Arch. Math. 122 (2024), 559–573

0003-889X/24/050559-15
published online April 12, 2024
https://doi.org/10.1007/s00013-024-01970-6

Archiv der Mathematik

Minimal periods for semilinear parabolic equations

Gerd Herzog and Peer Christian Kunstmann

Abstract. We show that, if $-A$ generates a bounded holomorphic semigroup in a Banach space X, $\alpha \in [0,1)$, and $f : D(A) \to X$ satisfies $\|f(x) - f(y)\| \le L\|A^\alpha(x-y)\|$, then a non-constant T-periodic solution of the equation $\dot{u} + Au = f(u)$ satisfies $LT^{1-\alpha} \ge K_\alpha$ where $K_\alpha > 0$ is a constant depending on α and the semigroup. This extends results by Robinson and Vidal-Lopez, which have been shown for self-adjoint operators $A \ge 0$ in a Hilbert space. For the latter case, we obtain - with a conceptually new proof - the optimal constant K_α, which only depends on α, and we also include the case $\alpha = 1$. In Hilbert spaces H and for $\alpha = 0$, we present a similar result with optimal constant where Au in the equation is replaced by a possibly unbounded gradient term $\nabla_H \mathscr{E}(u)$. This is inspired by applications with bounded gradient terms in a paper by Mawhin and Walter.

Mathematics Subject Classification. 34C25, 35B10, 34G20, 35K90, 47D06.

Keywords. Periodic orbits, Minimal period, Semilinear evolution equations, Gradient systems.

1. Introduction and main results. In this paper, we study periodic solutions of equations

$$\dot{u}(t) + Au(t) = f(u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $-A$ is the generator of a bounded analytic semigroup in a complex Banach space $(X, \|\cdot\|)$, $X \ne \{0\}$, with domain $D(A)$ and $f : D(A) \to X$ is a function which is Lipschitz continuous in the sense that

$$\|f(x) - f(y)\| \le L\|A^\alpha(x-y)\|, \qquad x, y \in D(A), \tag{1.2}$$

for a fixed $\alpha \in [0,1)$ and some constant $L \ge 0$. Here A^α denotes the fractional power of A of order α. We shall relate the minimal period of a non-constant T-periodic solution u of (1.1) to the Lipschitz constant L in (1.2).

The ODE case $A = 0$ and $\alpha = 0$, i.e.,

$$\dot{u}(t) = f(u(t)), \quad t \in \mathbb{R}, \quad \text{where } \|f(x) - f(y)\| \leq L\|x - y\|, \quad x, y \in X, \tag{1.3}$$

has been addressed in several papers: If u is a T-periodic solution to (1.3) and $LT < 6$, then u has to be constant [2]. The constant 6 is known to be optimal in general Banach spaces [10]. In a Hilbert space, a T-periodic solution u is constant if $LT < 2\pi$ and 2π is optimal [2,8,17]. These results rely on the estimates in Lemma 3.1 below, in particular, the Hilbert space result uses Wirtinger's inequality in Lemma 3.1(a). L^p-versions of Wirtinger's inequality with optimal constants have been established in [4]. They have been used in [11] to improve the constant 6 in case $X = L^p(\Omega)$ for p in a certain symmetric interval around 2, which is strictly contained in $(\frac{4}{3}, 4)$. For further details on the L^p-case, we refer to Remark 4.1(c) below. In strictly convex Banach spaces X, a T-periodic solution u to (1.3) is constant if $LT \leq 6$; see [11].

For the special case that $X = H$ is a Hilbert space and A is self-adjoint with $A \geq 0$, the problem has been studied in [13] under the additional restrictions $\alpha \in [0, \frac{1}{2})$ and A invertible with A^{-1} compact. These additional restrictions have been removed in [14]. It is shown in [13,14] that there exists a constant $K_\alpha > 0$ only depending on α such that $LT^{1-\alpha} < K_\alpha$ implies that a T-periodic solution u to (1.1) is constant. The proofs given there rely in an essential way on properties of spectral projections for A provided by the spectral theorem and study the mild formulation (4.1) of the abstract Cauchy problem corresponding to (1.1), also known as Duhamel's principle or variation-of-constants formula. In a remark [14, p. 4286], conditions are given that allow to extend this method of proof to Banach spaces. These conditions involve existence and certain estimates for spectral projections of the operator A and seem rather restrictive. Hence, the extension to the situation "when A is a sectorial operator, as treated by Henry [7]" ([13, p. 402]) is still missing.

The new contributions to the problem in the present paper are the following.

- We modify the argument in [14] in such a way that it works in arbitrary Banach spaces X under the sole assumption that $-A$ generates a bounded analytic semigroup. In particular, no assumptions on spectral projections are needed; see Theorem 1.1. We thus provide the extension conjectured on [13, p. 402].
- In case $X = H$ is a Hilbert space and A is self-adjoint with $A \geq 0$, we present a new argument, which yields the optimal constants for the result in [14]. Our proof is based on refined energy type estimates, inspired by the applications in [9], and we can also include the limit case $\alpha = 1$. See Theorem 1.2 and Sect. 2.
- For $X = H$ a real Hilbert space and $\alpha = 0$, we replace the term $Au(t)$ in (1.1) by a possibly unbounded nonlinear gradient term $\nabla_H \mathscr{E}(u(t))$. This is inspired by finite-dimensional applications in [9]. We get the same bound as for the case $\alpha = 0$ in Theorem 1.2, namely $LT < 2\pi$; see Theorem 1.4.

We recall that, if $-A$ is the generator of a bounded analytic semigroup, one can define fractional powers A^γ of A for any $\gamma \geq 0$ and has

$$c_\gamma := \sup_{t>0} \|t^\gamma A^\gamma e^{-tA}\| < \infty, \qquad \gamma \geq 0; \tag{1.4}$$

see, e.g., [6,12]. We denote by $[D(A)]$ the space $D(A)$ equipped with the graph norm. Our main results read as follows.

Theorem 1.1. *Let $-A$ be the generator of a bounded analytic semigroup in a Banach space X. Let $\alpha \in [0,1)$ and suppose that $f : D(A) \to X$ satisfies (1.2) for some $L \geq 0$. If $T \in (0,\infty)$ and $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, [D(A)])$ is a T-periodic solution of (1.1) with*

$$LT^{1-\alpha} < \left(1 - \frac{c_1}{6k}\right)\left(\frac{c_\alpha}{6k^\alpha} + \frac{c_\alpha k^{1-\alpha}}{1-\alpha}\right)^{-1} \tag{1.5}$$

for some $k \in \mathbb{N}$ with $k > \frac{c_1}{6}$, then u is constant. In a Hilbert space, the conclusion holds if

$$LT^{1-\alpha} < \left(1 - \frac{c_1}{2\pi k}\right)\left(\frac{c_\alpha}{2\pi k^\alpha} + \frac{c_\alpha k^{1-\alpha}}{1-\alpha}\right)^{-1} \tag{1.6}$$

for some $k \in \mathbb{N}$ with $k > \frac{c_1}{2\pi}$. Here we understand that the right hand side in (1.5) or (1.6) is $= \infty$ if $c_\alpha = 0$.

The proof of Theorem 1.1 is inspired in principle by the approach in [14]. In case $X = H$ is a Hilbert space and A is self-adjoint in H with $A \geq 0$, then $c_0 = 1$ and, for $A \neq 0$, the spectral theorem (see, e.g., [15]) allows us to calculate

$$c_\gamma = \sup_{t\geq 0, \lambda\in\sigma(A)} (t\lambda)^\gamma e^{-t\lambda} = \sup_{s\geq 0} \left(s^\gamma e^{-s}\right) = \gamma^\gamma e^{-\gamma}, \quad \gamma > 0,$$

where $\sigma(A)$ denotes the spectrum of A, which by $A \geq 0$ satisfies $\sigma(A) \subseteq [0,\infty)$. Hence $c_1 = e^{-1} < 1$, we can take $k = 1$, and the condition (1.6) reads

$$LT^{1-\alpha} < \left(1 - \frac{1}{2\pi e}\right)\left(\alpha^\alpha e^{-\alpha}\left(\frac{1}{2\pi} + \frac{1}{1-\alpha}\right)\right)^{-1}. \tag{1.7}$$

But one can do better.

Theorem 1.2. *Let $X = H$ be a Hilbert space and A be self-adjoint in H with $A \geq 0$. Let $\alpha \in [0,1]$ and suppose that $f : D(A) \to X$ satisfies (1.2) for some $L \geq 0$. If $T \in (0,\infty)$ and $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, [D(A)])$ is a T-periodic solution of (1.1) with*

$$LT^{1-\alpha} < \frac{(2\pi)^{1-\alpha}}{\sqrt{\alpha^\alpha (1-\alpha)^{1-\alpha}}}, \tag{1.8}$$

then u is constant. The bound (1.8) is sharp.

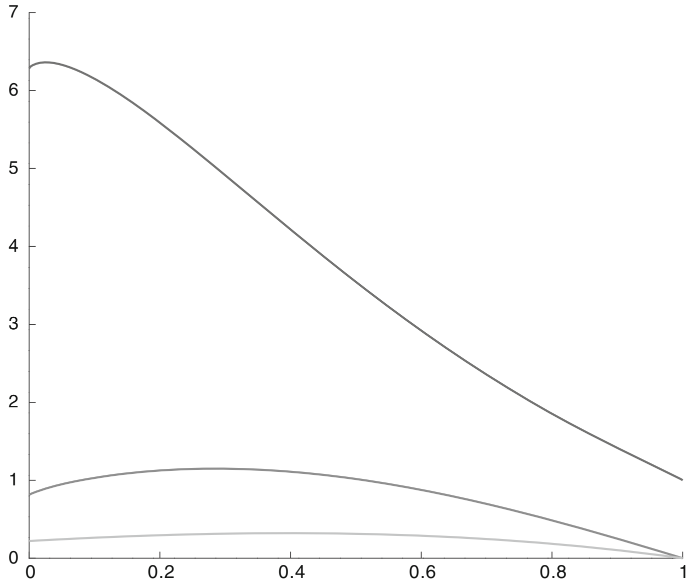

Figure 1. Different bounds for $LT^{1-\alpha}$ for a self-adjoint operator $A \geq 0$ in a Hilbert space H as a function of $\alpha \in [0, 1]$. The upper curve is the bound in (1.8), the intermediate curve is the bound in (1.7), and the lower curve is the bound in (1.9) from [14]

Remark 1.3. In the situation of Theorem 1.2, one can compare the constants in (1.7) and in (1.8) with the constant in [14, p. 4286] where (recalling γ from p. 4285) the corresponding condition reads

$$LT^{1-\alpha} < \left(2^{1-2\alpha} + \frac{\alpha^{\alpha} e^{-\alpha}}{(1-\alpha)(1-e^{-1/2})}\right)^{-1}; \tag{1.9}$$

see Fig. 1.

We can replace $Au(t)$ in (1.1) by a gradient term $\nabla_H \mathscr{E}(u(t))$. Here, H is a real Hilbert space, V is a Banach space that is densely and continuously embedded into H, and $\mathscr{E} : V \to \mathbb{R}$ is continuously differentiable. For the precise definition of the H-gradient $\nabla_H \mathscr{E}$ and some remarks on existence, we refer to Sect. 6. We look at periodic solutions of the equation

$$\dot{u}(t) + \nabla_H \mathscr{E}(u(t)) = f(u(t)), \quad t \in \mathbb{R}, \tag{1.10}$$

where $f : H \to H$ is Lipschitz continuous, i.e., there exists a constant $L \geq 0$ such that

$$\|f(x) - f(y)\|_H \leq L\|x - y\|_H, \qquad x, y \in H. \tag{1.11}$$

Then we have the following result.

Theorem 1.4. *In the situation described above, let $T \in (0, \infty)$ and let $u \in C^1(\mathbb{R}, V)$ be a T-periodic solution of (1.10). If $LT < 2\pi$, then u is constant. The bound 2π is sharp.*

Remark 1.5. (a) The regularity assumptions on u in our results are made for simplicity. We do not say much on existence in this paper. From the proofs, one can see that the natural assumption in Theorem 1.1 is $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}; X) \cap L^1_{\mathrm{loc}}(\mathbb{R}; [D(A)])$ for the statement in Banach spaces. For the Hilbert space statement in Theorem 1.1, the natural assumption here is $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; X) \cap L^2_{\mathrm{loc}}(\mathbb{R}; [D(A)])$, and in Theorem 1.2, it is $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; H) \cap L^2_{\mathrm{loc}}(\mathbb{R}; [D(A)])$. In Theorem 1.4, we can relax the condition to $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; V)$, provided the derivative $\mathscr{E}' : V \to V'$ maps bounded sets into bounded sets; we refer to Remark 6.1.

(b) A special case of the situation in Theorem 1.4 is given by $\mathscr{E}(v) = \|A^{1/2}v\|^2_H$ where A is a self-adjoint operator in H with $A \geq 0$ and $V = D(A^{1/2})$. However, in this case neither the regularity assumptions on u in Theorem 1.2 and Theorem 1.4 nor their respective relaxations in part (a) are comparable.

We remark that in [13,14], applications are given to the two-dimensional Navier–Stokes equation with periodic boundary conditions. In this context, we also refer to the recent existence results on time-periodic solutions in [1,5].

The paper is organized as follows. In Sect. 2, we show optimality of the bounds in Theorems 1.2 and 1.4. In Sect. 3, we collect the basic inequalities we shall use. Then we present the proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.4 in Sects. 4, 5, and 6, respectively.

2. Optimality in Theorem 1.2 and Theorem 1.4. We start with examples for the cases $\alpha = 0$ and $\alpha = 1$ in Theorem 1.2.

Example 2.1. (Case $\alpha = 0$) Let $A = 0$ in $X = H = \mathbb{C}$, $f(x) = 2\pi i x$. Then f is Lipschitz continuous with $L = 2\pi$ and the ordinary differential equation $\dot{u}(t) = 2\pi i u(t)$ has the 1-periodic solution $u(t) = e^{2\pi i t}$, $t \in \mathbb{R}$. Hence, for $\alpha = 0$, the condition $LT < 2\pi$ in Theorem 1.2 is optimal.

Example 2.2. For $\alpha = 1$, the condition $L < 1$ in Theorem 1.2 is optimal: Let $X = H = \mathbb{C}$ and $Ax = ax$ where $a > 0$. Let $f(x) = (a + ib)x$ where $b = \varepsilon a$ and $\varepsilon > 0$. Then the ordinary differential equation

$$\dot{u}(t) + au(t) = f(u(t)) = (a + ib)u(t) \iff \dot{u}(t) = ibu(t)$$

has the solution $u(t) = e^{ibt}$, which is T-periodic for $T = \frac{2\pi}{b} = \frac{2\pi}{\varepsilon a}$, and f satisfies (1.2) for $\alpha = 1$ with constant $L = \frac{\sqrt{a^2+b^2}}{a} = \sqrt{1+\varepsilon^2}$ which tends to 1 as $\varepsilon \to 0$. Observe that, by adjusting a for fixed $\varepsilon > 0$, we can arrange for any period $T > 0$. It seems unclear what happens for $L = 1$.

As a preparation for our examples on optimality in Theorem 1.2 for $\alpha \in (0, 1)$, we note the following consequence for linear f.

Corollary 2.3. *Let $X = H$ be a Hilbert space, A be self-adjoint with $A \geq 0$, U be unitary in H, and $L > 0$. If $\alpha \in (0,1)$ and $T > 0$ are such that (1.8) holds, then purely imaginary eigenvalues of $A + LUA^\alpha$ belong to $i(-\frac{2\pi}{T}, \frac{2\pi}{T})$.*

Proof. The linear operator $B : D(A) \to X$ given by $B = -LUA^\alpha$ satisfies (1.2) in place of f. Let $\lambda \in \mathbb{R}$ be such that $i\lambda$ is an eigenvalue of $A - B$ with eigenvector x. We may assume that $\lambda \neq 0$. Then $u(t) = e^{-i\lambda t}x$ is a non-constant $\frac{2\pi}{|\lambda|}$-periodic solution of $\dot{u}(t) + Au(t) = Bu(t)$. By Theorem 1.2 and (1.8), we infer that $\frac{2\pi}{|\lambda|} > T$, i.e., $|\lambda| < \frac{2\pi}{T}$. □

Example 2.4. Let $\alpha \in (0,1)$, $X = H = \mathbb{C}$, $Ax = ax$ where $a > 0$, $L = \frac{a^{1-\alpha}}{\sqrt{\alpha}}$, $f(x) = -Le^{i\varphi}a^\alpha x$, and $\lambda = a\sqrt{\frac{1-\alpha}{\alpha}}$. Then, f satisfies (1.2) and we have

$$|a - i\lambda|^2 = a^2 + \lambda^2 = a^2\left(1 + \frac{1-\alpha}{\alpha}\right) = \frac{a^2}{\alpha} = (La^\alpha)^2,$$

and we find $\varphi \in \mathbb{R}$ such that $a - i\lambda = -Le^{i\varphi}a^\alpha$ which means that $a + Le^{i\varphi}a^\alpha = i\lambda$. But then $u(t) = e^{-i\lambda t}$ defines a $\frac{2\pi}{\lambda}$-periodic solution of (1.1), and for $T = \frac{2\pi}{\lambda} = \frac{2\pi}{a}\sqrt{\frac{\alpha}{1-\alpha}}$, we have

$$(LT^{1-\alpha})^2 = \frac{a^{2(1-\alpha)}}{\alpha}\frac{(2\pi)^{2(1-\alpha)}}{a^{2(1-\alpha)}}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} = \frac{(2\pi)^{2(1-\alpha)}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}. \tag{2.1}$$

Thus the bound in (1.8) is sharp.

The example can be modified to work in the real Hilbert space $\mathbb{R}^2$. We simply use the representation of complex numbers $x + iy$ as matrices $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. To be more precise, let $I \in \mathbb{R}^{2\times 2}$ denote the identity matrix and set $A := aI$ where $a > 0$. Let L, λ be as before and $f(x) = -LU_\varphi(a^\alpha x)$, $x \in \mathbb{R}^2$, where $U_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then f satisfies (1.2) and we have

$$A + LU_\varphi(a^\alpha I) = a(I + \frac{1}{\sqrt{\alpha}}U_\varphi), \quad \sigma(A + LU_\varphi(a^\alpha I)) = \left\{a\left(1 + \frac{1}{\sqrt{\alpha}}e^{\pm i\varphi}\right)\right\}.$$

As before, we find φ such that $a(1 + \frac{1}{\sqrt{\alpha}}e^{i\varphi}) = ia\sqrt{\frac{1}{\alpha} - 1} =: i\lambda$. Finally, $u(t) = \begin{pmatrix} \cos(\lambda t) \\ -\sin(\lambda t) \end{pmatrix}$ is a $\frac{2\pi}{\lambda}$-periodic solution of (1.1), and (2.1) holds as before.

Finally, we present an example for sharpness of the bound in Theorem 1.4.

Example 2.5. We rephrase Example 2.1 in $H = \mathbb{R}^2$, a real Hilbert space. Let $V = \mathbb{R}^2$, $\mathscr{E}(x) = 0$, $f(x) = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$. Then f satisfies (1.2) for $\alpha = 0$ and $L = 2\pi$. We have $\nabla_H \mathscr{E}(x) = 0$ for all $x \in \mathbb{R}^2$. Hence $u(t) = \begin{pmatrix} \cos(2\pi t) \\ -\sin(2\pi t) \end{pmatrix}$ defines a 1-periodic solution of (1.10).

Concerning the context of Remark 6.1, observe that $\mathscr{E}'$ maps bounded sets of $V = \mathbb{R}^2$ into bounded sets of $V' = \mathbb{R}^2$.

3. Basic inequalities. The following lemma is a basic tool in the proofs.

Lemma 3.1. *Let $v : \mathbb{R} \to X$ be continuously differentiable and T-periodic.*

(a) If $X = H$ is a Hilbert space and $\int_0^T v(r)\,dr = 0$, then

$$\left(\int_0^T \|v(r)\|^2\,dr\right)^{1/2} \le \frac{T}{2\pi}\left(\int_0^T \|\dot v(r)\|^2\,dr\right)^{1/2}.$$

(b) In the general Banach space case, we have

$$\int_0^T\int_0^T \|v(t)-v(s)\|\,ds\,dt \le \frac{T}{6}\int_0^T\int_0^T \|\dot v(t)-\dot v(s)\|\,ds\,dt.$$

We include a proof for (a), which is called Wirtinger's inequality, and refer to [2] or [10] for the proof of (b).

Proof. (a): By scaling, it is sufficient to study the case $T = 1$. We expand v in a Fourier series

$$v(t) = \sum_{j\in\mathbb{Z}\setminus\{0\}} c_j e^{2\pi ijt}, \qquad \dot v(t) = \sum_{j\in\mathbb{Z}\setminus\{0\}} 2\pi ij c_j e^{2\pi ijt},$$

and use Plancherel in H, e.g., $\int_0^T \|v(r)\|^2\,dr = \sum_{j\neq 0}\|c_j\|^2$. □

Remark 3.2. If, in the situation of Lemma 3.1, $X = H$ is a Hilbert space, we also have

$$\left(\int_0^T\int_0^T \|v(t)-v(s)\|^2\,ds\,dt\right)^{1/2} \le \frac{T}{2\pi}\left(\int_0^T\int_0^T \|\dot v(t)-\dot v(s)\|^2\,ds\,dt\right)^{1/2}.$$

This follows easily from Lemma 3.1(a).

4. The general Banach space case. We recall the well-known fact that a solution $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, [D(A)])$ of the abstract Cauchy problem

$$\dot u(t) + Au(t) = f(u(t)), \quad t \in \mathbb{R},$$

satisfies, by Duhamel's formula, the equation

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A} f(u(s))\,ds, \quad t \ge 0. \tag{4.1}$$

Proof of Theorem 1.1. Suppose that $u \in C^1(\mathbb{R}, X)\cap C(\mathbb{R}, [D(A)])$ is a solution of (1.1) that is T-periodic. We write, for $s, t \in [0, T]$,

$$\begin{aligned} A^\alpha(u(t)-u(s)) &= A^\alpha e^{-kTA}(u(t)-u(s)) + A^\alpha(I - e^{-kTA})(u(t)-u(s)) \\ &=: v_1(t,s) + v_2(t,s), \end{aligned}$$

where $k \in \mathbb{N}$ is such that $\frac{c_1}{6k} < 1$. Observe that the operator $A^\alpha e^{-kTA}$ is bounded. We shall give an estimate on

$$d_T(u) := \int_0^T \int_0^T \|A^\alpha(u(t) - u(s))\| \, ds \, dt \tag{4.2}$$

from which our result will follow. Clearly,

$$d_T(u) \leq \int_0^T \int_0^T \|v_1(t,s)\| \, ds \, dt + \int_0^T \int_0^T \|v_2(t,s)\| \, ds \, dt =: I_1 + I_2,$$

and we start by applying Lemma 3.1(b) to $v(t) = A^\alpha e^{-kTA} u(t)$. By (1.1), we then obtain

$$\begin{aligned}
\frac{6}{T} I_1 &\leq \int_0^T \int_0^T \|A^\alpha e^{-kTA}(\dot{u}(t) - \dot{u}(s))\| \, ds \, dt \\
&\leq \int_0^T \int_0^T \|A^\alpha e^{-kTA} A(u(t) - u(s))\| \, ds \, dt \\
&\quad + \int_0^T \int_0^T \|A^\alpha e^{-kTA}(f(u(t)) - f(u(s)))\| \, ds \, dt \\
&\leq \frac{c_1}{kT} \int_0^T \|A^\alpha(u(t) - u(s))\| \, ds \, dt \\
&\quad + \frac{c_\alpha}{k^\alpha T^\alpha} \int_0^T \int_0^T \|f(u(t)) - f(u(s))\| \, ds \, dt \\
&\leq \left(\frac{c_1}{kT} + \frac{c_\alpha L}{k^\alpha T^\alpha} \right) \int_0^T \int_0^T \|A^\alpha(u(t) - u(s))\| \, ds \, dt.
\end{aligned}$$

Hence, we have shown

$$I_1 \leq \left(\frac{c_1}{6k} + \frac{c_\alpha}{6k^\alpha} L T^{1-\alpha} \right) d_T(u).$$

In order to get an estimate on I_2, we use (4.1) and the T-periodicity of u to write

$$u(t) = u(t + kT) = e^{-kTA} u(t) + \int_0^{kT} e^{-(kT-r)A} f(u(r+t)) \, dr.$$

Hence, we have

$$(I - e^{-kTA})(u(t) - u(s)) = \int_0^{kT} e^{-(kT-r)A} \left(f(u(r+t)) - f(u(r+s)) \right) dr.$$

For I_2, we thus obtain, by Minkowski's inequality,

$$\begin{aligned}
I_2 &= \int_0^T \int_0^T \|A^\alpha (I - e^{-kTA})(u(t) - u(s))\| \, ds \, dt \\
&\leq \int_0^T \int_0^T \left\| \int_0^{kT} A^\alpha e^{-(kT-r)A} (f(u(r+t)) - f(u(r+s))) \, dr \right\| \, ds \, dt \\
&\leq \int_0^{kT} \int_0^T \int_0^T \left\| A^\alpha e^{-(kT-r)A} (f(u(r+t)) - f(u(r+s))) \right\| \, ds \, dt \, dr \\
&\leq c_\alpha \int_0^{kT} (kT - r)^{-\alpha} \int_0^T \int_0^T \|f(u(r+t)) - f(u(r+s))\| \, ds \, dt \, dr \\
&\leq c_\alpha L \int_0^{kT} r^{-\alpha} \, dr \int_0^T \int_0^T \|A^\alpha (u(t) - u(s))\| \, ds \, dt \\
&= \frac{c_\alpha k^{1-\alpha}}{1-\alpha} L T^{1-\alpha} d_T(u).
\end{aligned}$$

Hence, we have proved

$$d_T(u) \leq \left(\frac{c_1}{6k} + \left(\frac{c_\alpha}{6k^\alpha} + \frac{c_\alpha k^{1-\alpha}}{1-\alpha} \right) L T^{1-\alpha} \right) d_T(u),$$

and conclude $d_T(u) = 0$ if

$$\left(\frac{c_\alpha}{6k^\alpha} + \frac{c_\alpha k^{1-\alpha}}{1-\alpha} \right) L T^{1-\alpha} < 1 - \frac{c_1}{6k}. \tag{4.3}$$

We see that (4.3) implies that $A^\alpha u$ is constant. Then $Au = A^{1-\alpha} A^\alpha u$ is constant and, by (1.2), $f(u)$ is constant. Hence (1.1) implies that also $\dot{u}$ is constant. But since u is periodic, it has to be constant, too.

In case $X = H$ is a Hilbert space, we can run a nearly identical argument, letting

$$d_T(u) = \left(\int_0^T \int_0^T \|A^\alpha (u(t) - u(s))\|^2 \, ds \, dt \right)^{1/2}$$

and using Remark 3.2 to obtain that u is constant if

$$\left(\frac{c_\alpha}{2\pi k^\alpha} + \frac{c_\alpha k^{1-\alpha}}{1-\alpha} \right) L T^{1-\alpha} < 1 - \frac{c_1}{2\pi k}. \tag{4.4}$$

Of course, we have to take $k \in \mathbb{N}$ with $\frac{c_1}{2\pi k} < 1$ here. □

Remark 4.1. If $X = L^p(\Omega)$ for a σ-finite measure space (Ω, μ) and $p \in (1, \infty)$, it is tempting to use

$$d_T(u) = \left(\int_0^T \int_0^T \|A^\alpha(u(t) - u(s))\|^p_{L^p(\Omega)} \, ds \, dt \right)^{1/p}$$
$$= \left\| \left(\int_0^T \int_0^T \left| A^\alpha(u(t) - u(s))(\cdot) \right|^p ds \, dt \right)^{1/p} \right\|_{L^p(\Omega)}$$

and an analogue of Lemma 3.1(a) or (b) in L^p for scalar-valued T-periodic functions in order to improve the constant $\frac{T}{6}$.

The best constant of the L^p-analogue of Lemma 3.1(a) is known (see [4]). This has been used in [11] to give estimates on minimal periods, but only leads to an improvement over the constant $\frac{T}{6}$ for p in an interval $I \ni 2$, which is strictly contained in $(\frac{4}{3}, 4)$ and is symmetric in the sense that $p \in I \Leftrightarrow \frac{p}{p-1} \in I$. For example, the best constant in the L^1-analogue of Lemma 3.1(a) is $\frac{T}{4}$.

To the best of our knowledge, the best constant in the L^p-analogue of Lemma 3.1(b) is not known apart from the cases $p = 1, 2, \infty$. Here, it is tempting to resort to interpolation. However, for such an inequality, one has to interpolate closed subspaces of L^p, which is possible in this case by a retraction-coretraction argument (see [16, 1.2.4]). The corresponding operators will bring in other constants which do not seem to lead to improvements over the L^p-result in [11].

5. Self-adjoint operators in a Hilbert space.

Proof of Theorem 1.2. We take a T-periodic solution $u \in C^1(\mathbb{R}, H) \cap C(\mathbb{R}, [D(A)])$ of (1.1) and put

$$v(t) := u(t) - u(t - \tau), \quad t \in \mathbb{R},$$

where $\tau \in (0, T)$ is arbitrary. Then $r \mapsto g(r) := \langle Av(r), v(r) \rangle$ is T-periodic and differentiable with

$$\frac{d}{dr} \langle Av(r), v(r) \rangle = 2 \operatorname{Re} \langle Av(r), \dot{v}(r) \rangle, \quad r \in \mathbb{R}.$$

For a proof, observe that, by the self-adjointness of A,

$$\frac{g(r+h) - g(r)}{h} = \left\langle Av(r+h), \frac{1}{h}(v(r+h) - v(r)) \right\rangle + \left\langle \frac{1}{h}(v(r+h) - v(r)), Av(r) \right\rangle,$$

and take the limit as $h \to 0$.

Hence, we have $\int_0^T \operatorname{Re} \langle Av(r), \dot{v}(r) \rangle \, dr = 0$, and we obtain

$$\int_0^T \|\dot v(r)\|^2 + \|Av(r)\|^2\,dr = \int_0^T \|\dot v(r) + Av(r)\|^2\,dr$$
$$= \int_0^T \|f(u(r)) - f(u(r-\tau))\|^2\,dr.$$

Thus, by (1.2), we have

$$\int_0^T \|\dot v(r)\|^2 + \|Av(r)\|^2\,dr \le L^2 \int_0^T \|A^\alpha v(r)\|^2\,dr. \tag{5.1}$$

We shall exploit (5.1) for the different cases of α.

If $\alpha = 0$, we obtain, by Lemma 3.1(a),

$$d_T(v) := \int_0^T \|\dot v(r)\|^2 + \|Av(r)\|^2\,dr \le \left(\frac{LT}{2\pi}\right)^2 \int_0^T \|\dot v(r)\|^2\,dr.$$

Hence, if $LT < 2\pi$, then $\dot v$ vanishes and, since τ was arbitrary, $\dot u$ is constant. Since u is periodic, $\dot u$ has to vanish and u is constant.

If $\alpha = 1$ and $L < 1$, we see from (5.1) that Av vanishes, and then also $\dot v$ vanishes. Again, u has to be constant. In other words, if $\alpha = 1$ and $L < 1$, any periodic solution has to be constant.

In case $\alpha \in (0, 1)$, we use the following two lemmata.

Lemma 5.1. *Let $v \in C^1(\mathbb{R}, H) \cap C(\mathbb{R}, [D(A)])$ be T-periodic with $\int_0^T v(r)\,dr = 0$ and $\alpha \in (0, 1)$. Then*

$$\|A^\alpha v\|_{L^2((0,T);H)} \le \left(\frac{T}{2\pi}\right)^{1-\alpha} \|\dot v\|^{1-\alpha}_{L^2((0,T);H)} \|Av\|^\alpha_{L^2((0,T);H)}.$$

Proof. We have the moment inequality $\|A^\alpha x\| \le \|x\|^{1-\alpha}\|Ax\|^\alpha$ for $x \in D(A)$ (using the spectral theorem (see, e.g., [15]) write $\|A^\beta x\|^2 = \int_0^\infty \lambda^{2\beta}\,d\mu_x$ for $\beta \in \{0, \alpha, 1\}$, where μ_x is the spectral measure for x and use Hölder's inequality with exponent $\frac{1}{\alpha}$ and dual exponent $(\frac{1}{\alpha})' = \frac{1}{1-\alpha}$). Thus we have

$$\int_0^T \|A^\alpha v(r)\|^2\,dr \le \int_0^T \|v(r)\|^{2(1-\alpha)} \|Av(r)\|^{2\alpha}\,dr.$$

We use Hölder again with exponent $\frac{1}{\alpha}$ and dual exponent $(\frac{1}{\alpha})' = \frac{1}{1-\alpha}$ and obtain

$$\int_0^T \|A^\alpha v(r)\|^2\,dr \le \left(\int_0^T \|v(r)\|^2\,dr\right)^{1-\alpha} \left(\int_0^T \|Av(r)\|^2\,dr\right)^\alpha.$$

Finally, we use Lemma 3.1(a). $\square$

Lemma 5.2. *For all $a, b \geq 0$ and $\alpha \in [0,1]$, we have*

$$a^\alpha b^{1-\alpha} \leq \alpha^\alpha (1-\alpha)^{1-\alpha}(a+b).$$

Proof. The assertion is clear for $\alpha \in \{0,1\}$, so let $\alpha \in (0,1)$. Letting $x = \frac{a}{\alpha}$, $y = \frac{b}{1-\alpha}$, the assertion is equivalent to $x^\alpha y^{1-\alpha} \leq \alpha x + (1-\alpha)y$, which again is clear if $0 \in \{x, y\}$. For $x, y \neq 0$, it is equivalent to

$$\alpha \ln x + (1-\alpha)\ln y \leq \ln(\alpha x + (1-\alpha)y),$$

and this holds since ln is concave. □

We continue the proof of Theorem 1.2. For $\alpha \in (0,1)$, we have, combining (5.1), Lemmas 5.1, and 5.2,

$$d_T(v) \leq L^2 \left(\frac{T}{2\pi}\right)^{2(1-\alpha)} \alpha^\alpha (1-\alpha)^{1-\alpha}\, d_T(v).$$

Hence, if

$$LT^{1-\alpha} < \frac{(2\pi)^{1-\alpha}}{\sqrt{\alpha^\alpha(1-\alpha)^{1-\alpha}}},$$

then $\dot{v}$ and Av vanish, and we conclude that u is constant as before. □

6. Gradient systems. In this section, we replace $Au(t)$ in (1.1) by a gradient term. For the setting, we follow [3]. So let H be a real Hilbert space and V be a Banach space with a dense and continuous embedding $V \hookrightarrow H$. Let $\mathscr{E} : V \to \mathbb{R}$ be differentiable with continuous derivative $\mathscr{E}' : V \to V'$, where V' denotes the dual space of V. Identifying $h \in H$ with the linear functional $v \mapsto \langle h, v\rangle_H$, we can consider H as a subspace of V'. Then the gradient $\nabla_H \mathscr{E}$ of $\mathscr{E}$ with respect to H is defined by

$$\begin{aligned} D(\nabla_H \mathscr{E}) &= \{u \in V : \exists h \in H\, \forall v \in V : \mathscr{E}'(u)v = \langle h, v\rangle_H\}, \\ \nabla_H \mathscr{E}(u) &= h \quad \text{for}\, u \in D(\nabla_H \mathscr{E}). \end{aligned}$$

We recall the usual solution concept from [3, Sect. 6] for the equation

$$\dot{u}(t) + \nabla_H \mathscr{E}(u(t)) = g(t), \quad t \in \mathbb{R}, \tag{6.1}$$

where $g \in L^2_{\text{loc}}(\mathbb{R}; H)$: u is a solution of (6.1) if $u \in W^{1,2}_{\text{loc}}(\mathbb{R}; H) \cap L^\infty_{\text{loc}}(\mathbb{R}; V)$ and

$$\langle \dot{u}(t), v\rangle_H + \mathscr{E}'(u)v = \langle f(t), v\rangle_H \quad \text{for all } v \in V \text{ and almost every } t \in \mathbb{R}.$$

Observe that this implies $u(t) \in D(\nabla_H \mathscr{E})$ for almost every $t \in \mathbb{R}$. One has existence and uniqueness of solutions to the corresponding initial value problem on finite time intervals if the following conditions (i)–(iii) on $\mathscr{E}$ hold (see [3, Theorem 6.1]):

(i) $\mathscr{E} : V \to \mathbb{R}$ is convex,
(ii) $\mathscr{E} : V \to \mathbb{R}$ is coercive, i.e., $\{u \in V : \mathscr{E}(u) \leq c\}$ is bounded in V for every $c \in \mathbb{R}$,
(iii) $\mathscr{E}' : V \to V'$ maps bounded sets into bounded sets.

Here, we do not go into details as in Theorem 1.4 we are interested in periodic solutions to (1.10) that are more regular, namely solutions $u \in C^1(\mathbb{R}, V)$.

Proof of Theorem 1.4. Let $u \in C^1(\mathbb{R}, V)$ be a solution to (1.10) that is T-periodic. Put $v(t) := u(t) - u_0$, where $u_0 = \frac{1}{T}\int_0^T u(r)\,dr$. Then $C^1(\mathbb{R}, V)$ is T-periodic as well and $\int_0^T v(r)\,dr = 0$. Since

$$\frac{d}{dr}\mathscr{E}(u(r)) = \mathscr{E}'(u(r))\dot{u}(r) = \nabla_H\mathscr{E}(u(r))\dot{u}(r)$$

and u is T-periodic, we have

$$\int_0^T \langle \nabla_H\mathscr{E}(u(r)) - f(u_0), \dot{u}(r)\rangle_H \; dr = 0.$$

Hence we obtain

$$\begin{aligned}
&\int_0^T \|\dot{v}(r)\|_H^2 + \|\nabla_H\mathscr{E}(u(r)) - f(u_0)\|_H^2\,dr \\
&= \int_0^T \|\dot{u}(r) + \nabla_H\mathscr{E}(u(r)) - f(u_0)\|_H^2\,dr \\
&= \int_0^T \|f(v(r)+u_0) - f(u_0)\|_H^2\,dr \\
&\le L^2 \int_0^T \|v(r)\|_H^2\,dr \le \left(\frac{LT}{2\pi}\right)^2 \int_0^T \|\dot{v}(r)\|_H^2\,dr.
\end{aligned}$$

If $LT < 2\pi$, then $\dot{v}$ vanishes, $u = u_0$ is constant, and $\nabla_H\mathscr{E}(u_0) = f(u_0)$. $\square$

Remark 6.1. We can relax the regularity of u in Theorem 1.4 to $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; V)$ provided the property (iii) above holds. Under this assumption, we can prove Lemma 6.2 below. With Lemma 6.2 at hand, we can run the same argument as before for $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; V)$.

Lemma 6.2. *Let $I \subseteq \mathbb{R}$ be an interval and $u \in W^{1,2}(I; V)$ be such that $u(r) \in D(\nabla_H\mathscr{E})$ for almost every $r \in I$. Then, for all $s, t \in I$ with $s < t$, we have*

$$\int_s^t \nabla_H\mathscr{E}(u(r))\dot{u}(r)\,dr = \mathscr{E}(u(t)) - \mathscr{E}(u(s)).$$

Proof. It clearly suffices to show that $r \mapsto \mathscr{E}'(u(r))\dot{u}(r)$ is the weak derivative of $r \mapsto \mathscr{E}(u(r))$. This is clear for $u \in C^1(I, V)$. Using mollifiers and passing to a subsequence if necessary, we approximate u by a sequence (u_n) in $C^1(I, V)$ such that $u_n \to u$ in $W^{1,2}(I; V)$ and such that we have pointwise almost everywhere $\dot{u}_n \to \dot{u}$ in V. By the inclusion $W^{1,2}(I; V) \hookrightarrow C_b(I, V)$, we have $u_n \to u$ uniformly on I. Then the sequence $(\mathscr{E}'(u_n))$ is bounded in V' by (iii) and converges pointwise in V' to $\mathscr{E}'(u)$. Since $(\dot{u}_n)$ converges to $\dot{u}$ in $L^2(I; V)$, we have $\mathscr{E}'(u_n)\dot{u}_n \to \mathscr{E}'(u)\dot{u} = \nabla_H\mathscr{E}(u)\dot{u}$ in $L^2(I; \mathbb{R})$. Since also $\mathscr{E}(u_n) \to \mathscr{E}(u)$ pointwise, the assertion follows. $\square$

Acknowledgements. The authors thank Patrick Tolksdorf for a discussion on the proof of Lemma 5.1.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

[1] Brandt, F., Hieber, M.: Strong periodic solutions to quasilinear parabolic equations: An approach by the Da Prato-Grisvard theorem. Bull. Lond. Math. Soc. **55**(4), 1971–1993 (2023)

[2] Busenberg, S.N., Fisher, D.C., Martelli, M.: Better bounds for periodic solutions of differential equations in Banach spaces. Proc. Amer. Math. Soc. **98**, 376–378 (1986)

[3] Chill, R., Fašangová, E.: Gradient Systems. Lecture Notes of the 13th International Internet Seminar. Matfyzpress, Prague (2010)

[4] Dacorogna, B., Gangbo, W., Subía, N.: Sur une généralisation de l'inégalité de Wirtinger. Ann. Inst. Henri Poincaré **9**(1), 29–50 (1992)

[5] Geissert, M., Hieber, M., Nguyen-Matthias, T.H.: A general approach to time periodic incompressible viscous fluid flow problems. Arch. Ration. Mech. Anal. **220**(3), 1095–1118 (2016)

[6] Haase, M.: The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, 169. Birkhäuser, Basel (2006)

[7] Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, 840. Springer, Berlin (1981)

[8] Lasota, A., Yorke, J.: Bounds for periodic solutions of differential equations in Banach spaces. J. Differential Equations **10**, 83–91 (1971)

[9] Mawhin, J., Walter, W.: A symmetry principle and some implications. J. Math. Anal. Appl. **186**, 778–798 (1994)

[10] Martelli, M.: Soluzioni periodiche di sistemi Lipschitziani e applicazioni. Rend. Sem. Mat. Fis. Milano **58**, 67–80 (1988)

[11] Nieuwenhuis, M.A.C., Robinson, J.C., Steinerberger, S.: Minimal periods for ordinary differential equations in strictly convex Banach spaces and explicit bounds for some L^p-spaces. J. Differential Equations **256**, 2846–2857 (2014)

[12] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, 44. Springer, New York (1983)

[13] Robinson, J.C., Vidal-López, A.: Minimal periods of semilinear evolution equations with Lipschitz nonlinearity. J. Differential Equations **220**, 396–406 (2006)

[14] Robinson, J.C., Vidal-López, A.: Minimal periods of semilinear evolution equations with Lipschitz nonlinearity revisited. J. Differential Equations **254**, 4279–4289 (2013)

[15] Rudin, W.: Functional Analysis. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg (1973)

[16] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York (1978)

[17] Yorke, J.: Periods for periodic solutions and the Lipschitz constant. Proc. Amer. Math. Soc. **22**, 509–512 (1969)

Gerd Herzog and Peer Christian Kunstmann
Institute for Analysis
Karlsruhe Institute of Technology (KIT)
Englerstr. 2
76128 Karlsruhe
Germany
e-mail: gerd.herzog2@kit.edu

Peer Christian Kunstmann
e-mail: peer.kunstmann@kit.edu

Received: 1 June 2023

Revised: 10 January 2024

Accepted: 21 January 2024