Research Article

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Gromov-Hausdorff limits of closed surfaces

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Abstract: We completely describe the Gromov-Hausdorff closure of the class of length spaces being homeomorphic to a fixed closed surface.

Keywords: Gromov-Hausdorff convergence; length spaces; surfaces; generalized cactoids; Peano spaces; ANR

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1 Introduction

Let M be a closed connected smooth manifold of dimension larger than two. Then, a theorem by Ferry and Okun implies that every simply connected compact absolute neighborhood retract (ANR) carrying a length metric can be obtained as the Gromov-Hausdorff limit of length spaces being homeomorphic to M (cf. [8, p. 1866]). In dimension two, the statement does not apply. For example, a space being homeomorphic to the 3-disk cannot be obtained as the limit of length spaces being homeomorphic to the 2-sphere (cf. [5, p. 269]). This observation leads to the following question: is there a two-dimensional version of the statement?

In this study, we completely describe the Gromov-Hausdorff closure of the class of length spaces being homeomorphic to a fixed closed surface. As a corollary, we derive an answer to our question.

We will see that the spaces of the closure have the following topological properties:

Theorem 1.1. *Let X be the Gromov-Hausdorff limit of a convergent sequence of length spaces being homeomorphic to a fixed closed surface. Then, the following statements apply*:
(1) *X is at most two-dimensional.*
(2) *X is locally simply connected.*
(3) *There are finitely many closed surfaces $S_1, \dots, S_n$ and $k \in \mathbb{N}_0$ such that $\pi_1(X)$ is isomorphic to $\pi_1(S_1) * \dots * \pi_1(S_n) * \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k\text{-times}}$.*

Before we state our main result, we introduce some definitions: A subset C of a Peano space X is called *cyclicly connected* if every pair of points in C can be connected by a simple closed curve in C. The subset is denoted as *maximal cyclic* provided that it is not degenerate to a point, cyclicly connected, and no proper subset of a cyclicly connected subset of X.

Definition. We say that a Peano space is a generalized cactoid if all its maximal cyclic subsets are closed surfaces and only finitely many of them are not homeomorphic to the 2-sphere.

In the case that all maximal cyclic subsets are homeomorphic to the 2-sphere, we just call it a *cactoid.* An example of a generalized cactoid is shown in Figure 1.

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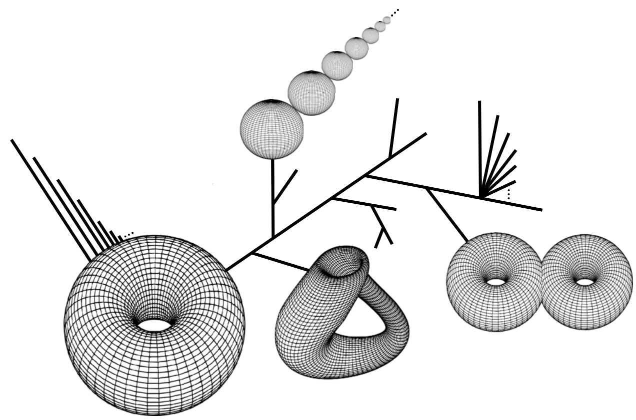

Figure 1: Generalized cactoid. All but one maximal cyclic subset are orientable and one maximal cyclic subset is homeomorphic to the Klein bottle. The illustration of the Klein bottle uses a parametrization taken from [22, p. 141].

A space being isometric to a metric quotient of X whose underlying equivalence relation identifies exactly two points is referred to as a *metric 2-point-indentification* of X.

The *connectivity number* of a closed surface is defined as its first Betti number with coefficients in $\mathbb{Z}_2$. Moreover, we call a surface carrying a length metric a *length surface*.

The following main result of our work completely describes the Gromov-Hausdorff closure of the class of closed length surfaces whose connectivity number is fixed:

Main Theorem. *Let $c \in \mathbb{N}_0$ and X be a compact length space. Then, the following statements are equivalent*:

(1) *X can be obtained as the Gromov-Hausdorff limit of closed length surfaces whose connectivity number is equal to c.*

(2) *X can be obtained by a successive application of k metric 2-point identifications to a geodesic generalized cactoid such that the sum of the connectivity numbers of its maximal cyclic subsets is less or equal to $c - 2k$.*

This result was partly conjectured by Young (cf. [28, p. 348], [21, p. 854]). Furthermore, the equivalence of the statements remains true even if we restrict the first statement to smooth Riemannian or polyhedral 2-manifolds (cf. [17, p. 1674], [20, p. 77]). We will also see how the result changes if we restrict the first statement to orientable or non-orientable surfaces.

The main theorem is illustrated in Figure 2.

In the 1930s Whyburn already proved that the limits of length spaces being homeomorphic to the 2-sphere are cactoids (cf. [26, p. 419]). Moreover, there is a related result about closed Riemannian 2-manifolds with

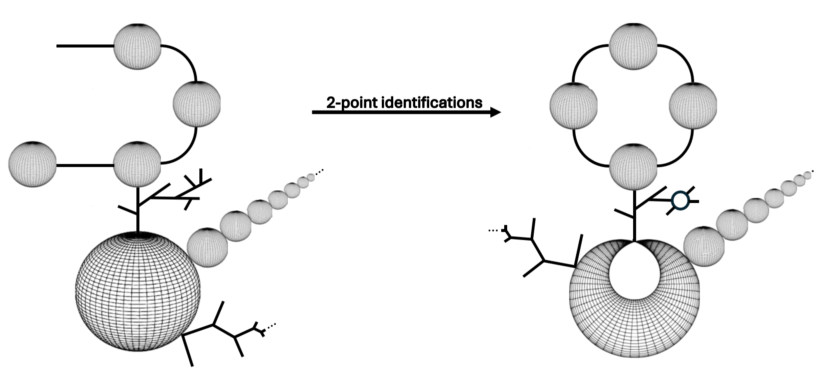

Figure 2: Illustration of the main theorem. The space on the right-hand side can be obtained by a successive application of three metric 2-point identifications to the geodesic cactoid situated on the left-hand side. Since the connectivity number of the 2-sphere is equal to zero, the main theorem implies that the space on the right-hand side can be obtained as the limit of closed length surfaces whose connectivity number is equal to six.

uniformly bounded total absolute curvature by Shioya (cf. [24, p. 1767]) and a sketch of a local description of the limit spaces by Gromov (cf. [13, S. 102]).

Finally, the following corollary answers the question from the beginning:

Corollary 1.2. *Let X be a metric space and S be a closed surface. Then, the following statements are equivalent*:
(1) *X is a simply connected ANR that can be obtained as the Gromov-Hausdorff limit of length spaces being homeomorphic to S.*
(2) *X is a geodesic cactoid having only finitely many maximal cyclic subsets.*

This work is organized as follows: we start with some preliminary notes on Gromov-Hausdorff convergence and the topology of Peano spaces, closed surfaces, and gluings.

Section 3 is devoted to the topological connection between maximal cyclic subsets and their ambient space. In particular, we derive a fundamental group formula for locally simply connected Peano spaces in terms of their maximal cyclic subsets. The section also provides first topological properties of generalized cactoids.

In Section 4, we show that the first statement of the main theorem implies the second. For this, we begin with a consideration of sequences with additional topological control. At the end of the section, we give a proof of Theorem 1.1.

The aim of the last section is to show the remaining direction of the main theorem. Then, we prove Corollary 1.2.

The final results of Sections 4 and 5 refine their corresponding statement of the main theorem. Together, they completely describe the Gromov-Hausdorff closure of the class of length spaces being homeomorphic to a fixed closed surface.

Finally, we sketch the key ideas for the proof of the main theorem: to show that the first statement implies the second, we distinguish two cases: either there is subsequence that admits a sequence of non-contractible simple closed curves whose diameters tend to zero or there is no such subsequence.

In the latter case, we start by showing that the limit space locally looks like a cactoid (Lemma 4.1). Using this topological bound, we finally derive that the limit space is a generalized cactoid as in the main theorem (Corollary 4.4).

Provided that the first case applies, the complexity reduces a problem dealing with lower connectivity numbers (Lemma 4.5). To see this, we apply a classification result for simple closed curves in closed surfaces (Proposition 2.9).

Finally, an induction over the connectivity number using Whyburn's result as base case yields the first direction of the main theorem.

For the converse direction, we successively reduce the complexity of the problem: first, we approximate geodesic generalized cactoids by suitable successive metric wedge sums of closed length surfaces (Corollary 3.8). Then, we see how to approximate metric wedge sums and metric 2-point identifications of closed length surfaces by suitable closed length surfaces (Lemmas 5.1 and 5.3). The corresponding constructions use an argument via the Kuratowski embedding. We also observe that it is possible to increase the connectivity number of the approximating surfaces (Corollary 5.5). Combining all these results, we inductively obtain the remaining direction of the main theorem.

2 Preliminaries

2.1 Gromov-Hausdorff convergence

In this subsection, we present results about Gromov-Hausdorff convergence. For basic definitions and results concerning the Gromov-Hausdorff distance, we refer the reader to [5, pp. 251–270]. Moreover, there is a corresponding notion of convergence for maps, which is treated in [19, pp. 401–402].

For the sake of simplicity, we note the following: if we consider a Gromov-Hausdorff convergent sequence, then there are isometric embeddings of the spaces and their limit into some compact metric space such that the induced sequence Hausdorff converges to the image of the limit (cf. [12, pp. 64–65]). Whenever we apply this statement, we will identify corresponding sets without mentioning the underlying space.

We introduce the concept of almost isometries: if $f : X \to Y$ is a map between metric spaces, we define its *distortion* by

$$\operatorname{dis}(f) \coloneqq \sup_{x_1, x_2 \in X} \{|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)|\}.$$

The map f is called an *ε-isometry* provided that $\operatorname{dis}(f) \leq \varepsilon$ and $f(X)$ is an ε-net in Y.

The next two results are tools to prove convergence:

Proposition 2.1. (cf. [5, p. 260]) *Let X be a compact metric space and $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces. Then, the following statements are equivalent*:
(1) *The sequence converges to X.*
(2) *For every $n \in \mathbb{N}$, there is an ε_n-isometry $f_n : X_n \to X$ and $\varepsilon_n \longrightarrow 0$.*

Moreover, the equivalence remains true if we interchange X_n and X in the second statement.

Proposition 2.2. (cf. [5, p. 264]) *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces. Then, there is a convergent subsequence if the following statements apply*:
(1) *There is some $D \in \mathbb{R}$ such that* $\operatorname{diam}(X_n) \leq D$ *for every $n \in \mathbb{N}$.*
(2) *There is a map $N : (0, \infty) \to \mathbb{R}$, which satisfies the following property: for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an ε-net in X_n with at most $N(\varepsilon)$ points.*

Now, we consider sequences with topological control: we say that a sequence $(X_n)_{n\in\mathbb{N}}$ of metric spaces is *uniformly semi-locally simply connected* if X_n does not contain non-contractible loops of diameter less than 2ε for every $n \in \mathbb{N}$.

In general, the fundamental group is not stable under Gromov-Hausdorff convergence. The upcoming result presents a case in which it is stable:

Theorem 2.3. (cf. [25, p. 3588]) *Let X be a semi-locally simply connected compact metric space and $(X_n)_{n\in\mathbb{N}}$ be a uniformly semi-locally simply connected sequence of compact length spaces. If $X_n \longrightarrow X$, then $\pi_1(X)$ is isomorphic to $\pi_1(X_n)$ for almost all $n \in \mathbb{N}$.*

From [5, p. 267], we deduce the following:

Proposition 2.4. *Every compact metric tree can be obtained as the limit of finite metric trees.*

Throughout this work, we denote a simple closed curve as a *Jordan curve*. We note that the class of compact metric trees is equal to the class of compact length spaces not containing Jordan curves (cf. [6, pp. 367–368]).

Finally, we state Whyburn's theorem about the limits of 2-spheres:

Theorem 2.5. (cf. [26, p. 419]) *A space that can be obtained as the limit of length spaces being homeomorphic to the 2-sphere is a cactoid.*

2.2 Topology of Peano spaces

This subsection is devoted to Peano spaces. Our main source about this topic is given by [27]:

A compact connected locally connected metric space is called a *Peano space*.

From a topological point of view, there is no difference between Peano spaces and compact length spaces:

Theorem 2.6. (cf. [3, p. 1109]) *Every Peano space is homeomorphic to a compact length space.*

A point x of a connected metric space X is denoted as a *cut point* of X if $X\backslash\{x\}$ is disconnected. Moreover, we say that x is a *local cut point* of X provided that there is some connected open neighborhood U of x such that $U\backslash\{x\}$ is disconnected.

The following result summarizes basic properties of Peano spaces:

Lemma 2.7. (cf. [27, pp. 65–71, 143]) *Let X be a Peano space and T be a maximal cyclic subset of X. Then, the following statements apply*:

(1) *There are only countably many maximal cyclic subsets in X.*
(2) *There are only countably many connected components in* $X\backslash T$.
(3) T *contains only countably many cut points of* X.
(4) *If* $(C_n)_{n\in\mathbb{N}}$ *is a sequence of pairwise distinct connected components of* $X\backslash T$, *then* $\operatorname{diam}(C_n) \longrightarrow 0$.
(5) *If* $(T_n)_{n\in\mathbb{N}}$ *is a sequence of pairwise distinct maximal cyclic subsets of X, then* $\operatorname{diam}(T_n) \longrightarrow 0$.
(6) *If C is a connected component of* $X\backslash T$, *then there is some* $x \in T$ *such that* $\partial C = \{x\}$.
(7) *The map* $r : X \to T$ *such that* $r(x) = x$ *for every* $x \in T$ *and* $r(x) \in \partial C$ *for every connected component* C *of* $X\backslash T$ *and* $x \in C$ *is continuous.*

2.3 Topology of closed surfaces

We want to fix two results concerning closed surfaces. The first one describes a property that is shared by their fundamental groups:

Proposition 2.8. (cf. [14, pp. 141, 264], [2, pp. 1480–1481]) *Let S be a closed surface. Then,* $\pi_1(S)$ *is not isomorphic to a free product of non-trivial groups.*

Finally, we have the following classification of non-contractible Jordan curves in closed surfaces:

Proposition 2.9. (cf. [16, pp. 54–55]) *Let S be a closed surface of connectivity number c and J be a non-contractible Jordan curve in S. Then, the topological quotient* $X := S/J$ *can be described in one of the following ways*:

(1) *There are* $c_1, c_2 \in \mathbb{N}$ *with* $c_1 + c_2 = c$, *a closed surface* S_1 *of connectivity number* c_1, *and a closed surface* S_2 *of connectivity number* c_2 *such that X is a wedge sum of* S_1 *and* S_2. *Moreover, at least one of the surfaces is non-orientable if and only if S is non-orientable.*
(2) *There is a closed surface of connectivity number* $c - 2$ *such that X is a topological 2-point identification of it. Moreover, the surface is orientable if S is orientable.*
(3) *X is a closed surface of connectivity number* $c - 1$ *and S is non-orientable.*

The aforementioned classification result is illustrated in Figure 3.

2.4 Fundamental groups of gluings

We present two results that are related to the Seifert-Van Kampen theorem.

Proposition 2.10. (cf. [11, p. 176]) *Let X and Y be locally simply connected and path connected metric spaces. Then, the fundamental group of a wedge sum of X and Y is isomorphic to* $\pi_1(X) * \pi_1(Y)$.

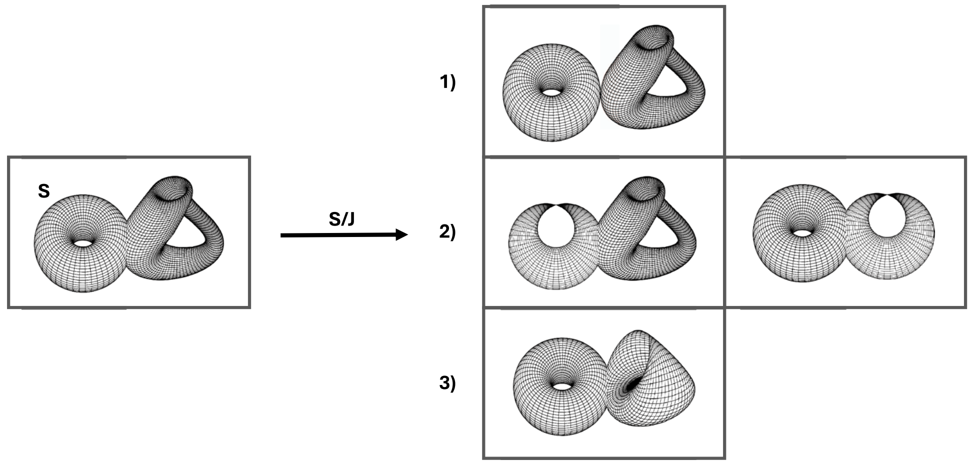

Figure 3: Classification of the Jordan curves in a non-orientable closed surface of connectivity number four. The surface S is situated on the left-hand side. On the opposite side, we see all possible quotient spaces S/J, where J is a Jordan curve in S. The numbering corresponds to the cases described in Proposition 2.9. In the third case, we see a connected sum of the torus and the real projective plane. The illustration of the real projective plane uses a parametrization taken from [10, p. 334].

Since metric 2-point identifications play an important role in our investigation, the next result provides a useful tool. It follows from the HNN-Seifert Van Kampen theorem in [9, pp. 1292–1293].

Proposition 2.11. *Let X be a locally simply connected and path-connected metric space. Then, the following statements apply*:
(1) *If Y is a topological 2-point identification of X, then the fundamental group $\pi_1(Y)$ is isomorphic to $\pi_1(X) * \mathbb{Z}$.*
(2) *If X contains a local cut point not being a cut point, then there is a group G such that the fundamental group $\pi_1(X)$ is isomorphic to $G * \mathbb{Z}$.*

Notation

$\mathcal{M}$ The class of compact metric spaces.
$\mathcal{S}(c)$ The class of closed length surfaces whose connectivity number is equal to c.
$\mathcal{S}(c, \varepsilon)$ The class of spaces in $\mathcal{S}(c)$ that do not contain non-contractible loops of diameter less than 2ε.
$\mathcal{G}(c)$ The class of geodesic generalized cactoids such that the connectivity numbers of their maximal cyclic subsets sum up to c.
$\mathcal{W}$ The class of successive metric wedge sums of non-degenerate cyclicly connected compact length spaces and finite metric trees.
$\mathcal{W}_0$ The class of successive metric wedge sums of closed length surfaces such that every wedge point is only shared by two of their surfaces.

We note that we allow a change of the wedge point in every construction step of a successive metric wedge sum. In Figure 4, we see a space in $\mathcal{W}_0$.

3 Topology via maximal cyclic subsets

In this section, we investigate the topological connection between maximal cyclic subsets and their ambient space. From the results, we derive first topological properties of generalized cactoids.

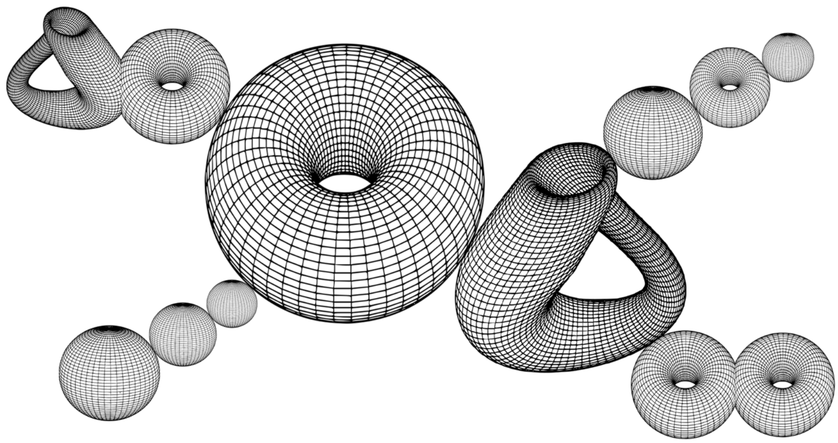

Figure 4: Space in W_0. The space is a successive metric wedge sum of ten closed length surfaces, and every wedge point is only shared by two of the surfaces.

3.1 Dimension bound

First, we consider the dimension of Peano spaces. Throughout this work, the term “dimension” refers to the covering dimension.

There are many properties that are satisfied by the whole Peano space provided that they are shared by all its maximal cyclic subsets (cf. [27, pp. 81–83]). The next result contains an example of such a property:

Proposition 3.1. *Let X be a Peano space. Then, the following statements apply*:
(1) *If all maximal cyclic subsets of X are at most n-dimensional, then X is at most n-dimensional.* (cf. [27, p. 82]).
(2) *If X is at most n-dimensional and Y is a metric 2-point identification of X, then Y is at most n-dimensional* (cf. [23, pp. 266, 271]).

As a consequence, we derive the following result about generalized cactoids:

Corollary 3.2. *Let X be a space that can be obtained by a successive application of metric 2-point identifications to a generalized cactoid. Then, X is at most two-dimensional.*

3.2 Local contractibility

Now, we consider the property of local contractibility.

A metric space X is called *locally strongly contractible* provided that every $x \in X$ has arbitrarily small open neighborhoods such that $\{x\}$ is a strong deformation retract of them. For example, X is locally strongly contractible if it is a manifold.

The following observation motivates the next result: we denote the circle of radius $1/n$ around the point $(1/n, 0) \in \mathbb{R}^2$ by C_n. Then, the subset $C \coloneqq \cup_{n\in\mathbb{N}} C_n$ is a Peano space whose maximal cyclic subsets are homeomorphic to the 1-sphere (Figure 5). Hence, the maximal cyclic subsets are locally strongly contractible. But C is also the Hawaiian earring, and it is well known that this space is not even semi-locally simply connected.

On the other hand, the union of finitely many circles in $\mathbb{R}^2$ is always locally strongly contractible.

Proposition 3.3. *Let X be a Peano space. Then, the following statements apply*:
(1) *If X contains infinitely many non-contractible maximal cyclic subsets, then X is not locally contractible.*
(2) *If X has only finitely many maximal cyclic subsets and all of them are locally strongly contractible, then X is locally strongly contractible.*

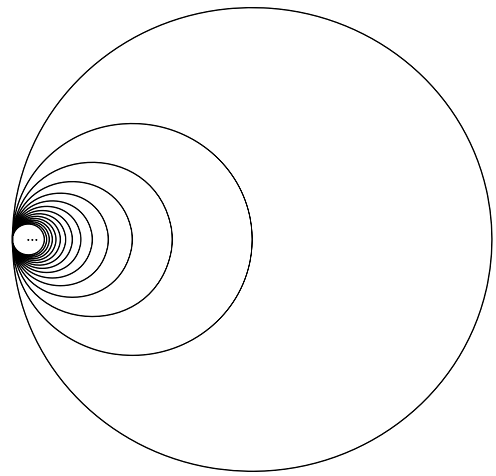

Figure 5: Hawaiian earring. A Peano space whose maximal cyclic subsets are locally strongly contractible but the space itself is not even semi-locally simply connected.

Proof. (1) There is a sequence of pairwise distinct non-contractible maximal cyclic subsets in X. From Lemma 2.7, it follows that there is a subsequence converging to some $p \in X$. Hence, an arbitrarily small open neighborhood of p contains a non-contractible maximal cyclic subset of X.

For the sake of contradiction, we assume that X is locally contractible. Then, there is a contractible open neighborhood U of p. This neighborhood contains some non-contractible maximal cyclic subset T of X. By Lemma 2.7, we have that T is a retract of U. Therefore, T is also contractible, which is a contradiction.

(2) We denote the number of maximal cyclic subsets in X by n.

If $n = 0$, then X is homeomorphic to a compact metric tree and hence locally strongly contractible (cf. [1, p. 20]).

If $n = 1$, we consider $p \in X$ and $\varepsilon > 0$. We may assume that p is contained in the only maximal cyclic subset T of X. Then, there is an open neighborhood U_0 of p in T with diameter less than ε such that $\{p\}$ is a strong deformation retract of U_0. Furthermore, we may assume that U_0 contains infinitely many cut points of X.

Let $(c_k)_{k\in\mathbb{N}}$ be an enumeration of these cut points. We write C_k for the closure of the union of all connected components of $X\setminus T$ whose boundaries are given by $\{c_k\}$. Then, C_k is a compact metric tree. Hence, there is an open neighborhood U_k of c_k in C_k with diameter less than ε such that $\{c_k\}$ is a strong deformation retract of U_k. In particular, there is a homotopy H_k between the identity map on U_k and the map sending every point of U_k to c_k such that $H_k(c_k, t) = c_k$ for every $t \in [0, 1]$.

Now, we set $U \coloneqq \cup_{k\in\mathbb{N}_0} U_k$ and define a map $H : U \times [0, 1] \to U$ by

$$H(x, t) = \begin{cases} H_k(x, t), & x \in U_k; \\ x, & x \in U_0. \end{cases}$$

The diameter of U is less than 2ε. From Lemma 2.7, we derive that U is an open neighborhood of p in X and H is continuous. Hence, U_0 is a strong deformation retract of U, and it follows that $\{p\}$ is a strong deformation retract of U. We conclude that X is locally strongly contractible.

If $n \geq 2$, then X is homeomorphic to a wedge sum of Peano spaces with less than n maximal cyclic subsets. If both spaces are locally strongly contractible, then so is X. Hence, the claim follows by induction. □

A metric space whose dimension is finite is an ANR if and only if it is locally contractible (cf. [23, pp. 347, 392]). Hence, Corollary 3.2 and the last proposition imply the following statement:

Corollary 3.4. *Let X be a generalized cactoid. Then, the following statements are equivalent*:
(1) *X is an ANR.*
(2) *X has only finitely many maximal cyclic subsets.*

3.3 Fundamental group formula

In this subsection, we present a formula for the fundamental group of a locally simply connected Peano space in terms of its maximal cyclic subsets. For this, we first reduce the complexity of the problem:

Lemma 3.5. *Let X be a compact length space and $(T_k)_{k=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then, X can be obtained as the limit of compact length spaces having only finitely many maximal cyclic subsets. Furthermore, the maximal cyclic subsets of the space with index n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^{n}$ for every $n \in \mathbb{N}$.*

Proof. Let $n, k \in \mathbb{N}$. We define an equivalence relation $\sim$ on X as follows: $x \sim y$ if and only if x and y lie in the same connected component of $\cup_{m=n+1}^{n+k} T_m$. Furthermore, we define $X_{n,k}$ as the metric quotient $X/\sim$ and denote the corresponding projection map by $p_{n,k}$. Then, $p_{n,k}$ is surjective and 1-lipschitz. Hence, we may assume that there is a space $X_n \in \mathcal{M}$ and a map $p_n : X \to X_n$ such that $X_{n,k} \longrightarrow X_n$ and $p_{n,k} \longrightarrow p_n$ uniformly.

Since the map $p_{n,k}$ is monotone and its restriction to T_m is distance preserving for every $m \in \{1, \dots, n\}$, the same applies to p_n (cf. [27, p. 174]).

Let T be a maximal cyclic subset of X_n. Then, we find some $k \in \mathbb{N}$ such that $T \subset p_n(T_k)$ (cf. [27, pp. 145–146]). Because p_n is constant on T_m for every $m \in \mathbb{N}$ with $m > n$, we have $k \leq n$. Hence, $p_n(T_k)$ is cyclicly connected, and we derive that the inclusion is an equality.

Due to the fact that for every non-degenerate cyclicly connected subset of X_n, there is a maximal cyclic subset containing it (cf. [27, p. 79]), we derive that $\{p_n(T_k)\}_{k=1}^{n}$ is the set of maximal cyclic subsets of X_n and has cardinality n.

Since p_n is surjective and 1-lipschitz, we may assume that there is $\tilde{X} \in \mathcal{M}$ with $X_n \longrightarrow \tilde{X}$. Choosing a diagonal sequence, we may assume that $X_{n,n} \longrightarrow \tilde{X}$ and $(p_{n,n})_{n \in \mathbb{N}}$ is uniformly convergent. Finally, the limit map is an isometry between X and $\tilde{X}$. □

Lemma 3.6. *Let X be a compact length space having only finitely many maximal cyclic subsets. Then, X can be obtained as the limit of spaces in $\mathcal{W}$. Furthermore, the maximal cyclic subsets of the spaces of the sequence are in isometric one-to-one correspondence with those of X.*

Proof. Let ε be the minimum of the diameters of the maximal cyclic subsets of X. If $n \in \mathbb{N}$ and T is a maximal cyclic subset of X, we denote the set of connected components of $X \setminus T$ having diameter less than ε/n by C_T. Moreover, we set $\mathcal{T}$ as the set of maximal cyclic subsets of X and $X_n \coloneqq X \setminus (\cup_{T \in \mathcal{T}} \cup_{C \in C_T} C)$.

The space X_n is a compact length space that has the same maximal cyclic subsets as X. Furthermore, it is an ε/n-net in X and it follows that the inclusion map from X_n to X is an ε/n-isometry. Hence, we have $X_n \longrightarrow X$.

Moreover, X_n is a successive metric wedge sum of its maximal cyclic subsets and finitely many compact metric trees. Let D be such a tree. We denote the wedge points lying in D by $p_1, \dots, p_N$. There is a sequence $(D_k)_{k \in \mathbb{N}}$ of finite metric trees converging to D. Furthermore, we may assume the existence of a sequence $(f_k)_{k \in \mathbb{N}}$ such that f_k is a $1/k$-isometry from D to D_k. We define $X_{n,k}$ as the successive metric wedge sum created by replacing D with D_k in X_n and the corresponding wedge points with $f_k(p_1), \dots, f_k(p_N)$.

This new space is again a compact length space such that its maximal cyclic subsets are in isometric one-to-one correspondence with those of X. In particular, we may assume that $X_{n,k} \in \mathcal{W}$. Otherwise, we repeat the argument above until every tree is replaced by a finite one. Moreover, we have that $X_{n,k} \longrightarrow X_n$ since f_k defines a $1/k$-isometry between X_n and $X_{n,k}$. Choosing a diagonal sequence, we may assume that $X_{n,n} \longrightarrow X$. □

Corollary 3.7. *Let X be a compact length space and $(T_k)_{k=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then, X can be obtained as the limit of spaces in $\mathcal{W}$. Furthermore, the maximal cyclic subsets of the space with index n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^{n}$ for every $n \in \mathbb{N}$.*

Every compact interval or point can be obtained as the limit of spaces in $\mathcal{S}(0)$. Hence, we can proceed as in the second half of the last proof to get rid of the finite metric trees first and then the cut points lying in more than one maximal cyclic subset. From this, we derive the following statement for later purposes:

Corollary 3.8. *Let X be a geodesic generalized cactoid and $(T_k)_{k=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then, X can be obtained as the limit of spaces in $\mathcal{W}_0$. Furthermore, the maximal cyclic subsets of the space with index n are in isometric one-to-one correspondence with $\{T_k\}_{k=1}^{n}$ and finitely many spaces in $\mathcal{S}(0)$ for every $n \in \mathbb{N}$.*

Now, we are able to state the formula:

Proposition 3.9. *Let X be a locally simply connected Peano space and $(T_n)_{n=1}^{\infty}$ be an enumeration of its maximal cyclic subsets. Then, $\pi_1(X)$ is isomorphic to $\pi_1(T_1) * \ldots * \pi_1(T_n)$ for almost all $n \in \mathbb{N}$.*

Proof. Since X is locally simply connected, the same applies to its maximal cyclic subsets. Moreover, there is some $\varepsilon > 0$ such that every loop in X of diameter less than ε lies in some simply connected subset of X. If T is a maximal cyclic subset of X, then also every loop in T of diameter less than ε lies in some simply connected subset of T.

By Theorem 2.6, we may assume that X is geodesic. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence as in Corollary 3.7. An application of Proposition 2.10 yields that the sequence is uniformly semi-locally simply connected. Hence, $\pi_1(X)$ is isomorphic to $\pi_1(X_n)$ for almost all $n \in \mathbb{N}$. From the same proposition, we also derive that $\pi_1(X_n)$ is isomorphic to $\pi_1(T_1) * \ldots * \pi_1(T_n)$ for every $n \in \mathbb{N}$. This completes the proof. □

3.4 Cactoids

As it will turn out, limits of spaces in $\mathcal{S}(c)$ locally look like cactoids (Corollary 4.7). Hence, we are interested in the topology of cactoids. Again, we first reduce the complexity of the problem:

Lemma 3.10. *Let $X \in \mathcal{G}(0)$. Then, X is homeomorphic to a space that can be obtained as the limit of compact length spaces whose maximal cyclic subsets are isometric to round 2-spheres.*

Proof. First, we may assume that there are infinitely many maximal cyclic subsets in X. Let $(T_n)_{n=1}^{\infty}$ be an enumeration of them. There is a homeomorphism f_n from T_n to the round 2-sphere of diameter $1/2^n$. We denote this 2-sphere by S_n.

The following condition naturally induces an equivalence relation ~ on the disjoint union of the closures of the connected components of $X \setminus T_1$ and S_1: $x \sim y$ if $f_1(x) = y$. We equip the disjoint union with its induced length metric and denote the corresponding metric quotient by Y_1. Analogously, we construct the space Y_2 using the disjoint union of the closures of the connected components of $Y_1 \setminus T_2$ and S_2. We continue this construction and derive a sequence $(Y_n)_{n\in\mathbb{N}}$ of metric spaces.

We note that there is an induced enumeration of the maximal cyclic subsets of Y_n. If $k \in \mathbb{N}$, we define the metric space $Y_{n,k}$ in the same way for Y_n as $X_{n,k}$ is defined for X in the proof of Lemma 3.5.

The maps $f_1, \ldots, f_n$ naturally induce a homeomorphism from X to Y_n. We denote the composition of this homeomorphism with the projection map from Y_n to $Y_{n,k}$ by $g_{n,k}$. Moreover, we find a map $p_{n,k} : Y_{n+1,k} \to Y_{n,k+1}$ such that the diagram together with the maps $g_{n+1,k}$ and $g_{n,k+1}$ commutes. This map is 1-lipschitz and has a distortion less than $1/2^n$. Furthermore, the sequence $(g_{n,k})_{k\in\mathbb{N}}$ is uniformly equicontinuous. We may assume that there is a space $\tilde{Y}_n$ such that $Y_{n,k} \longrightarrow \tilde{Y}_n$. In addition, we may assume the existence of maps $p_n : \tilde{Y}_{n+1} \to \tilde{Y}_n$ and $g_n : X \to \tilde{Y}_n$ such that $p_{n,k} \longrightarrow p_n$ and $g_{n,k} \longrightarrow g_n$ uniformly (cf. [19, p. 402]).

Using the proof of Lemma 3.5, we see that every maximal cyclic subset of $\tilde{Y}_n$ is isometric to a round 2-sphere. Furthermore, the diagram consisting of p_n, g_n and g_{n+1} commutes and p_n is a 1-lipschitz map that has a distortion less or equal to $1/2^n$. If we set $\varepsilon_k \coloneqq \sum_{m=k}^{\infty} 1/2^m$, then $p_k \circ \dots \circ p_{n-1}$ is an ε_k-isometry between $\tilde{Y}_n$ and $\tilde{Y}_k$ for every $n \in \mathbb{N}$ with $n > k$. Hence, $(\tilde{Y}_n)_{n\in\mathbb{N}}$ is convergent (cf. [19, p. 399]), and we denote its limit space by Y. Moreover, $(g_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous, and we may assume the existence of a map $g : X \to Y$ such that $g_n \longrightarrow g$ uniformly. Finally, g is bijective and hence a homeomorphism. □

Corollary 3.11. *Cactoids are locally simply connected and simply connected.*

Proof. In a successive metric wedge sum of round 2-spheres and finite metric trees, every open ball is simply connected. A theorem by Petersen implies that this property is stable under Gromov-Hausdorff convergence (cf. [18, p. 501]). Finally, the claim follows by Corollary 3.7 and Lemma 3.10. □

4 Limit spaces

The goal of this section is to show that the first statement of the main theorem implies the second. As a consequence, we derive Theorem 1.1.

4.1 Controlled convergence

First, we investigate the limit spaces of sequences with additional topological control.

A metric space X is called a *local cactoid* if every point in X has an open neighborhood being homeomorphic to an open subset of a cactoid.

Lemma 4.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c, \varepsilon)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then, X is a local cactoid.*

Proof. First, we may assume that all surfaces of the sequence are not homeomorphic to the 2-sphere. Otherwise, the claim follows by Theorem 2.5.

Let $x \in X$. Then, there is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ such that $x_n \longrightarrow x$. We define $\mathcal{D}$ as the set of all topological closed disks in X_n that are bounded by a Jordan curve in $B_\varepsilon(x_n)$. Moreover, we set A as the union of $B_\varepsilon(x_n)$ and the disks in $\mathcal{D}$.

It follows that A is open and connected. If J is a Jordan curve in $B_\varepsilon(x_n)$, then J is contractible in X_n since $X_n \in \mathcal{S}(c, \varepsilon)$. Hence, J bounds a topological closed disk in X_n (cf. [7, p. 85]), and we derive that J is contractible in A. This already yields that every loop in $B_\varepsilon(x_n)$ is contractible in A (cf. [15, p. 626]).

Let $\gamma : [0, 1] \to A$ be a loop. The set consisting of $B_\varepsilon(x_n)$ and the interiors of all disks in $\mathcal{D}$ is an open cover of $\gamma([0, 1])$. Since γ is uniformly continuous and its image is compact, there is a finite subdivision $t_0 \coloneqq 0 < t_1 < \dots < t_k \coloneqq 1$ of the unit interval such that $\gamma([t_i, t_{i+1}]) \subset B_\varepsilon(x_n)$ or we find some $D \in \mathcal{D}$ whose interior contains $\gamma([t_i, t_{i+1}])$. An induction over the number of subcurves not lying in $B_\varepsilon(x_n)$ finally shows that γ is homotopic to some loop in $B_\varepsilon(x_n)$.

We deduce that A is simply connected and conclude that A is homeomorphic to the plane or the 2-sphere (cf. [7, p. 85]). Because X_n is not homeomorphic to the 2-sphere, the first case applies. We derive that the metric quotient $Y_n \coloneqq X_n/A^c$ is homeomorphic to the 2-sphere. Especially, we have $Y_n \in \mathcal{S}(0)$.

Since the natural projection $p_n : X_n \to Y_n$ is surjective and 1-lipschitz, we may assume that there is a space $Y \in \mathcal{M}$ and a map $p : X \to Y$ such that $Y_n \longrightarrow Y$ and $p_n \longrightarrow p$. From Theorem 2.5, we obtain that Y is a cactoid. Furthermore, we may assume that the sequences $(\bar{B}_{\frac{\varepsilon}{2}}(x_n))_{n\in\mathbb{N}}$ and $(\bar{B}_{\frac{\varepsilon}{2}}(p_n(x_n)))_{n\in\mathbb{N}}$ are convergent. Their limits are given by $\bar{B}_{\frac{\varepsilon}{2}}(x)$ and $\bar{B}_{\frac{\varepsilon}{2}}(p(x))$. Because p_n defines an isometry between $\bar{B}_{\frac{\varepsilon}{2}}(x_n)$ and $\bar{B}_{\frac{\varepsilon}{2}}(p_n(x_n))$, the same applies to p with respect to $\bar{B}_{\frac{\varepsilon}{2}}(x)$ and $\bar{B}_{\frac{\varepsilon}{2}}(p(x))$. In particular, this also holds for the corresponding open balls. We finally conclude that X is a local cactoid. □

Corollary 4.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c,\varepsilon)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then, the following statements apply*:

(1) *X is locally simply connected.*

(2) *If $c = 0$, then the maximal cyclic subsets of X are simply connected.*

(3) *If $c > 0$, then there is a closed surface S of connectivity number c such that the fundamental group of one maximal cyclic subset of X is isomorphic to $\pi_1(S)$ and all other maximal cyclic subsets are simply connected. Moreover, S is orientable if and only if X_n is orientable for infinitely many $n \in \mathbb{N}$.*

Proof. Combining Corollary 3.11 and Lemma 4.1, we derive the first statement. Since the sequence is uniformly semi-locally simply connected, it follows that $\pi_1(X)$ isomorphic to $\pi_1(X_n)$ for almost all $n \in \mathbb{N}$. Hence, Propositions 2.8 and 3.9 close the proof. □

From the upcoming lemma follows that limits of spaces in $\mathcal{S}(c,\varepsilon)$ are generalized cactoids:

Lemma 4.3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c,\varepsilon)$, $X \in \mathcal{M}$ with $X_n \longrightarrow X$ and T be a maximal cyclic subset of X. Then, T is a closed surface.*

Proof. First we show that T is free of local cut points: For the sake of contradiction, we assume that T contains a local cut point. Due to Corollary 4.2, the space X is locally simply connected. From Proposition 2.11, it follows that there is a group G such that $\pi_1(T)$ is isomorphic to $G * \mathbb{Z}$. Moreover, Corollary 4.2 yields that $\pi_1(T)$ is isomorphic to the fundamental group of some closed surface. This contradicts Proposition 2.8.

Let $p \in T$. Then, Lemma 4.1 implies that there is a connected open neighborhood V of p in X and a homeomorphism f from V to an open subset of some cactoid C. Furthermore, Lemma 2.7 yields that $T \cap V$ is connected. Since T is free of local cut points, there is a Jordan curve J in $V \cap T$. In particular, $f(J)$ is contained in some maximal cyclic subset S of C.

The subset $f(V) \cap S$ is connected and S is free of local cut points. Therefore, we derive that $f(V \cap T) = f(V) \cap S$. Hence, $V \cap T$ is homeomorphic to an open subset of the 2-sphere.

We conclude that T is a surface. Especially, T is a closed surface because $\pi_1(T)$ is isomorphic to the fundamental group of some closed surface. □

Combining the last lemma and Corollary 4.2, we obtain the following result:

Corollary 4.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c,\varepsilon)$, where $c > 0$, and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then, one maximal cyclic subset T of X is a closed surface of connectivity number c, and all other maximal cyclic subsets are homeomorphic to the 2-sphere. Moreover, T is orientable if and only if X_n is orientable for infinitely many $n \in \mathbb{N}$.*

4.2 General case

Next, we see what happens if we omit the additional topological control.

If a sequence in $\mathcal{S}(c)$ has no uniformly semi-locally simply connected subsequence, then there is a subsequence and a sequence of Jordan curves as in the following lemma:

Lemma 4.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Furthermore, let $(J_n)_{n\in\mathbb{N}}$ be a sequence such that J_n is a non-contractible Jordan curve in X_n for every $n \in \mathbb{N}$ and $\operatorname{diam}(J_n) \longrightarrow 0$. Then, one of the following cases applies*:

(1) *There are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$, a convergent sequence of spaces in $\mathcal{S}(c_1)$ and a convergent sequence of spaces in $\mathcal{S}(c_2)$ such that X is a metric wedge sum of their limits. If X_n is non-orientable for infinitely many $n \in \mathbb{N}$, then the surfaces of at least one of the sequences may be chosen to be non-orientable.*

(2) *There is a convergent sequence of spaces in $\mathcal{S}(c-2)$ such that X is its limit or a metric 2-point identification of it.*
(3) *There is a sequence of spaces in $\mathcal{S}(c-1)$ converging to X.*

If X_n is orientable for infinitely many $n \in \mathbb{N}$, then always one of the first two cases applies and the surfaces of the corresponding sequences may be chosen to be orientable.

Proof. First, we may assume that all surfaces of the sequence are orientable or all are non-orientable. Moreover, we may assume that the Jordan curves all belong to the same class in the sense of Proposition 2.9. In particular, we have that the corresponding sequence $(Y_n)_{n\in\mathbb{N}}$ of metric quotients is convergent with limit X. We just go through the cases of the proposition:

In the first case, there are $c_1, c_2 \in \mathbb{N}$ with $c_1 + c_2 = c$ such that Y_n is a metric wedge sum of a space in $\mathcal{S}(c_1)$ and a space in $\mathcal{S}(c_2)$. Especially, at least one of the surfaces is non-orientable if and only if X_n is non-orientable. Moreover, we may assume that the sequence of the wedge points and the sequences of the surfaces considered as subsets of the wedge sums are convergent. We derive that X is a metric wedge sum of the limits (cf. [26, p. 412]).

Now, we consider the second case: then, there is a space Z_n in $\mathcal{S}(c-2)$ such that Y_n is a metric 2-point identification of Z_n. In particular Z_n is orientable if X_n is orientable. Furthermore, we may assume the sequence $(Z_n)_{n\in\mathbb{N}}$ to be convergent. It follows that X is the limit or a metric 2-point identification of it.

If we look at the third case, then we have that Y_n is a space in $\mathcal{S}(c-1)$ and X_n is non-orientable. □

The final result of this section refines a statement of the main theorem:

Theorem 4.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then, X can be obtained by a successive application of k metric 2-point identifications to some space $Y \in \mathcal{G}(c_0)$, where $c_0 \leq c - 2k$. Moreover, the following statements apply:*
(1) *If X_n is orientable for infinitely many $n \in \mathbb{N}$, then the maximal cyclic subsets of Y are orientable.*
(2) *If X_n is non-orientable for infinitely many $n \in \mathbb{N}$ and the maximal cyclic subsets of Y are orientable, then $c_0 < c$.*

Proof. The proof proceeds by induction over the connectivity number:

In the case $c = 0$, the claim directly follows by Theorem 2.5.

Now, we consider the case $c > 0$. Moreover, we assume that the claim is true, if the connectivity number is less than c. Provided that the sequence is uniformly semi-locally simply connected, the claim directly follows by Corollary 4.4. Otherwise, we may assume that there is a sequence $(J_n)_{n\in\mathbb{N}}$ such that J_n is a non-contractible Jordan curve in X_n and $\operatorname{diam}(J_n) \longrightarrow 0$. Hence, one of the cases of Lemma 4.5 applies. We note that the surfaces of the sequences occurring there have a connectivity number less than c.

Finally, an application of the induction hypothesis and the following observation yield the claim: let Y_1 and Y_2 be metric spaces. Furthermore, let Z_i be a space that can be obtained by a successive application of k_i metric 2-point identifications to Y_i. Then, every metric wedge sum of Z_1 and Z_2 is a space that can be obtained by a successive application of $k_1 + k_2$ metric 2-point identifications to a metric wedge sum of Y_1 and Y_2. Moreover, every metric 2-point identification of Z_1 is a space that can be obtained by a successive application of $k_1 + 1$ metric 2-point identifications to Y_1. □

The property of being a local cactoid is stable under applications of metric wedge sums and metric 2-point identifications. Hence, the aforementioned induction also yields the following result:

Corollary 4.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(c)$ and $X \in \mathcal{M}$ with $X_n \longrightarrow X$. Then, X is a local cactoid.*

Finally, we derive Theorem 1.1:

Proof of Theorem 1.1. From Corollary 3.2, it follows that X is at most two-dimensional. Furthermore, Corollaries 3.11 and 4.7 imply that X is locally simply connected. Finally, Propositions 2.11 and 3.9 yield the desired representation of $\pi_1(X)$. □

5 Approximation of generalized cactoids

The aim of this section is to prove that the second statement of the main theorem implies the first. At the end, we show Corollary 1.2.

We start by showing that metric wedge sums of closed length surfaces can be approximated by closed length surfaces:

Lemma 5.1. *Let S_1 and S_2 be closed length surfaces and X be a metric wedge sum of them. Then, X can be obtained as the limit of length spaces being homeomorphic to a connected sum of S_1 and S_2.*

Proof. We denote the wedged points by p_1 and p_2. For every $i \in \{1, 2\}$ and $n \in \mathbb{N}$, there is a topological closed disk $D_{i,n}$ of diameter less than $1/n$ in S_i that contains p_i in its interior. Moreover, we may assume $D_{i,n}$ to be bounded by a piecewise geodesic Jordan curve $J_{i,n} : [0, b_{i,n}] \to X$ (cf. [24, p. 1794], [26, pp. 413–415]).

Using the Kuratowski embedding, we identify S_i with a subset of $l^\infty(X)$. Furthermore, we define $\tilde{D}_{i,n}$ to be the union of all linear segments from p_i to a point of $\partial D_{i,n}$. By direct calculation using the embedding, we see that linear segments from p_i to distinct points of S_i only intersect in p_i. Therefore, we obtain that $F_{i,n} \coloneqq \tilde{D}_{i,n} \cup (S_i \backslash D_{i,n})$ is homeomorphic to S_i. Now, we equip $F_{i,n}$ with its induced length metric and denote the obtained space by $S_{i,n}$. It follows that the identity map is a homeomorphism between $F_{i,n}$ and $S_{i,n}$. Moreover, we have $S_{i,n} \longrightarrow S_i$.

For every $\lambda \in (0, 1]$, the map $\gamma_{i,\lambda}(t) \coloneqq \lambda J_{i,n}(\frac{t}{\lambda}) + (1 - \lambda)p_i$ is a piecewise geodesic Jordan curve in $S_{i,n}$. Moreover, the curve bounds a topological closed disk $B_{i,\lambda}$ that contains p_i in its interior. We have that $l_{i,\lambda} \coloneqq \mathrm{length}(\gamma_{i,\lambda}) \longrightarrow 0$, if $\lambda \longrightarrow 0$, and there are sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ in $(0, 1]$ converging to 0 such that $l_{1,\alpha_k} = l_{2,\beta_k}$.

We assume the subsets in the upcoming construction to be equipped with their induced length metric. There is a natural equivalence relation ~ on $(S_{1,n} \backslash \mathring{B}_{1,\alpha_k}) \sqcup (S_{2,n} \backslash \mathring{B}_{2,\beta_k})$ defined by the following condition: $x \sim y$ if there is some $t \in [0, l_{1,\alpha_k}]$ such that $x = \gamma_{1,\alpha_k}(t)$ and $y = \gamma_{2,\beta_k}(t)$. We denote the corresponding metric quotient by $\tilde{S}_{n,k}$ and note that this space is homeomorphic to a connected sum of S_1 and S_2 (cf. [4, p. 69]).

Now, $(\tilde{S}_{n,k})_{k\in\mathbb{N}}$ converges to a metric wedge sum W_n of $S_{1,n}$ and $S_{2,n}$ along p_1 and p_2. Since $W_n \longrightarrow X$, we may complete the proof. □

From Corollary 3.8 and the last lemma, we inductively obtain the following result:

Corollary 5.2. *Let $X \in \mathcal{G}(c)$. Then, there is a sequence of spaces in $\mathcal{S}(c)$ converging to X. Moreover, the following statements apply*:
(1) *If all maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.*
(2) *If there is a non-orientable maximal cyclic subset in X, then the surfaces of the sequence may be chosen to be non-orientable.*

A similar argument to that used in the proof of Lemma 5.1 shows the following result concerning metric 2-point identifications:

Lemma 5.3. *Let $S \in \mathcal{S}(c)$ and X be a metric 2-point identification of S. Then, there is a sequence of spaces in $\mathcal{S}(c + 2)$ converging to X. Moreover, the following statements apply*:
(1) *The surfaces of the sequence may be chosen to be non-orientable.*
(2) *If S is orientable, then the surfaces of the sequence may be chosen to be orientable.*

Now we combine the last two results:

Lemma 5.4. *Let Y be a space that can be obtained by a successive application of k metric 2-point identifications to a space $X \in \mathcal{G}(c)$. Then, there is a sequence of spaces in $\mathcal{S}(c+2k)$ converging to Y. Moreover, the following statements apply*:

(1) *If there is a non-orientable maximal cyclic subset in X or $k > 0$, then the surfaces of the sequence may be chosen to be non-orientable.*

(2) *If all maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.*

Proof. The proof proceeds by induction over k:

In the case $k = 0$, the claim directly follows by Corollary 5.2.

Now, we consider the case $k > 0$. Furthermore, we assume that the claim is true for every $k_0 \in \mathbb{N}$ with $k_0 < k$. There is a space Z that can be obtained by a successive application of $k-1$ metric 2-point identifications to X such that Y is a metric 2-point identification of Z. By the induction hypothesis, there is a sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(c+2(k-1))$ with $Z_n \longrightarrow Z$ as in the claim.

Let $z_1, z_2 \in Z$ be the points that are glued to construct Y. There is a sequence $(z_{i,n})_{n\in\mathbb{N}}$ with $z_{i,n} \in Z_n$ converging to z_i. In particular, we may assume that $z_{1,n}$ and $z_{2,n}$ are distinct. If Y_n denotes a metric 2-point identification of Z_n along $z_{1,n}$ and $z_{2,n}$, then we have $Y_n \longrightarrow Y$.

Choosing a diagonal sequence, the claim follows by Lemma 5.3. □

Using a metric wedge sum with a vanishing sequence of length spaces all being homeomorphic to the torus or all being homeomorphic to the real projective plane, we derive a further corollary of Lemma 5.1:

Corollary 5.5. *Let $S \in \mathcal{S}(c)$. Then, the following statements apply*:

(1) *S can be obtained as the limit of non-orientable closed length surfaces whose connectivity number is equal to $c+1$.*

(2) *If S is orientable, then S can be obtained as the limit of orientable closed length surfaces whose connectivity number is equal to $c+2$.*

Now, the last two results provide all tools to prove the final result of this section, which refines the remaining direction of the main theorem:

Theorem 5.6. *Let $c \in \mathbb{N}_0$ and Y be space that can be obtained by a successive application of k metric 2-point identifications to a space $X \in \mathcal{G}(c_0)$, where $c_0 \leq c - 2k$. Then, there is a sequence in $\mathcal{S}(c)$ converging to Y. Moreover, the following statements apply:*

(1) *If all maximal cyclic subsets of X are orientable, then the surfaces of the sequence may be chosen to be orientable.*

(2) *If there is a non-orientable maximal cyclic subset in X or $c_0 < c$, then the surfaces of the sequence may be chosen to be non-orientable.*

We note that the last theorem and Theorem 4.6 completely describe the Gromov-Hausdorff closure of the class of length spaces being homeomorphic to a fixed closed surface.

Finally, we are able to prove Corollary 1.2:

Proof of Corollary 1.2. Let Y be a space that can be obtained by a successive application of metric 2-point identifications to a generalized cactoid. By Theorem 1.1 and the main theorem all such spaces are locally simply connected. Hence, Propositions 2.11 and 3.9 imply that Y is simply connected if and only if Y is a cactoid.

From Theorem 5.6, it follows that every geodesic cactoid can be obtained as the limit of length spaces being homeomorphic to S. Now, the claim follows by the main theorem and Corollary 3.4. □

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