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Positivity

A Perron–Frobenius type result in Banach algebras via asymptotic closeness to a cone

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Abstract
For an element a of a Banach algebra (scaled to spectral radius 1) we prove that the spectral radius is contained in the spectrum, if the sequence of powers (a^k) is asymptotically not too far from a normal cone.

Keywords Banach algebras · Ordered Banach spaces · Perron–Frobenius theory

Mathematics Subject Classification 46B40 · 46H20

1 Introduction

In this paper we give sufficient conditions for the property $r(a) \in \sigma(a)$ where a is an element of a Banach algebra $\mathcal{A}$ with spectrum $\sigma(a)$ and spectral radius $r(a)$. That the spectral radius belongs to the spectrum is an aspect of what is usually subsumed under the term "Perron–Frobenius theory", classically for positive matrices on $\mathbb{R}^n$ or positive linear operators on Banach lattices. More general, we consider here elements of a Banach algebra that is ordered by a cone. We fix the relevant notation.

Let $\mathcal{A}$ be a complex Banach algebra with unit $\mathbf{1}$. A set $\emptyset \neq K \subseteq \mathcal{A}$ is called a *cone* if K is closed, $K + K \subseteq K$, $\lambda K \subseteq K$ $(\lambda \geq 0)$ and $K \cap (-K) = \{0\}$. By setting $a \leq b$ $:\Leftrightarrow b - a \in K$ we obtain a partial order on $\mathcal{A}$.

In the following we always assume that K is a *normal cone*, that is

$$\exists \gamma \geq 1 \; \forall a, b \in \mathcal{A} : \; 0 \leq a \leq b \; \Rightarrow \; \|a\| \leq \gamma \|b\|,$$

and we fix a γ with this property.

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Following the notation in Raubenheimer and Rode [10] we call K an *algebra cone* if K satisfies in addition $\mathbf{1} \in K$ and

$$a, b \in K \;\Rightarrow\; ab \in K.$$

In contrast to the usual definition of ordered Banach algebras we do *not* assume in general that K is an algebra cone, that is in our setting $\mathcal{A}$ is a Banach algebra and $\mathcal{A}$ is an ordered Banach space.

The problem of developing a Perron–Frobenius theory in Banach algebras ordered by a cone that is not fully invariant under multiplication was also addressed by Mouton and Muzundu in [7, 9]: In a *Commutatively Ordered Banach Algebra* (COBA) one only assumes that $\mathbf{1} \in K$ and that

$$a, b \in K,\; ab = ba \;\Rightarrow\; ab \in K.$$

This includes the important special case of a C^*-algebra ordered by the cone of positive semidefinite self-adjoint elements.

Let $\sigma(a)$, $\rho(a)$ and $r(a)$ denote the spectrum, resolvent set and spectral radius of $a \in \mathcal{A}$, respectively. For $r(a) > 0$, the property $r(a) \in \sigma(a)$ is invariant under scaling and we can always resort to $r(a) = 1$ by considering $a/r(a)$. Following the notation in [4] also in our setting, we call $a \in \mathcal{A}$ *asymptotically positive* if $r(a) > 0$ and

$$\lim_{k\to\infty} d_k(a) = 0 \text{ where } d_k(a) := \operatorname{dist}\left(\frac{a^k}{r(a)^k}, K\right) \quad (k \in \mathbb{N}_0).$$

Let us call a condition $C(a)$ a *Perron–Frobenius condition* if $C(a)$ implies $r(a) \in \sigma(a)$ for all $a \in \mathcal{A}$ with $r(a) > 0$. Note that $C(a) =[a \in K$ for a normal cone $K]$ is not a Perron–Frobenius condition, as can be seen by the trivial example $\mathcal{A} = \mathbb{C}$, $K = \{\lambda i : \lambda \geq 0\}$ and $a = i$.

There are many known Perron–Frobenius conditions in matrix algebras, operator algebras and Banach algebras. Without claiming completeness we refer to

1. $C(a) = [a \geq 0]$, i.e. a is *positive*, for general Banach algebras ordered by an algebra cone, see Raubenheimer and Rode [10, Theorem 5.2],
2. $C(a) = [\exists k_0 \forall k \geq k_0 : a^k \geq 0]$, i.e. a is *eventually positive*, for matrix algebras ordered by the cone of matrices with nonnegative entries, see Chaysri and Noutsos [2, Theorem 2.5],
3. $C(a) = [\lim_{k\to\infty} d_k(a) = 0]$, i.e. a is *asymptotically positive*, for operator algebras on ordered Banach lattices, see Glück [4, Theorem 4.1],
4. $C(a) = [\lim_{k\to\infty} \sqrt{k} d_k(a) = 0]$, i.e. a is *asymptotically positive* with a rate of convergence, for general Banach algebras ordered by an algebra cone, see [5],

and the references given in the cited literature. Note that the setting in [5] comprises the one in [4].

For a survey on spectral theory in ordered Banach algebras we refer to the paper of Mouton and Raubenheimer [8].

Our main result generalizes the above mentioned results from [4] and [5] in two ways. First, we relax the condition that K is an algebra cone, which already gives a new result for the matrix case. Second, we relax the condition of asymptotic positivity by showing that $C(a) = [\limsup_{k\to\infty} d_k(a)$ is "sufficiently small"] is a Perron–Frobenius condition in our setting.

2 Results

Theorem 1 *Let $a \in \mathcal{A}$ with $r(a) > 0$ such that*

$$\limsup_{k\to\infty} d_k(a) < \frac{1}{\pi\gamma + 1}.$$

Then $r(a) \in \sigma(a)$.

Remark 1 An inspection of the proof[1] shows, that the weaker but less manageable condition on Abel means of the sequence $(d_k(a))_{k\in\mathbb{N}_0}$,

$$AM(a) := \limsup_{t\to 1+} (t-1) \sum_{k=0}^{\infty} \frac{d_k(a)}{t^{k+1}} < \frac{1}{\pi\gamma + 1},$$

is still sufficient for $r(a) \in \sigma(a)$. This condition is weaker by Lemma 3.

Aiming at optimality of the constant in Remark 1 we can define $\tau(\mathcal{A})$ as the maximum of all $\tau > 0$ such that

$$\forall a \in \mathcal{A} : \; r(a) > 0, \; AM(a) < \tau \implies r(a) \in \sigma(a).$$

Remark 1 shows $\tau(\mathcal{A}) \geq (\pi\gamma + 1)^{-1}$. For $a = -\mathbf{1}$ we have $1 = r(a) \notin \sigma(a) = \{-1\}$ and, for any cone K, we have $\operatorname{dist}(\pm\mathbf{1}, K) \leq 1$, hence $AM(-\mathbf{1}) \leq 1$. This gives the trivial bound $\tau(\mathcal{A}) \leq 1$. If either $\mathbf{1} \in K$ or $-\mathbf{1} \in K$, then

$$d_k(-\mathbf{1}) = \operatorname{dist}((-1)^k \mathbf{1}, K)$$

is alternatingly 0 and ≤ 1. Thus $AM(-\mathbf{1}) \leq 1/2$ and we have $\tau(\mathcal{A}) \leq 1/2$.

As an introductory example consider the matrix-algebra $\mathcal{A} = \mathbb{C}^{m\times m}$ endowed with the row-sum norm $\|\cdot\|_\infty$ and ordered by the algebra cone $K := [0, \infty)^{m\times m}$ of all matrices with nonnegative entries. Here $\gamma = 1$. Thus, for $A \in \mathbb{C}^{m\times m}$, Theorem 1 reads

$$\limsup_{k\to\infty} d_k(A) < \frac{1}{\pi + 1} \;\Rightarrow\; r(A) \in \sigma(A),$$

[1] Look at the very end of the proof.

and we have

$$\forall m \in \mathbb{N}: \ 0,2415 \approx \frac{1}{\pi+1} \leq \tau(\mathbb{C}^{m\times m}) \leq \frac{1}{2}.$$

For the matrix

$$A = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

we have $\sigma(A) = \{-1, 0\}$ and $d_k(A) = 1/2$ ($k \in \mathbb{N}$). Hence

$$\limsup_{k\to\infty} d_k(A) = AM(A) = \frac{1}{2}.$$

In case $m = 1$ we have for $a \in \mathbb{C}$ that $r(a) = 1$ if and only if $a = e^{it}$ for some $t \in [0, 2\pi)$. For $t \in (0, 2\pi)$ one can check that

$$\limsup_{k\to\infty} d_k(e^{it}) = 1, \quad AM(e^{it}) \geq \frac{1}{2}.$$

In particular $\tau(\mathbb{C}) = 1/2$.

The matrix

$$A = \frac{1}{5+2\sqrt{5}}\begin{pmatrix} 9 & -2 \\ -2 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

is an example where, in this setting, A is not asymptotically positive but Theorem 1 applies. We have

$$\lim_{k\to\infty} A^k = \begin{pmatrix} \frac{1}{2}+\frac{1}{\sqrt{5}} & -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} & \frac{1}{2}-\frac{1}{\sqrt{5}} \end{pmatrix},$$

thus

$$\lim_{k\to\infty} d_k(A) = \frac{1}{2\sqrt{5}} < \frac{1}{\pi+1},$$

and $1 \in \sigma(A)$.

3 Proof

We will use the following lemmas. For Lemma 1 see, e.g., [1, Theorem 3.3.5.]. A result related to Lemma 2, though with the worse constant 4γ, can be found in the preprint [6, Lemma 4.6] by Huang, Jaffe, Liu, and Wu.

Lemma 1 *Let $a \in \mathcal{A}$. Then*

$$r((\lambda \mathbf{1} - a)^{-1}) = \frac{1}{\operatorname{dist}(\lambda, \sigma(a))} \quad (\lambda \in \rho(a)).$$

Lemma 2 *Let $(a_k)_{k \in \mathbb{N}_0}$ and $(\lambda_k)_{k \in \mathbb{N}_0}$ be sequences in K and $\mathbb{C}$, respectively, such that the series*

$$\sum_{k=0}^{\infty} \lambda_k a_k$$

is absolutely convergent. Then we have

$$\left\| \sum_{k=0}^{\infty} \lambda_k a_k \right\| \leq \pi \gamma \left\| \sum_{k=0}^{\infty} |\lambda_k| a_k \right\|.$$

Proof W.l.o.g. we may first assume $\lambda_k \neq 0$ $(k \in \mathbb{N}_0)$ and then $|\lambda_k| = 1$ $(k \in \mathbb{N}_0)$ since the modulus of λ_k can be absorbed into the vector a_k. Set $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ and let $h : S^1 \to \mathbb{R}$ be defined as

$$h(z) = \begin{cases} 1, & \Re(z) \geq 0 \\ 0, & \Re(z) < 0 \end{cases}.$$

First note that

$$\forall z \in S^1 : \; z = \frac{1}{2} \int_{-\pi}^{\pi} h(z e^{-it}) e^{it} dt.$$

Thus, for $l \geq 0$,

$$\begin{aligned} \left\| \sum_{k=0}^{l} \lambda_k a_k \right\| &= \frac{1}{2} \left\| \sum_{k=0}^{l} \left(\int_{-\pi}^{\pi} h\left(\lambda_k e^{-it}\right) e^{it} dt \right) a_k \right\| \\ &= \frac{1}{2} \left\| \int_{-\pi}^{\pi} \left(\sum_{k=0}^{l} h\left(\lambda_k e^{-it}\right) e^{it} a_k \right) dt \right\| \\ &\leq \frac{1}{2} \int_{-\pi}^{\pi} \left\| \sum_{k=0}^{l} h\left(\lambda_k e^{-it}\right) e^{it} a_k \right\| dt \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \left\| \sum_{k=0}^{l} h\left(\lambda_k e^{-it}\right) a_k \right\| dt \\ &\leq \frac{\gamma}{2} \int_{-\pi}^{\pi} \left\| \sum_{k=0}^{l} a_k \right\| dt = \pi \gamma \left\| \sum_{k=0}^{l} a_k \right\|. \end{aligned}$$

Now $l \to \infty$ proves the desired inequality. □

Lemma 3 *Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $(1,\infty)$ with limit 1, and let $(\beta_k)_{k\in\mathbb{N}_0}$ be a bounded sequence in $[0,\infty)$ and $s := \limsup_{k\to\infty}\beta_k$. Then*

$$\limsup_{n\to\infty}(t_n-1)\sum_{k=0}^{\infty}\frac{\beta_k}{t_n^{k+1}} \le s.$$

Proof Let $\varepsilon > 0$. Then we find $k_0 \in \mathbb{N}_0$ such that $\beta_k < s+\varepsilon$ $(k > k_0)$. Thus, for $n \in \mathbb{N}$,

$$\begin{aligned}(t_n-1)\sum_{k=0}^{\infty}\frac{\beta_k}{t_n^{k+1}} &\le (t_n-1)\sum_{k=0}^{k_0}\frac{\beta_k}{t_n^{k+1}} + (s+\varepsilon)(t_n-1)\sum_{k=k_0+1}^{\infty}\frac{1}{t_n^{k+1}}\\ &\le (t_n-1)\sum_{k=0}^{k_0}\frac{\beta_k}{t_n^{k+1}} + (s+\varepsilon)\frac{(t_n-1)}{t_n}\sum_{k=0}^{\infty}\frac{1}{t_n^{k}}\\ &= (t_n-1)\sum_{k=0}^{k_0}\frac{\beta_k}{t_n^{k+1}} + s+\varepsilon.\end{aligned}$$

We obtain

$$\limsup_{n\to\infty}(t_n-1)\sum_{k=0}^{\infty}\frac{\beta_k}{t_n^{k+1}} \le s+\varepsilon,$$

and $\varepsilon \to 0+$ proves the assertion. □

Proof of Theorem 1 We may assume $r(a) = 1$. Let $\lambda_0 \in \sigma(a)$ with $|\lambda_0| = 1$. By assumption there is a sequence $(b_k)_{k\in\mathbb{N}_0}$ in $\mathcal{A}$ such that $a^k + b_k \ge 0$ $(k \in \mathbb{N}_0)$ and

$$s := \limsup_{k\to\infty}\|b_k\| < \frac{1}{\pi\gamma+1}.$$

For $|\lambda| > 1$ we have

$$(\lambda\mathbf{1}-a)^{-1} = \sum_{k=0}^{\infty}\frac{a^k}{\lambda^{k+1}}.$$

We choose a sequence (t_n) in $(1,\infty)$ with limit 1. Now Lemma 2 yields

$$\begin{aligned}\|((t_n\lambda_0)\mathbf{1}-a)^{-1}\| = \left\|\sum_{k=0}^{\infty}\frac{a^k}{(t_n\lambda_0)^{k+1}}\right\| &\le \left\|\sum_{k=0}^{\infty}\frac{a^k+b_k}{(t_n\lambda_0)^{k+1}}\right\| + \left\|\sum_{k=0}^{\infty}\frac{b_k}{(t_n\lambda_0)^{k+1}}\right\|\\ &\le \pi\gamma\left\|\sum_{k=0}^{\infty}\frac{a^k+b_k}{t_n^{k+1}}\right\| + \left\|\sum_{k=0}^{\infty}\frac{b_k}{(t_n\lambda_0)^{k+1}}\right\|\end{aligned}$$

$$\le \pi\gamma \left\| \sum_{k=0}^{\infty} \frac{a^k}{t_n^{k+1}} \right\| + \pi\gamma \left\| \sum_{k=0}^{\infty} \frac{b_k}{t_n^{k+1}} \right\| + \left\| \sum_{k=0}^{\infty} \frac{b_k}{(t_n\lambda_0)^{k+1}} \right\|$$
$$= \pi\gamma \|(t_n\mathbf{1} - a)^{-1}\| + \pi\gamma \left\| \sum_{k=0}^{\infty} \frac{b_k}{t_n^{k+1}} \right\| + \left\| \sum_{k=0}^{\infty} \frac{b_k}{(t_n\lambda_0)^{k+1}} \right\|$$
$$\le \pi\gamma \|(t_n\mathbf{1} - a)^{-1}\| + (\pi\gamma + 1) \sum_{k=0}^{\infty} \frac{\|b_k\|}{t_n^{k+1}}.$$

By Lemma 3 with $\beta_k := \|b_k\|$ $(k \in \mathbb{N}_0)$ we have

$$\limsup_{n\to\infty} (t_n - 1) \sum_{k=0}^{\infty} \frac{\|b_k\|}{t_n^{k+1}} \le s < \frac{1}{\pi\gamma + 1}.$$

Moreover, by Lemma 1,

$$(t_n - 1)\|((t_n\lambda_0)\mathbf{1} - a)^{-1}\| \ge (t_n - 1) r(((t_n\lambda_0)\mathbf{1} - a)^{-1})$$
$$= \frac{t_n - 1}{\operatorname{dist}(t_n\lambda_0, \sigma(a))} = 1.$$

If we now assume that $1 \notin \sigma(a)$ we obtain

$$1 \le \limsup_{n\to\infty} (t_n - 1)\|((t_n\lambda_0)\mathbf{1} - a)^{-1}\|$$
$$\le \limsup_{n\to\infty} (t_n - 1) \left[\pi\gamma \|(t_n\mathbf{1} - a)^{-1}\| + (\pi\gamma + 1) \sum_{k=0}^{\infty} \frac{\|b_k\|}{t_n^{k+1}} \right]$$
$$= (\pi\gamma + 1) \limsup_{n\to\infty} (t_n - 1) \sum_{k=0}^{\infty} \frac{\|b_k\|}{t_n^{k+1}} < 1,$$

a contradiction. □

4 Applications

1. Let $\mathcal{A}$ be a Banach algebra and let $\phi : \mathcal{A} \to \mathbb{C}$ be a continuous linear functional on $\mathcal{A}$ (with $\|\phi\| \ge 1$ to avoid the trivial case). Then

$$K := \{c \in \mathcal{A} : \|c\| \le \Re(\phi(c))\}$$

is a normal cone with $\gamma = \|\phi\|$. By application of Theorem 1 we get the following corollaries.

Corollary 1 *Let $a \in \mathcal{A}$ with $r(a) = 1$. If there exists a sequence (c_k) in $\mathcal{A}$ and $k_0 \in \mathbb{N}$ with*

$$\forall k \geq k_0 : \ \|c_k\| \leq \Re(\phi(c_k)), \quad \limsup_{k\to\infty} \|a^k - c_k\| < \frac{1}{\pi\|\phi\| + 1},$$

then $1 \in \sigma(a)$.

A special case of the situation in Corollary 1 is that (a^k) is eventually in K. Then we can eventually choose $c_k = a^k$, and obtain:

Corollary 2 *Let $a \in \mathcal{A}$ with $r(a) = 1$. If*

$$\exists k_0 \in \mathbb{N}\, \forall k \geq k_0 : \ \|a^k\| \leq \Re(\phi(a^k)),$$

then $1 \in \sigma(a)$.

2. Let $\mathcal{A}$ be a C^*-algebra and let

$$K := \{c \in \mathcal{A} : c = c^*, \sigma(c) \subseteq [0, \infty)\}.$$

Then K is a normal cone with $\gamma = 1$ [3, 1.6.9] and we have the following.

Corollary 3 *Let $a \in \mathcal{A}$ with $r(a) = 1$. If*

$$\limsup_{k\to\infty} \left\| a^{2k} - \frac{1}{4}(a^k + (a^*)^k)^2 \right\| < \frac{1}{\pi + 1}$$

then $\sigma(a) \cap \{-1, 1\} \neq \emptyset$.

Proof Since $(b + b^*)^2 \geq 0$ for each $b \in \mathcal{A}$ we have

$$d_k(a^2) \leq \left\| a^{2k} - \frac{1}{4}(a^k + (a^*)^k)^2 \right\| \quad (k \in \mathbb{N}_0).$$

Application of Theorem 1 to a^2 yields $1 \in \sigma(a^2)$, hence $\sigma(a) \cap \{-1, 1\} \neq \emptyset$. □

Remark 2 Of course, other quite natural assumptions are possible, such as

$$\limsup_{k\to\infty} \left\| a^{2k} - (aa^*)^k \right\| < \frac{1}{\pi + 1}$$

or

$$\limsup_{k\to\infty} \left\| a^{2k} - \frac{1}{2}(a^k(a^*)^k + (a^*)^k a^k) \right\| < \frac{1}{\pi + 1}.$$

However, numerical experiments with random matrices indicate that the assumption in Corollary 3 is fulfilled more easily.

3. Let $a \in \mathcal{A}$ with $r(a) = 1$. Assume that $\{a^k : k \in \mathbb{N}\}$ is relatively compact, and that the sequence (a^k) has a finite number of accumulation points $b_1, \ldots, b_m$ which are linearly independent. To apply Theorem 1 set

$$K := \left\{ \sum_{j=1}^{m} \alpha_j b_j : \alpha_j \geq 0 \; (j = 1, \ldots, m) \right\}.$$

Then K is a cone since $b_1, \ldots, b_m$ are linearly independent, and K is normal since it is contained in a finite dimensional subspace of $\mathcal{A}$. Now a is asymptotically positive with respect to this cone, hence $1 \in \sigma(a)$. Without the assumption of linear independence this is trivially wrong as can be seen by considering $a = -\mathbf{1}$.

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Declarations

Conflict of interest The authors declare no competing interests.

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