathematical proof of it in [24]. The Efimov effect was considered by physicists as a purely theoretical curiosity, until it was observed experimentally in the early 2000's in an ultracold gas of caesium atoms [9]. Efimov's effect has since then be studied both by the physics and mathematics community, see for example the review [11], the PhD thesis [3] or the lecture notes [4] for further references.

One particularly interesting question was whether a similar effect could occur in configurations different from the classical situation of three particles in dimension three. It was proven in [21] that the Efimov effect does not exists for a system of three one- or two-dimensional bosons. Advances in experiments with ultra-cold Fermi-Fermi mixtures such as in [16] make it possible to study situations where different species of particles are confined to distinct subspaces of $\mathbb{R}^3$, see for example [10]. Nishida and Tan discussed the possible existence of a so called *confinement-induced Efimov effect* in [12] and [13]. In [11][p. 44, Table 1] the existence of the confinement-induced Efimov effect was predicted in various situations.

We consider the case where two particles can move along two parallel lines in a plane and the third particle moves in a line perpendicular to this plane. In [11][p. 44, Table 1] the existence of the confinement-induced Efimov effect was predicted for this configuration. We prove that this prediction is wrong. We show that the discrete spectrum of the operator describing this system is always finite, even when the two-body subsystems have a zero-energy resonance.

Our analysis is based on [26] where Zhislin formulated a useful condition on the finiteness of the discrete spectrum of many particle Schrödinger operators. His approach was further developed and applied to show absence of Efimov's effect in various situations for example in [19],[18],[22] and [23]. In [2] Barth, Bitter and Vugalter showed the absence of Efimov's effect in various systems of one- and two-dimensional particles. Our proof uses the techniques in that paper.

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Our paper is organised as follows. In Section 2, we describe the model and state our theorem. In Section 3, we give the proof of it in the case where the two-particle operators have no bound states. In Appendix A, we recall some of the results of previous works that we use and in Appendix B, we study the case where at least one of the two-particle systems has a bound state, which is mainly a direct adaptation of [26].

2. The Model

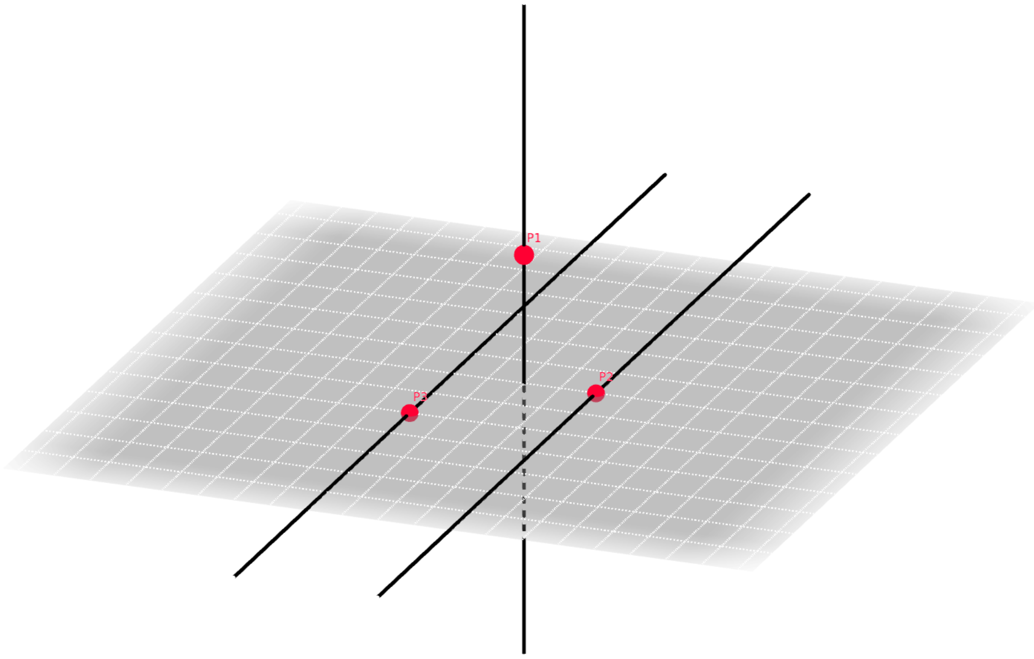

Figure 1. The spatial configuration of the system.

Inspired by recent predictions from physics for the *confinement-induced Efimov effect* we consider a system of three particles with identical masses interacting via short-range two-body potentials in $\mathbb{R}^3$. Two particles, called 2 and 3, are confined to parallel lines and the third particle called 1 is confined to a line perpendicular to the plane spanned by the two lines on which particles 2 and 3 are moving. Furthermore, the line on which particle 1 moves is assumed to not intersect with the two lines for particles 2 and 3. Compare this to Figure 1.

For simplicity we assume that the parallel lines have distance 1. The configuration of such a system is determined by three real numbers $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. For any $x \in \mathbb{R}^3$ we denote by $|x|$ the Euclidean norm of the vector. The particles have positions

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} x_2 \\ 1/2 \\ 0 \end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix} x_3 \\ -1/2 \\ 0 \end{pmatrix}. \tag{2.1}$$

The particles interact pairwise. For $i, j \in \{1, 2, 3\}$ with $i < j$, we denote by V_{ij} the potential describing the interaction between particles at $\mathbf{r}_i$ and $\mathbf{r}_j$. From physics we know that it is reasonable to assume that the potential V_{ij} solely depends on the distance between the particles

i and j. We consider the case where all potentials are short-range, *i.e.* that there exist $R_0 > 0$ and $C, \nu > 0$ such that for all $i, j \in \{1,2,3\}$ with $i < j$,

$$|V_{ij}(r)| \leq \frac{C}{r^{2+\nu}}, \text{ when } r \geq R_0. \tag{2.2}$$

Note that $|\mathbf{r}_i - \mathbf{r}_j| \geq 1/2$ for $i, j \in \{1,2,3\}$ with $i < j$. That is, all three particles have a minimal distance from each other. Thus we can, without loss of generality, assume that all potentials are bounded. The corresponding operator is

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \sum_{1 \leq i < j \leq 3} V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|). \tag{2.3}$$

Under this assumptions H is self-adjoint on the Sobolev space $H^2(\mathbb{R}^3)$ and its form domain is $H^1(\mathbb{R}^3)$. Such a system of three particles can be decomposed into three subsystems of two particles. For the pair of particles $(1, j)$ with $j \in \{2,3\}$ we consider the operator

$$h_{1j} := -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_j^2} + V_{1j}(|\mathbf{r}_1 - \mathbf{r}_j|). \tag{2.4}$$

While the full three-particle system is not invariant under any translation in $\mathbb{R}^3$ due to the geometric constraints, the subsystem consisting of particles 2 and 3 is invariant under translations in the e_1-direction. Thus for the subsystem of particles 2 and 3 we need to study it in the center of mass frame by introducing relative coordinates in $\mathbb{R}^2$. These are given by $q, \xi \in \mathbb{R}$ with

$$q := \frac{1}{\sqrt{2}}(x_3 - x_2), \quad \xi := \frac{1}{\sqrt{2}}(x_3 + x_2). \tag{2.5}$$

Up to a factor of $\sqrt{2}$ the coordinate ξ describes the position of the center of mass of the two particles 2 and 3, while the coordinate q describes the relative motion of these two particles in the e_1-direction.
Note that $|\mathbf{r}_2 - \mathbf{r}_3| = \sqrt{2q^2 + 1}$. The operator

$$\left[-\partial_q^2 + V_{23}\left(\sqrt{2q^2+1}\right)\right] \otimes \mathbb{1} + \mathbb{1} \otimes (-\partial_\xi^2) \tag{2.6}$$

corresponds to the pair $(2,3)$. In the center of mass frame we have

$$h_{23} := -\partial_q^2 + V_{23}\left(\sqrt{2q^2+1}\right). \tag{2.7}$$

Let

$$\Sigma := \min_{i<j\in\{1,2,3\}} \inf \sigma(h_{ij}). \tag{2.8}$$

Similar to the HVZ-Theorem [8, 17, 25] (see also [14][Thm. XIII.17]), one sees that $\Sigma \leq 0$. It follows from the same theorem that $\sigma_{ess}(H) = [\Sigma, \infty)$. The goal of this paper is to study the discrete spectrum of H. Our main result is

Theorem 2.1. *The operator H has at most a finite number of discrete eigenvalues below Σ.*

Remark 2.2. The theorem above allows any of the two-particle operators to have a virtual level. In [11] it was predicted that the system described by the operator H in equation (2.3) shows a confinement-induced Efimov effect. Our Theorem 2.1 disproves this claim, which was based on heuristic arguments from physics.

3. Proof of Theorem 2.1

According to the min-max principle, the spectrum of H below Σ is finite if there exists a finite dimensional space $M \subset L^2(\mathbb{R}^3)$ such that, for any $\psi \in L^2(\mathbb{R}^3)$ orthogonal to M,

$$\langle \psi, H\psi \rangle \geq \Sigma \|\psi\|^2 . \tag{3.1}$$

Due to [26] (see also Appendix A.1), such a space M exists whenever there exist $b, \beta > 0$ with

$$L[\psi] := \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} |\partial_{x_i}\psi|^2 + \sum_{i<j} V_{ij} |\psi|^2 \right) dx - \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^\beta} dx \geq \Sigma \|\psi\|^2 \tag{3.2}$$

for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$.

In the following, we set $\beta = 2+\nu$, where $\nu > 0$ is the exponent corresponding to the short-range property of the potentials as stated in equation (2.2). We first prove the theorem in the case $\Sigma = 0$. With a small modification, the case $\Sigma < 0$ is analogous to the one considered in [26]. For the convenience of the reader we give the proof for $\Sigma < 0$ in Appendix B.

Heuristics from physics predicts that the three-particle system breaks up if (at least) one particle is far away from the others. So following ideas of [21], we want to define, for all $i, j \in \{1, 2, 3\}$ with $i < j$, the set of geometric configurations where the particles i and j are close to each other and the third particle is far away. Observe that

$$|\mathbf{r}_2 - \mathbf{r}_3|^2 = 2q^2 + 1, \; |\mathbf{r}_1 - \mathbf{r}_j|^2 = x_1^2 + x_j^2 + 1/4, \; j \in \{2, 3\}. \tag{3.3}$$

Given parameters $b > 0$ and $\gamma \in (0, 1)$ we define the regions

$$\begin{aligned} K_{23}^b(\gamma) &:= \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |q| \leq \gamma \sqrt{\xi^2 + x_1^2}, |x| > b \text{ for } q, \xi \text{ defined in (2.5)} \right\}, \\ K_{1j}^b(\gamma) &:= \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \, |(x_1, x_j)| \leq \gamma \, |x_i| \, , \text{for } i \in \{2, 3\}, i \neq j, \; |x| > b \right\}. \end{aligned} \tag{3.4}$$

Here $|(x_1, x_j)|$ is the Euclidean length of the vector (x_1, x_j) in the (1-j)-plane. Note that outside of the ball $B_b(0)$ the above sets describe conical regions in $\mathbb{R}^3$. For example, outside of $B_b(0)$ the region $K_{13}^b(\gamma)$ is a conical region around the x_2-direction, where the particles 1 and 3 are close to each other and particle 2 is far away. Compare this to Figure 2.

We also define the set Ω_0 of configurations where all three particles are far apart:

$$\Omega_0 := \{x \in \mathbb{R}^3 | \, |x| > b\} \setminus \left(\bigcup_{1 \leq i < j \leq 3} K_{ij}^b(\gamma) \right). \tag{3.5}$$

For a measurable set $\Omega \subset \mathbb{R}^3$, we define

$$L[\psi, \Omega] := \int_{\Omega} \left(\sum_{i=1}^{3} |\partial_{x_i}\psi|^2 + \sum_{i<j} V_{ij} |\psi|^2 \right) dx - \int_{\Omega} \frac{|\psi(x)|^2}{|x|^{2+\nu}} dx. \tag{3.6}$$

We prove the bound (3.2), by estimating $L[\psi, K_{ij}^b(\gamma)]$ and $L[\psi, \Omega_0]$ from below. Notice that for γ small enough the sets $K_{ij}^b(\gamma)$ and Ω_0 are disjoint. In the following we shall assume that $\gamma < 1/4$ is small enough.

We start with $L[\psi, K_{1j}^b(\gamma)]$ for $j \in \{2, 3\}$.

Lemma 3.1. *For $j \in \{2, 3\}$, there exist $C, b_0 > 0$ such that for all $b \geq b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with* $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$

$$L[\psi, K_{1j}^b(\gamma)] \geq -C \int_{\partial K_{1j}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma. \tag{3.7}$$

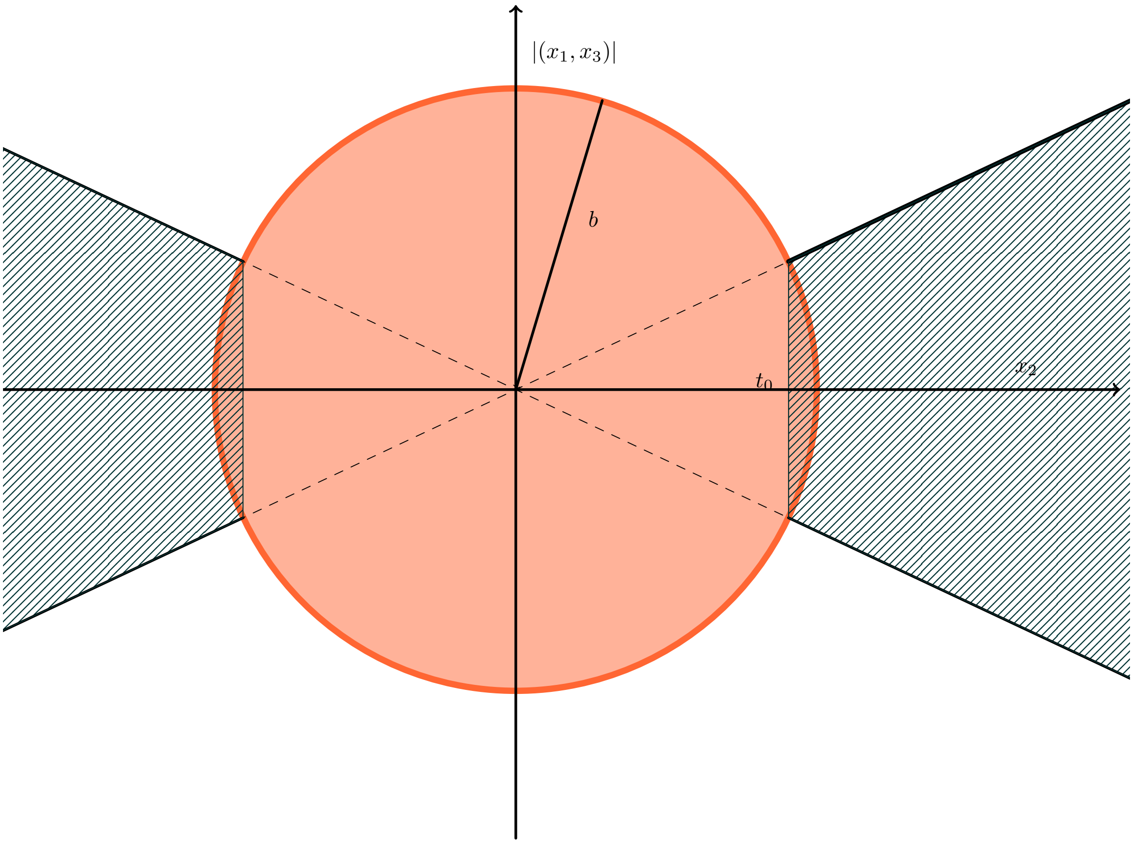

FIGURE 2. A sketch of the set $K^b_{13}(\gamma)$ defined in equation (3.4). The other sets look similar.

where $\nu > 0$ *is defined in equation* (2.2).

Proof. We prove the estimate for $j = 3$. The proof for $j = 2$ is similar. For any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ let

$$\zeta \coloneqq (x_1, x_3) \in \mathbb{R}^2. \tag{3.8}$$

Then

$$K^b_{13}(\gamma) = \{(\zeta, x_2) \in \mathbb{R}^3 : |\zeta| \leq \gamma\, |x_2|\, , |(\zeta, x_2)| \geq b\}\, . \tag{3.9}$$

Given $b > 0$ and $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^3|\, |x| > b\}$, we decompose

$$L[\psi, K^b_{13}(\gamma)] = L_3[\psi, K^b_{13}(\gamma)] + L_4[\psi, K^b_{13}(\gamma)] \tag{3.10}$$

where

$$\begin{aligned} L_3[\psi, K_{13}^b(\gamma)] &:= \int_{K_{13}^b(\gamma)} \left(|\nabla_\zeta \psi|^2 + V_{13}\left(\sqrt{|\zeta|^2 + \frac{1}{4}}\right) |\psi|^2 \right) d(\zeta, x_2), \\ L_4[\psi, K_{13}^b(\gamma)] &:= \int_{K_{13}^b(\gamma)} \left(|\partial_{x_2}\psi|^2 + (V_{23} + V_{12}) |\psi|^2 \right) d(\zeta, x_2) - \int_{K_{13}^b(\gamma)} \frac{|\psi|^2}{|x|^{2+\nu}} \, dx \,. \end{aligned} \tag{3.11}$$

Let $t_0 := b/\sqrt{1+\gamma^2}$, as it appears in Figure 2. Then, for all $(x_2, \zeta) \in K_{13}^b(\gamma)$, $|x_2| \geq t_0$ and, since ψ vanishes whenever $|\zeta|^2 + x_2^2 \leq b^2$,

$$L_3[\psi, K_{13}^b(\gamma)] = \int_{|x_2| \geq t_0} \iint_{|\zeta| \leq \gamma |x_2|} |\nabla_\zeta \psi|^2 + V_{13}\left(\sqrt{|\zeta|^2 + \frac{1}{4}}\right) |\psi|^2 \, d\zeta dx_2 \,. \tag{3.12}$$

We remark that V_{13} solely depends on $|\zeta|$ and is short-range. By assumption $-\Delta + V_{13} = h_{13} \geq 0$. Thus, by Lemma A.2 which is a restatement of [1][Lemma 6.6], there exists some $C_0 > 0$ such that, when b is large enough so that $\gamma t_0 \geq R_0$,

$$L_3[\psi, K_{13}^b(\gamma)] \geq -C_0 \int_{|x_2| \geq t_0} \frac{\int_0^{2\pi} |\psi(x_2, \gamma |x_2|, \theta)|^2 \, d\theta}{(\gamma |x_2|)^\nu} \, dx_2 \,. \tag{3.13}$$

We remark that

$$\begin{cases} (-\infty, -t_0] \cup [t_0, \infty) \times [0, 2\pi) & \to \mathbb{R}^3 \\ (x_2, \theta) & \mapsto (x_2, \gamma |x_2|, \theta) \end{cases} \tag{3.14}$$

is a parametrization of the two components of the surface $\partial K_{13}^b(\gamma)$ outside of $B_b(0)$. Since in addition ψ vanishes on $B_b(0)$ and $|x| = \sqrt{1+\gamma^2}\, |x_2|$ on that surface, we can rewrite equation (3.13) in

$$L_3[\psi, K_{13}^b(\gamma)] \geq -C_1 \int_{\partial K_{13}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma, \tag{3.15}$$

for some constant $C_1 > 0$. Here $d\sigma = |x| \, d|x| \, d\theta$ is the surface measure on the set $\partial K_{13}^b(\gamma)$, which explains the additional factor of $|x|$ in the denominator of equation (3.15).

Let us now bound $L_4[\psi, K_{13}^b(\gamma)]$. Note that for any $x \in K_{13}^b(\gamma)$

$$\begin{aligned} |\mathbf{r}_1 - \mathbf{r}_2| &\geq |(x_1, x_2)| \geq |x_2| \geq t_0, \\ |\mathbf{r}_2 - \mathbf{r}_3| &\geq |x_2 - x_3| \geq (1-\gamma) |x_2| \geq (1-\gamma) t_0 \,. \end{aligned} \tag{3.16}$$

Since $t_0 = b/\sqrt{1+\gamma^2}$, we can use the short-range property of the potentials V_{23}, V_{12} to find that there exists $C_2 > 0$ such that for $b \geq R_0\sqrt{1+\gamma^2}$, we have for any $x \in K_{13}^b(\gamma)$

$$|(V_{23} + V_{12})(x)| \leq \frac{C_2}{|x_2|^{2+\nu}}. \tag{3.17}$$

Thus

$$L_4[\psi, K_{13}^b(\gamma)] \geq \iint_{\mathbb{R}^2} \int_{|x_2| \geq \max\{t_0, |\zeta|/\gamma\}} \left(|\partial_{x_2}\psi|^2 - \frac{C_2 + 1}{|x_2|^{2+\nu}} |\psi|^2 \right) dx_2 d\zeta. \tag{3.18}$$

For any fixed $\zeta \in \mathbb{R}^2$, we define $a(\zeta) := \max\{t_0, |\zeta|/\gamma\}$ and apply [2][Lemma 6.3] (see Appendix A.5) to the innermost integral. Thus, there exists $C_3 > 0$ such that

$$L_4[\psi, K_{13}^b(\gamma)] \geq -C_3 \iint_{\mathbb{R}^2} \frac{1}{a(\zeta)^{1+\nu}} \left[|\psi(a(\zeta), \zeta)|^2 + |\psi(-a(\zeta), \zeta)|^2 \right] d\zeta \,. \tag{3.19}$$

By construction $\psi(\pm a(\zeta),\zeta)$ vanishes whenever $|\zeta| \le \gamma t_0$. Then

$$L_4[\psi, K^b_{13}(\gamma)] \ge -C_3\gamma^{1+\nu} \iint_{|\zeta|\ge\gamma t_0} \frac{1}{|\zeta|^{1+\nu}} \left[|\psi\left(|\zeta|/\gamma,\zeta\right)|^2 + |\tilde{\psi}\left(-|\zeta|/\gamma,\zeta\right)|^2 \right] d\zeta\,. \tag{3.20}$$

Note once more that $\zeta \mapsto (\pm|\zeta|/\gamma, \zeta)$ for $\zeta \in \mathbb{R}^2$ with $|\zeta| \ge \gamma t_0$ is a parametrization of the two components of the surface $\partial K^b_{13}(\gamma)$ outside of $B_b(0)$. Similarly to what we did in order to get from equation (3.13) to equation (3.15), we can rewrite equation (3.20) in

$$L_4[\psi, K^b_{13}(\gamma)] \ge -C_4 \int_{\partial K^b_{13}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma, \tag{3.21}$$

for some $C_4 > 0$. Inserting the bounds (3.15) and (3.21) into (3.10), we find

$$L[\psi, K^b_{13}(\gamma)] \ge -(C_4 + C_1) \int_{\partial K^b_{13}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}}\, d\sigma\,. \tag{3.22}$$

This proves the lemma. □

Next we provide a lower bound for $L[\psi, K^b_{23}(\gamma)]$. The techniques used are similar to the ones in the proof of Lemma 3.1 but the proof is slightly different due to the different geometry of $K^b_{23}(\gamma)$. We show

Lemma 3.2. *There exist $C, b_0 > 0$ such that, for all $b \ge b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with* $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 |\, |x| > b\}$

$$L[\psi, K^b_{23}(\gamma)] \ge -C \int_{\partial K^b_{23}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma. \tag{3.23}$$

Proof. Let $\eta := (\xi, x_1) \in \mathbb{R}^2$. Then the set $K^b_{23}(\gamma)$ is

$$K^b_{23}(\gamma) = \left\{ (\eta, q) \in \mathbb{R}^3 : |q| \le \gamma\,|\eta|\,, \sqrt{|\eta|^2 + q^2} \ge b \right\}. \tag{3.24}$$

Similar to Lemma 3.1, for some $b > 0$, let $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 |\, |x| > b\}$ and define

$$\begin{aligned} L_1[\psi, K^b_{23}(\gamma)] &:= \int_{K^b_{23}(\gamma)} \left(|\partial_q\psi|^2 + V_{23}\left(\sqrt{2q^2+1}\right) |\psi|^2 \right) d(q,\eta), \\ L_2[\psi, K^b_{23}(\gamma)] &:= \int_{K^b_{23}(\gamma)} \left(|\nabla_\eta\psi|^2 + \sum_{j=2}^{3} V_{1j}\, |\psi|^2 \right) d(q,\eta) - \int_{K^b_{23}(\gamma)} \frac{|\psi|^2}{|x|^{2+\nu}}\, dx. \end{aligned} \tag{3.25}$$

We decompose

$$L[\psi, K^b_{23}(\gamma)] = L_1[\psi, K^b_{23}(\gamma)] + L_2[\psi, K^b_{23}(\gamma)]. \tag{3.26}$$

Recall that $t_0 = b/\sqrt{1+\gamma^2}$, then

$$L_1[\psi, K^b_{23}(\gamma)] = \iint_{|\eta|\ge t_0} \int_{-\gamma|\eta|}^{\gamma|\eta|} |\partial_q\psi|^2 + V_{23}\,|\psi|^2 \; dq\, d\eta \tag{3.27}$$

since ψ vanishes whenever $|\eta|^2 + q^2 \le b^2$. Recall that $h_{23} \ge 0$ and that V_{23} is short-range. Then we can apply [2][Lemma 6.2] (see Appendix A.4) to conclude that there exists $D_0 > 0$ such that for all b with $\gamma t_0 = \gamma b/\sqrt{1+\gamma^2} \ge R_0$

$$L_1[\psi, K^b_{23}(\gamma)] \ge -D_0 \iint_{|\eta|\ge t_0} \frac{|\psi(\gamma\,|\eta|\,,\eta)|^2 + |\psi(-\gamma\,|\eta|\,,\eta)|^2}{(\gamma\,|\eta|)^{1+\nu}}\,. \tag{3.28}$$

Similarly as in the proof of Lemma 3.1, we can rewrite this in

$$L_1[\psi, K_{23}^b(\gamma)] \geq -D_1 \int_{\partial K_{23}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \tag{3.29}$$

for some $D_1 > 0$.

Let us now bound $L_2[\psi, K_{23}^b(\gamma)]$. On the set $K_{23}^b(\gamma)$

$$|\mathbf{r}_1 - \mathbf{r}_j| \geq \sqrt{1-4\gamma}\,|\eta| \geq \sqrt{1-4\gamma}t_0 \tag{3.30}$$

for $j \in \{2, 3\}$ by the construction of the set $K_{23}^b(\gamma)$. Recall that $\gamma < 1/4$ by assumption. Thus we can use that the potentials V_{12}, V_{13} are short-range. Then, there exists $D_2 > 0$ such that, for b large enough with $t_0 \geq R_0$,

$$L_2[\psi, K_{23}^b(\gamma)] \geq \int_{\mathbb{R}} \iint_{|\eta| \geq \max\{|q|/\gamma, t_0\}} |\nabla_\eta \psi|^2 - \frac{D_2}{|\eta|^{2+\nu}} |\psi|^2 \, d\eta dq\,. \tag{3.31}$$

Going into spherical coordinate and applying [2][Lemma 6.7] (see Appendix A.3) to the innermost integral above one can bound the above integral by a surface integral: there exists $D_3 > 0$ such that

$$L_2[\psi, K_{23}^b(\gamma)] \geq -D_3 \int_{|q| \geq \gamma t_0} \frac{\int_0^{2\pi} |\psi(q, |q|/\gamma, \theta)|^2 \, d\theta}{|q|^\nu} dq\,. \tag{3.32}$$

Using that ψ vanishes inside the ball $B_b(0)$ we conclude

$$\begin{aligned} L_2[\psi, K_{23}^b(\gamma)] &\geq -D_3 \int_{|q| \geq \gamma t_0} \frac{\int_0^{2\pi} |\psi(q, |q|/\gamma, \theta)|^2 \, d\theta}{|q|^\nu} dq \\ &\geq -D_4 \int_{\partial K_{23}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \end{aligned} \tag{3.33}$$

for some $D_4 > 0$, where in the last bound we used that $|q| = \sqrt{1+\gamma^{-2}}|x|$ on $\partial K_{23}^b(\gamma)$. Recall that $d\sigma$ is the surface measure on $\partial K_{23}^b(\gamma)$ similar to equation (3.15).

For the first to second line we used the fact that the integral is taken over the part of the surface of $\partial K_{23}^b(\gamma)$ where ψ does not vanish. Compare this to the previous case in the proof of Lemma 3.1. Inserting the inequalities in equation (3.29), (3.33) into equation (3.26) concludes the proof of the lemma. □

In the next step, we prove that we can compensate the integrals over the surface of the sets $K_{ij}^b(\gamma)$ in the equations (3.23), (3.7) by a small portion of the kinetic energy on Ω_0. We do so with the help of the trace theorem and Hardy's inequality on the half line.

Lemma 3.3. *For $1 \leq i < j \leq 3$ and $\gamma' \in (\gamma, 1)$, we define $\Omega_{ij}^b(\gamma, \gamma') := K_{ij}^b(\gamma') \backslash K_{ij}^b(\gamma)$. For all $\varepsilon > 0$, there exists $b_0 > 0$ such that, for all $b \geq b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with* $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$

$$\int_{\partial K_{ij}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \leq \varepsilon \int_{\Omega_{ij}^b(\gamma, \gamma')} |\nabla \psi|^2 \, dx. \tag{3.34}$$

Proof. Let us first prove the lemma for $(i, j) = (1, 3)$. We introduce spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}^+ \times [-\pi/2, \pi/2] \times [0, 2\pi)$:

$$(x_1, x_2, x_3) = (r \cos\theta \cos\varphi, r \sin\theta, r \cos\theta \sin\varphi)\,. \tag{3.35}$$

Defining $\theta_0 := \arctan(\gamma)$, we see that the set $K_{13}^b(\gamma)$ takes in those coordinates a very simple form:

$$\left\{(r, \theta, \varphi) : \varphi \in [0, 2\pi), |\theta| \geq \frac{\pi}{2} - \theta_0, r > b\right\}. \tag{3.36}$$

Hence,

$$\int_{\partial K_{13}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma = \int_b^\infty \frac{\int_0^{2\pi} \left|\psi(r, \frac{\pi}{2} - \theta_0, \varphi)\right|^2 + \left|\psi(r, -\frac{\pi}{2} + \theta_0, \varphi)\right|^2 d\varphi}{r^\nu} \sin\theta_0 dr, \tag{3.37}$$

where we did not write the part of the integral which is in $B_b(0)$ since ψ vanishes there.

For $\gamma' \in (\gamma, 1)$, we define $\theta_1 := \arctan(\gamma') > \theta_0$. Then $K_{23}^b(\gamma') \supset K_{23}^b(\gamma)$. We will use the well known trace theorem. For each $r \in [b, \infty)$, we apply [6][Theorem 1, p. 272] to the function

$$\theta \mapsto \int_0^{2\pi} |\psi(r,\theta,\varphi)|^2 \, d\varphi \tag{3.38}$$

on the interval $\theta \in (\pi/2 - \theta_1, \pi/2 - \theta_0) =: I(\theta_0, \theta_1)$. We find that there exists a constant $C_4(\gamma, \gamma')$ such that

$$\int_0^{2\pi} \left|\psi(r, \frac{\pi}{2} - \theta_0, \varphi)\right|^2 d\varphi \le C_4(\gamma,\gamma') \int_{\pi/2-\theta_1}^{\pi/2-\theta_0} \int_0^{2\pi} |\psi|^2 + |\partial_\theta \psi|^2 \, d\varphi d\theta. \tag{3.39}$$

A similar inequality holds in the interval $(-\pi/2 + \theta_0, -\pi/2 + \theta_1)$. Inserting (3.39) into (3.37), we find

$$\begin{aligned} \int_{\partial K_{13}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma &\le C_4(\gamma,\gamma') \sin\theta_0 \int_b^\infty \int_{|\theta| \in I(\theta_0,\theta_1)} \int_0^{2\pi} \frac{|\psi|^2 + |\partial_\theta \psi|^2}{r^\nu} d\varphi d\theta dr \\ &\le \frac{C_4(\gamma,\gamma') \sin\theta_0}{b^\nu} \int_b^\infty \int_{|\theta| \in I(\theta_0,\theta_1)} \int_0^{2\pi} \left(|\psi|^2 + |\partial_\theta \psi|^2\right) d\varphi d\theta dr. \end{aligned} \tag{3.40}$$

We remark that the domain of integration in the equation (3.40) is exactly $\Omega_{13}^b(\gamma, \gamma')$ as defined in the statement of Lemma 3.3. In order to conclude the proof, we transform the right-hand side of equation (3.34) to spherical coordinates:

$$\int_{\Omega_{13}^b(\gamma,\gamma')} |\nabla\psi|^2 \, dx \ge \int_b^\infty \int_{|\theta| \in I(\theta_0,\theta_1)} \int_0^{2\pi} \left(|\partial_r \psi|^2 + r^{-2} |\partial_\theta \psi|^2\right) r^2 \cos(\theta) d(r,\theta,\varphi). \tag{3.41}$$

For fixed (θ, φ) we want to apply Hardy's inequality on the half-line to the function $\psi(\cdot, \theta, \varphi)$. Note that $\liminf_{r\to\infty} |\psi(r,\theta,\varphi)| = 0$ since $\psi \in H^1(\mathbb{R}^3)$. Thus, we can apply [7][Theorem 2.65] to find

$$\int_{\Omega_{13}^b(\gamma,\gamma')} |\nabla\psi|^2 \, dx \ge \frac{\sin\theta_0}{4} \int_{\rho_0}^\infty \int_{|\theta| \in I(\theta_0,\theta_1)} \int_0^{2\pi} \left(|\psi|^2 + |\partial_\theta \psi|^2\right) d(r,\theta,\varphi). \tag{3.42}$$

Combining the inequalities in equations (3.40) and (3.42), we see that, for all $\varepsilon > 0$, equation (3.34) holds for $(i,j) = (1,3)$ for all b large enough. The proof in the case $(i,j) = (1,2)$ is similar, the only difference being that we exchange x_2 and x_3 in the definition of the spherical coordinates.

Concerning the case $(i,j) = (2,3)$, we recall the coordinates q and ξ introduced in (2.5). We define spherical coordinates by

$$(x_1, q, \xi) = (r\cos\theta\cos\varphi, r\sin\theta, r\cos\theta\sin\varphi). \tag{3.43}$$

In this set of coordinates the set $K_{23}^b(\gamma)$ reads

$$\{(r,\theta,\varphi) : \varphi \in [0, 2\pi), |\theta| \le \theta_0, r > b\} \tag{3.44}$$

and

$$\int_{\partial K_{23}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma = \int_b^\infty \frac{\int_0^{2\pi} |\psi(r,\theta_0,\varphi)|^2 + |\psi(r,-\theta_0,\varphi)|^2 d\varphi}{r^\nu} \cos\theta_0 dr. \tag{3.45}$$

From this point, one can follow the proof in the case $(i, j) = (1, 3)$. In this case we apply the trace theorem to the function

$$\theta \mapsto \int_0^{2\pi} |\psi(r,\theta,\varphi)|^2 \, d\varphi \tag{3.46}$$

on the intervals (θ_0, θ_1) and $(-\theta_1, -\theta_0)$ to conclude the statement of this lemma. □

We now complete the proof of Theorem 2.1 by proving that inequality (3.2) holds. We fix γ' in $(1, \gamma)$ such that the sets $\Omega_{ij}^b(\gamma, \gamma')$ do not intersect. Combining the results of Lemmas 3.1, 3.2 and 3.3 (for $\varepsilon = 1/2$), we find that there exists a $b_0 > 0$ such that, for any $b > b_0$ and $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$,

$$L[\psi] \geq L[\psi, \Omega_0] - \frac{1}{2} \sum_{1 \leq i < j \leq 3} \int_{\Omega_{ij}^b(\gamma,\gamma')} |\nabla \psi|^2 \, dx. \tag{3.47}$$

Let us fix some $b \geq b_0$. Remember the definition of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ in terms of $(x_1, x_2, x_3) \in \mathbb{R}^3$ in (2.1). By construction of the set Ω_0, we have that for all $x \in \Omega_0$ and all $1 \leq i < j \leq 3$

$$|\mathbf{r}_i - \mathbf{r}_j| \geq \frac{\gamma}{\sqrt{1+\gamma^2}} |x| \geq \frac{\gamma}{\sqrt{1+\gamma^2}} b. \tag{3.48}$$

Hence we can use that the short-range property of the potentials V_{ij} in Ω_0 and write for some $C > 0$ that

$$\begin{aligned} L[\psi] &\geq \int_{\Omega_0} |\nabla \psi|^2 - \frac{C}{|x|^{2+\nu}} dx - \frac{1}{2} \sum_{i<j} \|\nabla \psi\|^2_{L^2(\Omega_{ij}^b(\gamma,\gamma'))} \\ &= \int_{\Omega_0} \frac{1}{2} |\nabla \psi|^2 - \frac{C}{|x|^{2+\nu}} dx + \frac{1}{2} \|\nabla \psi\|^2_{L^2(\Omega_0)} - \frac{1}{2} \sum_{i<j} \|\nabla \psi\|^2_{L^2(\Omega_{ij}^b(\gamma,\gamma'))} . \end{aligned} \tag{3.49}$$

Remember that we constructed the $\Omega_{ij}^b(\gamma, \gamma')$ such that they are disjoint subsets of Ω_0. As a consequence,

$$\|\nabla \psi\|^2_{L^2(\Omega_0)} - \sum_{i<j} \|\nabla \psi\|^2_{L^2(\Omega_{ij}^b(\gamma,\gamma'))} \geq 0. \tag{3.50}$$

By construction of Ω_0, there exists a set $M \subset S^2$ such that $\Omega_0 = (b, \infty) \times M$. Therefore,

$$\int_{\Omega_0} \frac{1}{2} |\nabla \psi|^2 - \frac{C}{|x|^{2+\nu}} dx \geq \int_M \int_b^\infty \left(\frac{1}{2} |\partial_r \psi|^2 - \frac{C}{r^{2+\nu}} \right) r^2 dr d\omega \geq 0 \tag{3.51}$$

by Hardy's inequality on the half-line (see [7][Theorem 2.65]). Inserting equation (3.51) and equation (3.50) into equation (3.49) concludes the statement.

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Appendix A. Some Lemmas

For the convenience of the reader we repeat here some lemmas from other publications without proofs.

In [26] Zhislin gave the following criterion for the finiteness of the discrete spectrum of a Schrödinger operator. The following Lemma is a straight forward adaption of [2][Lemma C.1]

Lemma A.1. *Let $H = -\Delta + V$ in $L^2(\mathbb{R}^3)$ with V bounded. Let $\Sigma \le 0$ and assume there exist $\beta, b, \varepsilon > 0$ such, that*

$$\int_{\mathbb{R}^3} \left(|\nabla\psi|^2 + V |\psi|^2 \right) dx - \varepsilon \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^\beta} \ge \Sigma \|\psi\|^2 \tag{A.1}$$

for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$. Then the operator H has at most a finite number of eigenvalues below Σ.

In the proof of the main Theorem we make use of some Lemmas from [2]. For convenience we repeat these Lemmas here. The following is a restatement of [1][Lemma 6.6]

Lemma A.2. *Consider $h = -\Delta + V$ in $L^2(\mathbb{R}^2)$ with V bounded and V short-range (see equation (2.2)). Assume $h \ge 0$ and V radial symmetric then there exists a $c_0 > 0$ such, that for any $b_0 > R_0$*

$$\int_{|x| \ge b_0} \left(|\nabla\psi|^2 + V |\psi|^2 \right) dx \ge -c_0 b_0^{-\nu} \int_0^{2\pi} |\psi(b_0, \theta)|^2 \, d\theta \tag{A.2}$$

where R_0, ν are the constants from the short-range property of V.

The next Lemma is a restatement of [2][Lemma 6.7]

Lemma A.3. *Let $c_0 > 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R}^2)$*

$$\int_{|x| \ge b} \left(|\nabla\psi(x)|^2 - c_0 |x|^{-2-\nu} |\psi(x)|^2 \right) dx \ge -\frac{c_0 b^{-\nu}}{\pi} \int_0^{2\pi} |\psi(b, \theta)|^2 \, d\theta \, . \tag{A.3}$$

The next Lemma is a restatement of [2][Lemma 6.2]

Lemma A.4. *Consider $h = -\Delta + V$ in $L^2(\mathbb{R})$ with V bounded and V short-range (see equation (2.2)). Assume $h \ge 0$ then there exists $c > 0$, such that for any $b_0 \ge R_0$ and $\psi \in H^1(\mathbb{R})$*

$$\int_{b_0}^{b_0} (|\psi'(t)|^2 + V(t) |\psi(t)|^2) dt \ge -c b_0^{-1-\nu} (|\psi(b_0)|^2 + |\psi(-b_0)|^2) \tag{A.4}$$

where R_0, ν are the constants from the short-range property of V.

The next Lemma is a restatement of [2][Lemma 6.3]

Lemma A.5. *Let $c_0 \ge 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R})$*

$$\int_b^\infty \left(|\psi'(t)|^2 - c_0 t^{-2-\nu} |\psi(t)|^2 \right) dt \ge -2c_0 b^{-1-\nu} |\psi(b)|^2 \, . \tag{A.5}$$

We use several times Hardy's inequality on the half-line. Let us copy here the version of [7][Theorem 2.65]

Theorem A.6. *Let $\rho \in \mathbb{R} \setminus \{1\}$. Let u be weakly differentiable on $\mathbb{R}_+$ with $u' \in L^2(\mathbb{R}_+, r^\rho dr)$ and assume that*

$$\liminf_{r \to 0} |u(r)| = 0 \text{ if } \rho < 1, \quad \liminf_{r \to \infty} |u(r)| = 0 \text{ if } \rho > 1, \tag{A.6}$$

Then

$$\int_0^\infty |u(r)|^2 r^{-2+\rho} \, dr \le \left(\frac{2}{\rho - 1} \right)^2 \int_0^\infty |u'(r)|^2 r^\rho dr. \tag{A.7}$$

The constant on the right side is optimal.

The following Lemma is a straight forward adaptation of [20][Lemma 5.1]. Recall the definitions of $K_{ij}^b(\gamma)$ in equation (3.4), then

Lemma A.7. *Given $\varepsilon, \gamma > 0$ for all $i, j \in \{1, 2, 3\}$, $i < j$ there exist $\tilde{\gamma} \in (0, \gamma)$ and continuous functions $u_{ij}, v_{ij} : \mathbb{R}^3 \to \mathbb{R}$ with piecewiese continuous derivatives such that*

$$u_{ij}^2 + v_{ij}^2 = 1, \quad u_{ij}(x) = \begin{cases} 1 & x \in K_{ij}^0(\tilde{\gamma}) \\ 0 & x \notin K_{ij}^0(\gamma) \end{cases}, \quad |\nabla u_{ij}|^2 + |\nabla v_{ij}|^2 \leq \varepsilon \left(\frac{v_{ij}^2}{|x|^2} + \frac{u_{ij}^2}{|q_{ij}|^2} \right) \tag{A.8}$$

where $q_{23} = q = \frac{1}{\sqrt{2}}(x_2 - x_3) \in \mathbb{R}$ and $q_{1j} = (x_1, x_j) \in \mathbb{R}^2$.

Appendix B. Proof of Theorem 2.1 when $\Sigma < 0$

We now cover the case when Σ defined in equation (2.8) is negative. As in the previous case we will prove that there exists a $b > 0$ such that

$$L[\psi] := \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} |\partial_{x_i} \psi|^2 + \sum_{i<j} V_{ij} |\psi|^2 \right) dx - \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^{2+\nu}} dx \geq \Sigma \|\psi\| \tag{B.1}$$

for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$. As in the case $\Sigma = 0$, we consider the sets $K_{ij}^b(\gamma)$ and Ω_0 defined in (3.4) and (3.5). Contrary to the previous case, we cannot apply the lemmas stated in Appendix A, since we would need to integrate a constant function on a infinite-measure domain. For this reason, we localize the functional L with smooth cut-off functions. For fixed $\varepsilon < 1/8$ we consider the family of cut-off functions u_{ij} defined in Lemma A.7 together with $\mathcal{V} := \sqrt{1 - \sum_{i<j \in \{1,2,3\}} u_{ij}^2}$. The family $\{\mathcal{V}, u_{ij}$ for $i, j \in \{1, 2, 3\}$ with $i < j\}$ forms a partition of unity by construction. According to the IMS localization formula,

$$L[\psi] = \sum_{1 \leq i < j \leq 3} \Big(L[u_{ij}\psi] - \langle \psi, |\nabla u_{ij}|^2 \psi \rangle \Big) + L[\mathcal{V}\psi] - \langle \psi, |\nabla \mathcal{V}|^2 \psi \rangle. \tag{B.2}$$

Recall the definition of $\Omega_{ij}^b(\tilde{\gamma}, \gamma)$ in the statement of Lemma 3.3. Note that, on each $\Omega_{ij}^b(\tilde{\gamma}, \gamma)$, $\mathcal{V} = v_{ij} = \sqrt{1 - u_{ij}^2}$. Moreover, $\nabla \mathcal{V}$ vanishes outside of

$$\bigcup_{1 \leq i < j \leq 3} \Omega_{ij}^b(\tilde{\gamma}, \gamma)$$

by construction. Similarly, ∇u_{ij} vanishes outside of $\Omega_{ij}^b(\tilde{\gamma}, \gamma)$.

Consequently by Lemma A.8,

$$\begin{aligned} \sum_{1 \leq i < j \leq 3} \langle \psi, |\nabla u_{ij}|^2 \psi \rangle + \langle \psi, |\nabla \mathcal{V}|^2 \psi \rangle &= \sum_{1 \leq i < j \leq 3} \int_{\Omega_{ij}^b(\tilde{\gamma}, \gamma)} (|\nabla u_{ij}|^2 + |\nabla v_{ij}|^2) |\psi|^2 \, dx \\ &\leq \varepsilon \sum_{1 \leq i < j \leq 3} \int_{\Omega_{ij}^b(\tilde{\gamma}, \gamma)} \left(\frac{v_{ij}^2}{|x|^2} + \frac{u_{ij}^2}{|q_{ij}|^2} \right) |\psi|^2 \, dx. \end{aligned} \tag{B.3}$$

Therefore,

$$L[\psi] \geq \sum_{1 \leq i < j \leq 3} \left(L[u_{ij}\psi] - \varepsilon \int_{\Omega_{ij}^b(\tilde{\gamma}, \gamma)} \frac{u_{ij}^2}{|q_{ij}|^2} |\psi|^2 \, dx \right) + L[\mathcal{V}\psi] - \varepsilon \sum_{1 \leq i < j \leq 3} \int_{\Omega_{ij}^b(\tilde{\gamma}, \gamma)} \frac{\mathcal{V}^2}{|x|^2} |\psi|^2 \, dx. \tag{B.4}$$

We show the following

Lemma B.1. *There is a $b_0 > 0$ such that for all $b \geq b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp} \psi \subseteq \{x \in \mathbb{R}^3 | \, |x| > b\}$ with $i, j \in \{1, 2, 3\}$, $i \neq j$,*

$$L[u_{ij}\psi] - \varepsilon \int_{\Omega_{ij}^b(\tilde{\gamma}, \gamma)} \frac{u_{ij}^2}{|q_{ij}|^2} |\psi|^2 \, dx \geq \Sigma \|u_{ij}\psi\|^2 \tag{B.5}$$

where q_{ij} was defined in Lemma A.7.

Proof. Let

$$\xi_{ij} := \begin{cases} (\xi, x_1) \in \mathbb{R}^2, & (ij) = (23) \\ x_2 \in \mathbb{R}, & (ij) = (13) \\ x_3 \in \mathbb{R}, & (ij) = (12) \end{cases}, \quad \tilde{\psi} := \psi u_{ij} \,. \tag{B.6}$$

We define

$$\begin{aligned} \tilde{L}_1[\tilde{\psi}] &:= \langle \tilde{\psi}, (-\Delta_{q_{ij}} + V_{ij})\tilde{\psi} \rangle \,, \\ \tilde{L}_2[\tilde{\psi}] &:= \langle \tilde{\psi}, (-\Delta_{\xi_{ij}} + W_{ij})\tilde{\psi} \rangle - \int_{\mathbb{R}^3} \frac{\left|\tilde{\psi}(x)\right|^2}{|x|^{2+\nu}} dx. \end{aligned} \tag{B.7}$$

Here $W_{ij} := \left[\sum_{1\le l<m\le 3} V_{lm}\right] - V_{ij}$. Then

$$L[\tilde{\psi}] = \tilde{L}_1[\tilde{\psi}] + \tilde{L}_2[\tilde{\psi}] \,. \tag{B.8}$$

We distinguish between the cases $\inf \sigma(h_{ij}) > \Sigma$ and $\inf \sigma(h_{ij}) = \Sigma$. We start with the case $\inf \sigma(h_{ij}) > \Sigma$. Let $\kappa := \inf \sigma(h_{ij}) - \Sigma > 0$. Then

$$\tilde{L}_1[\tilde{\psi}] \geq (\Sigma + \kappa) \left\|\tilde{\psi}\right\|^2 . \tag{B.9}$$

Using the short-range property of W_{ij} and the fact that, on $\Omega^b_{ij}(\tilde{\gamma}, \gamma)$, $|q_{ij}| \geq \tilde{\gamma}\, |\xi_{ij}|$, we find that there exists $C > 0$ and $b_0 > 0$ such that for any $b \geq b_0$

$$\tilde{L}_2[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{u_{1j}^2}{|q_{ij}|^2} |\psi|^2 \, dx \geq -\langle \tilde{\psi}, \Delta_{\xi_{ij}} \tilde{\psi} \rangle - C \int_{\mathbb{R}^3} \frac{\left|\tilde{\psi}(x)\right|^2}{|\xi_{ij}|^2} dx \,. \tag{B.10}$$

Since $-\Delta_{\xi_{ij}} \geq 0$, and on $K^b_{ij}(\gamma)$, $|\xi_{ij}| \geq |x| / \sqrt{1+\gamma^2} \geq b/\sqrt{1+\gamma^2}$, we find

$$\tilde{L}_2[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{u_{1j}^2}{|q_{ij}|^2} |\psi|^2 \, dx \geq -C \int_{\mathbb{R}^3} \frac{\left|\tilde{\psi}(x)\right|^2}{|\xi_{ij}|^2} dx \geq -\frac{C_1}{b^2} \left\|\tilde{\psi}\right\|^2 \tag{B.11}$$

for some $C_1 > 0$ independent of b. Choose b large enough such that $\kappa \geq C_1/b^2$. Then, combining Equation (B.9) and Equation (B.11) concludes the statement in this case.

Next we consider the case $\inf \sigma(h_{ij}) = \Sigma$. In this case Σ is an eigenvalue of h_{ij} and thus there exists a $\varphi_0 \in L^2(dq_{ij})$ with $\|\varphi_0\| = 1$ such that $\langle \varphi_0, h_{ij}\varphi_0 \rangle = \Sigma$. We define

$$f(\xi_{ij}) := \langle \varphi_0, \tilde{\psi} \rangle_{L^2(dq_{ij})}, \quad g(q_{ij}, \xi_{ij}) := \tilde{\psi}(q_{ij}, \xi_{ij}) - f(\xi_{ij})\varphi_0(q_{ij}). \tag{B.12}$$

Note that φ_0 and g are orthogonal in the usual $L^2(dq_{ij})$-sense by construction. Since Σ is an isolated nondegenerate eigenvalue of h_{ij}, there exists some $\kappa' > 0$ such that

$$\tilde{L}_1[\tilde{\psi}] = \langle \tilde{\psi}, h_{ij}\tilde{\psi} \rangle \geq \Sigma \left\|\tilde{\psi}\right\|^2 + \kappa' \left\|g\right\|^2 . \tag{B.13}$$

One can choose κ' as the distance of Σ and the remaining spectrum of h_{ij}. Using the short-range property of W_{ij} and Lemma A.7, there exists $C > 0$ such that for $b > 0$ large enough

$$\begin{aligned} \tilde{L}_2[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{u_{ij}^2}{|q_{ij}|^2} |\psi|^2 \, dx \geq \frac{1}{2}\langle \tilde{\psi}, -\Delta_{\xi_{ij}} \tilde{\psi}\rangle - C \int \frac{\left|\tilde{\psi}\right|^2}{|\xi_{ij}|^{2+\delta}} d(q_{ij}, \xi_{ij}) \\ + \frac{1}{2}\langle \tilde{\psi}, -\Delta_{\xi_{ij}} \tilde{\psi}\rangle - \varepsilon \int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{\left|\tilde{\psi}\right|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) \, . \end{aligned} \tag{B.14}$$

Note that the variable ξ_{ij} is either one- or two-dimensional. Depending on its dimension we use Hardy's inequality on the half line or the two-dimensional Hardy's inequality (see [15]) to estimate the first line on the right hand side in equation (B.14). The inequality is applicable since $\tilde{\psi}$ vanishes whenever $|\xi_{ij}|$ is small enough by construction. Thus

$$\frac{1}{2}\langle \tilde{\psi}, -\Delta_{\xi_{ij}} \tilde{\psi}\rangle - C \int \frac{\left|\tilde{\psi}\right|^2}{|\xi_{ij}|^{2+\delta}} d(q_{ij}, \xi_{ij}) \geq 0. \tag{B.15}$$

It remains to prove the non negativity of the second line. Recall that φ_0 and g are orthogonal in the $L^2(dq_{ij})$-sense and thus

$$\langle \tilde{\psi}, -\Delta_{\xi_{ij}} \tilde{\psi}\rangle = \langle |\varphi_0|^2 f, -\Delta_{\xi_{ij}} f\rangle + \langle g, -\Delta_{\xi_{ij}} g\rangle \geq \langle f, -\Delta_{\xi_{ij}} f\rangle_{L^2(d\xi_{ij})}. \tag{B.16}$$

We use $(a+b)^2 \leq 2a^2 + 2b^2$ to find that there exists $C' > 0$ such that for b large enough

$$\begin{aligned} \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{\left|\tilde{\psi}\right|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) &\leq 2 \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|\varphi_0|^2 |f|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) + 2 \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|g|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) \\ &\leq 2 \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|q_{ij}|^2 |\varphi_0|^2 |f|^2}{|q_{ij}|^4} d(q_{ij}, \xi_{ij}) + \frac{C'}{b^2} \|g\|^2 , \end{aligned} \tag{B.17}$$

where in the last line, we used that, on $\Omega^b_{ij}(\tilde{\gamma},\gamma)$,

$$|q_{ij}|^2 \geq \frac{\tilde{\gamma}^2}{1+\tilde{\gamma}^2} |x|^2 \geq \frac{\tilde{\gamma}^2}{1+\tilde{\gamma}^2} b^2 \, . \tag{B.18}$$

Since φ_0 is an eigenfunction of h_{ij} associated with a discrete eigenvalue, it decays exponentially at infinity (*cf.* for example [14], Theorem XIII.39). Hence for any $\delta_1 > 0$ there exists $q_0 > 0$ such that $|q_{ij}|^2 |\varphi_0(q_{ij})|^2 \leq \delta_1$ for any $|q_{ij}| \geq q_0$. Thus for b large enough

$$\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|q_{ij}|^2 |\varphi_0|^2 |f|^2}{|q_{ij}|^4} d(q_{ij}, \xi_{ij}) \leq \frac{\delta_1}{\tilde{\gamma}^4} \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|f|^2}{|\xi_{ij}|^4} d(q_{ij}, \xi_{ij}) \, . \tag{B.19}$$

From equations (B.19) and (B.17), we find

$$\begin{aligned} \frac{1}{2}\langle \tilde{\psi}, -\Delta_{\xi_{ij}} \tilde{\psi}\rangle - \varepsilon \int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{\left|\tilde{\psi}\right|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) \\ \geq \frac{1}{2}\langle f, -\Delta_{\xi_{ij}} f\rangle_{L^2(d\xi_{ij})} - \varepsilon \frac{\delta_1}{\tilde{\gamma}^4} \int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{|f|^2}{|\xi_{ij}|^4} d(q_{ij}, \xi_{ij}) - \varepsilon \frac{C'}{b^2} \|g\|^2 \, . \end{aligned} \tag{B.20}$$

As in the proof of equation (B.15) we can again use Hardy's inequality depending on the dimension of the variable ξ_{ij} to find

$$\frac{1}{2}\langle f, -\Delta_{\xi_{ij}} f\rangle_{L^2(d\xi_{ij})} - \varepsilon\frac{\delta_1}{\tilde{\gamma}^4}\int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})}\frac{|f|^2}{|\xi_{ij}|^4}d(q_{ij},\xi_{ij}) \geq 0 \tag{B.21}$$

for $b > 0$ large enough. Combining equations (B.14), (B.20) and (B.21), we find

$$\tilde{L}[\tilde{\psi}] - \varepsilon\int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})}\frac{u_{ij}^2}{|q_{ij}|^2}|\psi|^2\, d(q_{ij},\xi_{ij}) \geq \Sigma\left\|\tilde{\psi}\right\|^2 + (\kappa' - \varepsilon\frac{C'}{b^2})\left\|g\right\|^2 . \tag{B.22}$$

The statement in equation (B.5) follows by taking b large enough. □

We have thus proved that

$$L[\psi] \geq \sum_{1\leq i<j\leq 3}\|u_{ij}\psi\|^2\,\Sigma + L[\mathcal{V}\psi] - \varepsilon\sum_{1\leq i<j\leq 3}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\frac{\mathcal{V}^2}{|x|^2}|\psi|^2\, dx. \tag{B.23}$$

By construction $\mathcal{V}$ is supported in

$$\Omega_0(\tilde{\gamma}) \coloneqq \mathbb{R}^3 \setminus \bigcup_{1\leq i<j\leq 3} K^b_{ij}(\tilde{\gamma}). \tag{B.24}$$

Therefore, by applying the short-range property to any of the potentials we can write

$$L[\mathcal{V}\psi] \geq \int_{\Omega_0(\tilde{\gamma})}|\nabla\mathcal{V}\psi|^2 - \frac{C}{|x|^{2+\nu}}|\mathcal{V}\psi|^2\, dx. \tag{B.25}$$

Hence we can estimate the remaining terms in equation (B.23). Combining the previous inequalities shows that for $b > 0$ large enough

$$L[\mathcal{V}\psi] - \varepsilon\sum_{1\leq i<j\leq 3}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\frac{|\mathcal{V}\psi|^2}{|x|^2}dx \geq \int_{\mathbb{R}^3}|\nabla\mathcal{V}\psi|^2 - \frac{2\varepsilon}{|x|^2}|\mathcal{V}\psi|^2\, dx \geq 0 \tag{B.26}$$

by Hardy's inequality. Thus we have shown

$$L[\psi] \geq \sum_{1\leq i<j\leq 3}\|u_{ij}\psi\|^2\,\Sigma \geq \|\psi\|^2\,\Sigma \tag{B.27}$$

and hence the statement in equation (B.1).

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