Karlsruher Institut für Technologie
Fakultät für Mathematik
Institut für Analysis

On Mixing and Resonances in Fluid Systems

Christian Zillinger

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Contents

CHAPTER 1

Introduction

The analysis of fluids and related systems is an area of great historical significance in mathematics, physics and engineering. Despite their long history (the Euler equations of fluid dynamics were introduced in 1755 and are sometimes referred to as the "second partial differential equation ever written" [**BČCK**, chapter 4]), many fundamental questions of existence, uniqueness and stability of the equations of fluid dynamics are still open problems [**NRŠ96, KŠ14, JŠ14**]. These famously include the Millennium problem of the wellposedness of the Navier-Stokes equations

$$
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \nu \Delta v, \\
\operatorname{div}(v) &= 0, \\
(t, x) &\in \mathbb{R}^+ \times \Omega,
\end{aligned}
$$

and the rigorous mathematical understanding of turbulence. In this context also questions of non-uniqueness using methods of convex integration [**Gro73, Spr10, MŠ03**] have been a very active field of research, culminating in the resolution of Onsager's conjecture in a series of works by Isett, Székelyhidi, De Lellis, Vicol and Buckmaster [**BDLSJV19, Ise17, BDLSJ16**].

This thesis concerns itself with the stability and long-time behavior of fluid flows. That is, we ask whether given a particular solution and a small perturbation at an initial time, does the perturbation remain small for all times or can small perturbations lead to large changes? More precisely, we consider kinetic equations such as the Euler, Navier-Stokes, Boussinesq or Vlasov-Poisson equations and study the long-time behavior of solutions which are initially close to stationary solutions. As we discuss in the following, here one observes a dichotomy: on the one hand *very strong* stability results hold for the linearized equations. On the other hand, nonlinear stability results require *very strong* assumptions (regularity between C^∞ and analyticity). This sharp contrast in stability is due to chains of resonances in the equations, where high and low frequency perturbations repeatedly interact (by *echoes*, see Chapter 2), causing large growth of the nonlinear dynamics as compared to the linear dynamics.

This habilitation thesis is a cumulative work based on the following articles:

(1) Christian Zillinger, On Echo Chains in Landau Damping: Traveling Wave-like Solutions and Gevrey 3 as a Linear Stability Threshold, Annals of PDE 7, 1 (2021), 2021. [**Zil21a**]
(2) Yu Deng, Christian Zillinger, Echo chains as a linear mechanism: Norm inflation, modified exponents and asymptotics, Archive for Rational Mechanics and Analysis (2021), 2021. [**DZ21**]

(3) Christian Zillinger, On the Boussinesq equations with non-monotone temperature profiles, Journal of Nonlinear Science, volume 31, Article number: 64 (2021), 2021. [**Zil21d**]
(4) Christian Zillinger, On Enhanced Dissipation for the Boussinesq Equations, Journal of Differential Equations, Volume 282, Pages 407-445, 2021. [**Zil21c**]
(5) Christian Zillinger, On stability estimates for the inviscid Boussinesq equations. arXiv preprint arXiv:2209.01950 (2022). [**Zil22**]
(6) Christian Zillinger, On echo chains in the linearized Boussinesq equations around traveling waves. arXiv preprint arXiv:2103.15441 (2021). To appear in SIAM Journal on Mathematical Analysis. [**Zil21b**]
(7) Xian Liao and Christian Zillinger, On variable viscosity and enhanced dissipation. arXiv preprint arXiv:2110.10976 (2021). [**LZ21**]
(8) Christian Zillinger, On geometric and analytic mixing scales: comparability and convergence rates for transport problems, Pure and Applied Analysis 1.4 (2019): 543-570, 2019. [**Zil19**]

Let us comment on some of the thesis' main achievements:

- *Traveling waves, sharp stability estimates and norm inflation.* As a novel perspective in investigating the above outlined dichotomy between the low and high regularity stability results for the linear and nonlinear dynamics, we introduce the notion of *traveling wave* solutions. These provide a major tool to systematically understand instabilities in fluid systems. In particular, we show that resonances or "echoes", which are commonly considered a purely *nonlinear* effect [**MV11, BM15a, DM23, Bed20**], are already a feature of the *linearized* problem around waves. Using these waves, we further obtain a sharp characterization of stability, norm inflation and even infinite time blow-up in optimal spaces.
- *A hierarchy of stability notions.* We show that there is a hierarchy of different notions of asymptotic stability (see Section 2.2) and construct solutions, which exhibit stability with respect to the "physical" notion of stability, but infinite time blow-up with respect to the "scattering" notion [**Zil21a, DZ21**]. Furthermore, we identify new resonance mechanisms in the Boussinesq equations with partial dissipation [**Zil21b**] and establish qualitatively improved stability estimates for the inviscid Boussinesq equations [**Zil22**].
- *Comparing geometric and analytic notions of mixing.* Complementing the investigation of mixing as a mechanism for nonlinear resonances, in [**Zil19**] we provide a new perspective on notions of mixing commonly used in the literature. In particular, we show that two notions which had previously been considered distinct are actually comparable after natural adjustments (see Chapter 3).
- *Mixing enhanced dissipation for fluid systems.* Last but not least, we investigate how mixing in combination with (full) dissipation can have major stabilizing effects, which are robust enough to persist in coupled systems and stratified regimes [**Zil21b, LZ21**].

Each article's main results are discussed and summarized in the respective sections of this thesis and a copy of each article is included in Chapter 6. The article [**Zil21b**] has been accepted for publication, but as of the time of writing has not yet appeared in print. An author's copy of the accepted version is thus attached. The recent articles [Zil22, LZ21] have been submitted, but as of the date of this thesis have not yet been accepted.

The remainder of this thesis is structured as follows:

- In Section 1.1 we provide a short review of classical linear and nonlinear stability results for the Euler equations. We also discuss the contrast between strong linear stability but apparent nonlinear instability of the Couette flow, which is known as the Sommerfeld paradox.
- Section 1.2 discusses (phase-)mixing as a stabilizing mechanism but also source of norm inflation.
- In Chapter 2, we introduce *traveling waves* as a key tool to understand instability in fluid systems. Here we summarize and connect the main results of a series of articles for the Boussinesq equation [Zil22, Zil21b], the Euler equations [DZ21] and Vlasov-Poisson equations [Zil21a].
- In Chapter 3 we discuss different notions of "mixing" used in various mathematical communities. In particular, as a main result of [Zil19] we show that "geometric" and "analytic" notions are comparable when adjusting for precision and low frequency oscillation.
- In Chapter 4 we discuss *enhanced dissipation* in systems [Zil21d, Zil21c, LZ21], where mixing and dissipation constructively interact to result in dissipation on improved time scales.
- Finally, in Chapter 5 we discuss some of the main results and provide an outlook on future research topics.

1.1. Classical stability theory and the Sommerfeld paradox

In this thesis, we investigate the long time behavior of fluid systems and, in particular, the effects of mixing and instability. As an important, prototypical fluid system, in this section we provide a short review of classical stability results for the Euler equations of fluid dynamics:

$$\begin{aligned}\partial_t v + v \cdot \nabla v + \nabla p &= 0,\\ \operatorname{div}(v) &= 0,\\ (t,(x,y)) &\in \mathbb{R} \times \Omega.\end{aligned}$$

Here Ω is a two (or three) dimensional domain, v denotes the velocity of the fluid inside this domain and p is the pressure. The condition $\operatorname{div}(v) = 0$ models the incompressibility of the fluid. In case of a bounded domain $\Omega \subset \mathbb{R}^d$ these equations have to be augmented with boundary conditions, where a common choice is given by non-penetration conditions. That is, there should be no flow in the normal direction n:

$$v|_{\partial\Omega} \cdot n = 0.$$

A natural starting point for the investigation of the long-time behavior of solutions is to first identify stationary solutions and then investigate the behavior of perturbations to these stationary solutions.

Here a distinguished class is given by "one-dimensional" solutions, i.e. solutions depending only on one spatial variable, which leads to so-called *shear flows*:

$$v(t,x,y) = (U(y),0), \ p = 0.$$

Different vertical layers of the fluid move with different constant speed and if these speeds are different, then these layers are sheared apart. A distinguished shear flow is given by the case when $U(y)$ is affine, that is, after normalization,

$$U(y) = y,$$

which is also a stationary solution to the Navier-Stokes equations for all values of the viscosity. We remark that for the inviscid problem, in contrast any profile $U(y)$ yields a stationary solution.

Heuristically, while any profile $U(y)$ is a stationary solution, in view of applications and experiments it seems reasonable that a particular focus should be on solutions which are "stable". In particular, in physical applications we expect to mainly observe such solutions which are stable under small perturbations. However, it is a priori not clear which notion of stability or smallness of perturbations should be used and different notions may be appropriate depending on the underlying (de)stabilizing effect. As three prototypical examples we may here consider:

- *Poiseuille* flow $U(y) = y^2$ as a strictly convex shear flow.
- *Kolmogorov* flow $U(y) = \cos(y)$, which satisfies a nonlinear convexity-like condition.
- *Couette* flow $U(y) = y$, which is strictly monotone, but not convex.

Stronger results may be obtained for flows which exhibit multiple of these properties such as $U(y) = e^y$ being monotone and convex.

Since shear flows are independent of time and of the horizontal spatial variable x, they are particularly amenable to linear stability theory (however, nonlinear stability turns out to remain a very challenging problem). Hence, classical stability results consider the stability of the linearized problem

$$\partial_t v + U(y)\partial_x v + v_2 U'' + \nabla p = 0. \tag{1}$$

More specifically, one makes the ansatz of "wave-like" perturbations [SH00]

$$v(t,x,y) = \tilde{v}(y)e^{i(kx-kct)}, \tag{2}$$

where $k \in \mathbb{R}$ denotes the spatial frequency and $c \in \mathbb{C}$ is the complex phase speed (and the physical perturbation is given by $\Re(v)$). Here, we for simplicity of notation consider the two-dimensional case. In three dimensions one may add an additional factor $e^{i\sigma z}$ with $\sigma \in \mathbb{R}$.

Classical stability results consider stability in terms of exponentially growing perturbations in this specific class (2).

Definition 1.1 (cf. [SH00], Section 2.2). *We say that a shear flow $U(y)$ is* linearly *or* spectrally *stable if there are no non-trivial solutions to* (1) *of the form* (2) *with* $\Im(kc) > 0$.

We remark that, while this convention is standard, such solutions should be understood as "not unstable in a specific sense" rather than being stable. Indeed, we will see in later results that such linearly stable shear flows may still exhibit growth of (linearized) perturbations at algebraic rates, which also underlies nonlinear norm inflation and instability results.

The two most famous classical stability theorems with respect to the notion of spectral stability are due to Rayleigh [Ray79] (in 1879) and a slightly improved version by Fjørtoft [Dra02] (1950) and argue by contradiction. The following theorem summarizes both results for the case of a finite channel $\Omega = \mathbb{R} \times [-1, 1]$:

THEOREM 1.2. *Let $U \in C^2([-1, 1])$ be a given shear flow. Then the following conditions are necessary conditions for U to be spectrally unstable:*

- *Rayleigh's criterion: there exists a point y_s with $U''(y_s) = 0$.*
- *Fjørtoft's criterion: there exists a point $\tilde{y}$ such that*

$$U''(\tilde{y})(U(\tilde{y}) - U(y_s)) < 0.$$

The first criterion, in particular, implies that any strictly convex (or concave) shear flow is spectrally stable, which for instance includes the Poiseuille flow

$$U(y) = y^2.$$

The criterion due to Fjørtoft further allows to deduce the stability of certain flows, which are not strictly convex such as

$$U(y) = y^{2j}$$

with $j \in \mathbb{N}$, $j \geq 2$. That is, Fjørtoft's criterion requires that critical points cannot be minima (but could be maxima or saddle points).

The following proof is adapted from Section 2.2 of [SH00].

PROOF OF THEOREM 1.2. For simplicity of presentation in the following we discuss the two-dimensional case in a finite channel $\mathbb{R} \times [-1, 1]$ with non-penetration boundary conditions. The case of infinite, periodic or three-dimensional channels can be treated analogously.

The linearized equations around the shear flow are given by

$$\partial_t v + U(y)\partial_x v + (v_2 U', 0) + \nabla p = 0.$$

Computing the divergence of this equation and using that $\mathrm{div}(v) = 0$, the pressure can thus be obtained as the solution of

$$\mathrm{div}(\nabla p) = -2U'\partial_x v_2.$$

Hence, we may rewrite (4) in self-contained form as

$$((\partial_t + U\partial_x)\Delta - U''\partial_x)v_2 = 0.$$

Inserting the ansatz (2)

$$v_2(t, x, y) = \tilde{v}(y)e^{ikx - ikct} \tag{3}$$

then yields the *Rayleigh equation*:

$$(U - c)(\partial_y^2 - k^2)\tilde{v} - U''\tilde{v} = 0, \tag{4}$$

where we restricted to the case $k \neq 0$ and the boundary conditions are given by non-penetration conditions:

$$\tilde{v}|_{y=-1,1} = 0.$$

We may thus interpret (4) as a (slightly non-standard) eigenvalue problem for a second order ordinary differential equation and spectral stability asks whether there exists an eigenvalue with positive imaginary part.

We remark that all coefficients in (4) are real valued. Hence if v_2 as in (3) yields a solution of (4) for c, then the complex conjugate $\overline{v}_2$ yields a solution with $\overline{c}$. Hence spectral stability corresponds to the case where the spectrum is purely real.

In order to prove the results of the theorem for the sake of contradiction suppose that a non-trivial solution $\tilde{v}$ with $\Im(c) > 0$ exists. Then $U - c$ is bounded away from 0 in all of $[-1, 1]$ and we may thus test (4) against $-\frac{1}{U-c}\tilde{v}$ and integrate by parts to obtain

$$\int |\partial_y \tilde{v}|^2 + (k^2 + \sigma^2)|\tilde{v}|^2 dy = -\int \frac{U''}{U-c}|\tilde{v}|^2 dy.$$

In particular, the left-hand-side is real-valued and hence the imaginary part of the right-hand-side has to vanish:

$$0 = \int \Im(\frac{U''}{U-c})|\tilde{v}|^2 dy = \int \frac{|\tilde{v}|^2}{|U-c|^2}\Im(c)U'' dy.$$

Since $\tilde{v}$ and $\Im(c)$ are assumed non-trivial, this can only be the case if U'' changes sign (and hence is zero somewhere by the intermediate value theorem) or if U'' vanishes on the support of $\tilde{v}$. This hence yields Rayleigh's criterion.

In order to prove Fjørtoft's criterion we instead consider the real parts:

$$\int \frac{U''(U - \Re(c))}{|U-c|^2}|\tilde{v}|^2 dy = -\int |\partial_y \tilde{v}|^2 + k^2|\tilde{v}|^2 dy.$$

By the above proof of Rayleigh's criterion we may replace $\Re(c)$ on the left-hand-side by an arbitrary constant without changing the value. Using the fact that the right-hand-side is strictly negative, it hence follows that

$$\int \frac{U''(U - U(y_s))}{|U-c|^2}|\tilde{v}|^2 dy < 0.$$

Thus the integrand has to be negative at a point in the domain, which concludes the proof. □

These prototypical results highlight the stabilizing effect of convexity and related sign conditions. We remark that the study of linear stability is a wide area of research, which we do not attempt to survey in this document. The interested reader is referred to the text books of Drazin, Reid [**DR04**, **Dra02**], Schmid, Hennigson [**SH00**] and Yaglom [**Yag12**].

Instead we stress that while linear stability is certainly a useful prerequisite, it a priori does *not* allow to deduce nonlinear stability results. However, in the case of shear flows a similar result due to Arnold [**Arn66**] is available and also exploits convexity-like properties.

As a preliminary step towards this criterion we observe that in two dimensions, for any stationary solution the vorticity $\omega = \partial_x v_2 - \partial_y v_1$ and velocity satisfy

$$v \cdot \nabla\omega = 0.$$

Thus introducing the stream function ϕ as the solution of

$$\Delta\phi = \omega,$$

and computing the velocity $v = \nabla^\perp\phi$, the gradients of ϕ and ω are co-linear, and at least locally, there exists a function F such that

$$\omega = F(\phi)$$

and

$$\nabla\omega = F'(\omega)\nabla\phi.$$

Arnold's theorems require suitable sign or size bounds on F', which in the following is used as a convexity condition on the primitive function of F. Alternatively, it can also be interpreted as a second variation of a geometric reformulation of the Euler equations (see [**KW09**, **AK98**]).

Before stating the theorem, we introduce the notion of (nonlinear) Lyapunov stability.

Definition 1.3 (Lyapunov stability, c.f [**SH00**]). *We say that an evolution equation is* Lyapunov stable *in the Banach spaces X, Y if for all $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ such that*

$$\|u(0)\|_X \leq \delta$$

implies that

$$\|u(t)\|_Y \leq \epsilon.$$

In most cases, we are interested in situations where the spaces X and Y are the same or at least closely related (e.g. slight loss of constants), and construct a so-called *Lyapunov functional.* This functional is decreasing in time and at all times provides an upper bound on $\|u(t)\|_Y$. In particular, the existence of a Lyapunov functional implies Lyapunov stability.

Theorem 1.4 ([**AK98**]). *Consider a stationary solution on a two-dimensional manifold and suppose that there globally exists a function F such that $\phi = F(\omega)$ and that there exist constants $0 < c \leq C < \infty$ such that*

$$c \leq F' \leq C. \tag{5}$$

Then, denoting the velocity and vorticity perturbations as v, ω for all times $t \geq 0$ it holds that

$$\int |v(t)|^2 + c|\omega(t)|^2 \leq \int |v(0)|^2 + C|\omega(0)|^2.$$

A similar result also holds if instead

$$c \leq -F' \leq C, \tag{6}$$

where it follows that

$$\int c|\omega(t)|^2 - |v|^2 \leq \int C|\omega(0)|^2 - |v(0)|^2.$$

We note that for the specific case of a shear flow, we may compute

$$F'(-U'(y)) = \frac{U(y)}{U''(y)}.$$

Hence, the requirement that F' is bounded requires that $U'' \neq 0$, unless $U(y)$ also vanishes at any such inflection point. The most important application of the variant (6) of Arnold's theorem is given by the analysis of Kolmogorov flow

$$U(y) = \sin(ly),$$

where we consider flows which are periodic in both x and y, but with possibly different periods. Then $F' = -l^2 < 0$ is constant and we may view

$$\int l^2|\omega|^2 - |v|^2$$

as a bilinear form in ω (with $\|v\|_{L^2}^2 = \|\omega\|_{H^{-1}}^2$). This form is definite when considering a short torus $\mathbb{T}_L \times \mathbb{T}_{2\pi}$ with $L < 2\pi$ (which is hence stable), semi-definite for a square torus, $L = 2\pi$, and indefinite for a long torus, $L > 2\pi$ (which is known to be unstable [**CZEW23**]). This criterion is optimal for this particular example.

The following proof is adapted from [**AK98**, page 94].

Proof of Theorem 1.4. While F' is a priori only defined on the range of $-U'$, we may extend it by a constant (and F in an affine way) to obtain a function that is defined on all of $\mathbb{R}$ and further define G as a primitive function of F so that

$$c \leq G'' \leq C.$$

Thus G is a *convex* function. Denoting the stationary solution by ϕ_s, ω_s, v_s and the perturbation by ϕ, ω, v, we then claim that

$$H_2(\phi) := \int |v|^2/2 + G(\omega_s + \omega) - G(\omega_s) - G'(\omega_s)\omega dxdy$$

is a conserved quantity of the Euler equations. Indeed, since ϕ_s and $\phi + \phi_s$ are solutions of the Euler equations, the following quantities are conserved:

$$H_1(\phi_s) = \int |v_s|^2/2 + G(\omega_s),$$
$$H_1(\phi + \phi_s) = \int |v + v_s|^2/2 + G(\omega + \omega_s).$$

In particular, we may compute $H_2(\phi)$ as

$$H_2(\phi) = H_1(\phi + \phi_s) - H_1(\phi) + \int \langle v, v_s \rangle + G'(\omega_s)\omega.$$

However, the latter integral is just the Euler equations for ω_s in weak formulation tested against v and hence this integral identically vanishes. Therefore, H_2 is given by the difference of two conserved quantities and is thus itself conserved.

It remains to use H_2 to prove the theorem, which however is just an application of the intermediate value theorem for the convex function G. That is, for any (fixed) value of ω_s and any $z \in \mathbb{R}$ it holds that

$$G(\omega_s + z) - G(\omega_s) - G'(\omega_s)z = G''(\tilde{z})|z|^2/2$$

for some intermediate value $\tilde{z}$. In particular the right-hand-side is bounded above and below by

$$\frac{c}{2}|z|^2$$

and

$$\frac{C}{2}|z|^2,$$

which concludes the proof. $\square$

While this short exposition for simplicity of presentation focused on the classical theory of the Euler equations, we emphasize that the (linear) stability of other related systems also has attracted significant research interest [**TWZZ20**, **TW19**, **BLW20**, **HXY18**]. This includes the Miles-Howard criterion [**How61**] for the Boussinesq equations, which we will revisit in Section 2.4.

We, however, stress that all preceding stability results require restrictive conditions on the shear flow and that linear stability results only rule out certain types of instability. Indeed, we note that the Couette flow

$$U(y) = y$$

or the related Taylor-Couette flow between rotating cylinders is a prototypical flow which is monotone rather than convex and is not covered by Arnold's criterion. Furthermore, while the stability criterion for Fjørtoft ensures linear stability for the inviscid problem and also the associated linearized viscous problem is stable for all values of $\nu > 0$, experimentally one observes that this flow is unstable for small values of $\nu > 0$. This discrepancy between linear stability and observed (nonlinear) instability is known as the *Sommerfeld paradox* [**Yag12**]. A full understanding of the nonlinear behavior of perturbations for large times is still an open problem. However, as was shown by [**LZ11**] one possible explanation is given by non-trivial unstable solutions, called Kelvin's cat eyes, which are close in H^s neighborhoods with $s < \frac{3}{2}$. Yet, as we discuss in the following section, the nonlinear stability of the Couette flow depends on the chosen regularity in a rather subtle way and highly regular perturbations (e.g. analytic) are not only stable, but even converge to a nearby stationary solutions as $t \to \infty$.

1.2. Mixing as a (de)stabilizing effect

The Euler equations in vorticity formulation may be viewed as an active scalar equation

$$\begin{aligned} \partial_t \omega + V \cdot \omega = 0, \\ V = V[\omega], \\ (t, (x, y)) \in \mathbb{R} \times \Omega, \end{aligned}$$

where the velocity is generated by the vorticity. As an easier but instructive setting and in order to obtain a first heuristic understanding of (phase-)mixing, in this section we investigate a *passive scalar problem* where the velocity field is instead viewed as given. More precisely, we consider the equation

$$\partial_t f + v \partial_x f = 0,$$

$$(t, x, v) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R},$$

in $1 + 1$ dimensions. We stress that here v denotes the second variable and more generally (x, v) is an element of a tangent bundle on a manifold. These equations model the evolution of a phase-space density $f(t, x, v)$, where the position changes according to the velocity and the velocity due to forces (which are trivial in this simple case). We remark that after a change of notation ($f \mapsto \omega$, $v \mapsto y$), these equations coincide with the linearized problem around Couette flow, but more generally also appear as a natural model in kinetic equations.

Using the method of characteristics, the solution may be explicitly computed in terms of the initial data f_0:

$$f(t, x, v) = f_0(x - tv, v).$$

In particular, we observe that all L^p norms of f remain conserved in time, which we may interpret as a stability result around $f = 0$ in Lebesgue spaces. However, we also note the following dual effects:

- If f_0 is not independent of x, then f will oscillate more and more rapidly in v and as $t \to \infty$, the solution $f(t)$ weakly converges to the x-average $\langle f_0 \rangle_x(v)$. Since the L^p norm of this average generally is strictly smaller than that of f_0 (and by the conservation property also of $f(t)$), there generally is no strongly convergent subsequence.
- If f_0 is in a *higher* Sobolev space $W^{s,p}$, $s > 0$, then also $f(t) \in W^{s,p}$ for all $t > 0$. However, generically $\|f(t)\|_{W^{s,p}}$ will grow as t^s as $t \to \infty$. Thus, the zero solution cannot be expected to be stable in positive Sobolev regularity (however note that there is no eigenvalue instability here).
- As an in a sense dual result we observe that *negative* Sobolev (semi-)norms may improve as $t \to \infty$. For instance for f_0 with vanishing x-average it holds that

  $$\|f(t)\|_{H^{-1}} \leq C(1 + t)^{-1} \|f_0\|_{H^1}.$$

  The shear dynamics convert regularity of the initial data into decay of the solution in *negative* norms.

In view of the time-reversible conservation law structure of the equations (conserving all L^p norms and other integrals of f) and time invertibility this behavior is at first sight very unexpected. The above decay mechanism underlies one of the major stabilizing effects of plasma dynamics, known as *Landau damping.* In order to investigate this stabilization and related effects in more detail, in the following we discuss this plasma setting.

The evolution of plasmas is in many regimes well described by a mean field model in terms of the phase-space density $f(t, x, v)$ known as the Vlasov-Poisson equations:

$$\begin{aligned}
\partial_t f + v \cdot \nabla_x f + F[\rho] \cdot \nabla_v f = 0,& \\
\rho = \int f dv,& \\
F[\rho] = \nabla_x W *_x \rho,& \\
(t, x, v) \in \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d.&
\end{aligned}$$

Here $f(t,x,v)$ models the phase-space density of (ions or electrons in) a plasma and (up to normalization) corresponds to the probability density of finding a particle at position x moving with velocity v. By Newton's second law, changes of the velocity v are due to a force field F, which is generated by f. Moreover, the force field $F[\rho](t,x)$ is independent of v and only depends on f in terms of the spatial density ρ. The precise choice of F is model dependent, with the most important cases in applications being given by *gravitational interaction*

$$F[\rho](t,x) = \nabla_x \Delta_x^{-1} \rho(t,x),$$

with an attractive force, and *Coulomb interaction*

$$F[\rho](t,x) = -\nabla_x \Delta_x^{-1} \rho(t,x),$$

with a repulsive force.

It is then an easy observation that formally any density $f = f_0(v)$ which only depends on v generates a trivial force field. We remark that here $F[\rho]$ possibly needs to be interpreted as a principal value integral in case ρ or $\nabla_x W$ are not sufficiently integrable. This is known in the physics community as the *Jeans swindle* (see [**MV10**]). In particular such solutions are stationary solutions of the Vlasov-Poisson equations and of the free transport equations.

If one now considers a small perturbation

$$g(t,x,v) = f(t,x,v) - f_0(v)$$

of this solution which depends on x but is initially very smooth, it seems reasonable to expect that at least up to some time scale the effect of the force field can be neglected and that g is well-approximated by a solution to the free transport equations:

$$g(t,x,v) \approx g_\infty(x - tv, v).$$

However, this in turn implies that the spatial density

$$\rho(t,x) \approx \int g_\infty(x - tv, v) dv$$

decays rapidly in time. Indeed, the method of non-stationary phase suggests that the right-hand-side decays at least like t^{-N} if $g_\infty \in W^{1,N+1}$. Furthermore, this in turn implies that the force field decays in time

$$F[\rho](t) \to 0$$

as $t \to \infty$ and even decays very quickly, if g_∞ is smooth. This very rapid decay for smooth perturbations was discovered by Lev Landau in 1946 [**Lan46**] and has been a highly influential result in plasma dynamics with an impact in research for the following decades. It is hence known as *Landau damping*. We remark that the above mechanism can be described as converting regularity into decay. That is, when viewed in coordinates not moving with the flow, bounds on higher Sobolev norms of $g_\infty(x - tv, v)$ deteriorate in time, while the decay of $\rho(t)$ is tied to quantitative control of the regularity of $g_\infty(x, v)$ (which in the nonlinear case also depends on t).

While the above sketch formally approximated the evolution of the perturbation by free transport, the work of Landau established the result for the linearized Vlasov-Poisson equations around densities $f(v)$ and also provided sufficient (and close to necessary) conditions on $f(v)$ for this stability to hold. However, as was pointed out already a short time later in what is known as *Backus's objection* [**Bac60**], there are some large obstacles to nonlinear stability results. Indeed, supposing for the moment that solutions of the nonlinear equations were well-approximated by the free transport equations, the second factor in the nonlinearity

$$F[\rho] \cdot \nabla f$$

grows in time. Hence, a priori one may only expect that the linear dynamics accurately predict the nonlinear dynamics on time scales $\mathcal{O}(\epsilon^{-1})$, where $\epsilon > 0$ is the size of the initial perturbation. In particular, results on the linearized equations for times tending to ∞ thus a priori have very limited predictive power for the nonlinear problem.

A second, more serious obstacle are (non-linear) resonances, which were experimentally observed [**MWGO68**] in 1968. We provide a short heuristic sketch of the experiment in the setting of a periodic infinite channel $\mathbb{T} \times \mathbb{R}$ (which is analogous to working in cylindrical coordinates):

- Beginning with a plasma at rest, at the initial time one introduces a small perturbation $\epsilon e^{ilx}\psi(v)$, where $l \in \mathbb{Z} \setminus \{0\}$ and $\psi \in \mathcal{S}(\mathbb{R})$ is frequency-localized near 0.
- According to the linearized dynamics, this perturbation approximately evolves by free transport:
$$\epsilon e^{ilx-itlv}\psi(v).$$
In particular, for large times this solution becomes more and more oscillatory and the corresponding force field perturbation is damped to zero.
- At a later time $\tau > 0$ one introduces a second perturbation $\epsilon e^{ikx}\psi(v)$ with $k \neq l$. Neglecting nonlinear interactions, this perturbation is then also expected to evolve by free transport:
$$\epsilon e^{ikx-i(t-\tau)kv}\psi(v),$$
and thus exhibit Landau damping. We remark that this solution agrees with the one starting as $\epsilon e^{ikx+i\xi v}\psi(v)$ with $\xi = k\tau$ at time 0.
- At a predictable later time depending on τ, k, l one observes a large perturbation of the force field at the frequency $k + l$. The interaction of both perturbations resulted in a *plasma echo*.

Since this echo is not predicted by the linearized problem around the stationary solution, this effect is commonly viewed as a *non-linear* phenomenon. A rough calculation for a toy model (see for instance [**MV10**]) suggests that, denoting $\xi = k\tau$, the size of this echo is comparable to

$$\epsilon^2 \frac{\xi}{(k+l)^3}$$

Choosing $l = -1$ and k large, it further seems possible that this correction at frequency $k-1$ in turn results in an echo at frequency $k-2$ at a later time and so on. In particular, we may thus expect that a *chain of echoes* (or *cascade of resonances*)

$$k \mapsto k-1 \mapsto k-2 \mapsto \cdots \mapsto 1$$

may result in growth compared to the size ϵ of the initial perturbation by

$$\prod_{j=1}^{k-1} \epsilon \frac{\xi}{j^3}.$$

Choosing k to maximize this product, Stirling's approximation yields an estimate of the possible norm inflation by

$$\exp(C\sqrt[3]{|\xi|})$$

Therefore one may a priori only expect stability in spaces requiring suitable bounds, which are given by Gevrey spaces.

DEFINITION 1.5. *Let $f \in L^2(\mathbb{T} \times \mathbb{R})$ and $1 \leq s < \infty$, then f is in the* Gevrey class *$\mathcal{G}_s$ if there exists a constant $C > 0$ such that the Fourier transform of f satisfies*

$$\sum_k \int \exp(C|\xi|^{1/s})|\mathcal{F}f(k,\xi)|^2 d\xi < \infty.$$

Alternatively, one may require that $f \in H^n$ for all $n \in \mathbb{N}$ and that there exists a constant $D > 0$ such that

$$\|f\|_{L^2_x H^n_v} \leq D^n (n!)^s.$$

Both definitions are equivalent up to losses in the constants C and D and quantify regularity between C^∞ and analyticity (any function in $\mathcal{G}_1$ is analytic). We remark that there exist non-trivial functions with compact support, which are Gevrey regular with $s > 1$.

As seen from the above sketched resonance mechanism any norm inflation due to echoes can be expected to be absorbed into a loss of the constant C (as t increases) provided $s \leq 3$. In a seminal result Mouhot and Villani [**MV11**] proved that this is indeed the case and that the Vlasov-Poisson equations are stable in (sub-critical) Gevrey regularity. However, we stress that nonlinear results available in the literature do not quite reach the critical regularity class of the above echo model, but rather consider Gevrey regular initial data with $1 \leq s < 3$ [**GNR21, BMM16**]. This stronger regularity allows to treat many contributions as error terms at the cost of a large loss of constant. Furthermore, while the above model seems to be a reasonable worst case estimate, it is a priori not clear whether such growth is actually attained or whether the model overestimates the possible norm inflation. Here Bedrossian [**Bed20**] could provide a first answer in that there exists special initial data, which exhibit norm inflation, when measured in Sobolev regularity.

THEOREM 1.6 (Adapted from [**Bed20**]). *Let $f_0 = \frac{4\pi\delta}{1+v^2}$, $R \geq 1$. Then there exists $\sigma_0(R) > 100R$ such that for all $\sigma \geq \sigma_0$ there exist $\epsilon_0, \delta_0 > 0$ and such that if $\epsilon \leq \epsilon_0, \delta \leq \delta_0$ there exists a real analytic function h_{in} with*

$$\|\sqrt{1+v^2}h_{in}\|_{H^\sigma} \leq \epsilon^{1-p},$$
$$\|\sqrt{1+v^2}h_{in}\|_{H^{\sigma-3}} \leq \epsilon,$$

and a finite time $t_\star = t_\star(\epsilon, R)$ with $\epsilon t_\star \to \infty$ as $\epsilon \downarrow 0$ such that for all $z \geq 0$ it holds that

$$\|h(t_\star) \circ \mathcal{T}_{t_\star}\|_{H^{\sigma-R+z}} \gtrsim \epsilon^{-z},$$

$$|\mathcal{F}[\rho](t_\star, \pm 1)| \gtrsim t_\star^{R-\sigma}.$$

Hence, this result shows that no uniform bound in Sobolev regularity is possible. However, we remark that the norm inflation is attained at finite time and subsequently the solution remains asymptotically stable as $t \to \infty$ (by the results for analytic perturbations of [MV11]). Thus this result still leaves open the question of whether such resonance chains can indeed break asymptotic stability and, if so, in which sense. We remark that a similar construction of initial data exhibiting norm inflation has also been used for the Euler equations [DM23]. However, due the methods used, these results only provide a solution exhibiting norm inflation up to a finite time T depending on the frequency of the perturbation. In particular, this result makes no statement on asymptotic behavior as $t \to \infty$.

These results motivate our works [DZ21, Zil21a] (discussed in Sections 2.1 and 2.2) which show that the echo mechanism should not only be seen as a nonlinear mechanism. Instead the improved stability of the linear problem is an artifact of linearizing around a "trivial" solution, which hides the mechanism in the linear dynamics. Linearizing around a "non-trivial" nearby nonlinear solution, the chains of echoes can be captured already in the linearized dynamics. Moreover, exploiting the simpler linear structure, we are able to reach critical spaces and to construct initial data (for the linear problem) which exhibit non-trivial dynamics as $t \to \infty$.

CHAPTER 2

Traveling waves, echoes and hierarchies of instability

In many equations of interest, such as the Vlasov-Poisson equations, the Euler equations or the Boussinesq equations, the linearized analysis of stationary states suggests good asymptotic stability properties of the equations in Sobolev spaces. However, at the nonlinear level this stability is not reflected and instead much higher, Gevrey regularity is required to ensure stability. The associated instability mechanism is hence commonly considered a purely nonlinear effect. In the following, we argue that this difference in regularity should more accurately be seen as an artifact of linearizing around *trivial* solutions (that is, with trivial x-dependence). If one instead considers the problem around (explicit) non-trivial solutions, the resonance mechanism is accurately captured in the linearized equations.

2.1. Plasma echoes and Landau damping

As discussed in Section 1.2, it is expected that the critical spaces for nonlinear stability of the Vlasov-Poisson equations

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f &= 0, \\ F &= \nabla W *_x \rho, \\ \rho &= \int f dv, \\ (t,x,v) &\in \mathbb{R} \times \mathbb{T} \times \mathbb{R}, \end{aligned} \tag{7}$$

are given by the Gevrey 3 class (see Definition 1.5). However, existing stability results [**MV11**] do not quite reach the critical regularity. Similarly, the result by Bedrossian [**Bed20**] achieves norm inflation after a finite time, but the constructed solution is asymptotically stable as $t \to \infty$.

In [**Zil21a**], we thus investigated whether these very strong assumptions are optimal and what behavior can be expected in the critical Gevrey 3 regular case as $t \to \infty$. In particular, in [**Zil21a**] we point out that there is the following *hierarchy* of results, which are all referred to as Landau damping, but actually correspond to successively stronger results:

[1] The force field converges to a (simpler) asymptotic profile as time tends to infinity. This is the physically observed phenomenon.
[2] The corresponding perturbation to the phase-space density remains uniformly bounded in a suitable L^p space and asymptotically converges weakly as time tends to infinity.
[3] The perturbation asymptotically behaves like a free solution of an associated linear problem. In the language of dispersive equations one says that the solution scatters with respect to the linear dynamics.

As a main result of [Zil21a], we show that the norm inflation is not necessarily a nonlinear mechanism, but rather a linear mechanism when linearizing around non-trivial (i.e. x-dependent) solutions. More precisely, in any neighborhood of the spatially homogeneous solution, there exist explicit nonlinear solutions, which we call traveling waves (in analogy to dispersive equations). Furthermore, for the linearized problem around these waves we are able to establish estimates in critical spaces and to construct data for which Landau damping in the physical sense [1] holds, but for which damping in the sense [3] fails. The main results of this article are summarized in the following theorem.

THEOREM 2.1 ([Zil21a]). *Traveling wave-like solutions: Consider the Vlasov-Poisson equations* (7) *on* $\mathbb{T}^d \times \mathbb{R}^d$*. Let* $\psi \in \mathcal{S}(\mathbb{R}^d)$ *with* $supp(\mathcal{F}_v\psi) \subset B_\delta(0)$, $0 < \delta < 0.1$*. Then for any* $\epsilon > 0$ *the function*

$$f_*(t,x,v) = f_0(v) + \epsilon \cos(e_1 \cdot (x - tv))\psi(v)$$

is a solution of the Vlasov-Poisson equations on $\mathbb{T}^d \times \mathbb{R}^d$ *for* $t \in (0.1, \infty)$*. We call such a solution a* traveling wave-like solution *in reference to the same term used in dispersive equations.*

Linear stability, blow-up and damping: Let $\epsilon > 0$, $d = 1$ *and* $\psi \in \mathcal{S}(\mathbb{R})$ *be given and consider the linearized Vlasov-Poisson equations with Coulomb or gravitational interaction on* $\mathbb{T} \times \mathbb{R}$ *around* f_* *with* $f_0 \equiv 0$ *on* $(0,\infty)$ *with initial data* h_0*:*

$$\text{(LVP)}\qquad \begin{aligned} &\partial_t h + F[\int h(t, x - tw, w)dw](\nabla_v - t\nabla_x)(\epsilon\cos(x)\psi(v)) = 0, \\ &F[\rho] = \pm\partial_x(\partial_x^2)^{-1}\rho,\ \rho(t,x) = \int h(t,x,v)dv \\ &h(0) = h_0. \end{aligned}$$

Then we have the following results:

- *Stability in Gevrey regularity: Suppose that the Fourier transform* $\tilde{h}_0 := \mathcal{F}_{x,v}h_0$ *of the initial data* h_0 *satisfies*

$$\sum_k \int |\tilde{h}_0(k,\xi)|^2 \exp(c\sqrt[3]{|\xi|})d\xi \leq C_0 < \infty,$$

  with a sufficiently large constant $c > 0$*. That is,* $h_0 \in \mathcal{G}_3$ *is Gevrey regular. Then there exists a constant* $C > 0$ *(independent of* h_0 *or* C_0*) such that for all times*

$$\sum_k \int |\tilde{h}(t,k,\xi)|^2 \exp(\sqrt[3]{|\xi|})d\xi \leq CC_0.$$

  Thus Landau damping in the strong sense [3] holds. This in turn implies that Landau damping in the physical sense [1] holds and that hence the force field decays to zero as $t \to \infty$*.*
- *Norm inflation: Suppose in addition that* $\mathcal{F}_v\psi \geq 0$*, then there exists initial data* $h_0 \in \mathcal{G}_3$*, supported in frequency in* $\{k_0\} \times (\xi_0 - 1/2, \xi_0 + 1/2)$ *such that the solution* $h(t)$ *with this initial data is stationary for* $t > \xi_0 + 1/2 + \delta k_0 =: T_1'$ *and there exist constants* c_1, c_2 *(proportional to* $\|\psi(v)\|_{L^\infty}$ *and independent of* ξ_0*) such that*

$$\exp(\sqrt[3]{c_1|\xi_0|}) \leq \|h(T_1')\|_{L^2} \leq \exp(\sqrt[3]{c_2|\xi_0|}).$$

There is norm inflation *due to* echo chains.

- *Physical damping: For every $s \in \mathbb{R}$ there exists $h_\infty \in H^s$ but $h_\infty \notin H^{s+\delta}$ for all $\delta > 0$ and $h_0 \in \mathcal{G}_3$ (with small constant) such that the solution $h(t)$ with initial datum h_0 converges to h_∞ in H^s as $t \to \infty$. In particular, the solution $h(t)$* diverges *in $H^{s+\delta}$ for all $\delta > 0$. However, if $s \geq 0$ then $F[h](t) \to_{L^2} 0$ as $t \to \infty$. For this data physical linear damping in the sense [1] holds, but strong damping to the free transport equations in the sense [3] fails.*

Hence, these results show that mixing in the physical sense [1] is a robust mechanism and can occur even if stability in the sense [3] does not hold. Moreover, we remark that the solutions exhibiting norm inflation at time T_1' are asymptotically stable as $t \to \infty$. Indeed, as sketched for the experiment any individual echo chain concludes after finitely many steps and hence after a finite time. Thus, the above result combines countably infinitely many chains to achieve divergent behavior as $t \to \infty$.

2.2. A hierarchy of inviscid damping in the Euler equations

Similarly to the case of the Vlasov-Poisson equations discussed in Section 2.1, also in the Euler equations nonlinear resonances have been observed experimentally [**YOD05**]. Here again a toy model suggests that for frequency-localized perturbations the possible norm inflation is comparable to

$$\exp(C\sqrt{\epsilon|\xi|})$$

and that thus the critical space is given by the Gevrey 2 class. Indeed, following the seminal results of [**BM15b**] in sub-critical regularity, the nonlinear stability in Gevrey 2 with large constant has been established in [**IJ20**]. Furthermore, in [**DM23**] special initial data has been constructed, for which the solution exhibits norm inflation on a large finite time interval. However, since the control of the nonlinear dynamics becomes difficult under large perturbations, no statement is made for larger times or indeed the behavior as $t \to \infty$. In particular, as in the Vlasov-Poisson setting, it might be the case that the dynamics are nevertheless asymptotically stable.

In a joint work with Yu Deng [**DZ21**], we thus investigated whether inviscid damping can fail in the critical Gevrey class with small constant. More precisely, as in Section 2.1 we showed that the answer to this question crucially depends on the definition of inviscid damping. Similarly to the case of plasma dynamics discussed in Section 2.1, we showed that in the literature there actually is a hierarchy of distinct notions, which have not previously been distinguished:

[1] The velocity field v converges (to another shear flow) as $t \to \infty$. We speak of *physical damping*.
[2] The vorticity ω converges weakly in L^2 as $t \to \infty$.
[3] There exists a map $\phi(t, y)$ such that $\omega(t, x - \phi(t, y), y)$ converges strongly in L^2 as $t \to \infty$. We speak of *strong scattering*.

As our first main results, we show that there exist global in time solutions of the Euler equations, which are initially arbitrarily close to the stationary solution, but exhibit linear stability properties matching the nonlinear equations. One such solution is of the form

$$\omega(t, x, y) = -1 + c\cos(x - ty).$$

The (slightly simplified) linearized problem around this wave-like solution in vorticity formulation and in coordinates $(x-ty, y)$ is given by

$$\begin{aligned} \partial_t \omega + c\sin(x)\partial_y \Delta_t^{-1}\omega = 0, \\ \Delta_t = \partial_x^2 + (\partial_y - t\partial_x)^2. \end{aligned} \tag{8}$$

This linear problem captures the above sketched resonance mechanism and matches the critical Gevrey 2 stability class of the nonlinear problems. Moreover, it allows us to achieve results in critical spaces with critical constants and to construct data such that physical damping holds, but strong scattering fails.

THEOREM 2.2 ([**DZ21**]). *Let $0 < c < 0.2$, then there exists $C > 0$ such that for any $s \in \mathbb{R}$ there exist solutions of (8) with $\omega_0 \in \mathcal{G}_{C,2}$ such that $\omega(t)$ converges to a limit ω_∞ in $H^\sigma, \sigma \leq s$, but diverges in $H^\sigma, \sigma > s$. In particular, in the case $s \geq 0$ we note that* damping*, that is asymptotic convergence of the velocity field, holds, while* asymptotic stability *in Gevrey regularity fails.*

Furthermore, the constant C is (almost) optimal in the sense that there exists a (larger) constant $C_0 > 0$ such that for any $C_1 > 2C_0$ if $\omega_0 \in \mathcal{G}_{C_1,2}$ (see Definition 1.5), then $\omega(t) \in \mathcal{G}_{C_1-C_0,2}$ for all times and $u(t)$ converges in $\mathcal{G}_{C_1-C_0,2}$.

Similarly as in the plasma setting of Section 2.1, we thus obtain a precise description of instability at the critical Gevrey 2 regularity. In particular, we can construct data which exhibits a prescribed norm blow-up as $t \to \infty$. We remark that for the Euler equations the Biot-Savart law is more challenging to control due to the lack of "lower dimensional" structure $(f(t,x,v) \mapsto \rho(t,x))$ and includes stronger coupling between different frequencies. Indeed, as a major new result of [**DZ21**] we showed that at large frequencies the norm inflation factors are qualitatively different than predicted by the toy model.

Building on these results on the effects of mixing in the Euler equations, in the following sections we consider stability and long-time behavior in systems. Here, the coupling and system structure is shown to play a crucial role and gives rise to new resonance mechanisms.

2.3. Thermal echoes in the viscous Boussinesq equations

While the Euler equations exhibit *fluid echoes* and norm inflation and are thus unstable in Sobolev regularity, this effect is suppressed by viscosity in the Navier-Stokes equations – at least for small data. Moreover, viscous dissipation and mixing may even constructively interact to result in improved damping estimates. As we discuss in Chapter 4, this stabilization by viscosity is sufficient to establish *linear* stability of the Boussinesq equations with viscosity but without thermal resistivity:

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \nabla p &= \nu\Delta v + \theta e_2, \\ \partial_t \theta + v\cdot\nabla\theta &= 0, \\ \operatorname{div}(v) &= 0, \\ (t,x,y) &\in (0,\infty)\times\mathbb{T}\times\mathbb{R}. \end{aligned}$$

However, nonlinear stability results [**MSHZ22**] require (better than) Gevrey 3 regularity. Considering that fluid echoes are suppressed and that this class is different from the one in the Euler equations, a key objective of [Zil21b] has been to identify the underlying resonance mechanism and corresponding critical spaces.

In particular, we show that the interaction of buoyancy and viscosity enforces a balance between the velocity and temperature in terms of the *good unknown*

$$G = \nu \partial_x \omega + \Delta^{-1} \partial_x^2 \theta,$$

which is rapidly damped by the viscous dissipation. If we for simplicity of presentation for the moment assume that $G \equiv 0$, this in turn implies that perturbations of the temperature θ can induce perturbations of the vorticity and hence velocity. These velocity perturbations in turn can become resonant. We thus speak of *thermal echoes*. Compared to the setting of the Euler equations considered in Section 2.2 this mechanism exhibits a strongly modified version of the Biot-Savart law (now in terms of $(\theta, G) \mapsto v$) and crucially relies on the system structure of the equations.

In a first step, we construct wave-type solutions of the form

$$\begin{aligned} v &= \begin{pmatrix} y \\ 0 \end{pmatrix} + \frac{f(t)}{1+t^2} \sin(x - ty) \begin{pmatrix} t \\ 1 \end{pmatrix}, \\ \omega &= \nabla v = -1 + f(t) \cos(x - ty), \\ \theta &= \alpha y + g(t) \sin(x - ty), \end{aligned} \tag{9}$$

where $f(t), g(t)$ satisfy a system of ordinary differential equations. Considering the long-time behavior of these coefficients for simplicity of discussion one may heuristically estimate

$$\begin{aligned} g(t) &\approx 2\nu c, \\ f(t) &\approx \frac{4c}{1+t^2}, \end{aligned}$$

where c is a small constant. The following theorem summarizes our main results for the linearized equations around these waves.

THEOREM 2.3 ([Zil21b]). *Consider the (simplified) linearized Boussinesq equations around the above wave and suppose that $\frac{g}{2\nu} =: c$ satisfies $c < 0.001$. Further define $G = \nu \partial_x \omega + \partial_x^2 \Delta^{-1} \theta$.*

- *There exists $C > 0$ such that if the (Fourier transform of the) initial data satisfy*
$$\int \exp(C \sqrt[3]{|\xi|})(1+k^2)^N (|\mathcal{F}\theta_0|^2 + |\xi||\mathcal{F}G_0|^2) < \infty,$$
then for all times $t > 0$ it holds that
$$\int \exp(\frac{C}{2} \sqrt[3]{|\xi - kt|})(1+k^2)^N (|\mathcal{F}\theta(t)|^2 + |\xi||\mathcal{F}G(t)|^2) < \infty.$$
The evolution preserves Gevrey 3 regularity *up to a loss of constant.*
- *For $c|\xi| > \nu^{-3/2}$ there exists initial data $\theta_0 \in L^2$ localized at frequency ξ and $G_0 = 0$, such that for all $t > 2\xi$ the solution satisfies*
$$\|\theta(t)\|_{L^2} \geq \exp(\sqrt[3]{c|\xi|}).$$
There exists frequency localized initial data which exhibits norm inflation*. However, after attaining this norm inflation the solution is stable for all future times.*

- *Moreover, for every $s_0 \in \mathbb{R}$ there exists $0 < C' < C$ and initial data with*

$$\int \exp(C' \sqrt[3]{|\xi|})(1+k^2)^N(|\mathcal{F}\theta_0|^2 + |\mathcal{F}\omega_0|^2) < \infty,$$

such that $\theta(t)$ converges in H^s for $s < s_0$ and diverges to infinity in H^s for $s > s_0$, as $t \to \infty$. Hence, the Gevrey 3 regularity class is a critical space for stability and damping may persist despite blow-up.

These results strongly exploit the system structure of the equations and need to control the interaction between buoyancy effects, viscous dissipation and the coupling of viscosity and temperature. In particular, the non-local operator connecting G, θ and the velocity exhibits a very different frequency-dependence than for the Euler equations or Vlasov-Poisson equations and hence estimates need to be specifically tailored to this system.

2.4. Resonances in the inviscid Boussinesq equations

The results of the previous Section 2.3 on the partially viscous Boussinesq equations strongly relied on the effects of viscosity to enforce a balance between the temperature and vorticity. This resulted in (in)stability estimates worse than for the Navier-Stokes equations (i.e. ignoring temperature) but better than for the Euler equations. Letting $\nu \downarrow 0$, it is thus a natural question to inquire about the long-time behavior and (in)stability of the *inviscid* Boussinesq equations

$$\begin{aligned} \partial_t\omega + v\cdot\nabla\omega &= \partial_x\theta, \\ \partial_t\theta + v\cdot\nabla\theta &= 0, \\ (t,x,y) &\in \mathbb{R}\times\mathbb{T}\times\mathbb{R} \end{aligned}$$

close to the stationary solution

$$\omega = -1, v = (y,0), \theta = \alpha y, \tag{10}$$

where we consider the regime of stable stratification $\alpha > \frac{1}{4}$ (see [How61]).

Unlike the Euler equations, for this system already the linearized problem is algebraically unstable in terms of ω, θ. However, if one introduces the following good unknowns following [BBCZD21]

$$\begin{aligned} Z(t,x,y) &:= \sqrt{\alpha}\left((\partial_x^{-2}\Delta)^{-1/4}\omega\right)(t,x-ty,y), \\ Q(t,x,y) &:= \left((\partial_x^{-2}\Delta)^{1/4}\partial_x\theta\right)(t,x-ty,y), \end{aligned} \tag{11}$$

then the linearized problem is known to be stable:

Lemma 2.4 ([**BBCZD21**, **BZD20**, **TWZZ20**, **Zil22**]). *Let $\alpha > \frac{1}{4}$. Then the linearized Boussinesq equations around the stationary solution (10) are stable in the sense that for any initial data ω, θ with $\int \omega dx = \int \theta dx = 0$ the energy*

$$\alpha\|((\partial_x^{-2}\Delta)^{-1/4}\omega)(t,x-ty,y)\|_{L^2}^2 + \|((\partial_x^{-2}\Delta)^{1/4}\theta)(t,x-ty,y)\|_{L^2}^2$$

is bounded above and below for all times, uniformly in terms of its initial value, with a constant depending only on α.

We remark that in the (partially) viscous setting other choices of unknowns are natural [**ABSPW22**, **ACW10**, **LWX⁺21**, **TWZZ20**, **TW19**, **DWZZ18**, **CW13**, **ZZ23**].

This linear stability result in terms of (Z, Q) also has implications for wave-type solutions of the full nonlinear problem.

Lemma 2.5 ([**Zil22**], see also Proposition 2.1 in [**Zil21b**], [**BZD20**] and [**BBCZD21**]). *Let $\alpha \geq 0$ be given, then there exist non-trivial functions $f(t)$ and $g(t)$ such that the triple (ω, θ, v) with*

$$\begin{aligned} \omega(t,x,y) &= -1 + f(t)\cos(x - ty), \\ \theta(t,x,y) &= \alpha y + g(t)\sin(x - ty), \\ v(t,x,y) &= (y, 0) + \frac{1}{1+t^2}\nabla^\perp \cos(x - ty), \end{aligned} \tag{12}$$

is a solution of the nonlinear inviscid Boussinesq equations for all times. We call these solutions traveling waves. *Moreover, if $\alpha > \frac{1}{4}$ it holds that*

$$E(t) := \frac{|\alpha|}{\sqrt{1+t^2}}|f(t)|^2 + \sqrt{1+t^2}|g(t)|^2$$

satisfies

$$cE(0) \leq E(t) \leq CE(0)$$

for some constants $0 < c < C < \infty$ depending on α.

In our construction, we further observe that $f(t)$ grows in time and that if $f(0) = \epsilon \ll 1$, the vorticity only remains small on time scales

$$0 < t < \epsilon^{-2}.$$

For this reason the nonlinear stability result of [**BBCZD21**] restricts itself to this time interval and establishes stability in (slightly stronger than) Gevrey 2 regularity – matching the regularity class of the Euler equations (for $0 < t < \infty$). Considering the rather different linear stability results, time scales and system structure, a key objective of [**Zil22**] has been to investigate whether this class actually is optimal, how the resonance mechanism differs and how it depends on the frequency. To this end, we consider the linearized Boussinesq equations around the traveling waves in terms of Z and Q. More precisely, we consider the following simplified equations in coordinates moving with the shear:

$$\begin{aligned} &\partial_t \begin{pmatrix} Z \\ Q \end{pmatrix} + A \begin{pmatrix} Z \\ Q \end{pmatrix} = -\begin{pmatrix} |\partial_x|^{1/2}\Delta_t^{-1/4}(\nabla^\perp|\partial_x|^{-1/2}\Delta_t^{-3/4}Z \cdot \nabla f(t)\cos(x)) \\ |\partial_x|^{-1/2}\Delta_t^{1/4}(\nabla^\perp|\partial_x|^{-1/2}\Delta_t^{-3/4}Z \cdot \nabla g(t)\cos(x)) \end{pmatrix}, \\ &\Delta_t = \partial_x^2 + (\partial_y - t\partial_x)^2. \end{aligned} \tag{13}$$

Compared to the full linearized problem, we here suppress changes of the underlying shear profile, which can be viewed as a small forcing term. As we discuss in [**Zil22**], this simplification is not required in our estimates of the main damping mechanism and also not for large times. For small times, the corresponding error can also be controlled at the cost of (assuming high) regularity. We expect that this is only a technical issue and that errors remain bounded in any regularity in coordinates moving with the perturbed shear. However, as the associated estimates are technically very involved and since our main interest lies in the resonance mechanism, we introduced this simplified model.

Our main results are summarized in the following theorem.

THEOREM 2.6 ([Zil22]). *Let $0 < \delta < 0.1$ and $0 < \epsilon < 0.1$ be given and let $f(t), g(t)$ correspond to traveling waves*

$$Z_{wave} = \frac{f(t)}{(1+t^2)^{1/4}} \cos(x),$$
$$Q_{wave} = g(t)(1+t^2)^{1/4} \sin(x),$$

with $f(0) = g(0) = \epsilon$. Let further (Z, Q) denote a solution of the simplified linearized Boussinesq equations (13) with initial data (Z_0, Q_0).

Then there exists $C > 0$ and $|\gamma| < \delta$ such that for any initial data (Z_0, Q_0) whose Fourier transform satisfies

$$\sum_k \int \exp\left(2C \min((\epsilon|\xi|^{1+\gamma})^{2/3-2\gamma}, \epsilon^{-2})\right) |\mathcal{F}(Z_0, Q_0)(k, \xi)|^2 d\xi \leq 1$$

the corresponding solution remains regular up to a loss in the constant C. That is, for all times $t > 0$ it holds that

$$\sum_k \int \exp\left(C \min((\epsilon|\xi|^{1+\gamma})^{2/3-2\gamma}, \epsilon^{-2})\right) (1 + \frac{\epsilon^{-2}}{|\xi|}) |\mathcal{F}(Z, Q)(t, k, \xi)|^2 d\xi \leq 1. \tag{14}$$

Here $k \in \mathbb{Z}$ denotes the frequency with respect to x and $\xi \in \mathbb{R}$ denotes the frequency with respect to y.

Moreover, there exists a constant $c = c(\alpha)$ such that if the Fourier transform of the initial data is supported in the region $|\xi| \geq c\epsilon^{-4}$ then the stability estimate improves to a uniform estimate

$$\|(Z, Q)(t)\|_{L^2} \leq 2\|(Z_0, Q_0)(t)\|_{L^2}. \tag{15}$$

We in particular highlight the following results:

- The bound by $\exp\left(2C \min((\epsilon|\xi|^{1+\gamma})^{2/3-2\gamma}, \epsilon^{-2})\right)$ or by a uniform constant, respectively, strongly differs from a Gevrey 2 bound by $\exp(C\sqrt{|\xi|})$. Furthermore, it identifies $|\xi| \sim \epsilon^{-4}$ as a distinguished region in frequency space for which both bounds may coincide. A key innovation here is to make use of precise growth behavior of the waves and to link the time-restriction to a frequency cut-off.
- Away from this frequency region bounds are drastically improved, highlighting the need for more precise spaces also for the nonlinear analysis.
- While the frequency-dependence and optimality of norm inflation bounds is very difficult to see in the nonlinear problem, the approach by traveling waves serves to clearly illustrate the interaction of the size of waves, the nonlinearity, frequencies and resonant times. In particular, it allows us to clearly identify expected critical regularity classes. We remark that the article establishes upper bounds, but for lower bounds restricts to a slightly simplified model. We expect that these lower bounds are also attained for the full model, but that a proof will be technically very challenging.

- While this threshold at first sight seems to match the one of the Euler equations, we emphasize that this is not the case. That is, when restricting the Euler equations to the finite time interval $(0, \epsilon^{-2})$ the expected optimal class for the Euler equations is given by Gevrey 3 instead, as we discuss in the following section.

2.5. Waves, resonances and critical spaces

The results summarized in the preceding sections 2.1 to 2.4 show that waves allow us to understand nonlinear instabilities in fluid systems as repeated resonances ("echoes") of a high frequency perturbation with an underlying low frequency wave. In this short section, we summarize the underlying general mechanism.

For a given fluid system, in order to obtain a first heuristic insight for the expected critical regularity classes, we may apply the following steps:

(1) Characterize the size $f(t)$ of waves which initially are of size $\epsilon > 0$. This also motivates the choice of good unknowns.
(2) Determine how resonances for a given high frequency perturbation depend on the frequency, both in terms of the size $R(\xi, k)$ of the resonance and the associated time $t_{\xi,k}$ and time interval.
(3) Chains of resonances yield a product formula for the possible norm inflation:

$$\prod_{k \in K} f(t_{\xi,k}) R(\xi, k),$$

which depends on the size of the wave, the frequency of the perturbation and the index set $K \subset \mathbb{Z}$. In particular, when considering bounded time intervals, K may not be chosen freely. Critical classes are determined by extremizers of this mechanism.

In order to outline this for an explicit example, let us consider the case of the Euler equations near Couette flow on the time interval $[0, \epsilon^{-2})$. In this special case, an explicit wave is given by

$$\omega(t, x, y) = -1 + \epsilon \cos(x - ty).$$

This wave's size is constant in time and hence

$$f(t) = \epsilon$$

for all $t > 0$.

Concerning the second step, as discussed in Section 2.2, the resonance mechanism for these equations is governed by the Biot-Savart law and results in a first heuristic estimate of the growth factor by

$$R(\xi, k) = \frac{\xi}{k^2},$$

for a resonance occurring at around the time $t_{\xi,k} = \frac{\xi}{k}$. We emphasize that in the full model this factor requires some further correction for very large values of ξ (see Section 2.2), which we here omit for simplicity of presentation.

Equation	size of waves	time scale	optimal Gevrey class
Vlasov-Poisson	ϵ	∞	3
Euler	ϵ	∞	2
		$t < \epsilon^{-2}$	3
Viscous Boussinesq	0, ϵ	∞	3
Inviscid Boussinesq	$\epsilon t^{+1/2}$, $\epsilon t^{-1/2}$	$t < \epsilon^{-2}$	2

TABLE 1. Growth and decay of waves and associated optimal Gevrey classes.

We thus obtain an estimate for the total possible norm inflation by

$$\prod_{k\in K} \epsilon \frac{\xi}{k^2},$$

with the set of resonances K to be chosen in the following. Here, we observe that in order to maximize the value of the product, the set K needs to satisfy two competing constraints:

- On the one hand, any resonance needs to yield a factor at least one, since otherwise omitting it increases the size of the product. Hence, any $k \in K$ should satisfy the *upper bound*
$$k \leq \sqrt{\epsilon|\xi|}.$$
- On the other hand, for a resonance to have occurred, the resonant time $t_{\xi,k} = \frac{\xi}{k}$ needs to have passed. Hence, when considering the time interval $[0, \epsilon^{-2})$ in our product we may only allow such $k \in K$, which satisfy the following *lower bound*:
$$|\frac{\xi}{k}| \leq \epsilon^{-2} \Leftrightarrow |\xi|\epsilon^2 \leq |k|.$$

Therefore, for any given frequency $\xi > 0$ we choose

$$K := \{k \in \mathbb{N} : |\xi|\epsilon^2 \leq k \leq \sqrt{\epsilon|\xi|}\}.$$

In particular, we note that this set is non-empty only if

$$|\xi|\epsilon^2 \leq \sqrt{\epsilon|\xi|} \Leftrightarrow |\xi| \leq \epsilon^{-3},$$

and in this case we obtain a bound of the norm inflation by

$$\prod_{k\in K} \epsilon \frac{\xi}{k^2} \lesssim \exp(\sqrt{\epsilon|\xi|}) \leq \exp(\sqrt{|\xi|^{2/3}}) = \exp(|\xi|^{1/3}).$$

If instead $|\xi| \gg \epsilon^{-4}$, then the set K is empty and one obtains a uniform bound on the possible norm inflation (see Section 2.4 for such a result in the case of the inviscid Boussinesq equations). Combining both estimates, we thus deduce that the heuristically expected critical regularity class (for estimates uniform in ϵ) for the Euler equations on the time interval $[0, \epsilon^{-2})$ is given by Gevrey 3.

The Table 1 summarizes the results of the preceding sections for various systems. We stress that in addition to identifying critical classes this wave model further allows us to identify the most important frequency regions. For instance, in the inviscid Boussinesq model it highlights the frequency region $\xi \approx \epsilon^{-4}$, which is not immediately visible from the nonlinear equations. Moreover, this model enables us to develop a precise understanding of the resonance mechanism and its frequency-dependent behavior. In this way we can construct solutions to the linearized problem around waves, which are exactly at the critical level of regularity and can exhibit precise convergence and divergence behavior as $t \to \infty$. These results hence provide an essential step towards understanding the full nonlinear problem at critical regularity and on long time scales.

CHAPTER 3

Ways to measure mixing and optimal decay rates

As a closely connected question to the stabilisation by (phase-)mixing, in recent years there has been intense interest in questions of quantifying mixing in *passive scalar problems*

$$\begin{aligned}\partial_t \rho + v \cdot \nabla \rho = 0,\\ (t,x) \in \mathbb{R} \times \Omega,\end{aligned}$$

where v is considered given and $\Omega \subset \mathbb{R}^d$ is an open set [**ACM19**, **ACM14**, **ACDL+08**, **EZ19**]. These serve as a natural first model of *active scalar problems*, where $v = v[\rho]$ may depend on the density, such as the Euler or Navier-Stokes equations in vorticity form. Such equations are also useful in settings where v might not be known explicitly (e.g. in turbulence). If v is sufficiently regular, we define the flow map X as the unique solution of the transport equation

$$\begin{aligned}\partial_t X(t,\alpha) = v(t, X(t,\alpha)),\\ X(0,\alpha) = \alpha.\end{aligned}$$

In particular, it follows that

$$\frac{d}{dt}\rho(t, X(t,\alpha)) = 0$$

and, hence, $\rho(t,x)$ can in principle be obtained from the initial data by inverting the map $\alpha \mapsto X(t,\alpha)$. For simplicity of presentation in this section we will assume that v is smooth and divergence-free and thus $X(t,\cdot)$ is volume-preserving and hence (locally) invertible.

Then $\rho(t,x) = \rho_0(X^{-1}(t,x))$ is only rearranged and thus for any (Borel) regular function f it holds that

$$\begin{aligned}\int f(\rho(t,x))dx &= \int f(\rho(t,X(t,\alpha))\det(DX)d\alpha\\ &= \int f(\rho(0,\alpha))d\alpha = \int f(0,x)dx.\end{aligned}$$

For instance, all L^p norms of ρ, its average or the entropy are conserved and moreover the evolution is time reversible (consider $-v(T-t,x)$ in place of $v(t,x)$)

Yet, as we have seen in Section 1.2 for the special case $v(x) = (x_2, 0)$, we observe that $\rho(t,x) \rightharpoonup \langle \rho \rangle_{x_1}$ converges to an average as $t \to \infty$ (or $t \to -\infty$). Thus, in general the L^p norm might strictly decrease in the infinite time limit. An aim of the study of "mixing" is to quantify this discrepancy between weak and strong convergence and to introduce a *scale* of how mixed ρ is at a time t. Given a scale it is then natural to ask for "best/worst case" choices of the vector field v transporting the density and how quickly

ρ might be mixed given (norm) constraints on v and how this answer depends on the definition of being "mixed".

As discussed in the work by Doering et al. [DWZZ18], in the literature there are multiple different notions of "mixing", which are popular in different communities. As a motivating example consider the case where ρ is a characteristic function and suppose that at time $t = 0$ and time $t = 1$ the level sets are given by a union of cubes as pictured in Figure 1. While any Lebesgue norm of both $\rho(0)$ and $\rho(1)$ is the same, a mixing scale

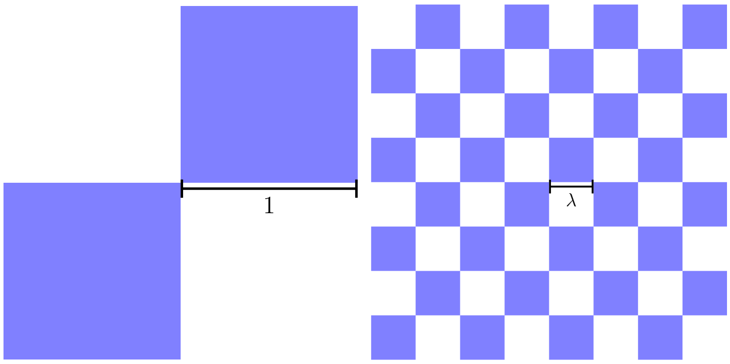

FIGURE 1. Characteristic function covering half the domain. Left: Squares of width 1. Right: Squares of width λ.

should be comparable to 1 and λ, respectively, and should behave as a length under rescaling.

Motivated by applications to active scalar problems, such as the Biot-Savart law in fluid dynamics or electrodynamics, *analytic mixing scales* are of the form

$$(\|\rho\|_{H^s}/\|\rho\|_{L^2})^{1/|s|},$$

with $s < 0$, where $s = -1$ and $s = -1/2$ are the most common choices. We remark that this definition extends to the case when ρ is not necessarily a characteristic function.

In the case that ρ is sufficiently regular, this scale is naturally related to $H^{|s|}$ by the interpolation inequality

$$\|\rho\|_{L^2} \leq \|\rho\|_{H^s}\|\rho\|_{H^{-s}},$$

but is well defined even if ρ is not sufficiently regular.

Another popular mixing scale is given by the *geometric mixing scales.* These scales are tailored towards working with characteristic functions and quantify that when averaging over large balls we mostly see the average (which is $1/2$ in the above example), while for small balls we see an approximation of ρ and thus values close to 0 and 1. After subtracting the average and introducing

$$\tilde{\rho} := 2(\rho - \int \rho),$$

we thus pick a *precision* $0 < \kappa < 1$ and say ρ is mixed at least at scale $r_0 > 0$ if

$$\frac{1}{|B_r|}\left|\int_{B_r} \tilde{\rho} dx\right| \leq \kappa$$

for all $r \geq r_0$ and all balls B_r in the domain. In analogy to the above description we can view this scale as associated to the inequality

$$\left\| (\frac{1}{|B_r|} 1_{B_r}) * \tilde{\rho} \right\|_{L^\infty} \leq \|\tilde{\rho}\|_{L^\infty}.$$

This notion of mixing is popular in geometric applications, since it works well for the description of sets (i.e. when ρ is a characteristic function). We remark that there are further popular notions in other communities, such as Wasserstein distances.

As one of the results of [**DWZZ18**] it is shown that these mixing scales are generally not equivalent by constructing two (counter)examples. However, in [**Zil19**] we argue that instead they are comparable when correcting for large scale structures and choosing the precision appropriately. More precisely, we argue that being geometrically mixed at level r_0 with precision κ only contains information on the structure of ρ at level $\max(r_0, \kappa)$. Thus κ and r_0 should be coupled.

Definition 3.1 (Mixing scales; c.f. [**Thi12, Zil19**]). *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a given measurable function. Then we call $\|\rho\|_{H^{-1}}$ the* analytic mixing scale*.*

Furthermore, for given $r > 0$, we define the geometric mixing functionals

$$\mathfrak{g}_r[\rho] := \sup_{B_R(\xi); R \geq r} \frac{1}{|B_R|} \left| \int_{B_R(\xi)} \rho \right|. \tag{16}$$

If further $\rho \in L^\infty$, then for each $\kappa \in (0,1)$ we define the geometric mixing scale *as*

$$\mathcal{G}_\kappa[\rho] := \inf\{r > 0 : \mathfrak{g}_r[\rho] \leq \kappa \|\rho\|_{L^\infty}\}. \tag{17}$$

As one of the main results of [**Zil19**], we show that while both notions are not equivalent, they are comparable if one takes into account low frequency components and connects the precision κ to the measured geometric mixing scale.

Theorem 3.2 (Comparison of mixing scales, [**Zil19**]). *Let $\rho \in L^2(\mathbb{R}^n)$ and $\|\rho\|_{L^2} \leq 1$. Then for all $0 < \epsilon \leq 1$ it holds that:*

(1) There exists a constant $C > 0$ depending only on the dimension n and $\alpha = \frac{2}{n+2}$ and $\beta = \frac{2}{n+4}$, such that if $\|\rho\|_{H^{-1}} \leq \epsilon$ and ρ is supported in B_1, then also $\mathfrak{g}_{\epsilon'}[\rho] \leq C\epsilon'$ for all $\epsilon' \geq \epsilon^\alpha$ and $\mathfrak{g}_{\tilde{\epsilon}}[\rho] \leq C$ for all $\tilde{\epsilon} \geq \epsilon^\beta$.

In particular, supposing additionally that $\|\rho\|_{L^\infty} = 1$, it follows that

$$\mathcal{G}_C[\rho] \leq \tilde{\epsilon},$$
$$\mathcal{G}_{C\epsilon'}[\rho] \leq \epsilon',$$

(2) If $\mathfrak{g}_\epsilon[\rho] \leq \epsilon$ and ρ is supported in a compact set K, then also $\|\rho\|_{H^{-1}} \leq C_K \epsilon$.

These estimates are optimal in the powers of ϵ.

As an application we derive bounds on mixing rates for the free transport equations.

Theorem 3.3 ([**Zil19**]). *In the following, let $0 < s \leq 1$, $\rho_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$ with $\fint_{\mathbb{T}^n} \rho_0(x,y) dx = 0$, and let*

$$\rho(t,x,y) = \rho_0(t, x - ty, y),$$

be the solution of the free transport problem. For $\sigma, s \in \mathbb{R}$ let $H^\sigma H^s = H^\sigma(\mathbb{T}^n; H^s(\mathbb{R}^n))$ denote the Hilbert space with norm

$$\|\rho\|_{H^\sigma H^s}^2 = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2\sigma} \int_{\mathbb{R}^n} \langle \xi\rangle^{2s} |\tilde{\rho}(k,\xi)|^2 d\xi.$$

(1) *There exists $C_s > 1$ such that for all $t \geq 1$ and all initial data*
$$\|\rho(t)\|_{L^2 H^{-1}} \leq C t^{-s} \|\rho_0\|_{H^{-s}H^s}.$$
(2) *Let $\alpha_j > 0$ with $\|(\alpha_j)_j\|_{l^2} = 1$. Then there exist $c > 0$, a sequence of times $t_j \to \infty$ and initial data ρ_0 such that*
$$\|\rho(t_j)\|_{L^2 H^{-1}} \geq c\alpha_j t_j^{-s} \|\rho_0\|_{H^{-s}H^s}.$$
(3) *There exists no non-trivial initial data $\rho_0 \in L^2(\mathbb{T}^n; H^s(\mathbb{R}^n))$ such that*
$$\|\rho(t_j)\|_{L^2 H^{-1}} \geq c t_j^{-s} \|\rho_0\|_{H^{-s}H^s}$$
along some sequence $t_j \to \infty$.

We, in particular, note that there is an improvement of the decay rate over the expected algebraic rate t^{-s}. Moreover, by the above results, we may connect these “analytic” estimates to “geometric” estimates.

CHAPTER 4

Enhanced dissipation and systems

Building on the mixing estimates for the inviscid problems, in two articles we investigated how mixing and dissipation interact and how this impacts the (asymptotic) stability of solutions. As a first, explicitly solvable model problem we may consider the Navier-Stokes equations on $\mathbb{T} \times \mathbb{R}$:

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= \nu \Delta \omega, \\ \operatorname{div}(v) &= 0, \end{aligned}$$

linearized around the Couette flow $v = (y, 0), \omega = -1$. These equations read

$$\partial_t \omega + y \partial_x \omega = \nu \Delta \omega.$$

Changing to coordinates moving with the shear, we obtain a constant coefficient, time-dependent differential operator

$$\partial_t \omega = \nu (\partial_x^2 + (\partial_y - t\partial_x)^2) \omega,$$

which hence decouples after a Fourier transform. We remark that this decoupling structure is lost upon considering any other shear flow to linearize around. We then obtain that

$$\begin{aligned} \tilde{\omega}(t, k, \xi) &= \exp(-\nu \int_0^t k^2 + (\xi - ks)^2 ds) \omega_0(k, \xi), \\ \tilde{v}(t, k, \xi) &= \frac{(-i(\xi - kt), ik)}{k^2 + (\xi - kt)^2} \exp(-\nu \int_0^t k^2 + (\xi - ks)^2 ds) \omega_0(k, \xi). \end{aligned}$$

Computing the integral, we observe that ω not only decays with a rate $\exp(-\nu k^2 t)$ and thus on time scales ν^{-1} as for heat flow, but rather with a rate $\exp(-\nu k^2 t^3/3)$ and thus already on a much shorter time scale $\nu^{-1/3}$. One thus speaks of *mixing enhanced dissipation*, which combines the following effects:

- Mixing pushes L^2 mass to higher frequencies.
- At higher frequencies dissipation is stronger.
- Both mechanisms thus constructively interact and result in enhanced damping rates $\nu^{1/3} \gg \nu^1$ (for small values of $\nu > 0$).

While results for the linearized problem around Couette flow can be obtained by explicit computation, obtaining results for other shear flows, other systems or the non-linear equations requires more robust methods of proof. More specifically, in the following sections we discuss these mechanisms for the stratified Navier-Stokes equations and the partially viscous Boussinesq equations.

4.1. Stratified viscosity and local mixing rates

In a joint work with Xian Liao [LZ21], we investigated the case of a shear flow and stratified viscosity in the Navier-Stokes equations. In particular, we asked how mixing enhanced dissipation changes if we allow the viscosity $\nu > 0$ to vary by many orders of magnitude. These investigations were motivated by physical settings with density-dependent viscosity, which have numerous industrial applications. In order to simplify the analysis, instead of the full compressible equations, we considered a stratified model, where $\rho(y)$ and $\mu(y)$ depend on the vertical direction only:

$$\begin{aligned}\partial_t v + v \cdot \nabla v &= \operatorname{div}(\mu(y) S v) - \nabla p, \\ \operatorname{div}(v) &= 0, \\ (t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}.\end{aligned}$$

Here, $Sv = \frac{1}{2}(\nabla v + (\nabla v)^T)$ denotes the symmetrized gradient. In this setting instead of Couette flow the distinguished shear flow solutions $v = (U(y), 0)$ are determined by the equation

$$\mu(y)\partial_y U(y) = \text{const}. \tag{18}$$

Hence, if in a neighborhood of $y = 0$ this solution satisfies $\partial_y U(y) \approx 1$, i.e. the flow is close to Couette flow, then if in another region we increase the viscosity by a factor 100, the slope of the shear will actually *decrease* by a factor $\frac{1}{100}$. Furthermore, considering the fact that by the calculation for the constant viscosity case we expect a "local decay rate" of

$$\sqrt[3]{\mu(y)(\partial_y U(y))^2},$$

this rate turns out to be proportional to $\mu^{-1/3}$. Therefore, an *increase* of the viscosity leads to a *decrease* of the mixing rate and vice versa.

THEOREM 4.1 ([LZ21]). *Let $\mu(y), U(y)$ be a given solution to (18) with $\mu(y) > 0$ and suppose $\ln(\mu)$ is slowly varying in the sense that*

$$\|\partial_y \ln(\mu)\|_{L^\infty} + \|\frac{1}{U'}\partial_y\partial_y \ln(\mu)\|_{L^\infty} \ll 1.$$

Then the linearized Navier-Stokes equations around this solution are stable in weighted Sobolev spaces. Moreover, they exhibit enhanced dissipation with a rate proportional to $(\mu(U')^2)^{1/3}$. More precisely, there exists time-dependent family of integro-differential operators $A(t)$, such that

$$\|A(t)\omega(t)\|_{L^2} \approx \|\omega(t)\|_{L^2}$$

with constants uniform in time and

$$\partial_t \|A(t)\omega(t)\|_{L^2}^2 \lesssim -\iint (\mu(U')^2)^{1/3}|\omega(t)|^2 + \mu|\partial_x \omega(t)|^2 dxdy$$

Let us briefly comment on these results.

- For simplicity of notation, these results are stated in L^2 spaces. They also extend to weighted, higher Sobolev spaces if μ is sufficiently regular.

- While $\log(\mu(y))$ is slowly varying, this *does not* imply that $U(y)$ is close to Couette or that $\partial_y^2 U(y)$ is small. Indeed, a prototypical example is given by an exponential profile $U(y) = \exp(cy)$ with a small constant $c \in \mathbb{R}$. We hence introduce *strongly weighted spaces* and require many localized changes of variables and adapted Fourier methods to deal with these highly non-affine profiles. The development of tools capable of dealing with this highly variable setting also seems of independent interest to the community.
- Since $\mu(y)$ may change by many orders of magnitude, many estimates need to be suitably localized. However, this is not easily achievable since the Biot-Savart law $\omega \mapsto v$ is non-local. This competition makes the analysis technically challenging and natural leads to requirements on $\log(\mu(y))$ to be able slowly varying, so that it is possible to *glue* together estimates.

4.2. Rayleigh-Bénard and enhanced dissipation

In two other works [**Zil21c, Zil21d**], we investigated enhanced dissipation phenomena for the linearized Boussinesq equations with partial dissipation. As discussed in Section 2.3, the Boussinesq equations are a common model for the evolution of a heat conducting fluid:

$$\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \nu \Delta v + \theta e_2, \\
\partial_t \theta + v \cdot \nabla \theta &= \kappa \Delta \theta, \\
\operatorname{div}(v) &= 0, \\
(t, x, y) &\in (0, \infty) \times \mathbb{T} \times \mathbb{R}.
\end{aligned}$$

Here the velocity v of the fluid and the pressure p satisfy a forced Navier-Stokes equation and the temperature density θ is advected by v. The term θe_2 causes hotter fluid to rise above colder fluid and models buoyancy effects. The coefficients $\nu \geq 0$ and $\kappa \geq 0$ take into account viscous dissipation and resistivity and, in many examples of interest, may be anisotropic and of highly different orders of magnitude (e.g. close to purely vertical dissipation or only viscous dissipation). In view of the dissipation, a family of distinguished solutions is given by the Couette flow

$$v = (y, 0),$$

and thermal stratification

$$\theta = \beta y.$$

The behavior of both solutions by themselves has been well-studied and includes the following physical effects:

- The Navier-Stokes equations (i.e. setting $\theta \equiv 1$) include mixing enhanced dissipation and corresponding asymptotic stability in Sobolev regularity (for small data) [**BVW18**].
- In the inviscid limit $\nu \downarrow 0$ we observe destabilization by mixing and resonances, but assuming sufficient (Gevrey) regularity never the less observe inviscid damping of the velocity field [**BM15b, DZ21**].
- Considering thermal stratification of a fluid initially at rest ($v = 0$), it is experimentally easily observed that a configuration with hotter fluid below colder

fluid is unstable. For instance, heating the bottom of a container, "plumes" of hotter liquid rise from the bottom and after sufficient time the flow becomes highly chaotic. The instability is known as *Rayleigh-Bénard convection* [DR04].

- In contrast, if $\beta > 0$, that is hotter fluid is above colder fluid, there is no such instability. Moreover, if $\beta > \frac{1}{4}$ [How61] (the so-called Miles-Howard criterion) certain algebraic instabilities are suppressed and we speak of stable stratification.

Both effects individually (shear at constant temperature or thermal stratification at rest) are well studied. However, their interaction had been scarcely investigated and we hence asked whether mixing enhanced dissipation can counteract Rayleigh-Bénard instability in the setting without thermal dissipation, $\kappa = 0$. Our main results are summarized in the following theorem.

THEOREM 4.2 ([Zil21c]). *Consider the two-dimensional Boussinesq equations on* $\mathbb{T} \times \mathbb{R}$ *with viscosity* $\nu > 0$ *and without thermal resistivity,* $\kappa = 0$. *Then any pair* $v = (y, 0), \theta = T(y)$ *is a stationary solution. Moreover, under suitable assumptions, which reduce to the requirement* $\beta \gtrsim -\nu^{-1/3}$ *in the case of affine stratification, the linearized Boussinesq equations are asymptotically stable and satisfy*

$$\nu^{1/3}\|\omega(t)\|_{H^N}^2 + \|\partial_x \theta\|_{H^N}^2 \leq C(1 + \nu^{-2/3})^2(\nu^{1/3}\|\omega_0\|_{H^N}^2 + \|\partial_x \theta_0\|_{H^N}^2 + \nu^{-2/3}\|\theta_0\|_{H^N}^2).$$

Let us briefly comment on these results.

- For simplicity of notation this result is stated in terms of Sobolev spaces. A more precise statement can be given using Fourier multipliers with cut-offs depending negative powers of ν.
- We stress that here β can be allowed to be *negative*. Thus, mixing enhanced dissipation can counteract Rayleigh-Bénard instability.
- In [Zil21d] we also establish stability for the full nonlinear Boussinesq equations with *full dissipation*, i.e. with $\kappa = \nu > 0$, in the regime of small data.
- As a related result, in [MSHZ22] the nonlinear problem with $\nu > 0 = \kappa$ is shown to be stable in Gevrey 3 regularity for small data. As we discuss in Section 2.3 the resonances of the Euler equations (associated with Gevrey 2 regularity) here are suppressed by the viscosity. However, we suggest that instead a new resonance mechanism, which call *thermal echoes*, appears and hence stability fails in Sobolev regularity (see Section 2.3). This mechanism is shown to crucially rely on the systems structure of the Boussinesq equations.

CHAPTER 5

Outlook

As shown by the results of this thesis, *mixing* is a central effect in the analysis of stability and instability of fluid flows and strongly depends on the system under consideration (see Section 2.5). In particular, new, challenging effects appear when studying data at critical regularity, such as infinite time blow-up or a strong discrepancy between different notions of stability. Moreover, we stress that traveling waves allow us to identify essential effects of the system structure of equations, which enabled us to study the inviscid Boussinesq equations at critical regularity and to establish qualitatively improved stability estimates for the associated linear problem [**Zil22**]. Building on these insights, we expect that also for the full nonlinear inviscid Boussinesq equations improved stability estimates can be established. Furthermore, the constructed waves are solutions of the nonlinear equations for all times and heuristic calculations suggest that a control of perturbations might be possible even past a time scale ϵ^{-2} (see Section 2.4).

More generally, we stress that while mathematically Gevrey classes appear as critical spaces in the study of stability, in physical observations the associated damping results seem to be much more robust. Here the results of [**DZ21**] and [**Zil21a**] provide an important first explanation, by constructing data for which stability in a “scattering” sense fails, but persists in a “physical” sense (for the linearized problem around waves; see Sections 2.1 to 2.5 for a discussion and other systems). A natural question thus concerns the construction of nonlinear solutions in critical spaces, which exhibit such instability and the challenging widely open problem of characterizing the behavior of solutions at lower regularity. Here, we stress that existing nonlinear echo chain constructions [**DM23, Bed20**] are not sufficient: A single echo chain concludes after a finite time and the evolution is subsequently stable. As a long term goal, we thus intend to construct nonlinear solutions exhibiting *multiple* chains, which will serve as an important step towards addressing non-trivial dynamics at large times.

Taking a higher level view of “mixing” in partial differential equations, in [**Zil19**], we have shown an unexpected connection between different notions of mixing used in different communities. As an application we highlighted connections between independently developed results on “mixing costs” and established lower bounds on achievable mixing rates. Furthermore, we introduced a discrete, dyadic model as a natural setting to study notions of mixing. In future work, we intend to explore analogues of Bressan’s conjecture on mixing costs [**Bre03**] in this model setting and to connect tools from “geometric” and “analytic” mixing communities.

Bibliography

[ABSPW22] Dhanapati Adhikari, Oussama Ben Said, Uddhaba Raj Pandey, and Jiahong Wu. Stability and large-time behavior for the 2d Boussineq system with horizontal dissipation and vertical thermal diffusion. *Nonlinear Differential Equations and Applications NoDEA*, 29(4):1–43, 2022.

[ACDL+08] Luigi Ambrosio, Gianluca Crippa, Camillo De Lellis, Felix Otto, Michael Westdickenberg, Luigi Ambrosio, and Gianluca Crippa. Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields. *Transport equations and multi-D hyperbolic conservation laws*, pages 3–57, 2008.

[ACM14] Giovanni Alberti, Gianluca Crippa, and Anna L Mazzucato. Exponential self-similar mixing and loss of regularity for continuity equations. *Comptes rendus mathematique*, 352(11):901–906, 2014.

[ACM19] Giovanni Alberti, Gianluca Crippa, and Anna Mazzucato. Exponential self-similar mixing by incompressible flows. *Journal of the American Mathematical Society*, 32(2):445–490, 2019.

[ACW10] Dhanapati Adhikari, Chongsheng Cao, and Jiahong Wu. The 2d Boussinesq equations with vertical viscosity and vertical diffusivity. *Journal of Differential Equations*, 249(5):1078–1088, 2010.

[AK98] Vladmir Igorevich Arnold and Boris Khesin. *Topological methods in hydrodynamics*, volume 125. Springer Verlag, 1998.

[Arn66] Vladimir Igorevich Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16(1):319–361, 1966.

[Bac60] George Backus. Linearized plasma oscillations in arbitrary electron velocity distributions. *Journal of Mathematical Physics*, 1(3):178–191, 1960.

[BBCZD21] Jacob Bedrossian, Roberta Bianchini, Michele Coti Zelati, and Michele Dolce. Nonlinear inviscid damping and shear-buoyancy instability in the two-dimensional Boussinesq equations. *arXiv preprint arXiv:2103.13713*, 2021.

[BČCK] Tristan Buckmaster, Sunčica Čanić, Peter Constantin, and Alexander A Kiselev. Progress in mathematical fluid dynamics. *Lecture Notes in Mathematics. Fondazione CIME/CIME Foundation Subseries*, 2272.

[BDLSJ16] Tristan Buckmaster, Camillo De Lellis, and László Székelyhidi Jr. Dissipative Euler flows with Onsager-critical spatial regularity. *Communications on Pure and Applied Mathematics*, 69(9):1613–1670, 2016.

[BDLSJV19] Tristan Buckmaster, Camillo De Lellis, László Székelyhidi Jr, and Vlad Vicol. Onsager's conjecture for admissible weak solutions. *Communications on pure and applied mathematics*, 72(2), 2019.

[Bed20] Jacob Bedrossian. Nonlinear echoes and Landau damping with insufficient regularity. *Tunisian Journal of Mathematics*, 3:121–205, 2020.

[BLW20] Nicki Boardman, Hongxia Lin, and Jiahong Wu. Stabilization of a background magnetic field on a 2 dimensional magnetohydrodynamic flow. *SIAM Journal on Mathematical Analysis*, 52(5):5001–5035, 2020.

[BM15a] Jacob Bedrossian and Nader Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.*, 122:195–300, 2015.

[BM15b] Jacob Bedrossian and Nader Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2d Euler equations. *Publications mathématiques de l'IHÉS*, 122(1):195–300, 2015.

[BMM16] Jacob Bedrossian, Nader Masmoudi, and Clément Mouhot. Landau damping: paraproducts and Gevrey regularity. *Annals of PDE*, 2(1):4, 2016.

[Bre03] Alberto Bressan. A lemma and a conjecture on the cost of rearrangements. *Rendiconti del Seminario Matematico della Universita di Padova*, 110:97–102, 2003.

[BVW18] Jacob Bedrossian, Vlad Vicol, and Fei Wang. The Sobolev stability threshold for 2D shear flows near Couette. *Journal of Nonlinear Science*, 28(6):2051–2075, 2018.

[BZD20] Roberta Bianchini, Michele Coti Zelati, and Michele Dolce. Linear inviscid damping for shear flows near Couette in the 2d stably stratified regime. *arXiv preprint arXiv:2005.09058*, 2020.

[CW13] Chongsheng Cao and Jiahong Wu. Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation. *Archive for Rational Mechanics and Analysis*, 208(3):985–1004, 2013.

[CZEW23] Michele Coti Zelati, Tarek M Elgindi, and Klaus Widmayer. Stationary structures near the kolmogorov and poiseuille flows in the 2d euler equations. *Archive for Rational Mechanics and Analysis*, 247(1):12, 2023.

[DM23] Yu Deng and Nader Masmoudi. Long time instability of the Couette flow in low Gevrey spaces. *Communications on Pure and Applied Mathematics*, page 0001–0084, 2023.

[DR04] Philip G. Drazin and William Hill Reid. *Hydrodynamic stability.* Cambridge Univ Pr, 2004.

[Dra02] Philip G. Drazin. *Introduction to Hydrodynamic Stability.* Cambridge University Press, 2002.

[DWZZ18] Charles R Doering, Jiahong Wu, Kun Zhao, and Xiaoming Zheng. Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. *Physica D: Nonlinear Phenomena*, 376:144–159, 2018.

[DZ21] Yu Deng and Christian Zillinger. Echo chains as a linear mechanism: Norm inflation, modified exponents and asymptotics. *Archive for rational mechanics and analysis*, 242(1):643–700, 2021.

[EZ19] Tarek M Elgindi and Andrej Zlatoš. Universal mixers in all dimensions. *Advances in Mathematics*, 356:106807, 2019.

[GNR21] Emmanuel Grenier, Toan T Nguyen, and Igor Rodnianski. Landau damping for analytic and Gevrey data. *Math. Res. Lett*, 28(6):1679–1702, 2021.

[Gro73] Mikhail Leonidovich Gromov. Convex integration of differential relations. i. *Izvestiya: Mathematics*, 7(2):329–343, 1973.

[How61] Louis N Howard. Note on a paper of John W. Miles. *Journal of Fluid Mechanics*, 10(4):509–512, 1961.

[HXY18] Ling-Bing He, Li Xu, and Pin Yu. On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves. *Annals of PDE*, 4(1):1–105, 2018.

[IJ20] Alexandru D Ionescu and Hao Jia. Nonlinear inviscid damping near monotonic shear flows. *arXiv preprint arXiv:2001.03087*, 2020. to appear in Acta Mathematica.

[Ise17] Philip Isett. On the endpoint regularity in Onsager's conjecture. *arXiv preprint arXiv:1706.01549*, 2017.

[JŠ14] Hao Jia and Vladimír Šverák. Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions. *Inventiones mathematicae*, 196:233–265, 2014.

[KŠ14] Alexander Kiselev and Vladimir Šverák. Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Annals of mathematics*, 180(3):1205–1220, 2014.

[KW09] Boris Khesin and Robert Wendt. *The geometry of infinite-dimensional groups*, volume 51. Springer Verlag, 2009.

[Lan46] Lev Davidovich Landau. On the vibration of the electronic plasma. *J. Phys. USSR 10*, 1946.

[LWX+21] Suhua Lai, Jiahong Wu, Xiaojing Xu, Jianwen Zhang, and Yueyuan Zhong. Optimal decay estimates for 2d Boussinesq equations with partial dissipation. *Journal of Nonlinear Science*, 31(1):1–33, 2021.

[LZ11] Zhiwu Lin and Chongchun Zeng. Inviscid dynamical structures near Couette Flow. *Archive for rational mechanics and analysis*, 200(3):1075–1097, 2011.

[LZ21] Xian Liao and Christian Zillinger. On variable viscosity and enhanced dissipation. *arXiv preprint arXiv:2110.10976*, 2021.

[MŠ03] Stefan Müller and Vladimir Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. *Annals of mathematics*, 157(3):715–742, 2003.

[MSHZ22] Nader Masmoudi, Belkacem Said-Houari, and Weiren Zhao. Stability of Couette flow for 2d Boussinesq system without thermal diffusivity. 245(2):645–752, 2022.

[MV10] Clément Mouhot and Cédric Villani. Landau damping. *Notes de cours, CEMRACS*, 2010.

[MV11] Clément Mouhot and Cédric Villani. On Landau damping. *Acta mathematica*, 207(1):29–201, 2011.

[MWGO68] J. H. Malmberg, C. B. Wharton, R. W. Gould, and T. M. O'Neil. Plasma wave echo experiment. *Physical Review Letters*, 20(3):95–97, 1968.

[NRŠ96] Jindřich Nečas, Michael Růžička, and Vladimir Šverák. On Leray's self-similar solutions of the Navier-Stokes equations. 1996.

[Ray79] Lord Rayleigh. On the stability, or instability, of certain fluid motions. *Proceedings of the London Mathematical Society*, 1(1):57, 1879.

[SH00] Peter J. Schmid and Dan S. Henningson. *Stability and Transition in Shear Flows*. Springer, 2000.

[Spr10] David Spring. *Convex integration theory: solutions to the h-principle in geometry and topology*. Springer Science & Business Media, 2010.

[Thi12] Jean-Luc Thiffeault. Using multiscale norms to quantify mixing and transport. *Nonlinearity*, 25(2):R1, 2012.

[TW19] Lizheng Tao and Jiahong Wu. The 2d Boussinesq equations with vertical dissipation and linear stability of shear flows. *Journal of Differential Equations*, 267(3):1731–1747, 2019.

[TWZZ20] Lizheng Tao, Jiahong Wu, Kun Zhao, and Xiaoming Zheng. Stability near hydrostatic equilibrium to the 2d Boussinesq equations without thermal diffusion. *Archive for Rational Mechanics and Analysis*, 237(2):585–630, 2020.

[Yag12] Akiva M Yaglom. *Hydrodynamic instability and transition to turbulence*, volume 100. Springer Science & Business Media, 2012.

[YOD05] J. H. Yu, T. M. O'Neil, and C. F. Driscoll. Fluid echoes in a pure electron plasma. *Physical review letters*, 94(2):025005, 2005.

[Zil19] Christian Zillinger. On geometric and analytic mixing scales: comparability and convergence rates for transport problems. *Pure and Applied Analysis*, 1(4):543–570, 2019.

[Zil21a] Christian Zillinger. On echo chains in Landau damping: Traveling wave-like solutions and Gevrey 3 as a linear stability threshold. *Annals of PDE*, 7(1):1–29, 2021.

[Zil21b] Christian Zillinger. On echo chains in the linearized Boussinesq equations around traveling waves. *arXiv preprint arXiv:2103.15441*, 2021. to appear in SIAM Journal on Mathematical Analysis.

[Zil21c] Christian Zillinger. On enhanced dissipation for the Boussinesq equations. *Journal of Differential Equations*, 282:407–445, 2021.

[Zil21d] Christian Zillinger. On the Boussinesq equations with non-monotone temperature profiles. *Journal of Nonlinear Science*, 31(4):64, 2021.

[Zil22] Christian Zillinger. On stability estimates for the inviscid Boussinesq equations. *arXiv preprint arXiv:2209.01950*, 2022.

[ZZ23] Cuili Zhai and Weiren Zhao. Stability threshold of the Couette flow for Navier-Stokes Boussinesq system with large Richardson number $\gamma > 1/4$. *SIAM Journal on Mathematical Analysis*, 55(2):1284–1318, 2023.

CHAPTER 6

Articles

[Copies of the articles are included in the printed version. See Section 1 for links to the journal versions.]