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Edge-minimum saturated k-planar drawings

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Abstract

For a class $\mathcal{D}$ of drawings of loopless (multi-)graphs in the plane, a drawing $D \in \mathcal{D}$ is *saturated* when the addition of any edge to D results in $D' \notin \mathcal{D}$—this is analogous to saturated graphs in a graph class as introduced by Turán and Erdős, Hajnal, and Moon. We focus on k-planar drawings, that is, graphs drawn in the plane where each edge is crossed at most k times, and the classes $\mathcal{D}$ of all k-planar drawings obeying a number of restrictions, such as having no crossing incident edges, no pair of edges crossing more than once, or no edge crossing itself. While saturated k-planar drawings are the focus of several prior works, tight bounds on how sparse these can be are not well understood. We establish a generic framework to determine the minimum number of edges among all n-vertex saturated k-planar drawings in many natural classes. For example, when incident crossings, multicrossings and selfcrossings are all allowed, the sparsest n-vertex saturated k-planar drawings have $\frac{2}{k-(k \bmod 2)}(n-1)$ edges for any $k \geq 4$, while if all that is forbidden, the sparsest such drawings have $\frac{2(k+1)}{k(k-1)}(n-1)$ edges for any $k \geq 6$.

KEYWORDS

k-planar graphs, multigraphs, saturated drawings

1 | INTRODUCTION

Graph *saturation problems* concern the study of edge-extremal n-vertex graphs under various restrictions. They originate in the works of Turán [36] and Erdős et al. [17]. For a family $\mathcal{F}$ of graphs, a graph G without loops or parallel edges is called $\mathcal{F}$*-saturated* when no subgraph of G belongs to $\mathcal{F}$ and for every $u, v \in V(G)$, where $uv \notin E(G)$, some subgraph of the graph $G + uv$ belongs to $\mathcal{F}$. Turán [36] described, for each t, the n-vertex graphs that are $\{K_t\}$-saturated and have the maximum number of edges—this led to the introduction of the Turán numbers where the setting moves from graphs to hypergraphs, see for example the surveys [26, 34]. Analogously, Erdős et al. [17] studied the n-vertex graphs G that are $\{K_t\}$-saturated and have the minimum number of edges. This sparsest saturation view has also received much subsequent study [18], and our work fits into this latter direction but concerns "drawings of (multi-)graphs," also called *topological (multi-)graphs*.

There has been increasing interest in saturation problems on drawings of (multi-)graphs in addition to the abstract graphs above. A *drawing* is a graph together with a cyclic order of edges around each vertex and the sequence of crossings along each edge so that it can be realized in the plane (or on another specified surface). The saturation conditions usually concern the crossings (which can be thought of as avoiding certain topological subgraphs). The majority of work has been on *Turán-type* results regarding the maximum number of edges which can occur in an n-vertex drawing (without loops and homotopic parallel edges) of a particular drawing style. For example, n-vertex planar (crossing-free) drawings are well known to have at most $3n - 6$ edges for any $n \geq 3$. In the case of planar drawings, the sparsest saturation version (as in Erdős et al. [17]) is also equal to the Turán version: Every saturated planar drawing has $3n - 6$ edges.

However, for drawing styles that allow crossings in a limited way, these two measures become nontrivial to compare and can indeed be quite different, as first observed by Brandenburg et al. [10]. This interesting phenomenon happens for example for k*-planar* drawings where at most k crossings on each edge are allowed; and which are the focus of the present paper. The left of Figure 1 depicts a drawing of the 8-cycle C_8 in which each edge is crossed exactly four times and one cannot add a ninth (nonloop) edge to the drawing while maintaining 4-planarity, that is, this is a saturated 4-planar drawing of C_8. On the other hand, note that even the complete graph K_8 in fact admits 3-planar drawings as shown in the middle of Figure 1.

In this sense, we call a drawing that attains the Turán-type maximum number of edges a *max-saturated* drawing, while a sparsest saturated drawing is called *min-saturated* (max-saturated drawings are also called *optimal* in the literature [9] while saturated drawings in general are also called *maximal*). The target of this paper is to determine the number of

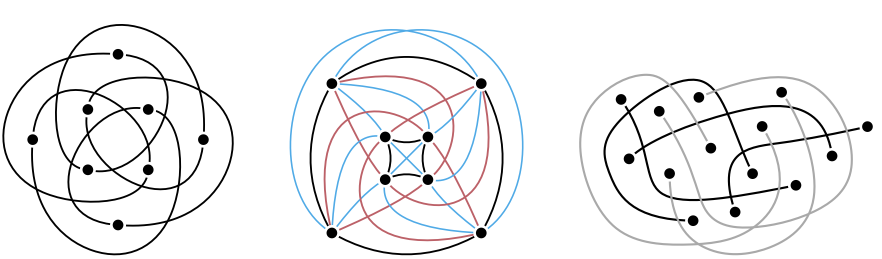

FIGURE 1 Saturated 4-planar drawing of the 8-cycle (left), 3-planar drawing of the 8-clique (middle), and saturated 6-planar drawing of the 7-matching (right). [Color figure can be viewed at wileyonlinelibrary.com]

edges in min-saturated k-planar drawings of loopless (multi-)graphs, that is, the *smallest* number of edges among all saturated k-planar drawings with n vertices. Somewhat mysteriously, the answer for the cases considered here will always be of the form $\alpha_k \cdot (n-1)$. However, it turns out that the precise value of α_k depends on numerous subtleties of what precisely we allow in the considered k-planar drawings. Such subtleties are formalized by drawing styles Γ later, each one with its own constant α_Γ. As we always require k-planarity, we omit k from the notation α_Γ.

For example, by restricting to connected graphs, we have at least $n-1$ edges on n vertices, that is, $\alpha_\Gamma \geq 1$. And in fact we also have $\alpha_\Gamma \leq 1$ for all $k \geq 4$ as testified by entangled drawings of cycles like in the left of Figure 1. Allowing disconnected graphs but restricting to contiguous[1] drawings, we immediately have $\alpha_\Gamma \geq 1/2$ since we have minimum degree at least 1 in that case. And again we also have $\alpha_\Gamma \leq 1/2$ for all $k \geq 6$ as one can find saturated k-planar drawings of matchings like in the right of Figure 1. Other subtleties occur when we distinguish whether selfcrossing edges, repeatedly crossing edges, crossing incident edges, and so on, are allowed or forbidden. We enable a concise investigation by first deriving lower bounds on α_Γ for any drawing style that satisfies only some mild assumptions. We can then consider specific drawing styles Γ given by combinations of the crossing restrictions mentioned above and swiftly determine the exact value of α_Γ, thus determining the smallest number of edges among all k-planar drawings of that style on n vertices. Our results for multigraphs are summarized in Table 1.

An extended abstract of this paper appeared and won the best paper award at the 29th International Symposium on Graph Drawing and Network Visualization held in 2021 [13].

1.1 | Related work

Many results in the literature concern *simple* drawings. In such a drawing any two edges share at most one point which implies that there are no parallel edges. For k-planar graphs the Turán-type question, the edge count in max-saturated drawings, is well studied. Any k-planar simple drawing on n vertices contains at most $3.81\sqrt{k}\,n$ edges [1], and better (and tight) bounds are known for small k [1, 30, 31]. Specifically 1-planar drawings contain at most $4n-8$ edges which is tight [31]. For $k \leq 3$, any k-planar drawing with the fewest crossings (among all k-planar drawings of the abstract graph) is necessarily simple [30]. Therefore the tight bounds for $k \leq 3$ also hold for drawings that are not necessarily simple. However, already for $k=4$, Schaefer [33, p. 58] has constructed k-planar graphs having no k-planar simple drawings, and these easily generalize to all $k > 4$. Pach et al. [30] conjectured that for every k there is a max-saturated k-planar graph with a simple k-planar drawing. For $k = 2, 3$, the max-saturated k-planar homotopy-free multigraphs have been characterized [9].

In the sparsest saturation setting not only min-saturated k-planar drawings are of interest but also min-saturated k-planar (abstract) graphs: sparse k-planar graphs that are no longer k-planar after adding any edge. Brandenburg et al. [10] and independently Eades et al. [16] constructed saturated 1-planar n-vertex graphs with only $2.64n$ edges and saturated 1-planar drawings with $2.33n$ edges. Barát and Tóth [7] show that any saturated 1-planar n-vertex drawing ($n \geq 4$) has at least $\frac{20}{9}n - \frac{10}{3} \approx 2.22n$ edges, but they remark that their bounds seem suboptimal. For $k=2$, Auer et al. [4] construct saturated 2-planar drawings with $1.33n$ edges,

[1]The planarization (defined in Section 1.2) is connected.

TABLE 1 Overview of results (see also Theorem 5): The minimum number of edges of saturated k-planar drawings on n vertices of a drawing style defined by a set of restrictions.

k	Restrictions	Minimum number of edges of saturated k-planar drawings on n vertices	Tight example
$k \geq 4$	no restriction **I** no incident crossings	$\frac{2}{k-(k \bmod 2)} \cdot (n-1)$	Figure 2
$k \geq 4$	**S** no selfcrossings **S** no self- and **I** no incident crossings	$\frac{2}{k-1} \cdot (n-1)$	Figure 3
$k \geq 4$	**M** no multicrossings	$\frac{2(k-1)}{(k-1)(k-2)+2} \cdot (n-1)$	Figure 4
$k \geq 4$	**S** no self- and **M** no multicrossings	$\frac{2(k+1)}{k(k-1)} \cdot (n-1)$	Figure 5
$k = 4$		$\frac{4}{5} \cdot (n-1)$	
$k \geq 5$	**I** no incident and **M** no multicrossings	$\frac{2(k-1)}{(k-1)(k-2)+2} \cdot (n-1)$	Figures 6 and 7
$k \geq 6$	**S** no self-, **M** no multi-, and **I** no incident crossings **S** no self-, **M** no multi-, **I** no incident crossings, and **H** homotopy-free	$\frac{2(k+1)}{k(k-1)} \cdot (n-1)$	Figure 1 (right) and Figure 8

Note: To attain the stated bounds via the constructions given in the respective figures, insert an isolated vertex in each empty cell.

while Barát and Tóth [8] show that any saturated simple 2-planar drawing has at least $n-1$ edges. For saturated 2-planar (abstract) graphs, Hoffmann and Reddy [22] show that any such graph has at least $2n$ edges and construct a saturated 2-planar graph with $2n + O(1)$ edges. The questions we address in this work have also been explicitly asked [24, section 3.2].

Recently, the case of saturation problems for simple drawings has come into focus. The Turán-type question is trivial here as all complete graphs have simple drawings. However, knowing when a given simple drawing is saturated turns out to be rather complex as it has recently been shown that it is NP-complete to decide whether a given simple drawing is saturated [2]. In fact, it is NP-complete to decide whether a single edge can be inserted into a simple drawing [3]. Contrary to the simple drawings of complete graphs, there are constructions of saturated simple drawings (and generalizations thereof) with only $O(n)$ edges [21, 28]. The Turán-type question was studied also for simple drawings of multigraphs [20, 25, 32] where the results distinguish between various drawing restrictions.

In the case of *k-quasiplanar*[2] graphs the focus has been on the Turán-type question. It was conjectured that, for every fixed k, every k-quasiplanar graph has $O(n)$ edges. This has been verified for $k \leq 4$, but the best general upper bound for simple k-quasiplanar graphs is $c_k n \log n$ for some constant c_k and has been improved slightly for some special cases [35]. For t-simple[3] k-quasiplanar graphs a general bound $2^{\alpha(n)^c} n \log n$ is known where c is a

[2] A drawing is k-quasiplanar if every k-set of edges contains a pair of edges that do not cross each other.

[3] A drawing is t-simple if any two edges share at most t points.

constant depending on k and t and $\alpha(n)$ is the inverse of the Ackermann function [35]. Seemingly the only min-saturation results for k-quasiplanar drawings concern so-called *outer* or *convex* drawings in which all of the vertices occur on the boundary of a single face of the drawing. Here, as in planar drawings, the min-saturated drawings and max-saturated outer k-quasiplanar n-vertex drawings coincide [12, 15, 29]. In particular, Capoyleas and Pach [11] show that every saturated outer k-quasiplanar drawing on $n \geq 2k + 1$ vertices has exactly $2(k-1)n - \binom{2k-1}{2}$ edges.

Also for the concept of *gap-planarity* [5], which generalizes the notion of k-planarity, the focus so far has been on the Turán-type question.

For further results, consider the surveys and book on *beyond planar* graph classes [14, 23, 27], the report on sparsest saturation [24, section 3.2].

1.2 | Drawings, crossing restrictions, and drawing types

Throughout the paper, we consider topological drawings in the plane, that is, vertices are represented by distinct points in $\mathbb{R}^2$ and edges are represented by continuous curves connecting their respective endpoints. We allow parallel edges but forbid loops. As usual, edges do not pass through vertices, any two edges have only finitely many interior points in common, each of which is a proper crossing, and no three edges cross in a common point. An edge may cross itself but it uses any crossing point at most twice. Also, each of these selfcrossings are counted twice when considering the number of times that edge is crossed.

The *planarization* of a drawing D is the planar drawing obtained from D by making each crossing into a new vertex, thereby subdividing the edges involved in the crossing. Although we forbid loops in D its planarization might have loops due to selfcrossing edges. In a drawing, an edge involved in at least one crossing is a *crossed* edge, while those involved in no crossing are the *planar* or *uncrossed* edges. The *cells* of a drawing are the connected components of the plane after the removal of every vertex and edge in D. In other words, the cells of D are the faces of its planarization. A vertex v is incident to a cell c if v is contained in the boundary of c. Thus, in this case one could at least start drawing an uncrossed edge from v into cell c.

Two distinct parallel edges e and f in a drawing D are called *homotopic*, if there is a homotopy of the sphere between e and f, that is, the curves of e and f can be continuously deformed into each other along the surface of the sphere while all vertices of D are treated as holes.

In what follows, we investigate drawings that satisfy a specific set of restrictions, where we focus on those with frequent appearance in the literature:

- k-*planar*: Each edge is crossed at most k times.
- **H** *homotopy-free*: No two distinct parallel edges are homotopic.
- **M** *single-crossing*: Any pair of edges crosses at most once and any edge crosses itself at most once (edges with $t \in \{0, 1, 2\}$ common endpoints have at most $t + 1$ common points). In this case we say that there are no *multicrossings*.
- **I** *locally starlike*: Incident edges do not cross (selfcrossing edges are allowed). In this case we say that there are no *incident crossings*. In other papers this is also called *star simple* or *semi simple* [6, 19] and may not allow selfcrossing edges.
- **S** *selfcrossing-free*: No edge crosses itself.
- *branching*: The drawing is **M** single-crossing, **I** locally starlike, **S** selfcrossing-free, and **H** homotopy-free.

A *drawing style* is just a class Γ of drawings, that is, a predicate whether any given drawing D is in Γ or not. A drawing style Γ is *monotone* if removing any edge or vertex from any drawing $D \in \Gamma$ results again in a drawing $D' \in \Gamma$, i.e., Γ is closed under edge/vertex removal.

Let X be a subset of $\{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ and $k \geq 0$ be an integer. Then we define $\Gamma_{X,k}$ to be the drawing style given by all k-planar drawings of finite, loopless multigraphs obeying the subset X of the restrictions above. We focus on the restrictions **M** single-crossing, **S** selfcrossing-free, **I** locally starlike, and **H** homotopy-free. Note that the k-planar drawing style is monotone, and so is $\Gamma_{X,k}$ for each $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$. However, the style of all homotopy-free drawings is not monotone, as removing a vertex may render two edges homotopic.

We are interested in k-planar drawings in $\Gamma_{X,k}$ to which no further edge can be added without either violating k-planarity or any of the restrictions in X, and particularly in how sparse these drawings can be; namely, the sparsest saturated such drawings.

Definition A drawing D is Γ-*saturated* for drawing style Γ if $D \in \Gamma$ and the addition of any new edge to D results in a drawing $D' \notin \Gamma$.

1.3 | Outline of our results

To determine the sparsest k-planar $\Gamma_{X,k}$-saturated drawings for restrictions in X, we introduce in Section 2 the concept of *filled* drawings in general monotone drawing styles and give lower bounds on the number of edges in these. Using the lower bounds for filled drawings and constructing particularly sparse $\Gamma_{X,k}$-saturated drawings, we then give in Section 3 the precise answer for all $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ and for the branching style, that is, $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$, leaving open only a few cases for $k \in \{4, 5\}$. Our results for multigraphs are summarized in Table 1 and formalized in Theorem 5. In Section 4 we discuss saturated drawings of simple graphs instead of multigraphs. Finally, in Section 5 we discuss further extensions.

2 | LOWER BOUNDS AND FILLED DRAWINGS

Recall that Γ is monotone if it is closed under the removal of vertices and/or edges. Throughout this section, let Γ be an arbitrary monotone drawing style; not necessarily k-planar or defined by any of the restrictions in Section 1.2.

Definition A drawing D is *filled* if any two distinct vertices that are incident to the same cell c of D are connected by an uncrossed edge that lies completely in the boundary of c.

For example, the filled crossing-free homotopy-free drawings are exactly the planar drawings of loopless multigraphs with every face bounded by three edges. Using Euler's formula, such drawings on $n \geq 3$ vertices have exactly $m = 3n - 6$ edges. In this section we derive lower bounds on the number of edges in n-vertex filled drawings in drawing style Γ. Another important example of filled drawings are those in which every cell has at most one incident vertex. Note that every cell in a filled drawing has at most three incident vertices. Generally, for a drawing D we use the following notation:

$$n_D = \#\text{ vertices} \qquad c_i(D) = \#\text{ cells with exactly } i \text{ incident vertices}, i \geq 0$$
$$m_D = \#\text{ edges} \qquad c_2'(D) = \#\text{ cells with 2 uncrossed edges in their boundary.}$$

For a drawing D, let G be its underlying graph and P be its planarization. A *component* of D is a connected component of P. A *cut-vertex* of D is a cut-vertex of G that is also a cut-vertex of P. And finally, D is *essentially* 2*-connected* if it contains at least one edge, and if we remove all isolated vertices, the remaining drawing does not have a cut-vertex. This means that for each simple closed curve that intersects D in exactly one vertex (of G)[4] or not at all, either the interior or the exterior contains no edges from D.

Lemma 1. *For every monotone drawing style* Γ *and every filled drawing* $D \in \Gamma$ *we have* $m_D \geq \alpha_\Gamma \cdot (n_D + c_0(D) - 1)$ *where*

$$\alpha_\Gamma = \min\left\{\frac{m_{D'}}{n_{D'} + c_0(D') - 1} : D' \in \Gamma \textit{ is filled and essentially 2-connected}\right\}.$$

Proof. We proceed by induction on the number n_D of vertices in D. The desired inequality $m_D \geq \alpha_\Gamma(n_D + c_0(D) - 1)$ clearly holds if D itself is essentially 2-connected. Otherwise D has a cut-vertex or two components with an edge. In both cases we can choose a simple closed curve C with at least one vertex of D in its interior and at least one vertex in its exterior such that $D \cap C$ is either empty or a single vertex. Let D' and D'' denote the drawings obtained from D by removing every edge and vertex of D in the exterior of C, respectively interior of C. Observe that D', D'' are filled and in Γ, as Γ is monotone. Further observe that $m_{D'} + m_{D''} = m_D$, as every edge of D lies on one side of C.

Now if $C \cap D \neq \emptyset$, then $C \cap D$ consists of exactly one vertex and $n_{D'} + n_{D''} = n_D + 1$. Moreover $c_0(D') + c_0(D'') = c_0(D)$, since the vertex in $C \cap D$ is incident to the cell containing curve C in both drawings D' and D''. Hence, using induction on D' and D'' we conclude

$$\begin{aligned} m_D = m_{D'} + m_{D''} &\geq \alpha_\Gamma(n_{D'} + c_0(D') - 1) + \alpha_\Gamma(n_{D''} + c_0(D'') - 1) \\ &= \alpha_\Gamma(n_{D'} + n_{D''} - 1 + c_0(D') + c_0(D'') - 1) = \alpha_\Gamma(n_D + c_0(D) - 1). \end{aligned}$$

On the other hand, if $C \cap D = \emptyset$, then $n_{D'} + n_{D''} = n_D$. Moreover $c_0(D') + c_0(D'') = c_0(D) + 1$, since the cell of D containing curve C can have incident vertices only on one side of C, as the drawing is filled. Similar as before, we have

$$\begin{aligned} m_D = m_{D'} + m_{D''} &\geq \alpha_\Gamma(n_{D'} + c_0(D') - 1) + \alpha_\Gamma(n_{D''} + c_0(D'') - 1) \\ &= \alpha_\Gamma(n_{D'} + n_{D''} + c_0(D') + c_0(D'') - 1 - 1) = \alpha_\Gamma(n_D + c_0(D) - 1). \end{aligned}$$ □

As suggested by Lemma 1, we shall now focus on filled drawings that are essentially 2-connected. Our goal is to determine the parameter α_Γ. First, we give an exact formula for the number of edges in any filled essentially 2-connected drawing. The parameter k in the

[4]Throughout, we shall generally only refer to vertices of G. And we shall mention it explicitly in the few situations when we refer to vertices of P.

following lemma will later be the k for the k-planar drawings in Section 3. However, we do not require any drawing to be k-planar here.

Lemma 2. *For any $k > 2$, if D is a filled, essentially 2-connected drawing with $n_D \geq 3$ vertices, then $m_D = \frac{2}{k-2}(n_D + c_0(D) - 2 + \varepsilon(D))$, where*

$$\varepsilon(D) = \left(\frac{k}{2}m_{\mathrm{x}} - \mathrm{cr}\right) + \frac{k-4}{4}m_{\mathrm{p}} + c_2' + c_3, \textit{such that}$$
$$m_{\mathrm{p}} = \#\textit{planar edges}, \ \mathrm{cr} = \#\textit{crossings}, \text{ and } m_{\mathrm{x}} = \#\textit{crossed edges}.$$

Proof. First observe that, since D is filled, no cell has four or more incident vertices. Hence, $\#\text{cells} = c_0 + c_1 + c_2 + c_3$. By counting along the angles around each vertex, we see that

$$\#\text{isolated} + 2m_D = \#\text{isolated} + \sum_v \deg(v) = c_1 + 2c_2 + 3c_3. \tag{1}$$

Note that this relies on the assumption that D is essentially 2-connected, as this guarantees that each nonisolated vertex v lies on the boundary of exactly $\deg(v)$ cells.

As D is filled, each cell with exactly two vertices on its boundary is incident to either one or two planar edges and each cell with three vertices on its boundary is incident to exactly three planar edges. Moreover, each planar edge is contained in the boundary of exactly two distinct such cells since D has no cut-vertices and $n_D \geq 3$. By counting along the sides of the planar edges, we see that $2m_{\mathrm{p}} = c_2 + c_2' + 3c_3$, which together with (1) gives

$$\#\text{isolated} + 2m_{\mathrm{x}} = c_1 + c_2 - c_2'. \tag{2}$$

Consider the planarization P of D. Since D is essentially 2-connected, P has exactly $(1 + \#\text{isolated})$ many connected components. Moreover we have

$$|V(P)| = n_D + \mathrm{cr} \quad \text{and} \quad |E(P)| = m_D + 2\mathrm{cr} \quad \text{and} \quad \#\text{cells} = c_0 + c_1 + c_2 + c_3. \tag{3}$$

Applying Euler's formula to P we have

$$\begin{aligned} 2 &= |V(P)| - |E(P)| + \#\text{cells} - \#\text{isolated} \\ &\overset{(3)}{=} \mathrm{cr} + n_D - m_D - 2\mathrm{cr} + c_0 + c_1 + c_2 + c_3 - \#\text{isolated} \\ &\overset{(2)}{=} n_D - m_D - \mathrm{cr} + 2m_{\mathrm{x}} + \#\text{isolated} + c_2' + c_0 + c_3 - \#\text{isolated} \\ &= n_D + m_{\mathrm{x}} - m_{\mathrm{p}} - \mathrm{cr} + c_2' + c_0 + c_3 \\ &= n_D + \frac{2-k}{2}(m_{\mathrm{x}} + m_{\mathrm{p}}) + \frac{k-4}{2}m_{\mathrm{p}} + \left(\frac{k}{2}m_{\mathrm{x}} - \mathrm{cr}\right) + c_2' + c_0 + c_3. \end{aligned}$$

Solving for m_D we have:

$$m_D = \frac{2}{k-2} \cdot \left(n_D + c_0 - 2 + \left(\frac{k}{2}m_{\mathrm{x}} - \mathrm{cr}\right) + \frac{k-4}{4}m_{\mathrm{p}} + c_2' + c_3\right). \qquad \square$$

Lemmas 1 and 2 together imply that for any filled drawing $D \in \Gamma$ we have

$$\frac{m_D}{n_D - 1} \geq \frac{m_D}{n_D + c_0(D) - 1} \geq \min_{D'} \frac{m_{D'}}{n_{D'} + c_0(D') - 1}$$
$$= \min_{D'} \frac{2}{k-2} \cdot \frac{n_{D'} + c_0(D') - 2 + \varepsilon(D')}{n_{D'} + c_0(D') - 1},$$

where both minima are taken over all filled, essentially 2-connected drawings $D' \in \Gamma$ and $\varepsilon(D')$ can be thought of as an error term for the drawing D', which we seek to minimize. Indeed, if D' is k-planar, that is, each edge is crossed at most k times, then $2\mathrm{cr} \leq k m_{\mathrm{x}}$. Thus for $k \geq 4$ we have $\varepsilon(D') \geq 0$. In the next section, we shall see that (in many cases) the minimum is indeed attained by drawings D' with $\varepsilon(D') = 0$.

3 | EXACT BOUNDS AND SATURATED DRAWINGS

Recall that we seek to find the sparsest k-planar, $\Gamma_{X,k}$-saturated drawings in a drawing style $\Gamma_{X,k}$ that is given by a set $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$ of additional restrictions. These $\Gamma_{X,k}$-saturated drawings are related to the filled drawings from Section 2.

Lemma 3. *For any $k \geq 0$ and any $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$, as well as for $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$, every k-planar, $\Gamma_{X,k}$-saturated drawing is filled.*

Proof. Consider a k-planar drawing $D \in \Gamma_{X,k}$ and a cell c in D with two incident vertices u, v, such that u and v are not connected by an uncrossed edge in the boundary of c. That is, D is not filled and we shall show that it is not $\Gamma_{X,k}$-saturated. We add a new uncrossed edge $e = uv$ in that cell, resulting in a new drawing D'. Clearly, the introduction of e did not create any new selfcrossings, incident crossings, multicrossings, or edges being crossed more than k times. Hence, for $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$, drawing D' lies in $\Gamma_{X,k}$ and D was not $\Gamma_{X,k}$-saturated.

It remains to consider $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$ and rule out that e is homotopic to another edge to show that $D' \in \Gamma_{X,k}$. So let e' be an edge parallel to e which is closest to e in the cyclic order of edges incident to u. Since incident crossings and selfcrossings are forbidden, e and e' together form a simple closed curve C. If e' is uncrossed, then e' is not in the boundary of cell c. Since e' is a parallel edge closest to e and since incident crossings are forbidden we find edges f and f' connecting u to a vertex in the interior and a vertex in the exterior of C, respectively. Hence e and e' are not homotopic. On the other hand, suppose that e' is crossed by some edge e''. As incident crossings are forbidden, neither u nor v is an endpoint of e''. As multicrossings are forbidden, the two endpoints of e'' lie in the exterior and the interior of C, respectively. Hence e and e' are not homotopic. □

To determine the exact edge-counts for min-saturated drawings, we shall find for each drawing style some essentially 2-connected, $\Gamma_{X,k}$-saturated drawings that attain the minimum in Lemma 1. Motivated by the error term $\varepsilon(D) = \left(\frac{k}{2} m_{\mathrm{x}} - \mathrm{cr}\right) + \frac{k-4}{4} m_{\mathrm{p}} + c_2' + c_3$ in Lemma 2, we define *tight drawings* as those k-planar drawings in which (1) every edge is crossed exactly k times $\left(\text{so } \frac{k}{2} m_{\mathrm{x}} = \mathrm{cr}\right)$ and (2) every cell contains exactly one vertex (so $m_{\mathrm{p}} = c_0 = c_2' = c_3 = 0$). Observe that tight drawings are indeed $\Gamma_{X,k}$-saturated and filled and exist only in case $k \geq 4$. Note that, to aid readability, isolated vertices are omitted from the drawings in the figures. Namely, the actual drawings have one isolated vertex in each cell shown empty in the figures. This is also mentioned in the figure captions.

Lemma 4. *For every $k \geq 4$ and every monotone drawing style Γ of k-planar drawings, if $D \in \Gamma$ is a tight drawing, then $\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_D - 2}{n_D - 1} < 1$.*

Proof. If D is not essentially 2-connected, then there is a closed curve C containing edges of D in the interior as well as exterior, such that $C \cap D$ is either empty or a single vertex. Then the drawing obtained by removing everything inside C (and adding an isolated vertex if the resulting cell is empty) is again in Γ by monotonicity and again tight, but has fewer vertices. As $\frac{n-2}{n-1}$ is monotone increasing in n, it thus suffices to prove the claim for any essentially 2-connected tight drawing D_0.

Clearly, D_0 is filled, as there are no two vertices incident to the same cell. We immediately get $c_0(D_0) = 0, m_\mathrm{p} = 0, c_2' = c_3 = 0, 2\mathrm{cr} = km_\mathrm{x}$, and it follows that $\varepsilon(D_0) = \left(\frac{k}{2}m_\mathrm{x} - \mathrm{cr}\right) + \frac{k-4}{4}m_\mathrm{p} + c_2' + c_3 = 0$. As there is at least one edge in D_0 and this is crossed $k \geq 4$ times, Euler's formula implies that there are at least three cells. Hence $n_{D_0} \geq 3$ and Lemma 2 gives

$$\alpha_\Gamma \leq \frac{m_{D_0}}{n_{D_0} + c_0(D_0) - 1} = \frac{m_{D_0}}{n_{D_0} - 1} = \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} < \frac{2}{k-2} \leq 1.$$

□

Theorem 5 (See also Table 1). *Let $k \geq 4$, $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$ be a set of restrictions, and $\Gamma = \Gamma_{X,k}$ be the corresponding drawing style of k-planar drawings.*

For infinitely many values of n, the minimum number of edges in any n-vertex Γ-saturated drawing is

$$\begin{array}{ll}
\frac{2}{k-(k \bmod 2)}(n-1) & \text{for } X = \{\mathbf{I}\} \text{ or } X = \emptyset. \\
\frac{2}{k-1}(n-1) & \text{for } X = \{\mathbf{S}\} \text{ or } X = \{\mathbf{S}, \mathbf{I}\}. \\
\frac{2(k-1)}{(k-1)(k-2)+2}(n-1) & \text{for } X = \{\mathbf{M}\}. \\
\frac{2(k+1)}{k(k-1)}(n-1) & \text{for } X = \{\mathbf{S}, \mathbf{M}\}. \\
\frac{4}{5}(n-1) & \text{for } X = \{\mathbf{I}, \mathbf{M}\} \text{ and } k = 4. \\
\frac{2(k-1)}{(k-1)(k-2)+2}(n-1) & \text{for } X = \{\mathbf{I}, \mathbf{M}\} \text{ and } k \geq 5. \\
\frac{2(k+1)}{k(k-1)}(n-1) & \text{for } X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}\} \text{ and } k \geq 6. \\
\frac{2(k+1)}{k(k-1)}(n-1) & \text{for } X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\} \text{ and } k \geq 6.
\end{array}$$

Proof. We start with the cases when $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$. Here the drawing style $\Gamma_{X,k}$ is monotone and every $\Gamma_{X,k}$-saturated drawing is filled by Lemma 3. Thus, by Lemma 4, we have $\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0}-2}{n_{D_0}-1}$ for every tight drawing D_0. This gives the smallest bound when n_{D_0} is minimized. In this case D_0 is essentially 2-connected and $m_{D_0} = \frac{2}{k-2}(n_{D_0} - 2)$ by Lemma 2, since $n_{D_0} \geq 3$ for tight drawings. So it suffices to consider a tight drawing D_0 with the smallest possible number m_{D_0} of edges.

Next, we shall go through the possible subsets X of $\{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ and determine exactly the value α_Γ for $\Gamma = \Gamma_{X,k}$ in two steps.

- First, we present a tight (hence filled) drawing D_0 with the smallest possible number m_{D_0} of edges, which gives by Lemma 4 the upper bound

$$\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0}-2}{n_{D_0}-1}.$$

- Second, we argue that for every filled (hence also every $\Gamma_{X,k}$-saturated), essentially 2-connected drawing $D' \in \Gamma_{X,k}$ we have

$$\frac{n_{D'} + c_0(D') - 2 + \varepsilon(D')}{n_{D'} + c_0(D') - 1} \geq \frac{n_{D_0}-2}{n_{D_0}-1}, \tag{4}$$

which by Lemmas 1 and 2 then proves the matching lower bound:

$$\alpha_\Gamma = \min_{D'o} \frac{m_{D'}}{n_{D'} + c_0(D') - 1} = \min_{D'} \frac{2}{k-2} \cdot \frac{n_{D'} + c_0(D') - 2 + \varepsilon(D')}{n_{D'} + c_0(D') - 1} \overset{(4)}{\geq} \frac{2}{k-2} \cdot \frac{n_{D_0}-2}{n_{D_0}-1}.$$

To verify (4), observe that if $\varepsilon(D') \geq 1$, then the lefthand side is at least 1, while the righthand side is less than 1. Thus it is enough to verify (4) when $\varepsilon(D') < 1$. In particular we may assume $c_2' = c_3 = 0$ and $2\mathrm{cr} \geq km_\times - 1$ for D'. Similarly, as $\varepsilon(D') \geq 0$, we may assume that $n_{D'} + c_0(D') \leq n_{D_0} - 1$. Altogether this implies that (4) is fulfilled unless

$$m_{D'} = \frac{2}{k-2}(n_{D'} + c_0(D') - 2 + \varepsilon(D')) < \frac{2}{k-2}(n_{D_0} - 1 - 2 + 1) = m_{D_0}.$$

In summary, for each X we shall give a tight drawing D_0 with as few edges as possible, and argue that every filled, essentially 2-connected drawing D' with fewer edges satisfies the inequality (4). Note that $m_{D'} \geq 1$ as essentially 2-connected drawings have at least one edge. In fact, we may assume that D' contains at least one crossed edge. Otherwise D' is filled, planar and hence connected. Thus $m_{D'} \geq n_{D'} - 1$ and $c_0(D') = 0$ which verifies (4) as follows:

$$\frac{n_{D'} - c_0(D') - 2 + \varepsilon(D')}{n_{D'} - c_0(D') - 1} = \frac{k-2}{2} \cdot \frac{m_{D'}}{n_{D'} - 1} \geq 1 > \frac{n_{D_0}-2}{n_{D_0}-1}.$$

Case 1. $X = \{\mathbf{I}\}$ and $X = \varnothing$.

Figure 2 shows drawings D_0 with $m_{D_0} = 1$ edge when k is even, and $m_{D_0} = 2$ edges when k is odd, which are tight for $\Gamma = \Gamma_{X,k}$ for both $X = \{\mathbf{I}\}$ and $X = \varnothing$, as incident edges

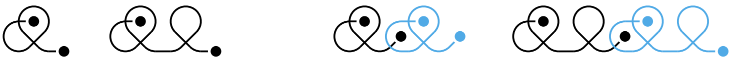

FIGURE 2 Smallest tight drawings for even $k \geq 4$ (left, cases $k = 4$ and $k = 6$ depicted) and odd $k \geq 4$ (right, cases $k = 5$ and $k = 7$ depicted) in case $X = \varnothing$ and $X = \{\mathbf{I}\}$, that is, nothing, resp. incident crossings, are forbidden. (Isolated vertices in empty cells are omitted.) [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 3 Smallest tight drawings for $k \geq 4$ in case $X = \{\mathbf{S}\}$ and $X = \{\mathbf{S}, \mathbf{I}\}$, that is, selfcrossings, resp. also incident crossings, are forbidden. (Isolated vertices in empty cells are omitted.) [Color figure can be viewed at wileyonlinelibrary.com]

do not cross. Thus $m_{D_0} = 1 + (k \bmod 2)$ and $n_{D_0} = \frac{k+2}{2}$ for k even, respectively, $n_{D_0} = k$ for k odd. Together this gives $\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0}-2}{n_{D_0}-1} = \frac{2}{k-(k \bmod 2)}$.

On the other hand, let $D' \in \Gamma_{X,k}$ be any filled, essentially 2-connected drawing. As argued above, we may assume that $1 \leq m_\mathrm{x} \leq m_{D'} < m_{D_0}$. For even k, there is nothing to show as $m_{D'} \geq 1 = m_{D_0}$. For odd k, we may assume that D' consists of exactly one edge, which has exactly $(k-1)/2$ selfcrossings (since $2\mathrm{cr} \geq km_\mathrm{x} - 1$), and some of the resulting cells may contain an isolated vertex. In particular, $\varepsilon(D') \geq \frac{k}{2}m_\mathrm{x} - \mathrm{cr} = 1/2$. Applying Euler's formula to the planarization of D' we get $n_{D'} + c_0(D') = (k+1)/2$, which verifies (4) as follows:

$$\frac{n_{D'} + c_0(D') - 2 + \varepsilon(D')}{n_{D'} + c_0(D') - 1} \geq \frac{(k+1)/2 - 2 + 1/2}{(k+1)/2 - 1} = \frac{k-2}{k-1} = \frac{n_{D_0} - 2}{n_{D_0} - 1}.$$

Case 2. $X = \{\mathbf{S}\}$ and $X = \{\mathbf{S}, \mathbf{I}\}$.

Figure 3 shows drawings D_0 with $m_{D_0} = 2$ edges which are tight for $\Gamma = \Gamma_{X,k}$ for both $X = \{\mathbf{S}\}$ and $X = \{\mathbf{S}, \mathbf{I}\}$, as there are neither incident crossings nor selfcrossings. Thus $m_{D_0} = 2$ and $n_{D_0} = k$, which gives $\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0}-2}{n_{D_0}-1} = \frac{2}{k-1}$.

On the other hand, let D' be any drawing in $\Gamma_{X,k}$, and assume again that $1 \leq m_\mathrm{x} < m_{D_0} = 2$. In particular, D' has exactly one crossed edge, which however is impossible as selfcrossings are forbidden.

Case 3. $X = \{\mathbf{M}\}$.

Figure 4 shows tight drawings D_0 with $m_{D_0} = k - 1$ edges. Thus $n_{D_0} = \frac{k-2}{2}m_{D_0} + 2 = \binom{k-1}{2} + 2$, which gives

$$\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} = \frac{2}{k-2} \cdot \frac{\binom{k-1}{2}}{\binom{k-1}{2} + 1} = \frac{2(k-1)}{(k-1)(k-2) + 2}.$$

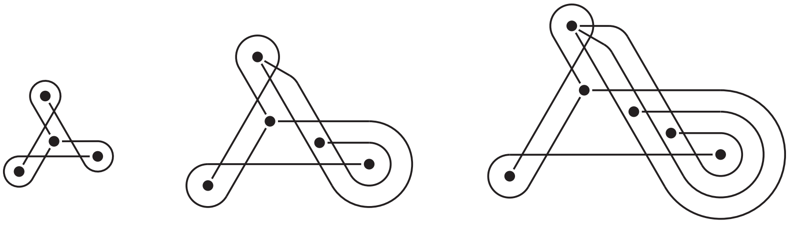

FIGURE 4 Smallest tight drawings for $k \geq 4$ in case $X = \{\mathbf{M}\}$, that is, multicrossings are forbidden. (Isolated vertices in empty cells are omitted.)

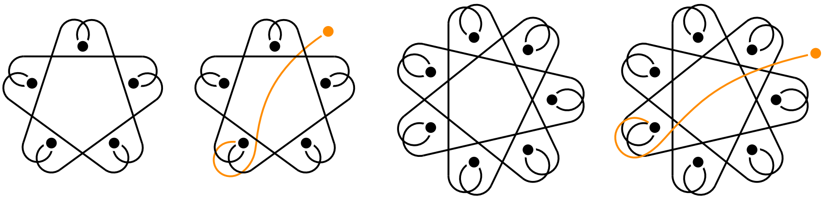

FIGURE 5 Smallest tight drawings for $k \geq 4$ in case $X = \{\mathbf{S}, \mathbf{M}\}$, that is, selfcrossings and multicrossings are forbidden. (Isolated vertices in empty cells are omitted.) [Color figure can be viewed at wileyonlinelibrary.com]

On the other hand, let D' be any drawing in $\Gamma_{X,k}$. As argued above the desired inequality (4) holds, unless $km_{\mathrm{x}} - 1 \leq 2\mathrm{cr}$ and $1 \leq m_{\mathrm{x}} \leq m_{D'} < m_{D_0} = k - 1$. As there are no multicrossings, the crossed edges may pairwise cross at most once, and additionally each crossed edge may cross itself at most once, that is, $\mathrm{cr} \leq m_{\mathrm{x}} + \binom{m_{\mathrm{x}}}{2} = \binom{m_{\mathrm{x}}+1}{2}$. However, this would imply

$$km_{\mathrm{x}} - 1 \leq 2\mathrm{cr} \leq (m_{\mathrm{x}} + 1)m_{\mathrm{x}} \leq (k - 2 + 1)m_{\mathrm{x}} = km_{\mathrm{x}} - m_{\mathrm{x}},$$

and thus $m_{\mathrm{x}} = 1$. However, then $2\mathrm{cr} \geq km_{\mathrm{x}} - 1 = k - 1 \geq 3$, which contradicts that there are no multicrossings.

Case 4. $X = \{\mathbf{S}, \mathbf{M}\}$.

Figure 5 shows tight drawings D_0 with $m_{D_0} = k + 1$ edges. Thus $n_{D_0} = \frac{k-2}{2} m_{D_0} + 2 = \binom{k}{2} + 1$, which gives

$$\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} = \frac{2}{k-2} \cdot \frac{\binom{k}{2} - 1}{\binom{k}{2}} = \frac{2(k+1)}{k(k-1)}.$$

On the other hand, let D' be any drawing in $\Gamma_{X,k}$. Again (4) holds, unless $km_{\mathrm{x}} - 1 \leq 2\mathrm{cr}$ and $1 \leq m_{\mathrm{x}} \leq m_{D'} < m_{D_0} = k + 1$. As there are no multicrossings and no selfcrossings, we have $\mathrm{cr} \leq \binom{m_{\mathrm{x}}}{2}$. However, this would imply $km_{\mathrm{x}} - 1 \leq 2\mathrm{cr} \leq m_{\mathrm{x}}(m_{\mathrm{x}} - 1) \leq k(m_{\mathrm{x}} - 1) = km_{\mathrm{x}} - k \leq km_{\mathrm{x}} - 4$, which is a contradiction.

Case 5. $X = \{\mathbf{I}, \mathbf{M}\}$.

Figures 6 and 7 show tight drawings D_0 with $m_{D_0} = 4$ edges for $k = 4$, and $m_{D_0} = k - 1$ edges for $k \geq 5$.

For $k = 4$ we have $n_{D_0} = \frac{k-2}{2} m_{D_0} + 2 = 6$, which gives

$$\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} = \frac{2}{4-2} \cdot \frac{6-2}{6-1} = \frac{4}{5}.$$

For $k \geq 5$ we have analogous to Case 3 $n_{D_0} = \binom{k-1}{2} + 2$, which gives

$$\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} = \frac{2}{k-2} \cdot \frac{\binom{k-1}{2}}{\binom{k-1}{2} + 1} = \frac{2(k-1)}{(k-1)(k-2) + 2}.$$

On the other hand, let D' be any drawing in $\Gamma_{X,k}$. Clearly, $D' \in \Gamma_{X,k} \subset \Gamma_{\{\mathbf{M}\}}$ for $\{\mathbf{M}\} \subset X = \{\mathbf{I}, \mathbf{M}\}$. However, we already argued in Case 3 that there is no drawing D' in $\Gamma_{\{\mathbf{M}\}}$ with $km_\text{x} - 1 \leq 2\text{cr}$ and $1 \leq m_\text{x} < k - 1$. This already seals the deal for $k \geq 5$.

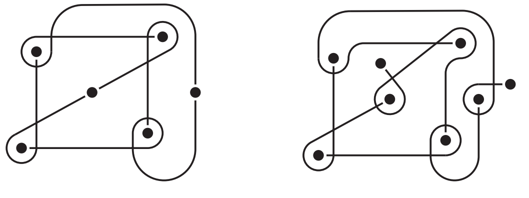

FIGURE 6 Smallest tight drawings for $k = 4$ (left) and $k = 5$ (right) in case $X = \{\mathbf{I}, \mathbf{M}\}$, that is, incident crossings and multicrossings are forbidden. (Isolated vertices in empty cells are omitted.)

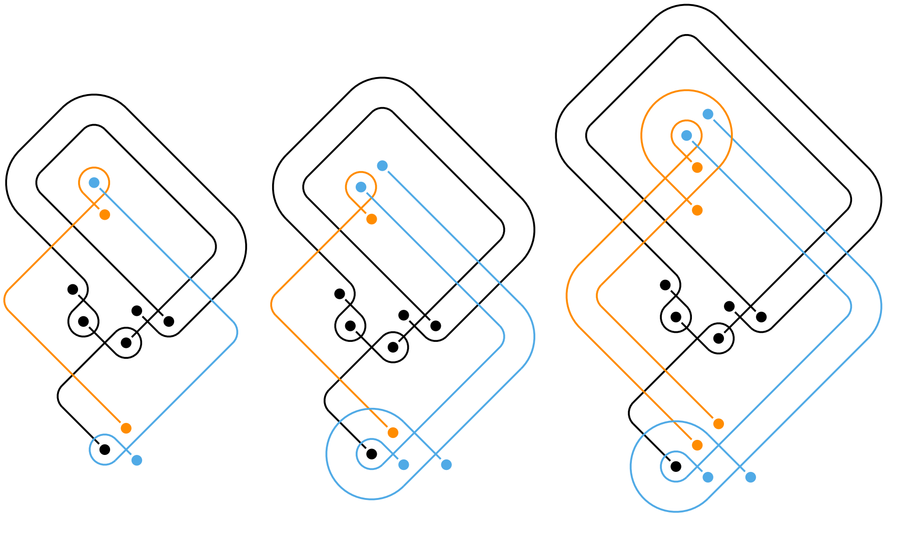

FIGURE 7 Smallest tight drawings for $k \geq 6$ in case $X = \{\mathbf{I}, \mathbf{M}\}$, that is, incident crossings and multicrossings are forbidden. (Isolated vertices in empty cells are omitted.) [Color figure can be viewed at wileyonlinelibrary.com]

For $k = 4$, assume that D' is a filled, essentially 2-connected drawing in $\Gamma_{X,k}$. As before, we may assume that $2\mathrm{cr} \geq km_{\mathrm{x}} - 1 = 4m_{\mathrm{x}} - 1$, that is, there are at least $2m_{\mathrm{x}}$ crossings. On the other hand, as multicrossings are forbidden, we have again $\mathrm{cr} \leq m_{\mathrm{x}} + \binom{m_{\mathrm{x}}}{2}$, which together implies that $m_{\mathrm{x}} \geq 3$. We are done if $m_{\mathrm{x}} + m_{\mathrm{p}} \geq 4 = k$. Otherwise, we have $m_{\mathrm{x}} = 3$ and $\mathrm{cr} \geq 2m_{\mathrm{x}} = 6$. Now if two of the three crossed edges were incident, they would not cross each other as $\mathbf{I} \in X$, which would give at most three selfcrossings and two crossings of independent edges, contradicting $\mathrm{cr} \geq 6$. Thus we may assume that all three crossed edges are crossing themselves, pairwise crossing and pairwise independent, that is, $n_{D'} = 6$ and $\mathrm{cr} = 6$.

Now let us consider the subdrawing H of the planarization of D' obtained by removing all vertices of D'. That is, H is planar, connected, $|V(H)| = \mathrm{cr} = 6$, and $|E(H)| = (k-1)m_{\mathrm{x}} = 3m_{\mathrm{x}} = 9$. Applying Euler's formula shows that the number of faces of H is $|E(H)| - |V(H)| + 2 = 9 - 6 + 2 = 5$. As $n_{D'} \geq 6$, some face of H contains two vertices of D' in its interior, showing that $m_{\mathrm{p}} \geq 1$, as D' is filled. Thus $m_{\mathrm{x}} + m_{\mathrm{p}} \geq 3 + 1 = 4 = k$, as desired.

Case 6. $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$.

The right of Figure 1 (with isolated vertices added to both empty cells) and Figure 8 show tight drawings D_0 with $m_{D_0} = k + 1$ edges for $k \geq 6$. Analogous to Case 4 $n_{D_0} = \binom{k}{2} + 1$, which gives

$$\alpha_\Gamma \leq \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} = \frac{2}{k-2} \cdot \frac{\binom{k}{2} - 1}{\binom{k}{2}} = \frac{2(k+1)}{k(k-1)}.$$

On the other hand, any drawing $D' \in \Gamma_{X,k}$ is also a drawing in $\Gamma_{\{\mathbf{S},\mathbf{M}\}}$ for $\{\mathbf{S}, \mathbf{M}\} \subset X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$. However, we already argued in Case 4 that there is no drawing $D' \in \Gamma_{\{\mathbf{S},\mathbf{M}\}}$ with $km_{\mathrm{x}} - 1 \leq \mathrm{cr}$ and $m_{\mathrm{x}} < m_{D_0} = k + 1$.

Case 7. $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$.

We can not proceed with $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}, \mathbf{H}\}$ as before, since $\Gamma_{X,k}$ is not monotone in that case, since removing a vertex can make two edges homotopic. However, we see that the tight drawings D_0 in Figure 1 (right) and Figure 8 for drawing style $\Gamma_{\{\mathbf{S},\mathbf{I},\mathbf{M}\}}$ are also in $\Gamma_{X,k}$ as there are no parallel edges and hence no homotopic edges. Thus

$$\begin{aligned}
\frac{m_{D_0}}{n_{D_0} - 1} &\geq \min\left\{\frac{m_D}{n_D - 1} : D \in \Gamma_{X,k} \text{ is } \Gamma_{X,k}\text{-saturated}\right\} \\
&\geq \min\left\{\frac{m_D}{n_D - 1} : D \in \Gamma_{X,k} \text{ is filled}\right\} \\
&\geq \min\left\{\frac{m_D}{n_D - 1} : D \in \Gamma_{\{\mathbf{S},\mathbf{I},\mathbf{M}\}} \text{ is filled}\right\} \\
&= \min\left\{\frac{m_D}{n_D + c_0(D) - 1} : D \in \Gamma_{\{\mathbf{S},\mathbf{I},\mathbf{M}\}}\right. \\
&\qquad\qquad \left. \text{is filled and essentially 2-connected}\right\} \\
&= \alpha_{\Gamma_{\{\mathbf{S},\mathbf{I},\mathbf{M}\}}} = \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} = \frac{m_{D_0}}{n_{D_0} - 1}
\end{aligned}$$

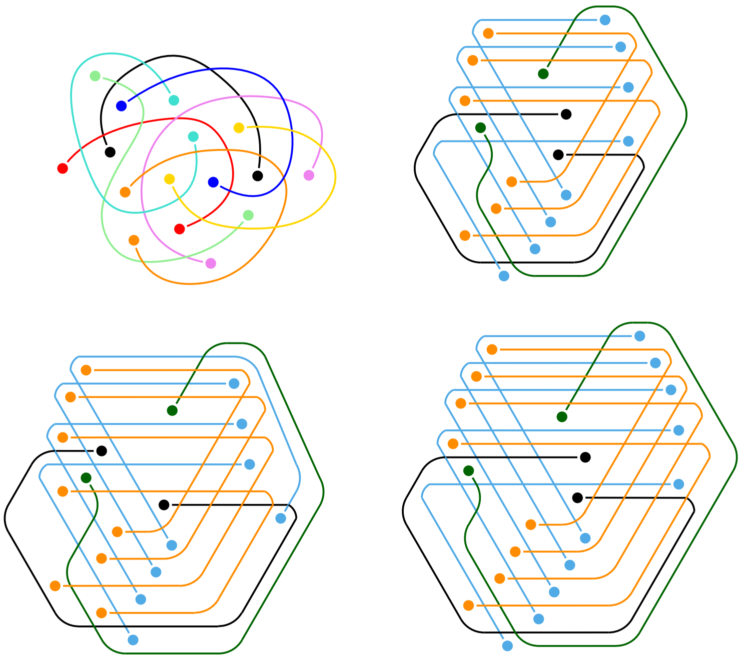

FIGURE 8 Smallest tight drawings for $k \geq 7$ (for $k = 6$, see Figure 1 (right)) in case $X = \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$, that is, selfcrossings, incident crossings, and multicrossings are forbidden. Top-Left: The 8-matching for $k = 7$. Top-Right: The 9-matching for $k = 8$. Bottom-Left: The 10-matching for $k = 9$. Bottom-Right: The 11-matching for $k = 10$. (Isolated vertices in empty cells are omitted.) [Color figure can be viewed at wileyonlinelibrary.com]

and equality holds throughout. Hence, for every filled and every $\Gamma_{X,k}$-saturated drawing D in $\Gamma_{X,k}$ we have $m_D \geq \alpha_{\Gamma_{\{\mathbf{S},\mathbf{I},\mathbf{M}\}}} \cdot (n_D - 1) = \frac{2(k+1)}{k(k-1)} \cdot (n - 1)$.

In Cases 1–6 we have determined exactly α_Γ for each considered drawing style $\Gamma = \Gamma_{X,k}$. By Lemmas 1 and 3 every Γ-saturated drawing D satisfies $m_D \geq \alpha_\Gamma(n_D - 1)$. For Case 7 we have shown this inequality directly. Moreover, we presented in each case a tight drawing D_0 attaining this bound:

$$m_{D_0} = \frac{2}{k-2}(n_{D_0} - 2) = \frac{2}{k-2} \cdot \frac{n_{D_0} - 2}{n_{D_0} - 1} \cdot (n_{D_0} - 1) = \alpha_\Gamma(n_{D_0} - 1)$$

It remains to construct an infinite family of Γ-saturated drawings attaining this bound. To this end it suffices to take tight drawings with $\alpha_\Gamma(n - 1)$ edges and iteratively glue these at single vertices. This again results in a tight drawing.

Formally, for vertices v_1, v_2 in two copies of (not necessarily distinct) tight drawings D_1 and D_2, respectively, with $m_{D_i} = \alpha_\Gamma(n_{D_i} - 1)$ for $i = 1, 2$, we consider the drawing D obtained from D_1, D_2 by identifying v_1 and v_2 into a single vertex and putting D_2 completely inside a cell of D_1 incident to v_1. Then D is again tight and thus Γ-saturated. Moreover we have $n_D = n_{D_1} + n_{D_2} - 1$ and

$$m_D = m_{D_1} + m_{D_2} = \alpha_\Gamma(n_{D_1} - 1) + \alpha_\Gamma(n_{D_2} - 1) = \alpha_\Gamma(n_D - 1). \qquad \square$$

4 | BOUNDS FOR SIMPLE GRAPHS

We define a *simple filled* drawing D of a simple graph G as a drawing in which any two vertices that are incident to the same cell c of D are connected. In contrast to filled drawings (according to Section 2) the connecting edge may (partially or completely) lie outside of the boundary of c. With this definition in mind, Lemmas 1 and 3 directly translate to the simple graph setting (note that $\mathbf{H} \notin X$ for any drawing style $\Gamma_{X,k}$ in this setting). Lemma 2 though does not translate and consequently neither does the bound in Lemma 4. We obtain the following bound on m_D.

Lemma 6. *For any $k \geq 0$, any k-planar simple filled and essentially 2-connected drawing D it holds that $m_D \geq \frac{2}{k+2}(n_D - 1)$.*

Proof. Consider the planarization P of D. As in the proof of Lemma 2 we find that with P being essentially 2-connected it has has exactly #isolated + 1 connected components where #isolated is the number of isolated vertices. Moreover, for the number of vertices and edges of P it holds that $|V(P)| = n_D + \mathrm{cr}$ and $|E(P)| = m_D + 2\mathrm{cr}$, with cr being the number of crossings in D. Let #cells be the number of faces of P. Since D is simple filled it holds that #cells $\geq$ #isolated + 1. By applying Euler's formula we obtain

$$\begin{aligned}
2 &= |V(P)| - |E(P)| + \#\mathrm{cells} - \#\mathrm{isolated} \\
&= n_D - m_D - \mathrm{cr} + \#\mathrm{cells} - \#\mathrm{isolated} \\
&\geq n_D - m_D - \mathrm{cr} + \#\mathrm{isolated} + 1 - \#\mathrm{isolated} \\
&= n_D - m_D - \mathrm{cr} + 1.
\end{aligned}$$

Hence, $m_D + \mathrm{cr} \geq n_D - 1$ and with $\mathrm{cr} \leq \frac{k}{2} m_D$ we obtain the desired bound. □

Consequently, any simple filled drawing $D \in \Gamma$ (and hence every saturated k-planar drawing of a simple graph) satisfies

$$\frac{m_D}{n_D - 1} \geq \min_{D'} \frac{m_{D'}}{n_{D'} - 1} \geq \min_{D'} \frac{2}{k+2} \cdot \frac{n_{D'} - 1}{n_{D'} - 1} = \frac{2}{k+2},$$

where both minima are taken over all k-planar simple filled, essentially 2-connected drawings $D' \in \Gamma$.

Considering upper bounds on the minimum number of edges in any $\Gamma_{X,k}$-saturated k-planar drawing of a simple graph, we show in the following theorem that for any drawing style $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ there exist sparser drawings than for multigraphs. Moreover, for $X = \emptyset$ and $X = \{\mathbf{I}\}$ the resulting bound is tight.

Theorem 7. *Let $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ be a set of restrictions, and $\Gamma = \Gamma_{X,k}$ be the corresponding drawing style of k-planar drawings of simple graphs. For infinitely many values of n, the minimum number of edges in any n-vertex Γ-saturated drawing is upper bounded by*

$$\begin{array}{ll} \frac{2}{k+2((k+1) \bmod 2)}(n-1) & \textit{if } X \subseteq \{\mathbf{I}\} \textit{ and } k \geq 2, k \neq 3. \quad \text{(Figure 9)} \\ \frac{2}{k+((k+1) \bmod 2)}(n-1) & \textit{if } X \subseteq \{\mathbf{S}, \mathbf{I}\} \textit{ and } k \geq 4. \quad \text{(Figure 9)} \\ \frac{2}{k-1}(n-1) & \textit{if } X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\} \textit{ and } k \geq 1. \quad \text{(Figure 10)} \end{array}$$

Proof. For $X = \varnothing$ and $X = \{\mathbf{I}\}$, as well as $X = \{\mathbf{S}\}$ and $X = \{\mathbf{S}, \mathbf{I}\}$ we modify the constructions used in Theorem 5. In Figure 9 we show the modifications. Taking disjoint unions of one of these drawings by placing instead of an isolated vertex the whole drawing again into an empty cell leads a saturated k-planar drawing of the respective drawing style with $m_D = \frac{m_{D_0}}{n_{D_0} + c_0(D_0) - 1}(n_D - 1)$ edges. In some sense, this is exactly the first argument of Lemma 4 in reverse, that is, instead of replacing parts of the drawing by isolated vertices, we replace an isolated vertex by a drawing.

In case of $X = \varnothing$ and $X = \{\mathbf{I}\}$ this leads to $m_D = \frac{1}{2+\frac{k}{2}-1}(n_D - 1) = \frac{2}{k+2}(n_D - 1)$ if k is even (recall that selfcrossings are counted twice per visible crossing) and $m_D = \frac{2}{4+2\frac{k-3}{2}-1}(n_D - 1) = \frac{2}{k}(n_D - 1)$ if k is odd. For $X = \{\mathbf{S}\}$ and $X = \{\mathbf{S}, \mathbf{I}\}$ we get $m_D = \frac{2}{4+k-2-1}(n_D - 1) = \frac{2}{k+1}(n_D - 1)$ if k is even and $m_D = \frac{2}{4+k-3-1}(n_D - 1) = \frac{2}{k}(n_D - 1)$ if k is odd.

For any $X \subseteq \{\mathbf{S}, \mathbf{I}, \mathbf{M}\}$ we modify a construction originally presented by Brandenburg et al. [10] for 1-planar drawings and adapted by Auer et al. [4] for 2-planar drawings. Here, we generalize the construction to k-planar drawings. See Figure 10 for an illustration for the cases $k = 1$, 2, and 3. The construction is more easily imagined on a

FIGURE 9 Modifications of the constructions used in Theorem 5 for $X \subseteq \{\mathbf{I}\}$ on the left, showing the cases $k = 2$ and $k = 5$, and $X \subseteq \{\mathbf{S}, \mathbf{I}\}$, $\mathbf{S} \in X$ on the right, showing the cases $k = 4$ and $k = 5$. (Isolated vertices in empty cells are omitted.) [Color figure can be viewed at wileyonlinelibrary.com]

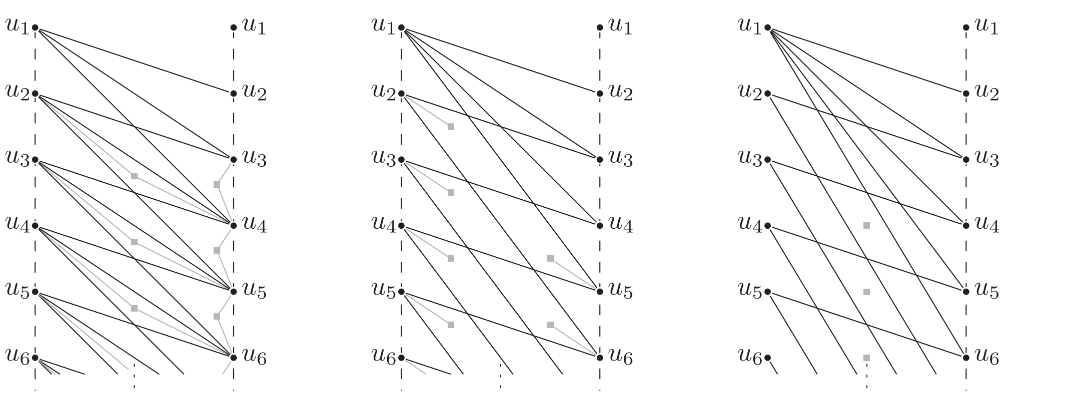

FIGURE 10 Construction for saturated simple k-plane drawings in case $k = 1$ (left), $k = 2$ (middle), and $k \geq 3$ (right). The dashed left and right sides of the drawings are identified.

cylinder. We describe it here for $k \geq 3$. Let D' be the drawing, it consists of a path on $n_{D'}$ vertices $u_1, \ldots, u_n$ with the vertices being laid out along a vertical line from the top to the bottom of the cylinder and each vertex u_i with $i \notin \{1, n\}$ being connected to u_{i+1} and u_j with $j = i + k + 2$ where we do not add an edge if u_j does not exist. Finally, we add all edges $u_1 u_j$ for $j = 2, \ldots, k + 2$ and $u_j u_n$ for $j = n_{D'} - (k + 2)$. As a result we obtain a series of cells with no vertex on their boundaries in which we add isolated vertices. Clearly this drawing is k-planar and no edge can be added without crossing another edge more than k times. Moreover, there are no multiple-, incident-, or selfcrossings and hence D' is $\Gamma_{X,k}$-saturated. Moreover, for $n_{D'} \geq k + 3$ the drawing has $m_{D'} = 2n_{D'} + k - 3$ many edges. It remains to count how many isolated vertices we can add. For every edge $u_i u_{i+1}$ with $i = k, \ldots, n_{D'} - k - 1$ bounds $k - 2$ cells in which we can place an isolated vertex. Additionally, the edges $u_i u_{i+1}$ with $i = 3, \ldots, k - 1$ and $i = n_{D'} - 4, \ldots, n_{D'} - k$ bound $1, \ldots, k - 3$ many cells in which we can place one isolated vertex each. In total we get that

$$n_D = n_{D'} + n_{D'}(k - 2) - 2k(k - 2) + (k - 3)(k - 2).$$

Solving for $n_{D'}$ we obtain that

$$n_{D'} = \frac{n_D + k^2 + k - 6}{k - 1}.$$

Plugging the above into $m_D = 2n_{D'} + k - 3$ we finally obtain

$$m_D = 2\frac{n_D + k^2 + k - 6}{k - 1} + k - 3 = \frac{2}{k - 1}(n_D - 1) + \frac{3k^2 - 2k - 9}{k - 1}. \qquad \square$$

5 | CONCLUDING REMARKS

Regarding multicrossings, we either disallowed their existence (**M**) or did not restrict their number. It is possible to make a more fine-grained analysis and consider the maximum number of times that a pair of edges (or an edge with itself) is allowed to cross as a parameter μ. Modifications of our constructions, for example retracing a side of each edge in the construction in Figure 8 from both endpoints, yield tight bounds for arbitrarily many values of k and μ.

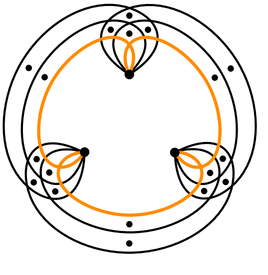

FIGURE 11 $\Gamma_{X,k}$-saturated 6-planar drawing for $X = \{\mathbf{S}, \mathbf{M}, \mathbf{H}\}$. [Color figure can be viewed at wileyonlinelibrary.com]

Our analysis of homotopy-free drawings (**H**) is restricted to the branching style (i.e., to simple drawings). Figure 11 shows a 6-planar drawing that is $\Gamma_{X,k}$-saturated with $X = \{\mathbf{S}, \mathbf{M}, \mathbf{H}\}$. The drawing can be easily generalized to a saturated k-planar drawing in this style for any even $k \geq 4$. By taking disjoint unions of this (and an isolated vertex) we obtain an n-vertex $\Gamma_{X,k}$-saturated k-planar drawing with $\frac{2}{k+1}(n-1)$ edges, which is less than any of the tight bounds in Theorem 5 and behaves more like the results for simple graphs.

Our drawings typically contain many isolated vertices. We discussed the case that isolated vertices are not desired already in the introduction: in this case the sparsest graphs possible are matchings and saturated k-planar drawings of matchings indeed exist. For $k \geq 6$, Figures 1 and 8 show saturated k-planar drawings of matchings that are simple and hence contained in all specific drawing styles that we consider. Disjoint unions of these drawings also yield arbitrarily large saturated drawings of matchings for any fixed $k \geq 6$. For $k \leq 1$ there are no saturated k-planar drawings of matchings in any drawing style considered here (besides the degenerate case of a single edge being saturated in many possible drawings if homotopic parallel edges are forbidden). For $2 \leq k \leq 5$ the existence of saturated k-planar drawings of matchings depends on the drawing style. The constructions from Figures 2, 3, and 9 give saturated k-planar drawings of matchings for this range of k for some drawing styles. Disjoint unions of these drawings also yield arbitrarily large saturated drawings of matchings in the respective style. For $k \leq 3$ saturated k-planar drawings of matchings do not exist whenever parallel edges are allowed, since a cell with two incident vertices is unavoidable.

For simple graphs, it is a relevant open question to determine the minimum number of edges in a saturated k-planar simple drawing. Finally, our techniques only work for fixed drawings. It remains open to determine the min-saturated k-planar (abstract) graphs and the sizes of their edge sets.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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