Extremizers and Stability of the Betke–Weil Inequality

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To the memory of Ulrich Betke and Wolfgang Weil

Abstract. Let K be a compact convex domain in the Euclidean plane. The mixed area $A(K,-K)$ of K and $-K$ can be bounded from above by $1/(6\sqrt{3})L(K)^2$, where $L(K)$ is the perimeter of K. This was proved by Ulrich Betke and Wolfgang Weil [5]. They also showed that if K is a polygon, then equality holds if and only if K is a regular triangle. We prove that among all convex domains, equality holds only in this case, as conjectured by Betke and Weil. This is achieved by establishing a stronger stability result for the geometric inequality $6\sqrt{3}A(K,-K) \leq L(K)^2$.

1. Introduction

For convex domains K, M (compact convex sets with nonempty interior) in $\mathbb{R}^2$, let $L(K)$ be the perimeter of K, let $A(K)$ be the area of K, and let $A(K,M)$ denote the mixed area of K and M (see Schneider [17, Section 5.1] or Section 2). Betke and Weil [5] proved the following theorem.

Theorem 1.1 (Betke and Weil [5]). *If $K, M \subset \mathbb{R}^2$ are convex domains, then*

$$L(K)L(M) \geq 8A(K,M) \tag{1.1}$$

with equality if and only if K and M are orthogonal (possibly degenerate) segments.

This result has been generalized to higher dimensions in [7], where also various improvements in the sense of stability results have been obtained. What makes the variational analysis of (1.1) convenient is the fact that K and M can be varied

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F. Bartha was supported by the NKFIH (National Research, Development and Innovation Office, Hungary) grants [KKP 129877, 2020-2.1.1-ED-2020-00003, TUDFO/47138-1/2019-ITM], by the EU-funded Hungarian grant [EFOP-3.6.2-16-2017-0015], by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and by the New national excellence program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation fund [UNKP-20-5]. F. Bencs was supported by the NKFIH (National Research, Development and Innovation Office, Hungary) grant [KKP-133921]. K. J. Böröczky was supported by the NKFIH (National Research, Development and Innovation Office, Hungary) grant [K 132002]. D. Hug was supported by the German Science Foundation (DFG) grant [HU 1874/5-1].

independently of each other and the dependence on K and M is Minkowski linear (in the Euclidean plane). Betke and Weil [5] also considered the case where $M = -K$ and found the following sharp geometric inequality.

THEOREM 1.2 (Betke and Weil [5]). *If K is a convex domain in $\mathbb{R}^2$, then*

$$L(K)^2 \geq 6\sqrt{3}A(K, -K). \tag{1.2}$$

In addition, if K is a polygon, then equality holds if and only if K is a regular triangle.

It is clear from the continuity of the involved functionals that it is sufficient to establish this inequality for convex polygons to deduce it for general convex domains in the plane. However, it has remained an open problem to characterize the equality case in (1.2) among all convex domains. We resolve this problem by proving more generally a stability version of Theorem 1.2. We refer to [7] (and in particular to the literature cited there) for a brief introduction to stability improvements of geometric inequalities.

THEOREM 1.3. *If K is a convex domain in $\mathbb{R}^2$ and*

$$L(K)^2 \leq (1+\varepsilon)6\sqrt{3}A(K, -K)$$

for some $\varepsilon \in [0, 2^{-28}]$, then there exists a regular triangle T with centroid z such that

$$T - z \subset K - z \subset \big(1 + 400\sqrt{\varepsilon}\big)(T - z).$$

The optimality of the stability exponent $\frac{1}{2}$ of ε can be seen by considering a regular triangle T of edge length 2. Then we add over each edge E_i of T an isosceles triangle with height $\sqrt{\varepsilon}$ that has the side E_i in common with T (for $i = 1, 2, 3$). For the resulting hexagon H, we have $L(H)^2 - 6\sqrt{3}A(H, -H) = 36\varepsilon$. However, if $d_{\mathrm{tr}}(H)$ is the minimal number $\rho \geq 0$ for which there is a regular triangle T_0 with centroid z such that $T_0 - z \subset H - z \subset (1 + \rho)(T_0 - z)$, then it is easy to check that $d_{\mathrm{tr}}(H) \geq \sqrt{\varepsilon}$.

COROLLARY 1.4. *Equality holds in (1.2) if and only if K is a regular triangle.*

Betke and Weil [5] also discuss an application of their Theorem 1.2 to an inequality for characteristics of a planar Boolean model. As a consequence of Corollary 1.4, the equality condition for the lower bound provided in [5, Theorem 3] now turns into an "if and only if" statement.

For the proof of inequality (1.2), it is sufficient to consider (convex) polygons with at most k vertices for any fixed $k \geq 3$. This task was accomplished by Betke and Weil, and we add the observation (extracted from an adaptation of their argument) that for a polygon P that is not a regular k-gon with an odd number of sides, there exist polygons P' with at most k vertices arbitrarily close to P such that

$$\frac{L(P')^2}{A(P', -P')} < \frac{L(P)^2}{A(P, -P)};$$

see Proposition 5.1. Although this can be used to prove the inequality, it does not give control over the equality cases. To determine all extremal sets, we show that if K is a convex domain that is not too far from a regular triangle, then inequality (1.2) can be strengthened to a stability result, that is, we show that if K also satisfies

$$L(K)^2 \leq (1+\varepsilon)6\sqrt{3}A(K,-K),$$

then K is ε-close to a regular triangle (if $\varepsilon > 0$ is small enough). This local stability result is stated and proved in Section 4 (see Proposition 4.1). An outline of the proof of Proposition 4.1, which is divided into six steps, is given at the beginning of the proof. A major geometric idea underlying the argument is to approximate K from inside by a triangle $T \subset K$ with maximal area. With T and K we associate hexagons H_1, H_2 such that $H_2 \supset K$ and $T \subset H_1 \subset K$. Another hexagon H_0 is derived from H_1 so that the perimeter is minimized. Then we show that

$$L(K)^2 - 6\sqrt{3}A(K,-K) \geq L(H_0)^2 - 6\sqrt{3}A(H_2,-H_2) \geq 0.$$

The fact that the right-hand side is nonnegative is far from obvious. More generally, we use a variational argument and validated numerics to establish a lower bound that involves five parameters, which determine the shapes of T and H_2 (see Lemma 3.1). In the course of the proofs, we have to determine various mixed areas of polygons. These mixed areas are obtained by a classical formula due to Minkowski and by a more recent one, which is due to Betke [4] and was first applied in [5].

Validated numerics is a well-established field of mathematics, which provides rigorous results by controlling both rounding and discretization errors in computer-aided proofs. For an introduction, we refer to [1; 14; 18; 19] or [3, Section 2.1]. A slightly weaker version of Theorem 1.3 can be proved without using validated numerics; namely, when the positive constant 2^{-28} in Theorem 1.3 is replaced by an unknown positive constant (see the remark after the proof of Lemma 3.1). Naturally, this weaker version also yields Corollary 1.4.

2. Notation and Mixed Area

For $p_1, \dots, p_\ell \in \mathbb{R}^2$, $\ell \in \mathbb{N}$, we denote by $[p_1, \dots, p_\ell]$ the convex hull of the point set $\{p_1, \dots, p_\ell\}$. In particular, $[p_1, p_2]$ is the segment connecting p_1 and p_2, and if p_1, p_2, p_3 are not collinear, then $[p_1, p_2, p_3]$ is the triangle with vertices p_1, p_2, p_3. In addition, the positive hull of $p_1, p_2 \in \mathbb{R}^2$ is given by $\mathrm{pos}\{p_1, p_2\} = \{\alpha_1 p_1 + \alpha_2 p_2 : \alpha_1, \alpha_2 \geq 0\}$. The scalar product of $x, y \in \mathbb{R}^2$ is denoted by $\langle x, y\rangle$, and the corresponding Euclidean norm of x is $\|x\| = \langle x, x\rangle^{1/2}$. The angle enclosed by two unit vectors $u, v \in \mathbb{R}^2$ is denoted by $\angle(u, v) \in [0, \pi]$ and satisfies $\cos\angle(u, v) = \langle u, v\rangle$. In addition, the determinant of a 2×2 matrix with columns $x, y \in \mathbb{R}^2$ is denoted by $\det(x, y)$.

The space of compact convex sets in $\mathbb{R}^2$ is equipped with the Hausdorff metric. In the following, by a polygon we always mean a convex set. For a (convex) polygon P in $\mathbb{R}^2$, let $\mathcal{U}(P)$ denote the finite set of exterior unit normals to the sides of P. For $u \in \mathcal{U}(P)$, we denote by $S_P(u)$ the length of the side of P with exterior normal u. As usual, the support function h_K of a compact convex set $K \subset \mathbb{R}^2$ is defined by $h_K(x) = h(K, x) = \max\{\langle x, y\rangle : y \in K\}$ for $x \in \mathbb{R}^2$.

We recall that for compact convex sets K and M in $\mathbb{R}^2$, the mixed area $A(K, M)$ of K and M is determined by the polynomial expansion $A(\lambda K + \mu M) = \lambda^2 A(K) + \mu^2 A(M) + 2\lambda\mu A(K, M)$ for $\lambda, \mu \geq 0$ (see [17, Section 5.1] or [13, Section 3.3]), where $A(K)$ is the area of K. In particular, the perimeter of K is $L(K) = 2A(K, B^2)$, where B^2 is the unit circular disk centered at the origin. We will use repeatedly two formulas which allow us to calculate and analyze the mixed area $A(P, Q)$ of two polygons P and Q. The first is due to Minkowski (see [17, (5.23)] or [13, (4.1)]) and states that

$$A(P, Q) = \frac{1}{2} \sum_{u \in \mathcal{U}(P)} h_Q(u) S_P(u). \tag{2.1}$$

Since $h_{-P}(u) = h_P(-u)$ for $u \in \mathbb{R}^2$, (2.1) implies that

$$A(P, -P) = \frac{1}{2} \sum_{u \in \mathcal{U}(P)} h_P(-u) S_P(u). \tag{2.2}$$

Since $A(P, -P) = A(-P, P)$ (see also below) and $S_{-P}(u) = S_P(-u)$, we also have

$$A(P, -P) = \frac{1}{2} \sum_{u \in \mathcal{U}(P)} h_P(u) S_P(-u).$$

For instance, if P is a triangle, then it follows from (2.2) that $A(P, -P) = 2A(P)$.

Another useful formula was established much later by Betke [4]. If w is a unit vector such that $w \notin \mathcal{U}(P) \cup \mathcal{U}(-Q)$, then

$$2 \cdot A(P, Q) = \sum_{\substack{u \in \mathcal{U}(P), v \in \mathcal{U}(Q) \\ w \in \mathrm{pos}\{u, -v\}}} |\det(u, v)| S_P(u) S_Q(v). \tag{2.3}$$

In particular, (2.3) yields that if $w \notin \mathcal{U}(P)$ for a (fixed) unit vector w, then

$$A(P, -P) = \sum_{\substack{\{u, v\} \subset \mathcal{U}(P) \\ w \in \mathrm{pos}\{u, v\}}} |\det(u, v)| S_P(u) S_P(v), \tag{2.4}$$

where the factor 2 from the preceding formula cancels, since we do not consider ordered pairs. Formula (2.4) was used in a clever way by Betke and Weil [5] to prove the Betke–Weil inequality stated in Theorem 1.2.

For compact convex sets $K, K_1, K_2, M \subset \mathbb{R}^2$, Minkowski proved the following properties of mixed areas (see [17; 13]):

$$\begin{aligned}
A(K, M) &= A(M, K), \\
A(K + z_1, M + z_2) &= A(K, M) \quad \text{for } z_1, z_2 \in \mathbb{R}^2, \\
A(\Phi K, \Phi M) &= |\det \Phi| \cdot A(K, M) \quad \text{for } \Phi \in \mathrm{GL}(2, \mathbb{R}), \\
A(K, K) &= A(K), \\
A(\alpha_1 K_1 + \alpha_2 K_2, M) &= \alpha_1 A(K_1, M) + \alpha_2 A(K_2, M) \quad \text{for } \alpha_1, \alpha_2 \geq 0, \\
A(K_1, M) &\leq A(K_2, M) \quad \text{if } K_1 \subset K_2.
\end{aligned}$$

We note that it is a subtle issue to decide under which conditions on convex polygons $P \subset Q$ the inequality $A(P,-P) \le A(Q,-Q)$ is strict. For example, let P be a triangle with its centroid at the origin o. Let v_1, v_2, v_3 denote the vertices of P, and let Q be the hexagon with vertices v_1, v_2, v_3, $-v_1$, $-v_2$, $-v_3$. Then $P \subset Q$ and $P \neq Q$, actually $A(Q) = 2A(P)$, and we still have $A(P,-P) = 2A(P) = A(Q) = A(Q,-Q)$.

Finally, we recall that $A(\cdot,\cdot)$ is additive (a valuation) in both arguments. By this we mean that if K, M, L are compact convex sets in the plane and $K \cup M$ is also convex, then

$$A(K \cup M, L) + A(K \cap M, L) = A(K, L) + A(M, L).$$

By symmetry the same property holds for the second argument.

Following the usual convention, the interior and boundary of a set $K \subset \mathbb{R}^2$ are denoted by $\operatorname{int} K$ and ∂K, respectively.

3. An Auxiliary Result for Associated Hexagons

Let $T \subset K$ denote a triangle of maximal area contained in K. Let v_1, v_2, v_3 be the vertices of T, let a_i be the side opposite to v_i, whose length is also denoted by a_i for $i = 1, 2, 3$, and let h_i be the height of T corresponding to a_i. Then we have

$$2A(T) = a_1 h_1 = a_2 h_2 = a_3 h_3.$$

We observe that v_i is a point of K of maximal orthogonal distance from the side a_i by the maximality of the area of T. Therefore the line passing through v_i and parallel to the side a_i is a supporting line to K. The width of K orthogonal to a_i can be expressed in the form $(1+t_i)h_i$ for some $t_i \in [0,1]$, where $t_i \le 1$ since T has the maximal area among all triangles in K. (The width of K orthogonal to a_i equals the length of the projection of K onto a line orthogonal to a_i.) It follows that K is contained in a circumscribed hexagon H_2 such that, for $i = 1, 2, 3$, H_2 has two sides parallel to a_i, one of which contains v_i and has length $(t_j + t_k)a_i$, $\{i, j, k\} = \{1, 2, 3\}$, and the opposite side has length $(1 - t_i)a_i$ (see Figure 1). The vertices of H_2 in clockwise order are denoted by w_{31}, w_{32}, w_{12}, w_{13}, w_{23}, w_{21} with $v_i \in [w_{ij}, w_{ik}]$ for $\{i, j, k\} = \{1, 2, 3\}$. Moreover, we denote by w'_{ij} the intersection point of the line through w_{ij} and w_{kj} and the line through v_i and v_j. The preceding statements follow by elementary geometry from the similarity of corresponding triangles. In fact, with the notation from Figure 1, we have

$$\|w'_{32} - w_{32}\| = \frac{a_2}{a_3} \cdot \|w_{32} - v_3\| \quad \text{and} \quad \|w'_{32} - w_{32}\| = \frac{a_2}{h_2} \cdot t_2 h_2,$$

hence

$$\|w'_{32} - w_{32}\| = t_2 a_2, \qquad \|w_{32} - v_3\| = t_2 a_3,$$

and similarly for permutations of the indices. Moreover,

$$\frac{\|w'_{32} - w'_{12}\|}{a_2} = \frac{h_2(1+t_2)}{h_2},$$

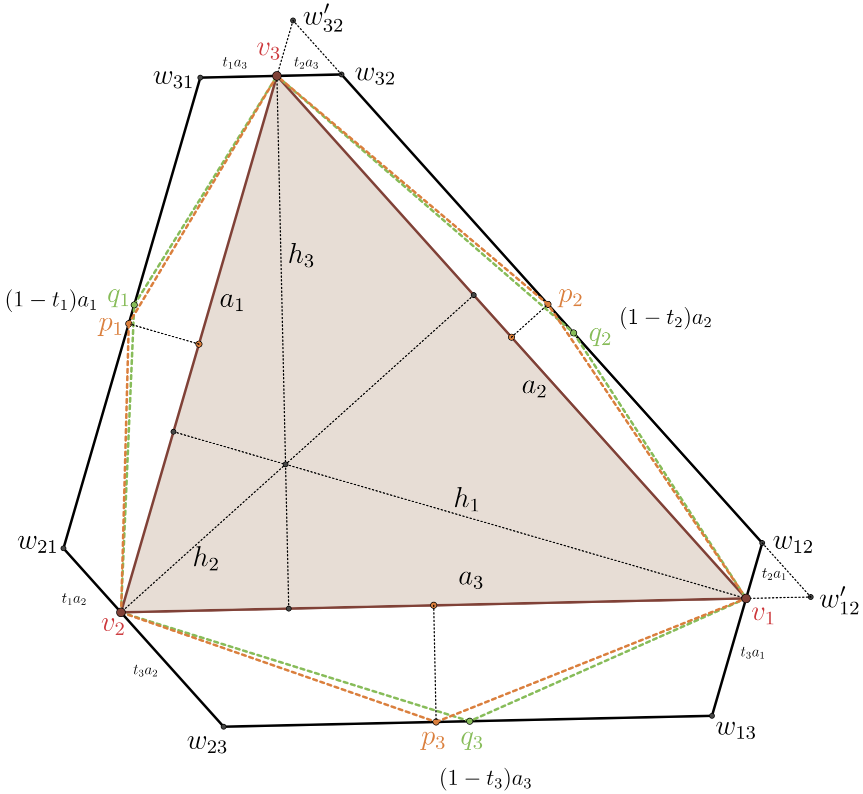

Figure 1 Illustration of the geometric construction with the triangle $T = [v_1, v_2, v_3]$, the outer hexagon $H_2 = [w_{32}, w_{31}, w_{21}, w_{23}, w_{13}, w_{12}]$, the inscribed hexagon $H_1 = [v_1, q_2, v_3, q_1, v_2, q_3]$, and the hexagon $H_0 = [v_1, p_2, v_3, p_1, v_2, p_3]$. In particular, we have $\|w_{ij} - v_i\| = t_j a_i$ and $\|w_{ij} - w_{kj}\| = (1-t_j)a_j$ for $\{i, j, k\} = \{1, 2, 3\}$.

so that $\|w'_{32} - w'_{12}\| = (1 + t_2)a_2$, and hence

$$\|w_{32} - w_{12}\| = (1 + t_2)a_2 - \|w'_{32} - w_{32}\| - \|w'_{12} - w_{12}\| = (1 - t_2)a_2.$$

Next, for $i = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$, we choose a point $q_i \in [w_{ji}, w_{ki}] \cap K$. Then we define the (possibly degenerate) hexagon $H_1 = [v_1, q_2, v_3, q_1, v_2, q_3] \subset K$. In addition, let p_i be the point on the line determined by $[w_{ji}, w_{ki}]$ (and parallel to a_i) that lies on the perpendicular bisector of the side a_i of T, and let H_0 be the hexagon with vertices v_1, p_2, v_3, p_1, v_2, p_3 (see Figure 1). Note that in general H_0 may not be convex, but the restricted choice of parameters encountered in the following will always ensure that H_0 is convex.

As $H_1 \subset K \subset H_2$ and $L(H_0) \le L(H_1)$, we have

$$\begin{aligned} &L(K)^2 - 6\sqrt{3}A(K,-K) \\ &\quad \ge L(H_1)^2 - 6\sqrt{3}A(H_2,-H_2) \ge L(H_0)^2 - 6\sqrt{3}A(H_2,-H_2). \end{aligned} \tag{3.1}$$

Clearly, T is also a triangle of maximal area contained in H_1. As among convex domains of given area, the maximal area of an inscribed triangle is the smallest for ellipses (see Blaschke [6], Sas [16], and Schneider [17, Theorem 10.3.3]), we have

$$A(T) \ge \frac{3\sqrt{3}}{4\pi} \cdot A(H_1) > 0.4 \cdot A(H_1) = 0.4(1 + t_1 + t_2 + t_3)A(T),$$

and hence

$$t_1 + t_2 + t_3 < 1.5. \tag{3.2}$$

The following lemma is the basis for obtaining better bounds on t_1, t_2, t_3 if we know that $L(K)^2 - 6\sqrt{3}A(K,-K)$ is small.

LEMMA 3.1. *If* $a_1 = 2, a_2, a_3 \in [2, 2 + \frac{1}{6}]$, *and* $t_1, t_2, t_3 \in [0, \frac{1}{6}]$, *then the hexagons* H_0 *and* H_2 *constructed as above satisfy*

$$\begin{aligned} &L(H_0)^2 - 6\sqrt{3}A(H_2,-H_2) \\ &\quad \ge (a_2 - 2)^2 + (a_3 - 2)^2 + (t_1 - t_0)^2 + (t_2 - t_0)^2 + (t_3 - t_0)^2 \end{aligned}$$

for $t_0 = (t_1 + t_2 + t_3)/3$.

REMARK. In the lemma, we do not need K, only the triangle T and $t_1, t_2, t_3 \ge 0$ are required to define H_0 and H_2. Moreover, although H_0 will be convex in the situation of the lemma, this will not be needed in the argument.

Proof. By the translation invariance of the mixed area we can assume that v_2 is the origin. Then from (2.2) we obtain that

$$\begin{aligned} A(H_2,-H_2) = &\frac{1}{2}\{(1-t_2)a_2 \cdot 0 + (t_1+t_2)a_3t_3h_3 + (1-t_1)a_1h_1 \\ &+ (t_1+t_3)a_2h_2(1+t_2) + (1-t_3)a_3h_3 + (t_2+t_3)a_1h_1t_1\} \\ = &2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1). \end{aligned}$$

By Heron's formula,

$$\begin{aligned} 2A(T) &= a_1h_1 = a_2h_2 = a_3h_3 \\ &= \frac{1}{2}\sqrt{(a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3)}, \end{aligned}$$

and in addition we have

$$L(H_0) = \sum_{i=1}^{3} \sqrt{a_i^2 + 4t_i^2h_i^2}.$$

Setting $b_i = 2t_i h_i = \frac{4A(T)t_i}{a_i}$ for $i = 1, 2, 3$, it follows from the Minkowski inequality (or, equivalently, the triangle inequality for (a_i, b_i), $i = 1, 2, 3$) that

$$\begin{aligned} L(H_0)^2 &= \left(\sum_{i=1}^{3} \sqrt{a_i^2 + b_i^2}\right)^2 \\ &\geq (a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2 \\ &= (a_1 + a_2 + a_3)^2 + 16A(T)^2(t_1/a_1 + t_2/a_2 + t_3/a_3)^2 \\ &=: f_1(a_2, a_3, t_1, t_2, t_3). \end{aligned}$$

Recall that $a_1 = 2$. For the subsequent analysis, we set $f_2(a_2, a_3, t_1, t_2, t_3) := 16A(T)^2$ (which is independent of t_1, t_2, t_3), hence

$$f_2(a_2, a_3, t_1, t_2, t_3) = (a_1 + a_2 + a_3)(-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)(a_1 + a_2 - a_3),$$

and we consider

$$\begin{aligned} f_1(a_2, a_3, t_1, t_2, t_3) = (a_1 + a_2 + a_3)^2 \\ + f_2(a_2, a_3, t_1, t_2, t_3)(t_1/a_1 + t_2/a_2 + t_3/a_3)^2. \end{aligned}$$

Finally,

$$\begin{aligned} f(a_2, a_3, t_1, t_2, t_3) := f_1(a_2, a_3, t_1, t_2, t_3) \\ - 3\sqrt{3}\sqrt{f_2(a_2, a_3, t_1, t_2, t_3)}(1 + t_1t_2 + t_2t_3 + t_3t_1). \end{aligned}$$

Thus we obtain

$$L(H_0)^2 - 6\sqrt{3} \cdot A(H_2, -H_2) \geq f(a_2, a_3, t_1, t_2, t_3). \tag{3.3}$$

In the following, we consider

$$W := \left\{(a_2, a_3, t_1, t_2, t_3)^\top \in \mathbb{R}^5 : a_2, a_3 \in \left[2, 2 + \frac{1}{6}\right] \text{ and } t_1, t_2, t_3 \in \left[0, \frac{1}{6}\right]\right\},$$

$$z_t := (2, 2, t, t, t)^\top, \quad t \in \left[0, \frac{1}{6}\right],$$

and the orthonormal basis

$$e_1 = (1, 0, 0, 0, 0)^\top, \qquad e_2 = (0, 1, 0, 0, 0)^\top, \qquad e_3 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)^\top,$$

$$e_4 = \left(0, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)^\top, \qquad e_5 = \left(0, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^\top.$$

We write $Df(x) : \mathbb{R}^5 \to \mathbb{R}$ for the derivative (a linear map) and $D^2 f(x) : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ for the second derivative (a symmetric bilinear form) of f at x. With respect to a given scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^5$, we can identify $Df(x)$ with the gradient of f at x (a vector) and $D^2 f(x)$ with a symmetric linear map $\mathbb{R}^5 \to \mathbb{R}^5$ via $D^2 f(x)(a, b) = \langle a, D^2 f(x)(b)\rangle$ for $a, b \in \mathbb{R}^5$. Moreover, we also write $D^2 f(x)$ for the Hessian matrix with respect to the standard basis $e_1^\circ, \dots, e_5^\circ$ of $\mathbb{R}^5$.

Using a computer algebra system (for convenience) or direct calculations, we obtain that if $t \in [0, \frac{1}{6}]$, then

$$f(z_t) = 0, \qquad Df(z_t) = o. \tag{3.4}$$

It follows from the Taylor formula and (3.4) that for any $x \in W$, there exists $\xi = \xi(x) \in (0, 1)$ such that

$$f(x) = \frac{1}{2}\langle x - z_t, D^2 f(z_t + \xi(x - z_t))(x - z_t)\rangle.$$

By relation (3.3) Lemma 3.1 follows once we have shown that

$$f(x) \geq \|x - z_t\|^2 \tag{3.5}$$

for $x = (a_2, a_3, t_1, t_2, t_3)^\top \in W$ and $t = (t_1 + t_2 + t_3)/3$.

Since for $x = (a_2, a_3, t_1, t_2, t_3)^\top \in W$, we have $x - z_t \in e_5^\perp$ (we write $e_5^\perp$ for the orthogonal complement of e_5) with $t = (t_1 + t_2 + t_3)/3$ and $z_t + \xi(x - z_t) \in W$, the proof will be finished if we can verify that

$$\langle v, D^2 f(\bar{x})v\rangle \geq 2\|v\|^2, \quad \bar{x} \in W, v \in e_5^\perp. \tag{3.6}$$

Using a computer algebra system (such as SageMath or Maple) or tedious calculations, for the Hessian matrix of f at $(2, 2, 0, 0, 0)^\top$, we obtain that

$$D^2 f(2, 2, 0, 0, 0) = \begin{pmatrix} 12 & -6 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 & 0 \\ 0 & 0 & 24 & -12 & -12 \\ 0 & 0 & -12 & 24 & -12 \\ 0 & 0 & -12 & -12 & 24 \end{pmatrix}$$

has the eigenvalues 6, 18, 36, 36, 0, and as associated pairwise orthogonal eigenvectors, we can choose $(1, 1, 0, 0, 0)^\top$ to correspond to the eigenvalue 6, $(1, -1, 0, 0, 0)^\top$ to correspond to 18, e_3 and e_4 to correspond to 36, and, finally, e_5 to correspond to 0.

Define the orthogonal matrix $S := (e_1 \dots e_5) \in \mathrm{O}(5)$ and write $y = (y_1, \dots, y_5)^\top \in \mathbb{R}^5$. Further, we define

$$\tilde{f}(y_1, \dots, y_5) := f\left(\sum_{i=1}^{5} y_i e_i\right) = f(Sy).$$

It follows that $f(x) = \tilde{f}(S^\top x)$ for $x \in \mathbb{R}^5$. By the chain rule, for $x \in W$ and $v \in \mathbb{R}^5$, we have

$$\langle v, D^2 f(x)v\rangle = \langle S^\top v, D^2 \tilde{f}(S^\top x)S^\top v\rangle.$$

In addition, note that $\|S^\top v\|^2 = \|v\|^2$ and

$$S^\top(W) \subset \left[2, 2 + \frac{1}{6}\right]^2 \times \left[-\frac{1}{6}\sqrt{\frac{2}{3}}, \frac{1}{6}\sqrt{\frac{2}{3}}\right]^2 \times \left[0, \frac{\sqrt{3}}{6}\right]$$
$$\subset \widetilde{W} := \left[2, 2 + \frac{1}{6}\right]^2 \times [-0.14, 0.14]^2 \times [0, 0.3].$$

Here we use that for $y = S^\top (a_2, a_3, t_1, t_2, t_3)^\top$,

$$\begin{aligned} y_3^2 + y_4^2 &= \frac{1}{2}(t_1 - t_2)^2 + \frac{1}{6}(t_1 + t_2 - 2t_3)^2 \\ &= \frac{2}{3}\frac{1}{2}[(t_3 - t_2)^2 + (t_3 - t_1)^2 + (t_2 - t_1)^2] \\ &\leq \frac{2}{3}\max\{t_1, t_2, t_3\}^2 = \frac{2}{3} \cdot \frac{1}{6^2} \end{aligned}$$

and

$$0 \leq y_5 = \frac{1}{\sqrt{3}}(t_1 + t_2 + t_3) \leq \sqrt{3} \cdot \frac{1}{6}.$$

Moreover, $v \in e_5^\perp$ if and only if $\langle S^\top v, e_5^\circ \rangle = 0$, where $e_5^\circ = (0, 0, 0, 0, 1)^\top$. Hence (3.6) follows if we can verify that

$$\langle \tilde{v}, D^2 \tilde{f}(y)\tilde{v} \rangle \geq 2\|\bar{v}\|^2, \quad y \in \widetilde{W}, \tilde{v} \in (e_5^\circ)^\perp.$$

Writing $H(y) := (D^2 \tilde{f}(y)_{ij})_{i,j=1}^4$ for the 4×4 matrix (principal minor) obtained from the 5×5 Hessian matrix representing $D^2 \tilde{f}(y)$ with respect to the standard basis $e_1^\circ, \ldots, e_5^\circ$ of $\mathbb{R}^5$, we want to verify that

$$\langle \bar{v}, H(y)\bar{v} \rangle \geq 2\|\bar{v}\|^2, \quad y \in \widetilde{W}, \bar{v} \in \mathbb{R}^4, \tag{3.7}$$

that is, all eigenvalues of $H(y)$ are at least 2. Let

$$\Xi := \{(\bar{v}_1, \ldots, \bar{v}_4)^\top \in [-1, 1]^4 : \bar{v}_i = 1 \text{ for some } i \in \{1, \ldots, 4\}\}.$$

By the scaling invariance of (3.7) with respect to $\bar{v} \in \mathbb{R}^4$, (3.7) is equivalent to

$$\langle \bar{v}, H(y)\bar{v} \rangle \geq 2\|\bar{v}\|^2, \quad y \in \widetilde{W}, \bar{v} \in \Xi. \tag{3.8}$$

Since all eigenvalues of $H((2, 2, 0, 0, 0)^\top)$ are positive, this holds if and only if all eigenvalues of $H(y)^2$ are at least 4 for $y \in \widetilde{W}$. The latter means that we have to show that $\langle \bar{v}, H(y)^2\bar{v} \rangle \geq 4\|\bar{v}\|^2$ for $y \in \widetilde{W}$ and $\bar{v} \in \mathbb{R}^4$ or, equivalently,

$$\|H(y)\bar{v}\|^2 \geq 4\|\bar{v}\|^2, \quad y \in \widetilde{W}, \bar{v} \in \mathbb{R}^4. \tag{3.9}$$

Again by the scaling invariance of (3.9) with respect to $\bar{v}$, (3.9) is in turn equivalent to

$$\|H(y)\bar{v}\|^2 \geq 4\|\bar{v}\|^2, \quad y \in \widetilde{W}, \bar{v} \in \Xi. \tag{3.10}$$

Direct rigorous numerical analysis of the eigendecomposition of the Hessian $D^2 \tilde{f}(y)$ for $y \in \widetilde{W}$ may be challenging due to requiring too many subdivisions of $\widetilde{W}$ to achieve the required precision [15; 12; 9; 11]. As both, $\widetilde{W}$ and Ξ are compact and finite-dimensional, and as the desired inequalities, either (3.8) or (3.10), are expected to be strict, they are well suited for being studied by rigorous numerics [14; 1; 19].

Namely, for small $\widetilde{W}' \subset \widetilde{W}$ and $\Xi' \subset \Xi$, we perform the following procedure with all computations being carried out rigorously using interval arithmetic and automatic differentiation [14; 1; 19; 10]. First, we bound the jet of $\tilde{f}$ up to Taylor coefficients of degree 6 over $\widetilde{W}'$. To increase precision and eliminate some of the dependency issues, for a given $\widetilde{W}'$, the degree 6 jet of f is bound both over

$\widetilde{W}'$ and over the midpoint of $\widetilde{W}'$. Hence, using the multivariate Taylor expansions with appropriate remainder term, we obtain enhanced bounds on the Taylor coefficients of $\tilde{f}$ over $\widetilde{W}'$ and, in turn, a better enclosure of the Hessian matrix $H(y) = D^2\tilde{f}(y)$. Second, we test if we can guarantee inequality (3.8) or (3.10) for all $v \in \Xi'$. If that is not the case, then an adaptive bisection scheme of $\widetilde{W}' \times \Xi'$ is used, and the arising subsets of $\widetilde{W}' \times \Xi'$ are processed separately.

We have implemented our software using the package CAPD [8] and verified both inequalities independently and successfully. The required number of subsets (of $\widetilde{W} \times \Xi$) and the associated computational times (without parallelization on an i7-9750) were

- (3.8): 25880 subsets, 8 m 14 s;
- (3.10): 2440 subsets, 46 s.

We note that the increased complexity of (3.8) is most likely just an artefact of the naive computation of the inner product and could be decreased (to that of (3.10)) by choosing a more efficient evaluation scheme. The source code and output logs are available at [2].

In particular, both (3.8) and (3.10) and, in turn, Lemma 3.1 have been verified. □

Remark. We note that a slightly weaker version of Lemma 3.1 can be proved without validated numerics; namely, when the positive constant $\frac{1}{6}$ in Lemma 3.1 is replaced by an unknown positive constant $c \leq \frac{1}{6}$. The point is that the eigenvalues of the restriction of $D^2 f(2,2,0,0,0)$ to $e_5^{\perp}$ are all larger than 2. Therefore (3.6) holds by the continuity of $D^2 f$ if $v \in e_5^{\perp}$ and $\bar{x}$ lies in a small but unknown neighborhood of $(2,2,0,0,0)^{\top}$ instead of lying in W. In turn, using the same argument as below, this slightly weaker version of Lemma 3.1 still yields a somewhat weaker version of Theorem 1.3 where the positive constant 2^{-28} is replaced by an unknown positive constant.

4. Local Stability

In the proof of Proposition 4.1, we will use the following two claims.

Claim 4.1. *If the regular triangle T_0 of side length b contains a triangle T that has a side of length at most a, where $\frac{b}{2} \leq a \leq b$, then $A(T) \leq \frac{a}{b} A(T_0)$.*

Proof. Let ℓ be the line containing a side of T of length at most a. We may assume that the vertex v of T opposite to $\ell \cap T$ is also a vertex of T_0.

If the distance of ℓ from v (that is, the height of T) is at most $b\sqrt{3}/2$, then we are done. Therefore we may now assume that the distance of ℓ from v is larger than $b\sqrt{3}/2$. Let D_v be the circular disc with center v and radius $b\sqrt{3}/2$. Since the side $\ell \cap T$ is disjoint from D_v, it lies in one of the two connected components of $T_0 \setminus D_v$. It follows that T is contained in one of the two triangles obtained by cutting T_0 into two subtriangles by the height emanating from v, and thus $A(T) \leq \frac{1}{2} A(T_0)$.

CLAIM 4.2. *If $\varrho_1, \varrho_2, \varrho_3$ are the side lengths of a triangle T and $A(T) \geq \xi\varrho_1$ for some $\xi \in [0, \varrho_1]$, then*

$$\varrho_2 + \varrho_3 \geq \varrho_1 + \frac{\xi^2}{\varrho_1}.$$

Proof. The height h of T corresponding to ϱ_1 is at least 2ξ, and $\varrho_2 + \varrho_3$ is minimized under this condition if $\varrho_2 = \varrho_3$; thus

$$\varrho_2 + \varrho_3 \geq 2\sqrt{\left(\frac{\varrho_1}{2}\right)^2 + h^2} \geq \sqrt{\varrho_1^2 + 16\xi^2} = \varrho_1\sqrt{1 + \frac{16\xi^2}{\varrho_1^2}} \geq \varrho_1 + \frac{\xi^2}{\varrho_1},$$

which proves the claim. □

For any convex domain K, let $d_{\mathrm{tr}}(K)$ be the minimal $\rho \geq 0$ such that there exists a regular triangle T with centroid z satisfying

$$T - z \subset K - z \subset (1 + \rho)(T - z).$$

In particular, $d_{\mathrm{tr}}(K)$ measures how close K is to a suitable regular triangle.

PROPOSITION 4.1. *Suppose that K is a convex domain with $d_{\mathrm{tr}}(K) \leq 6^{-2}$ and*

$$L(K)^2 \leq (1 + \varepsilon)6\sqrt{3}A(K, -K) \tag{4.1}$$

for some $\varepsilon \in [0, (6 \cdot 180)^{-2}]$. Then $d_{\mathrm{tr}}(K) \leq 400\sqrt{\varepsilon}$.

Proof. Let $d_{\mathrm{tr}}(K) = \eta \leq 6^{-2}$, and let ε be as in the statement of the proposition. There exists a regular triangle T_0 of side length b containing K such that a translate of $\frac{1}{1+\eta}T_0$ is contained in K. For a triangle $T \subset K$ of maximal area contained in K, we have

$$A(T) \geq \frac{A(T_0)}{(1 + \eta)^2} \geq \frac{A(T_0)}{1 + 3\eta}. \tag{4.2}$$

From now on, we use the notions and auxiliary constructions introduced for K and T at the beginning of Section 3, including the hexagons H_0, H_1, H_2, the parameters $t_1, t_2, t_3 \geq 0$, etc.

The main part of the proof is divided into several steps. In Step 1, we prove that T is $\sqrt{\varepsilon}$-close to a regular triangle, and Step 2 shows that $A(H_2, -H_2) - A(K, -K)$ is ε small. Based on these findings, Step 3 verifies that if H_2 is close to T in the sense that $\max\{t_1, t_2, t_3\} \leq 100\sqrt{\varepsilon}$ (see (4.21)), then Proposition 4.1 holds.

The rest of the argument is indirect. Starting from Step 4, we assume that the assumption $\max\{t_1, t_2, t_3\} > 100\sqrt{\varepsilon}$ (see (4.24)) is satisfied and derive a contradiction. Under this assumption, we prove in Step 4 that H_1 is reasonably close to H_0 in the sense that $\|p_i - q_i\|$ is reasonably small for $i = 1, 2, 3$ (see (4.26)). Then Step 5 verifies that $K \subset D = \frac{1}{2}H_1 + \frac{1}{2}H_2$ and clearly $D \subset H_2$. Finally, in Step 6, we prove that the gap between $A(H_2, -H_2)$ and $A(D, -D)$ (and hence the gap between $A(H_2, -H_2)$ and $A(K, -K)$ by Step 5) is too large, which yields the desired contradiction.

STEP 1. *T is $\sqrt{\varepsilon}$-close to a regular triangle.*

By the scaling invariance of the statement (and symmetry) we may assume that the side lengths of T satisfy

$$a_1 \leq a_2 \leq a_3 \quad \text{and} \quad a_1 = 2.$$

Assuming that $a_1 < \frac{b}{2}$, we can apply Claim 4.1 with $a = b/2$ and get

$$A(T) \leq \frac{1}{2} A(T_0) \leq \frac{1}{2}(1+\eta)^2 A(T)$$

by (4.2), which is a contradiction, since $\eta \leq 6^{-2}$. Hence we have $\frac{b}{2} \leq a_1 \leq b$, and another application of Claim 4.1 now yields that

$$A(T) \leq \frac{a_1}{b} A(T_0) \leq \frac{a_1}{b}(1+3\eta) A(T),$$

so that

$$\frac{b}{1+3\eta} \leq a_1 = 2 \leq a_3 \leq b,$$

and hence

$$a_3 \leq b \leq 2(1+3\eta) = 2 + 6\eta \leq 2 + \frac{1}{6}. \tag{4.3}$$

It also follows from (4.2) that

$$(1 + t_1 + t_2 + t_3) A(T) = A(H_1) \leq A(K) \leq A(T_0) \leq (1+3\eta) A(T),$$

and therefore

$$t_i \leq \frac{1}{12} < \frac{1}{6} \quad \text{for } i = 1, 2, 3. \tag{4.4}$$

Thus the condition $d_{\mathrm{tr}}(K) \leq 6^{-2}$ ensures that Lemma 3.1 can be applied.

Hence by a combination of (3.1) with Lemma 3.1 and assumption (4.1) of the proposition we see that

$$(a_i - 2)^2 \leq 6\sqrt{3} A(K, -K)\varepsilon \leq 11 A(K, -K)\varepsilon \quad \text{for } i = 2, 3, \tag{4.5}$$

$$\begin{aligned}(t_i - t)^2 &\leq 6\sqrt{3} A(K, -K)\varepsilon \\ &\leq 11 A(K, -K)\varepsilon \quad \text{for } t = \frac{1}{3}(t_1 + t_2 + t_3), i = 1, 2, 3.\end{aligned} \tag{4.6}$$

To estimate $A(K, -K)$, we observe that the height h_1 of T corresponding to the side $a_1 = 2$ satisfies $h_1 \leq \sqrt{a_3^2 - 1} < 2$ (since $a_2 \geq a_1$), thus $A(T) \leq \sqrt{a_3^2 - 1} < 2$ by (4.3), and hence

$$\begin{aligned}A(K, -K) &\leq A(T_0, -T_0) = 2A(T_0) \leq 2(1+3\eta) A(T) \\ &\leq \left(2 + \frac{1}{6}\right) \cdot 2 = \frac{13}{3}.\end{aligned} \tag{4.7}$$

Now (4.5)–(4.7) imply that

$$2 \leq a_i \leq 2 + 7\sqrt{\varepsilon} \quad \text{for } i = 2, 3, \tag{4.8}$$

$$|t_i - t| \leq 7\sqrt{\varepsilon} \quad \text{for } i = 1, 2, 3. \tag{4.9}$$

We observe that $\frac{a_1^2+a_2^2-a_3^2}{a_2}$ is an increasing function of $a_2 \geq 2$ as $a_3 \geq a_1$. Writing α_i to denote the angle of T opposite to the side a_i and using that $7^2\sqrt{\varepsilon} \leq 1$, we have

$$\cos\alpha_3 = \frac{a_1^2+a_2^2-a_3^2}{2a_1a_2} \geq \frac{8-(2+7\sqrt{\varepsilon})^2}{8} \geq \frac{4-29\sqrt{\varepsilon}}{8} \geq \frac{1}{2} - 4\sqrt{\varepsilon} > 0.$$

In particular, we have $\alpha_1 \leq \alpha_2 \leq \alpha_3 < \pi/2$, $\alpha_3 \geq \pi/3$, and $\alpha_1 \leq \pi/3$.

Since $\cos'(s) = -\sin s \leq -\sqrt{3}/2$ for $s \in [\frac{\pi}{3}, \frac{\pi}{2}]$, the mean value theorem implies that

$$\alpha_3 \leq \frac{\pi}{3} + \frac{2}{\sqrt{3}}4\sqrt{\varepsilon} \leq \frac{\pi}{3} + 5\sqrt{\varepsilon}.$$

Since $\pi \leq \alpha_1 + 2\alpha_3 \leq \alpha_1 + 2(\frac{\pi}{3} + 5\sqrt{\varepsilon})$, we conclude that

$$\frac{\pi}{3} - 10\sqrt{\varepsilon} \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \frac{\pi}{3} + 5\sqrt{\varepsilon}. \tag{4.10}$$

As a first estimate, we deduce $\tan\alpha_3 \leq 2$, and thus $\tan'(s) = 1 + (\tan s)^2 \leq 5$ for $s \in [\frac{\pi}{3}, \alpha_3]$. In particular, (4.10) yields

$$\sqrt{3} - 50\sqrt{\varepsilon} \leq \tan\alpha_1 \leq \tan\alpha_2 \leq \tan\alpha_3 \leq \sqrt{3} + 25\sqrt{\varepsilon}. \tag{4.11}$$

Now let T_1' be the regular triangle of edge length 2 positioned in such a way that the side a_1 is common with T and $\operatorname{int} T \cap \operatorname{int} T_1' \neq \emptyset$. Recalling that v_2 and v_3 are the endpoints of a_1, we denote by v_1' the third vertex of T_1', by z_1 the centroid of T_1', and by $m = \frac{1}{2}(v_2+v_3)$ the midpoint of a_1. As $a_3 \geq a_2$ and $\frac{\pi}{2} > \alpha_3 \geq \frac{\pi}{3}$, there exists a point q such that $v_1' \in [m,q]$ and $v_1 \in [v_3,q]$. In addition, we consider the intersection point $\{p\} = [q,m] \cap [v_1,v_2] \neq \emptyset$ (see Figure 2).

We deduce from (4.11) that

$$\|q - v_1'\| = \tan(\alpha_3) - \sqrt{3} \leq 25\sqrt{\varepsilon}$$

and

$$\|p - m\| = \tan(\alpha_2) \geq \tan(\alpha_1) \geq \sqrt{3} - 50\sqrt{\varepsilon},$$

and if $p \in [v_1', m]$, then also

$$\|p - v_1'\| = \|v_1' - m\| - \|p - m\| = \sqrt{3} - \tan(\alpha_2) \leq 50\sqrt{\varepsilon}.$$

Therefore $\|v_1' - z_1\| = \frac{2}{\sqrt{3}}$ implies that in any case

$$\left(1 - 25\sqrt{3}\sqrt{\varepsilon}\right)(T_1' - z_1) \subset T - z_1 \subset \left(1 + \frac{25\sqrt{3}}{2}\sqrt{\varepsilon}\right)(T_1' - z_1). \tag{4.12}$$

For the regular triangle

$$T_1 = z_1 + \left(1 - 25\sqrt{3}\sqrt{\varepsilon}\right)(T_1' - z_1)$$

with centroid at z_1, we have

$$T_1 - z_1 \subset T - z_1 \subset \left(1 + 70\sqrt{\varepsilon}\right)(T_1 - z_1). \tag{4.13}$$

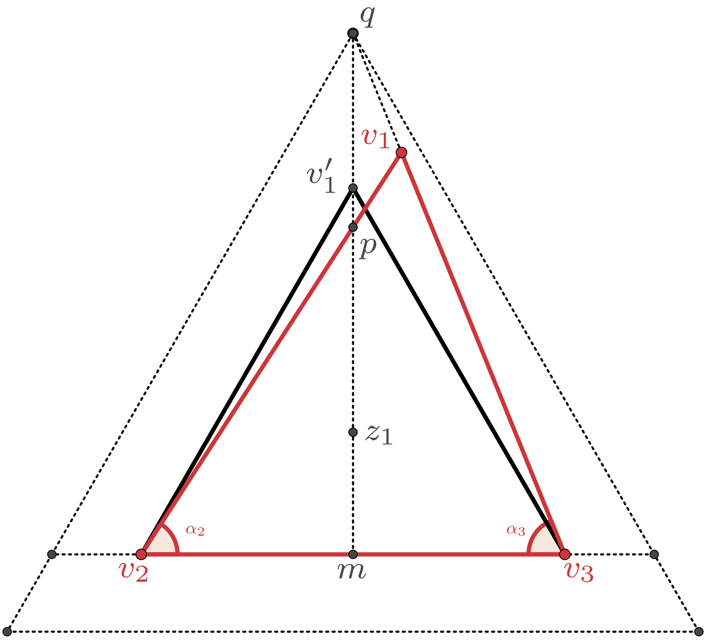

Figure 2 Illustration of the triangles $T = [v_1, v_2, v_3]$, $T_1' = [v_1', v_2, v_3]$ together with the auxiliary points m, p, q, z_1 in the case where $p \in [v_1', m]$. In the case $p \notin [v_1', m]$ (not shown in the figure), we have $T_1' \subset T$ and $p = q = v_1$.

In addition, let z be the centroid of T. The argument above also shows that

$$v_1 \in v_1' + \frac{25\sqrt{3}}{2}\sqrt{\varepsilon}(T_1' - z_1).$$

Since $z = \frac{1}{3}(v_1 + v_2 + v_3)$ and $z_1 = \frac{1}{3}(v_1' + v_2 + v_3)$, we get

$$\begin{aligned} z - z_1 &\in \frac{25\sqrt{3}}{2 \cdot 3}\sqrt{\varepsilon} \cdot (T_1' - z_1) \subset 10\sqrt{\varepsilon} \cdot (T_1 - z_1), \\ z_1 - z &\in 20\sqrt{\varepsilon} \cdot (T_1 - z_1), \end{aligned} \tag{4.14}$$

where for the second containment, we used that $-(T_1 - z_1) \subset 2(T_1 - z_1)$. In summary, (4.13) and (4.14) show that the regular triangle T_1 is a very good approximation of T.

Note that p_i and q_i lie on the same side of H_2 (parallel to and near a_i) for $i = 1, 2, 3$. This follows from $\alpha_i \le \frac{\pi}{2}$, $a_i \ge 2$, (4.3), and (4.4), which imply that $t_i a_j \le \frac{1}{12}(2 + \frac{1}{6}) < 0.2 < 1 \le \frac{a_k}{2}$ for $i, j, k \in \{1, 2, 3\}$. Moreover, for $i = 1, 2, 3$, we have

$$1.77 > \left(1+4\sqrt{\varepsilon}\right)\left(\sqrt{3}+25\sqrt{\varepsilon}\right) \geq \frac{a_i}{2}\tan(\alpha_3)$$
$$\geq h_i \geq \frac{a_i}{2}\tan(\alpha_1) \geq \sqrt{3}-50\sqrt{\varepsilon} > 1.68, \tag{4.15}$$

and hence in particular

$$2 > 1.77 > A(T) = h_1 > 1.68 > 1.5. \tag{4.16}$$

STEP 2. *$A(K,-K)$ and $A(H_2,-H_2)$ are ε-close, and $L(H_0)$, $L(H_1)$, and $L(K)$ are ε-close.*

We deduce from (4.7) that $6\sqrt{3}A(K,-K) \leq 26\sqrt{3} < 50$. Therefore (4.1), Lemma 3.1, $H_1 \subset K \subset H_2$, and $L(H_0) \leq L(H_1)$ imply that

$$50\varepsilon \geq L(K)^2 - 6\sqrt{3}A(K,-K) \geq L(K)^2 - 6\sqrt{3}A(H_2,-H_2)$$
$$\geq L(H_1)^2 - 6\sqrt{3}A(H_2,-H_2) \geq L(H_0)^2 - 6\sqrt{3}A(H_2,-H_2) \geq 0. \tag{4.17}$$

In particular,

$$L(K)^2 - 6\sqrt{3}A(K,-K) \leq 50\varepsilon,$$
$$6\sqrt{3}A(H_2,-H_2) - L(K)^2 \leq 0,$$

which yields

$$A(H_2,-H_2) \leq A(K,-K) + 5\varepsilon. \tag{4.18}$$

Using $L(K) \geq L(H_1) \geq L(H_0) \geq L(T) \geq 6$ and

$$L(K)^2 - 6\sqrt{3}A(H_2,-H_2) \leq 50\varepsilon,$$
$$6\sqrt{3}A(H_2,-H_2) - L(H_1)^2 \leq 0,$$

we get

$$50\varepsilon \geq L(K)^2 - L(H_1)^2 = (L(K)-L(H_1))(L(K)+L(H_1))$$
$$\geq 12(L(K)-L(H_1)).$$

We deduce that

$$L(K) \leq L(H_1) + 5\varepsilon. \tag{4.19}$$

A similar argument shows that

$$L(H_1) \leq L(H_0) + 5\varepsilon. \tag{4.20}$$

This completes Step 2.

For the remaining part of the proof, we set $\gamma = 10^2$. In the following, we distinguish whether

$$\max\{t_1,t_2,t_3\} \leq \gamma\sqrt{\varepsilon} \tag{4.21}$$

is satisfied or not. If (4.21) holds, then H_2 is $\sqrt{\varepsilon}$-close to K (see the argument below).

STEP 3. *If* (4.21) *holds, then* $d_{\mathrm{tr}}(K) \le 4\gamma\sqrt{\varepsilon}$.

It follows from (4.13) and $T \subset K$ that

$$T_1 - z_1 \subset K - z_1. \tag{4.22}$$

Using (4.21) and recalling that z is the centroid of T, we get

$$K - z \subset H_2 - z \subset (1 + 3\max\{t_1, t_2, t_3\})(T - z) \subset \big(1 + 3\gamma\sqrt{\varepsilon}\big)(T - z).$$

Therefore (4.13) and (4.14) imply that if (4.21) holds, then

$$\begin{aligned} K - z_1 = K - z + (z - z_1) &\subset \big(1 + 3\gamma\sqrt{\varepsilon}\big)(T - z_1) + 3\gamma\sqrt{\varepsilon}(z_1 - z) \\ &\subset \big[1 + 70\sqrt{\varepsilon} + 3\gamma\sqrt{\varepsilon} + 210\gamma\varepsilon + 60\gamma\varepsilon\big](T_1 - z_1) \\ &\subset \big(1 + 4\gamma\sqrt{\varepsilon}\big)(T_1 - z_1). \end{aligned} \tag{4.23}$$

In view of (4.22), we conclude Proposition 4.1 if (4.21) holds.

It remains to consider the case where

$$\max\{t_1, t_2, t_3\} > \gamma\sqrt{\varepsilon}, \tag{4.24}$$

which will finally lead to a contradiction.

It follows from (4.9) and (4.24) that

$$\min\{t_1, t_2, t_3\} \ge 86\sqrt{\varepsilon}. \tag{4.25}$$

STEP 4. *Assuming* (4.24), H_1 *is reasonably close to* H_0.

More precisely, we claim that

$$\|p_i - q_i\| < 0.06 \quad \text{for } i = 1, 2, 3, \tag{4.26}$$

which is what we mean by saying that H_1 is "reasonably close" to H_0.

Let $\{i, j, k\} = \{1, 2, 3\}$, and assume that $q_i \ne p_i$ (otherwise, (4.26) readily holds). For the line ℓ_i through p_i and q_i and parallel to a_i, let $\tilde{v}_j$ be the reflection of v_j through ℓ_i; hence p_i is the midpoint of $[\tilde{v}_j, v_k]$. For the triangle $\widetilde{T}_i = [q_i, \tilde{v}_j, v_k]$, we have

$$A(\widetilde{T}_i) = 2A([q_i, p_i, v_k]) = \|p_i - q_i\| t_i h_i.$$

Recall from (4.15) that $1.68 < h_i < 1.77$. Using (4.8) and $t_i \le \frac{1}{12}$, by (4.4) we get

$$\begin{aligned} \|\tilde{v}_j - v_k\| = 2\|p_i - v_k\| \le 2\left(\frac{a_i}{2} + \frac{1}{12}h_i\right) \le a_i + \frac{1}{6}h_i \\ \le 2 + \frac{7}{6 \cdot 180} + \frac{1}{6} \cdot 1.77 \le 2.31. \end{aligned}$$

Therefore

$$A(\widetilde{T}_i) \ge \|p_i - q_i\| t_i \cdot 1.68 \ge \frac{\|p_i - q_i\| t_i}{2.31} \cdot 1.68 \cdot \|\tilde{v}_j - v_k\|.$$

Since

$$\frac{\|p_i - q_i\| t_i}{2.31} \cdot 1.68 \le \frac{a_i}{12 \cdot 2.31} \cdot 1.68 < a_i \le \|\tilde{v}_j - v_k\|,$$

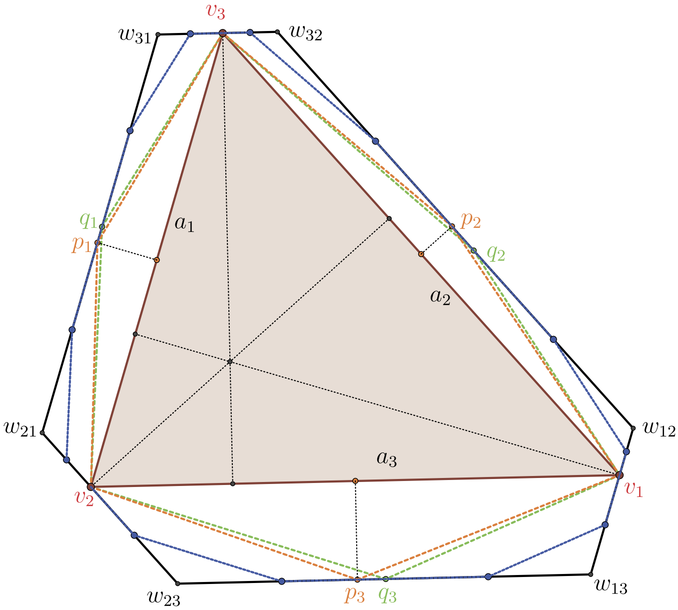

Figure 3 Illustration for $D = \frac{1}{2}H_1 + \frac{1}{2}H_2$.

Claim 4.2 can be applied. In combination with $t_i \geq 86\sqrt{\varepsilon}$ (by (4.25)), this leads to

$$\begin{aligned}
\|q_i - v_j\| + \|q_i - v_k\| &= \|q_i - \tilde{v}_j\| + \|q_i - v_k\| \\
&\geq \|\tilde{v}_j - v_k\| + \frac{1}{2.31}\left(\frac{\|p_i - q_i\| t_i \cdot 1.68}{2.31}\right)^2 \\
&= \|p_i - v_j\| + \|p_i - v_k\| + \|p_i - q_i\|^2 t_i^2 \frac{1.68^2}{2.31^3} \\
&\geq \|p_i - v_j\| + \|p_i - v_k\| + \|p_i - q_i\|^2 \frac{86^2 \cdot 1.68^2}{2.31^3}\varepsilon.
\end{aligned}$$

We deduce from (4.20) that

$$5\varepsilon \geq \|p_i - q_i\|^2 \frac{86^2 \cdot 1.68^2}{2.31^3}\varepsilon,$$

and hence (4.26) follows.

STEP 5. *Assuming* (4.24), *we have* $K \subset D := \frac{1}{2}H_1 + \frac{1}{2}H_2$.

The polygon $D \subset H_2$ has twelve sides (see Figure 3).

Six of these sides are subsets of the six sides of H_2 as $h_{H_1}(u) = h_{H_2}(u)$ for all $u \in \mathcal{U}(H_2)$, and the other six sides of D are parallel to the sides of H_1. We prove

$$K \subset D \tag{4.27}$$

indirectly, so we assume that there exists $x \in K \setminus D$ and seek a contradiction. Then there exists $u \in \mathcal{U}(D)$ such that $h_K(u) \geq \langle x, u\rangle > h_D(u)$. Since $x \in K \subset H_2$, u is normal to a side of D parallel to a side of H_1. Let u be normal to $[v_i, q_j]$, $i \neq j$, so that w_{ij} is the vertex of H_2 where u is a normal to H_2. For

$$\Delta = h_{H_2}(u) - h_{H_1}(u) = \langle u, w_{ij} - q_j\rangle,$$

we have

$$\Delta \cdot \|q_j - v_i\| = 2A([q_j, v_i, w_{ij}]) = t_j h_j \|w_{ij} - q_j\|.$$

We have $h_j \geq 1.68$ by (4.15), and since $t_i \leq \frac{1}{12}$ by (4.4) and $h_i < 2$, we obtain using (4.26) that

$$\|q_j - v_i\| \leq \|p_i - v_i\| + 0.1 \leq \frac{a_i}{2} + \frac{1}{12}h_i + 0.1 \leq \frac{3}{2} + \frac{1}{6} + 0.1 < 2$$

and

$$\|w_{ij} - q_j\| \geq \|w_{ij} - p_j\| - 0.1 \geq \frac{a_j}{2} - 0.1 \geq 1 - 0.1 = 0.9.$$

Thus we get

$$\Delta \geq \frac{1}{2}t_j \cdot 1.68 \cdot 0.9 \geq 60\sqrt{\varepsilon}.$$

We deduce that

$$\langle u, x - q_j\rangle > h_D(u) - h_{H_1}(u) = \frac{1}{2}(h_{H_2}(u) - h_{H_1}(u)) = \frac{\Delta}{2} \geq 30\sqrt{\varepsilon},$$

and therefore

$$A([q_j, v_i, x]) = \frac{\langle u, x - q_j\rangle \cdot \|q_j - v_i\|}{2} \geq 15\sqrt{\varepsilon} \cdot \|q_j - v_i\|.$$

Note that $15\sqrt{\varepsilon} \leq 0.02$ and $\|q_j - v_i\| \geq \|p_j - v_i\| - 0.1 \geq 0.9$. It follows from Claim 4.2 and $\|q_j - v_i\| \leq 2$ that

$$\|x - v_i\| + \|x - q_j\| \geq \|q_j - v_i\| + \frac{1}{\|q_j - v_i\|} \cdot \left(15\sqrt{\varepsilon}\right)^2 \geq \|q_j - v_i\| + 15\varepsilon.$$

Denoting by $\widetilde{H}$ the polygon that is the convex hull of x and H_1, we have that $[v_i, x]$ and $[x, q_j]$ are sides of $\widetilde{H}$, and hence

$$L(K) \geq L(\widetilde{H}) \geq L(H_1) + 15\varepsilon,$$

contradicting (4.19). In turn, we conclude (4.27).

STEP 6. *Assuming* (4.24), $A(H_2, -H_2) - A(K, -K)$ *is too large.*

To calculate $A(D, -D)$ for $D = \frac{1}{2}H_1 + \frac{1}{2}H_2$, we claim that $D_0 = \frac{1}{2}T + \frac{1}{2}H_2$ (see Figure 4) satisfies

$$A(D, -D) = A(D_0, -D_0) + A(T)(t_1t_2 + t_2t_3 + t_1t_3). \tag{4.28}$$

We prove (4.28) by applying Betke's formula (2.4) three times. First, we introduce some notation for any i, j, k with $\{i, j, k\} = \{1, 2, 3\}$, where i, j, k are fixed for this paragraph. Let b_i be the side of D containing q_i (and hence b_i is contained in the "long" side of H_2 parallel to a_i), and let $\tilde{u}_i \in \mathcal{U}(D)$ be the normal to D at the vertex v_i of T, and hence $-\tilde{u}_i$ is the exterior unit normal to b_i. In addition, let

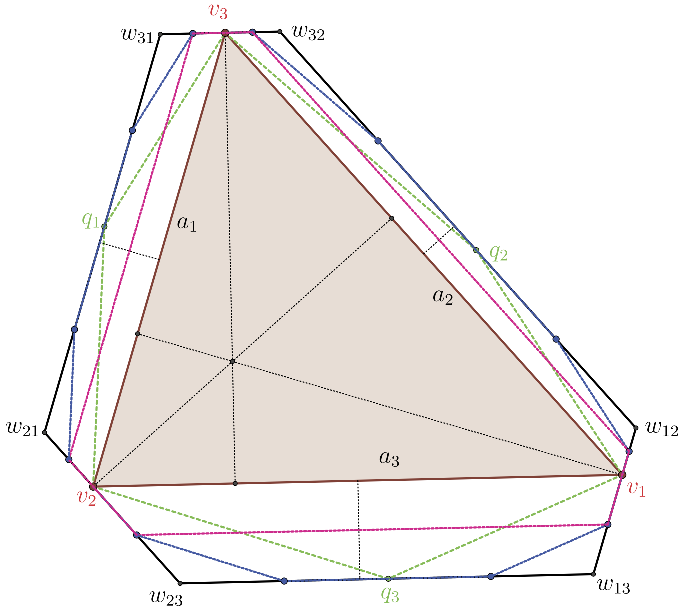

Figure 4 Illustration for $D = \frac{1}{2}H_1 + \frac{1}{2}H_2$ and $D_0 = \frac{1}{2}T + \frac{1}{2}H_2$.

d_i be the diagonal of D that cuts off b_i and the two sides neighboring b_i from D, and hence d_i is parallel to b_i. There is a side of D that is cut off by $[q_i, v_j]$ and is parallel to $[q_i, v_j]$, and we denote by ν_{ij} the exterior unit normal to that side of D. In particular, ν_{ij} and ν_{ik} are the normals to the two sides neighboring b_i.

Now d_1 dissects D into a trapezoid and a polygon D_1 with 10 sides, and on the way to verify (4.28), we first claim that

$$A(D, -D) = A(D_1, -D_1) + S_D(\tilde{u}_1)S_D(\nu_{12})|\det(\tilde{u}_1, \nu_{12})|. \tag{4.29}$$

To prove (4.29), we choose a unit vector $u_1 \neq \tilde{u}_1$ very close to $\tilde{u}_1$ and such that $\langle u_1, v_2 - v_3\rangle > 0$, where very close means that $\angle(u_1, \tilde{u}_1) < \angle(u, \tilde{u}_1)$ for any (exterior or interior) unit normal $u \neq \tilde{u}_1$ to a side of D. When we apply Betke's formula (2.4) to calculate the difference $A(D, -D) - A(D_1, -D_1)$ using u_1 as the reference vector, we deduce after cancelation of summands common to $A(D, -D)$ and $A(D_1, -D_1)$ that the exterior unit normal $-\tilde{u}_1$ to the sides b_1 of D and d_1 of D_1 does not occur in either term and precisely one of the two exterior unit normals of the two sides of D neighboring b_1, in this case ν_{12}, occurs. To see this, we observe that if $\{i, j, k\} = \{1, 2, 3\}$, then

$$\angle(-\tilde{u}_i, \nu_{ij}) < 0.16 \tag{4.30}$$

because $\angle(-\tilde{u}_i, \nu_{ij}) = \angle(q_i - v_j, v_k - v_j)$ satisfies $\tan\angle(-\tilde{u}_i, \nu_{ij}) \leq (t_i h_i)/(1 - 0.06) < 0.16$ using the estimates (4.26), $h_i < 1.77$, and $t_i \leq \frac{1}{12}$ (cf. (4.4)). On the one hand, if $\{i, j\} = \{2, 3\}$, then (4.10), (4.30), and $\sqrt{\varepsilon} < (6 \cdot 180)^{-1}$ yield

$$\begin{aligned}\angle(\tilde{u}_1, v_{i1}) &= \angle(\tilde{u}_1, -\tilde{u}_i) - \angle(-\tilde{u}_i, v_{i1}) = \alpha_j - \angle(-\tilde{u}_i, v_{i1}) \\ &\geq \frac{\pi}{3} - 10\sqrt{\varepsilon} - 0.16 > 0.87;\end{aligned}$$

therefore the angle of $\tilde{u}_1$ with any other exterior unit normal to D or D_1 is at least 0.87. On the other hand, $\angle(-\tilde{u}_1, v_{1j}) < 0.16$ for $j = 2, 3$ by (4.30), concluding the proof of (4.29).

Since

$$S_D(v_{12}) = \frac{1}{2}\|q_1 - v_2\|, \qquad S_D(\tilde{u}_1) = \frac{1}{2}(t_2 + t_3)a_1 = t_2 + t_3,$$
$$|\det(\tilde{u}_1, v_{12})| = \frac{t_1 h_1}{\|q_1 - v_2\|},$$

we deduce from (4.29) that

$$A(D, -D) = A(D_1, -D_1) + \frac{1}{2}t_1(t_2 + t_3)A(T). \tag{4.31}$$

Next, we observe that d_2 dissects D_1 into a trapezoid and a polygon D_2 with eight sides. Choosing a unit vector $u_2 \neq \tilde{u}_2$ close enough to $\tilde{u}_2$ and applying Betke's formula (2.4) to calculate $A(D_1, -D_1) - A(D_2, -D_2)$ with u_2 as a reference vector, we conclude as before that

$$A(D_1, -D_1) = A(D_2, -D_2) + \frac{1}{2}t_2(t_1 + t_3)A(T).$$

Finally, the analogous argument for a unit vector $u_3 \neq \tilde{u}_3$ close enough to $\tilde{u}_3$ implies that

$$A(D_2, -D_2) = A(D_0, -D_0) + \frac{1}{2}t_3(t_1 + t_2)A(T).$$

Thus we arrive at

$$A(D, -D) = A(D_0, -D_0) + \frac{1}{2}[t_1(t_2 + t_3) + t_2(t_1 + t_3) + t_3(t_1 + t_2)]A(T),$$

which completes the proof of (4.28).

We recall that $A(T, -T) = 2A(T)$ and $A(H_2, -H_2) = 2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1)$ and observe that $A(H_2, -T) = A(T, -H_2) = A(T, -T)$ by the symmetry and rigid motion invariance of the mixed area and Minkowski's formula (2.1). Thus (4.28) and the linearity of the mixed area imply

$$\begin{aligned}A(D, -D) &= A(D_0, -D_0) + A(T)(t_1t_2 + t_2t_3 + t_1t_3) \\ &= A\left(\frac{1}{2}T + \frac{1}{2}H_2, -\frac{1}{2}T - \frac{1}{2}H_2\right) + A(T)(t_1t_2 + t_2t_3 + t_1t_3) \\ &= \frac{1}{4} \cdot 2A(T) + \frac{2}{4} \cdot 2A(T) + \frac{1}{4} \cdot 2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1) \\ &\quad + A(T)(t_1t_2 + t_2t_3 + t_1t_3) \\ &= 2A(T) + \frac{3}{2}A(T)(t_1t_2 + t_2t_3 + t_3t_1).\end{aligned}$$

We deduce from (4.18), $K \subset D$ (see (4.27)), (4.25), and $A(T) \geq 3/2$ that

$$\begin{aligned} 5\varepsilon &\geq A(H_2, -H_2) - A(K, -K) \geq A(H_2, -H_2) - A(D, -D) \\ &= 2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1) - 2A(T) - \frac{3}{2}A(T)(t_1t_2 + t_2t_3 + t_3t_1) \\ &= \frac{1}{2}A(T)(t_1t_2 + t_2t_3 + t_3t_1) \geq \frac{1}{2} \cdot \frac{3}{2} \cdot 2 \cdot (86 \cdot \gamma)\sqrt{\varepsilon}^2 \geq \gamma^2\varepsilon, \end{aligned}$$

which is a contradiction, proving that (4.24) does not hold. Therefore the argument in Step 3 proves Proposition 4.1. □

5. Proof of Theorem 1.3

Before starting the actual proof of Theorem 1.3, we recall Proposition 5.1 and Lemma 5.2 proved in essence by Betke and Weil as Lemmas 1 and 2 in [5]. We slightly modified the argument from [5] and added a useful observation.

PROPOSITION 5.1 (Betke and Weil [5]). *If P is a polygon with $k \geq 3$ sides, and P is not a regular polygon with an odd number of sides, then there exists a polygon P' with k sides and arbitrarily close to P such that*

$$\frac{L(P')^2}{A(P', -P')} < \frac{L(P)^2}{A(P, -P)}.$$

Proof. For a vertex v of the polygon P, we write $N_P(v)$ for the normal cone of P at v; that is, if $u_1, u_2 \in \mathcal{U}(P)$ are the exterior unit normals of the two sides meeting at v, then $N_P(v) = \text{pos}\{u_1, u_2\}$.

Case 1. *There exist vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$.*

In this case, P is not a triangle. Let v_3, v_4 be the neighbors of v_1. For $v_1' \in [v_1, v_4] \backslash \{v_1, v_4\}$, let P' be obtained from P by replacing the vertex v_1 by v_1'. In particular, there exists a unit vector $w \in -\text{int}\, N_P(v_1)$ orthogonal to $[v_1', v_3]$. Using this vector w in (2.4) and the property $-N_P(v_1) \subset N_P(v_2)$, we get $A(P', -P') = A(P, -P)$, whereas obviously $L(P') < L(P)$ by strict containment.

Case 2. *There exist no vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$ as in Case 1, but there exist a vertex v and a side e with exterior normal $u_0 \in \mathcal{U}(P)$ such that $-u_0 \in N_P(v)$ and $-u_0$ does not halve the angle of $N_P(v)$.*

In this case, P does not have any parallel sides, since we are not in Case 1. The line ℓ through v parallel to e is a support line of P. Let v_4 denote the point preceding v and v_3 the point following v on ∂P in the clockwise order. Let ν_0 denote the unit vector orthogonal to u_0 such that $(v_4 - v)/\|v_4 - v\|$, $(v_3 - v)/\|v_3 - v\|$, and ν_0 are in counterclockwise order on the unit circle. Let α_4 denote the angle enclosed by $v_4 - v$ and $-\nu_0$, and let α_3 denote the angle enclosed by $v_3 - v$ and ν_0. Let u_i denote the exterior unit normal of $[v_i, v]$ for $i = 3, 4$. Since $-u_0$ does not halve the angle enclosed by u_3, u_4, we have $\alpha_3 \neq \alpha_4$. We may assume that $\alpha_4 > \alpha_3$. By Fermat's principle, moving v along ℓ an arbitrarily small amount in the direction of ν_0 to v' and denoting by P' the polygon obtained from P by replacing v by v', we get $L(P') < L(P)$. Clearly, we thus still get a convex k-gon if v' is sufficiently close to v.

To prove $A(P, -P) = A(P', -P')$, we denote by Q the (nonempty) convex hull of the (common) vertices of P and P' (thus v, v' are removed). Note that in the case where P (and hence also P') is a triangle, Q is a segment. We consider the triangles $\triangle = [v_3, v_4, v]$ and $\triangle' = [v_3, v_4, v']$ that satisfy $P = Q \cup \triangle$ and $P' = Q \cup \triangle'$. For the segment $I = [v_3, v_4]$, we have $A(\triangle, -I) = A(\triangle, I) = A(\triangle)$. Using this and the additivity of the mixed area in both arguments, we obtain

$$A(P, -P) = A(Q, -Q) + 2A(Q, -\triangle) - 2A(Q, -I).$$

A similar expression is obtained for P' with $\triangle$ replaced by $\triangle'$.

We choose v_5, v_6 such that $e = [v_5, v_6]$ and $v_6 - v_5$ is a positive multiple of $v' - v$. If v' is sufficiently close to v, then the first assumption in Case 2 ensures that for the exterior unit normals u_j of the sides of Q between v_4 and v_5 (in counterclockwise order), we have

$$h_\triangle(u_j) = \langle v_5, u_j\rangle = h_{\triangle'}(u_j)$$

and that for the exterior unit normals u_j of the sides of Q between v_6 and v_3 (in counterclockwise order), we have

$$h_\triangle(u_j) = \langle v_6, u_j\rangle = h_{\triangle'}(u_j).$$

Then (writing e also for the length of the edge e)

$$A(Q, -\triangle) - A(Q, -\triangle') = e\langle u_0, -v\rangle - e\langle u_0, -v'\rangle = e\langle u_0, v' - v\rangle = 0,$$

which proves the statement.

Case 3. *There exist no vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$, and for any vertex v and side e with exterior normal $u_0 \in \mathcal{U}(P)$ such that $-u_0 \in N_P(v)$, the vector u_0 halves the angle of $N_P(v)$, but P is not a regular polygon with an odd number of sides.*

Let $u_1, \dots, u_k$ be the exterior unit normals to the sides of P in clockwise order. We observe that no closed half-plane having the origin on its boundary contains $u_1, \dots, u_k$. We set $u_{i+k} = u_i$ for $i = 1, \dots, k$.

We can assume that $\angle(u_1, u_2) = \min\{\angle(u_i, u_{i+1}) : i \in \{1, \dots, k\}$. By the first assumption of Case 3, $u_i \neq -u_j$ for $i, j \in \{1, \dots, k\}$. Hence there is a unique $m \in \{2, \dots, k-1\}$ such that $-u_1 \in \operatorname{int}\operatorname{pos}\{u_m, u_{m+1}\}$. By the second assumption of Case 3 we have $\angle(-u_1, u_m) = \angle(-u_1, u_{m+1}) =: \alpha_1$. Moreover, applying twice the first assumption of Case 3, it follows that $-u_2 \in \operatorname{int}\operatorname{pos}\{u_{m+1}, u_{m+2}\}$, and by the second assumption we then get $\angle(-u_2, u_{m+1}) = \angle(-u_2, u_{m+2}) =: \alpha_2$. Thus we obtain $\alpha_1 + \alpha_2 = \angle(-u_1, -u_2) = \angle(u_1, u_2)$. By the minimality of $\angle(u_1, u_2)$ we conclude that $\alpha_1 = \alpha_2$, and hence $\angle(u_m, u_{m+1}) = \angle(u_{m+1}, u_{m+2}) = \angle(u_1, u_2)$.

Considering successively in this way the normal vectors $u_3, u_{m+3}, \dots, u_{m-1}$, u_k, u_m, u_1, we conclude that $k = 2m - 1$ is odd, $-u_i \in \operatorname{int}\operatorname{pos}\{u_{i+m-1}, u_{i+m}\}$ halves the angle enclosed by u_{i+m-1} and u_{i+m}, and $\angle(u_1, u_2) = \angle(u_i, u_{i+1})$ for $i \in \{1, \dots, k\}$. In turn, we find that all exterior angles of P are $2\pi/k$.

For $i \in \{1, \dots, k\}$, we denote by e_i both the side corresponding to u_i of P and its length, and we denote by ℓ_i the line containing e_i. We set $\mathsf{e}_{i+k} = \mathsf{e}_i$ for $i = 1, \dots, k$. Since P is not regular, the side lengths of P are not the same. Therefore there exist $2 \le p < q \le k+1$ such that

$$\mathsf{e}_{p-1} + \mathsf{e}_p \neq \mathsf{e}_{q-1} + \mathsf{e}_q. \tag{5.1}$$

Fix $i \in \{1, \dots, k\}$ for the moment. If $|t|$ is small, then let $P_{i,t}$ be the k-gon bounded by the lines ℓ_j, $j \in \{1, \dots, k\} \backslash \{i\}$, and $\ell_i + t u_i$. Set $\kappa = 2\frac{1-\cos\frac{2\pi}{k}}{\sin\frac{2\pi}{k}}$. Then we get

$$L(P_{i,t}) = L(P) + 2t \cdot \left(\frac{1}{\sin\frac{2\pi}{k}} - \frac{1}{\tan\frac{2\pi}{k}} \right) = L(P) + \kappa \cdot t.$$

Choosing $w = -u_i$ in (2.4), we see that

$$A(P_{i,t}, -P_{i,t}) = A(P, -P) + (\mathsf{e}_{i+m} + \mathsf{e}_{i+m-1}) \cdot \varrho \cdot t,$$

where $\varrho = \frac{\sin\frac{\pi}{k}}{\sin\frac{2\pi}{k}}$. We deduce that

$$\frac{d}{dt} \frac{A(P_{i,t}, -P_{i,t})}{L(P_{i,t})^2} \Big|_{t=0} = \frac{(\mathsf{e}_{i+m} + \mathsf{e}_{i+m-1}) \cdot \varrho \cdot L(P) - A(P, -P) \cdot 2\kappa}{L(P)^3}. \tag{5.2}$$

From (5.1) and (5.2) we conclude the existence of $i \in \{1, \dots, k\}$ such that

$$\frac{d}{dt} \frac{A(P_{i,t}, -P_{i,t})}{L(P_{i,t})^2} \Big|_{t=0} \neq 0.$$

In particular, we can choose a $t \neq 0$ with arbitrarily small absolute value such that

$$\frac{A(P_{i,t}, -P_{i,t})}{L(P_{i,t})^2} > \frac{A(P, -P)}{L(P)^2},$$

and hence we can choose $P' = P_{i,t}$, completing the proof of Proposition 5.1. □

For regular polygons with an odd number of sides, we have the following estimates.

LEMMA 5.2. *If P is a regular polygon with an odd number $k \ge 5$ of sides, then*

$$d_{\mathrm{tr}}(P) > 0.25, \tag{5.3}$$

$$\frac{L(P)^2}{A(P, -P)} \ge 20 \sin\frac{\pi}{5} > 1.1 \cdot 6\sqrt{3}. \tag{5.4}$$

Proof. We may assume that $L(P) = 1$, and hence $A(P, -P) = (4k \sin\frac{\pi}{k})^{-1}$ according to (2.2), proving (5.4). For (5.3), assuming that the origin is the centroid of P, we have $-P \subset (\cos\frac{\pi}{k})^{-1} P$, and hence $A(P, -P) \le (\cos\frac{\pi}{k})^{-1} A(P)$. On the other hand, if $T_0 \subset P$ is a regular triangle with centroid z_0 and $P - z_0 \subset (1+d)(T_0 - z_0)$, then

$$A(P, -P) \ge A(T_0, -T_0) = 2A(T_0) \ge 2(1+d)^{-2} A(P),$$

and hence $d_{\mathrm{tr}}(P) \ge \sqrt{2\cos\frac{\pi}{5}} - 1 > 0.25$.

Proof of Theorem 1.3. Let $\varepsilon \in [0, 2^{-28}]$, and let $K \subset \mathbb{R}^2$ be a convex domain with $L(K)^2 \leq (1+\varepsilon)6\sqrt{3}A(K,-K)$. Then $0 \leq \varepsilon < (6 \cdot 2400)^{-2} < (6 \cdot 180)^{-2}$. We distinguish two cases.

Case 1: $d_{\mathrm{tr}}(K) \leq 6^{-2}$. Then Proposition 4.1 implies that $d_{\mathrm{tr}}(K) \leq 400\sqrt{\varepsilon}$, and the proof is finished.

Case 2: $d_{\mathrm{tr}}(K) > 6^{-2}$. We will show that in fact this case does not occur. We fix a number ε' such that $0 \leq \varepsilon < \varepsilon' < (6 \cdot 2400)^{-2}$. Since $d_{\mathrm{tr}}(\cdot)$, $A(\cdot,\cdot)$, and $L(\cdot)$ are continuous, there is a polygon $P \subset \mathbb{R}^2$ with $d_{\mathrm{tr}}(P) > 6^{-2}$ and $L(P)^2 \leq (1+\varepsilon')6\sqrt{3}A(P,-P)$. Since $d_{\mathrm{tr}}(\cdot)$, $A(\cdot,\cdot)$, and $L(\cdot)$ are translation invariant, $d_{\mathrm{tr}}(\cdot)$ is scaling invariant, and $K \mapsto L^2(K)/A(K,-K)$ (for convex domains $K \subset \mathbb{R}^2$) is also scaling invariant, we can assume that $o \in P$ and $A(P,-P) = 1$. Let $k \geq 3$ be the number of vertices of P. We write $\mathcal{P}_k$ for the set of all polygons $G \subset \mathbb{R}^2$ with at most k vertices, $d_{\mathrm{tr}}(G) \geq 6^{-2}$, $o \in G$, $A(G,-G) = 1$, and $L(G)^2 \leq (1+\varepsilon')6\sqrt{3}$. Then in particular we have $P \in \mathcal{P}_k \neq \emptyset$.

We claim that there is $P_0 \in \mathcal{P}_k$ such that $L(P_0) = \inf\{L(G) : G \in \mathcal{P}_k\}$. For the proof, it is sufficient to consider a minimizing sequence $P_i \in \mathcal{P}_k$ with $L(P_i) \leq L(P)$ for $i \in \mathbb{N}$. Let B^2 denote the unit disc with center at the origin. Then $P_i \subset L(P)B^2$ for $i \in \mathbb{N}$. An application of Blaschke's selection theorem (see [17, Theorem 1.8.7] or [13, Theorem 3.4]) shows that the sequence P_i, $i \in \mathbb{N}$, has a convergent subsequence with limit $P_0 \subset \mathbb{R}^2$. Since all conditions involved in the definition of $\mathcal{P}_k$ are preserved under limits and $L(\cdot)$ is continuous, we conclude that $P_0 \in \mathcal{P}_k$ realizes the infimum.

Since $d_{\mathrm{tr}}(P_0) \geq 6^{-2}$, P_0 is not a regular triangle. Assuming (for the moment) that P_0 is a regular r-gon with r odd, we have $k \geq r \geq 5$. Then Lemma 5.2 shows that $L(P_0)^2 \geq 1.1 \cdot 6\sqrt{3}$. Since also $L(P_0)^2 \leq (1+\varepsilon')6\sqrt{3}$, we get $\varepsilon' \geq 0.1$, a contradiction.

Hence $P_0 \in \mathcal{P}_k$ is a k-gon but not a regular polygon with an odd number of edges. Assume (for the moment) that $d_{\mathrm{tr}}(P_0) > 6^{-2}$. Proposition 5.1 then shows that there is a k-gon P_1 such that $d_{\mathrm{tr}}(P_1) > 6^{-2}$ and

$$\frac{L(P_1)^2}{A(P_1,-P_1)} < \frac{L(P_0)^2}{A(P_0,-P_0)} = L(P_0)^2.$$

Again by scaling and translation invariance we obtain a k-gon P_2 for which $d_{\mathrm{tr}}(P_2) > 6^{-2}$, $A(P_2,-P_2) = 1$, $o \in P_2$, and $L(P_2)^2 < L(P_0)^2 \leq (1+\varepsilon')6\sqrt{3}$, that is, $P_2 \in \mathcal{P}_k$. This contradicts the minimality of $L(P_0)$. Therefore we conclude that $d_{\mathrm{tr}}(P_0) = 6^{-2}$. Since $L(P_0)^2 \leq (1+\varepsilon')6\sqrt{3}A(P_0,-P_0)$, it follows from another application of Proposition 4.1 that

$$6^{-2} = d_{\mathrm{tr}}(P_0) \leq 400\sqrt{\varepsilon'} < 400 \cdot (6 \cdot 2400)^{-1} = 6^{-2},$$

a contradiction. This finally shows that the present case does not occur, which completes the argument.

Acknowledgment. The authors are grateful for helpful discussions with Martin Henk on the subject of the paper.

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