Numerical Stability Analysis with Polymorphic Uncertainty Modelling under Consideration of Spatial Varying Imperfections

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Abstract. Imperfections have a significant influence on critical buckling loads of structures. These imperfections can be variations of the nominal geometry or a spatial variability of the material parameters. Regulations are not often available for complex structures like cylindrical shells. The numerical stability analysis of structures requires the consideration of data uncertainty. The spatial variation of uncertain geometric and material properties can be described by random fields. A wide variety of methods for the simulation of Gaussian stochastic processes are available. Often, the Karhunen-Loève Expansion or - for an optimal discretization of the random field - the EOLE method is used. In all methods a correlation parameter has to be defined to control the shape of the random field and hence the shape of imperfections. The proper determination of such a correlation parameter is problematic to represent imperfections close to reality. If measured results are available a correlation length can be approximately determined. This choice is more difficult in case of material imperfections, where the spatial variation of stiffness values is often unknown. In fact, that the uncertain input parameters are fuzzy, a better way to determine a realistic correlation length is the application of polymorphic uncertainty models. The stochastic analysis is extended to the fuzzy randomness. The polymorphic uncertainty models are introduced in [6] and a good overview is given in [2]. In this paper, the correlation length for geometrical imperfections is described as a fuzzy input value. The fuzzification of the correlation parameter leads to a more realistic consideration of it's uncertainty. The goal is a robust fuzzy computational model for a better representation of the uncertainty in the lack of input data for numerical buckling analyses of thin walled structures.

1 Introduction

In structural engineering, the need of thin-walled structures is becoming increasingly important. Because of the slim design, the loss of stability becomes more relevant and imperfections have a major impact on the critical buckling load. In Eurocode 3 (DIN EN 1993), the standard for design and construction of steel structures a conventional deterministic approach is proposed. Here, the material and geometrical imperfections should be taken into account in the form of critical eigenmodes with a specified amplitude for the FE-Model. Beside the accuracy in computing methods, a detailed representation of the imperfections for the numerical model is needed. Because of their random character, the spatial variation of uncertain geometric and material properties can be described by random fields. If a structure is loaded with imperfections, no distinct stability point occurs, which is shown in Figure 1. The different postcritical behaviour between beam and plate structures can be seen clearly. The critical displacement u_{crit} and load p_{crit} are calculated with an eigenvalue analysis. The tangent stiffness matrix can be divided in linear $\boldsymbol{K}_{lin}$ and nonlinear parts $\boldsymbol{K}_{nlin}$. If available, the nonlinear terms $\boldsymbol{K}_{nlin}$ can be identified as $\boldsymbol{K}_u$, the initial displacement matrix and $\boldsymbol{K}_\sigma$, the geometrical matrix,

$$\boldsymbol{K}_T = \boldsymbol{K}_{lin} + \boldsymbol{K}_{nlin} = \boldsymbol{K}_{lin} + \boldsymbol{K}_u + \boldsymbol{K}_\sigma \quad . \tag{1}$$

Consequently, an eigenvalue problem for a linear buckling analysis can be constructed via

$$[\boldsymbol{K}_{lin} + \Lambda(\boldsymbol{K}_u + \boldsymbol{K}_\sigma)]\,\boldsymbol{\varphi} = \boldsymbol{0} \quad . \tag{2}$$

If a linear pre-buckling behaviour is observed, the result of the eigenvalue problem is a critical load factor Λ and the critical load can be calculated with the relation

$$\boldsymbol{P}_{crit} = \Lambda\, \boldsymbol{P} \quad . \tag{3}$$

After the stability point of the perfect structure is calculated, random imperfections are

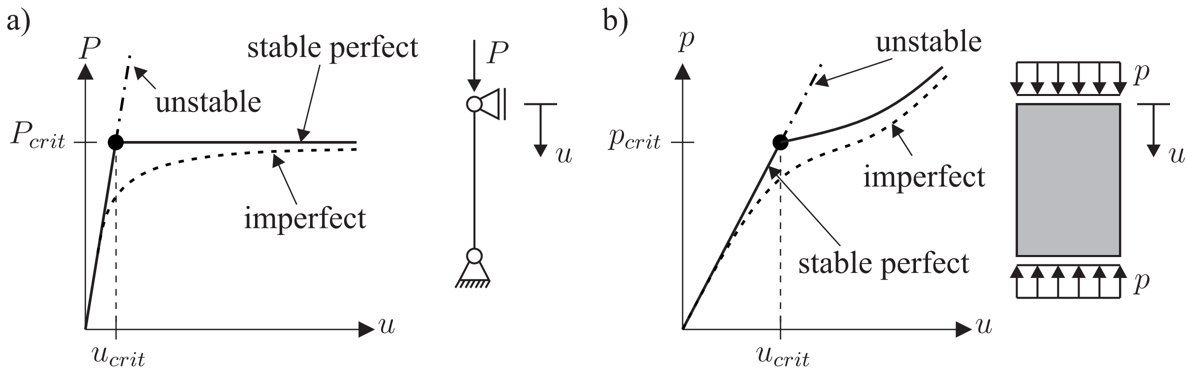

Figure 1: Elastic buckling behavior of a beam a) and a plate b) with imperfections.

applied and the imperfect structure is loaded based on a displacement control until the critical displacement u_{crit} is reached (see also [1]). This will be the evaluation point for the fuzzy structural analysis.

2 Representation of Imperfections as Random Fields

2.1 Introduction to Random Fields

A random field $H(\boldsymbol{x}, \theta)$ is a collection of random variables indexed by a continuous parameter $\boldsymbol{x} \in \Omega$ [8],

$$\{H(\boldsymbol{x}, \theta) : \boldsymbol{x} \in \Omega, \theta \in \Theta\} \quad . \tag{4}$$

This means that a random field describes a scalar field, where for a given $\boldsymbol{x}_0$, $H(\boldsymbol{x}_0, \theta)$ is a random variable. Θ containing all possible outcomes θ. Thus, for a given outcome θ_0,

$$h_0(\boldsymbol{x}) := H(\boldsymbol{x}, \theta_0) \tag{5}$$

is a realization of the random field. One visualisation of a random field realization is illustrated in Figure 2. A random field is Gaussian, if for every given point $\boldsymbol{x}_n$ the random variable is Gaussian,

$$H_0(\theta) := H(\boldsymbol{x}_0,\ \theta\ ) \sim \mathcal{N}(\mu, \sigma^2) \quad . \tag{6}$$

A Gaussian field is completely defined by it's mean function,

$$\mu(\boldsymbol{x}) = E[H(\boldsymbol{x})] \tag{7}$$

and it's auto-correlation function,

$$C(\boldsymbol{x}_i, \boldsymbol{x}_j) = E[(H(\boldsymbol{x}_i) - \mu(\boldsymbol{x}_i))(H(\boldsymbol{x}_j) - \mu(\boldsymbol{x}_j))] \quad , \tag{8}$$

which controls the shape of the random field and hence the shape of imperfections.

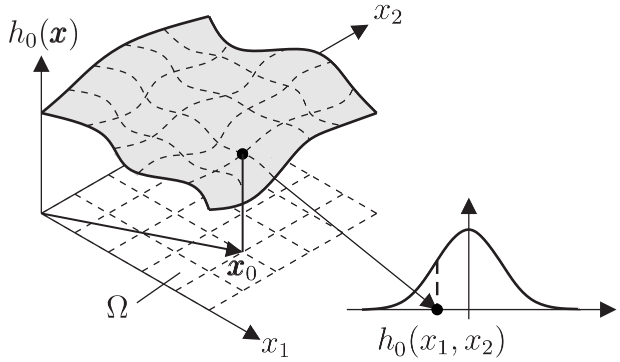

Figure 2: One realization of a random field

2.2 Optimal Discretization of Random Fields using the EOLE-Method

The numerical implementation requires a discretization of the random field

$$H(\boldsymbol{x}) \xrightarrow{Discretization} \hat{H}(\boldsymbol{x}) \quad . \tag{9}$$

In this paper, for an optimal representation of a random field, the EOLE-method (Expansion Optimal Linear Estimation) from [3] is used. The advantage of the method is it's ability to represent the random field with only a few random variables by minimising the variance error. Here, the covariance matrix is only required on a sub-set of field nodes (random field mesh). This aspect is very interesting for the fuzzy structural analysis, because a lot of computational time can be saved. The random field is represented by

$$\hat{H}(\boldsymbol{x}, \theta) = \mu(\boldsymbol{x}) + \sigma \cdot \left(\sum_{i=1}^{M} \frac{\xi_i(\theta)}{\sqrt{\lambda_i}}\ \boldsymbol{\varphi}_i(\boldsymbol{x}^S) \right) \cdot C(\boldsymbol{x}^S, \boldsymbol{x}) \quad , \tag{10}$$

where the vector $\boldsymbol{x}^S = [\boldsymbol{x}_1 \ ... \ \boldsymbol{x}_i^S \ ... \ \boldsymbol{x}_M^S]$ contains the M-nodes of the random field and $\boldsymbol{x} = [\boldsymbol{x}_1 \ ... \ \boldsymbol{x}_j \ ... \ \boldsymbol{x}_N]$ the N-nodes in full space (e.g. FE-nodes). Consequently, $C(\boldsymbol{x}^S, \boldsymbol{x})$ is the correlation matrix between the random field nodes and FE-nodes,

$$C(\boldsymbol{x}^S, \boldsymbol{x}) = \begin{bmatrix} C(\boldsymbol{x}_1^S, \boldsymbol{x}_1) & \dots & C(\boldsymbol{x}_1^S, \boldsymbol{x}_j) & \dots & C(\boldsymbol{x}_1^S, \boldsymbol{x}_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C(\boldsymbol{x}_i^S, \boldsymbol{x}_1) & \dots & C(\boldsymbol{x}_i^S, \boldsymbol{x}_j) & \dots & C(\boldsymbol{x}_i^S, \boldsymbol{x}_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C(x_M^S, \boldsymbol{x}_1) & \dots & C(\boldsymbol{x}_M^S, \boldsymbol{x}_j) & \dots & C(\boldsymbol{x}_M^S, \boldsymbol{x}_N) \end{bmatrix} \quad . \tag{11}$$

$\boldsymbol{\varphi}_i$ and λ_i are the eigenfunctions and eigenvalues of a given autocovariance function $C(\boldsymbol{x}_i^S, \boldsymbol{x}_j^S)$ on the random field mesh. The variable $\xi_i(\theta)$ is the uncorrelated Gaussian random variable with zero mean and unit standard deviation. Herein, a quadratic exponential covariance function is used for numerical analysis,

$$C(\boldsymbol{x}_i^S, \boldsymbol{x}_j^S) = \exp\left[-\frac{d^2(i,j)}{l_c}\right] \quad , \tag{12}$$

wherein $d(i,j)$ is the distance between two nodes $\boldsymbol{x}_i^S$, $\boldsymbol{x}_j^S$ and l_c is the correlation length. This kind of correlation function represents a differentiable process and leads to a smoother random field. The correlation length controls how quickly the covariance falls off. For l_c tending to infinity or a zero distance between two separated points, the exponential covariance function converges to the value one, that means two points are full dependent. The effect on the random field is shown in Figure 3 for a 2-D structure with 8×8 nodes. The shape becomes more uniform for great values of l_c.

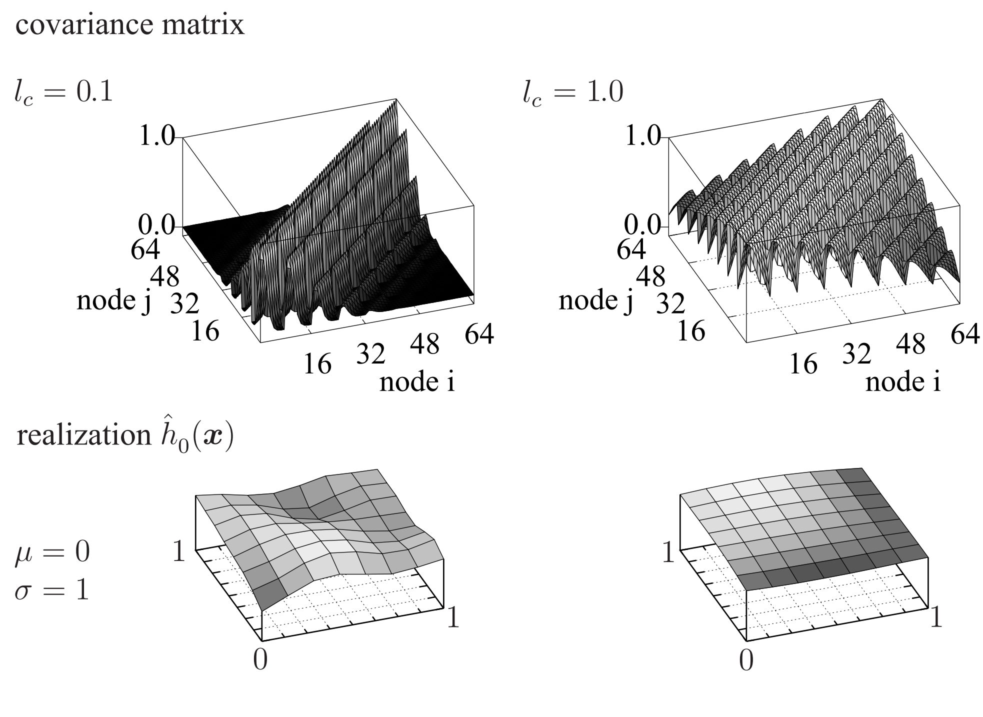

Figure 3: Different covariance matrices with realizations

3 Polymorphic Uncertainty Modelling

3.1 Introduction to Fuzzy Randomness

A real random variable X is the mappping,

$$X : \Theta \rightarrow \mathbb{R}^n \quad . \tag{13}$$

Fuzzy randomness is described by a fuzzy random variable, which is the fuzzy result of an uncertain mapping,

$$\tilde{X} : \Theta \xrightarrow{\sim} \mathcal{F}(\mathbb{R}^n) \quad , \tag{14}$$

with $\mathcal{F}(\mathbb{R}^n)$ the set of all fuzzy numbers on the fundamental set $\mathbb{R}^n$ [2]. Each fuzzy number of a fuzzy realization is defined as a normalized fuzzy set,

$$\tilde{x}_i = \{(x, \mu_{x_i}(x)) \mid x \in \mathbb{R}^1\}; \quad \mu_x(x) \geq 0 \quad \text{(one-dimensional)}, \tag{15}$$

with $\mu_{x_i}(x)$, the normalized membership function [6]. Therefore, similar to real random field definition and the notation in section 2.1, a fuzzy random field can be defined by a collection of fuzzy random variables,

$$\{\tilde{H}(\boldsymbol{x}, \theta) : \boldsymbol{x} \in \Omega, \theta \in \Theta\} \quad , \tag{16}$$

where $\tilde{H}(\boldsymbol{x}_0, \theta)$ is a fuzzy random variable for a given $\boldsymbol{x}_0$. Furthermore $\tilde{h}_0(\boldsymbol{x})$ is a realization of a fuzzy random field to an outcome $\theta_0 \in \Theta$, which also describes a fuzzy function [5]. Figure 4, according to a representation in [6], shows one realization of a fuzzy random field with the fuzzy number $\tilde{h}_0$ for a given point $\boldsymbol{x}_0$.

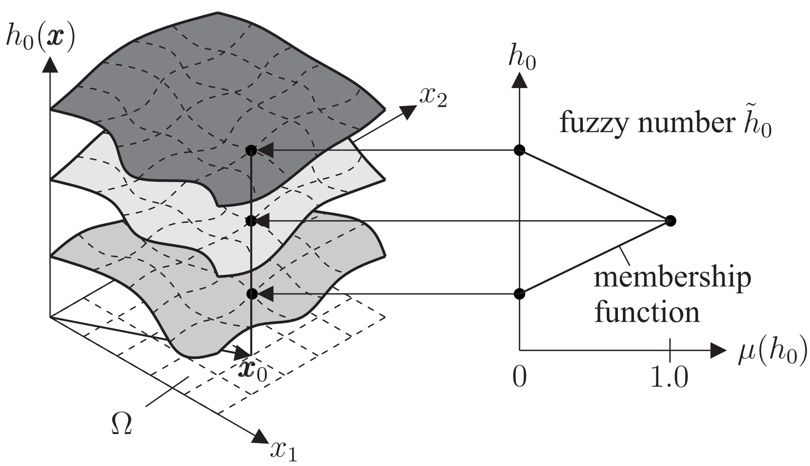

Figure 4: One realization of a two-dimensional fuzzy random field [6]

3.2 Fuzzy Structural Analysis

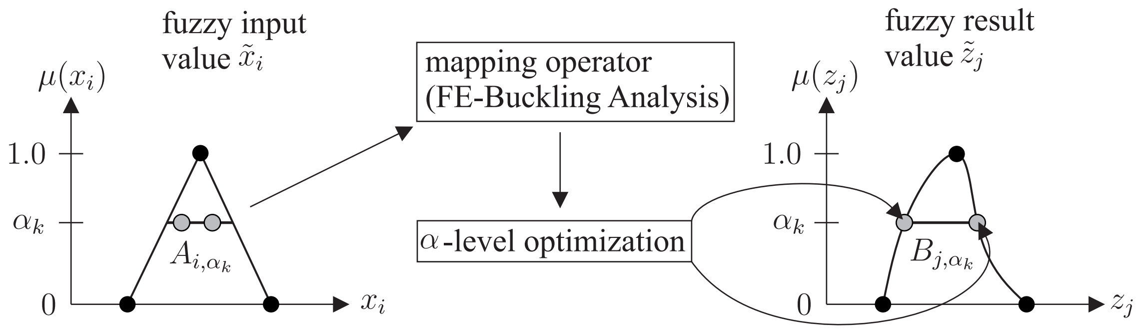

Figure 5: Fuzzy structural analysis scheme [6]

The fuzzy structural analysis describes a mapping of fuzzy input values $\tilde{x}_i$ onto result values $\tilde{z}_j$, which are also fuzzy values. In the presented examples, the fuzzy input is the correlation length l_c, which becomes to a fuzzy correlation length $\tilde{l}_c$ and the fuzzy output is the mean value of the buckling load $\tilde{P}_{crit}$. The mapping operator is the deterministic fundamental solution, which is represented here by a finite element model or more exactly a numerical buckling analysis. So, the fuzzy structural analysis changes to a fuzzy finite element method (FFEM) [6]. For the numerical implementation, the fuzzy values have to be discretized with the help of so called α-level sets $A_i, \alpha_k, i = 1, ..., n$ and $B_i, \alpha_k, i = 1, ..., n$. The search for the smallest and largest result element on the α-level represents an optimization problem. The entire procedure is shown in Figure 5. For the α-level optimization, the modified evolution strategy from [7] is used. This algorithm works reliably for a Monte Carlo Simulation. As part of this work a program was developed for fuzzy structural analysis, named FuFEAP (A Fuzzy Finite Element Analysis Program). It's a MATLAB [4] program with an interface to run FEAP [9] to get the fundamental solution. Besides, the fuzzy and stochastic analysis are done by MATLAB. For numerical buckling analyses of shell structures a non-linear four-node isoparametric shell element from [10] is used.

4 Numerical Examples

The correlation is described by a fuzzy correlation length $\tilde{l}_c$ with the corresponding fuzzy correlation function $\tilde{C}$. Other parameters such as the variance and mean value are deterministic. The fuzzification of the correlation length can be fitted, if measurements are available. In addition, expert knowledge can be taken into account. Here, the correlation lengths are only chosen due to plausible representation of random field realizations. Figure 6 (left) shows the chosen correlation lengths by a fuzzy triangular number. For the numerical analysis, four α-levels are investigated, $\alpha = [0,\ 0.33,\ \ 0.66,\ \ 1]$. On the right side of the figure, the corresponding homogeneous correlation functions with the quadratic exponential form according to equation (12) are plotted. The fuzzy covariance is a fuzzy function [6]. Thereby, a fuzzy number $\tilde{C}(d_i)$ can be specified for every distance d_i.

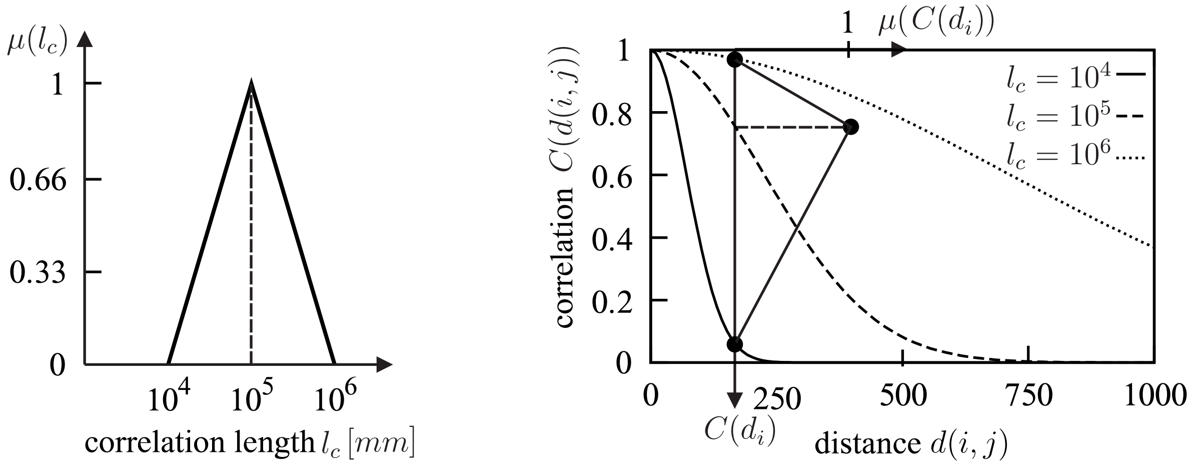

Figure 6: Fuzzy correlation length $\tilde{l}_c$ and fuzzy correlation function $\tilde{C}(d(i,j))$

4.1 Buckling of an Elastic Bar

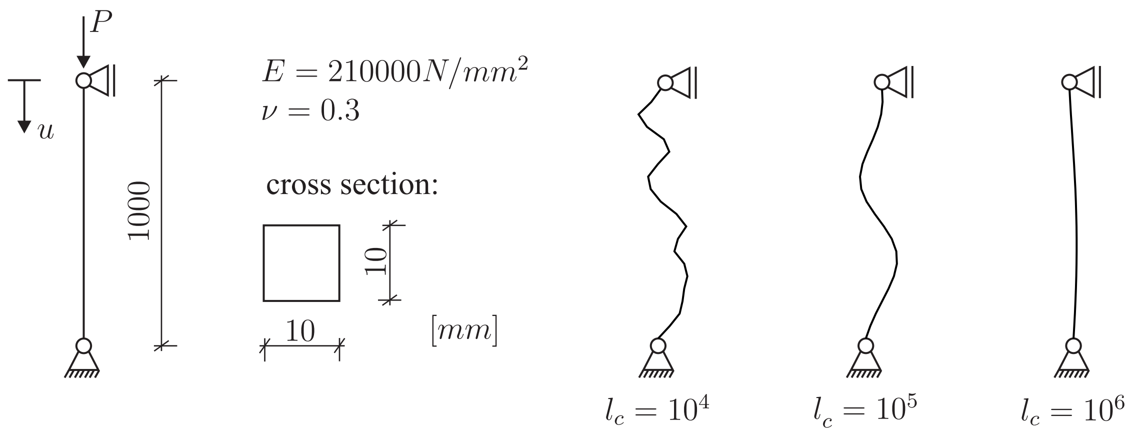

Figure 7: Elastic bar with Euler case 2 boundary conditions and visualisation of the random geometrical imperfections (enlarged)

Geometrical imperfections are applied on an elastic bar, see Figure 7. The figure also shows three realizations of the imperfections for the chosen range of correlation lengths. A 2-node 2D Timoshenko beam element with linear shape functions and finite rotations is used. The bar is discretised with 20 finite and stochastic elements. The critical displacement $u_{crit} = 0.08239\,mm$ and load $P_{crit} = 1730\,N$ are calculated with an eigenvalue analysis of the perfect structure which is described in section 1. The evaluation point for the fuzzy structural analysis is the corresponding load at u_{crit} of the imperfect structure.

In equation (10), the variance is $\sigma = 5\,mm$, the mean value is $\mu = 0$ and the random geometrical imperfections are scaled to an amplitude $L/200 = 5\,mm$ with an exceeding probability of 5% (2,5% on each side). The fundamental solution is obtained in each case by a Monte Carlo Simulation based on 500 simulations. Figure 8 shows a test run for different correlation lengths with the chosen fuzzy correlation lengths contained in the grey area. For correlation lengths tending to infinity, the mean value of the buckling load converges to the critical load P_{crit}. In this case, the imperfections vanish. The minimum and maximum of the mean value in the grey area fit to the left and right limit of the zero α-level in Figure 9, where the fuzzy expected value of the buckling load is presented. The fuzzy result shows a convex shape, wherein each α-level includes the higher levels. Both figures show the great impact of geometrical imperfections on the buckling load, a decrease of the mean value of ≈ 45 %.

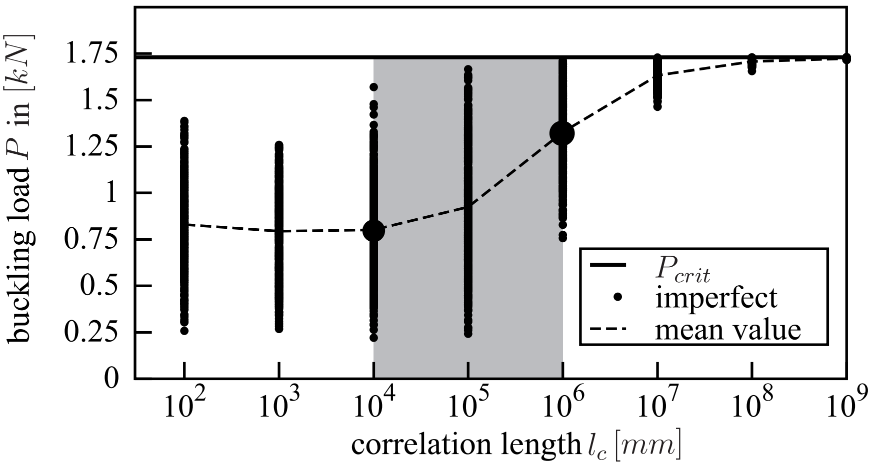

Figure 8: Monte Carlo Simulation for the elastic bar with 500 samples for each correlation length

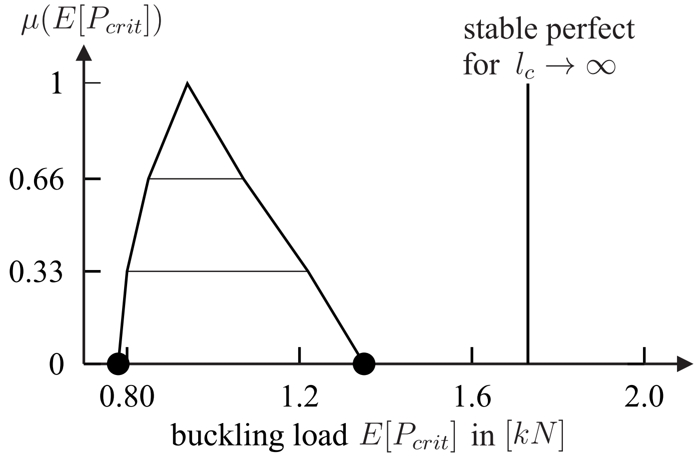

Figure 9: Fuzzy expected value of the buckling load P_{crit}

4.2 Buckling of an Axially Compressed Steel Plate

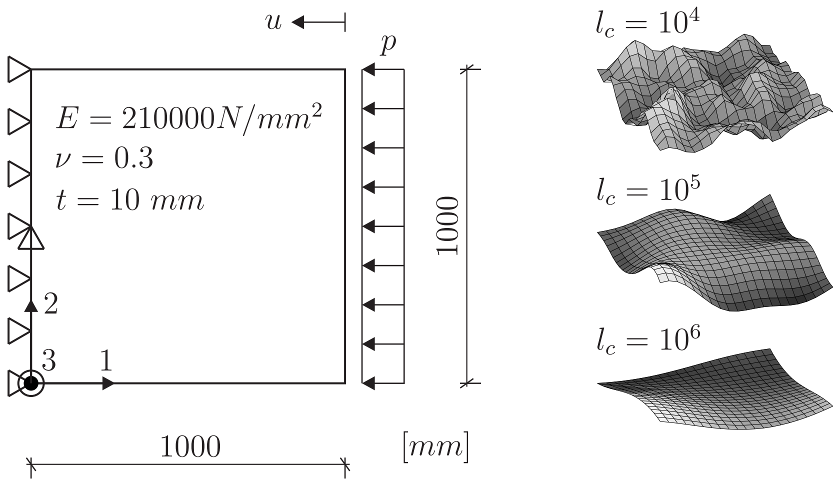

Figure 10: Steel plate (simply supported) and visualisation of the random geometrical imperfections

Now, 2D random geometrical imperfections are applied on an axially compressed steel plate as illustrated in Figure 10. For the FE-mesh and random field mesh, 20 × 20 elements are used. The illustrated slab is simply supported, whereby all edges are free to rotate. The critical displacement of the perfect structure is $u_{crit} = 0.361\,mm$ together with an associated critical load $p_{crit} = 758{,}73\,N/mm$. The scaling and sto hastic parameters are the same as in the previous example. To determine the mean value of the buckling loads in each case, the Monte Carlo Simulation is again based on 500 samples.

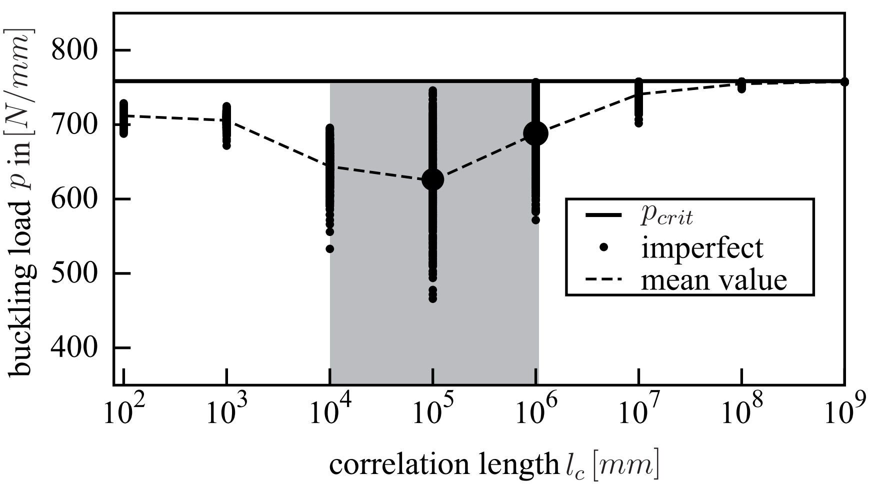

Figure 11: Monte Carlo Simulation for the steel plate with 500 samples for each correlation length

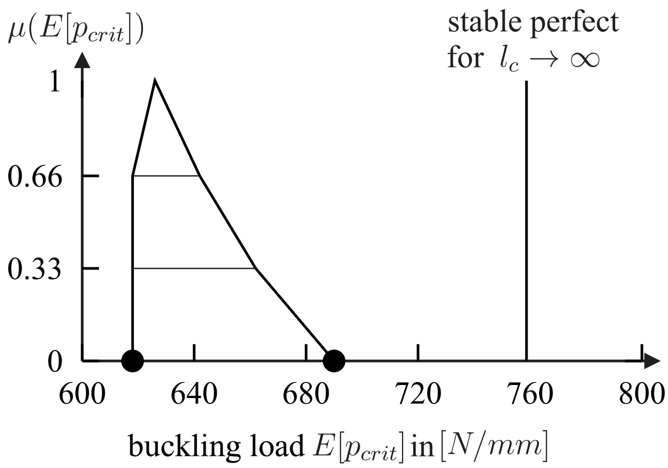

Figure 12: Fuzzy expected value of the buckling load p_{crit}

The mean value curve in Figure 11 differs from the curve in Figure 8. A decrease of the mean value of the buckling loads of approximately 20 % can be observed. The implemented optimization algorithm works and finds the correct minimum and maximum. The fuzzy result in Figure 12 has also a convex shape. Summarizing, the results clearly show the impact of the correlation lengths on the buckling load.

5 Conclusions

In this paper, the fuzzy analysis is used for a more realistic representation of imperfections in buckling analyses. In the conventional stochastic analysis the definition of a correlation length is quite difficult, because also very few measurements are available. Polymorphic uncertainty modelling with the fuzzification of the correlation length allows a more realistic representation of spatial varying imperfections. If experiments are available, the fuzzy input can be verified. Furthermore, other parameters can be considered, e.g. the correlation length for spatial material imperfections, the mean value, variance or the loading. However, the fuzzy analysis requires a lot of computational time, especially the α-level optimization. Therefore, the program needs to be developed, e.g. testing other optimization algorithms, parallelization of the program code and a data reduction. The goal is to investigate more complex structures like cylindrical shells or dynamic problems.

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