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Curvature bounds on length-minimizing discs

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Abstract
We show that a length-minimizing disk inherits the upper curvature bound of the target. As a consequence we prove that harmonic discs and ruled discs inherit the upper curvature bound from the ambient space.

Keywords Alexandrov geometry · Harmonic surfaces · Ruled surfaces

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1 Introduction

We extend the results of [24, 27] to the case of curvature bounds different from 0 and provide full proofs of some consequences of the main theorems, which are mentioned in [27] in the case of CAT(0) spaces.

We need some notation, in order to state the main result. Let $f : \mathbb{D} \to Y$ be a continuous map from the closed unit disc $\mathbb{D}$ into a CAT(κ) space Y. For $x, z \in \mathbb{D}$ we define the *length pseudodistance* $\langle x - z \rangle_f \in [0, \infty]$ *induced by* f as

$$\langle x - z \rangle_f := \inf_{\gamma} \ell_Y(f \circ \gamma) \in [0, \infty] , \tag{1}$$

where $\ell_Y(\eta)$ denotes the length of a curve η in the metric space Y, and the infimum in (1) is taken over all curves γ in $\mathbb{D}$ connecting x and z.

We say that f is *length-connected* if the value $\langle x - z \rangle_f$ is finite, for all $x, z \in \mathbb{D}$. In this case, the length pseudodistance defines a metric space by identifying points with length pseudodistance 0. This space is denoted by $\langle \mathbb{D} \rangle_f$ and is called the *length metric space induced by* f.

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We say that f is *length-continuous* if the canonical projection $\hat{\pi}_f : \mathbb{D} \to \langle \mathbb{D} \rangle_f$ is continuous. Length-continuity always holds if f is a composition of a homeomorphism $\Phi : \mathbb{D} \to \mathbb{D}$ and a Lipschitz map $g : \mathbb{D} \to Y$.

A continuous map $f : \mathbb{D} \to Y$ is *length-minimizing* if for all continuous $g : \mathbb{D} \to Y$ with $g|_{\mathbb{S}^1} = f|_{\mathbb{S}^1}$ and

$$\ell_Y(g \circ \gamma) \leq \ell_Y(f \circ \gamma)\,, \tag{2}$$

for all curves γ in $\mathbb{D}$, equality must hold in (2), for all curves γ.

Now we can state our main result:

Theorem 1.1 *Let $f : \mathbb{D} \to Y$ be a length-continuous and length-minimizing map from the disc $\mathbb{D}$ to a* CAT(κ) *space Y. Then $\mathbb{D}$ with the length metric induced by f is a* CAT(κ) *space.*

We use this result in order to find upper curvature bounds on discs satisfying more common minimality assumptions. For the first consequence, we assume some familiarity with the notion of harmonic maps due to Korevaar–Schoen [11].

Corollary 1.2 *Let Y be a* CAT(κ) *space and let the continuous map $f : \mathbb{D} \to Y$ be harmonic. If the boundary curve $f : \mathbb{S}^1 \to X$ has finite length then $\mathbb{D}$ with the length metric induced by f is a* CAT(κ) *space.*

If f is *conformal*, the conclusion of Corollary 1.2 appears in [17] and a closely related statement appears in [19]. For non-conformal harmonic discs it seems barely possible to prove Corollary 1.2 by purely analytic means.

Another application concerns ruled discs:

Corollary 1.3 *Let $\eta_0, \eta_1 : [0, 1] \to Y$ be rectifiable curves in a* CAT(κ) *space Y. If $\kappa > 0$, we assume that the distance between $\eta_0(a)$ and $\eta_1(a)$ is less than $\frac{\pi}{\sqrt{\kappa}}$, for all $a \in [0, 1]$. For any $a > 0$, consider the geodesic $\gamma_a : [0, 1] \to Y$ between $\eta_0(a)$ and $\eta_1(a)$ parametrized proportionally to arclength. Then $[0, 1] \times [0, 1]$ with the length metric induced by the map $f(a, t) := \gamma_a(t)$ is* CAT(κ).

The statement about the inheritance of upper curvature bounds by ruled discs appeared in somewhat different generality with a sketchy proof in [2], the paper that gave birth to the theory of CAT(κ) spaces. Missing details in Alexandrovs proof were recently provided by Nagano–Shioya–Yamaguchi in [23].

It seems possible to extend our proof to the case of non-rectifiably boundary curves and to dispense of the assumption on the parametrization of the geodesics. However, it would require rather technical considerations. The generality we have chosen is sufficient for most applications and follows directly from Theorem 1.1.

We finish the introduction with several comments.

- Our notion of length-minimality corresponds to the notion called *metric-minimality* in [24]. In [27] a new, stronger and less natural notion of metric-minimality was introduced. With this new stronger notion, a version of our main theorem for CAT(0) spaces was proved only under the assumption of length-connectedness of f, instead of the stronger length-continuity. We have only been able to verify the validity of this stronger metric-minimality assumption in cases covered by our Theorem 1.1. In order to find a clearer and better comprehensible way to Corollaries 1.2, 1.3, we have decided to work with the original, more natural notion of metric-minimality used in [24]. In order to distinguish it from metric-minimality used in [27], we have given it a different name.
- In the last section we formulate several questions concerning generalizations of our main results.

- The length metrics arising in Theorem 1.1 and, therefore, in Corollaries 1.2, 1.3 are homeomorphic to deformation retracts of $\mathbb{D}$.
- A general existence result in Sect. 5 shows that length-minimizing discs are abundant, beyond harmonic and ruled discs.
- The main argument closely follows [27]. A central technical step in the proof is the reduction of the theorem to the case, where the map f has totally disconnected fibers. In order to achieve this reduction, another metric induced by f on the disc is investigated, the so-called *connecting pseudometric*. It has better topological and metric properties than the more natural *induced length metric*. Roughly speaking, while the induced length metric collapses curves sent by f to a point, the connecting pseudometric also collapses *pseudocurves* sent by f to a point. In general, the precise relation between the two metrics seems to be rather complicated. However, as verified in [27], the length-minimality together with the length-connectedness, imply that the induced length metric is just the intrinsic metric induced by the connecting pseudometric.
- The heart of the proof is an approximation of the induced length metric by polyhedral metrics, whose restriction to the 1-skeleton is length-minimizing. This idea presented in [24] works in our setting with minor modifications. The case of $\kappa > 0$ requires some additional considerations, due to the absence of the theorem of Cartan–Hadamard.

2 Preliminaries

2.1 Topology

A *curve* in a topological space X is a continuous map $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval. A *Jordan curve* in X is a subset homeomorphic to $\mathbb{S}^1$.

A topological space X is a *Peano continuum* if X is metrizable, compact, connected and locally connected. In this case X is arcwise connected.

A *disc retract* is a compact space homeomorphic to a subset X' of the closed unit disc $\mathbb{D}$ such that X' is a homotopy retract of $\mathbb{D}$. A Peano continuum is a disc retract if and only if it is homeomorphic to a non-separating subset of the plane, [21, p. 27]. We will only need the following statement about disc retracts, a variant of a classical theorem of Moore, [20], [27, Proposition 3.3]:

If $f : \mathbb{D} \to X$ is a surjective, continuous map, such that all fibers are connected and non-separating subsets of $\mathbb{R}^2$, then X is a disc retract.

Remark 2.1 The following properties will not be used below. The reader might find them helpful, since disc retracts are central objects of this paper.

A space X is a disc retract if and only if there exists a closed curve $\gamma : \mathbb{S}^1 \to X$, such that the mapping cylinder $([0, 1] \times \mathbb{S}^1) \cup_\gamma X$ is homeomorphic to $\mathbb{D}$, [8, 27]. The image of γ is the *boundary* of X, defined as the set of points at which X is not a 2-manifold.

Any maximal subset of a disc retract X which is not a point and does not contain cut points is a closed disc [21, p.27], [27]. The number of these *cyclic components* of X is at most countable.

By approximation, it follows that any disc retract admits a CAT(-1) metric. The space of (isometry classes of) disc retracts with a geodesic metric, uniformly bounded $\mathcal{H}^2$-measure and $\mathcal{H}^1$-measure of the boundary and a *quadratic isoperimetric inequality* is compact in the Gromov–Hausdorff topology, [8]. In particular, the space of isometry classes of CAT(κ) disc retracts with boundary of $\mathcal{H}^1$-measure at most $r < \frac{2\pi}{\sqrt{\kappa}}$ is compact in the Gromov–Hausdorff topology.

2.2 Metric geometry

We stick to the notations and conventions used in [27], and refer to [4] and [26] for introductions to metric geometry.

We denote the distance in a metric space Y by $|*-*|_Y = \langle * - * \rangle = \langle * - * \rangle_Y$.

The length of a curve γ in Y will be denoted by $\ell_Y(\gamma) = \ell(\gamma)$. A *geodesic* is a curve γ connecting points y_1, y_2 in Y, such that the length of γ equals $\langle y_1 - y_2 \rangle_Y$. We make no a priori assumption on the parametrization of a geodesic.

The space Y is a *length space* if the distance between any pair of its points equals the infimum of lengths of curves connecting these points. The space is a *geodesic space* if any pair of points is connected by a geodesic.

A *pseudometric space* is a metric space in which the distance can also assume values 0 and ∞. If the value ∞ is not assumed by a pseudometric on Y, we identify subsets of points of Y with pseudo-distance equal to 0 and obtain the *induced metric space* [26, Sect. 1.C].

We assume some familiarity with the Gromov–Hausdorff convergence and, for readers interested in the non-locally compact targets Y in our main results, with ultralimits. We refer to [26, Sects. 5,6] and [5, Sect. I.5].

2.3 CAT(κ) spaces

We assume familiarity with properties of CAT(κ) spaces, the reader is refered to [1, 4, 5] and [3]. We will stick to the convention that CAT(κ) spaces are complete and geodesic.

By M_κ we denote the model surface of constant curvature κ and by R_κ its diameter. Thus, $R_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and $R_\kappa = \infty$ if $\kappa \leq 0$.

3 Metrics induced by maps

We recall some notions and notations introduced explicitly or implicitly in [27, Sect. 2]. All statements in this section are proved there.

3.1 Length metric induced by a map

Let X be a topological space and Y a metric space. A continuous map $f : X \to Y$ induces a pseudometric on X by

$$\langle x - z \rangle_f = \inf\{\ell_Y(f \circ \gamma) : \gamma \text{ is a curve in } X \text{ joining } x \text{ to } z\}.$$

This pseudometric is called the *length pseudometric on X induced by f*.

As in the introduction, we call a continuous map $f : X \to Y$ *length-connected* if the length pseudometric induced by f is finite for all pairs of points. Thus, f is length-connected if any $x, z \in X$ are connected in X by a curve γ, whose image $f \circ \gamma$ has finite length in Y.

If $f : X \to Y$ is length-connected then the metric space arising from the pseudometric $\langle x - z \rangle_f$ by identifying points at pseudistance 0 will be called *the length metric space induced by f*. We denote it by $\langle X \rangle_f$.

The canonical surjective projection from X onto $\langle X\rangle_f$ will be denoted by

$$\hat{\pi}_f : X \to \langle X\rangle_f.$$

We will say that a continuous map $f : X \to Y$ is *length-continuous* if f is length-connected and the induced projection $\hat{\pi}_f : X \to \langle X\rangle_f$ is continuous.

For a compact metric space X, a map $f : X \to Y$ is length-continuous if and only if for any $\varepsilon > 0$ there exists $\delta > 0$, such that any pair of points $x, z \in X$ with $\langle x - z\rangle_X < \delta$ is connected by some curve γ in X with $\ell_Y(f \circ \gamma) < \varepsilon$.

If X is a length metric space and $f : X \to Y$ is locally Lipschitz continuous then f is length-continuous.

For any length-connected $f : X \to Y$, there exists a unique 1-Lipschitz map

$$\hat{f} : \langle X\rangle_f \to Y,$$

such that $\hat{f} \circ \hat{\pi}_f = f$. The following is stated between the lines in [27].

Lemma 3.1 *Let $f : X \to Y$ be length-connected. Let γ be a curve in X. The curve $f \circ \gamma$ has finite length in Y if and only if $\hat{\pi}_f \circ \gamma$ is a curve of finite length in $\langle X\rangle_f$. In this case*

$$\ell_Y(f \circ \gamma) = \ell_{\langle X\rangle_f}(\hat{\pi}_f \circ \gamma).$$

Proof If $\hat{\pi}_f \circ \gamma$ is a curve of finite length, then $\ell_Y(f \circ \gamma) \le \ell_{\langle X\rangle_f}(\hat{\pi}_f \circ \gamma)$, since $\hat{f}$ is 1-Lipschitz.

Assume now that $f \circ \gamma$ is of finite length. Then, for any t, s in the interval I of definition of γ, we have

$$\langle \gamma(t) - \gamma(s)\rangle_f = \langle (\hat{\pi}_f \circ \gamma)(t) - (\hat{\pi}_f \circ \gamma(s)\rangle_{\langle X\rangle_f} \le \ell_Y(f \circ \gamma|_{[t,s]}).$$

Since the length of $f \circ \gamma$ on small intervals around a fixed point t goes to 0 with the length of the interval, $\hat{\pi}_f \circ \gamma$ is continuous. Applying the above inequality to arbitrary partitions of I, we deduce $\ell_{\langle X\rangle_f}(\hat{\pi}_f \circ \gamma) \le \ell_Y(f \circ \gamma)$. □

3.2 Connecting pseudometric

Another pseudometric on X associated with a continuous map $f : X \to Y$ is the *connecting pseudometric* $|* - *|_f$ defined as

$$|x - z|_f = \inf\{\operatorname{diam} f(C) : C \subset X \text{ connected and } x, z \in C\}.$$

Whenever the connecting pseudometric assumes only finite values, we consider the associated metric spaces and denote it by $\big|X\big|_f$.

In this case we have the canonical projection map, denoted by

$$\bar{\pi}_f : X \to \big|X\big|_f.$$

Moreover, there exist a uniquely defined 1-Lipschitz map

$$\bar{f} : \big|X\big|_f \to Y,$$

such that $\bar{f} \circ \bar{\pi}_f = f$.

If $f : X \to Y$ is length-connected then the connecting pseudometric assumes only finite values and there exists a uniquely defined, surjective 1-Lipschitz map

$$\tau_f : \langle X\rangle_f \to \big|X\big|_f,$$

such that

$$\bar{\pi}_f = \tau_f \circ \hat{\pi}_f .$$

3.3 Basic properties

Recall that a map $f : X \to Y$ between topological spaces is called *monotone* (respectively, *light*) if any fiber of f is connected (respectively, totally disconnected). We recall from [27, Sect. 2]:

Lemma 3.2 *Let X be a Peano continuum and Y a metric space. Let $f : X \to Y$ be continuous. Then*

(a) The map $\bar{\pi}_f : X \to \left|X\right|_f$ is continuous. Hence, $\left|X\right|_f$ is a Peano continuum.
(b) $\ell_{|X|_f}(\gamma) = \ell_Y(\bar{f} \circ \gamma)$, for every curve γ in $\left|X\right|_f$.
(c) The map $\bar{\pi}_f : X \to \left|X\right|_f$ is monotone.
(d) The map $\bar{f} : \left|X\right|_f \to Y$ is light.

Lemma 3.2(b) implies that $\left|X\right|_f$ with the induced length metric is isometric to $\left|X\right|_f$ with the length metric induced by $\bar{f}$.

From Lemmas 3.1 and 3.2 we obtain:

Lemma 3.3 *Let X be a Peano continuum, Y a metric space and $f : X \to Y$ be length-connected. Then, for all curves γ in X,*

$$\ell_Y(f \circ \gamma) = \ell_{|X|_f}(\bar{\pi}_f \circ \gamma) = \ell_{|X|_f}(\tau_f \circ \hat{\pi}_f \circ \gamma).$$

If, in addition, $\ell_Y(f \circ \gamma)$ is finite or if f is length-continuous then $\hat{\pi}_f \circ \gamma$ is continuous and

$$\ell_Y(f \circ \gamma) = \ell_{\langle X\rangle_f}(\hat{\pi}_f \circ \gamma).$$

For a Peano continuum X and a length-continuous map $f : X \to Y$, the natural 1-Lipschitz map $\tau_f : \langle X\rangle_f \to \left|X\right|_f$ may be non-injective [25, Example 4.2]. Since every connected subset of a finite graph is arcwise connected, this pathology cannot occur if X is a finite, connected graph. The following result proven in [27, Lemma 3.3, Proposition 9.3] is much less trivial:

Lemma 3.4 *Let $f : \mathbb{D} \to Y$ be length-connected and let any fiber $f^{-1}(y)$ be a non-separating subset in $\mathbb{R}^2$. Then $|\mathbb{D}|_f$ is a disc retract. If, in addition, f is length-continuous then the map $\tau_f : \langle\mathbb{D}\rangle_f \to |\mathbb{D}|_f$ is a homeomorphism, which preserves the length of all curves in $\langle\mathbb{D}\rangle_f$.*

4 Length-minimizing maps

4.1 Definition and first properties

Let X be a topological space, Z a closed subset of X. Let $f : X \to Y$ be a continuous map into a metric space. For another map $g : X \to Y$ we will write $f \trianglerighteq g$ (rel Z) if f and g coincide on Z and

$$\ell_Y(f \circ \gamma) \geq \ell_Y(g \circ \gamma)\,, \quad \text{for every curve } \gamma \text{ in } X. \tag{3}$$

We will call f *length-minimizing relative to* Z if for every map $f \trianglerighteq g$ (rel Z) equality holds in (3), for every curve γ.

If X is arcwise connected and Z is empty then length-minimizing maps relative to Z are constant. From now on, we will always assume that Z is not empty.

Lemma 4.1 *Let X be a topological space, Y be a metric space and $Z \subset X$ be closed. Let $f : X \to Y$ be length-connected. A continuous map $g : X \to Y$ satisfies $f \trianglerighteq g$ (rel. Z) if and only if g is length-connected, coincides with f on Z and*

$$\langle x_1 - x_2 \rangle_f \geq \langle x_1 - x_2 \rangle_g,$$

for all $x_1, x_2 \in X$.

Proof Assume $f \trianglerighteq g$ (rel. Z). By definition, f and g coincide on Z. For any $x_1, x_2 \in X$, we find a curve γ in X connecting x_1 and x_2, such that $f \circ \gamma$ has finite length arbitrarily close to $\langle x_1 - x_2 \rangle_f$. Since $\ell_Y(f \circ \gamma) \geq \ell_Y(g \circ \gamma)$ we deduce that $\langle x_1 - x_2 \rangle_g$ is finite and not larger than $\langle x_1 - x_2 \rangle_f$.

Assume on the other hand, that g satisfies the conditions in the statement of the Lemma. In order to prove $f \trianglerighteq g$ (rel. Z), consider an arbitrary curve γ in X. If $f \circ \gamma$ has infinite length, then $\ell_Y(f \circ \gamma) \geq \ell_Y(g \circ \gamma)$.

If $f \circ \gamma$ has finite length, then $\ell_Y(f \circ \gamma) = \ell_{\langle X \rangle_f}(\hat{\pi}_f \circ \gamma)$ by Lemma 3.1. By assumption, the canonical map $\mu : \langle X \rangle_f \to \langle X \rangle_g$ is 1-Lipschitz and commutes with the projections $\hat{\pi}_f$ and $\hat{\pi}_g$. Thus, $\hat{\pi}_g \circ \gamma$ is continuous and has length at most $\ell_{\langle X \rangle_f}(\hat{\pi}_f \circ \gamma)$. Applying Lemma 3.1 twice, we deduce $\ell_Y(f \circ \gamma) \geq \ell_Y(g \circ \gamma)$. □

The property of being length-minimizing is inherited by restrictions:

Lemma 4.2 *Let X be a topological space, let $Z, S \subset X$ be closed. Let Y be a metric space and $f : X \to Y$ be length-minimizing relative to Z. Then the restriction $f : S \to Y$ of f is length-minimizing relative to $Z_S := \partial S \cup (Z \cap S)$.*

Proof Assume the contrary. Then there exists a map $h : S \to Y$ such that $f\big|_S \trianglerighteq h$ (rel Z_S) together with a curve $\gamma_0 : I_0 \to S$ such that $\ell_Y(f \circ \gamma_0) > \ell_Y(h \circ \gamma_0)$.

Define $g : X \to Y$ by setting $g = h$ on S and $g = f$ on $X \setminus S$. The maps h and f agree on ∂S, hence g is continuous. By construction, $g = f$ on Z. Moreover,

$$\ell_Y(h \circ \gamma_0) = \ell_Y(g \circ \gamma_0) < \ell_Y(f \circ \gamma_0).$$

Let now $\gamma : I \to X$ an arbitrary curve. Then $f \circ \gamma$ and $g \circ \gamma$ agree on the closed set $C = \gamma^{-1}(\overline{X \setminus S})$. The complement $I \setminus C$ is a countable union of open intervals I_j. For any j, the length of the restriction of $g \circ \gamma$ to I_j does not exceed the length of $f \circ \gamma\big|_{I_j}$, since $f\big|_S \trianglerighteq h$. Computing the length via the 1-dimensional Hausdorff measure $\mathcal{H}^1$, see [4, Exercise 2.6.4], we deduce

$$\ell(g \circ \gamma) \leq \ell(f \circ \gamma).$$

This contradicts the length-minimality of f. □

As a consequence we obtain the following *non-bubbling* property:

Corollary 4.3 *Let X be a Peano space and let $f : X \to Y$ be length-minimizing relative to a closed subset $Z \subset X$. Then, for any $y \in Y$ and any connected component U of $X \setminus f^{-1}(y)$ the intersection $U \cap Z$ is not empty.*

Proof Assume $U \cap Z = \emptyset$ and let S be the closure of U in X. Since connected components are always closed in the ambient space, the boundary ∂S of S in X, satisfies $\partial S \subset f^{-1}(y)$. Moreover, $S \cap Z = \partial S \cap Z$.

By Lemma 4.2, $f : S \to Y$ is length-minimizing relative to ∂S. The constant map $g : S \to Y$, which sends every point to y satisfies $f \trianglerighteq g$. The length-minimality of f implies that the image of every curve in U has length 0. Therefore, f is constant on U, hence on S. Thus, U is contained in $f^{-1}(y)$, which is impossible. □

Lemma 4.4 *Let X be a topological space, $Z \subset X$ be closed. Let Y be a* $\mathrm{CAT}(\kappa)$ *space. Let $K \subset Y$ be closed, convex and contained in a ball of radius $< \frac{R_\kappa}{2}$ in Y.*

Let $f : X \to Y$ be a length-connected, length-minimizing map relative to Z. If $f(Z)$ is contained in K then $f(X)$ is contained in K.

Proof Assume that there exists a point $x \in X$ with $f(x) \notin K$. We find a curve γ_0 in X between a point $z \in Z$ and x such that $f \circ \gamma_0$ has finite length.

Due to [12, Theorem 1.1], there exists a 1-Lipschitz retraction $\Psi : Y \to K$, which decreases the length of any rectifiable curve of positive length not completely contained in K. Hence, the composition $g := \Psi \circ f$ satisfies $f \trianglerighteq g$ (rel. Z) and $\ell_Y(g \circ \gamma_0) < \ell_Y(f \circ \gamma_0)$. This contradicts the length-minimality of f. □

Lemma 4.5 *Under the assumptions of Lemma 4.4, let $g : X \to Y$ satisfy $f \trianglerighteq g$ (rel. Z). Then $f = g$.*

Proof By definition, g is length-minimizing relative to Z as well. By Lemma 4.4, the images of f and of g are contained in K.

Denote by $\Delta \subset K \times K$ the diagonal. We apply [12, Corollary 1.2] and obtain a 1-Lipschitz retraction $\Pi : K \times K \to \Delta$, such that Π decreases the length of any curve of finite positive length not completely contained in Δ.

We identify Δ with K, rescaled by $\sqrt{2}$ and consider the map $h : X \to Y$

$$h(x) := \Pi(f(x), g(x)).$$

Then h coincides with f and with g on Z. For any curve γ in X, we have

$$\ell_Y(h \circ \gamma) \leq \ell_Y(f \circ \gamma) = \ell_Y(g \circ \gamma).$$

Whenever $\ell_Y(f \circ \gamma)$ is finite and positive, and $f \circ \gamma$ does not coincide with $g \circ \gamma$, the above inequality is strict.

If f does not coincide with g on X, we find a curve γ_0 in X, such that $f \circ \gamma_0$ and $g \circ \gamma_0$ are different curves of finite length, since f is length-connected. We infer $f \trianglerighteq h$ (rel. Z) and that f is not metric minimizing. Contradiction. □

We will apply Lemmas 4.5, 4.4 only for balls $K = B_r(y) \subset Y$ with $r < \frac{R_\kappa}{2}$. However, these lemmas might be useful in other cases as well, cf. [28].

4.2 Length-minimizing graphs

The results in this subsection are contained explicitly or between the lines [27, Sect. 5].

By a *finite graph* we mean a connected, finite 1-dimensional CW-complex. Thus, we allow multiple edges and edges connecting a vertex with itself.

Proposition 4.6 *Let Y be a geodesic metric space, Γ a finite graph and A a set of its vertices. If $f : \Gamma \to Y$ is length-minimizing relative to A then the restriction of f to any edge is a geodesic.*

Proof By Lemma 4.2, the restriction $f_0 : E \to Y$ of f to any edge $E \subset \Gamma$ is length-minimizing relative to its endpoints. We can identify E with the interval $[0, a]$, such that a is the distance $a = \langle f(0) - f(a) \rangle_Y$ between the endpoints. Consider a geodesic $\gamma : [0, a] \to Y$ with the same endpoints as f, parametrized by arclength. Then $C := \gamma([0, a])$ is an isometric embedding of the interval $[0, a]$. Since intervals are injective metric spaces, cf. [26, Lecture 3], there exists a 1-Lipschitz retraction $\Phi : Y \to C$. Hence, for $g := \Phi \circ f_0$ we have $f \trianglerighteq g$ (rel. $\{0, a\}$).

Hence, also the map $g : [0, a] \to C$ is length-minimizing relative to $\{0, a\}$. Due to the non-bubbling result, Corollary 4.3, the map g provides a monotone parametrization of the simple arc C. Hence, the length of the curve g is exactly the length of the geodesic γ. Since f_0 is length-minimizing, the length of f_0 cannot be larger than the length of g. Hence, $f_0 : E \to C$ is a geodesic. Since the edge E was arbitrary, this finishes the proof. □

It follows that any length-minimizing graph $f : \Gamma \to Y$ as in Proposition 4.6 is length-continuous. Any connected subset of Γ is arcwise connected. Therefore, the canonical map $\tau_f : \langle \Gamma \rangle_f \to |\Gamma|_f$ is a homeomorphism. Moreover, the space $\langle \Gamma \rangle_f$ is a finite union of geodesics intersecting only at their endpoints. Thus, $\langle \Gamma \rangle_f$ is finite topological graph with a geodesic metric. A vertex in $\langle \Gamma \rangle_f$ corresponds to connected subgraphs of Γ sent by f to a single point. Any edge $\hat{E}$ of $\langle \Gamma \rangle_f$ is the image of an edge E in Γ and the restricton $\hat{\pi}_f : E \to \hat{E}$ is monotone. The induced map $\hat{f} : \langle \Gamma \rangle_f \to Y$ sends any edge isometric onto a geodesic in Y. Finally, any curve in $\langle \Gamma \rangle_f$ lifts to a curve in Γ. By definition, the last statement implies:

Corollary 4.7 *Let $f : \Gamma \to Y$ be length-minimizing relative to A, as in Proposition 4.6. Then $\hat{f} : \langle \Gamma \rangle_f \to Y$ is length-minimizing relative to $\hat{\pi}_f(A)$.*

The following result is proved in detail in [27, Proposition 5.2] for $\kappa = 0$. Since the proof only requires the first variational inequality for distances [1, Inequality 6.7] and Reshetnyak's majorization theorem [1, Theorem 9.56] in spaces of directions, the proof applies literally in CAT(κ) spaces:

Lemma 4.8 *Let Y be* CAT(κ). *Let Γ be a finite geodesic graph and A be a subset of its vertices. Let $f : \Gamma \to Y$ be length-minimizing relative to A and assume that the restriction of f to any edge is an isometry.*

Let $p \in \Gamma \setminus A$ be a vertex. Let $\gamma_1, \ldots, \gamma_n$ be the images in Y of the edges in Γ starting in p and enumerated in an arbitrary order. Then the sum of the n consecutive angles satisfies

$$\measuredangle_p(\gamma_1, \gamma_2) + \ldots + \measuredangle_p(\gamma_{n-1}, \gamma_n) + \measuredangle_p(\gamma_n, \gamma_1) \geq 2\pi.$$

In particular, any vertex $p \in \Gamma \setminus A$ is contained in at least 2 edges of Γ.

4.3 Length-minimizing discs

A *length-minimizing disc* will denote a map $f : \mathbb{D} \to Y$ length-minimizing relative to $\mathbb{S}^1$.

Due to Corollary 4.3 and Lemma 3.4, for any length-connected and length-minimizing disc $f : \mathbb{D} \to Y$, the spaces $|\mathbb{D}|_f$ is a disc retract. If, in addition, f is length-continuous, then the map $\tau_f : \langle \mathbb{D} \rangle_f \to |\mathbb{D}|_f$ is a homeomorphism preserving the length of all curves.

Lemma 4.9 *Let $f : \mathbb{D} \to Y$ be length-connected and length-minimizing disc in a* CAT(κ) *space Y. Let G be a Jordan curve of length $l < 2 \cdot R_\kappa$ in the disc retract $|\mathbb{D}|_f$. Denote by J the closed disc bounded by G in $|\mathbb{D}|_f$. Then*

(1) The restriction $\bar{f} : J \to Y$ is a length-connected and length-minimizing disc.
(2) The image $\bar{f}(J)$ is contained in a ball of radius $r < \frac{R_\kappa}{2}$ in Y.
(3) If f is length-continuous then so is $\bar{f} : J \to Y$.

Proof The map $\bar{f}$ is 1-Lipschitz, hence length of the closed curve $\bar{f}(G)$ is less than $2 \cdot R_\kappa$. Reshetnyak's majorization theorem [1, Theorem 9.56] implies that $\bar{f}(G)$ is contained in a ball B of radius $r < \frac{R_\kappa}{2}$ in Y.

Denote by $\tilde{G}$ and $\tilde{J}$ the preimages of G and J in $\mathbb{D}$, respectively.

For any point $p \in J$ consider any preimage $\tilde{p} \in \tilde{J}$ of p. We find a curve γ in $\mathbb{D}$, which connects $\tilde{p}$ with a point in $\tilde{G}$ and such that $f \circ \gamma$ has finite length. By cutting the curve, if needed, we may assume that γ is contained in $\tilde{J}$. Then the projection $\bar{\gamma} = \bar{\pi}_f \circ \gamma$ is a curve connecting p to a point on G, such that $\bar{f} \circ \bar{\gamma}$ has finite length.

Since G has finite length and $\bar{f}$ preserves all lengths, $\langle p - q \rangle_{\bar{f}}$ is finite, for any $p \in J$ and any $q \in G$. By the triangle inequality, $\bar{f}$ is length-connected.

Assume now that $g : J \to Y$ satisfies $\bar{f} \trianglerighteq g$ (rel. G). Set

$$\tilde{g} := g \circ \bar{\pi}_f : \tilde{J} \to Y.$$

Then $\tilde{g}$ coincides with f on $\tilde{G}$ and $\tilde{J} \setminus \tilde{G}$ does not intersect the boundary $\mathbb{S}^1$. Due to Lemma 4.2, the restriction $f : \tilde{J} \to Y$ is length-minimizing relative to $\tilde{G}$.

The assumption $\bar{f} \trianglerighteq g$ (rel. G) implies $f|_{\tilde{J}} \trianglerighteq \tilde{g}$. Due to Lemma 4.4, $f(\tilde{J}) = \bar{f}(J)$ is contained in the ball B, proving (2).

Due to Lemma 4.5, f equals $\tilde{g}$ on $\tilde{J}$. Hence, $\bar{f} = g$. This proves (1).

Let, finally, f be length-continuous. By Lemma 3.4, $\langle X \rangle_f$ is a geodesic metric space and $\tau_f : \langle X \rangle_f \to |X|_f$ is a homeomorphism. Consider the disc $\hat{J} = \tau_f^{-1}(J) \subset \langle X \rangle_f$. Its boundary $\hat{G} = \tau_f^{-1}(G)$ has finite length, therefore, the length metric of the subset $\hat{J}$ of $\langle X \rangle_f$ induces the subset topology of $\hat{J}$, cf. [18, Lemma 2.1]. The restriction $\hat{f} : \hat{J} \to Y$ is 1-Lipschitz, if $\hat{J}$ is equipped with its length metric. Hence, $\hat{f} : \hat{J} \to Y$, and therefore also $\bar{f} : J \to Y$, is length-continuous. □

5 An existence result

The following result is essentially [27, Proposition 5.1].

Lemma 5.1 *Let X be a Peano continuum, Z a closed subset of X and Y a* CAT(κ) *space. Let the map $f : X \to Y$ be length-continuous. If $f(Z)$ is a contained in a closed ball $B \subset Y$ of radius $\leq \frac{R_\kappa}{2}$ then there exists a map $g : X \to Y$, length-minimizing relative to Z, with $f \trianglerighteq g$ (rel. Z).*

Proof We consider the set $\mathcal{F}$ of all maps $h : X \to Y$ which coincide with f on Z and satisfy $f \trianglerighteq h$ (rel. Z). We are looking for a minimal element in the set $\mathcal{F}$ with the partial ordering given by $\trianglerighteq$ (rel. Z).

By [12, Theorem 1.1] there exists a 1-Lipschitz retraction $\Phi : Y \to B$. Hence, for any $h \in \mathcal{F}$, we have $\Phi \circ h \in \mathcal{F}$ and $h \trianglerighteq \Phi \circ h$.

We replace $\mathcal{F}$ by its subset $\mathcal{F}_0$ of all maps $h \in \mathcal{F}$ whose image is contained in the ball B. The existence of the retraction Φ implies that an element g in $\mathcal{F}_0$ minimal with respect to the partial ordering $\trianglerighteq$ (rel. Z) will be length-minimizing.

In order to find such a minimal element in $\mathcal{F}_0$ we apply Zorn's lemma. It thus suffices to show that for any totally ordered subset $\mathcal{T}$ of $\mathcal{F}_0$ there exists an element $g \in \mathcal{F}_0$ with $h \trianglerighteq g$ (rel. Z) for all $h \in \mathcal{T}$.

For any $h \in \mathcal{T}$, the identity $id : Y \to Y$ induces a 1-Lipschitz map $\mu : \langle Y \rangle_f \to \langle Y \rangle_h$. The map h then factorizes as $h = \hat{h} \circ \mu \circ \hat{\pi}_f$. Since the maps $\hat{h}$ and μ are 1-Lipschitz and $\hat{\pi}_f$ is continuous by assumption, the family $\mathcal{T}$ is equicontinuous.

Fix a dense sequence (x_n) in the compact space X. For any pair (n, m), set

$$d_{m,n} := \inf\{\langle x_n - x_m \rangle_h \ \mid \ h \in \mathcal{T}\}.$$

For any triple of natural numbers (n, m, k) we find some $h = h_{k,n,m}$ in $\mathcal{T}$ such that

$$d_{m,n} + \frac{1}{k} > \langle x_n - x_m \rangle_h.$$

We enumerate the set of all triples and use a diagonal sequence argument in order to find a sequence $h_1 \trianglerighteq h_2 \trianglerighteq h_3 \trianglerighteq \ldots$ of elements in $\mathcal{T}$ such that for any x_n, x_m

$$\lim_{i \to \infty} \langle x_n - x_m \rangle_{h_i} = d_{m,n}.$$

If Y is proper, hence B compact, we take a convergent subsequence of the equi-continuous sequence h_i and obtain a uniform limit $g := \lim_i h_i : X \to B$.

If Y is non-proper, we take the ultralimit

$$h_\omega = \lim_\omega h_i : X \to B^\omega$$

from Y to the ultracompletion B^ω of B. Then we apply [12, Theorem 1.1] again and find a 1-Lipschitz retraction $\Psi : B^\omega \to B$. Set now

$$g := \Psi \circ h_\omega.$$

Since all h_i agree with f on Z, so does g. Since length of curves is semi-continuous under Gromov–Hausdorff convergence and under ultraconvergence, we have

$$\lim \ell_Y(h_i \circ \gamma) \geq \ell_{B^\omega}(h_\omega \circ \gamma) \geq \ell_B(g \circ \gamma),$$

for any curve γ in X. In particular, g is contained in $\mathcal{F}_0$. Moreover, for any n, m, i

$$d_{m,n} \leq \langle x_n - x_m \rangle_g \leq \langle x_n - x_m \rangle_{h_i}.$$

Thus, equality holds on the left. By the assumption on h_i, for any $h \in \mathcal{T}$,

$$\langle x_n - x_m \rangle_h \geq \langle x_n - x_m \rangle_g$$

Since the sequence x_m is dense in X and, for all $h \in \mathcal{T}$, the distance $\langle * - * \rangle_h$ is continuous on X, we deduce

$$\langle x - z \rangle_h \geq \langle x - z \rangle_g,$$

for any pair of points $x, z \in X$ and any $h \in \mathcal{T}$.

Thus, $h \trianglerighteq g$ (rel. Z), for all $h \in \mathcal{T}$. This finishes the proof. $\square$

The CAT(κ)-assumption and the assumption that the image is contained in a small ball is only used for the existence of a 1-Lipschitz retraction $\Pi : Y^\omega \to Y$. Thus, in the proper setting the proof provides the following result, which we state for the sake of completeness:

Proposition 5.2 *Let X be a Peano space, $Z \subset X$ closed. Let $f : X \to Y$ be a length-continuous map into a proper metric space Y. Then there exists a length-continuous map $g : X \to Y$, which is length-minimizing relative to Z and satisfies $f \trianglerighteq g$ (rel Z).*

The first part of the proof of Proposition 4.6, shows that for any finite graph Γ and any continuous map $f : \Gamma \to Y$ into a geodesic space Y there exists a map $g : \Gamma \to Y$ such that $f \trianglerighteq g$ and such that the restriction of g to any edge is a geodesic in Y. Since any such g is length-continuous, we deduce from Proposition 5.1

Corollary 5.3 *Let Γ be a finite graph and $A \subset \Gamma$ a set of vertices. Let B be a ball of radius $\leq \frac{R_\kappa}{2}$ in a CAT(κ) space Y. For any continuous map $f : \Gamma \to B$, there exists a length-minimizing map $h : \Gamma \to B$ relative to A such that $f \trianglerighteq h$.*

6 Main result

The following proposition is the technical heart of all results in this paper. This is a slight generalization of [27, Key-Lemma 6.2]. The proof follows [27] filling some details and some additional arguments in positive curvature.

Proposition 6.1 *Let Γ be a finite graph, which is the 1-skeleton of a triangulation of the disc $\mathbb{D}$. Let $\mathcal{V}$ be the set of vertices of Γ and $A := \mathcal{V} \cap \mathbb{S}^1$. Let B be a ball of radius $< \frac{R_\kappa}{2}$ in some CAT(κ) space Y. Let $f : \Gamma \to B$ be continuous. Let $0 < \varepsilon < \frac{R_\kappa}{2}$ be such that, for the vertices x_1, x_2, x_3 of any triangle T in the triangulation of $\mathbb{D}$, the distances satisfy $\langle f(x_i) - f(x_j)\rangle_Y \leq \varepsilon$.*

Then there exists some CAT(κ) disc retract W, a 1-Lipschitz map $q : W \to Y$ and a map $p : \mathbb{D} \to W$ with the following properties

(1) All fibers of p are contractible and $p(\mathcal{V})$ is ε-dense in W.
(2) $q \circ p|_A = f|_A$.
(3) For every curve γ in Γ, we have $\ell_Y(f \circ \gamma) \geq \ell_W(p \circ \gamma) = \ell_Y(q \circ p \circ \gamma)$.

Proof We may replace Y by B and assume $Y = B$. Due to Corollary 5.3, we find a map $h : \Gamma \to Y$ length-minimizing relative to A such that $f \trianglerighteq h$ (rel. A).

Consider the induced length-metric space $\langle\Gamma\rangle_h$, which is a geodesic graph. Due to Corollary 4.7, the map $\hat{h} : \langle\Gamma\rangle_h \to B$ is length-minimizing relative to $\hat{\pi}_h(A)$. By Proposition 4.6, the restriction of $\hat{h}$ to any edge of $\langle\Gamma\rangle_h$ is an isometry and the projection $\hat{\pi}_h : \Gamma \to \langle\Gamma\rangle_h$ is monotone. Thus, the conclusions of Lemma 4.8 are valid for the map $\hat{h} : \langle\Gamma\rangle_h \to Y$.

For any triangle $T \subset \mathbb{D}$ bounded by 3 edges of Γ with vertices x_1, x_2, x_3 we consider the unique, possibly degenerated triangle T^κ in the surface M^2_κ of constant curvature κ, such that the sides of T^κ have distances equal to

$$\langle x_i - x_j\rangle_{\langle\Gamma\rangle_h} = \langle h(x_i) - h(x_j)\rangle_Y \leq \varepsilon.$$

We glue all these triangles T^κ (possibly degenerated to a vertex or to an edge) together in the same way as the corresponding triangles T are glued in $\mathbb{D}$. We denote the obtained metric space by W.

The metric space W is glued from triangles of curvature κ and, possibly, some edges, cf. [1, definition 12.1]. The gluing maps are simplicial. A priori, edges or vertices of a single triangle T may be identified in W.

We have a tautological length-preserving map $\iota : \langle\Gamma\rangle_h \to W$. Some caution is in order: two non-degenerate edges in a triangle whose third side degenerates to a point are sent by ι to the same curve in W, hence ι may be non-injective. However, all maps defined below respect the identifications on $\langle\Gamma\rangle_h$ given by ι. Thus, we will identify $\langle\Gamma\rangle_h$ with $\iota(\langle\Gamma\rangle_h)$ below.

For any triangle $T \subset \mathbb{D}$, the restriction of $\hat{\pi}_h : \Gamma \to \langle\Gamma\rangle_h$ to ∂T defines a map $f_T : \partial T \to \partial T^\kappa$. We claim that f_T admits an extension to a map $\tilde{f}_T : T \to T^\kappa$, such that any fiber of $\tilde{f}_T$ is topologically a point, an interval or a disc.

If T^κ degenerates to a point $w \in W$ we choose the constant map $\tilde{f}_T \to \{w\}$.

If T^κ degenerates to an edge E, the map f_T sends one side of ∂T to a point and the two other edges monotonically to E. Identifying T with a Euclidean triangle, we then obtain a unique extension $\tilde{f}_T : T \to E$ of f_T with all fibers convex subsets. This finishes the construction in the degenerated case.

If none of the three sides of T^κ degenerates to a point, T^κ is a topological disc with boundary ∂T^κ. Identifying the two triangles T and T^κ with the disc and coning the map f_T, we obtain an extension map $\tilde{f}_T : T \to T^\kappa$ of f_T. The preimages of points under the map $\tilde{f}_T$ are (possibly degenerated) compact intervals.

All the maps $\tilde{f}_T$ constructed above coincide on common edges and points of different intersecting triangles T. Thus, all $\tilde{f}_T$ glue together to a continuous map $p : \mathbb{D} \to W$. By construction, the preimage $P := p^{-1}(w)$ of any point $w \in W$ is a Peano continuum and the intersection of P with any triangle T is contractible.

We claim that any such preimage $P = p^{-1}(w)$ is contractible. Otherwise, we would find a non-contractible circle S in P. Denote by O the open Jordan domain of S in $\mathbb{D}$. Since the intersection of P with each triangle T is empty or contractible, S is not contained in any triangle. Hence, O has a non-empty intersection with Γ. The open disc O does not contain points in A. By Corollary 4.3, applied to the length-minimizing map h, the subset $O \cap \Gamma$ must be contained in $h^{-1}(w)$, hence in P. Since any triangle intersects P in an empty or in a contractible set, we deduce $T \cap O \subset P$, for all triangles T, with $T \cap O \neq \emptyset$. Therefore, $O \subset P$. This contradicts the non-contractibility of S.

Hence, p has contractible fibers. Therefore, W is a disc-retract [27, Lemma 3.3]. In particular, W is contractible.

The diameter of any triangle T^κ equals to the maximum of the distances between its vertices. Hence, this diameter is at most ε, by assumption. Therefore, the vertices of T^κ are ε-dense in T^κ and $p(\mathcal{V})$ is ε-dense in W.

For any edge E in $\langle\Gamma\rangle_h$, the map $\hat{h} : E \to Y$ is an isometric embedding. For any degenerated triangle $T^\kappa \subset W$, this gives an isometric embedding $\tilde{h}_T := \hat{h} : T^\kappa \to Y$.

If T^κ is non-degenerated, then the boundary ∂T^κ is the *comparison triangle* of the triangle $\hat{h}(\partial T^\kappa)$ in Y. Reshetnyak's majorization theorem, [1, Theorem 9.56, Proposition 9.54] implies that $\hat{h}$ extends to a 1-Lipschitz map $\tilde{h}_T : T^\kappa \to Y$.

The maps $\tilde{h}_T$ coincide on common edges and vertices, where they are given by $\hat{h}$. Hence, these maps glue together to a 1-Lipschitz map $q : W \to Y$, whose restriction to $\langle\Gamma\rangle_h \subset W$ is $\hat{h}$.

By construction, $q \circ p|_A = h|_A = f|_A$. For every curve γ in Γ, we have

$$\ell_Y(f \circ \gamma) \geq \ell_Y(h \circ \gamma) = \ell_W(p \circ \gamma).$$

It remains to show that the space W is CAT(κ). By construction, W is obtained by gluing together (simplically) some number of triangles of constant curvature κ. Since the space W

is a disc retract, the space of directions $\Sigma_w W$ at any point $w \in W$ topologically embeds into a circle. Hence, $\Sigma_w W$ is topologically either a circle or a disjoint union of (possibly degenerated) intervals.

If W is not locally CAT(κ), we find a point $w \in W$ and a closed local geodesic of length less than 2π in $\Sigma_w W$ [1, Theorem 12.2]. Then the whole space $\Sigma_w W$ must be a circle of length less than 2π. Hence, w must be a vertex of $\langle\Gamma\rangle_h$ and the cyclic sum of angles α_j (in triangles T_j^κ) adjacent to w is less than 2π.

By construction and angle comparison, the angles in T_κ^j are not smaller than the corresponding angles in $\hat{h}(\partial T_i^\kappa) \subset Y$.

Thus, the cyclic sum of all angles $\angle_{\hat{h}(w)}(\hat{h}(e_i), \hat{h}(e_{i+1}))$ in Y is less than 2π, where e_i run over all edges in $\langle\Gamma\rangle_h$ adjacent to w in $\langle\Gamma\rangle_h$. This contradicts the length-minimality of $\hat{h}$ and Lemma 4.8.

Therefore, the space W is locally CAT(κ). Since W is simply connected, we may apply the Cartan–Hadamard theorem, [1, Theorem 9.6], and deduce that W is CAT(κ), if $\kappa < 0$. The case $\kappa > 0$ requires an additional argument.

Thus, we assume now $\kappa > 0$.

The fact that the total angle at any interior point is at least 2π implies that no two sides of a non-degenerate triangle T^κ are identified in W. Therefore, the triangles T^κ define a triangulation of W with 1-skeleton $\iota(\langle\Gamma\rangle_h)$.

If W is not CAT(κ), we find an isometrically embedded circle S of length $< 2R_\kappa$ in W [6, Theorem 2.2.11]. Moreover, we find and fix such a circle with the smallest possible length. Upon rescaling we may assume that S has length 2π. Then $\kappa < 1$. Since there are no closed geodesics of length $< 2\pi$, the space W is CAT(1) and locally CAT(κ).

Consider the subdisc W_0 of W bounded by S, which is a convex subset of W. We are going to construct a 1-Lipschitz map $\tilde{q} : W_0 \to Y$, which equals q on S and such that $\tilde{q}$ shortens the length of some curve $\gamma \subset \langle\Gamma\rangle_h \cap W_0$. Once $\tilde{q}$ is constructed, the map

$$j : \langle\Gamma\rangle_h \to Y$$

given by q on $\langle\Gamma\rangle_h \setminus W_0$ and by $\tilde{q}$ on W_0 would satisfy $\hat{h} \trianglerighteq j$ (rel. A) and contradict the length-minimality of $\hat{h}$.

It remains to construct the 1-Lipschitz $\tilde{q} : W_0 \to Y$, which shortens the length of some curve in $\langle\Gamma\rangle_h \cap W_0$.

Denote by H the hemisphere of curvature 1 with pole o. Glue H to W_0 identifying S with the boundary of the hemisphere H. The arising space

$$W_1 := W_0 \cup_S H$$

is CAT(1) by Reshetnyak's gluing theorem. There is a canonical 1-Lipschtitz retraction $\Pi : W_1 \to H$, defined as follows (compare [10, Lemma 1.3] [12]):

- $\Pi(x) = x$ if $|o - x|_{W_1} \leq \frac{\pi}{2}$.
- $\Pi(x) = o$ is $|o - x|_{W_1} \geq \pi$.
- $\Pi(x) = \eta_x(\pi - |o - x|_{W_1})$ if $\pi > |o - x|_{W_1} > \frac{\pi}{2}$, where η_x is the geodesic from o to x in W_1 parametrized by arclength.

The hemisphere H has constant curvature $1 > \kappa$. We apply the Kirzsbraun–Lang–Schroeder extension theorem, [13], [1, Theorem 10.14], and find a 1-Lipschitz map $g : H \to Y$ which extends the 1-Lipschitz map $q : S \to Y$.

We define the 1-Lipschitz map $\tilde{q} : W_0 \to Y$ as the composition $\tilde{q} = g \circ \Pi$. Then $\tilde{q}$ coincides with q on S. It remains to find a curve in $G := \langle \Gamma \rangle_h \subset W_0$, whose length is strictly contracted by $\tilde{q}$.

Assume on the contrary, that $\tilde{q}$ preserves the length of all curves in G. The closed geodesic S cannot be contained in a single triangle T^κ nor can it intersect twice the same side of the same triangle T^κ. It follows that G together with S constitute the 1-skeleton of a triangulation of W_0.

By assumption, $\tilde{q}$ is 1-Lipschitz on G. On the other hand $\hat{h}$ is length-preserving and length-minimizing on G relative to $G \cap S$. Therefore, $\tilde{q}$ is length-minimizing on G relative to $G \cap S$ as well. Hence, $\tilde{q}$ is an isometry on every edge of the graph G. Since g is 1-Lipschitz, we deduce that the 1-Lipschitz map $\Pi : W_0 \to H$ restricts to an isometry to every edge of G.

We find a triangle $T_0 = xpz$ in the triangulation of W_0 defined by $G \cup S$, such that one side xz of T_0 lies on S. The definition of the map Π and the equality

$$|p - x|_{W_0} = |p - x|_{W_1} = |\Pi(p) - \Pi(x)|_H$$

imply that $|p - x|_{W_0} \leq \frac{\pi}{2}$ and that the triangle opx is of constant curvature 1. Since W_0 is locally CAT(κ), it implies that the triangle opx is degenerated. Hence px meets S at x orthogonally. The same is true for the geodesic pz.

Since W_0 is CAT(1), we deduce that p has distance $\frac{\pi}{2}$ to x and to z and that the triangle pxz has constant curvature 1. Since W_0 is locally CAT(κ), this implies that the triangle T_0 is degenerated, which is impossible.

This contradiction shows that W is CAT(κ) and finishes the proof. □

We are now in position to provide

Proof of Theorem 1.1 Thus, let Y be CAT(κ) and $f : \mathbb{D} \to Y$ be length-continuous and length-minimizing relative to $\mathbb{S}^1$. We need to prove that $\langle \mathbb{D} \rangle_f$ is CAT(κ).

Due to Lemma 3.4, $\langle \mathbb{D} \rangle_f$ is a disc retract. It is sufficient to prove that any Jordan triangle G (thus a Jordan curve, built by 3 geodesics) of length $< 2R_\kappa$ in $\langle \mathbb{D} \rangle_f$ is not thicker than its comparison triangle in M_κ^2. We fix G and denote by J the closed disc bounded by G in the disc retract $\langle \mathbb{D} \rangle_f$.

It is sufficient to prove that J with its intrinsic metric is CAT(κ). Since the boundary G of J has finite length, J is a geodesic metric space in its intrinsic metric.

The restriction $\hat{f} : J \to Y$ is length-minimizing relative to G and its image is contained in a ball $B \subset Y$ of radius $r < \frac{R_\kappa}{2}$, by Lemma 4.9. Moreover, $\hat{f} : J \to Y$ is a light map, which preserves the length of all curves in J, by Lemma 3.4 and Lemma 3.2.

We thus have a geodesic metric space J homeomorphic to a closed disc, whose boundary G is a geodesic triangle of perimeter less than $2R_\kappa$. Upon renaming B into Y, we have a light map $\hat{f} : J \to Y$ into a CAT(κ) space Y which is contained in a ball of radius less than $\frac{R_\kappa}{2}$ around some of its points. The map $\hat{f}$ preserves the length of all curves in J and $\hat{f}$ is length-minimizing relative to G. We need to verify that J is CAT(κ).

For any natural n we find a finite, connected, piecewise geodesic graph $\tilde{\Gamma}_n$ in J, which contains G and satisfies the following conditons, see [7, Theorem 1.2] and [22, Proposition 5.2]:

The boundary of any connected component T_0 of $J \setminus \tilde{\Gamma}_n$ is a Jordan triangle and the closure T of T_0 has diameter less than $\frac{1}{n}$. The embedding of $\tilde{\Gamma}_n$ with its intrinsic metric into J is a $\frac{1}{n}$-isometry, thus,

$$|x - z|_{\tilde{\Gamma}_n} \leq |x - z|_J + \frac{1}{n},$$

for all $x, z \in \tilde{\Gamma}_n$.

Note that the graph $\tilde{\Gamma}_n$ provided by [22, Proposition 5.2] does not need to be the 1-skeleton of a triangulation of J: A vertex of one triangle may lie on the side of another triangle.

We refine $\tilde{\Gamma}_n$ by adding one vertex o_T inside of each triangle T defined by $\tilde{\Gamma}_n$ and by connecting o_T by pairwise disjoint curves inside T to all vertices of $\tilde{\Gamma}_n$ on the sides of T. Denote the arising graph by Γ_n.

By construction, Γ_n defines a triangulation of J and each triangle of the triangulation has diameter at most $\frac{1}{n}$.

We now apply Proposition 6.1 and obtain a CAT(κ) disc retract W_n, maps $p_n : J \to W_n$ and $q_n : W_n \to Y$ with the properties (1), (2), (3) stated there, for $\varepsilon = \frac{1}{n}$ and $f = \hat{f}$.

Hence, p_n is surjective and has contractible fibers, and q_n is 1-Lipschitz.

Denote by $\mathcal{V}_n$ the vertices of the subgraph $\tilde{\Gamma}_n$ of Γ_n. Any vertex of Γ_n lies at distance at most $\frac{1}{n}$ to some vertex of $\tilde{\Gamma}_n$. Since $\hat{f}$ is length-preserving and therefore 1-Lipschitz, this distance estimate holds for the images of the vertices in Y and, therefore for vertices of the triangulation of W_n. Together with the property (2), this implies that $p_n(\mathcal{V}_n)$ is $\frac{2}{n}$-dense in W_n.

For every pair of points $x, z \in \mathcal{V}_n$, we choose a curve $\gamma \subset \Gamma_n$ realizing the distance between x and z in Γ_n. Then

$$|x - z|_J + \frac{1}{n} \geq \ell_J(\gamma) \geq \ell_Y(\hat{f} \circ \gamma) \geq \ell_{W_n}(p_n \circ \gamma) \geq |p_n(x) - p_n(z)|_{W_n}.$$

After choosing a subsequence we obtain a Gromov–Hausdorff converging sequence of images $p_n(\Gamma_n)$ to a compact metric space W. Moreover, the spaces W_n converge to the space W as well and the maps p_n converge to a surjective 1-Lipschitz map $p : J \to W$. Since the spaces W_n are CAT(κ), the limit space W is a CAT(κ) space as well.

Set $A_n := \mathcal{V}_n \cap G$. Then A_n is $\frac{1}{n}$-dense in G and, by Proposition 6.1, the restriction of $q_n \circ p_n$ to A_n coincides with $\hat{f}|_{A_n}$. In the limit we obtain

$$\hat{f}|_G = q \circ p|_G.$$

By assumption, $\hat{f}$ preserves the length of all curves. Since $q \circ p$ is 1-Lipschitz, we deduce $\hat{f} \trianglerighteq q \circ p$ (rel. G). By assumption, $\hat{f}$ is length-minimizing relative to G, hence $\hat{f} = q \circ p$, by Lemma 4.5. Since p and q are 1-Lipschitz, the map p must preserve the length of any curve.

Since $\hat{f}$ is a light map, also p must be a light map. The fibers of p_n converge to subsets of the corresponding fibers of p. The fibers of p_n are connected, hence any limit set is connected as well. Since all fibers of p are totally disconnected, we deduce that any limit of fibers $p_n^{-1}(w_n)$ must be a singleton. Denoting by ε_n the maximal diameter of fibers of p_n, we deduce $\lim \varepsilon_n = 0$.

We claim that p is injective. Otherwise, we find some $x \neq z \in p^{-1}(w)$. Then $|p_n(x) - p_n(z)|_{W_n}$ converge to 0. Consider the geodesic e_n between $p_n(x)$ and $p_n(z)$ in W_n and set $E_n := p_n^{-1}(e_n)$. Since p_n has connected fibers, E_n is a connected subset of J.

Taking a subsequence, we obtain in the limit a connected subset E of J, which contains x and z and is sent by p to the point w. Since p is a light map, this is a contradiction.

Therefore, $p : J \to W$ is injective, hence bijective. Since J is compact, p is a homeomorphism. Since p preserves the length of any curve, p is an isometry and $J = W$ is a CAT(κ) space. □

7 Ruled discs

We are going to prove Corollary 1.3 in this section. Thus, let $\eta_0, \eta_1 : [0,1] \to Y$ be rectifiable curves in a CAT(κ) space Y. For $a \in [0,1]$, let $\gamma_a(t) : [0,1] \to Y$ be geodesics of length $< R_\kappa$ parametrized proportionally to arclength and connecting $\eta_0(a)$ with $\eta_1(a)$.

We consider the square $\mathcal{Q} := [0,1] \times [0,1]$ as a topological disc. We define $f : \mathcal{Q} \to Y$ by $f(a,t) := \gamma_a(t)$. We need to verify that $\langle \mathcal{Q} \rangle_f$ is CAT(κ).

Since geodesics of length $< R_\kappa$ depend continuously on the endpoints, we deduce that f is continuous. By continuity, we find some $\delta > 0$, such that all geodesics γ_a have length at most $R_\kappa - \delta$. By the quadrangle comparison, we find some $L, \rho > 0$ with the following property:

Whenever $|\eta_i(a) - \eta_i(b)|_Y \le \rho$, for $i = 0, 1$, then, for all $t \in [0,1]$:

$$|\gamma_a(t) - \gamma_b(t)|_Y \le L \cdot \big(|\eta_0(a) - \eta_0(b)|_Y + |\eta_1(a) - \eta_1(b)|_Y\big).$$

Therefore, for any $s \in [0,1]$, the curve $\eta_s(t) := f(t,s)$ is of finite length, bounded from above by $L \cdot (\ell_Y(\eta_0) + \ell_Y(\eta_1))$. Moreover, once η_0, η_1 have length at most r on an interval $[a,b] \subset [0,1]$, then any of the horizontal curves η_t, $t \in [0,1]$ has length less than $2 \cdot L \cdot r$.

Hence, any point $(a,t) \in \mathcal{Q}$ sufficiently close to a given point (a_0, t_0) can be connected to (a_0, t_0) by a concatenation of a vertical and horizontal segments in $\mathcal{Q}$, which is mapped to a curve of a small length. Therefore, f is length-continuous.

Therefore, Corollary 1.3 is a direct consequence of Theorem 1.1 and the following:

Lemma 7.1 *The map $f : \mathcal{Q} \to Y$ is length-minimizing relative to $\partial \mathcal{Q}$.*

Proof Let a continuous map $g : \mathcal{Q} \to Y$ satisfy $f \trianglerighteq g$ (rel. $\partial \mathcal{Q}$). For the vertical segment $t \to (a,t) \in \mathcal{Q}$, the image curve $t \to g(a,t)$ has length not larger than the geodesic γ_a and connects the same boundary points.

Hence $t \to g(a,t)$ coincides with γ_a up to parametrization. Since also

$$\ell_Y(f \circ \gamma) \ge \ell_Y(g \circ \gamma)$$

for all subsegments γ of the segment $t \to (a,t)$, parametrizations must coincide, hence $f(a,t) = g(a,t)$, for all a and t.

Thus, $f \trianglerighteq g$ (rel. $\partial \mathcal{Q}$) implies $f = g$. Hence f is length-minimizing. □

8 Harmonic disks

8.1 Basics on Sobolev discs

In this section, we assume some knowledge on Sobolev and harmonic maps with values in metric spaces. We refer the reader to [9, 11, 15, 29], for introductions to this subject.

Throughout the section let Y be a CAT(κ) space and Ω be a bounded, open subset of $\mathbb{R}^2$. Any Sobolev map $f \in W^{1,2}(\Omega, Y)$ has a (Korevaar–Schoen) energy $E(f) \in [0, \infty)$ [11], [15, Proposition 4.6].

If Ω is a Lipschitz domain, hence the boundary $\partial\Omega$ is a union of Lipschitz curves, then for any Sobolev map $f \in W^{1,2}(\Omega, Y)$ there is a *trace* of f, $tr(f) \in W^{1,2}(\partial\Omega, Y)$, [11, Theorem 1.12.2]. If $f : \bar{\Omega} \to Y$ is continuous, then the trace is just the restriction of f to the boundary $\partial\Omega$.

If $\partial\Omega$ is not Lipschitz we say that $f, g \in W^{1,2}(\Omega, Y)$ have *equal traces*, if $\langle f - g \rangle_Y$ is contained in $W_0^{1,2}(\Omega, \mathbb{R})$, see [29, Sect. 1.4], [16, Sect. 4.1].

A map $f \in W^{1,2}(\Omega, Y)$ is called *harmonic* if f has the smallest energy among all maps $g \in W^{1,2}(\Omega, Y)$ with the same trace as f.

For any $f \in W^{1,2}(\Omega, B)$, where B is a ball in Y of radius $< \frac{R_\kappa}{2}$, there exists a unique harmonic map $g \in W^{1,2}(\Omega, Y)$ with the same trace as f, [29, Theorem 1.16]. Moreover, the image of this harmonic g is contained in B as well and g has a (unique) locally Lipschitz representative [29, Theorem 3.1]. We will always use this representative below. Finally, the restriction of a harmonic map to any subdomain $\tilde{\Omega} \subset \Omega$ is harmonic as well, [16, Lemma 4.2].

Assume that $f, g : \bar{\Omega} \to Y$ are continuous and that $f \in W^{1,2}(\Omega, Y)$. Assume further that $f \trianglerighteq g$. Then $\ell_Y(f \circ \gamma) \geq \ell_Y(g \circ \gamma)$, for any curve γ in Ω. Using the upper-gradient definition of Sobolev maps [9] this implies that $g \in W^{1,2}(\Omega, Y)$ as well. Moreover, at almost all points of Ω the *approximate metric differential* of f is not less than the corresponding *approximate metric differential* of g, [15, Proposition 4.10]. Then also the energies satisfy $E(f) \geq E(g)$, [15, Proposition 4.6]. In particular, if f is harmonic then g is harmonic as well.

If $f \in W^{1,2}(\mathbb{D}, Y)$ is harmonic and $tr(f) : \mathbb{S}^1 \to Y$ is continuous, then f is continuous on the closed disc $\mathbb{D}$, [14, Proposition 4.4]. The following Lemma is essentially contained in the proof of [14, Proposition 4.4].

Lemma 8.1 *Let $f : \mathbb{D} \to Y$ be a harmonic map such the trace $f : \mathbb{S}^1 \to Y$ is a curve of finite length. Then f is length-continuous.*

Proof The map f is locally Lipschitz on $\mathbb{D}_0 := \mathbb{D} \setminus \mathbb{S}^1$. Hence, $\hat{\pi}_f : \mathbb{D} \to \langle \mathbb{D} \rangle_f$ is continuous on $\mathbb{D}_0$.

Let $z \in \mathbb{S}^1$ and $\varepsilon > 0$ be arbitrary. We find an arbitrary small ball $B_r(z)$ such that the lengths of the two curves $f(\partial B_r(z) \cap \mathbb{D})$ and $f(\mathbb{S}^1 \cap B_r(z))$ are smaller than ε. Moreover, if r is small enough, the energy of the restriction of f to $B_0 := B_r(z) \cap \mathbb{D}$ is also smaller than ε.

Hence $f : B_0 \to Y$ is a continuous, harmonic and of energy smaller than ε. We claim that there exists some $\delta(\varepsilon) > 0$ which goes to 0 with ε, such that the following holds true: Any point $p \in B_0$ can be connected by a curve γ with a point on ∂B_0 such that $\ell_Y(f \circ \gamma)$ is at most δ.

Once the claim is verified, we can use the fact that the restriction of f to ∂B_0 has length $< \varepsilon$ to deduce, for every $p \in B_0$,

$$\langle z - p \rangle_f \leq \delta + \varepsilon.$$

The right hand side goes to 0 as r converges to 0. Hence the map $\hat{\pi}_f$ would be continuous at z. Thus, $\hat{\pi}_f$ would continuous on all of $\mathbb{D}$.

It remains to verify the claim above. Since precomposing with a conformal diffeomorphism between domains in $\mathbb{C}$ does not change the energy (and the induced intrinsic distances), we use the Riemann mapping theorem and may replace B_0 by $\mathbb{D}$ and p by the origin $0 \in \mathbb{D}$.

We denote by $\eta_\theta(t) = t \cdot \theta$ the radial segment in the direction $\theta \in \mathbb{S}^1$, which connects 0 with the point $\theta \in \mathbb{S}^1 = \partial\mathbb{D}$. Denote by η_θ^+ the first half of the segment connecting 0 with $\frac{1}{2}\theta$ and by η_θ^- the second half of η_θ.

The restriction of f to the smaller ball $B_{\frac{1}{2}}(0)$ is $L \cdot \varepsilon$-Lipschitz, where L is some uniform constant, see [29, Theorem 2.2]. Hence, for any segment η_θ^+, the length of $f \circ \eta_\theta^+$ is at most $\frac{1}{2} \cdot L \cdot \cdot \varepsilon$.

On the other hand, integrating the energy in polar coordinates, we find that the length of $f \circ \eta_\theta^-$ is at most $\sqrt{\varepsilon}$, for at least one $\theta \in \mathbb{S}^1$. Indeed, denoting the absolute gradient of u by

$\rho \in L^2(\mathbb{D})$, we obtain:

$$\varepsilon = E(u) \geq \frac{1}{2}\int_{\mathbb{D}} \rho^2 \geq \frac{1}{4}\int_{\theta\in\mathbb{S}^1}\left(\int_{\eta_\theta^-}\rho^2\right)d\theta \geq \frac{1}{4}\int_{\mathbb{S}^1}\ell_Y^2(f\circ\eta_\theta^-)\,d\theta.$$

Thus, for this θ, the length of $f \circ \eta_\theta$ is at most $\frac{1}{2}\cdot L\cdot\varepsilon + \sqrt{\varepsilon}$. This finishes the proof of the claim and of the lemma. □

It is possible to prove (but requires some rather technical considerations in the case of positive κ) that any harmonic disc $f : \mathbb{D} \to Y$ is length-minimizing. We arrive at the proof of Corollary 1.2 faster restricting the domain of definition:

Proof of Corollary 1.2 Thus, let $f : \mathbb{D} \to Y$ be continuous and harmonic and such that $f(\mathbb{S}^1)$ has finite length. If, for some $y \in Y$, the set $\mathbb{D} \setminus f^{-1}(y)$ has a connected component O not intersecting $\mathbb{S}^1$, then the restriction of f to this component O could not be harmonic. Hence, this is impossible. Therefore, the geodesic space $\langle\mathbb{D}\rangle_f$ is a disc retract by Lemma 8.1 and Lemma 3.4.

Let G be a Jordan triangle in $\langle\mathbb{D}\rangle_f$ of length less than $2R_\kappa$ and let J be the closed disc bounded by G in the disc retract $\langle\mathbb{D}\rangle_f$. As in the proof of Theorem 1.1, it suffices to show that $\langle J\rangle_{\hat{f}}$ with the induced length metric is CAT(κ). Due to Theorem 1.1 it suffices to prove that $\hat{f} : J \to Y$ is length-minimizing relative to G.

Let $\tilde{G}$ be the preimage of G in $\mathbb{D}$ and let $O \subset \mathbb{D}$ be the preimage of $J \setminus G$. Then $f(\tilde{G}) = \hat{f}(G)$ is a curve of length less than $2R_\kappa$, hence it is contained in a ball B of radius less than $\frac{R_\kappa}{2}$ in Y. Applying a strictly 1-Lipschitz retraction to B, [12], we deduce that the image of the harmonic map $f : O \to Y$ is contained in B.

Assume now that $\hat{f}$ is not length-minimizing relative to G and let $g : J \to Y$ with $\hat{f} \trianglerighteq g$ (rel. G) be given. Then the map $\tilde{g} := g \circ \hat{\pi}_f : \bar{O} \to Y$ satisfies $f \trianglerighteq \tilde{g}$ (rel. $\tilde{G}$). As seen above, $\tilde{g}$ is harmonic on O. By the uniqueness of harmonic maps with values in B, we deduce $\tilde{g} = f$. Hence $\hat{f} = g$. This finishes the proof of the fact that $\hat{f}$ is length-minimizing and of the Corollary. □

9 Questions

Here is the promised list of (mostly technical) questions.

Question 9.1 *What does it mean in differential-geometric terms for a smooth map $f : \mathbb{D} \to M$ into a Riemannian manifold M to be length-minimizing?*

See [27, Sect. 10] for partial answers to this question.

Question 9.2 *Does our main theorem hold true for length-minimizing discs which are length-connected but not length-continuous?*

This question was the essential motivation for the definition of metric-minimality created in [27].

Question 9.3 *Does the conclusion of Lemma 3.4 hold true without the non-bubbling assumption? For which topological spaces X does the conclusion of Lemma 3.4 hold true, for all length-continuous maps $f : X \to Y$? In particular, does Lemma 3.4 hold true for Euclidean balls X of dimension larger than 2.*

The last part of this question appears in the arXiv-version of [25].

Question 9.4 *Let $f : \mathbb{D} \to Y$ be length-connected. Can f be not length-continuous if $\langle \mathbb{D} \rangle_f$ is compact? What about more general Peano spaces X instead of $\mathbb{D}$?*

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