

Poset Ramsey number $R(P, Q_n)$. III. Chain compositions and antichains

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ABSTRACT

An induced subposet (P_2, \leq_2) of a poset (P_1, \leq_1) is a subset of P_1 such that for every two $X, Y \in P_2$, $X \leq_2 Y$ if and only if $X \leq_1 Y$. The Boolean lattice Q_n of dimension n is the poset consisting of all subsets of $\{1, \dots, n\}$ ordered by inclusion.

Given two posets P_1 and P_2 the poset Ramsey number $R(P_1, P_2)$ is the smallest integer N such that in any blue/red coloring of the elements of Q_N there is either a monochromatically blue induced subposet isomorphic to P_1 or a monochromatically red induced subposet isomorphic to P_2 .

We provide upper bounds on $R(P, Q_n)$ for two classes of P : parallel compositions of chains, i.e. posets consisting of disjoint chains which are pairwise element-wise incomparable, as well as subdivided Q_2 , which are posets obtained from two parallel chains by adding a common minimal and a common maximal element. This completes the determination of $R(P, Q_n)$ for posets P with at most 4 elements. If P is an antichain A_t on t elements, we show that $R(A_t, Q_n) = n + 3$ for $3 \leq t \leq \log \log n$. Additionally, we briefly survey proof techniques in the poset Ramsey setting P versus Q_n .

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1. Introduction

1.1. Basic setting and background

A *partially ordered set*, or *poset* for short, is a pair (P, \leq_P) of a set P and a partial order \leq_P on this set, i.e. a binary relation that is transitive, reflexive and anti-symmetric. Usually we refer to a poset (P, \leq_P) just as P . The elements of P are often called *vertices*. If two vertices A and B are *incomparable*, i.e. if $A \not\leq B$ and $A \not\geq B$, we write $A \parallel B$. A poset (P_2, \leq_{P_2}) is an (*induced*) *subposet* of a poset (P_1, \leq_{P_1}) if $P_2 \subseteq P_1$ and for every two $X, Y \in P_2$, $X \leq_{P_2} Y$ if and only if $X \leq_{P_1} Y$. If such a P_2 is isomorphic to some poset P' , we say that P_2 is a *copy* of P' in P_1 . Equivalently, P_2 is a copy of P' in P_1 if P_2 is the image of an *embedding* $\phi: P' \rightarrow P_1$, i.e. an injective function such that for every two $X, Y \in P'$, $X \leq_{P'} Y$ if and only if $\phi(X) \leq_{P_1} \phi(Y)$.

The *Boolean lattice* Q_n is the poset whose vertices are the subsets of an n -element *ground set* ordered by inclusion. In this paper we consider colorings of the vertices of posets. A *blue/red coloring* of a poset P , is a mapping $c: P \rightarrow \{\text{blue}, \text{red}\}$. We say that a poset is *monochromatic* if all its vertices have the same color. If all vertices are blue, we say that the poset is *blue*. Similarly, if all vertices are red, the poset is *red*.

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Ramsey-type problems are widely studied for graphs and hypergraphs and were extended to posets by Nešetřil and Rödl [19] in a general form. A special case of their studies asks to find for a fixed poset P , a hosting poset W such that every blue/red-coloring of the elements of W contains a monochromatic copy of P . Kierstead and Trotter [16] considered this setting with the goal of minimizing $p(W)$ for all posets P with fixed $p(P)$, where p is a poset parameter such as size, height or width. Axenovich and Walzer [1] introduced a closely related Ramsey setting which recently attracted the attention of various researchers. For fixed posets P_1 and P_2 the *poset Ramsey number* of P_1 versus P_2 is

$$R(P_1, P_2) = \min\{N \in \mathbb{N} : \text{every blue/red coloring of } Q_N \text{ contains either a blue copy of } P_1 \text{ or a red copy of } P_2\}.$$

For the diagonal setting $P_1 = P_2 = Q_n$, the bounds on $R(Q_n, Q_n)$, $n \geq 3$, were gradually improved to $2n + 1 \leq R(Q_n, Q_n) \leq n^2 - n + 2$, see chronologically Axenovich and Walzer [1], Cox and Stolee [8], Lu and Thompson [17], and Bohman and Peng [4]. The asymptotic behavior in terms of n remains an open problem. Note that for any P_1 and P_2 the poset Ramsey number is well-defined: It is easy to see that every two posets P_1 and P_2 are induced subposets of Q_n for large n , thus an upper bound on $R(Q_n, Q_n)$ implies the existence of $R(P_1, P_2)$. For further results on diagonal poset Ramsey numbers $R(P, P)$ see e.g. Chen, Chen, Cheng, Li, and Liu [6] and Walzer [22].

Another actively investigated setting of poset Ramsey numbers considers $R(Q_m, Q_n)$ for m fixed and n large. It is trivial to see that $R(Q_1, Q_n) = n + 1$. In the case $m = 2$ it was shown that $R(Q_2, Q_n) = n + \Theta(\frac{n}{\log(n)})$ with upper bound due to Grósz, Methuku, and Tompkins [13] and lower bound due to Axenovich and the author [2]. For $m \geq 3$ only rough estimates are known, see Lu and Thompson [17]. This open field of research as well as Erdős-Hajnal-type questions on posets motivated a detailed study of the off-diagonal poset Ramsey number $R(P, Q_n)$ for fixed P and large n which is presented in a series of papers [23], [3] including the present paper.

Other extremal problems on posets include rainbow Ramsey problems, see Chang, Gerbner, Li, Methuku, Nagy, Patkós, and Vizer [5]; Turan-type, most notable see Methuku and Pálvölgyi [18]; and saturation-type questions, which are discussed in a recent survey by Keszezh, Lemons, Martin, Pálvölgyi, and Patkós [15].

1.2. Summary of results

In order to understand the general framework of bounds on $R(P, Q_n)$ we consider two special posets. The V -shaped poset V_2 has three distinct vertices X, Y , and Z with $Z \leq X, Z \leq Y$, and $X \parallel Y$. Its symmetric counterpart is the poset Λ_2 which has three distinct vertices X, Y , and Z where $Z \geq X, Z \geq Y$, and $X \parallel Y$. We say that a poset P is *non-trivial* if P contains a copy of either V_2 or Λ_2 , otherwise we say that P is *trivial*. It was shown by Axenovich and the author [2] that two different asymptotic behaviors of $R(P, Q_n)$ emerge depending on whether the fixed P is trivial or not. Combined with a general lower bound of Walzer [22] and a general upper bound by Axenovich and Walzer [1] the following is known:

Theorem 1 ([1][2][22]). *Let P be a trivial poset. Then for every n ,*

$$n + h(P) - 1 \leq R(P, Q_n) \leq n + h(P) + \alpha(w(P)) + 1.$$

Let P be a non-trivial poset. Then for sufficiently large n ,

$$n + \frac{1}{15} \frac{n}{\log n} \leq R(P, Q_n) \leq h(P)n + \dim_2(P).$$




















Throughout this paper ‘log’ refers to the logarithm base 2. The height $h(P)$ is the size of the largest set of pairwise comparable vertices in P , while the width $w(P)$ denotes the size of the largest set of pairwise incomparable vertices in P . The 2 -dimension $\dim_2(P)$ is the smallest dimension N of a Boolean lattice Q_N which contains a copy of P . It is a basic observation that this poset parameter is well-defined. The *Sperner number* $\alpha(t)$ is the minimal integer N such that $\binom{N}{\lfloor N/2 \rfloor} \geq t$. It was determined almost exactly in an explicit form by Habib, Nourine, Raynaud and Thierry [14] who showed that

$$\alpha(t) \in \left\{ \left\lfloor \log t + \frac{\log \log t}{2} \right\rfloor + 1, \left\lfloor \log t + \frac{\log \log t}{2} \right\rfloor + 2 \right\}. \tag{*}$$

We remark that posets P of height $h(P) = 1$, i.e. *antichains*, are trivial, however for such P the general upper bound $R(P, Q_n) \leq h(P)n + \dim_2(P)$ by Axenovich and Walzer [1] is stronger than the bound $R(P, Q_n) \leq n + h(P) + \alpha(w(P)) + 1$ given in Theorem 1.

Another important contribution of Walzer [22] in the analysis of $R(P, Q_n)$ is providing a bound for *parallel compositions* P . Given a poset P , two subposets $P_1, P_2 \subseteq P$ are *parallel* if they are element-wise incomparable. We denote by $P_1 + P_2$ the *parallel composition* of two posets P_1 and P_2 , that is the poset consisting of a copy of P_1 and a copy of P_2 which are disjoint and parallel. In the literature the parallel composition is also referred to as *independent union*. Note that this operation is commutative and associative.

Table 1
Off-diagonal poset Ramsey bounds for small P and reference to the proofs of lower bound (LB) and upper bound (UB).

poset P	$R(P, Q_n)$	proof
	$C_1 = Q_0$ $n + 0$	trivial
	$C_2 = Q_1$ $n + 1$	Theorem 8
	$A_2 = 2C_1$ $n + 2$	Theorem 8
	C_3 $n + 2$	Theorem 8
	$C_{2,1}$ $n + 3$	Theorem 8
	$A_3 = 3C_1$ $n + 3$	Theorem 5
	$V_2 = K_{1,2}$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 1 + o(1)$	[2]
	C_4 $n + 3$	Theorem 8
	$C_{2,2} = 2C_2$ $n + 3$	Theorem 8
	$A_4 = 4C_1$ $n + 3$	Theorem 5
	$C_{3,1}$ $n + 4$	Theorem 8
	$C_{2,1,1}$ $n + 4$	Theorem 9
	$V_2 + C_1$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 1 + o(1)$	LB: Theorem 1 ([2]), UB: [2], Theorem 2 ([22])
	$K_{1,3} = V_3$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 1 + o(1)$	LB: Theorem 1 ([2]), UB: [23]
	$\boxtimes = \mathcal{N}$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 1 + o(1)$	LB: Theorem 1 ([2]), UB: [3]
	$Q_2 = K_{1,2,1}$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 2 + o(1)$	LB: Theorem 1 ([2]), UB: [13]
	$K_{1,1,2} = Y$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 2 + o(1)$	LB: Theorem 1 ([2]), UB: [23]
	Q_2^- $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 2 + o(1)$	LB: Theorem 1 ([2]), UB: Corollary 4
	$K_{2,2} = \boxtimes$ $n + \frac{c(n)n}{\log(n)}, \frac{1}{15} \leq c(n) \leq 4 + o(1)$	LB: Theorem 1 ([2]), UB: [23]

Theorem 2 ([22]). Let $\ell \geq 2$ and let P_1, P_2, \dots, P_ℓ , and Q be arbitrary posets. Let $P = P_1 + P_2 + \dots + P_\ell$ be the parallel composition of P_1, \dots, P_ℓ . Then

$$R(P, Q) \leq \max_{j \in [\ell]} \{R(P_j, Q)\} + \alpha(\ell) \leq \max_{j \in [\ell]} \{R(P_j, Q)\} + \log(\ell) + \frac{1}{2} \log \log(\ell) + 2.$$

Collecting known results and providing several new bounds in this paper, we obtain bounds on $R(P, Q_n)$ which are asymptotically tight in the two leading additive terms for all posets P on at most 4 vertices (of which there are 19 up to symmetry). Moreover, we exactly determine $R(P, Q_n)$ for all trivial P on at most 4 vertices. An overview of the bounds is given in Table 1. In every row of the table a poset P is defined by its Hasse diagram and labeled using the notation of this paper. Formal definitions of all posets are stated in Sections 1.3 and 2.1. Some posets have alternative names used in the literature, these are additionally mentioned in the table.

1.3. New results

A chain C_t of length t is a poset on t vertices forming a linear order. For $s, t \in \mathbb{N}$, let $SD_{s,t}$ denote the (s, t) -subdivided diamond, the poset obtained from two disjoint and element-wise incomparable chains of length s and t , respectively, by

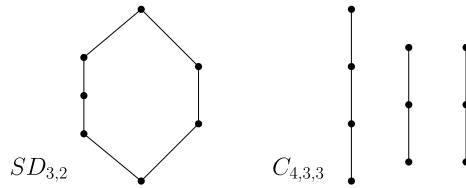


Fig. 1. Hasse diagrams of $SD_{3,2}$ and $C_{4,3,3}$.

adding a common minimal vertex and a common maximal vertex, i.e. a vertex which is smaller than all others and a vertex which is larger than all others, see Fig. 1. Note that $SD_{1,1} = Q_2$. Our first result shows that for subdivided diamonds the lower bound obtained from Theorem 1 is asymptotically tight in a strong sense.

Theorem 3. *Let s and t be fixed natural numbers. Then for sufficiently large n ,*

$$R(SD_{s,t}, Q_n) \leq n + \frac{(2 + o(1))n}{\log n}.$$

Note that the poset Q_2^- , as defined in the penultimate line of Table 1, is an induced subposet of $SD_{1,2}$. Thus, Theorem 3 implies:

Corollary 4. *For n sufficiently large, $R(Q_2^-, Q_n) \leq R(SD_{1,2}, Q_n) \leq n + (2 + o(1))\frac{n}{\log n}$.*

We remark that Theorem 3 can be generalized, but the argument is rather technical, so we omit it here. Generalizing subdivided diamonds we can show that for every poset P with width $w(P) = 2$ and which contains no copy of the N-shaped 4-element poset \bowtie ,

$$R(P, Q_n) = n + O\left(\frac{n}{\log n}\right).$$

This bound follows from Theorem 3 of the present article, Theorem 1 and Corollary 6 of [23] as well as a characterization by Valdes [21] stating that a poset is \bowtie -free if and only if it is series-parallel.

An antichain A_t is the parallel composition of t single vertices, i.e. the poset consisting of t pairwise incomparable vertices. Since $h(A_t) = 1$ and $\dim_2(A_t) = \alpha(t)$ by the famous Sperner's theorem [20], the bounds in [1] and (*) imply that

$$n \leq R(A_t, Q_n) \leq n + \alpha(t) \leq n + \log t + \frac{\log \log t}{2} + 2.$$

In this paper we exactly determine $R(A_t, Q_n)$, not only for fixed t but also if t grows at most double-logarithmic in terms of n . We remark that the case $t \leq 2$ is covered by Theorem 8.

Theorem 5. *For every two integers t and n with $3 \leq t \leq \log \log n$,*

$$R(A_t, Q_n) = n + 3.$$

In fact, our result holds for $n \geq 2^{2^{t-2}} - 2$, which is a slightly weaker precondition than $t \leq \log \log n$. Besides that, we prove that if t is large in terms of n , the poset Ramsey number $R(A_t, Q_n)$ exceeds $n + 3$.

Theorem 6. *Let $n, r, t \in \mathbb{N}$ such that $t > \binom{n+2r+1}{r}$. Then $R(A_t, Q_n) \geq n + 2r + 2$. In particular, if $t \geq n + 4$, then $R(A_t, Q_n) \geq n + 4$.*

Stated explicitly in terms of n and t , Theorems 1 and 6, and (*) provide the following.

Corollary 7. *For $n, t \in \mathbb{N}$ with $n \geq 3$ and $t \geq 2$,*

$$n + \frac{2 \log t}{3 + \log n} \leq R(A_t, Q_n) \leq n + \log t + \frac{\log \log t}{2} + 2.$$

It is a simple observation that a poset is trivial if and only if it is a parallel composition of chains $C_{t_1}, \dots, C_{t_\ell}$. We say that this is the chain composition with parameters t_1, \dots, t_ℓ ,

$$C_{t_1, t_2, \dots, t_\ell} = C_{t_1} + C_{t_2} + \dots + C_{t_\ell}.$$

Throughout the paper we use the convention that $t_1 \geq t_2 \geq \dots \geq t_\ell$. Theorem 1 provides that $n + t_1 - 1 \leq R(C_{t_1, t_2, \dots, t_\ell}, Q_n) \leq n + t_1 + \alpha(\ell) + 1$. Here we show exact bounds for chain compositions of width $\ell \leq 3$.

Theorem 8. Let $n, t_1, t_2 \in \mathbb{N}$ such that $t_1 \geq t_2$. Then

$$R(C_{t_1}, Q_n) = n + t_1 - 1 \quad \text{and} \quad R(C_{t_1, t_2}, Q_n) = n + t_1 + 1.$$

Theorem 9. Let $n, t_1, t_2, t_3 \in \mathbb{N}$ with $t_1 \geq t_2 \geq t_3$. If $t_1 \leq t_2 + 1$, then $R(C_{t_1, t_2, t_3}, Q_n) = n + t_1 + 2$. If $t_1 > t_2 + 1$, then $R(C_{t_1, t_2, t_3}, Q_n) = n + t_1 + 1$.

Theorems 2 and 8 imply an improvement of the general bound for trivial posets.

Corollary 10. Let $n \in \mathbb{N}$. Let P be a trivial poset. If $w(P) = 1$, then $R(P, Q_n) = n + h(P) - 1$. If $w(P) \geq 2$, then $n + h(P) + 1 \leq R(P, Q_n) \leq n + h(P) + \alpha(w(P)) - 1$.

Note that every poset with $w(P) = 1$ is trivial. By Theorem 1, the bounds in Corollary 10 for $w(P) \geq 2$ also hold for non-trivial posets if n is large. However, for small n , Corollary 10 does not extend to non-trivial posets, for example it can be easily checked that $R(V_2, Q_1) = 3 < 4$.

Our paper is structured as follows. In Section 2 we introduce basic definitions and notation, and discuss preliminary observations. Section 3 gives a proof of Theorem 3. Section 4 focuses on antichains, and gives a proof of Theorems 5 and 6 as well as Corollary 7. In Section 5, we present proofs of Theorems 8 and 9. Finally, in Section 6 we summarize known proof techniques for bounding off-diagonal poset Ramsey numbers and collect some open problems.

In this paper we omit floors and ceilings where appropriate. The set of the first n natural numbers is denoted by $[n] = \{1, \dots, n\}$.

2. Preliminaries

2.1. Poset notation and classic results

In the previous section we stated formal definitions of the Boolean lattice Q_n , the V -shaped poset V_2 , the chain C_t , the antichain A_t , the (s, t) -subdivided diamond $SD_{s,t}$ and the chain composition $C_{t_1, t_2, \dots, t_\ell}$. Besides those we use the following notation for posets. Examples of all mentioned posets are given in Table 1.

The N -shaped poset \bowtie consists of four distinct vertices W, X, Y , and Z such that $W \leq Y, Y \geq X, X \leq Z, W \parallel X, W \parallel Z$, and $Y \parallel Z$. The hook-shaped poset Q_2^- has distinct vertices W, X, Y , and Z where $X \leq Y \leq Z, W \geq X, W \parallel Y$, and $W \parallel Z$.

The complete ℓ -partite poset K_{t_1, \dots, t_ℓ} is a poset on $\sum_{i=1}^\ell t_i$ many vertices defined as follows. For each index $i \in [\ell]$, there is a set of t_i distinct vertices, called layer i . Every pair of vertices in the same layer is incomparable. For any two vertices X and Y belonging to different layers i_X and i_Y with $i_X < i_Y$, we have $X \leq Y$.

In this paper, we commonly consider a Boolean lattice Q_n with a specified ground set. Given a set \mathbf{X} , the Boolean lattice $\mathcal{Q}(\mathbf{X})$ is the poset on all subsets of \mathbf{X} equipped with set inclusion relation. The dimension of $\mathcal{Q}(\mathbf{X})$ is $|\mathbf{X}|$. Let $N = |\mathbf{X}|$. For $\ell \in \{0, \dots, N\}$, we say that layer ℓ of $\mathcal{Q}(\mathbf{X})$ is the set of elements in $\{Z \in \mathcal{Q}(\mathbf{X}) : |Z| = \ell\}$. Note that $\mathcal{Q}(\mathbf{X})$ consists of $N + 1$ layers and that each layer induces an antichain in $\mathcal{Q}(\mathbf{X})$. A blue/red coloring of a Boolean lattice is layered if within each layer every vertex receives the same color.

The following classic result is known as Dilworth's theorem [11].

Theorem 11 (Dilworth [11]). A poset P contains no antichain of size t if and only if all vertices of P can be covered by $t - 1$ chains.

A chain in an N -dimensional Boolean lattice is said to be symmetric if it consists of vertices $X_\ell \subset \dots \subset X_{N-\ell}$ for some non-negative integer ℓ such that $|X_i| = i$ for all $i \in \{\ell, \dots, N - \ell\}$. De Bruijn, Tengbergen and Krusywijk [9] showed the following decomposition result.

Theorem 12 ([9]). The vertices of an N -dimensional Boolean lattice can be decomposed into pairwise disjoint, symmetric chains.

2.2. Red Boolean lattice versus blue chain

In this paper we often consider the Boolean lattice $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ where \mathbf{X} and \mathbf{Y} are two disjoint sets with $\mathbf{Y} \neq \emptyset$. We denote a linear ordering τ of \mathbf{Y} where $y_1 <_\tau y_2 <_\tau \dots <_\tau y_k$ by a sequence $\tau = (y_1, \dots, y_k)$ implying that $\mathbf{Y} = \{y_1, \dots, y_k\}$. Given a linear ordering $\tau = (y_1, \dots, y_k)$ of \mathbf{Y} , a \mathbf{Y} -chain corresponding to τ is a $(k + 1)$ -element chain in $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ on vertices

$$X_0 \cup \emptyset, X_1 \cup \{y_1\}, X_2 \cup \{y_1, y_2\}, \dots, X_k \cup \mathbf{Y},$$

where $X_0 \subseteq X_1 \subseteq \dots \subseteq X_k \subseteq \mathbf{X}$. Note that \mathbf{Y} -chains corresponding to distinct linear orderings of \mathbf{Y} are distinct. The following *Chain Lemma* was proved implicitly by Chen, Cheng, Li and Liu [7] as well as Grósz, Methuku and Tompkins [13]. Since it is a fundamental lemma in determining $R(P, Q_n)$, we restate its self-contained proof by following the lines of Axenovich and the author, see Lemma 8 in [2].

Lemma 13 ([13], *Chain Lemma*). *Let \mathbf{X} and \mathbf{Y} be disjoint sets with $|\mathbf{X}| = n$ and $|\mathbf{Y}| = k$. Let $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ be a blue/red colored Boolean lattice. Fix a linear ordering $\tau = (y_1, \dots, y_k)$ of \mathbf{Y} . Then there exists in $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ either a red copy of Q_n , or a blue \mathbf{Y} -chain corresponding to τ .*

Proof. We denote the set of the first i elements of \mathbf{Y} with respect to τ by $\mathbf{Y}[i] = \{y_1, \dots, y_i\}$ for $i \in [k]$ and let $\mathbf{Y}[0] = \emptyset$. Assume that there does not exist a blue \mathbf{Y} -chain corresponding to τ . We show that there is a red copy of Q_n . Recall that a copy of P in Q is the image of an embedding $\phi: P \rightarrow Q$. In the following for every $X \subseteq \mathbf{X}$, we recursively define a label $\ell_X \in \{0, \dots, k\}$ such that the function

$$\phi: \mathcal{Q}(\mathbf{X}) \rightarrow \mathcal{Q}(\mathbf{X} \cup \mathbf{Y}), \phi(X) = X \cup \mathbf{Y}[\ell_X]$$

is an embedding with monochromatic red image. The image of such an embedding is a red copy of Q_n as required. We choose labels $\ell_X, X \subseteq \mathbf{X}$, with the following properties:

- (i) For any $U \subseteq X, \ell_U \leq \ell_X$.
- (ii) There is a blue chain \mathcal{C}^X of length ℓ_X “below” the vertex $X \cup \mathbf{Y}[\ell_X]$, i.e. in the Boolean lattice $Q^X := \mathcal{Q}(X \cup \mathbf{Y}[\ell_X])$ there is a blue $\mathbf{Y}[\ell_X]$ -chain corresponding to the linear ordering (y_1, \dots, y_{ℓ_X}) .
- (iii) The vertex $X \cup \mathbf{Y}[\ell_X]$ is colored red.

First, consider the subset $\emptyset \subseteq \mathbf{X}$. Let ℓ_\emptyset be the smallest index $\ell, 0 \leq \ell \leq k$, such that the vertex $\emptyset \cup \mathbf{Y}[\ell]$ is red. If there is no such ℓ , then the vertices $\emptyset \cup \mathbf{Y}[0], \dots, \emptyset \cup \mathbf{Y}[k]$ form a blue \mathbf{Y} -chain corresponding to τ , a contradiction. It is immediate that Properties (i) and (iii) hold for ℓ_\emptyset . If $\ell_\emptyset = 0$, then (ii) holds trivially. If $\ell_\emptyset \geq 1$, then vertices $\emptyset \cup \mathbf{Y}[0], \dots, \emptyset \cup \mathbf{Y}[\ell_\emptyset - 1]$ form a blue chain of length ℓ_\emptyset as required for (ii).

Now consider an arbitrary non-empty $X \subseteq \mathbf{X}$ and suppose that for every $U \subset X$ we already defined ℓ_U with Properties (i)-(iii). Let $\ell'_X = \max_{U \subset X} \ell_U$ and fix some $W \subset X$ with $\ell_W = \ell'_X$. Let \mathcal{C}^W be the blue chain obtained by Property (ii) for W . We define ℓ_X as the smallest integer ℓ with $\ell'_X \leq \ell \leq k$ such that the vertex $X \cup \mathbf{Y}[\ell]$ is red in the coloring of $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$. If such an ℓ does not exist, i.e. if there is no red $X \cup \mathbf{Y}[\ell]$, the vertices $X \cup \mathbf{Y}[\ell'_X], \dots, X \cup \mathbf{Y}[k]$ form a blue chain of length $k - \ell'_X + 1$. The chain \mathcal{C}^W is a blue chain “below” $W \cup \mathbf{Y}[\ell_W]$. Note that $W \cup \mathbf{Y}[\ell_W] \subset X \cup \mathbf{Y}[\ell'_X]$, thus both chains combine to a chain \mathcal{C}^X of length $k + 1$. It is easy to see that \mathcal{C}^X is a \mathbf{Y} -chain corresponding to τ , so we arrive at a contradiction. Thus, ℓ_X is well-defined.

It is immediate that Property (iii) holds for ℓ_X . Furthermore, for $U \subset X \subseteq \mathbf{X}$ we have $\ell_U \leq \ell'_X \leq \ell_X$, thus (i) holds. It remains to verify Property (ii) for ℓ_X . Recall that $W \subset X$ such that $\ell_W = \ell'_X$. If $\ell_X = \ell'_X$, the chain $\mathcal{C}^X := \mathcal{C}^W$ is as required. If $\ell_X \neq \ell'_X$, the chain \mathcal{C}^W together with vertices $X \cup \mathbf{Y}[\ell'_X], \dots, X \cup \mathbf{Y}[\ell_X - 1]$ is a blue chain of length ℓ_X , which verifies Property (ii).

We use the labels $\ell_X, X \subseteq \mathbf{X}$, to define an embedding of a Boolean lattice Q_n in $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$. Let $\phi: \mathcal{Q}(\mathbf{X}) \rightarrow \mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ with $\phi(X) = X \cup \mathbf{Y}[\ell_X]$. Property (iii) implies that the image of ϕ is colored monochromatically red. We show that for any two $X_1, X_2 \subseteq \mathbf{X}$, it holds that $X_1 \subseteq X_2$ if and only if $\phi(X_1) \subseteq \phi(X_2)$. Indeed, if $\phi(X_1) \subseteq \phi(X_2)$, it is immediate that $X_1 = \phi(X_1) \cap \mathbf{X} \subseteq \phi(X_2) \cap \mathbf{X} = X_2$. Conversely, if $X_1 \subseteq X_2$, then by Property (i) we see that $\ell_{X_1} \leq \ell_{X_2}$. Thus $X_1 \cup \mathbf{Y}[\ell_{X_1}] \subseteq X_2 \cup \mathbf{Y}[\ell_{X_2}]$. Therefore, ϕ is an embedding of $\mathcal{Q}(\mathbf{X})$ with red image, so in particular $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ contains a red copy of Q_n . \square

The red copy of Q_n obtained in this lemma has a strong additional property with respect to \mathbf{X} which is not needed here. For further details we refer the reader to Lemma 8 of [2]. The following corollary is a simplified version of the Chain Lemma.

Corollary 14. *Let $n \geq 1$ and $k \geq 0$ be integers. Let Q be a blue/red colored Boolean lattice of dimension $n + k$. Then Q contains a red copy of Q_n or a blue chain of length $k + 1$.*

Note that Corollary 14 immediately implies the upper bound for $R(C_{t_1}, Q_n)$ in Theorem 8.

2.3. Embedding of a Boolean lattice

Axenovich and Walzer [1] showed that an embedding of a small Boolean lattice into a larger Boolean lattice has the following nice property, see Theorem 8 of [1].

Lemma 15 ([1]). *Let $n \in \mathbb{N}$ and let \mathbf{Z} be a set with $|\mathbf{Z}| > n$. If there is an embedding $\phi: Q_n \rightarrow \mathcal{Q}(\mathbf{Z})$, then there exist a subset $\mathbf{X} \subset \mathbf{Z}$ with $|\mathbf{X}| = n$, and an embedding $\phi': \mathcal{Q}(\mathbf{X}) \rightarrow \mathcal{Q}(\mathbf{Z})$ with the same image as ϕ such that $\phi'(X) \cap \mathbf{X} = X$ for all $X \subseteq \mathbf{X}$.*

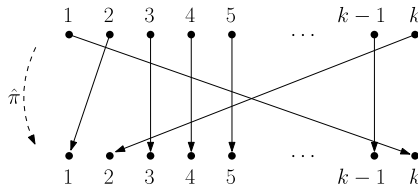


Fig. 2. The permutation $\hat{\pi} : [k] \rightarrow [k]$.

In other words, for any copy Q' of Q_n in the hosting lattice $\mathcal{Q}(\mathbf{Z})$ there is some $\mathbf{X} \subset \mathbf{Z}$ such that the vertex-wise restriction of Q' to \mathbf{X} is equal to $\mathcal{Q}(\mathbf{X})$.

3. Upper bound on $R(SD_{s,t}, Q_n)$

3.1. Counting permutations

In this subsection we bound the number of permutations with a special property, in preparation for our proof of Theorem 3. A permutation $\pi : [k] \rightarrow [k]$ is called r -proper if for every $j \in [k]$, $|\{\ell \leq j : \pi(\ell) \geq j - 1\}| \leq r$. For example, the permutation $\hat{\pi}$ given by $(\hat{\pi}(1), \dots, \hat{\pi}(k)) = (k, 1, 3, 4, 5, \dots, k - 1, 2)$, see Fig. 2, is not 1-proper because at $j = 3$, $\{\ell \leq 3 : \hat{\pi}(\ell) \geq 2\} = \{1, 3\}$. However, $\hat{\pi}$ is 2-proper.

Lemma 16. *Let $r, k \in \mathbb{N}$. There are at most $2^{(r+\log r)k}$ distinct r -proper permutations $\pi : [k] \rightarrow [k]$.*

Proof. For an r -proper permutation π , we say that an index $i \in [k]$ is *bad* if $\pi(i) \geq i$, and *good* if $\pi(i) \leq i - 1$. Let B_π and G_π denote the set of indices that are bad and good, respectively, i.e. the sets partition $[k]$. Again considering the example $(\hat{\pi}(1), \dots, \hat{\pi}(k)) = (k, 1, 3, 4, 5, \dots, k - 1, 2)$, we have that $B_{\hat{\pi}} = \{1\} \cup \{3, 4, \dots, k - 1\}$ and $G_{\hat{\pi}} = \{2, k\}$.

Given an r -proper permutation π , the *proper restriction* ρ of π is the restriction of π to its bad indices, i.e. $\rho : B_\pi \rightarrow [k]$ with $\rho(i) = \pi(i)$ for every i . For example, the proper restriction of $\hat{\pi}$ is $\hat{\rho} : [k - 1] \setminus \{2\} \rightarrow [k]$ with $\hat{\rho}(1) = k$ and $\hat{\rho}(i) = i$ for $3 \leq i \leq k - 1$. Note that ρ does not depend on r . Observe that a function ρ can be the proper restriction of distinct r -proper permutations. Let Π be the set of all r -proper permutations $\pi : [k] \rightarrow [k]$. If a function ρ is the proper restriction of some $\pi \in \Pi$, we say that ρ is a Π -restriction. To avoid ambiguity, we denote the domain of ρ by B_ρ . Inheriting the properties of an r -proper permutation, ρ is injective and

$$|\{\ell \in B_\rho : \ell \leq j, \rho(\ell) \geq j - 1\}| \leq r.$$

In the following we bound $|\Pi|$ by first estimating $|\{\rho : \rho \text{ is a } \Pi\text{-restriction}\}|$, and then bounding $|\{\pi \in \Pi : \rho \text{ is the proper restriction of } \pi\}|$ for every fixed ρ .

Claim 1. *There are at most 2^{rk} distinct Π -restrictions.*

Proof of Claim 1. We show that every Π -restriction has a distinct representation as a collection of r vectors $V_1, \dots, V_r \in \{0, 1\}^k$, which implies that there are at most 2^{rk} Π -restrictions. Let ρ be a Π -restriction with domain B_ρ . For every $i \in B_\rho$ we define an integer interval $I_i = [i, \rho(i) + 1]$. Consider the *interval graph* H given by intervals I_i , i.e. the graph on vertex set B_ρ where $\{i, j\}$ is an edge if and only if $i \neq j$ and $I_i \cap I_j \neq \emptyset$. In the following we use terminology common in graph theory, for a formal introduction we refer the reader to Diestel [10].

Next we bound the maximal size of a clique in H . Suppose that vertices i_1, \dots, i_m form a clique in H , then the intervals I_{i_1}, \dots, I_{i_m} pairwise intersect. Thus there exists an integer $j \in [k]$ such that $j \in I_{i_1} \cap \dots \cap I_{i_m}$. Now

$$m = |\{\ell \in B_\rho : j \in I_\ell\}| = |\{\ell \in B_\rho : \ell \leq j, \rho(\ell) + 1 \geq j\}| \leq r,$$

where the last inequality holds since ρ is a proper restriction. Thus there is no clique of size $r + 1$ in H . It is common knowledge that interval graphs are perfect, so there exists a proper vertex coloring of H using at most r colors. Fix such a coloring c of H with set of colors $[r]$. Note that for each color class the corresponding intervals are pairwise disjoint.

For every fixed $s \in [r]$, let the set of indices with color s be $B_s = \{i \in B_\rho : c(I_i) = s\}$. We define a vector $V_s \in \{0, 1\}^k$ as follows. Let

$$V_s(i) = \dots = V_s(\rho(i)) = 1 \text{ for any } i \in B_s \quad \text{and} \quad V_s(j) = 0 \text{ for all other } j \in [k].$$

Since the intervals I_i , $i \in B_s$, are pairwise disjoint, V_s is well-defined. Moreover, we obtain that $V_s(\rho(i) + 1) = 0$ for every $i \in B_s$. This implies that $V_s(i - 1) = 0$, if defined, for $i \in B_s$. Observe that the vector V_s encodes all indices in B_s and their

respective functional values $\rho(i), i \in B_s$: If for some $j \in [k], V_s(j) = 1$ and $V_s(j - 1) = 0$, then $j \in B_s$ and $\rho(j)$ is given by the maximal index j' such that $V_s(j) = \dots = V_s(j') = 1$.

We obtain a vector representation V_1, \dots, V_r of ρ . It is easy to see that distinct Π -restrictions have distinct representations. There are at most $(2^k)^r$ distinct such vector representations, which proves the claim.

Claim 2. Given a fixed Π -restriction ρ , the number of r -proper permutations π with proper restriction ρ is at most r^k .

Proof of Claim 2. Let ρ be a fixed Π -restriction and let $G = [k] \setminus B_\rho$. We count the possible assignments of good indices $i \in G$ in an r -proper permutation given that $\pi(\ell) = \rho(\ell)$ for every $\ell \in B_\rho$. For this purpose we iterate through all good indices $i \in G$ in increasing order while counting the choices for each $\pi(i)$. Observe that $1 \notin G$ since $\pi(1) \geq 1$ for any permutation π . Fix an $i \in G$, i.e. $i \geq 2$. Suppose that all indices $\ell \in G \cap [i - 1]$ are already assigned to an integer $\pi(\ell) \leq \ell - 1$ and all $\ell \in B_\rho$ are assigned to $\pi(\ell) = \rho(\ell)$. There are two conditions on the choice of $\pi(i)$: On the one hand i is a good index, so we require $\pi(i) \in [i - 1]$. On the other hand π is injective, thus $\pi(i) \neq \pi(\ell)$ for all $\ell < i$. Therefore $\pi(i) \in [i - 1] \setminus \{\pi(\ell) \in [i - 1] : \ell < i\}$. We evaluate the size of this set using the fact that $|\{\ell < i : \pi(\ell) \geq i - 1\}| \leq r$:

$$\begin{aligned} |\{\pi(\ell) \in [i - 1] : \ell < i\}| &= |\{\ell < i : \pi(\ell) \leq i - 1\}| = |[i - 1] \setminus \{\ell < i : \pi(\ell) > i - 1\}| \\ &\geq |[i - 1] \setminus \{\ell < i : \pi(\ell) \geq i - 1\}| \geq (i - 1) - r. \end{aligned}$$

Thus also

$$|[i - 1] \setminus \{\pi(\ell) \in [i - 1] : \ell < i\}| \leq (i - 1) - (i - 1 - r) = r.$$

Hence, there are at most r choices for selecting $\pi(i)$ for each $i \in G$. Note that $|G| \leq k$, consequently the number of r -proper permutations with proper restriction ρ is at most r^k .

Combining both claims, the number of r -proper permutations is at most

$$\sum_{\rho \text{ is a } \Pi\text{-restriction}} |\{\pi \in \Pi : \rho \text{ is a proper restriction of } \pi\}| \leq 2^{rk} r^k = 2^{(r+\log r)k}. \quad \square$$

The bound provided here is not best possible. With a more careful approach the number $N(k, r)$ of r -proper permutations $\pi : [k] \rightarrow [k]$ can be bounded between

$$r^k \leq N(k, r) \leq (2r)^{2k}.$$

Studying this extremal function might be of independent interest.

3.2. Proof of Theorem 3

Before presenting the proof of Theorem 3 we give a lemma which is purely computational and follows the lines of a similar claim by Grósz, Methuku and Tompkins [13].

Lemma 17. Let $n \in \mathbb{N}$, and let $c \in \mathbb{R}$ be a positive constant. Let $k = \frac{(2+\epsilon)n}{\log n}$ where $\epsilon = 3(\log \log n + \log e + c + 2)(\log n)^{-1}$. Then for sufficiently large n ,

$$k! > 2^{ck} \cdot 2^{2(n+k)}.$$

Proof. By Stirling's formula $k! > \left(\frac{k}{e}\right)^k = 2^{k(\log k - \log e)}$. We shall show that $k(\log k - \log e) > ck + 2(n + k)$. Using the fact that $k = \frac{(2+\epsilon)n}{\log n}$, we obtain

$$\begin{aligned} &k(\log k - \log e - c - 2) - 2n \\ &\geq \frac{(2 + \epsilon)n}{\log n} (\log(2 + \epsilon) + \log n - \log \log n - \log e - c - 2) - 2n \\ &\geq \epsilon n - \frac{n}{\log n} (2 + \epsilon)(\log \log n + \log e + c + 2) > 0, \end{aligned}$$

where the last inequality holds for sufficiently large n . \square

Proof of Theorem 3. For any $s \leq t$, note that $SD_{s,t}$ is an induced subposet of $SD_{t,t}$, so it suffices to show the Ramsey bound for $s = t$. Let $k = \frac{(2+\epsilon)n}{\log n}$ where $\epsilon = 3(\log \log n + \log e + c + 2)(\log n)^{-1}$ and $c = 2t + 2 + \log(2t + 2)$. Let \mathbf{X} and \mathbf{Y} be disjoint sets with $|\mathbf{X}| = n$, $|\mathbf{Y}| = k$. Consider an arbitrary blue/red coloring of $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ with no red copy of Q_n . We shall show that there is a blue copy of $SD_{t,t}$ in this coloring.

There are $k!$ linear orderings of \mathbf{Y} . For every linear ordering τ of \mathbf{Y} , Lemma 13 provides a blue \mathbf{Y} -chain C^τ in $\mathcal{Q}(\mathbf{X} \cup \mathbf{Y})$ corresponding to τ , say on vertices $Z_0^\tau \subset Z_1^\tau \subset \dots \subset Z_k^\tau$. Consider the smallest vertex Z_0^τ as well as the largest vertex Z_k^τ . Both vertices are subsets of $\mathbf{X} \cup \mathbf{Y}$, so there are at most $2^{2(n+k)}$ distinct pairs (Z_0^τ, Z_k^τ) . By pigeonhole principle there is a collection τ_1, \dots, τ_m of $m = \frac{k!}{2^{2(n+k)}}$ distinct linear orderings of \mathbf{Y} such that all of the corresponding \mathbf{Y} -chains C^{τ_i} have both $Z_0^{\tau_i}$ and $Z_k^{\tau_i}$ in common. Lemma 17 shows that $m > 2^{ck}$.

Fix an arbitrary $\sigma \in \{\tau_1, \dots, \tau_m\}$. By relabeling \mathbf{Y} we can suppose that $\sigma = (1, \dots, k)$, i.e. $1 <_\sigma \dots <_\sigma k$. Consider a linear ordering τ_j , $j \in [m]$, (allowing that $\tau_j = \sigma$) and let $\tau_j = (y_1, \dots, y_k)$. Then we say that τ_j is t -close to σ for some $t \in \mathbb{N}$ if for every $i \in [k - t]$, either $[i] \subseteq \{y_1, \dots, y_{i+t}\}$ or $\{y_1, \dots, y_i\} \subseteq [i + t]$. For example the linear ordering $(4, 5, \dots, k, 1, 2, 3)$ is 3-close to σ since the first i elements of this linear ordering are contained in $[i + 3]$, for any $i \in [k - 3]$. However, our example is not 2-close to σ , because neither $\{1\} \subseteq \{4, 5, 6\}$ nor $\{4\} \subseteq [3]$. In the remaining proof we distinguish two cases. If there is a linear ordering τ_j which is not t -close to σ , we build a copy of $SD_{t,t}$ from the \mathbf{Y} -chains corresponding to σ and τ_j . If every linear ordering τ_1, \dots, τ_m is t -close to σ , we find $m > 2^{ck}$ permutations fulfilling the property of Lemma 16, thus we arrive at a contradiction.

Case 1: There is a linear ordering $\tau \in \{\tau_1, \dots, \tau_m\}$ which is not t -close to σ .

Suppose that the \mathbf{Y} -chains corresponding to σ and τ are given by $Z_0^\sigma, \dots, Z_k^\sigma$ and $Z_0^\tau, \dots, Z_k^\tau$, respectively. Recall that $Z_0^\sigma = Z_0^\tau$ and $Z_k^\sigma = Z_k^\tau$. Since τ is not t -close to σ there is an index $i \in [k - t]$ such that neither $[i] \subseteq \{y_1, \dots, y_{i+t}\}$ nor $\{y_1, \dots, y_i\} \subseteq [i + t]$. Note that $i < k - t$ because for $i = k - t$ we have $[k - t] \subseteq [k] = \mathbf{Y} = \{y_1, \dots, y_k\}$. The definition of \mathbf{Y} -chains provides that $Z_i^\sigma \cap \mathbf{Y} = [i]$ and $Z_{i+t}^\tau \cap \mathbf{Y} = \{y_1, \dots, y_{i+t}\}$, thus $Z_i^\sigma \not\subseteq Z_{i+t}^\tau$. By transitivity, $Z_j^\sigma \not\subseteq Z_j^\tau$ for any two $j, j' \in \{i, \dots, i + t\}$. Similarly, $Z_i^\tau \not\subseteq Z_{i+t}^\sigma$ and so $Z_j^\tau \not\subseteq Z_{j'}^\sigma$. This implies that the set

$$\mathcal{P} = \left\{ Z_j^\sigma, Z_j^\tau : j \in \{0, k\} \cup \{i, \dots, i + t\} \right\}$$

induces a copy of $SD_{t,t}$. Furthermore, every vertex of \mathcal{P} is included in a blue \mathbf{Y} -chain and thus colored blue. This completes the proof for Case 1.

Case 2: Every linear ordering $\tau \in \{\tau_1, \dots, \tau_m\}$ is t -close.

Here we use the fact that every linear ordering τ_j , $j \in [m]$, is obtained by permuting the linear ordering σ . Fix an arbitrary $\tau \in \{\tau_1, \dots, \tau_m\}$, and let $\tau = (y_1, \dots, y_k)$. We say that the permutation corresponding to τ is $\pi : [k] \rightarrow [k]$ with $\pi(\ell) = y_\ell$. We show that π has the following property.

Claim. For every $j \in [k]$, $|\{\ell \leq j : \pi(\ell) > j + t\}| \leq t$.

The statement is trivially true if $j + t > k$. Now fix some $j \in [k - t]$. By t -closeness of τ either $\{\pi(1), \dots, \pi(j)\} = \{y_1, \dots, y_j\} \subseteq [j + t]$ or $[j] \subseteq \{y_1, \dots, y_{j+t}\} = \{\pi(1), \dots, \pi(j + t)\}$.

If $\{\pi(1), \dots, \pi(j)\} \subseteq [j + t]$, then for every $\ell \leq j$ we have $\pi(\ell) \leq j + t$. Therefore $\{\ell \leq j : \pi(\ell) > j + t\} = \emptyset$ and the statement holds.

If $[j] \subseteq \{\pi(1), \dots, \pi(j + t)\}$, then let $I = \{\pi(1), \dots, \pi(j + t)\} \setminus [j]$, note that $|I| = t$. Observe that for every $\ell \leq j$ with $\pi(\ell) > j + t$, we know in particular that $\pi(\ell) \notin [j]$, thus $\pi(\ell) \in I$. Since π is bijective,

$$|\{\ell \leq j : \pi(\ell) > j + t\}| = |\{\pi(\ell) : \ell \leq j, \pi(\ell) > j + t\}| \leq |I| = t.$$

This proves the claim.

In particular, π has the property that $|\{\ell \leq j : \pi(\ell) \geq j - 1\}| \leq 2t + 2$ for every $j \in [k]$, i.e. π is $(2t + 2)$ -proper. Note that distinct linear orderings τ_i , $i \in [m]$, correspond to distinct permutations $\pi_i : [k] \rightarrow [k]$. Lemma 16 provides that the number of $(2t + 2)$ -proper permutations π_i is at most

$$m \leq 2^{(2t+2+\log(2t+2))k} = 2^{ck}.$$

Recall that by Lemma 17, $m > 2^{ck}$, so we arrive at a contradiction. \square

4. Bounds on $R(A_t, Q_n)$

4.1. Erdős-Szekeres variant

In preparation for the proof of Theorem 5 we reshape the following well-known result of Erdős and Szekeres [12].

Theorem 18 (Erdős-Szekeres [12]). Let $m \in \mathbb{N}$. For an $(m^2 + 1)$ -element set \mathbf{Z} , let $S = (a_1, a_2, \dots, a_{m^2+1})$ be a sequence consisting of distinct elements of \mathbf{Z} . Let τ be an arbitrary linear ordering of $\mathbf{Z} = \{a_1, \dots, a_{m^2+1}\}$. Then there exists a subsequence $(a_{i_1}, \dots, a_{i_{m+1}})$ of S of length $m + 1$ such that $a_{i_1} <_{\tau} \dots <_{\tau} a_{i_{m+1}}$ or $a_{i_{m+1}} <_{\tau} \dots <_{\tau} a_{i_1}$.

In the following we refer to a finite sequence of distinct elements as a *permutation sequence* on the set of these elements. We say that a sequence $(b_1, b_2, \dots, b_{\ell})$ is an *undirected subsequence* of some sequence S if either $(b_1, b_2, \dots, b_{\ell})$ or $(b_{\ell}, b_{\ell-1}, \dots, b_1)$ is a subsequence of S . Let $\{S^1, \dots, S^m\}$, $m \in \mathbb{N}$, be a collection of permutation sequences on the same set. If $(b_1, b_2, \dots, b_{\ell})$ is an undirected subsequence of every S^i , it is referred to as a *common undirected subsequence* of S^1, \dots, S^m .

Corollary 19. Let S and T be two permutation sequences on an $(m^2 + 1)$ -element set \mathbf{Z} . Then there exists a common undirected subsequence of S and T which has length $m + 1$.

Proof. Let $S = (a_1, a_2, \dots, a_{m^2+1})$, i.e. $\mathbf{Z} = \{a_1, \dots, a_{m^2+1}\}$. For $\ell \in [m^2 + 1]$, let j_{ℓ} be indices such that $T = (a_{j_1}, \dots, a_{j_{m^2+1}})$. Consider the linear ordering τ given by $a_{j_1} <_{\tau} \dots <_{\tau} a_{j_{m^2+1}}$. Then the Erdős-Szekeres theorem, Theorem 18, yields a subsequence $(a_{i_1}, \dots, a_{i_{m+1}})$ of S which is also an undirected subsequence of T . In particular, $(a_{i_1}, \dots, a_{i_{m+1}})$ is a common undirected subsequence of S and T . \square

By iteratively applying Corollary 19 we obtain a lemma that we need later on.

Lemma 20. Let $d \in \mathbb{N}$ and $N \geq 2^{2^{d-1}} + 1$. Let \mathbf{Z} be an N -element set. Let τ_1, \dots, τ_d be arbitrary linear orderings of \mathbf{Z} . Then there exist pairwise distinct $x, y, z \in \mathbf{Z}$ such that for every $i \in [d]$,

$$x <_{\tau_i} y <_{\tau_i} z \quad \text{or} \quad z <_{\tau_i} y <_{\tau_i} x.$$

Proof. For each $i \in [d]$, say that τ_i is given by $a_1^i <_{\tau_i} a_2^i <_{\tau_i} \dots <_{\tau_i} a_N^i$. Then let $S(\tau_i)$ be the sequence $(a_1^i, a_2^i, \dots, a_N^i)$. Note that $S(\tau_i)$ is a permutation sequence on \mathbf{Z} . We show that there is a common undirected subsequence of $S(\tau_1), \dots, S(\tau_d)$ of length 3, because for such a subsequence (x, y, z) either $x <_{\tau_i} y <_{\tau_i} z$ or $z <_{\tau_i} y <_{\tau_i} x$ for every $i \in [d]$. We proceed with an iterative argument.

Let $T^1 = S(\tau_1)$; note that $|T^1| \geq 2^{2^{d-1}} + 1$. For $i \in [d - 1]$, suppose that T^i is a common undirected subsequence of all $S(\tau_j)$, $j \in [i]$, and has length at least $2^{2^{d-i}} + 1$. Let \mathbf{Z}^i be the underlying set of T^i , and let S^i be the restriction of $S(\tau_{i+1})$ to \mathbf{Z}^i . Then both T^i and S^i are permutation sequences on \mathbf{Z}^i . By Corollary 19, there is a common undirected subsequence T^{i+1} of T^i and S^i of length at least $(2^{2^{d-i}})^{\frac{1}{2}} + 1 = 2^{2^{d-(i+1)}} + 1$. Since T^{i+1} is an undirected subsequence of T^i , T^{i+1} is also an undirected subsequence of every $S(\tau_j)$, $j \in [i]$. Furthermore, because T^{i+1} is an undirected subsequence of S^i , it is also an undirected subsequence of $S(\tau_{i+1})$.

After $d - 1$ steps, we obtain a sequence T^d of length at least $2^{2^0} + 1 = 3$ which is a common undirected sequence of all S^j , $j \in [d]$. Choose an arbitrary 3-element subsequence (x, y, z) of T^d . Then x, y and z have the desired properties. \square

We remark that the bound $N \geq 2^{2^{d-1}} + 1$ in Lemma 20 is tight: It is widely known that there is a sequence of m^2 distinct elements which does not meet the property in the Erdős-Szekeres Theorem. Given such a sequence we can straightforwardly construct a collection of d linear orderings of a $2^{2^{d-1}}$ -element set such that no triple x, y, z has the property of Lemma 20.

4.2. Proof of Theorem 5

Proof of Theorem 5. Note that $R(A_3, Q_n) \leq R(A_t, Q_n) \leq R(A_{\lceil \log \log n \rceil}, Q_n)$ for all $3 \leq t \leq \log \log n$, so it suffices to show that $R(A_3, Q_n) \geq n + 3$ and $R(A_{\lceil \log \log n \rceil}, Q_n) \leq n + 3$. Let $N = n + 3$.

For the lower bound, we consider the following coloring of the Boolean lattice $\mathcal{Q} = \mathcal{Q}([N - 1])$: For all $i \in [N - 1]$, color $[i]$ and $[N - 1] \setminus [i]$ in blue and color all remaining vertices in red. Observe that all blue vertices can be covered by the two chains $\{[i]: i \in [N - 1]\}$ and $\{[N - 1] \setminus [i]: i \in [N - 1]\}$, thus among any three distinct blue vertices we find two vertices contained in the same chain, i.e. there is no blue copy of A_3 in \mathcal{Q} . Now, assume towards contradiction that there is a red copy of Q_n in \mathcal{Q} . Lemma 15 provides that there is a set $\mathbf{X} \subset [N - 1]$, $|\mathbf{X}| = n$, and an embedding $\phi: \mathcal{Q}(\mathbf{X}) \rightarrow \mathcal{Q}$ such that the image of ϕ is monochromatic red and $\phi(X) \cap \mathbf{X} = X$ for all $X \subseteq \mathbf{X}$. Note that $\phi(\emptyset) \neq \emptyset$ as \emptyset is colored blue in \mathcal{Q} , so say that $a \in \phi(\emptyset)$. Then $a \notin \mathbf{X}$ since $\phi(\emptyset) \cap \mathbf{X} = \emptyset$. Similarly, we know $\mathbf{X} \subseteq \phi(\mathbf{X}) \neq [N - 1]$, so we find an element $b \in [N - 1] \setminus \phi(\mathbf{X})$ with $b \notin \mathbf{X}$. Observe that $a \in \phi(\emptyset) \subseteq \phi(\mathbf{X})$, but $b \notin \phi(\mathbf{X})$, hence $a \neq b$. Recall that $|\mathbf{X}| = n = (N - 1) - 2$, thus $\mathbf{X} = [N - 1] \setminus \{a, b\}$. Because ϕ is an embedding of $\mathcal{Q}(\mathbf{X})$ and every vertex of $\mathcal{Q}(\mathbf{X})$ is comparable to \emptyset and \mathbf{X} , we obtain that $a \in \phi(X)$ and $b \notin \phi(X)$ for every $X \subseteq \mathbf{X}$. Now the fact that $\phi(X) \cap \mathbf{X} = X$ implies $\phi(X) = X \cup \{a\}$ for all $X \subseteq \mathbf{X}$.

If $b \in \{a + 1, \dots, N - 1\}$, then $[a - 1] \subseteq \mathbf{X}$. Thus $\phi([a - 1]) = [a - 1] \cup \{a\} = [a]$, but this vertex is colored blue in \mathcal{Q} . This is a contradiction to the fact that ϕ has a monochromatic red image.

If $b \in \{1, \dots, a - 1\}$, then $[N - 1] \setminus \{a\} \subseteq \mathbf{X}$. Now $\phi([N - 1] \setminus \{a\}) = ([N - 1] \setminus \{a\}) \cup \{a\} = [N - 1] \setminus \{a - 1\}$ which is colored blue in \mathcal{Q} , so again we reach a contradiction.

For the upper bound, let \mathbf{Z} be an arbitrary N -element set. Consider an arbitrary coloring of the Boolean lattice $\mathcal{Q}(\mathbf{Z})$ which contains no blue copy of A_t where $t \leq \log \log n$. We shall show that there is a red copy of Q_n in $\mathcal{Q}(\mathbf{Z})$.

By Dilworth's theorem, Theorem 11, there is a collection of $t - 1$ chains C_1, \dots, C_{t-1} which cover all blue vertices. Without loss of generality, we assume that every C_i is a full chain, i.e. a chain on $N + 1$ vertices of the form $\emptyset, \{a_1\}, \{a_1, a_2\}, \dots, \{a_1, \dots, a_N\} = \mathbf{Z}$. Then we say that each C_i , $i \in [t - 1]$, corresponds to the unique linear ordering τ_i of \mathbf{Z} given by $a_1 <_{\tau_i} a_2 <_{\tau_i} \dots <_{\tau_i} a_N$. Now we apply Lemma 20 to the collection of linear orderings τ_i , $i \in [t - 1]$, and obtain three distinct elements $x, y, z \in \mathbf{Z}$ such that for every $i \in [t - 1]$, $x <_{\tau_i} y <_{\tau_i} z$ or $z <_{\tau_i} y <_{\tau_i} x$. In particular, if for some blue vertex $Z \in \mathcal{Q}(\mathbf{Z})$ we have $x \in Z$ and $z \in Z$, then also $y \in Z$. This is because Z is covered by chain C_j for some $j \in [t - 1]$ and in the corresponding linear ordering τ_j either $y <_{\tau_j} z$ or $y <_{\tau_j} x$.

Now assume towards a contradiction that there is no red copy of Q_n in $\mathcal{Q}(\mathbf{Z})$, then let $\mathbf{Y} = \{x, y, z\}$ and let $\mathbf{X} = \mathbf{Z} \setminus \mathbf{Y}$. Let τ be the linear ordering of \mathbf{Y} given by $x <_{\tau} z <_{\tau} y$. Applying Lemma 13 we obtain that there exists a chain containing a blue vertex of the form $X \cup \{x, z\}$ for some $X \subseteq \mathbf{X}$. Then $y \notin X$, so this blue vertex contains both x and z but does not contain y , which is a contradiction. \square

4.3. Proofs of Theorem 6 and Corollary 7

If t is large in terms of n , an improved lower bound on $R(A_t, Q_n)$ holds, which is shown using a layered construction, i.e. a coloring of the hosting lattice where each layer is colored monochromatically.

Proof of Theorem 6. Let $n, r, t \in \mathbb{N}$ with $t > \binom{n+2r+1}{r}$. Let $\mathcal{Q} = \mathcal{Q}(\mathbf{Z})$ be the Boolean lattice on an arbitrary ground set \mathbf{Z} on $n + 2r + 1$ elements. Consider the following layered blue/red coloring of \mathcal{Q} : Color all vertices $X \in \mathcal{Q}$ such that $|X| \leq r$ or $|X| \geq n + r + 1$ in blue and all other vertices in red. We see that \mathcal{Q} consists of $n + 2r + 2$ layers of which $2r + 2$ are colored monochromatically blue and all remaining n layers are colored monochromatically red. Since $h(Q_n) = n + 1$, there is no red copy of Q_n in this coloring. It remains to show that there is no blue copy of A_t in \mathcal{Q} .

We fix a symmetric chain decomposition of \mathcal{Q} as provided by Theorem 12. Let Γ be the family of only those symmetric chains which contain a vertex of size exactly r . For any vertex X of size at most r or at least $n + r + 1$, there is some chain $C(X)$ in the decomposition which covers X . Using the properties of a symmetric chain we obtain $C(X) \in \Gamma$, thus the chains in Γ cover all vertices of size at most r or at least $n + r + 1$. Thus all blue vertices are covered by chains in Γ , whereat $|\Gamma| = \binom{n+2r+1}{r} < t$. Then Dilworth's theorem implies that there is no blue copy of A_t in \mathcal{Q} . \square

Proof of Corollary 7. The upper bound is obtained from the general bound by Axenovich and Walzer [1] and (*) in Section 1.2. For given $n \geq 3$ and $t \geq 2$, let r be the largest non-negative integer with $t > \binom{n+2r+1}{r}$. In Theorem 6 we showed that $R(A_t, Q_n) \geq n + 2r + 2$. Now we bound r in terms of n and t . Note that by the maximality of r ,

$$t \leq \binom{n + 2r + 3}{r + 1} \leq \left(\frac{e(n + 2r + 3)}{r + 1} \right)^{r+1} \leq \left(\frac{en}{r + 1} + 3e \right)^{r+1} \leq (2en)^{r+1}.$$

Thus, $r + 1 \geq \frac{\log t}{\log(2e) + \log n} \geq \frac{\log t}{3 + \log n}$. \square

5. Proofs of Theorems 8 and 9

Proof of Theorem 8. The lower bound on $R(C_{t_1}, Q_n)$ follows from Theorem 1. For the upper bound, fix an arbitrary blue/red coloring of the Boolean lattice $\mathcal{Q}^1 := \mathcal{Q}([n + t_1 - 1])$. By Corollary 14 there is a blue copy of C_{t_1} or a red copy of Q_n in \mathcal{Q}^1 , so $R(C_{t_1}, Q_n) \leq n + t_1 - 1$.

Next we consider the poset Ramsey number $R(C_{t_1, t_2}, Q_n)$. Let $N = n + t_1 + 1$. In order to prove that $R(C_{t_1, t_2}, Q_n) \leq N$, we consider an arbitrarily blue/red colored Boolean lattice $\mathcal{Q}^2 := \mathcal{Q}([N])$ which contains no red copy of Q_n . We show that there is a blue copy of C_{t_1, t_2} in this coloring. Corollary 14 yields the existence of a blue chain C of length $t_1 + 2$, say on vertices $Z_0 \subset Z_1 \subset \dots \subset Z_{t_1+1}$. Consider the subposet C' of C on vertices Z_1, \dots, Z_{t_1} , i.e. obtained by discarding the minimum and maximum vertex of C . The poset C' is a chain of length t_1 . Note that there exists an element $a \in Z_1$ since Z_1 has a proper subset Z_0 . Similarly, we find an element $b \in [N] \setminus Z_{t_1}$. We obtain that $\{a\} \subseteq Z_1 \subset \dots \subset Z_{t_1} \subseteq [N] \setminus \{b\}$.

We consider the subposet $\mathcal{Q}^3 := \{Z \in \mathcal{Q}^2 : b \in Z, a \notin Z\}$, see Fig. 3. Note that \mathcal{Q}^3 is a Boolean lattice of dimension $N - 2 = n + t_1 - 1$ with a blue/red coloring induced by the coloring of \mathcal{Q}^2 . Then \mathcal{Q}^3 contains no red copy of Q_n , so since $R(C_{t_2}, Q_n) = n + t_2 - 1 \leq n + t_1 - 1$ there is a blue copy \mathcal{D} of C_{t_2} in \mathcal{Q}^3 . For every two $X \in C'$ and $Y \in \mathcal{D}$, we know that $a \in X \setminus Y$ and $b \in Y \setminus X$, thus $X \parallel Y$. We conclude that the vertices of C' and \mathcal{D} induce a blue copy of C_{t_1, t_2} , so $R(C_{t_1, t_2}, Q_n) \leq N$.

It remains to show that $R(C_{t_1, t_2}, Q_n) \geq N = n + t_1 + 1$. We verify this lower bound by introducing a layered coloring of $\mathcal{Q}^4 := \mathcal{Q}([N - 1])$ which neither contains a blue copy of C_{t_1, t_2} nor a red copy of Q_n . Consider the coloring of \mathcal{Q}^4 where layer 0 and layer $N - 1$ (i.e. both one-element layers) are monochromatically blue, $t_1 - 1$ arbitrary additional layers are blue, and all remaining $N - (t_1 + 1) = n$ layers are red.

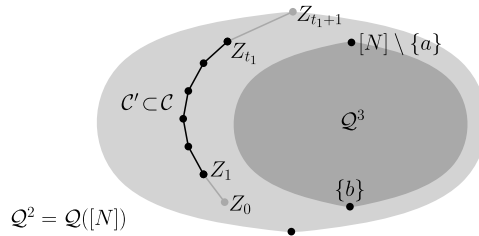


Fig. 3. C' and Q^3 are parallel.

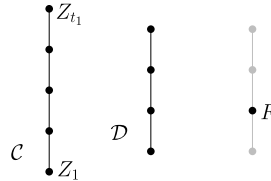


Fig. 4. Chains C and D and vertex F in a copy of C_{t_1, t_2, t_3} .

Since Q_n has height $n + 1$, there is no monochromatic red copy of Q_n in this coloring. Assume that there is a blue copy \mathcal{P} of C_{t_1, t_2} in Q^4 . Note that $\emptyset \notin \mathcal{P}$ and $[N - 1] \notin \mathcal{P}$, because each of these two vertices is comparable to all other vertices of Q^4 and in \mathcal{P} no vertex comparable to all other vertices of \mathcal{P} . Other than \emptyset and $[N - 1]$, Q^4 has only $t_1 - 1$ layers containing blue vertices, but $h(\mathcal{P}) = h(C_{t_1, t_2}) = t_1$, so there can not be a blue copy \mathcal{P} of C_{t_1, t_2} . \square

We remark that the upper bound on $R(C_{t_1}, Q_n)$ alternatively follows from Lemma 4 in Axenovich and Walzer [1].

Proof of Theorem 9. It is a consequence of Theorem 8 and Corollary 10 that

$$n + t_1 + 1 = R(C_{t_1, t_2}, Q_n) \leq R(C_{t_1, t_2, t_3}, Q_n) \leq n + t_1 + \alpha(3) - 1 = n + t_1 + 2.$$

Let $N = n + t_1 + 1$. Now suppose that $t_1 \geq t_2 + 2$. We shall show that $R(C_{t_1, t_2, t_3}, Q_n) \leq N$. Fix an arbitrary blue/red coloring of $Q^1 := Q([N])$ which contains no red copy of Q_n . We shall find a blue copy of C_{t_1, t_2, t_3} in Q^1 . By Corollary 14 there is a blue chain C of length $t_1 + 2$. Let C have vertices $Z_0 \subset \dots \subset Z_{t_1+1}$. Consider the subposet C' of C on vertices Z_1, \dots, Z_{t_1} , which is a chain of length t_1 . Note that $Z_1 \neq \emptyset$ and $Z_{t_1} \neq [N]$, thus we find some $a, b \in [N]$ such that $\{a\} \subseteq Z_1 \subseteq \dots \subseteq Z_{t_1} \subseteq [N] \setminus \{b\}$.

Now consider the subposet $Q^2 := \{Z \in Q^1 : b \in Z, a \notin Z\}$, which is a Boolean lattice of dimension $N - 2 \geq n + t_2 + 1$ with a blue/red coloring induced by Q^1 . In particular, Q^2 contains no red copy of Q_n , so Theorem 8 yields that there is a blue copy \mathcal{P} of C_{t_2, t_3} . Thus, for every two $Z \in \mathcal{P}$ and $U \in C'$ we know that $a \in U \setminus Z, b \in Z \setminus U$, so $Z \parallel U$. Consequently, the vertices of \mathcal{P} and C' induce a blue copy of C_{t_1, t_2, t_3} .

From now on, suppose that $t_1 \leq t_2 + 1$. It remains to show that $R(C_{t_1, t_2, t_3}, Q_n) > n + t_1 + 1 = N$. If $t_1 = 1$, this is a consequence of Theorem 5, so say that $t_1 \geq 2$. We construct a blue/red coloring of $Q^3 := Q([N])$ which neither contains a red copy of Q_n nor a blue copy of C_{t_1, t_2, t_3} as follows. Color all vertices in the four layers $\ell \in \{0, 1, N - 1, N\}$ monochromatically in blue. Color $t_1 - 2$ arbitrary additional layers blue and all remaining $(N + 1) - 4 - (t_1 - 2) = n$ layers red. Clearly, this coloring contains no red copy of Q_n since $h(Q_n) = n + 1$. Assume for a contradiction that there is a blue copy \mathcal{P} of C_{t_1, t_2, t_3} in Q^3 . In \mathcal{P} denote a blue chain of length t_1 by C , say on vertices $Z_1 \subset \dots \subset Z_{t_1}$. Furthermore there is a chain D of length t_2 in \mathcal{P} which is parallel to C , see Fig. 4. Let F be a vertex of \mathcal{P} which is neither in C nor in D , i.e. F is incomparable to every vertex in C and D .

It is easy to see that neither \emptyset nor $[N]$ are in \mathcal{P} , because both of these vertices are comparable to every other vertex in Q^3 which does not occur in \mathcal{P} . Excluding the two vertices \emptyset and $[N]$, there are exactly t_1 layers containing blue vertices, including layer 1 and layer $N - 1$.

Recall that C is a blue chain of length t_1 , therefore the smallest vertex Z_1 of C is in layer 1, while the largest vertex Z_{t_1} of C is in layer $N - 1$. Thus we find $a, b \in [N]$ such that $Z_1 = \{a\}$ and $Z_{t_1} = [N] \setminus \{b\}$. Let $Q^4 := \{Z \in Q^3 : b \in Z, a \notin Z\}$, this poset is a Boolean lattice. Since F is incomparable to every vertex in C , we obtain that $a \notin F$ and $b \in F$. Thus $F \in Q^4$ and in particular $F \supseteq \{b\}$ and $F \subseteq [N] \setminus \{a\}$. Similarly, $D \subseteq Q^4$.

The coloring of Q^3 induces a layered coloring of Q^4 where exactly t_1 of the layers are blue. Two of these blue layers in Q^4 are the one-element layers given by $\{b\}$ and $[N] \setminus \{a\}$. Since the chain D has height $h(D) \geq t_1 - 1$, either $\{b\} \in D$ or $[N] \setminus \{a\} \in D$. This is a contradiction, because F is incomparable to every vertex in D but both $\{b\}$ and $[N] \setminus \{a\}$ are comparable to F . \square

6. Summary of known approaches

6.1. Proof techniques

In order to give an insight on research of off-diagonal poset Ramsey numbers, we briefly survey known proof methods for bounding $R(P, Q_n)$ when P is fixed. For some easy-to-analyze posets, e.g. several trivial posets, no advanced tools are required to get an exact bound. An example for this is our proof of Theorem 8 in Section 5, which only relies on the basic observation Corollary 14. Many other posets require more involved proof techniques, and there are three methods which provides **upper bounds** on $R(P, Q_n)$.

A *blob approach* is used in particular for Boolean lattices $P = Q_m$. The idea behind this method is to consider a blue/red colored hosting lattice in which we define many *blobs*, i.e. small sublattices that are pairwise disjoint, arranged in a product structure. If any blob is monochromatically blue, we obtain a blue copy of P . Otherwise, we find a red copy of Q_n by choosing one red vertex in each blob. This proof technique was introduced in a related setting by Kierstead and Trotter [16], and provided a first general upper bound for poset Ramsey numbers in Axenovich and Walzer [1]. For a more refined version, see Lu and Thompson [17].

Grósz, Methuku and Tompkins [13] introduced another proof technique, the *chain approach*. Here we consider the large number of blue chains obtained by the Chain Lemma, Lemma 13, and use counting arguments to force the existence of a monochromatically blue copy of P . Exemplary for this method are the proof of Theorem 3 in Section 3 and Theorem 1 in Winter [23].

A more technical proof method is the *blocker approach*. It tries to strengthen the Chain Lemma in order to get a precise picture on how the blue chains in the coloring are located relative to each other, in fact they are grouped into structures called *blockers*. This proof technique is described in another part of this series, see Axenovich and the author [3].

A general **lower bound** on $R(P, Q_n)$ for every P is obtained from a trivial layered coloring, see Theorem 1. For bounding $R(P, Q_n)$ from below, most constructions slightly refine a layered coloring according to the fixed P , see e.g. the proofs of Theorems 8 and 9 or Grósz, Methuku and Tompkins [13]. Most known proofs for the lower bound strengthen the trivial lower bound just by a constant. The only non-marginal improvement of the trivial lower bound is given by Axenovich and the author [2] building on the structural insights obtained from the *blocker approach*.

6.2. Open problems

In our series of papers we discussed the asymptotic behavior in the off-diagonal poset Ramsey setting $R(P, Q_n)$ for P fixed and n large. We close this study by collecting some open problems. For trivial P , i.e. posets containing neither V_2 nor Λ_2 , Theorem 1 bounds $R(P, Q_n) = n + \Theta(1)$ tight up to an additive constant. We improved the bounds on this constant in Corollary 10, but a general exact bound for trivial P with $w(P) \geq 4$ remains to be determined. For non-trivial P the picture is more unclear. Theorem 1 provides that $n + \frac{1}{15} \frac{n}{\log n} \leq R(P, Q_n) \leq c_P \cdot n$. Axenovich and the author [2] conjectured that the true value of $R(P, Q_n)$ is closer to the lower bound.

Conjecture 21 ([2]). For every fixed poset P , $R(P, Q_n) = n + o(n)$.

The lower bound $R(P, Q_n) = n + \Omega(\frac{n}{\log n})$ is known to be asymptotically tight in the two leading additive terms for some non-trivial P , i.e. $R(P, Q_n) = n + \Theta(\frac{n}{\log n})$. We say that such a poset P is *modest*. Note that V_2 and Λ_2 are modest, and every non-trivial poset contains either V_2 or Λ_2 as a subposet. Belonging to the class of modest posets are e.g. subdivided diamonds, see Theorem 3, complete multipartite posets [23] and the N-shaped poset \times [3]. Notably, it remains open whether there exists a non-trivial poset which is *not* modest.

Conjecture 22. There is a fixed poset P with $R(P, Q_n) = n + \omega\left(\frac{n}{\log n}\right)$.

Known modest posets differ in various poset parameters, for example $SD_{t,t}$ has large height and $K_{1,t}$ has large width. However, every known modest poset has order dimension 2. The *order dimension* of P is the minimal number of linear orderings such that P is the intersection of these linear orderings. Natural candidates for proving Conjecture 22 are either Q_3 or the standard example S_3 , the 6-element poset induced by layers 1 and 2 of Q_3 . Both posets have order dimension 3.

Determining $R(Q_m, Q_n)$ for $m \in \mathbb{N}$ is one of the most interesting open problems regarding poset Ramsey numbers. While well-understood for $m \in \{1, 2\}$, the asymptotic behavior of $R(Q_m, Q_n)$ for fixed $m \geq 3$ is only bounded up to a constant linear factor, see Lu and Thompson [17]. Conjectures 21 and 22 are equivalent to the following.

Conjecture 23. For every fixed $m \in \mathbb{N}$, $R(Q_m, Q_n) = n + o(n)$. Furthermore there is a fixed integer $m \in \mathbb{N}$ with $R(Q_m, Q_n) = n + \omega\left(\frac{n}{\log n}\right)$.

Note that Conjecture 23 is a strengthening of an open conjecture raised by Lu and Thompson [17].

Declaration of competing interest

The author has no competing interests to declare.

Data availability

No data was used for the research described in the article.

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