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Correction to: On the two-dimensional Boussinesq equations with temperature-dependent thermal and viscosity diffusions in general Sobolev spaces

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We make the following corrections:

1. We remove the endpoint results concerning the propagation of the initial regularities $(\theta_0, u_0) \in H^2(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$ resp. $H^1(\mathbb{R}^2) \times (H^2(\mathbb{R}^2))^2$ by the two dimensional viscous Boussinesq flow in [1, Theorem 1.2], which correspond to the two endpoints $(2, 0)$ resp. $(1, 2)$ in [1, Figure 1]. See the grey points in Fig. 1 below for the corrected admissible regularity exponent set $\{(s_\theta, s_u) \in [1, \infty) \times [0, \infty) \,|\, s_u - 1 \leqslant s_\theta \leqslant s_u + 2\} \backslash \{(2, 0), (1, 2)\}$.
We remove the two corresponding estimates [1, (1.18) and (1.20)] and their proofs in [1, Subsection 2.3.2 and Subsection 2.3.3] for the two endpoint cases.
2. We correct the proof of the uniqueness result in [1, Theorem 1.2] under the initial data assumption $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, which corresponds to the endpoint $(1, 0)$ in Fig. 1.
We remove the second paragraph concerning the optimality of the uniqueness result in [1, Remark 1.3]. We correct $\nu \in (s - 1, 1) \subset (-1, 1)$ into $\nu \in (\max\{0, s - 1\}, 1)$ in [1, (2.34)].

The proofs of the results for the three endpoint regularity cases $(s_\theta, s_u) = (2, 0)$, or $(1, 2)$, or $(1, 0)$ stated in [1, Theorem 1.2] were wrong due to the failure of the embedding $L^1(\mathbb{R}^2) \not\hookrightarrow H^{-1}(\mathbb{R}^2)$. We sketch the corrected proofs below, using the same notations as well as the numbering of equations as in [1].

1. Since we remove the two endpoints $(2, 0)$ and $(1, 2)$ in Fig. 1, that is, the two (technical) inequalities in [1, (1.18) and (1.20)]:
   - H^2-Estimate for θ, if $u \in L^\infty_{\text{loc}}([0, \infty); (L^2(\mathbb{R}^2))^2) \cap L^2_{\text{loc}}([0, \infty); (H^1(\mathbb{R}^2))^2)$;
   - H^2-Estimate for u, if $\theta \in L^\infty_{\text{loc}}([0, \infty); H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}([0, \infty); H^2(\mathbb{R}^2))$,

   we have to show
   - H^2-Estimate for θ, if $u \in L^\infty_{\text{loc}}([0, \infty); (H^{0+}(\mathbb{R}^2))^2) \cap L^2_{\text{loc}}([0, \infty); (H^{1+}(\mathbb{R}^2))^2)$;
   - H^2-Estimate for u, if $\theta \in L^\infty_{\text{loc}}([0, \infty); H^{1+}(\mathbb{R}^2)) \cap L^2_{\text{loc}}([0, \infty); H^{2+}(\mathbb{R}^2))$,

   such that the vertical line $\{(2, s_u) \,|\, s_u \in (0, 2]\}$ and the horizontal line $\{(s_\theta, 2) \,|\, s_\theta \in (1, 2]\}$ are included in the admissible regularity exponent set.
   More precisely, if $u \in L^\infty_{\text{loc}}([0, \infty); (H^\varepsilon(\mathbb{R}^2))^2) \cap L^2_{\text{loc}}([0, \infty); (H^{1+\varepsilon}(\mathbb{R}^2))^2)$ for some $\varepsilon \in (0, 1)$, then we have (instead of [1, (1.18)])

$$\begin{aligned} \|\theta\|^2_{L^\infty_T H^2_x} + \|\nabla\theta\|^2_{L^2_T H^2_x} &\leqslant C(\kappa_*, \|a\|_{C^2}, \kappa^*)\|\theta_0\|^2_{H^2}(1 + \|\nabla\theta_0\|^2_{L^2}) \\ &\quad \times \exp\Big(C(\kappa_*, \varepsilon, \|a\|_{\text{Lip}})(\|u\|^2_{L^2_T H^{1+\varepsilon}_x} + \|u\|^4_{L^4_T L^4_x} + \|\nabla\theta\|^4_{L^4_T L^4_x})\Big), \end{aligned} \tag{(1.18)'}$$

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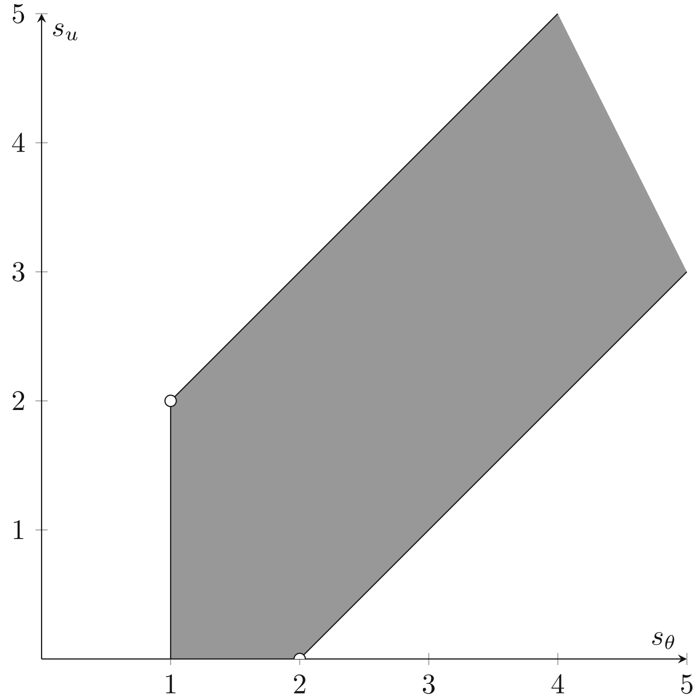

FIG. 1. Admissible regularity exponents

which follows from the same argument as in [1, Subsection 2.3.2], but with the following inequality (instead of [1, the inequality at the top of Page 16])

$$\int_{\mathbb{R}^2} |\nabla\Delta\eta\cdot\nabla u\cdot\nabla\eta|\,dx \leqslant \frac{\kappa^*}{4}\|\nabla\Delta\eta\|_{L^2_x}^2 + C(\kappa^*)\|\nabla u\cdot\nabla\eta\|_{L^2_x}^2$$

$$\leqslant \frac{\kappa^*}{4}\|\nabla\Delta\eta\|_{L^2_x}^2 + C(\kappa^*)\|\nabla u\|^2_{L_x^{\frac{2}{1-\varepsilon}}}\|\nabla\eta\|^2_{L_x^{\frac{2}{\varepsilon}}}$$

$$\leqslant \frac{\kappa^*}{4}\|\nabla\Delta\eta\|_{L^2_x}^2 + C(\kappa^*,\varepsilon)\|\nabla u\|^2_{H^\varepsilon_x}\|\nabla\eta\|^2_{H^1_x}.$$

Similarly, we have (instead of [1, (1.20)])

$$\begin{aligned}\|u\|^2_{L^\infty_T H^2_x} + \|\nabla u\|^2_{L^2_T H^2_x} &\leqslant (\|u\|^2_{L^\infty_T H^1_x} + \|\nabla u\|^2_{L^2_T H^1_x}) \\ &+ C\Big(\|\Delta u_0\|^2_{L^2_x} + \|u\|^2_{L^\infty_T H^1_x\cap L^2_T\dot H^2_x}(\|u\|^2_{L^\infty_T H^1_x\cap L^2_T\dot H^2_x} + \|\nabla\theta\|^2_{L^2_T H^{1+\varepsilon}_x}) \\ &+ \|\Delta\theta\|_{L^2_T L^2_x}\|\Delta u\|_{L^2_T L^2_x}\Big)\times\exp\Big(C\big(\|(u,\nabla\theta)\|^4_{L^4_T L^4_x} + \|\nabla^2\theta\|^2_{L^2_T H^\varepsilon_x}\big)\Big).\end{aligned} \tag{(1.20)'}$$

where the constant C depends on $\mu_*, \varepsilon, \|b\|_{C^2}, \|\theta\|_{L^\infty_T H^{1+\varepsilon}_x}, \|\nabla\theta\|_{L^2_T H^1_x}$.

We remark here that in general we can not show $\nabla\Delta\eta\cdot\nabla u\cdot\nabla\eta \in L^1_{\mathrm{loc}}L^1_x$ if $(\eta,u)\in L^\infty_{\mathrm{loc}}(H^2_x\times(L^2_x)^2)\cap L^2_{\mathrm{loc}}(H^3_x\times(H^1_x)^2)$ because of the failure of the Sobolev embedding $H^1(\mathbb{R}^2)\not\hookrightarrow L^\infty(\mathbb{R}^2)$.

2. We now correct the proof of the uniqueness result with initial data $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$ given in [1, Section 2.2]. We are going to show $H^{1+\delta} \times H^\delta$, $\delta \in (-1,0)$-Estimates (instead of $H^1 \times L^2$-Estimates in [1]) for the difference $(\dot\theta, \dot u)$ of two weak solutions (θ_1, u_1) and (θ_2, u_2) satisfying

$$\theta_1, \theta_2 \in C([0,\infty); H^1(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty); H^2(\mathbb{R}^2)),$$
$$u_1, u_2 \in C([0,\infty); (L^2(\mathbb{R}^2))^2) \cap L^2_{\mathrm{loc}}([0,\infty); (H^1(\mathbb{R}^2))^2),$$

following the arguments in [1, Subsection 2.3.1].

More precisely, we first observe that by virtue of the estimates in [1, (1.13) and (1.14)],

$$B(t) := 1 + \|(\nabla u_1, \nabla u_2)\|^2_{L^2_x} + \|(u_1, u_2, \nabla\eta_1, \nabla\eta_2)\|^4_{H^{\frac12}_x} + \|(\nabla\eta_1, \nabla\eta_2)\|^2_{H^1_x}$$
$$\in L^1_{\mathrm{loc}}([0,\infty)),$$

where $\eta = A(\theta) := \int_0^\theta a(\alpha)d\alpha$ is the function introduced in [1, (2.10)]. Following the arguments in the proof of [1, Lemma 2.2, Subsection 2.3.1], we derive the $H^{\delta+1} \times H^\delta$-Estimates for $(\dot\eta, \dot u)$ which satisfies the equations in [1, (2.19)], in the following three steps:

Step 1. We use the commutator estimate in [1, (2.28)] for $\delta \in (-1,0)$ and $j \geqslant -1$

$$\|[u_1, \Delta_j]\nabla\dot\eta\|_{L^2_x} \leqslant C l_j 2^{-j\delta} \|\nabla u_1\|_{L^2_x} \|\nabla\dot\eta\|_{H^\delta_x},$$
$$\|[\kappa_1, \Delta_j]\Delta\dot\eta\|_{L^2_x} \leqslant C l_j 2^{-j\delta} \|\nabla\kappa_1\|_{H^{\frac12}_x} \|\Delta\dot\eta\|_{H^{\delta-\frac12}_x},$$

with $(l_j)_{j\geqslant -1} \in \ell^1$, and the product estimates

$$\|\dot\kappa\Delta\eta_2\|^2_{L^2_T H^\delta_x} \lesssim \int_0^T \|\dot\kappa\|^2_{H^{\delta+1}_x} \|\Delta\eta_2\|^2_{L^2_x}\mathrm{dt},$$

$$\|\dot u \cdot \nabla\eta_2\|^2_{L^2_T H^\delta_x} \lesssim \int_0^T \Big(\|\dot u\|^2_{H^\delta_x}\|\nabla\eta_2\|^2_{H^1_x} + \|\nabla\dot u\|^2_{H^{\delta-\frac12}_x}\|\nabla\eta_2\|^2_{H^{\frac12}_x}\Big)\mathrm{dt}.$$

By use of interpolation inequalities, Gagliardo-Nirenberg's inequalities, Young's inequalities and Hölder's inequalities, we derive

$$\|\dot\eta\|^2_{L^\infty_T H^{\delta+1}_x} + \|\nabla\dot\eta\|^2_{L^2_T H^{\delta+1}_x} \leqslant C(\delta, \kappa_*, \|a\|_{C^2}, \|(\theta_1,\theta_2)\|_{L^\infty_T H^1_x})$$
$$\times \int_0^T (\|\dot u\|^2_{H^\delta_x} + \|\dot\eta\|^2_{H^{\delta+1}_x})B(t)\mathrm{dt} + \frac12\int_0^T \|\nabla\dot u\|^2_{H^\delta_x}\mathrm{dt}.$$

Step 2. Similarly as Step 1, we derive the following estimate

$$\|\dot u\|^2_{L^\infty_T H^\delta_x} + \|\nabla\dot u\|^2_{L^2_T H^\delta_x}$$
$$\leqslant C(\delta, \mu_*, \|b\|_{C^2}, \|(\theta_1,\theta_2)\|_{L^\infty_T H^1_x}) \int_0^T (\|\dot\eta\|^2_{H^{\delta+1}_x} + \|\dot u\|^2_{H^\delta_x})B(t)\mathrm{dt}.$$

Step 3. We sum the above two estimates up, to derive

$$\|\dot\eta\|^2_{L^\infty_T H^{\delta+1}_x} + \|\nabla\dot\eta\|^2_{L^2_T H^{\delta+1}_x} + \|\dot u\|^2_{L^\infty_T H^\delta_x} + \|\nabla\dot u\|^2_{L^2_T H^\delta_x}$$
$$\leqslant C(\delta, \kappa_*, \mu_*, \|(a,b)\|_{C^2}, \|(\theta_1,\theta_2)\|_{L^\infty_T H^1_x}) \int_0^T (\|\dot\eta\|^2_{H^{\delta+1}_x} + \|\dot u\|^2_{H^\delta_x})B(t)\mathrm{dt}.$$

Finally, the Gronwall's inequality implies $\dot{\eta} = 0$ and $\dot{u} = 0$. The uniqueness result follows.

We remark here that by virtue of the definition of $B(t)$ we do not expect the uniqueness result below the regularity assumption $(\theta_0, u_0) \in H^1(\mathbb{R}^2) \times (L^2(\mathbb{R}^2))^2$, which is critical by view of the Navier–Stokes-type equation for u and the temperature-dependent diffusion coefficients.

References

[1] He, Z., Liao, X.: On the two-dimensional Boussinesq equations with temperature-dependent thermal and viscosity diffusions in general Sobolev spaces. Z. Angew. Math. Phys. **73**, Paper No. 16, 25 (2022)

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