Magnetohydrodynamic Equations Around Couette Flow

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung: 24.07.2024
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Abstract

This thesis studies the stability of the magnetohydrodynamic equations around an affine shear flow and constant magnetic field. The dynamics of the equations change drastically depending on the fluid viscosity $\nu \geq 0$ and magnetic resistivity $\kappa \geq 0$. The main goal of this thesis is to establish results on (nonlinear) stability and instability for selected dissipation regimes. These stability and instability results are established in Chapters 3-5.

In the first part, we consider the inviscid $\nu = 0$ and resistive $\kappa > 0$ case. We linearize around explicit low-frequency solutions of traveling waves to infer the main growth model. Small data in Gevrey 2 spaces are necessary and sufficient for this main growth model to ensure stability.

In the second part, we consider the MHD equations with viscosity and horizontal resistivity $\nu = \kappa_x > 0$ and $\kappa_y = 0$. We show that small initial data in Sobolev spaces ensure stability. Furthermore, we show that for the viscous $\nu > 0$ and non-resistive $\kappa = 0$ case, for the linearized MHD equations, there exists initial data such that the magnetic field grows unbounded as $t \to \infty$. Thus, global in time stability of the magnetic field in Sobolev spaces cannot hold for the nonlinear system.

In the third part, we consider the case when resistivity is smaller than viscosity $0 < \kappa \leq \nu$. We show that small initial data in Sobolev spaces yield stability. Furthermore, the stability properties qualitatively differ for the cases $\kappa \geq \nu^3$ and $\kappa \leq \nu^3$. For the case $\kappa \leq \nu^3$ we obtain norm inflation of order $\nu\kappa^{-\frac{1}{3}}$ due to growth in the magnetic field.

Acknowledgements

First and foremost, I would like to thank my advisor, Christian Zillinger, for supervising me during my PhD. For all the helpful discussions that built my understanding of fluid dynamics, for his guidance during my PhD and for his patience with my writing. I also would like to thank Xian Liao for being a valuable member of our working group and for agreeing to act as a referee for my thesis. Next, I want to thank Diego Alonso-Orán for advising my Master's thesis and for his guidance.

Furthermore, I would like to thank the CRC 1173 for the opportunity to write my thesis in the Project AP6. I also want to thank my working group of non-linear differential equations for enriching mathematical and non-mathematical conversations. In particular, Wolfgang Reichel, Gerd Herzog, Björn, Sebastian, Jannis and Julia.

Furthermore, I would like to thank my colleagues and friends from the second floor. The coffee break at 4 p.m. was always a good opportunity for a break. I especially want to thank Marvin, Gianmichele, Lucrezia, Alex, Friedrich, and Max for being awesome.

Finally, many thanks to my Family, my brother Moritz and my parents, Achim and Petra, for always supporting me.

Contents

Chapter 1

Mixing and Stability of the Magnetohydrodynamic Equations

The magnetohydrodynamic (MHD) equations are a common model used in astrophysics, planetary magnetism and controlled nuclear fusion [Dav16]. They model the interaction of the velocity and magnetic field of a conducting non-magnetic fluid. The MHD equations are derived from the Navier-Stokes equations coupled with the Maxwell equations assuming vanishing charge density (see Appendix B for a detailed derivation).

In this thesis, we consider the two-dimensional MHD equations in a finite periodic channel,

$$\begin{aligned} \partial_t V + V \cdot \nabla V + \nabla \Pi &= \nu \Delta V + B \cdot \nabla B, \\ \partial_t B + V \cdot \nabla B &= \kappa \Delta B + B \cdot \nabla V, \\ \nabla \cdot V &= \nabla \cdot B = 0, \\ B\big|_{t=0} &= B_{in}, \quad V\big|_{t=0} = V_{in}, \\ (t, x, y) &\in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R} =: \Omega. \end{aligned} \tag{1.1}$$

Here $V : \Omega \to \mathbb{R}^2$ is the velocity of an incompressible conducting fluid interacting with the magnetic field $B : \Omega \to \mathbb{R}^2$. The quantity $\Pi : \Omega \to \mathbb{R}$ denotes the pressure. The dissipation parameters $\nu, \kappa \geq 0$ represent fluid viscosity and magnetic resistivity.

The MHD equations (1.1) are notoriously challenging. Already in the 2D case, global-in-time wellposedness is open for general initial data. Stability results for vanishing resistivity in the literature are often either local-in-time or limited to small perturbations around special solutions (mostly around a large constant magnetic field). Therefore, weak or vanishing resistivity poses great challenges. For the resistive $\kappa > 0$ and viscous $\nu > 0$ case we obtain global-in-time wellposedness [ST83]. Furthermore, a strong magnetic field is stable for small initial data [WZ17]. Also for the inviscid $\nu = 0$ and resistive $\kappa > 0$ MHD equations, one obtains global-in-time wellposedness [Koz89] for initial data in Sobolev spaces. For the viscous $\nu > 0$ and non-resistive $\kappa = 0$ MHD equations, one obtains local-in-time wellposedness [CMRR16, FMRR17], but global-in-time wellposedness is remains open. Furthermore, one obtains stability of a large enough constant magnetic field for small initial data [LXZ15, RWXZ14]. For the ideal MHD equations $\nu = \kappa = 0$, only local-in-time wellposedness is established [Koz89, Sch88, Sec93, MY06]. In [BSS88], it is shown that the dynamics of small initial perturbations of the ideal MHD equations around a large constant magnetic field behave like the linearized system.

An essential challenge for the MHD equations is the solutions' stability and long-time behavior. Specifically, we aim to understand the stability and instability of specific global in-time solutions that combine the effects of mixing and magnetic (de)stabilization. This dissertation focuses on the combination of an affine shear flow, called Couette flow, and a constant magnetic field

$$\begin{aligned} V_s(t,x,y) &= \begin{pmatrix} y \\ 0 \end{pmatrix}, \\ B_s(t,x,y) &= \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \qquad \alpha \in \mathbb{R}. \end{aligned} \tag{1.2}$$

This stationary solution combines the effects of mixing due to shear and coupling by the magnetic field. The Couette flow mixes perturbations, which leads to damping. The constant magnetic field induces a strong coupling of the magnetic and velocity perturbations. Therefore, perturbations of (1.2) highlight the velocity and magnetic field interaction.

The main goal of this thesis is to contribute to the understanding of stability and instability of perturbations of (1.2). Here, stability is understood in the Lipschitz sense, which we state in the following. We consider the perturbative unknowns

$$\begin{aligned} v(t,x,y) &= V(t,x+yt,y) - V_s(t,x+yt,y), \\ b(t,x,y) &= B(t,x+yt,y) - B_s(t,x+yt,y). \end{aligned} \tag{1.3}$$

The change of variables $x \mapsto x + yt$ follows the characteristics of the Couette flow. Then the equations (1.1) become

$$\begin{aligned} \partial_t v + v_2 e_1 &= \nu \Delta_t v + \alpha \partial_x b + b \cdot \nabla_t b - v \cdot \nabla_t v - \nabla_t \pi, \\ \partial_t b - b_2 e_1 &= \kappa \Delta_t b + \alpha \partial_x v + b \cdot \nabla_t v - v \cdot \nabla_t b, \\ \nabla_t \cdot v &= \nabla_t \cdot b = 0, \\ b\big|_{t=0} &= b_{in}, \quad v\big|_{t=0} = v_{in}, \\ (t,x,y) &\in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \end{aligned} \tag{1.4}$$

Here we write $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The spatial derivatives become time-dependent due to the change of variables and change to $\partial_y^t :=$

$\partial_y - t\partial_x$, $\nabla_t := (\partial_x, \partial_y^t)^T$ and $\Delta_t := \partial_x^2 + (\partial_y^t)^2$. For different values of ν and κ, we obtain different linear behavior and as a result, the nonlinear terms yield different interactions.

Problem 1.1. *Let f be a choice of unknowns which describes a solution of (1.4). Let X_1 and X_2 be two Banach spaces, then we aim to find a bound on the initial data $\varepsilon_0 = \varepsilon_0(f, X_i, \nu, \kappa)$ and a Lipschitz constant $L = L(f, X_i, \nu, \kappa)$ such that for initial data which satisfies*

$$\|f_{in}\|_{X_1} = \varepsilon \leq \varepsilon_0,$$

the corresponding solution is globally bounded in time by

$$\|f(t)\|_{X_2} \leq L\varepsilon.$$

Furthermore, the stability threshold are $\gamma_1, \gamma_2 \in \mathbb{R}$, such that for $\varepsilon_0 = c_0\nu^{\gamma_1}\kappa^{\gamma_2}$ with a $c_0 > 0$ we obtain

$$\begin{aligned} \|f_{in}\|_{X_1} &\leq c_0\nu^{\gamma_1}\kappa^{\gamma_2} \rightarrow \text{ stability,} \\ \|f_{in}\|_{X_1} &\gg c_0\nu^{\gamma_1}\kappa^{\gamma_2} \rightarrow \text{possible instability.} \end{aligned}$$

Stability thresholds are a typical approach and are also used for related equations such as the Navier-Stokes, Euler or Boussinesq equations. Our definition generalizes previous definitions [BVW18, BGM19, Lis20] to allow for two parameters γ_1 and γ_2.

In the following, we give an overview of the literature. First, we give an overview of the closely connected cases of the Navier-Stokes ($\nu > 0$) equations and the Euler equations ($\nu = 0$) around Couette flow. This corresponds to the MHD equations around Couette flow without a magnetic field ($b = 0 \in \mathbb{R}^2$ and $\alpha = 0 \in \mathbb{R}$) and has similar properties but behaves qualitatively differently. We refer to [BGM19] for a more comprehensive overview.

For the Euler equations around Couette flow, Gevrey 2 spaces (see Appendix A) are expected to be optimal for stability. In [BM15a], stability is shown for small initial data in Gevrey σ spaces for $1 \leq \sigma < 2$. When linearizing around small explicit low-frequency solutions, one obtains stability and norm inflation in Gevrey 2 spaces, depending on the radius of convergence [DZ21]. For the nonlinear system, one obtains nonlinear instability in a space slightly weaker than Gevrey 2 [DM23].

In the case of the Navier-Stokes equations around Couette flow, small initial data in Sobolev spaces with respect to ν are sufficient to obtain stability. In [BVW18] it is shown that initial data in Sobolev spaces smaller than $\nu^{\frac{1}{2}}$ yield stability. Masmoudi and Zhao improved this to $\nu^{\frac{1}{3}}$ [MZ22], which is expected to be optimal. In [BMV16], it is shown that small initial data in Gevrey σ spaces for $1 \leq \sigma < 2$ are sufficient to ensure stability independent of ν. Furthermore, one can trade smallness in terms of ν and Gevrey regularity to obtain stability [LMZ22].

The MHD equations around Couette flow and constant magnetic field recently received substantial attention [Lis20, KZ1, ZZ24, Dol24, KZ2, K, CZ24].

This includes the three publications and preprints which are the basis for this thesis:

1. Niklas Knobel and Christian Zillinger, *On Echoes in Magnetohydrodynamics with Magnetic Dissipation*, Journal of Differential Equations, 367:625–688, 2023, [KZ1]. Main result: Theorem 3.3.

2. Niklas Knobel and Christian Zillinger, *On the Sobolev Stability Threshold for the 2D MHD Equations with Horizontal Magnetic Dissipation* arXiv preprint, arXiv:2309.00496, 2023, [KZ2]. Main results: Proposition 4.2 and Theorem 4.3

3. Niklas Knobel, *Sobolev Stability for the 2D MHD Equations in the Non-Resistive Limit* arXiv preprint, arXiv:2401.12548, 2024, [K]. Main result: Theorem 5.1

For the MHD equations around Couette flow and constant magnetic field, the linear dynamics change depending on the values of the dissipation parameters ν and κ. As a consequence, the nonlinear terms exhibit different behaviors. For an intuition on linear and nonlinear effects on the MHD equations, we refer to Chapter 2. Here we give an overview of the various results.

In 2020, Liss proved the first stability result for the MHD equations around Couette flow and constant magnetic field [Lis20]. He considered the fully dissipative regime, $\kappa = \nu > 0$, and established stability of the three-dimensional MHD equations for initial data small in Sobolev spaces for a large enough constant magnetic field. For the analogous two-dimensional problem, Chen and Zi considered in [CZ24] the stability of the 2D MHD equations around a shear flow close to Couette for small initial data in Sobolev spaces. Dolce [Dol24] proved stability for small initial data in Sobolev spaces for the more general setting of $0 < \kappa^3 \lesssim \nu \leq \kappa$.

In [KZ1], which is part of this thesis in Chapter 3, Zillinger and the author consider the regime of vanishing viscosity $\nu = 0$ and resistivity $\kappa > 0$. In the linearized dynamics, the equations decouple with respect to frequency and are stable in Sobolev spaces. However, the nonlinear terms yield an interaction between different frequencies, possibly leading to norm inflation. In order to capture this effect, we consider traveling waves, which are explicit low-frequency solutions, from which we deduce the main growth model. There we showed that resonance chains can lead to Gevrey 2 norm inflation. On the one hand, Gevrey 2 spaces with a large enough radius of convergence ensure stability. On the other hand, there are specific initial data in Gevrey 2 spaces with a small radius of convergence such that the solutions grow unbounded in Sobolev spaces. Thus, Gevrey 2 spaces are sufficient and necessary for stability. In a corresponding nonlinear stability result, Zhao and Zi [ZZ24] proved the almost matching nonlinear stability result for small perturbations in Gevrey σ spaces for $1 \leq \sigma < 2$.

In [KZ2], which is a part of this thesis in Chapter 4, Zillinger and the author consider the case of horizontal resistivity and full viscosity, $\nu = \kappa_x > 0$ and $\kappa_y =$

0. We establish stability for small initial data in Sobolev spaces. Furthermore, we prove that for the non-resistive MHD equations, $\nu > 0$ and $\kappa = 0$, the linearised dynamics yield growth of the Sobolev norm by νt. Therefore, stability in Sobolev spaces cannot hold for the magnetic field. In this sense, horizontal resistivity is indeed necessary for stability.

In [K], which is a part of this thesis in Chapter 5, the author considers the case when resistivity is smaller than viscosity $0 < \kappa \leq \nu$. The author proves stability for small enough initial data in Sobolev spaces. The behavior qualitatively differs between the cases $\nu^3 \leq \kappa$ and $\nu^3 \geq \kappa$. The relation $\nu^3 \leq \kappa$ corresponds to the case when resistivity and viscosity are close to each other and thus yield stability and no norm inflation. In the case when $\nu^3 \geq \kappa$, the resistivity is very small compared to the viscosity. Thus the velocity field gets damped down on time scales ν^{-1} leading to growth in the magnetic field until times scales of order $\kappa^{-\frac{1}{3}}$. Therefore, this case yields stability with norm inflation of order $\nu\kappa^{-\frac{1}{3}}$.

To summarize the previously mentioned results for the MHD equations around Couette flow, the stability of the cases $\nu \approx \kappa > 0$ [Lis20, Dol24, CZ24, K] and $\nu = 0 < \kappa$ [KZ1, ZZ24] are by now well understood, with key contribution being part of this dissertation. The case $\kappa = 0 < \nu$ is still open due to the growth of the magnetic field and we cannot expect global-in-time stability for the magnetic field [KZ2]. Furthermore, [KZ2, K] contribute to the understanding of the non-resistive limit or the case of partially horizontal, anisotropic resistivity. For the ideal case, no results exist yet. We remark that the non-resistive and ideal MHD equations, which remain open in our setting, are exactly the two regimes where global-in-time wellposedness is still open for the MHD equations in general, as mentioned at the beginning. The main results of this thesis are summarized in the following table:

Paper	Ch	Result	Dissipation	Description
[KZ1]	3	Thm 3.3	$\nu = 0$, $\kappa > 0$	(In)stability of traveling waves
[KZ2]	4	Thm 4.3	$\nu = \kappa_x > 0$, $\kappa_y = 0$	Nonlinear stability
[KZ2]	4	Prop 4.2	$\nu > 0$, $\kappa = 0$	Linear growth of magnetic field
[K]	5	Thm 5.1	$\nu \geq \kappa \gtrsim \nu^{\frac{1}{3}} > 0$	Nonlinear stability
[K]	5	Thm 5.1	$\nu^{\frac{1}{3}} \gtrsim \kappa > 0$	Nonlinear stability, norm inflation

The remainder of this dissertation is structured as follows:

- In Chapter 2, we give a short overview of the main mathematical effects of the linear and nonlinear dynamics.

- In Chapter 3, we consider the inviscid $\nu = 0$ and resistive case $\kappa > 0$. While the linearised dynamics is stable in Sobolev spaces, we cannot expect stability for the nonlinear dynamics due to resonances in the nonlinear terms. We model these resonances by deriving a model from explicit low-frequency solutions. For this model, we show that Gevrey 2 spaces are necessary for stability. This chapter is based on a joint publication with Christian Zillinger [KZ1], published in the Journal of Differential Equations.

- In Chapter 4, we consider the case with viscosity and horizontal resistivity $\nu = \kappa_x > 0$ and vanishing vertical resistivity $\kappa_y = 0$. Smallness of initial data in Sobolev norms is sufficient for nonlinear stability. Furthermore, we show that for the viscous $\nu > 0$ and non-resistive $\kappa = 0$ case, for specific initial data, the magnetic field grows linearly in time. This is based on a joint preprint with Christian Zillinger [KZ2].

- In Chapter 5, we consider the case when resistivity is smaller than viscosity $0 < \kappa \leq \nu$. We obtain nonlinear stability for initial data small in Sobolev spaces. There are two different behaviors. If resistivity is close to viscosity, we obtain stability without norm inflation. If the resistivity is much smaller than viscosity, we obtain norm inflation depending on the resistivity and viscosity. In particular, stability estimates degenerate in the non-resistive limit. This is based on the author's preprint [K].

- In Appendix A, we give an overview of the necessary mathematical background.

- In Appendix B, we derive the magnetohydrodynamic equations from the Navier-Stokes and Maxwell equations.

Chapter 2

Mathematical Effects of the MHD Equations

In this chapter, we give a short overview of the leading mathematical effects of the MHD equations around Couette flow and constant magnetic field (1.4), for the unknowns (1.3),

$$\begin{aligned} \partial_t v + v_2 e_1 &= \nu \Delta_t v + \alpha \partial_x b + b \cdot \nabla_t b - v \cdot \nabla_t v - \nabla_t \pi, \\ \partial_t b - b_2 e_1 &= \kappa \Delta_t b + \alpha \partial_x v + b \cdot \nabla_t v - v \cdot \nabla_t b, \\ \nabla_t \cdot v &= \nabla_t \cdot b = 0, \\ b\big|_{t=0} &= b_{in}, \quad v\big|_{t=0} = v_{in}, \\ (t, x, y) &\in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \end{aligned} \tag{2.1}$$

Choosing the right unknowns is crucial for equation (2.1). Depending on the parameters ν, $\kappa \geq 0$ and $\alpha \in \mathbb{R}$, switching to different unknowns adapted to the linearized dynamics is natural. In particular, in this chapter, we will often use the adapted unknowns $p = \Lambda_t^{-1}(\nabla_t^\perp \cdot u, \nabla_t^\perp \cdot b)$, the vorticity $w = \nabla_t^\perp \cdot v$ and the current $j = \nabla_t^\perp \cdot b$. The fractional Laplacian is defined as $\Lambda_t := (-\Delta_t)^{\frac{1}{2}}$. Furthermore, we perform a Fourier transform such that for the variables $(x, y) \in \mathbb{T} \times \mathbb{R}$ we obtain $(k, \xi) \in \mathbb{Z} \times \mathbb{R}$ in Fourier space. With slight abuse of notation, we omit writing the hat of the Fourier transform.

We briefly outline the structure of this chapter. In the first three sections, we describe the linearized effects of the MHD equations. Section 2.1 considers the case when the magnetic field vanishes and corresponds to the Navier Stokes equation. Section 2.2 discusses effects in the ideal MHD (i.e. $\nu = \kappa = 0$) equations. In Section 2.3, we consider the dissipative MHD equations and give an overview of the different dissipation regimes. Section 2.4, provides an overview of all the useful unknowns. In Section 2.5 we describe the leading nonlinear effects in terms of toy models.

2.1 Linear Effects of the Navier-Stokes Equations with $b = 0$ and $\alpha = 0$

This section describes the linear effects of the MHD equations without a magnetic field. In particular, these are the linearised Navier-Stokes equations around Couette flow

$$\begin{aligned} \partial_t v + v_2 e_1 &= \nu \Delta_t v - v \cdot \nabla_t v - \nabla_t \pi, \\ \nabla_t \cdot v &= 0. \end{aligned}$$

The effects described in this section are well-known (see for example [Orr07, BM15a, BVW18, Zil17]). In vorticity $w = \nabla_t^\perp \cdot v$ formulation, the linearized Navier-Stokes equations read

$$\begin{aligned} \partial_t w &= \nu \Delta_t w, \\ v &= \nabla_t^\perp \Delta_t^{-1} w. \end{aligned} \tag{2.2}$$

Assuming vanishing x-average, we obtain two important stabilizing effects in linearized dynamics: inviscid damping and enhanced dissipation. After a Fourier transformation (as announced we omit the hat in the Fourier transform), we obtain

$$\begin{aligned} \partial_t w &= -\nu (k^2 + (\xi - kt)^2) w, \\ v &= -\tfrac{i}{k^2 + (\xi - kt)^2} \begin{pmatrix} \xi - kt \\ -k \end{pmatrix} w. \end{aligned} \tag{2.3}$$

The equation decouples in frequency space. Due to the assumption of vanishing x-average we obtain $k \neq 0$, since $k \in \mathbb{Z}$ that means $|k| \geq 1$.

Inviscid Damping with $\nu = 0$

The Couette flow induces a mixing of vorticity. This leads to a damping of the velocity field independent of ν, often called inviscid damping. For $\nu = 0$ the solution of (2.2) stays constant

$$w(t, k, \xi) = w_{in}(k, \xi)$$

and thus, for the velocity field, we infer

$$v(t, k, \xi) = \tfrac{i}{k} \tfrac{1}{1 + (t - \frac{\xi}{k})^2} \begin{pmatrix} t - \frac{\xi}{k} \\ 1 \end{pmatrix} w_{in}(k, \xi). \tag{2.4}$$

For fixed frequencies, the absolute value of the velocity reaches its maximum at time $t = \frac{\xi}{k}$ and is damped as t increases further. For norm estimates, lower Sobolev norms will decay in time [Orr07]

$$\langle t \rangle \| v_1 \|_{H^{N+1}} + \langle t \rangle^2 \| v_2 \|_{H^N} \lesssim \| w_{in} \|_{H^{N+1}} = \| v_{in} \|_{H^{N+2}}. \tag{2.5}$$

Enhanced Dissipation with $0 < \nu \ll 1$

Mixing by the Couette flow leads to faster dissipation rates, often called enhanced dissipation. In particular, at times $t \approx \nu^{-\frac{1}{3}}$ the dissipation leads to fast decay and compares to $t \approx \nu^{-1}$ for the heat equation. For $\nu > 0$

$$w(t,k,\xi) = \exp\left(-\nu k^2 \int_0^t (1 + (t - \tfrac{\xi}{k})^2) d\tau\right) w_{in}(k,\xi)$$

solves equation (2.3). We estimate this exponential term by

$$\exp\left(-\nu k^2 \int_0^t (1 + (t - \tfrac{\xi}{k})^2) d\tau\right) \lesssim \exp(-c\nu^{\frac{1}{3}} t)$$

and thus, for $\nu \ll 1$, we obtain the enhanced dissipation estimate

$$\|w\|_{H^{N+1}} \leq e^{-c\nu^{\frac{1}{3}} t} \|w_{in}\|_{H^{N+1}}.$$

For the velocity field, we obtain both enhanced dissipation and inviscid damping

$$\langle t \rangle \|v_1\|_{H^{N+1}} + \langle t \rangle^2 \|v_2\|_{H^N} \lesssim e^{-c\nu^{\frac{1}{3}} t} \|w_{in}\|_{H^{N+2}}.$$

2.2 Linear Effects of the Ideal MHD Equations with $\nu = \kappa = 0$

This section describes the linear effects of the ideal MHD equations, $\nu = \kappa = 0$. Here, there is competition between the growth of the magnetic field, the decay of the velocity field, and their interaction. We will describe these effects and then show stability for a large constant magnetic field $\alpha > \frac{1}{2}$. To show these effects, we consider the adapted unknowns

$$p = \Lambda_t^{-1}(\nabla_t^\perp \cdot v, \nabla_t^\perp \cdot b).$$

These unknowns are established in [KZ2] (Chapter 4) and similar unknowns are established independently in [Dol24]. Then (2.1) changes to

$$\begin{aligned} \partial_t p_1 &= \partial_x \partial_y^t \Delta_t^{-1} p_1 + \alpha \partial_x p_2, \\ \partial_t p_2 &= -\partial_x \partial_y^t \Delta_t^{-1} p_2 + \alpha \partial_x p_1, \\ p\big|_{t=0} &= p_{in}. \end{aligned} \tag{2.6}$$

We apply a Fourier transformation, only consider modes $k \neq 0$ and, by slight abuse of notation, we replace p_2 by $-ip_2$ to obtain the equation

$$\begin{aligned} \partial_t p_1 &= -\tfrac{t - \frac{\xi}{k}}{1 + (t - \frac{\xi}{k})^2} p_1 - \alpha k p_2, \\ \partial_t p_2 &= \tfrac{t - \frac{\xi}{k}}{1 + (t - \frac{\xi}{k})^2} p_2 + \alpha k p_1, \\ p\big|_{t=0} &= p_{in}. \end{aligned} \tag{2.7}$$

For simplicity of notation, we introduce the new variable $s = t - \frac{\xi}{k}$ and the initial time $s_{in} = -\frac{\xi}{k}$. Then equation (2.7) reads

$$\begin{aligned} \partial_s p_1 &= -\tfrac{s}{1+s^2} p_1 - \alpha k p_2, \\ \partial_s p_2 &= \tfrac{s}{1+s^2} p_2 + \alpha k p_1, \\ p\big|_{s=s_{in}} &= p_{in}. \end{aligned} \tag{2.8}$$

In the following, we will discuss all the different effects appearing in (2.8).

Decay and Growth of the Adapted Unknowns with $\alpha = 0$

As the first toy model, we consider

$$\begin{aligned} \partial_s p_1 &= -\tfrac{s}{1+s^2} p_1, \\ \partial_s p_2 &= \tfrac{s}{1+s^2} p_2. \end{aligned} \tag{2.9}$$

This corresponds to the case $\alpha = 0$, when there is no coupling between p_1 and p_2. Equation (2.9) has an explicit solution

$$\begin{aligned} p_1(s) &= \tfrac{\langle s_{in} \rangle}{\langle s \rangle} p_{1,in}, \\ p_2(s) &= \tfrac{\langle s \rangle}{\langle s_{in} \rangle} p_{2.in}. \end{aligned}$$

So in this model the p_2 grows by $\langle s \rangle$ and p_1 decays by $\langle s \rangle^{-1}$. The decay in p_1 corresponds to the inviscid damping of the Navier-Stokes equations discussed in (2.4). However, the p_2 unknown grows in time, which is a major challenge for longtime stability. It destabilizes the nonlinear equations in the settings of small magnetic field $0 \leq \alpha \ll \frac{1}{2}$ or for small values of the resistivity $0 \leq \kappa \ll \nu$.

Constant Magnetic Field and Circular Movement with $\alpha \in \mathbb{R}$ and without Couette Flow

In this subsection, we highlight the effect of the constant magnetic field α in (2.8). This is very similar to the effect of Alfén waves [Alf42, Dav16]. We consider the toy model

$$\begin{aligned} \partial_s p_1 &= -\alpha k p_2, \\ \partial_s p_2 &= \alpha k p_1, \end{aligned} \tag{2.10}$$

which corresponds to the second part of (2.8). This is solved by

$$p(s) = \begin{pmatrix} \cos(\alpha k(s - s_{in})) & -\sin(\alpha k(s - s_{in})) \\ \sin(\alpha k(s - s_{in})) & \cos(\alpha k(s - s_{in})) \end{pmatrix} p_{in}.$$

We call this *circular movement*, which leads to an exchange between p_1 and p_2.

Stability and Loss of Inviscid Damping $\alpha > \frac{1}{2}$

We consider the linearized ideal MHD equations

$$\begin{aligned} \partial_s p_1 &= -\tfrac{s}{1+s^2} p_1 - \alpha k p_2, \\ \partial_s p_2 &= \tfrac{s}{1+s^2} p_2 + \alpha k p_1. \end{aligned} \tag{2.11}$$

In particular, this is the combination of (2.9) and (2.10). For a large constant magnetic field $\alpha > \frac{1}{2}$, the circular movement averages the growth of the magnetic field and the decay in the velocity field. We define the energy

$$E = |p|^2 + \tfrac{2}{\alpha k} \tfrac{s}{1+s^2} p_1 p_2. \tag{2.12}$$

Using an energy of this form is a common approach used similarly in [BBZD23, Dol24, KZ2]. The energy E is strictly positive definite since $\alpha > \frac{1}{2}$ and satisfies

$$(1 - \tfrac{1}{2\alpha|k|})|p|^2 \leq E \leq (1 + \tfrac{1}{2\alpha|k|})|p|^2 \tag{2.13}$$

globally in time. Then we estimate

$$\partial_s E = \tfrac{2}{|\alpha k|} \tfrac{1-s^2}{(1+s^2)^2} |p_1 p_2| \leq \tfrac{2}{2\alpha|k|-1} \tfrac{1}{1+s^2} E$$

and thus we infer the energy estimate

$$\exp(-\tfrac{2\pi}{2\alpha|k|-1}) E(s_{in}) \leq E(s) \leq \exp(\tfrac{2\pi}{2\alpha|k|-1}) E(s_{in}).$$

Using (2.13), this implies stability

$$|p|(s) \approx |p_{in}|.$$

That means we are stable in the p variables. However, this comes with a cost to the inviscid damping: For the velocity field, we obtain

$$\begin{aligned} v &= \tfrac{i}{\sqrt{1+s^2}} \begin{pmatrix} s \\ 1 \end{pmatrix} p_1, \\ b &= \tfrac{i}{\sqrt{1+s^2}} \begin{pmatrix} s \\ 1 \end{pmatrix} p_2. \end{aligned} \tag{2.14}$$

Hence, using $t = s + \frac{\xi}{k}$ we infer the estimate

$$\|(v_1, b_1)\|_{H^{N+1}} + \langle t \rangle \|(v_2, b_2)\|_{H^N} \lesssim \|(v_{in}, b_{in})\|_{H^{N+1}}. \tag{2.15}$$

In particular, compared to the Euler equations (2.5) both components lose decay of order $\langle t \rangle^{-1}$. This happens due to the previously mentioned growth in the magnetic field.

2.3 Linear Effects of the Dissipative MHD Equations with $\nu > 0$ or $\kappa > 0$

The MHD equations behave very differently depending on the relation of the dissipation parameters. We consider the linearised MHD equations with dissipation

$$\begin{aligned} \partial_t p_1 &= \partial_x \partial_y^t \Delta_t^{-1} p_1 + \alpha \partial_x p_2 + \nu \Delta_t p_1, \\ \partial_t p_2 &= -\partial_x \partial_y^t \Delta_t^{-1} p_2 + \alpha \partial_x p_1 + \kappa \Delta_t p_2, \\ p\big|_{t=0} &= p_{in}. \end{aligned} \tag{2.16}$$

After a Fourier transform, replacing p_2 by $-ip_2$ and the variable change $s = t - \frac{\xi}{k}$ and $s_{in} = -\frac{\xi}{k}$ (similar as in the previous section), we obtain

$$\begin{aligned} \partial_s p_1 &= -\tfrac{s}{1+s^2} p_1 - \alpha k p_2 - \nu k^2 (1+s^2) p_1, \\ \partial_s p_2 &= \tfrac{s}{1+s^2} p_2 + \alpha k p_1 - \kappa k^2 (1+s^2) p_2, \\ p\big|_{s=s_{in}} &= p_{in}, \end{aligned} \tag{2.17}$$

for $k \neq 0$. From the stability and asymptotic of (2.17) in the p unknowns the stability of (2.16) and (2.1) follows directly, similar as in the previous section. This section focuses on the various effects appearing in (2.17). For a rigorous calculation, we refer to Chapters 3, 4 and 5. We saw in the inviscid case that for $\alpha > \frac{1}{2}$, the p unknowns are stable. That is not necessarily true for the dissipative regime since the imbalance of the dissipation counteracts the circular movement. In particular, we distinguish between different cases, $\nu = \kappa$, $\nu \approx \kappa$, close to resonant $s = 0$, $\nu \ll \kappa$, $\kappa \ll \nu$ and ($\nu = \kappa_x > 0$ and $\kappa_y = 0$). If both dissipation parameters are close to each other $\nu \approx \kappa$, a strong constant magnetic field $\alpha > \frac{1}{2}$ is sufficient to ensure stability. Both p get damped by a rate close to each other, ensuring stability. However, if one dissipation is much larger than the other, one component is damped to zero very fast. In contrast, in the other component, we obtain behavior similar to Subsection 2.2 for $\alpha = 0$ with some additional terms.

The Case '$\nu = \kappa$'

This case was first considered by Liss in [Lis20]. In the case when $\kappa = \nu$ and $\alpha > \frac{1}{2}$, the MHD equations are stable as in the ideal case with additional enhanced dissipation decay

$$|p|(s) \lesssim \exp(-c\nu^{\frac{1}{3}}(s - s_{in}))|p_{in}|.$$

With the energy (2.12) we obtain

$$\partial_s E + 2\nu k^2 (1+s^2) E \leq \tfrac{2}{\alpha} \tfrac{1}{1+s^2} p_1 p_2.$$

Thus we infer for E

$$
\begin{aligned}
E(s) &\le \exp\left(\int_{s_{in}}^{s} \tfrac{2}{\alpha}\tfrac{1}{1+s^2} - 2\nu k^2(1+s^2)\right)E(s_{in}) \\
&\lesssim \exp(-2c\nu^{\frac{1}{3}}(s-s_{in}))E(s_{in})
\end{aligned}
$$

for some $c = c(\alpha) > 0$. With $E \approx |p|^2$ for $\alpha > \frac{1}{2}$ we obtain

$$
|p|(s) \lesssim \exp(-c\nu^{\frac{1}{3}}(s-s_{in}))|p_{in}|.
$$

The Case '$\nu \approx \kappa$'

The regime $\nu \le \kappa \le \nu^{\frac{1}{3}}$ was first considered by Dolce in [Dol24]. When ν and κ are close with $\nu^3 \le \kappa \le \nu^{\frac{1}{3}}$, for a strong magnetic field $\alpha > \frac{1}{2}$, the circular movement ensures linear stability. Our enhanced dissipation changes to $\exp(-c\min(\nu,\kappa)^{\frac{1}{3}}t)$ with the minimum of the dissipation parameters. We obtain the estimate

$$
|p|(s) \lesssim \exp(-c\min(\nu,\kappa)^{\frac{1}{3}}(s-s_{in}))|p_{in}|.
$$

In the following, we show this for the regime $\kappa \le \nu$, the opposite regime is analog. We derive the energy (2.12)

$$
\begin{aligned}
&\partial_s E + 2\nu k^2(1+s^2)|p_1|^2 + 2\kappa k^2(1+s^2)|p_2|^2 \\
&\quad \le \tfrac{2}{\alpha k}\tfrac{1-s^2}{(1+s^2)^2}p_1p_2 - \tfrac{2}{\alpha}sk(\nu+\kappa)p_1p_2.
\end{aligned}
$$

Similarly to the previous cases, the first term will lead to finite growth. The second term appears due to the imbalance of dissipation and is relevant for stability. We estimate and split the second term

$$
\begin{aligned}
-\tfrac{2}{\alpha}sk(\nu+\kappa)p_1p_2 &\le \nu k^2(1+s^2)p_1^2 + \tfrac{1}{\alpha}\nu p_2^2 \\
&\le \nu k^2(1+s^2)p_1^2 + \kappa k^2(1+s^2)p_2^2 + (\tfrac{1}{\alpha}\nu - \kappa k^2(1+s^2))p_2^2.
\end{aligned}
$$

We absorb the first two terms into the dissipation. Thus we infer

$$
\begin{aligned}
&\partial_s E + 2c\kappa k^2(1+s^2)E \\
&\quad \le \tfrac{2}{\alpha}\tfrac{1}{1+s^2}p_1p_2 + (\tfrac{1}{\alpha}\nu - \kappa k^2(1+s^2))_+ p_1p_2 \\
&\quad \lesssim (\tfrac{1}{1+s^2} + (\tfrac{1}{\alpha}\nu - \kappa k^2(1+s^2))_+)E
\end{aligned}
$$

for some $c = c(\alpha) > 0$. At this point we require the assumption $\nu^3 \le \kappa$, to bound

$$
\int (\tfrac{1}{\alpha}\nu - \kappa k^2(1+\tau^2))_+ d\tau \le 2\sup_s(\tfrac{1}{\alpha}s\nu - \tfrac{\kappa}{3}k^2s^3) \le 4\tfrac{4}{\alpha}.
$$

Thus we deduce the estimate

$$
E(s) \lesssim \exp(-2c\min(\nu,\kappa)^{\frac{1}{3}}(s-s_{in}))E(s_{in}).
$$

Thus, with $\alpha > \frac{1}{2}$ we infer the estimate

$$
|p|(s) \lesssim \exp(-c\min(\nu,\kappa)^{\frac{1}{3}}(s-s_{in}))|p_{in}|.
$$

Frequencies Close to Resonant

Before discussing the cases where there is a large imbalance of the dissipation, we consider the case close to resonant which ensures stability in the p variables. For the non-resistive limit of Chapter 5 this is particularly important for stability. For two times s, s_0 (s_0 is any time, possibly different from s_{in}) satisfying $|s - s_0| \leq \min(\nu^{-1}, \kappa^{-1})$, a large constant magnetic field $\alpha > \frac{1}{2}$ ensures that the circular movement is stronger than the imbalance of dissipation yielding stability

$$|p|(s) \lesssim |p(s_0)|.$$

The derivative of the energy (2.12) satisfies

$$\begin{aligned}
&\partial_s E + 2\nu k^2(1+s^2)|p_1|^2 + 2\kappa k^2(1+s^2)|p_2|^2 \\
&= \tfrac{2}{\alpha k}\tfrac{1-s^2}{(1+s^2)^2} p_1 p_2 + \tfrac{2}{\alpha}|\nu+\kappa| s k p_1 p_2 \\
&\lesssim (\tfrac{1}{1+s^2} + \tfrac{2}{\alpha}|\nu+\kappa| s k) p_1 p_2.
\end{aligned}$$

We estimate the second term by

$$\begin{aligned}
\tfrac{2}{\alpha}|\nu+\kappa| s k p_1 p_2 &\leq \tfrac{4}{\alpha^2}\max(\nu,\kappa)|p_1|^2 + \max(\nu,\kappa) k^2(1+s^2)|p_2|^2, \\
\tfrac{2}{\alpha}|\nu+\kappa| s k p_1 p_2 &\leq \tfrac{4}{\alpha^2}\max(\nu,\kappa)|p_2|^2 + \max(\nu,\kappa) k^2(1+s^2)|p_1|^2,
\end{aligned}$$

which yields

$$\tfrac{2}{\alpha}|\nu+\kappa| s k p_1 p_2 \leq \tfrac{4}{\alpha^2}\max(\nu,\kappa)|p|^2 + \nu k^2(1+s^2)|p_1|^2 + \kappa k^2(1+s^2)|p_2|^2.$$

So we infer the estimate

$$\begin{aligned}
&\partial_s E + c\min(\kappa,\nu) k^2(1+s^2) E \\
&\quad\lesssim (\tfrac{1}{1+s^2} + \max(\nu,\kappa)) p_1 p_2 \\
&\quad\lesssim (\tfrac{1}{1+s^2} + \max(\nu,\kappa)) E
\end{aligned} \tag{2.18}$$

for some $c = c(\alpha)$. By Growall's lemma, we obtain

$$E(s) \lesssim \exp(\max(\nu,\kappa)(s-s_0)) E(s_{in}) \lesssim E(s_0),$$

and thus with $\alpha > \frac{1}{2}$ we deduce

$$|p|(s) \lesssim |p(s_0)|.$$

We note that, from (2.18) one obtains an enhanced dissipation rate of $\exp(-c(\min(\kappa,\nu))^{\frac{1}{3}}(s-s_0)$. However, this effect only appears if $|s - s_0| \gg \min(\kappa,\nu)^{-\frac{1}{3}}$. Therefore, to reach time scales where enhanced dissipation is relevant, we need the assumption $\nu^3 \leq \kappa \leq \nu^{\frac{1}{3}}$ of the previous section.

The Effect of Strong Resistivity with $\nu \ll \kappa$

In this case the viscosity is much smaller than the resistivity, see Chapter 3 for more details. In the first model, we formally set $p_2 = 0$. Then we obtain the behavior of the Euler and Navier-Stokes equations of Section 2.1. The assumption $p_2 = 0$ is too strong for a more precise analysis. We consider the vorticity and current formulation

$$\begin{aligned} \partial_t w &= -\alpha k j - \nu k^2(1+s^2)w, \\ \partial_t j &= (\tfrac{2s}{1+s^2} - \kappa k^2(1+s^2))j + \alpha k w. \end{aligned} \tag{2.19}$$

We define the good unknown $G = j + \frac{\alpha}{\kappa}\frac{1}{1+s^2}w$ and so (2.19) changes to

$$\begin{aligned} \partial_t w &= -\tfrac{\alpha^2}{\kappa}\tfrac{1}{1+s^2}w - \nu k^2(1+s^2)w + \alpha G, \\ \partial_t G &= (\tfrac{2s}{1+s^2} - \kappa k^2(1+s^2))G + \tfrac{\alpha}{\kappa}\tfrac{s}{(1+s^2)^2}w + \tfrac{\alpha}{\kappa}\partial_s(\tfrac{1}{1+s^2}w). \end{aligned}$$

The resistivity is large enough for high frequencies k to ensure $G \approx 0$. Therefore we obtain for $\beta = \frac{\kappa}{\alpha^2}$

$$\partial_t w = -\tfrac{1}{\beta}\tfrac{1}{1+s^2}w - \nu k^2(1+s^2)w$$

compared to the Navier-Stokes case, the vorticity w obtains additional damping of $-\frac{1}{\beta}\frac{1}{1+s^2}w$.

The Effect of Strong Viscosity with $\kappa \ll \nu$

In this case the resistivity is much smaller than the viscosity, see Chapter 5 for more details. At first glance, one would expect that large values of viscosity lead to better stability. This doesn't hold since for $\nu \gg \kappa^{\frac{1}{3}}$ we obtain growth in the p_2 unknown which corresponds to growth in the magnetic field. Due to the viscosity, for large times $s \gtrsim \nu^{-1} \approx s_0$ we obtain $p_1 \approx 0$ and so p_1 can be neglected (In Chapter 5, Proposition 5.2 and 5.3 we prove the following with the p_1 unknown). For times $s \gtrsim \nu^{-1} \approx s_0$, we reduce (2.19) to the toy model

$$\partial_t p_2 = (\tfrac{s}{1+s^2} - \kappa k^2(1+s^2))p_2. \tag{2.20}$$

We obtain the estimate

$$|p_2|(s) \lesssim \nu\kappa^{-\frac{1}{3}} e^{-c\kappa^{\frac{1}{3}}(s-s_0)}|p_2|(s_0), \tag{2.21}$$

which is optimal in the sense that for $k = -1$ we obtain the norm inflation

$$p_2(\kappa^{-\frac{1}{3}}, k=-1) \approx \nu\kappa^{-\frac{1}{3}} p_2(s_0, k=-1). \tag{2.22}$$

In particular, the p_2 unknown exhibit growth of the size $\nu\kappa^{-\frac{1}{3}}$.

The first term in (2.20) leads to linear growth until the resistivity is strong enough for the second term to take over. This is seen in the explicit solution of (2.20)

$$p_2(s) = \tfrac{\langle s \rangle}{\langle s_0 \rangle} \exp\left(-\kappa k^2 \int_{s_0}^{s} 1 + \tau^2 \, d\tau\right) p_2(s_0).$$

Then (2.22) and (2.21) follow directly. The reader may expect that the enhanced dissipation timescale $\nu^{-\frac{1}{3}}$ would be the relevant timescale, but the combination of circular movement and the viscosity gives enough decay for p_2 such that the linear growth is suppressed until the time ν^{-1}.

The Effect of Horizontal Resistivity with $\kappa_x = \nu > 0$ and $\kappa_y = 0$

We consider the effect of anisotropic resistivity when $\mu := \kappa_x = \nu > 0$ and $\kappa_y = 0$ for a strong constant magnetic field $\alpha > \frac{1}{2}$, see Chapter 4 for more details. Then we obtain the linearized equations

$$\begin{aligned} \partial_t p_1 &= -\tfrac{s}{1+s^2} p_1 - \alpha k p_2 - \mu k^2 (1+s^2) p_1, \\ \partial_t p_2 &= \tfrac{s}{1+s^2} p_2 + \alpha k p_1 - \mu k^2 p_2. \end{aligned} \tag{2.23}$$

The horizontal resistivity is sufficient to ensure stability

$$|p|(s) \lesssim e^{-c\mu(s-s_{in})} |p_{in}|.$$

This is shown with the energy E in (2.12)

$$\begin{aligned} \partial_s E + 2\mu k^2 (1+s^2)|p_1|^2 + 2\mu k^2 |p_2|^2 \\ = \tfrac{2}{\alpha k} \tfrac{1-s^2}{(1+s^2)^2} p_1 p_2 + \tfrac{2}{\alpha} \mu k s (1 + \tfrac{1}{1+s^2}) p_1 p_2. \end{aligned}$$

The last term is estimated by

$$\begin{aligned} \tfrac{2}{\alpha} \mu k s p_1 p_2 &\leq 2\mu k^2 s^2 p_1^2 + \tfrac{1}{2\alpha^2} \mu k^2 p_2^2, \\ \tfrac{2}{\alpha} \mu k s \tfrac{1}{1+s^2} p_1 p_2 &\leq \mu k p_1^2 + \tfrac{4}{1+s^2} p_2^2. \end{aligned}$$

This yields

$$\partial_s E + 2c\mu k^2 E \leq \tfrac{8}{1+s^2} p_1 p_2,$$

for some $c = c(\alpha) > 0$ and so with Gronwall's Lemma, we infer

$$E(s) \lesssim e^{-2c\mu(s-s_{in})} E(s_{in}).$$

Finally, we deduce

$$|p|(s) \lesssim e^{-c\mu(s-s_{in})} |p_{in}|.$$

2.4 Tailored Unknowns

As seen in the previous section, we use several notions of unknowns. In this section, we summarize the most important unknowns and their relations. Usually, the different unknowns are used if we consider the part of the equation without x-average. For the x-average, we use the original equations. Then, the different unknowns are equivalent formulations of the MHD equations. In the following, we only consider the part without x-average and omit writing the projection. The unknowns satisfy the following relation

$$
\begin{aligned}
p_1 &\xrightleftharpoons[-\Lambda_t^{-1}\nabla_t^{\perp}]{\Lambda_t^{-1}\nabla_t^{\perp}} v,\\
p_2 &\xrightleftharpoons[-\Lambda_t^{-1}\nabla_t^{\perp}]{\Lambda_t^{-1}\nabla_t^{\perp}} b,
\end{aligned}
$$

and

$$
\begin{aligned}
p_1 &\xrightleftharpoons[\Lambda_t^{-1}]{\Lambda_t} w,\\
\phi \xrightleftharpoons[-\Lambda_t^{-1}]{-\Lambda_t} p_2 &\xrightleftharpoons[\Lambda_t^{-1}]{\Lambda_t} j.
\end{aligned}
$$

Furthermore, if one dissipation parameter is much stronger, the symmetry of the p unknowns breaks. The dissipation and the linear forcing by the other unknown lead to a balance. Good unknowns are a typical approach to use this balance to obtain better estimates [Zil23, MSHZ22]. For the MHD equations, the relevant good unknowns are

$$
\begin{aligned}
G_w &= \partial_x w + \tfrac{\alpha}{\nu}\partial_x^2\phi,\\
G_\phi &= \partial_x\Delta_t\phi + \tfrac{\alpha}{\kappa}\partial_x^2\Delta_t^{-1}w.
\end{aligned}
$$

All these different unknowns lead to different equations. In the following, we summarize the most relevant equations:

Velocity and magnetic field formulation: The standard form of the MHD equations is written in terms of the velocity and magnetic field

$$
\begin{aligned}
\partial_t v + v_2 e_1 &= \nu\Delta_t v + \alpha\partial_x b + b\cdot\nabla_t b - v\cdot\nabla_t v - \nabla_t\pi,\\
\partial_t b - b_2 e_1 &= \kappa\Delta_t b + \alpha\partial_x v + b\cdot\nabla_t v - v\cdot\nabla_t b,\\
\nabla_t\cdot v &= \nabla_t\cdot b = 0.
\end{aligned}
\tag{2.24}
$$

The p unknowns formulation: The p unknowns reduce from the vectorial unknowns (v, b) to two scalar unknowns (p_1, p_2). The equations for the p unknown read

$$
\begin{aligned}
\partial_t p_1 - \partial_x\partial_y^t\Delta_t^{-1}p_1 - \alpha\partial_x p_2 &= \nu\Delta_t p_1 + \Lambda_t^{-1}\nabla_t^{\perp}(b\cdot\nabla_t b - v\cdot\nabla_t v),\\
\partial_t p_2 + \partial_x\partial_y^t\Delta_t^{-1}p_2 - \alpha\partial_x p_1 &= \kappa\Delta_t p_2 + \Lambda_t^{-1}\nabla_t^{\perp}(b\cdot\nabla_t v - v\cdot\nabla_t b).
\end{aligned}
\tag{2.25}
$$

This formulation is more useful for calculations and energy estimates. In particular, as seen in the previous sections, the p unknowns allow us to quantify the relation between circular movement, inviscid damping and different dissipation regimes. In Sobolev spaces, they are equivalent to the velocity and magnetic field.

Vorticity and magnetic potential formulation: If there is a strong imbalance between resistivity and viscosity, linear stability will shift from the p unknowns to the vorticity w and magnetic potential ϕ

$$\begin{aligned} \partial_t w + v\cdot\nabla w &= \nu\Delta_t w + \alpha\partial_x\Delta_t\phi + \nabla^\perp\phi\cdot\nabla\Delta_t\phi, \\ \partial_t\phi + v\cdot\nabla\phi &= \kappa\Delta_t\phi + \alpha\partial_x\Delta_t^{-1}w. \end{aligned} \tag{2.26}$$

Vorticity and magnetic good-unknown formulation: For the resistive $\kappa > 0$ and inviscid $\nu = 0$ case, the resistivity damps the magnetic potential to a balance with linear forcing by the vorticity. The good unknown G_ϕ corresponds to this balance

$$\begin{aligned} \partial_t w &= -\tfrac{\alpha^2}{\kappa}\partial_x^2\Delta_t^{-1}w + \alpha G_\phi + b\cdot\nabla_t j - v\cdot\nabla_t w, \\ \partial_t G_\phi &= \kappa\Delta_t G_\phi + \partial_x\Delta_t(v\cdot\nabla_t\phi) + \tfrac{\alpha}{\kappa}\partial_x(\Delta_t^{-1}w). \end{aligned} \tag{2.27}$$

Fluid good-unknown and magnetic potential formulation: In the viscous $\nu > 0$ and non-resistive case $\kappa = 0$, the viscosity damps the magnetic potential to a balance with the linear forcing by the magnetic potential. The good unknown G_w corresponds to this balance

$$\begin{aligned} \partial_t G_w &= \nu\Delta_t G_w + \tfrac{\alpha}{\nu}\partial_t\partial_x^2\phi + \partial_x(\nabla^\perp\phi\cdot\nabla\Delta_t\phi - v\cdot\nabla w), \\ \partial_t\phi &= -\tfrac{\alpha^2}{\nu}\partial_x^2\Delta_t^{-1}\phi + \alpha\Delta_t^{-1}G_w - v\cdot\nabla\phi. \end{aligned} \tag{2.28}$$

2.5 Nonlinear Behavior

For the MHD equations, the linearized dynamics decouple in frequency space and we obtain stability for a suitable choice of adapted unknowns. The nonlinear terms differ depending on the choice of unknowns and yield an interaction between different frequencies. Understanding and estimating the nonlinear interaction is the major challenge for establishing nonlinear stability. In the following, we give a simplified model to get an intuition for the main growth mechanism in nonlinear terms. We consider the toy model of an active scalar equation

$$\begin{aligned} \partial_t f &= (a\cdot\nabla_t)f, \\ a &= \nabla_t^\perp\Lambda_t^{-m}f. \end{aligned} \tag{2.29}$$

For f we assume vanishing x-average. The m corresponds to different choices of unknowns, for $m = 2$ this is the nonlinearity of (2.27) and $m = 1$ is a simplification of the nonlinearities in (2.25) (i.e. $\Lambda_t^{-1}\nabla_t^\perp(b\nabla_t b)$ replaced by $b\nabla_t(\Lambda_t^{-1}\nabla_t^\perp b) = b\nabla_t p$). After a Fourier transform of (2.29) we obtain

$$\partial_t f(k,\xi) = \sum_{k-l,l\neq 0} \int d\eta \frac{\xi l - k\eta}{((\xi-\eta-(k-l)t)^2+(k-l)^2)^{\frac{m}{2}}} f(k-l,\xi-\eta) f(l,\eta). \tag{2.30}$$

Nonlinear Toy Model: Reaction Term

We apply a paraproduct decomposition to the quadratic nonlinearity to isolate the main resonance mechanism in (2.30). This is a common approach, see for example [BM14, Zil23]. We split the solution into high and low frequencies $f = f_{lo} + f_{hi}$. This can be made precise with the help of a Littlewood-Paley decomposition [Gra14]. We thus obtain the system

$$\begin{aligned} \partial_t f_{hi} &= (a_{lo}\nabla_t) f_{hi} + (a_{hi}\nabla_t) f_{lo} + (a_{hi}\nabla_t) f_{hi}, \\ \partial_t f_{lo} &= (a_{lo}\nabla_t) f_{lo}. \end{aligned}$$

For this splitting, the $a_{lo}\nabla_t$ term acts as a transport and the $(a_{hi}\nabla_t)p_{hi}$ term is small in high Sobolev spaces. Our main contribution is then the term

$$\partial_t f_{hi} \approx (a_{hi}\nabla_t) f_{lo}. \tag{2.31}$$

We model this effect by replacing

$$f_{lo} \;\to\; -2c\cos(x).$$

These are called a traveling wave solution. The traveling in the name comes from the fact that our coordinates follow the characteristic of the Couette flow, which means we 'travel' in the original coordinates by $x - yt$. The cos is a solution of frequency $k = 1$ and yields a nearest-neighbor interaction between the k and $k \pm 1$ modes. This is a very good model for the largest possible growth in the full nonlinear system. More generally, we could use any periodic and smooth function in x. With this low frequent solution, we obtain

$$\begin{aligned} \partial_t f &= -\nabla^\perp \Lambda_t^{-m} f \cdot \nabla 2c\cos(x) \\ &= 2c\sin(x)\partial_y \Lambda_t^{-m} f, \end{aligned}$$

which is our main growth model. After a Fourier transform, we obtain

$$\begin{aligned} \partial_t f(k,\xi) &= c\frac{\xi}{(k+1)^m} \frac{1}{(1+(\frac{\xi}{k+1}-t)^2)^{\frac{m}{2}}} f(k+1,\xi) \\ &\quad - c\frac{\xi}{(k-1)^m} \frac{1}{(1+(\frac{\xi}{k-1}-t)^2)^{\frac{m}{2}}} f(k-1,\xi). \end{aligned}$$

We simplify this further by looking at the k mode acting on $k-1$ mode

$$\partial_t f(k-1,\xi) = c\frac{\xi}{k^m} \frac{1}{(1+(\frac{\xi}{k}-t)^2)^{\frac{m}{2}}} f(k,\xi).$$

We distinguish between the two main cases for m.

- Let $m = 2$, this corresponds to the main model for the resistive MHD equations in Chapter 3

$$\partial_t f(k-1,\xi) = c\tfrac{\xi}{k^2}\tfrac{1}{1+(\frac{\xi}{k}-t)^2} f(k,\xi)$$

The main effect of $f(k)$ acting on $f(k-1)$ will appear close to the time $\tilde{t}_k = \frac{\xi}{k}$, which we call resonant time. Close to this time, the $f(k)$ stays approximately constant and thus after integrating in time

$$f(t_{k-1}, k-1, \xi) \approx c\pi \tfrac{\xi}{k^2} f(t_k, k, \xi)$$

for times $t_l = \frac{1}{2}(\tilde{t}_l + \tilde{t}_{l+1})$ between the resonant times. If we iterate this growth $k \to k-1 \to \dots \to 1$ for initial data $f(t_k, l, \xi) = \delta_{kl}$ we obtain

$$f(t_1, 1, \xi) \approx \prod_{l=1}^{k} c\pi \tfrac{\xi}{l^2} = \tfrac{(c\pi\xi)^k}{(k!)^2}.$$

This is maximized by Sterling's approximation at $k \approx \sqrt{c\xi}$ to

$$f(t_1, 1, \xi) \approx \sqrt{\xi}\exp(\tilde{C}\sqrt{\xi}) f(t_k, k, \xi)$$

which is a loss if Gevrey 2 regularity of $\tilde{C} = \tilde{C}(c) > 0$ in the radius of convergence. Which corresponds to our result in Chapter 3.

- The case $m = 1$ correspnds to the main model of the p unknowns in Chapter 5

$$\partial_t f(k-1,\xi) = c\tfrac{\xi}{k}\tfrac{1}{(1+(\frac{\xi}{k}-t)^2)^{\frac{1}{2}}} f(k,\xi).$$

With the previous approach, we would need regularity stronger than Gevrey 1, which would impose problems in the nonlinear estimates. Here we perform a different approach, we show under which circumstances we can suppress the nonlinear effect with dissipation. At times $t \geq \min(\nu,\kappa)^{-\frac{1}{3}}$, the enhanced dissipation appears and damps the equation down quickly. Thus we consider times $t \leq \mu^{-\frac{1}{3}}$ with $\mu = \min(\nu,\kappa)$. We write

$$\tfrac{\xi}{k}\tfrac{1}{(1+(\frac{\xi}{k}-t)^2)^{\frac{1}{2}}} = 1 + \tfrac{t}{(1+(\frac{\xi}{k}-t)^2)^{\frac{1}{2}}}$$

and after integrating in time

$$\int_0^t \tfrac{\xi}{k}\tfrac{1}{(1+(\frac{\xi}{k}-\tau)^2)^{\frac{1}{2}}} d\tau \lesssim t\ln(2\sqrt{1+t^2}) \leq \mu^{-\frac{1}{3}}\ln(4\mu^{-\frac{1}{3}}).$$

If we assume that $f(k)$ is constant for $t \approx \frac{\xi}{k}$, we obtain then

$$f(t_{k-1}k-1,\xi) \leq 2c\mu^{-\frac{1}{3}}\ln(4\mu^{-\frac{1}{3}}) f(t_k,k,\xi).$$

Thus for $c < \frac{1}{2}\mu^{\frac{1}{3}}\left(\ln(4\mu^{-\frac{1}{3}})\right)^{-1}$ these resonances get supresed. The idea here corresponds to what is done in the nonlinear estimates if $\nu \approx \kappa > 0$. However, the nonlinear estimates are more challenging since the supremum norm in time and the L^2 norm in space do not commute. Therefore, the nonlinear estimates are worse than conjectured here for the toy model.

Chapter 3

On Echoes in Magnetohydrodynamics with Magnetic Dissipation

This chapter consists of the paper [KZ1], published in the Journal of Differential Equations and is a joint work with Christian Zillinger. In this Chapter, we often refer to Gevrey spaces in the sense of (A.1) since the important effects are visible there. We note here that the space (3.19) with (3.9) also includes the Gevrey spaces in the sense of (A.2).

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Abstract. We study the long time asymptotic behavior of the inviscid magnetohydrodynamic equations with magnetic dissipation near a combination of Couette flow and a constant magnetic field. Here we show that there exist nearby explicit global in time low frequency solutions, which we call waves. Moreover, the linearized problem around these waves exhibits resonances under high frequency perturbations, called echoes, which result in norm inflation Gevrey regularity and infinite time blow-up in Sobolev regularity.

3.1 Introduction and Main Results

In this article we consider the two-dimensional magnetohydrodynamic (MHD) equations with magnetic resistivity $\kappa > 0$ but without viscosity

$$\begin{aligned} \partial_t V + V \cdot \nabla V + \nabla p &= B \cdot \nabla B, \\ \partial_t B + V \cdot \nabla B &= \kappa \Delta B + B \cdot \nabla V, \\ \operatorname{div}(B) = \operatorname{div}(V) &= 0, \\ (t, x, y) &\in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}, \end{aligned} \tag{3.1}$$

near the stationary solution

$$\begin{aligned} V(t,x,y) &= (y,0), \\ B(t,x,y) &= (\alpha,0). \end{aligned} \tag{3.2}$$

The MHD equations are a common model of the evolution of conducting fluids interacting with (electro-)magnetic fields in regimes where the magnetization of the fluid can be neglected. They describe the evolution of the fluid in terms of the fluid velocity V, pressure p and magnetic field B. The constant mass and charge densities are normalized to 1. Here particular examples of applications range from the modeling of solar dynamics to geomagnetism and the earths molten core to using liquid metals in industrial applications or in fusion applications [Dav16].

A main aim of this article is to analyze the long-time asymptotic behavior of solutions to this coupled system and, in particular, the interaction of instabilities, partial dissipation and the system structure of the equations. Here we note that due to the affine structure of the stationary solution (3.2), the corresponding linearized problem around this solution decouples in Fourier space and can be shown to be stable in arbitrary Sobolev (or even analytic) regularity, as we prove in Section 3.2.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$ be given and consider the linear problem*

$$\begin{aligned} \partial_t V + y\partial_x V + (V_2,0) &= \alpha\partial_x B, \\ \partial_t B + y\partial_x B - (B_2,0) &= \kappa\Delta B + \alpha\partial_x V, \\ \mathrm{div}(B) = \mathrm{div}(V) &= 0, \\ (t,x) &\in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \end{aligned}$$

Then these equations are stable in H^s for any $s \in \mathbb{R}$ in the sense that there exists a constant $C > 0$ such that for any choice of initial data and all times $t > 0$ it holds that

$$\begin{aligned} &\|(\nabla^\perp \cdot V)(t,x-ty,y)\|_{H^s}^2 + \|(\nabla^\perp \cdot B)(t,x-ty,y)\|_{H^s}^2 \\ &\quad \leq (1+\kappa^{-2/3})^2(\|\nabla^\perp \cdot V|_{t=0}\|_{H^s}^2 + \|\nabla^\perp \cdot B|_{t=0}\|_{H^s}^2). \end{aligned}$$

Here $\nabla^\perp \cdot V =: W$ is the vorticity of the fluid and $\nabla^\perp \cdot B =: J$ is the (magnetically induced) current.

In contrast to this to this very strong linear stability result, the stability results for the inviscid nonlinear equations are expected to crucially rely on very high, Gevrey regularity (see Section 3.2.2 for a definition). More precisely, similarly to the nonlinear Euler equations [DM23, DZ21, Zil23] or Vlasov-Poisson equations [Bed20, Zil21a, MV11] the nonlinear equations are not a priori expected to not remain close to the linear dynamics due to "resonances" or "echoes"[MWGO68, YOD05], which may lead to unbounded norm inflation of any Sobolev norm. It is the main aim of this article to identify and capture this resonance mechanism for the resistive MHD equations. In particular, we ask to which extent magnetic dissipation can stabilize the dynamics. As we discuss in Section 3.2.2 the main nonlinear resonance mechanism is expected to be given by the repeated interaction of a high frequency perturbation with an underlying low frequency perturbation of (3.2). In this article we thus explicitly construct such low frequency nonlinear solutions, called *traveling waves* (a combination of an Alfvén waves and shear dynamics; see Section 3.2 and Lemma 3.5 for further discussion).

Lemma 3.2. *Let $\kappa > 0$ and $\alpha \in \mathbb{R}$ and let $(f_0, g_0) \in \mathbb{R}^2$. Then there exist smooth global in time solutions of the nonlinear, resistive MHD equations* (3.1)*, which are of the form*

$$V(t,x,y) = (y,0) - \frac{f(t)}{1+t^2}\nabla^\perp \cos(x-ty),$$
$$B(t,x,y) = (\alpha,0) + \frac{g(t)}{1+t^2}\nabla^\perp \sin(x-ty),$$

with $(f(0), g(0)) = (f_0, g_0)$. Furthermore, for a suitable choice of f_0, g_0 it holds that

$$f(t) \to 2c,$$
$$g(t) \to 0,$$

as $t \to \infty$.

In view of the underlying shear dynamics it is natural to change to coordinates

$$(x - ty, y).$$

In these coordinates the corresponding vorticity $W = \nabla^\perp \cdot V$ and current $J = \nabla^\perp \cdot B$ read

$$W = -1 + f(t)\cos(x),$$
$$J = 0 - g(t)\sin(x).$$

Unlike the stationary solution (3.2) these waves have a non-trivial x-dependence. As we discuss in Section 3.2.2 this x-dependence allows resonances to propagate in frequency and underlies the nonlinear instability of the stationary solution (3.2). More precisely, we show that the (simplified) *linearized* equations around these waves exhibit the above mentioned *nonlinear* resonance mechanism (in terms of both upper and lower bounds on solutions). In particular, we aim to obtain a precise understanding of the dependence of the resonance mechanism on the resistivity $\kappa > 0$ and the frequency-localization of the initial perturbation. The research on well-posedness and asymptotic behavior of the magnetohydrodynamic equations is a very active field of research and we in particular mention the recent work [Lis20], which considers a related, fully dissipative setting in 3D, as well as the articles [JW22, ZZ23, WZ21, BLW20, FL19, DYZ19, HXY18, LCZL18, WZ17]. More precisely, in [Lis20] Liss studied the nonlinear, fully dissipative, three-dimensional MHD equations around the same stationary solution (3.2) in a doubly-periodic three-dimensional channel $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ and established bounds on the Sobolev stability threshold as $\nu = \kappa \downarrow 0$. In contrast, this article considers the 2D setting with partial dissipation $\nu = 0$, $\kappa > 0$ in Gevrey regularity. Similar questions on the stability of systems with partial dissipation in critical spaces are also a subject of active research in other (fluid) systems, such as the Boussinesq equations [CW13, EW15, DWZZ18].

For simplicity of presentation and to simplify the analysis in this article we modify the linearized equations for the vorticity and current perturbations w, j

$$\begin{aligned} \partial_t w &= \alpha \partial_x j - (2c\sin(x)\partial_y \Delta_t^{-1} w)_{\neq} \\ \partial_t j &= \kappa \Delta_t j + \alpha \partial_x w - 2\partial_x \partial_y^t \Delta_t^{-1} j, \\ \Delta_t &= \partial_x^2 + (\partial_y - t\partial_x)^2, \end{aligned} \tag{3.3}$$

and fix the x-averages of w and j, which also fixes the underlying shear flow. Here, for simplicity we have also replaced $f(t), g(t)$ by $2c$ and 0, respectively. In analogy to other fluid systems [BBZD23, BM15b], a similar structure of the equations can be achieved by considering the coordinates

$$(x - \int_0^t \int V_1 dxdt, \frac{1}{t}\int_0^t \int V_1 dxdt) =: (X, Y),$$

which however makes estimates of Δ_t^{-1} technically more involved and less transparent [Zil17]. In the interest of a clear presentation of the resonance mechanism we hence instead fix $Y = y$ by a small forcing.

Theorem 3.3. *Let $0 < \alpha < 10$ and $0 < \kappa < 1$ with $\beta := \frac{\kappa}{\alpha^2}$ and $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$ be given. Consider the (simplified) linearized equations (3.3) around the wave of Lemma 3.2.*

Then there exists a constant C such that for any initial data w_0, j_0 whose Fourier transform satisfies

$$\sum_k \int \exp(C\sqrt{|\xi|})(|\mathcal{F}w_0(k,\xi)|^2 + |\mathcal{F}j_0(k,\xi)|^2) d\xi < \infty$$

the corresponding solution stays regular for all times up to a loss of constant in the sense that for all $t > 0$ it holds that

$$\sum_k \int \exp(\tfrac{C}{2}\sqrt{|\xi|})(|\mathcal{F}w(t,k,\xi)|^2 + |\mathcal{F}j(t,k,\xi)|^2) d\xi < \infty.$$

Moreover, there exists initial data w_0, j_0 and $0 < C^* < C$ such that

$$\sum_k \int \exp(C^*\sqrt{|\xi|})(|\mathcal{F}w_0(k,\xi)|^2 + |\mathcal{F}j_0(k,\xi)|^2) d\xi < \infty,$$

but so that the corresponding solution w, j grows unbounded in Sobolev regularity as $t \to \infty$.

Let us comment on these results:

- As we discuss in Section 3.2.2 the linearized equations around a traveling wave closely resemble the interaction of high and low-frequency perturbations in the nonlinear equations. These equations thus are intended to serve as slightly simplified model of the nonlinear resonance mechanism. We remark that in the full nonlinear problem the x-averages and hence the underlying shear dynamics change with time and the corresponding change of coordinates has to be controlled. For simplicity and clarity the present model instead fixes this change of coordinates.

- The Fourier integrability with a weight $\exp(C\sqrt{|\xi|})$ corresponds to Gevrey 2 regularity with respect to y. For simplicity of presentation the above results are stated with $L^2(\mathbb{T})$ regularity in x. All results also extend to more general Fourier-weighted spaces, such as H^N for any $N \in \mathbb{N}$ or suitable Gevrey or analytic spaces (see Definitions 3.8 and 3.9).

- The stability and norm inflation in Gevrey 2 regularity matches the regularity classes of the (nonlinear) Euler equations. In particular, the magnetic field and magnetic dissipation are shown to not be strong enough to suppress this growth. We remark that our choice of coupling between the size of the magnetic field and magnetic dissipation is made so that both effects are "of the same magnitude" and hence their interaction plays a more crucial role (see Section 3.2.3 for a discussion).

- These results complement the work of Liss [Lis20] on the Sobolev stability threshold in $3D$ with full dissipation. Indeed, the above derived upper and lower bounds establish Gevrey 2 as the optimal regularity class of the linearized problem in $2D$ with partial dissipation. We expect that as for the Euler [BM15a] or Vlasov-Poisson equations [BMM16] nonlinear stability results match the regularity classes of the linearized problem around appropriate traveling waves.

We further point out that the instability result of Theorem 3.3 also implies a norm inflation result for the nonlinear problem around each wave in slightly different spaces (see Corollary 3.26). In particular in any arbitrarily small analytic neighborhood around the stationary solution (3.2) there exist nonlinearly unstable solutions (with respect to lower than Gevrey 2 regularity) .

The remainder of the article is structured as follows:

- In Section 3.2 we discuss the linearized problem around the stationary state (3.2) and introduce waves as low-frequency solutions of the nonlinear problem.

- In Section 3.2.2 we discuss the resonance mechanism for a toy model. In particular, we discuss optimal spaces for norm inflation and (in)stability results as well as the time- and frequency-dependence of resonances.

- The main results of this article are contained in Section 3.4, where we establish upper bounds and lower bounds on the norm inflation.
- The 3.5 contains some auxiliary estimates of a growth factor in Section 3.4. In the second 3.6 we prove a nonlinear instability result for the traveling waves.

3.2 Linear Stability, Traveling Waves and Echo Chains

In this section we establish the linear stability of the resistive MHD equations (3.1)

$$\begin{aligned}\partial_t V + (V \cdot \nabla)V + \nabla p = (B \cdot \nabla)B, \\ \partial_t B + (V \cdot \nabla)B = \kappa\Delta B + (B \cdot \nabla)V, \\ \nabla \cdot B = \nabla \cdot V = 0,\end{aligned} \tag{3.4}$$

around the stationary solution (3.2) as stated in Lemma 3.1. Furthermore, we sketch the nonlinear resonance mechanism underlying the norm inflation result of Theorem 3.3, which is given by the repeated interaction of high and low frequency perturbations. This mechanism motivates the construction of the traveling wave solutions of Lemma 3.5 and the corresponding (simplified) linearized equations around these waves, which are studied in the remainder of the article.

In order to simplify notation we may restate the MHD equations with respect to other unknowns. That is, since we consider vector fields in two dimensions and V and B are divergence-free, we may introduce the magnetic potential Φ, magnetic current J and fluid vorticity W by

$$\begin{aligned}J &= \nabla^{\perp} \cdot B, \\ \Delta\Phi &= J, \\ W &= \nabla^{\perp} \cdot V.\end{aligned}$$

Under suitable decay assumptions (or asymptotics) in infinity the equations can then equivalently be expressed as

$$\begin{aligned}\partial_t W + (V \cdot \nabla)W = (B \cdot \nabla)\Delta\Phi, \\ \partial_t \Phi + (V \cdot \nabla)\Phi = \kappa\Delta\Phi,\end{aligned} \tag{3.5}$$

or in terms of J:

$$\begin{aligned}\partial_t W + (V \cdot \nabla)W = (B \cdot \nabla)J, \\ \partial_t J + (V \cdot \nabla)J = \kappa\Delta J + (B \cdot \nabla)W - 2(\partial_i V \cdot \nabla)\partial_i \Phi.\end{aligned} \tag{3.6}$$

With these formulations we are now ready to establish the linear stability of the stationary solution (3.2).

Proof of Lemma 3.1. Consider the formulation of the MHD equations as (3.5), then the linearization around $V = (y, 0), W = -1, B = (\alpha, 0), \Phi = \alpha y$ is given by

$$\begin{aligned} \partial_t W + y\partial_x W &= \alpha \partial_x \Delta \Phi, \\ \partial_t \Phi + y\partial_x \Phi + V_2 \alpha &= \kappa \Delta \Phi. \end{aligned}$$

We note that all operators other than $y\partial_x$ are constant coefficient Fourier multipliers. Hence we apply a change of variables

$$(x, y) \mapsto (x - ty, y)$$

to remove this transport term and obtain

$$\begin{aligned} \partial_t w &= \alpha \partial_x \Delta_t \phi, \\ \partial_t \phi &= -\alpha \partial_x \Delta_t^{-1} w + \kappa \Delta_t \phi, \end{aligned}$$

where w, ϕ denote the unknowns with respect to these variables and $\Delta_t = \partial_x^2 + (\partial_y - t\partial_x)^2$. We note that this system decouples in Fourier space and for simplicity of notation express it in terms of the (Fourier transform of the) current $j = \Delta_t \phi$:

$$\begin{aligned} \partial_t w &= ik\alpha j, \\ \partial_t j &= \frac{2k(kt - \xi)}{k^2 + (\xi - kt)^2} j - \kappa(k^2 + (\xi - kt)^2) j + ik\alpha w, \end{aligned}$$

where $k \in Z$ and $\xi \in \mathbb{R}$ denote the Fourier variables with respect to $x \in \mathbb{T}$ and $y \in \mathbb{R}$, respectively. Here and in the following, with slight abuse of notation, we reuse w and j to refer to the Fourier transforms of the vorticity and current perturbation. For $k = 0$ these equations are trivial and we hence in the following we may assume without loss of generality that $k \neq 0$. Furthermore, we note that the right-hand-side depends on ξ only in terms of $\frac{\xi}{k} - t$. Hence, by shifting time we may further assume that $\xi = 0$.

With this reduction we first note that by anti-symmetry for all $\alpha \in \mathbb{R}$ it holds that

$$\partial_t (|w|^2 + |j|^2)/2 = (\tfrac{2t}{1+t^2} - \kappa k^2 (1 + t^2))|j|^2.$$

We make a few observations:

- If $\kappa k^2 \geq 1$ the horizontal dissipation is sufficiently strong to absorb growth for all times.
- If $\kappa k^2 \leq 1$ is small, then for sufficiently large times $|t| \geq (k^2 \kappa)^{-1/3}$ the right-hand-side is non-positive.
- It thus only remains to estimate the growth on the time interval $|t| \leq (k^2 \kappa)^{-1/3}$, where

$$\partial_t (|w|^2 + |j|^2) \leq \frac{4(t)_+}{1 + t^2} (|w|^2 + |j|^2).$$

The latter case can be bounded by an application of Gronwall's lemma and after shifting back in time it yields

$$|w(t)|^2 + |j(t)|^2 \leq (1 + (k^2\kappa)^{-2/3})^2(|w(0)|^2 + |j(0)|^2)$$

for all $t > 0$.

$\square$

While the ground state is thus linearly stable in arbitrary Sobolev or even analytic regularity, nonlinear stability poses to be a much more subtle question with stronger regularity requirements.

3.2.1 Wave-type Perturbations

In order to investigate the stability of the MHD equations, it is a common approach to consider wave-type perturbation. Here a classical result considers perturbations around a constant magnetic field and a vanishing velocity field.

Lemma 3.4 ((c.f. [Alf42, Dav16])). *Consider the ideal MHD equations (i.e. $\kappa = 0$) in three dimensions linearized around a constant magnetic field $B = B_0 e_z$ and vanishing velocity field $V = 0$. Then a particular solution is of the form*

$$B = (B_1(t, z), 0, 0), V = (V_1(t, z), 0, 0)$$

where B_1 and V_1 are solutions of the wave equation

$$\begin{aligned} \partial_t^2 B_1 - B_0^2 \partial_z^2 B_1 = 0, \\ \partial_t^2 V_1 - B_0^2 \partial_z^2 V_1 = 0, \end{aligned}$$

The linearized problem thus admits wave-type solutions propagating in the direction e_z of the constant magnetic field and pointing into an orthogonal direction. These solutions are known as *Alfvén waves* [Alf42].

Proof of Lemma 3.4. We make the ansatz that B and v only depend on t and z and express the linearized equations in terms of the current $J = \nabla \times B$ and vorticity $W = \nabla \times V$. Then the equations reduce to

$$\begin{aligned} \partial_t J = B_0 \partial_z W, \\ \partial_t W = B_0 \partial_z J. \end{aligned}$$

These equations are satisfied if both J and W solve a wave equation and are chosen compatibly. More precisely, two linearly independent solutions are given by

$$W = f(z + B_0 t) = J$$

and

$$W = g(z - B_0 t) = -J,$$

where f and g are arbitrary smooth function.

We remark that since $B = B(t, z)$ and $V = V(t, z)$ point into a direction orthogonal to the z-axis, they are divergence-free for all times. Finally, since both functions are independent of x it follows that all nonlinearities $V \cdot \nabla V, B \cdot \nabla V, V \cdot \nabla B, B \cdot \nabla B$ identically vanish, so these are also nonlinear solutions. □

In the following we consider the two-dimensional setting and extend this construction to also include an underlying affine shear flow. We call the resulting solutions *traveling waves* in analogy to dispersive equations and related constructions for fluids and plasmas [DZ21, Bed20, Zil21a, Zil23, DM23]. As we sketch in Section 3.2.2 the non-trivial x-dependence of these waves will allow us to capture the main nonlinear norm inflation mechanism in the linearized equations around these waves (as opposed to linearizing around the stationary solutions (3.2)).

Lemma 3.5. *Let $\alpha \in \mathbb{R}$ and $\kappa \geq 0$ be given. Then for any choice of parameters $(f(0), g(0)) \in \mathbb{R}^2$ there exists a solution of (3.6) of the form*

$$\begin{aligned} W &= -1 + f(t)\cos(x - yt) \\ J &= -g(t)\sin(x - yt). \end{aligned} \tag{3.7}$$

We call such a solution a traveling wave.

We remark that this construction also allows for general profiles $h(t, x - ty)$ in place of $\cos(x - ty)$. This particular choice is made so that for $f(0)$ and $g(0)$ small, such a wave is an initially small, analytic perturbation of the stationary solution (3.2) and localized at low frequency.

Proof of Lemma 3.5. For easier reference we note that for this ansatz, we obtain

$$\begin{aligned} V &= (y, 0) + \tfrac{f(t)}{1+t^2}\sin(x - yt)(t, 1) \\ W &= -1 + f(t)\cos(x - yt) \\ B &= (\alpha, 0) + \tfrac{g(t)}{1+t^2}\cos(x - yt)(t, 1) \\ J &= -g(t)\sin(x - yt) \\ \Phi &= \alpha y + \tfrac{g(t)}{1+t^2}\sin(x - yt). \end{aligned}$$

Inserting this into the equation (3.6) the nonlinearities vanish due to the one-dimensional structure of the waves. Therefore, this ansatz yields a solution if and only if f and g solve the ODE system

$$\begin{aligned} f'(t) &= -\alpha g(t), \\ g'(t) &= -\kappa(1 + t^2)g(t) + \alpha f(t) + \tfrac{2t}{1+t^2}g(t). \end{aligned} \tag{3.8}$$

Thus by classical ODE theory for any choice of initial data there indeed exists a unique traveling wave solution. □

Given such a traveling wave we are interested in its behavior, in particular for large times, and how it depends on the choices of κ and α.

Lemma 3.6. *Let $\alpha > 0$ and $\kappa > 0$ then for any choice of initial data the solutions $f(t), g(t)$ of the ODE system* (3.8)

$$\begin{aligned} &f'(t) = -\alpha g(t), \\ &g'(t) = -\kappa(1+t^2)g(t) + \alpha f(t) + \tfrac{2t}{1+t^2} g(t), \end{aligned} \tag{3.9}$$

satisfy the following estimates:

$$\begin{cases} |f(t)|^2 + |g(t)|^2 \leq (1+t^2)^2(|f(0)|^2 + |g(0)|^2) & \text{if } 0 < t < \kappa^{-1/3} \\ |f(t)|^2 + |g(t)|^2 \leq |f(\kappa^{-1/3})|^2 + |g(\kappa^{-1/3})|^2 & \text{if } t > \kappa^{-1/3}. \end{cases} \tag{3.10}$$

Furthermore, for a specific choice of initial data it holds

$$|f(t) - \epsilon| \leq \frac{1}{2}\epsilon,$$

for all $t \geq 4\beta^{-1}$ and

$$|g(t)| \to 0$$

as $t \to \infty$.

Proof of Lemma 3.6. We first observe that by anti-symmetry of the coefficients it holds that

$$\partial_t(|f|^2 + |g|^2) = 2|g|^2 \left(-\kappa(1+t^2) + \tfrac{2t}{1+t^2}\right).$$

In particular, for $t > \kappa^{-1/3}$ the last factor is negative and hence $|f|^2 + |g^2|$ is non-increasing. For times smaller than this, we may derive a first rough bound from the estimate

$$\partial_t(|f|^2 + |g|^2) \leq (|f|^2 + |g|^2)\tfrac{4t}{1+t^2},$$

which yields an algebraic lower and upper bound growth bound. We next turn to the case of special data, due to lower and upper norm bounds (3.9) is time reversible. Therefore, we can obtain

$$f(t_0) = 1, \qquad g(t_0) = 0$$

for $t_0 = 4\beta^{-1}$. Then we deduce

$$g(t) = \alpha \int_{t_0}^{t} d\tau \ \exp(-\kappa(t - \tau + \tfrac{1}{3}(t^3 - \tau^3)))\tfrac{1+t^2}{1+\tau^2} f(\tau)$$

and thus

$$\begin{aligned}
f(t) &= 1 - \alpha \int_{t_0}^{t} d\tau_1 g(\tau_1) \\
&= 1 - \tfrac{\kappa}{\beta} \int_{t_0}^{t} d\tau_1 (1+\tau_1^2) \int_{t_0}^{\tau_1} d\tau_2 \ \exp(-\kappa(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3))) \tfrac{1}{1+\tau_2^2} f(\tau_2) \\
&= 1 - \tfrac{1}{\beta} \int_{t_0}^{t} d\tau_2 \tfrac{1}{1+\tau_2^2} f(\tau_2)(1 - \exp(-\kappa(t - \tau_2 + \tfrac{1}{3}(t^3 - \tau_2^3)))).
\end{aligned}$$

This gives the estimate

$$0 \leq 1 - f(t) \leq \tfrac{2}{\beta t_0}, \tag{3.11}$$

which implies that after time t_0 the value of f satisfies the same bound. Similarly, for g we recall that

$$\partial_s g = (\tfrac{2t}{1+t^2} - \kappa(1+t^2))g(t) + \alpha f(t)$$

and hence for $t_1 = 2\kappa^{-\frac{1}{3}}$ it holds that

$$g(t_1) \leq \alpha \tfrac{t_1^2}{t_0^2}.$$

Furthermore, this implies that for times $t \geq t_1$ it holds that

$$\begin{aligned}
g(t) &\leq \alpha \tfrac{t_1^2}{t_0^2} \exp(-\tfrac{\kappa}{3} t^2 (t - t_1)) + \alpha \int_{t_1}^{t} \exp(-\tfrac{\kappa}{3}(1+t^2)(t-\tau)) \\
&\leq \alpha \tfrac{t_1^2}{t_0^2} \exp(-\tfrac{\kappa}{3} t^2 (t - t_1)) + \tfrac{3\alpha}{\frac{\kappa}{3}(1+t)^2}.
\end{aligned}$$

Finally, for times $t \gg \kappa^{-\frac{1}{3}}$ we may estimate

$$g \leq \tfrac{4\alpha}{\kappa t^2}. \tag{3.12}$$

□

3.2.2 Paraproducts and an Echo Model

As mentioned in Section 3.1 the main mechanism for nonlinear instability is expected to be given by the repeated interaction of high- and low-frequency perturbations of the stationary solution (3.2). In the following we introduce a model highlighting the role of the traveling waves and discuss what stability and norm inflation estimates can be expected.

For this purpose we note that the nonlinear MHD equations (3.5) for the perturbations w, ϕ of the groundstate (3.2) in coordinates $(x - ty, y)$ can be expressed as

$$\begin{aligned}
\partial_t w + \nabla^\perp \Delta_t^{-1} w \cdot \nabla w &= \alpha \partial_x \Delta \phi + \nabla^\perp \phi \cdot \nabla \Delta_t \phi, \\
\partial_t \phi + \nabla^\perp \Delta_t^{-1} w \cdot \nabla \phi &= \alpha \partial_x \Delta^{-1} w + \kappa \Delta_t \phi, \\
\Delta_t &= \partial_x^2 + (\partial_y - t\partial_x)^2,
\end{aligned} \tag{3.13}$$

where we used cancellation properties of $\nabla^\perp \cdot \nabla$. The stability result of Lemma 3.1 considered the linearized problem around the trivial solution $(0,0)$, which removes all effects of the nonlinearities. In order to incorporate these effects into our model we thus consider the nonlinear equations as a coupled system for the low frequency part of the solution (w_{low}, ϕ_{low}) (defined as the Littlewood-Payley projection to frequencies $< N/2$ for some dyadic scale N) and the high frequency part (w_{hi}, ϕ_{hi}). If we for the moment consider the low frequency part as given then the action of the nonlinearities on the high frequency perturbation of the vorticity can be decomposed as

$$\nabla^\perp \Delta_t^{-1} w_{low} \cdot \nabla w_{hi} + \nabla^\perp \Delta_t^{-1} w_{hi} \cdot \nabla w_{low} + \nabla^\perp \Delta_t^{-1} w_{hi} \cdot \nabla w_{hi}. \tag{3.14}$$

Here the first term is of transport type and hence unitary in L^2 and we expect $\nabla^\perp \Delta_t^{-1} w_{low}$ to decay sufficiently quickly in time that this term should not yield a large contribution to possible norm inflation. Similarly for the last term we note that both factors are at comparable frequencies and that we by assumption consider a small high frequency perturbation and thus this term is also not expected to have a large impact on the evolution. The main norm inflation mechanism thus is expected to be given by the high frequency velocity perturbation interacting with a non-trivial low frequency vorticity perturbation.

In order to build our toy model we thus focus on this part and formally replace w_{low}, ϕ_{low} by the traveling waves, which are solutions of the nonlinear problem. Furthermore, as a simplification by a similar reasoning as above we also fix the underlying shear flow for our model. Then the equations for the (high frequency part of the) current perturbation $j = \Delta\phi$ and vorticity perturbation w read

$$\begin{aligned} \partial_t w &= \alpha \partial_x j - (2c \sin(x) \partial_y \Delta_t^{-1} w)_{\neq} \\ \partial_t j &= \kappa \Delta_t j_{\neq} + \alpha \partial_x w - 2\partial_x \partial_y^t \Delta_t^{-1} j, \end{aligned} \tag{3.15}$$

where we also simplified to $f(t) = 2c, g(t) = 0$.

We note that compared to the linearization around the stationary solution these equations break the decoupling in Fourier space. Indeed taking a Fourier transform and relabeling $j \mapsto -ij$ we arrive at

$$\begin{aligned} \partial_t w(k) &= -\alpha k j(k) - c\tfrac{\xi}{(k+1)^2} \tfrac{1}{1+(\frac{\xi}{k+1}-t)^2} w(k+1) + c\tfrac{\xi}{(k-1)^2} \tfrac{1}{1+(\frac{\xi}{k-1}-t)^2} w(k-1), \\ \partial_t j(k) &= (2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(\tfrac{\xi}{k}-t)^2)) j(k) + \alpha k w(k). \end{aligned} \tag{3.16}$$

Furthermore, if $t \approx \frac{\xi}{k}$ then the additional term is of size $c\frac{\xi}{(k)^2}$ and hence can possibly lead to a very large change of the dynamics. In reference to the experimental results mentioned in Section 3.1 we can interpret this as the low frequency and high frequency perturbation resulting in an "echo" around the time $t \approx \frac{\xi}{k}$. For the following toy model we neglect all modes except those at frequency k and $k-1$ and only include the action of the resonant mode k on the non-resonant mode $k-1$.

Lemma 3.7 (Toy model)**.** *Let c, κ, α be as in Theorem 3.3 such that $\beta = \frac{\kappa}{\alpha^2} \geq \pi$ and consider the Fourier variables $k \geq 2$ and $\xi \geq 10 \max(\kappa^{-1}, \frac{k^2}{c})$. Then for*

$$t_k := \frac{1}{2}(\frac{\xi}{k+1} + \frac{\xi}{k}) < t < \frac{1}{2}(\frac{\xi}{k} + \frac{\xi}{k-1}) =: t_{k-1}$$

we consider the toy model

$$\begin{aligned}
\partial_t w(k) &= -\alpha k j(k), \\
\partial_t j(k) &= (2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(\tfrac{\xi}{k}-t)^2))j(k) + \alpha k w(k), \\
\partial_t w(k-1) &= -\alpha(k-1)j(k-1) + c\tfrac{\xi}{k^2}\tfrac{1}{1+(\frac{\xi}{k}-t)^2}w(k), \\
\partial_t j(k-1) &= -\kappa\frac{\xi^2}{k^2}j(k-1) + \alpha(k-1)w(k-1).
\end{aligned} \tag{3.17}$$

Then for initial data $w(k, t_k) = 1$ and $w(k-1, t_k) = j(k, t_k) = j(k-1, t_k) = 0$ we estimate

$$(|w(k)| + |w(k-1)| + \alpha k|j(k)| + \alpha(k-1)|j(k-1)|)|_{t=t_{k-1}} \leq 2\pi c\frac{\xi}{k^2}.$$

Furthermore, this bound is attained up to a loss of constant in the sense that

$$|w(k-1, t_{k-1})| \geq \tfrac{\pi}{2}c\frac{\xi}{k^2}.$$

Proof of Lemma 3.7. We perform a shift in time such that $t = \frac{\xi}{k} + s$ and thus $s_0 := -\frac{\xi}{2}\frac{1}{k^2+k} \leq s \leq \frac{\xi}{2}\frac{1}{k^2-k} =: s_1$. Integrating the equations in time, for our choice of initial data we obtain that

$$j(s,k) = \alpha k \int_{s_0}^{t} \tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa k^2(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3)))w(k,\tau_2)d\tau_2$$

and thus

$$\begin{aligned}
w(k,s) &= 1 - \alpha k \int j(\tau_1, k)\ d\tau_1 \\
&= 1 - \alpha^2 k^2 \int_{s_0}^{s}\int_{s_0}^{\tau_1} \tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa k^2(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3)))w(k,\tau_2)d\tau_2 d\tau_1.
\end{aligned}$$

For the second term we insert $\alpha^2 = \frac{\kappa}{\beta}$ and deduce that

$$\begin{aligned}
&\frac{\kappa}{\beta}k^2\int_{s_0}^{s}\int_{s_0}^{\tau_1} \tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa k^2(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3)))w(k,\tau_2)d\tau_2 d\tau_1 \\
&\qquad = \frac{1}{\beta}\int_{s_0}^{s}\tfrac{1}{1+\tau_2^2}\int_{\tau_2}^{s}\kappa k^2(1+\tau_1^2)\exp(-\kappa k^2(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3)))w(k,\tau_2)d\tau_1 d\tau_2 \\
&\qquad = \frac{1}{\beta}\int_{s_0}^{s}\tfrac{w(k,\tau_2)}{1+\tau_2^2}\left[\exp(-\kappa k^2(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3)))\right]_{\tau_1=\tau_2}^{\tau_1=s} d\tau_2 \\
&\qquad = \frac{1}{\beta}\int_{s_0}^{s}\tfrac{w(k,\tau_2)}{1+\tau_2^2}(1 - \exp(-\kappa k^2(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3))))d\tau_2.
\end{aligned}$$

This further yields that

$$w(k,s) = 1 - \frac{1}{\beta}\int_{s_0}^{s} d\tau_2 \tfrac{w(k,\tau_2)}{1+\tau_2^2}(1-\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))). \quad (3.18)$$

Therefore, if $1 \geq w(k,s) \geq 0$ we obtain

$$|w(k,s)-1| \leq \frac{1}{\beta}\int_{s_0}^{s} \tfrac{1}{1+\tau_2^2} d\tau_2 \leq \tfrac{1}{\beta}(\arctan(s)+\tfrac{\pi}{2}).$$

and by bootstrap this assumption holds for all times if $\beta \geq \pi$. For the current $j(k)$ we similarly estimate

$$\begin{aligned}
&\int_{s_0}^{s_1} \tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))w(k,\tau_2)d\tau_2 \\
&\qquad \leq (\int_{s_0}^{\frac{\xi}{5k^2}} + \int_{\frac{\xi}{5k^2}}^{s_1})\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))d\tau_2 \\
&\qquad \leq \tfrac{1}{\kappa k^2}(\exp(-\kappa\xi(\tfrac{1}{3}+3^{-4}\tfrac{\xi^2}{k^4})) + \tfrac{4}{\eta^2}) \leq \tfrac{c}{\kappa k^2},
\end{aligned}$$

which yields

$$\alpha k j(s,k) \leq (\alpha k)^2 \tfrac{c}{\kappa k^2} = \tfrac{c}{\beta}.$$

Concerning the $k-1$ mode we argue similarly and write

$$j(k-1) = \alpha(k-1)\int \exp(\kappa\tfrac{\xi^2}{k^2}(s-\tau))w(k-1)d\tau$$

and

$$\begin{aligned}
w(k-1) &= c\tfrac{\xi}{k^2}\int \tfrac{1}{1+\tau^2}w(k)d\tau - \alpha(k-1)\int j(k-1)\ d\tau_1 \\
&= c\tfrac{\xi}{k^2}\int \tfrac{1}{1+\tau^2}w(k)d\tau \\
&- \alpha^2(k-1)^2\iint \exp(\kappa\tfrac{\xi^2}{k^2}(\tau_1-\tau_2))w(\tau_2,k-1)d\tau_2 d\tau_1 \\
&= c\tfrac{\xi}{k^2}\int_{s_0}^{s_1} \tfrac{1}{1+\tau^2}w(k)d\tau - \tfrac{\alpha^2k^2(k-1)^2}{\kappa\xi^2}\int d\tau_2 w(\tau_2,k-1).
\end{aligned}$$

Since

$$|\tfrac{\alpha^2k^2(k-1)^2}{\kappa\xi^2}\int d\tau_2| \leq \tfrac{1}{\beta\frac{\xi}{k^2}}$$

we deduce by bootstrap that

$$|w(k-1)| \leq 2\pi c\tfrac{\xi}{k^2}$$

and thus

$$
\begin{aligned}
|w(k-1) - \pi c \tfrac{\xi}{k^2}| &\leq c \tfrac{\xi}{k^2} \int_{\tau \notin [s_0, s_1]} \tfrac{1}{1+\tau^2} d\tau \\
&\quad + c \tfrac{\xi}{k^2} \tfrac{1}{\beta} \int_{s_0}^{s_1} \tfrac{1}{1+\tau^2} (\arctan(\tau) - \tfrac{\pi}{2}) d\tau + \tfrac{1}{\beta \frac{\xi}{k^2}} 2\pi c \tfrac{\xi}{k^2} \\
&\leq \pi c \tfrac{\xi}{k^2} (\tfrac{2}{\pi} \tfrac{k^2}{\xi} + \tfrac{\pi}{2\beta} + \tfrac{2}{\beta \frac{\xi}{k^2}}) \\
&\leq \tfrac{\pi}{2} c \tfrac{\xi}{k^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(k-1) j(k-1) &\leq 2\pi c \tfrac{\xi}{k^2} (\alpha(k-1))^2 \int \exp(\kappa \tfrac{\xi^2}{k^2} (s-\tau)) d\tau \\
&= 2\pi c \tfrac{\xi}{k^2} (\alpha(k-1))^2 \tfrac{1}{\kappa \frac{\xi^2}{k^2}} \\
&= \pi c \tfrac{1}{\beta \xi} \ll \pi c \tfrac{\xi}{k^2}.
\end{aligned}
$$

□

Based on this model we may thus expect that a repeated interaction or *chain of resonances* starting at k_0

$$k_0 \mapsto k_0 - 1 \mapsto \cdots \mapsto 1$$

results in a possible growth

$$|w(1, t_1)| \geq |w(k_0, t_{k_0})| \prod_{k=1}^{k_0} C'(1 + c\frac{\xi}{k^2}),$$

where $C' = C'(\beta)$. Choosing $k_0 \approx \sqrt{C' c \xi}$ to maximize this product and using Stirling's approximation formula we may estimate this growth by an exponential factor:

$$\prod_{k=1}^{k_0} C' c \frac{\xi}{k^2} = \frac{(C' c \xi)^{k_0}}{(k_0!)^2} \approx \exp(\sqrt{C' c \xi})$$

This suggests that stability can only be expected if the initial decays in Fourier space with such a rate, which is naturally expressed in terms of Gevrey spaces.

Definition 3.8. *Let $s \geq 1$, then a function $u \in L^2(\mathbb{T} \times \mathbb{R})$ belongs to the Gevrey class $\mathcal{G}_s$ if its Fourier transform satisfies*

$$\sum_k \int \exp(C|\xi|^{1/s}) |\mathcal{F}(u)(k, \xi)|^2 d\xi < \infty$$

for some constant $C > 0$.

In view of the more prominent role of the frequency with respect to y and for simplicity of notation this definition only includes $|\xi|^{1/s}$ as opposed to $(|k| + |\xi|)^{1/s}$ in the exponent. All results in this article also extend to more general Fourier weighted spaces X (see Definition 3.9) with respect to x with norms

$$\sum_k \int \exp(C|\xi|^{1/s})\lambda_k |\mathcal{F}(u)(k,\xi)|^2 d\xi. \tag{3.19}$$

We remark that any Gevrey function is also an element of H^N for any $N \in \mathbb{N}$ and that Gevrey classes are nested with the strongest constraint, $s = 1$, corresponding to analytic regularity with respect to y.

As the main result of this article and as summarized in Theorem 3.3 we show that the above heuristic model's prediction is indeed accurate and that the optimal regularity class for the (simplified) linearized MHD equations around a traveling wave are given by Gevrey 2.

3.2.3 Magnetic Dissipation, Coupling and the Influence of β

In the preceding proof we have seen that the interaction of interaction of $w(k)$ and $j(k)$ is determined by the combination of the action of the underlying magnetic field of size α and magnetic resistivity $\kappa > 0$ through the parameter

$$\beta = \tfrac{\kappa}{\alpha^2}.$$

More precisely, we recall that ignoring the influence of neighboring modes $w(k)$ and $j(k)$ are solutions of a coupled system:

$$\begin{aligned}
\partial_t w(k) &= -\alpha k j(k), \\
\partial_t j(k) &= (2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(\tfrac{\xi}{k}-t)^2))j(k) + \alpha k w(k).
\end{aligned}$$

Hence starting with data $w(k, s_0) = 1, j(k, s_0) = 0$ three different mechanisms interact to determine the size of $w(k, s)$:

- The vorticity $w(k, s)$ by means of the constant magnetic field generates a current perturbation $j(k, s)$.
- The current perturbation $j(k, s)$ is damped by the magnetic resistivity.
- The current $j(k, s)$ in turn by means of the constant magnetic field acts on the vorticity and damps it.

In this system several interesting regimes may arise, which are distinguished by the parameter β.

In the limit of infinite dissipation, $\beta \to \infty$, the current is rapidly damped and the system hence formally reduces to the Euler equations

$$\begin{aligned}
\partial_t w(k) &= 0, \\
j(k) &= 0,
\end{aligned}$$

where $w(k,s)$ remains constant in time.

As the opposite extremal case, if $\beta \downarrow 0$ we obtain the inviscid MHD equations and the system

$$\begin{aligned} \partial_s w(k) &= -\alpha k j(k), \\ \partial_s j(k) &= 2\tfrac{s}{1+s^2} j(k) + \alpha k w(k). \end{aligned}$$

Hence at least for $|s|$ large this suggests that

$$w(k) \approx c_1(1+s)\cos(\alpha k s), \quad j(k) \approx c_1(1+s)\sin(\alpha k s).$$

In particular, in stark contrast to the Euler equations (i.e. $\alpha = 0$) for the inviscid MHD equations with a magnetic field the vorticity $w(k)$ and current perturbations $j(k)$ cannot be expected to remain close to 1 and 0, respectively.

This article considers the regime $0 < \beta < \infty$, where the interaction of both extremal phenomena results in behavior which is qualitatively different from both limiting cases. Indeed, recall that by a repeated application of Duhamel's formula $w(k,s)$ satisfies the integral equation (3.18):

$$w(k,s) = 1 - \frac{1}{\beta}\int_{s_0}^{s} \tfrac{w(k,\tau_2)}{1+\tau_2^2}(1-\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3))))d\tau_2$$

Hence, as a first case which we also discussed in the toy model of Lemma 3.7, if we restrict to $\beta \geq \pi$ then the integral term is bounded and small

$$\frac{1}{\beta}\int_{s_0}^{s} d\tau_2 \tfrac{1}{1+\tau_2^2} \leq 1.$$

Hence, for large β the integral term can be treated as a perturbation and $w(k,s)$ remains comparable to 1 uniformly in s and thus close to the Euler case. However, unlike for the Euler equations the evolution of the current remains non-trivial.

If instead $0 < \beta < \pi$ we obtain different behaviour depending on the dissipation κk^2, the size of the magnetic field and the frequencies considered, whose interaction determines the behavior of the solution. More precisely, considering the integrand

$$\frac{1}{1+\tau_2^2}(1-\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))),$$

we observe that for $\kappa k^2 \gg 1$ large the magnetic dissipation is very strong and hence the integrand is well-approximated by $\frac{1}{1+\tau_2^2}$. In particular, this suggests that for these s it holds that

$$\begin{aligned} w(k,s) &\approx 1 - \frac{1}{\beta}\int_{s_0}^{s} d\tau_2 \tfrac{w(k,\tau_2)}{1+\tau_2^2}, \\ \Leftrightarrow \ \partial_s w(k,s) &\approx -\tfrac{1}{\beta}\tfrac{1}{1+s^2} w(k,s), \\ \Leftrightarrow \quad w(k,s) &\approx \exp(-\tfrac{1}{\beta}(\arctan(s)+\tfrac{\pi}{2}))w(k,s_0), \end{aligned}$$

and hence $w(k,s)$ might decay by a factor comparable to $\exp(-\frac{\pi}{\beta})$.

If instead $\kappa k^2 \leq 1$ is small, different effects interact and involve the following natural time scales:

- Mixing enhanced magnetic dissipation becomes relevant on time scales $(\kappa k^2)^{-1/3} \gg 1$.
- The resonant interval I^k is of size about $\frac{\xi}{k^2}$.
- Within this resonant interval most of the L^1 norm of $\frac{1}{1+\tau_2^2}$ is achieved on a much smaller sub-interval of size about 1.

Hence, for times $|s| < s^* \ll (\kappa k^2)^{-1/3}$ which are small compared to the disspation time scale the integrand is small and we may therefore expect that

$$w(k,s) \approx 1$$

remains constant. If we instead consider very large times $|s| \gg (\kappa k^2)^{-1/3} \gg s^*$ in view of the exponential factor and the decay of $\frac{1}{1+\tau_s^2}$ the size of $w(k,s)$ should largely be determined by the action of the time interval $(-s^*, s^*)$, that is

$$\begin{aligned} w(k,s) &\approx 1 - \frac{1}{\beta}\int_{-s^*}^{s^*} \tfrac{1}{1+\tau_2^2} d\tau_2 \\ &\approx 1 - \tfrac{\pi}{\beta}, \end{aligned}$$

provided such such s exist, that is if the size $\frac{\xi}{k^2}$ of I^k is much bigger than the dissipative time scale. In particular, the size of $w(k,s)$ transitions from being close to 1 for $|s| < s^*$ to being very far from 1 for $|s| \gg (\kappa k^2)^{-1/3}$ and further needs to be controlled on intermediate time scales. These different regimes all have to be considered in the upper and lower bounds of Section 3.4 and we in particular need to control the size of $w(k,s)$ in order to estimate the resulting norm inflation due to resonances. For this purpose we estimate $w(k,s)$ in terms of a growth factor L such that

$$|w(k,s)| \leq L w(k,s_0),$$

as we discuss in 3.5. For our upper bounds we will require that $cL \ll 1$ is sufficiently small to control back-coupling estimates.

3.3 Stability for Small and Large Times

In this section we establish some general estimates on the (simplified) linearized MHD equations (3.16). We note that these equations decouple with respect to ξ. In the following we hence treat ξ as an arbitrary but fixed parameter of the equations and consider (3.16) as an evolution equation for the sequences $w(\cdot,\xi,t)$ and $j(\cdot,\xi,t)$. As mentioned following the statement of Theorem 3.3 in addition to $\ell^2(\mathbb{Z})$ all proofs in the remainder of the article hold for a rather general family of weighted spaces:

Definition 3.9. *Consider a weight function $\lambda_l > 0$ such that*

$$\sup_l \tfrac{\lambda_{l\pm 1}}{\lambda_l} =: \hat{\lambda} < 10.$$

Then we define the Hilbert space X associated to this weight function as the set of all sequences $u: \mathbb{Z} \to \mathbb{C}$ such that $(u_l \lambda_l)_l \in \ell^2$.

This definition for instance includes ℓ^2 ($\lambda_l = 1$), (Fourier transforms of) Sobolev spaces H^s ($\lambda_l = 1 + C|l|^{2s}$ with $C > 0$ sufficiently small) or Gevrey regular or analytic functions with a suitable radius of convergence.

As sketched in Section 3.2.2 for a given frequency $\xi \in \mathbb{R}$ we expect the norm inflation for evolution by (3.16) to be concentrated around times $t_k \approx \frac{\xi}{k}$ for suitable $k \in \mathbb{Z}$. In particular, if the time is too large, $t > 2\xi$, there exists no such k and we expect the evolution to be stable. Similarly, if t is small also the size of the resonance predicted by the toy model is small and we again expect the evolution to be stable. The results of this section show that this heuristic is indeed valid and establish stability for "small" and "large" times. The essential difficulty in proving Theorem 3.3 thus lies in control the effects of resonances in the remaining time intervals, which are studied in Section 3.4. In the following we will often write L^∞_t as the supremum norm till time t.

Lemma 3.10 (Large time)**.** *Consider the equation* (3.16) *on the time interval* $(2\xi, \infty)$*. Then the possible norm inflation is controlled uniformly in time*

$$\|w, j\|_X(t) \leq \tfrac{1}{1-4c} \tfrac{1}{1-2c\hat{\lambda}} \|w, j\|_X(2\xi),$$

where $\hat{\lambda} = \max_l \frac{\lambda_l}{\lambda_{l\pm 1}}$ is as in Definition 3.9.

Proof. Let $\hat{w}(k) = |(w(k), j(k))|$. Then we infer

$$\tfrac{1}{2}\partial_t \hat{w}^2(k) \leq (a(k-1)w(k-1) - a(k+1)w(k+1))w(k) + b(k)j(k)^2,$$

where we introduced the short-hand notation a, b for the coefficient functions. Since $b(t, k) \leq 0$ for $t \geq 2\xi$, we further deduce that

$$\begin{aligned} &\tfrac{1}{2}\partial_t \hat{w}^2(k) \leq c\tfrac{\xi}{1+(t-\xi)^2}(\hat{w}(k+1) + \hat{w}(k-1) + 2\hat{w}(k))\hat{w}(k) \\ \rightsquigarrow\ &\hat{w}^2(k,t) < \hat{w}^2(k, 2\xi) + 2c(|\hat{w}^2(k)|_{L^\infty_t} + \tfrac{1}{2}|\hat{w}^2(k+1)|_{L^\infty_t} + \tfrac{1}{2}|\hat{w}^2(k-1)|_{L^\infty_t}). \end{aligned}$$

Hence by a bootstrap argument we control

$$\hat{w}^2(k,t) \leq \tfrac{1}{1-4c} \sum_l (2c)^{|k-l|} \hat{w}^2(l, 2\xi).$$

Summing this estimate with the weight λ_k then concludes the proof:

$$\begin{aligned} \|w, j\|_X(t) &\leq \tfrac{1}{1-4c} \sqrt{\sum_k \lambda_k \sum_l (2c)^{|k-l|} \hat{w}^2(l, 2\xi)} \\ &\leq \tfrac{1}{1-4c} \sqrt{\sum_l \lambda_l \hat{w}^2(l, 2\xi) \sum_k (2c\hat{\lambda})^{|k-l|}} \\ &\leq \tfrac{1}{1-4c} \tfrac{1}{1-2c\hat{\lambda}} \|w, j\|_X(2\xi) \end{aligned}$$

□

Thus it suffices to study the evolution for times $t < 2\xi$. In view of the estimates of Section 3.2.2 it here is convenient to partition $(0, 2\xi)$ into intervals where $t \approx \frac{\xi}{k}$ for some $k \in \mathbb{Z}$.

Definition 3.11. *Let $\xi > 0$ be given, then for any $k \in \mathbb{N}$ we define*

$$t_k = \frac{1}{2}(\tfrac{\xi}{k+1} + \tfrac{\xi}{k}) \textit{ if } k > 0,$$
$$t_0 = 2\xi.$$

We further define the time intervals $I^k = (t_k, t_{k-1})$, for $\xi < 0$ we define t_k analogously for $-k \in \mathbb{N}$.

Note that

$$t_k < \tfrac{\xi}{k} < t_{k-1}$$

and

$$t_{k-1} - t_k = \frac{1}{2}\left(\tfrac{\xi}{k+1} - \tfrac{\xi}{k-1}\right) = \tfrac{\xi}{k^2-1}.$$

Hence I_k is an interval containing the time of resonance $\frac{\xi}{k}$ and is of size about $\frac{\xi}{k^2}$.

The next lemma provides a very rough energy-based estimate, which will allow us to control the evolution for small times and frequencies. That is, we show that it is easy to obtain a energy estimate with ξt in the exponent. If the time t or the frequency ξ are small this rough estimate is sufficient. However, for Gevrey 2 norm estimates it will be necessary to improve this control to a $C\sqrt{\xi}$ term in the exponent in subsequent estimates. Furthermore, we remark that also the magnetic part needs to be handled adequately, since it may give an additional growth by $\exp(\frac{4}{3}\kappa^{-\frac{1}{2}})$.

Lemma 3.12 (Rough estimate)**.** *Consider a solution of* (3.16)*, then for fixed ξ and for all times $t > 0$ it holds that*

$$\|w, j\|_X(t) \leq \exp(\tfrac{4}{3}\kappa^{-\frac{1}{2}}) \exp((1+\hat{\lambda})c\xi t)\|w, j\|_X(0).$$

Proof. We define $\hat{w}(k) = |w, j|(k)$, then

$$\tfrac{1}{2}\partial_t \hat{w}^2(k) = (a(k+1)w(k+1) - a(k-1)w(k-1))w(k) + b(k)j(k)^2$$

with $b(k,t) = (2\frac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1 + (\frac{\xi}{k} - t)^2))$. We further define the define the growth factor

$$M(k,t) = \begin{cases} 1 & \text{if } t - \frac{\xi}{k} \leq 0 \;\text{ or }\; \kappa k^2 \geq 1, \\ \frac{1}{1+(\frac{\xi}{k}-t)^2} & \text{if } 0 \leq t - \frac{\xi}{k} \leq (\frac{2}{\kappa k^2})^{\frac{1}{3}} \;\text{ and }\; \kappa k^2 \leq 1, \\ \frac{1}{1+(\frac{2}{\kappa k^2})^{\frac{1}{3}}} & \text{if } (\frac{2}{\kappa k^2})^{\frac{1}{3}} \leq t - \frac{\xi}{k} \text{ and } \;\kappa k^2 \leq 1. \end{cases}$$

We note that this weight satisfies $b(k,t) + \frac{M'}{M}(k,t) \leq 0$. Hence, defining the energy

$$E = (\prod_l M(l,t))^2 \sum_k \lambda_k \hat{w}(k,t)^2,$$

we deduce that

$$\begin{aligned}
\tfrac{1}{2}\partial_t E &\leq \left(\prod_l M(l,t)\right)^2 \sum_k \lambda_k \left(a(k+1)\hat{w}(k+1,t) + a(k-1)\hat{w}(k-1,t)\right)\hat{w}(k) \\
&\leq (\prod_l M(l,t))^2 \sum_k (\lambda_k a(k) + \tfrac{a(k-1)\lambda_{k-1} + a(k+1)\lambda_{k+1}}{2})\hat{w}^2(k,t) \\
&= (1+\hat{\lambda})c\xi E.
\end{aligned}$$

Applying Gronwall's inequality thus yields

$$E(t) \leq \exp(2(1+\hat{\lambda})c\xi t)E(0).$$

This in turn leads to the estimate

$$\begin{aligned}
\|w,j\|_X &\leq \exp((1+\hat{\lambda})c\xi t) \prod_l |M(l,t)|^{-1} \|w_0, j_0\|_X \\
&\leq \exp((1+\hat{\lambda})c\xi t) \prod_{l=0}^{\kappa^{-\frac{1}{2}}} (1 + (\tfrac{2}{\kappa l^2})^{\frac{1}{3}}) \|w_0, j_0\|_X.
\end{aligned}$$

Finally, we can use Stirling's approximation of the factorial, which results in the desired estimate:

$$\|w,j\|_X(t) \leq \exp(\tfrac{4}{3}\kappa^{-\frac{1}{2}}) \exp((1+\hat{\lambda})c\xi t)\|w_0, j_0\|_X.$$

□

In the following we establish upper and lower bounds for small times. Here we use that for modes k such that $\frac{\xi}{k^2}$ is small nay possible resonance will not produce large enough norm inflation and the evolution can hence be treated perturbatively. More precisely, we consider the evolution on the time interval

$$I = [0, \tfrac{\xi}{2}(\tfrac{1}{k_0} + \tfrac{1}{k_0 - 1})]$$

for fixed k_0 to be determined later. For this purpose we introduce the parameter $\eta_0 := \frac{\xi}{k_0^2}$ which later will be chosen as $\eta_0 \approx \frac{1}{10c}$.

Lemma 3.13. *Let w, j be a solution of (3.16), define $d := c^{-1}$ and let ξ, k_0 be such that $\eta_0 \leq d^2$. Then for all times $0 \leq t \leq t_{k_0}$ it holds that*

$$\begin{aligned}
\|w(t), j(t)\|_X^2 &\leq \exp(2(1+\hat{\lambda})\max(c\eta_0, 1)\sqrt{\xi\eta_0})\|w_0, j_0\|_X^2 \\
&\leq \exp(C\sqrt{\xi})\|w_0, j_0\|_X^2.
\end{aligned}$$

Furthermore, suppose that $k_0 \geq \kappa^{-\frac{1}{2}}$ *and* $10d \leq \frac{\xi}{k_0^2} \leq \frac{1}{100c^2}$*, then for the initial data* $w(k,0) = \delta_{k_0,k}$ *and* $j(k,0) = 0$ *we obtain that*

$$\begin{aligned} w(k_0, t_{k_0}) &\geq \tfrac{1}{2} \max(1, w(k, t_{k_0}), j(k, t_{k_0})) \\ j(k_0, t_{k_0}) &\leq \tfrac{1}{\alpha_{k_0} \xi \eta_0}. \end{aligned}$$

Proof. Computing the time derivative, we obtain

$$\tfrac{1}{2}\partial_t \|w, j\|_X^2 = \sum_l (a(l+1)w(l+1) + a(l-1)w(l-1))\lambda_l w(l) + b(l)\lambda_l j(l)^2,$$

where the coefficient functions satisfy

$$\begin{aligned} a(l) &\leq \begin{cases} c\eta_0 & l \geq k_0 \\ 4c\frac{1}{1+\eta_0} & l \leq k_0 \end{cases} \\ &\leq \max(c\eta_0, 4c), \\ b(l) &\leq 1. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \partial_t \|w, j\|_X^2 &\leq 2(1+\hat{\lambda}) \max(c\eta_0, 1) \|w, j\|_X^2, \\ \|w, j\|_X^2(t) &\leq \exp(2(1+\hat{\lambda}) \max(c\eta_0, 1)t) \|w_0, j_0\|_X^2, \\ \|w, j\|_X^2(t_{k_0}) &\leq \exp(2(1+\hat{\lambda}) \max(c\eta_0, 1)\sqrt{\xi\eta_0}) \|w_0, j_0\|_X^2. \end{aligned}$$

To prove lower bounds on the norm inflation we further need to show that for $w(k,0) = \delta_{k_0,k}$ and $j(k,0) = 0$, the mode $w(k_0, t_{k_0})$ will stay the largest mode. Therefore, we introduce the short-hand notation

$$\hat{w}(k,t) = |w, j|(k,t)$$

and have to estimate the growth of $\hat{w}(\cdot, t)$. Since on the interval $[0, t_{k_0}]$ it holds that $b(k) \leq 0$ as $k_0 \geq \kappa^{-\frac{1}{2}}$, we obtain the system

$$\begin{aligned} \partial_t \hat{w}(k) &\leq a(k+1)\hat{w}(k+1) + a(k-1)\hat{w}(k-1) \\ \hat{w}(k,0) &= \delta_{k,k_0}. \end{aligned}$$

Let $\sqrt{\xi\pi} = l_0 \geq l \geq k_0$ to be fixed later. We want to prove by induction that

$$\begin{aligned} \hat{w}(m, t_{l-1}) &\leq 6\pi c\eta_0 (2c)^{|m-k_0|} \\ \hat{w}(l, t_{l-1}) &\leq 4(2c)^{|l-k_0|} \\ \hat{w}(n, t_{l-1}) &\leq 2(2c)^{|m-k_0|} \end{aligned} \tag{3.20}$$

for all $m > l > n$.
Induction start:

We integrate a in time to estimate

$$\begin{aligned}\int_0^{t_{l_0-1}} a(k) &= c\tfrac{\xi}{k^2}\int_0^{t_{l_0-1}} \tfrac{1}{1+(\frac{\xi}{k}-t)^2}\\ &\leq \begin{cases} \pi c\frac{\xi}{k^2} & k > l_0 \\ c & k \leq l_0 \end{cases}\\ &\leq c.\end{aligned}$$

Thus we obtain that

$$\hat{w}(k, t_{l_0-1}) < \delta_{k_0,k} + c(|\hat{w}(k+1)|_{L_t^\infty} + |\hat{w}(k-1)|_{L_t^\infty}),$$

which by a bootstrap argument yields that

$$\hat{w}(k, t_{l_0-1}) \leq \tfrac{1}{1-2c}(2c)^{|k_0-k|}$$

for all k which satisfy (3.20).
Induction step: We fix l and we assume that (3.20) holds for all $\tilde{l}$ with $l_0 \geq \tilde{l} \geq l+1 \geq k_0+1$ and then prove that it holds also for l. We here argue by bootstrap. That is, we show that the estimate (3.20) at least holds up until a time t^* with $t_l \leq t^* \leq t_{l-1}$ and that the maximal time with this property is given by $t^* = t_{l-1}$. For $n < l$ we estimate

$$\begin{aligned}\hat{w}(n, t_{l-1}) &\leq \delta_{n,k_0} + \int_0^{t_{l-1}} a(n\pm 1, \tau) w(n \pm 1, \tau)\\ &\leq \delta_{n,k_0} + c(4(2c)^{|n+1-k_0|} + 2(2c)^{|n-1-k_0|})\\ &< 2(2c)^{|n-k_0|}.\end{aligned}$$

To estimate the l mode we estimate the integral between t_l and t_{l-1} to deduce

$$\begin{aligned}\hat{w}(l, t_{l-1}) &\leq \hat{w}(l, t_l) + \int_{t_l}^{t_{l-1}} a(l\pm 1, \tau) w(l\pm 1, \tau)\\ &\leq 2(2c)^{l-k_0} + 6\pi c\eta_0 (2c)^{l+1-k_0} + 2c(2c)^{|l-1-k_0|}\\ &\leq 4(2c)^{l-k_0}.\end{aligned}$$

For $m > l$ we split the integrals as

$$\begin{aligned}\hat{w}(m, t_{l-1}) \leq \hat{w}(m, t_{m-1}) &+ \int_{t_{m-1}}^{t_l} a(m+1, t)\hat{w}(m+1, \tau)\\ &+ \int_{t_{m-1}}^{t_{m-2}} a(m-1, t)\hat{w}(m-1, \tau) + \int_{t_{m-2}}^{t_l} a(m-1, t)\hat{w}(m-1, \tau)\\ &\leq 4(2c)^{m-k_0} + 12\pi c^2 \eta_0 (2c)^{|m-k_0|} + 4\pi\eta_0(2c)^{m-k_0} + 6\pi c\eta_0 (2c)^{m-k_0}\\ &\leq 6\pi\eta_0 (2c)^{m-k_0}.\end{aligned}$$

So we finally deduce that

$$\hat{w}(k, t_{k_0}) \leq (2c)^{|k-k_0|} \begin{cases} 4 & k \leq k_0 + 1 \\ 6\pi\eta_0 & k > k_0 + 1 \end{cases}$$

Thus we established an upper bound for all modes, the next step is to show that for w indeed the k_0 mode is one of the largest modes. Therefore, we estimate $j(k_0)$ by

$$j(k_0, t) = \alpha k_0 \int_0^t d\tau \; \frac{1+(\frac{\xi}{k}_0 - t)^2}{1+(\frac{\xi}{k}_0 - \tau)^2} \exp\left(-\kappa k_0^2 (t - \tau + \tfrac{1}{3}((\tfrac{\xi}{k_0} - t)^3 - (\tfrac{\xi}{k_0} - \tau)^3)\right) w(k_0, \tau)$$

and hence obtain that

$$\begin{aligned} j(t) &\leq \alpha k_0 \int \exp(-\kappa k_0^2 \eta^2 (t - \tau) \\ &\leq \tfrac{1}{\sqrt{\beta\kappa\xi\eta_0^3}} = \tfrac{1}{\alpha_{k_0}\xi\eta_0} \end{aligned}$$

and

$$\begin{aligned} \alpha k_0 \int j(k_0, \tau_2) d\tau_1 = \tfrac{\kappa k_0^2}{\beta} &\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; \frac{1+(\frac{\xi}{k}_0 - \tau_1)^2}{1+(\frac{\xi}{k}_0 - \tau_2)^2} \\ &\times \exp\left(-\kappa k_0^2 (\tau_1 - \tau_2 + \tfrac{1}{3}((\tfrac{\xi}{k}_0 - \tau_1)^3 - (\tfrac{\xi}{k}_0 - \tau_2)^3)\right) \\ \leq \tfrac{1}{\beta} &\int d\tau_1 \frac{1}{1+(\frac{\xi}{k_0} - \tau_1)^2} \\ \leq \tfrac{4}{\beta\eta_0}&. \end{aligned}$$

With this we conclude that

$$\begin{aligned} |w(k_0, t_{k_0}) - 1| &\leq \int a(k_0+1)w(k_0+1) + a(k_0-1)w(k_0-1) + \alpha k_0 j(t) \\ &\leq 16\pi\eta_0 c^2 + 8c^2 + \tfrac{4}{\beta\eta_0} \leq \tfrac{1}{2}, \end{aligned}$$

which in turn yields

$$|w(k_0, t_{k_0})| \geq \tfrac{1}{2} \geq \max_{k \neq k_0} (\hat{w}(k, t_{k_0}), |j(k_0, t_{k_0})|).$$

□

3.4 Resonances and Norm Inflation

Having discussed the evolution for small times and large times in Section 3.3 it remains to discuss the evolution on the interval

$$(t_{k_0}, 2\xi) = \bigcup_{1 \leq k \leq k_0} I^k$$

with I^k as in Definition 3.11.

Based on the heuristics of the toy model of Section 3.2.2 our aim here is to establish both upper lower and upper bounds on the norm inflation on each resonant interval I^k, where the resonant mode $w(k)$ can possibly lead to a large growth of its neighboring modes $w(k \pm 1)$. In order to simplify notation we introduce the growth factor

$$L = L(\alpha, \kappa, k),$$

which estimates the maximal growth of $w(k)$ due to its interaction with the current $j(k)$, see 3.5. In particular, we show that $L = 1$ if $\beta \geq \pi$ and if $\beta < \pi$ we obtain an estimate $L = L(\alpha, \kappa, k) \leq \sqrt{c}$. We define M and M_n as

$$\begin{aligned} M &= \sum_m 10^{-|m|}(w + \tfrac{1}{\alpha_{k_m}} j)(k_m, s_0) \\ M_n &= \sum_m 10^{-|m-n|+\chi}(w + \tfrac{1}{\alpha_{k_m}} j)(k_m, \tilde{s}_0) \end{aligned}$$

where $\chi = -|\mathrm{sgn}(m) - \mathrm{sgn}(n)|$. We note that

$$\sum_{l \neq 0} (\tfrac{3}{\eta})^{|k_l - k_0|} M_l \leq \tfrac{3}{\eta} M.$$

With these notations the main results of this section are summarized in the following theorem:

Theorem 3.14. *Let* $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$, $\xi \geq 10\kappa^{-1}(1 + \beta^{-1})$ *and* $\eta = \frac{\xi}{k^2} \geq 10d$ *and* $t_k = \frac{\xi}{2}(\frac{1}{k} + \frac{1}{k+1})$*, then it holds that*

$$\|w, j\|_X(t_{k-1}) \leq 18\pi L \hat{\lambda}(c\eta)^{\gamma} \|w, j\|_X(t_k).$$

Furthermore, let $\kappa_k \min(\beta, 1) \geq \frac{1}{c}$ *and* $\beta \geq \frac{1}{5}$

$$w(k, t_k) \geq \tfrac{1}{2} \max(w(l, t_k), j(l, t_k)). \tag{3.21}$$

Then $w(k-1, t_{k-1})$ *satisfies* (3.21) *with* k *replaced by* $k-1$ *and*

$$|w(k-1, t_{k-1})| \geq \min(\beta, \pi)(c\eta)^{\gamma} w(k, t_k).$$

To prove the estimates of Theorem 3.14 it is convenient to rescale $\tilde{j}(k) = \alpha_k j(k)$ in (3.16) to obtain

$$\begin{aligned} \partial_t w(k) = &-\tilde{j}(k) \\ &- c\tfrac{\xi}{(k+1)^2} \tfrac{1}{1 + (\frac{\xi}{k+1} - t)^2} w(k+1) \\ &+ c\tfrac{\xi}{(k-1)^2} \tfrac{1}{1 + (\frac{\xi}{k-1} - t)^2} w(k-1) \\ \partial_t \tilde{j}(k) = &(2\tfrac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} - \kappa k^2(1 + (t - \tfrac{\xi}{k}^2))\tilde{j}(k) + \tfrac{\kappa k^2}{\beta} w(k), \end{aligned}$$

where we used that $\kappa = \beta\alpha^2$. With respect to these unknowns the norm on our space X changes slightly

$$\begin{aligned}\|w,j\|_X^2 &= \sum \lambda_k (w^2(k) + \tfrac{\beta}{\kappa k^2}\tilde{j}^2(k)) \\ &=: \|w,\tilde{j}\|_{\tilde{X}}^2.\end{aligned}$$

In the following sections, with slight abuse of notation we omit writing the tilde symbols both for j and X.

Given a choice of time interval I_{k_0}, considering k_0 as arbitrary but fixed (and unrelated to k_0 of Section 3.3) we further introduce the relative frequencies

$$k_n := k_0 + n,$$

where $n \in \mathbb{Z}_{>-k_0}$ and also shift our time variable

$$t = \tfrac{\xi}{k_0} + s.$$

Introducing the coefficient functions

$$\begin{aligned}a(k) &= c\eta \tfrac{k_0^2}{(k)^2} \tfrac{1}{1+(\eta\frac{k_0(k_0-k)}{k}-s)^2}, \\ b(k) &= 2\tfrac{(s-\eta\frac{k_0(k_0-k)}{k})}{1+(\eta\frac{k_0(k_0-k)}{k}-s)^2} - \kappa_k(1+(\eta\tfrac{(k_0-k)k_0}{k}-s)^2),\end{aligned} \tag{3.22}$$

the system (3.16) then reads

$$\begin{aligned}\partial_s w(k) &= -j(k) \\ &\quad - a(k+1)w(k+1) \\ &\quad + a(k-1)w(k-1), \\ \partial_t j(k) &= \tfrac{\kappa_k}{\beta} w(k) + b(k)j(k).\end{aligned} \tag{3.23}$$

For later reference, we note that the coefficient function a satisfies the following estimates:

$$\begin{aligned}a(k_0) &= c\eta\tfrac{1}{1+s^2}, \\ a(k_{\pm 1}) &\leq 4\tfrac{c}{\eta}, \\ a(k_n) &\leq \tfrac{c}{\eta},\end{aligned} \tag{3.24}$$

for all $|n| \geq 2$.

Finally, in view of cancellations of $-a(k-1)$ and $+a(k+1)$ on any given time interval I^k it is convenient to work with the unknowns

$$u_1 = w(k_0), u_2 = w(k_1) - w(k_{-1}), u_3 = w(k_1) + w(k_{-1}).$$

We then consider (3.23) as a forced system for these three modes (and a separate equation for all other modes):

$$\partial_s \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ j(k_0) \end{pmatrix} = \begin{pmatrix} 0 & -a_1 & a_2 & -1 \\ 2c\eta\frac{1}{1+s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\kappa_k}{\beta} & 0 & 0 & \frac{2s}{1+s^2} - \kappa_k(1+s^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ j(k_0) \end{pmatrix} + \begin{pmatrix} 0 \\ -a(k_{\pm 2})w(k_{\pm 2}) \mp j(k_{\pm 1}) \\ \mp a(k_{\pm 2})w(k_{\pm 2}) - j(k_{\pm 1}) \\ 0 \end{pmatrix} \tag{3.25}$$

where $a_1 = \frac{1}{2}(a(k_1) + a(k_{-1}))$ and $a_2 = \frac{1}{2}(a(k_1) - a(k_{-1}))$.

The analysis of this system is split into multiple subsections, where we also split the time interval I^k as

$$I^{k_0} = [s_0, -d] \cup [-d, d] \cup [d, s_1] =: I_1 \cup I_2 \cup I_3,$$

where $s_0 = -\frac{\eta}{2}\frac{k_0-1}{k_0}$ and $s_1 = \frac{\eta}{2}\frac{k_0+1}{k_0}$. Similarly to the setting of the Euler equations [DZ21] here the interaction between growth and decay of various modes interacts to determine the over all norm inflation.

3.4.1 Proof of Theorem 3.3

Before proceeding to the proof of Theorem 3.14, in this subsection we discuss how it can be used to establish Theorem 3.3. We split the proof into two auxiliary theorems.

Theorem 3.15 (technical statement)**.** *Let $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$, $\xi \geq 10\kappa^{-1}(1+\beta^{-1})$ and $\frac{\xi}{k^2} \geq 10d$. Then there exists exists a constant $C = C(\kappa, \alpha, c)$ such that for a fixed ξ we obtain*

$$\|w, j\|_X(t, \xi) \leq \exp(C\sqrt{\xi})\|w, j\|_X(0, \xi).$$

Furthermore, let $\xi \geq 10^4\frac{d^2}{\beta\kappa}$, $\beta \geq \frac{1}{5}$, $k_0 \approx \frac{c}{10}\sqrt{\xi}$ and $k_1 \approx \frac{4}{\sqrt{\beta\kappa}}$, then there exists a constant $C^* = C^*(\kappa, \alpha, c)$ such that for initial data $w(k, 0) = \delta_{k_0,k}$ and $j(k, 0) = 0$ we obtain

$$w(k_1, t) \geq \exp(\tilde{C}\sqrt{\xi}).$$

for $t \in [t_{k_1} - 1, t_{k_1} + 1]$.

Proof of Theorem 3.15. For fixed ξ, t and $k_0 =: 10d\sqrt{\xi}$ we consider $w(\cdot, \xi, t)$ as an element in X. On X we define the operator $S_{\tau_1,\tau_2} : X \to X$ as the solution operator of (3.16) on $[\tau_1, \tau_2]$, i.e.

$$S_{\tau_1,\tau_2}[w(\cdot, \xi, \tau_1)] = w(\cdot, \xi, \tau_2)$$
$$S_{\tau_1,\tau_2} \circ S_{\tau_2,\tau_3} = S_{\tau_1,\tau_3}.$$

By Lemma 3.13, Theorem 3.14 and Lemma 3.10 this S then satisfies the following norm estimates:

$$\begin{aligned}\|S_{0,t_{k_0+1}}\|_{X\to X} &= \exp(C_1\sqrt{\xi}),\\ \|S_{t_k,t_{k-1}}\|_{X\to X} &= 3\pi c(\tfrac{\xi}{k^2})^\gamma,\\ \|S_{t_1,t}\|_{X\to X} &= 2\tfrac{1}{1-c\sqrt{\hat{\lambda}}}.\end{aligned}$$

Combining these estimates with Stirling's approximation formula we thus obtain the desired upper bound:

$$\begin{aligned}\|S_{0,t}\|_{X\to X} &\leq 2\exp(C_1\sqrt{\xi})\prod_{k=1}^{k_0} 3\pi c(\tfrac{\xi}{k^2})^\gamma\\ &\leq \exp(C\sqrt{\xi}).\end{aligned}$$

Concerning the lower bound, we use first use Lemma 3.13 and then Theorem 3.14 to deduce that

$$\begin{aligned}w(k_0,t_{k_0}) &\geq \tfrac{1}{2}\\ w(k-1,t_{k-1}) &\geq (c\tfrac{\xi}{k^2})^\gamma \min(\beta,\pi) w(k,t_k)\end{aligned}$$

for $\sqrt{\frac{c}{10}\xi} \approx k_0 \geq k \geq k_1 \approx \frac{4}{\sqrt{\beta\kappa}}$. Thus, by again using Stirling's approximation, we conclude that

$$\begin{aligned}w(k_1,t_{k_1}) &\geq \tfrac{1}{2}\prod_{k=k_1}^{k_2}(c\tfrac{\xi}{k^2})^\gamma \min(\beta,\pi)\\ &\approx \exp(\tilde{C}\sqrt{\xi}).\end{aligned}$$

□

Theorem 3.16 (Stability and blow-up)**.** *Let $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$ and w, j be a solution to* (3.16) *, then there exists a constant $C = C(\kappa,\alpha,c)$ such that for all $C_1 > C$ and initial data which satisfy*

$$\int \exp(C_1\sqrt{\xi})\|w_0,j_0\|_X^2(\xi)\ d\xi < \infty,$$

the solution remains Gevrey 2 regular in the sense that

$$\sup_t \int \exp(C_2\sqrt{\xi})\|w,j\|_X^2(\xi,t)\ d\xi \leq \tilde{C}\int \exp(C_1\sqrt{\xi})\|w_0,j_0\|_X^2(\xi)\ d\xi,$$

where $C_2 = C_1 - C$ and $\tilde{C} > 0$ is a universal constant.

Furthermore, additionally suppose that $\beta \geq \frac{1}{5}$, then there exist a constant $0 < C^* < C$ and initial data w_0, j_0 which satisfy

$$\int \exp(C^*\sqrt{\xi})\|w_0,j_0\|_X^2(\xi)\ d\xi < \infty,$$

such that for a subsequence $k_{n,1}$ the solution diverges in L^2:

$$\|w(\cdot, t_{k_{n,1}})\|_{L^2\ell^2} \to \infty.$$

Proof of Theorem 3.16. The first part follows directly from Theorem 3.15. For the second part we fix $\xi_1 = 10^4 \frac{d^2}{\beta\kappa}$ and define the sequence $\xi_n = n\xi_1$ with the associated $k_0^{\xi_n} \approx \frac{c}{10}\sqrt{\xi_n}$ and $k_1 \approx \frac{4}{\sqrt{\beta\kappa}}$. Note that the starting mode $k_0^{\xi_n}$ is ξ_n-dependent, but the final mode k_1 is independent of ξ_n. Furthermore, let $z_n(\xi)$ be a function in $C^\infty \cap L^2$, such that

$$\begin{aligned} \operatorname{supp} z_n(\cdot) \subset [\xi_n - 1, \xi_n + 1] \\ \int z_n(\xi)^2 \, d\xi = 1. \end{aligned}$$

We then define the initial data

$$w(k, \xi, 0) = \sum_{n=1}^{\infty} \tfrac{1}{n} z_n(\xi) \exp(-\tfrac{1}{2} C^* \sqrt{\xi}) \delta_{k_{\xi_n,0},k}.$$

We observe that it satisfies the estimates

$$\begin{aligned} \|w(\cdot, \xi, 0)\|_{l^2}^2 &= \sum_{n=1}^{\infty} \tfrac{1}{n^2} z_n(\xi)^2 \exp(-C^* \sqrt{\xi}), \\ \int \exp(C^* \sqrt{\xi}) \|w(\cdot, \xi, 0)\|_{l^2}^2 \, d\xi &= 2 \sum \tfrac{1}{n^2} = \tfrac{\pi^2}{3}. \end{aligned}$$

Furthermore, by the norm inflation results for each mode at each time $t_{k_{n,1}}$ we obtain that

$$\|w(k_{n,1}, \xi, t_{k_{n,1}})\|_{l^2} \geq \tfrac{9}{10} \tfrac{1}{n^2} z(\xi, n) \exp((\tilde{C} - C^*)\sqrt{\xi}),$$

and integrating in ξ we conclude that

$$\|w(\cdot, t_{k_{n,1}})\|_{L^2 l^2} \geq \tfrac{9}{10} \tfrac{1}{n^2} \exp((\tilde{C} - C^*)\sqrt{\xi_n}) \to \infty.$$

□

3.4.2 Asymptotic Behavior on the Intervals I_1 and I_3

In this section we consider the equation (3.25) on the outer intervals $I_1 = [s_0, -d]$ and $I_3 = [d, s_1]$. Since a lot of calculations are similar on both intervals we in general write the interval as $[\tilde{s}_0, \tilde{s}_1]$, where we only need to distinguish the two cases on a few occasions and in the statement of the conclusions. For the interval I_1 we prove the following proposition:

Proposition 3.17 (Interval I_1)**.** *Let $c < \max(10^{-4}, 10^{-1}\beta)$ and $\xi \geq 10\max(\kappa^{-1}(1+\beta^{-1}), k_0^2 d)$. Then for a solution of (3.25) on the interval I_1 the following estimates hold at the time d:*

$$\begin{aligned}
|u_1(d)| &\leq 2M(c\eta)^{-\gamma_2},\\
|u_2(d)| &\leq 2M(c\eta)^{\gamma_1},\\
|u_3|(-d) &\leq 2M_1,\\
|w(k_n,-d)| &\leq 2M_n,\\
|j|(k_0,-d) &\leq \tfrac{c}{\beta}(c\eta)^{-\gamma_2} M \inf(c, \kappa_{k_0} c^{-2}),\\
|j|(k_{\pm 1},-d) &\leq \tfrac{4}{\beta\eta^2} M,\\
|j|(k_n,-d) &\leq \tfrac{4}{\beta\eta^2} M_n.
\end{aligned}$$

If we additionally assume that

$$w(k_0, t_{k_0}) \geq \tfrac{1}{2}\sup_l (w(l,t_l), j(l,t_l)), \tag{3.26}$$

we obtain that

$$\begin{aligned}
|u_1(-d) - (c\eta)^{-\gamma_2} u_1(s_0)| &= 50 c u_1(\tilde{s}_0)(c\eta)^{-\gamma_2}\\
|u_2(-d)| &\leq 50 c u_1(\tilde{s}_0)(c\eta)^{\gamma_1},\\
|u_3|(-d), |w|(k_m,-d) &\leq 2|u|(s_0) \qquad \text{for } |m| \geq 2,\\
|j|(k_m,-d) &\leq \tfrac{4}{\eta}|u|(s_0) \qquad \text{for } |m| \geq 1,\\
|j|(k_0,-d) &\leq \tfrac{2c^2}{\beta}|u|(s_0)(c\eta)^{-\gamma_2}.
\end{aligned} \tag{3.27}$$

The proof of this proposition is split into several lemmas and concludes at the end of this subsection. For the interval I_3 in a first step we only establish asymptotic estimates. The final conclusion for interval I_3 will be postponed to the proof of Theorem 3.14. On both intervals I_1 and I_2 the interaction of u_1 and u_2 is the main effect to be analyzed. Therefore, we consider the equations for u_1 and u_2 as an inhomogeneous linear system

$$\partial_s \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{c}{\eta} \\ 2c\eta\frac{1}{s^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + F, \tag{3.28}$$

where F is a force term. Equation (3.28) with $F = 0$ has a explicit homogeneous solution and we aim to show that (3.25) can be treated as a perturbation. In the following we denote $\tilde{u}$ as the homogeneous solution of (3.28). Furthermore, we split the forcing as

$$F =: F_{all} = F_{3mode} + F_j + F_{u_3} + F_{j(k_0 \pm 1)} + F_{\tilde{w}}$$

where we define

$$F_{3mode} = (\tfrac{c}{\eta} - a_1) e_1 u_2 - 2c\eta \tfrac{1}{s^2(s^2+1)} e_2 u_1,$$

as the 3 mode forcing

$$F_j = -e_1 j(k_0)$$

as the k_0-th current forcing and

$$F_{u_3} + F_{j(k_0 \pm 1)} + F_{\tilde{w}} = e_1 a_2 u_3 \mp e_2 j(k_{\pm 1}) - e_2 a(k_{\pm 2}) w(k_{\pm 2})$$

as the forcings due to u_3, $j(k_0 \pm 1)$ and $\tilde{w}$, respectively. The corresponding $R[F_*]$ are the called r changes. We also define $\gamma = \sqrt{1 - 8c^2}$ and $\gamma_1 = \frac{1}{2}(1+\gamma)$ and $\gamma_2 = \frac{1}{2}(1-\gamma)$ and note the following equalities:

$$\begin{aligned} \gamma_1 \gamma_2 &= 2c^2, \\ \gamma_1 + \gamma_2 &= 1, \\ \gamma &= 1 + \mathcal{O}(c^2), \\ \gamma_1 &= 1 + \mathcal{O}(c^2), \\ \gamma_2 &= \tfrac{1}{\gamma_1} 2c^2 = 2c^2 + \mathcal{O}(c^4). \end{aligned}$$

Lemma 3.18. *Consider* (3.28) *with $F = 0$, then the solution is given by*

$$\tilde{u}(s) = S(s) r$$

with

$$S(s) = \begin{pmatrix} |\frac{s}{\eta}|^{\gamma_1} & |\frac{s}{\eta}|^{\gamma_2} \\ -\frac{\gamma_1}{c} \frac{s}{\eta} |\frac{s}{\eta}|^{\gamma_1 - 2} & -\frac{\gamma_2}{c} \frac{s}{\eta} |\frac{s}{\eta}|^{\gamma_2 - 2} \end{pmatrix},$$

and $r = S^{-1}(\tilde{s}_0) \tilde{u}(s_0)$.

Furthermore, we define the operator S^* as

$$S^*(s) = \begin{pmatrix} |\frac{s}{\eta}|^{\gamma_1} & |\frac{s}{\eta}|^{\gamma_2} \\ \frac{\gamma_1}{c} |\frac{s}{\eta}|^{\gamma_1 - 1} & \frac{\gamma_2}{c} |\frac{s}{\eta}|^{\gamma_2 - 1} \end{pmatrix},$$

which gives the estimate

$$|S(s) r| \leq S^* r \qquad \forall r \in (\mathbb{R}_+)^2$$

The inverse of S can be computed as

$$\begin{aligned} S^{-1}(s) &= \operatorname{sgn}(s) c \gamma^{-1} \begin{pmatrix} -\frac{\gamma_2}{c} \frac{s}{\eta} |\frac{s}{\eta}|^{\gamma_2 - 2} & -|\frac{s}{\eta}|^{\gamma_2} \\ \frac{\gamma_1}{c} \frac{s}{\eta} |\frac{s}{\eta}|^{\gamma_1 - 2} & |\frac{s}{\eta}|^{\gamma_1} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\gamma_2}{\gamma} |\frac{s}{\eta}|^{\gamma_2 - 1} & -\frac{c}{\gamma} \frac{s}{\eta} |\frac{s}{\eta}|^{\gamma_2 - 1} \\ \frac{\gamma_1}{\gamma} |\frac{s}{\eta}|^{\gamma_1 - 1} & \frac{c}{\gamma} \frac{s}{\eta} |\frac{s}{\eta}|^{\gamma_1 - 1} \end{pmatrix}. \end{aligned}$$

Lemma 3.19. *Let u_1, u_2 be a solution to* (3.28) *with given $F = (F_1, F_2)$, then for*

$$R_1[F] = (1 + 10c^2)2c^2\eta^{1-\gamma_2} \int_{\tilde{s}_0}^{s} \tau^{\gamma_2 - 1} F_1(\tau)\ d\tau + c\eta^{-\gamma_2} \int_{\tilde{s}_0}^{s} \tau^{\gamma_2} F_2(\tau)\ d\tau,$$
$$R_2[F] = (1 + 10c^2)\eta^{1-\gamma_1} \int_{\tilde{s}_0}^{s} \tau^{\gamma_1 - 1} F_1(\tau)\ d\tau + c\eta^{-\gamma_1} \int_{\tilde{s}_0}^{s} \tau^{\gamma_1} F_2(\tau)\ d\tau,$$

we estimate

$$|u - \tilde{u}| \leq S^*(s)R[F].$$

Proof. Since S has an inverse, we write

$$u = S(s)r(s)$$

and our aim is to control the evolution of $r(s)$. Therefore, we calculate

$$\begin{aligned} |\partial_s r| &= S^{-1}F \\ |\partial_s r_1| &\leq 2c^2|\tfrac{s}{\eta}|^{\gamma_2 - 1}F_1 + c|\tfrac{s}{\eta}|^{\gamma_2}F_2 \\ |\partial_s r_2| &\leq |\tfrac{s}{\eta}|^{\gamma_1 - 1}F_1 + c|\tfrac{s}{\eta}|^{\gamma_1}F_2 \end{aligned}$$

and so

$$\begin{aligned} |r_1(s) - r_1(d)| &\leq 2c^2(1 + 10c^2)\eta^{1-\gamma_2} \int \tau^{\gamma_2 - 1}F_1(\tau) + c\eta^{-\gamma_2} \int \tau^{\gamma_2}F_2(\tau) \\ |r_2(s) - r_2(d)| &\leq (1 + 10c^2)\eta^{1-\gamma_1} \int \tau^{\gamma_1 - 1}F_1(\tau) + c\eta^{-\gamma_1} \int \tau^{\gamma_1}F_2(\tau) \end{aligned}$$

□

In the following we always assume that there exists $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \geq 0$ such that

$$\begin{aligned} |u| &\leq S^*(s)C(s) \\ C_1(s) &= c_1 + \tilde{c}_1(\tfrac{s}{\eta})^{-\gamma} \\ C_2(s) &= c_2 + \tilde{c}_2(\tfrac{s}{\eta})^{\gamma} \end{aligned} \tag{3.29}$$

on a maximal interval $[\tilde{s}_0, s^*]$. We will establish some estimates on the R_i depending on c_i and $\tilde{c}_i$ and then we will determine specific c_i and $\tilde{c}_i$ such that we prove that the maximal s^* will be greater than $\tilde{s}_1$. Later it will be sufficient to choose $\tilde{c}_1 = 0$ on I_1 and $\tilde{c}_2 = 0$ on I_3. We thus deduce

$$\begin{aligned} |u_1(s)| &\leq (c_1 + \tilde{c}_2)|\tfrac{s}{\eta}|^{\gamma_1} + (\tilde{c}_1 + c_2)|\tfrac{s}{\eta}|^{\gamma_2} \\ |u_2(s)| &\leq (\tfrac{\gamma_1}{c}c_1 + \tfrac{\gamma_2}{c}\tilde{c}_2)|\tfrac{s}{\eta}|^{\gamma_1 - 1} + (\tfrac{\gamma_1}{c}\tilde{c}_1 + \tfrac{\gamma_2}{c}c_2)|\tfrac{s}{\eta}|^{\gamma_2 - 1} \\ &\leq c_1^*|\tfrac{s}{\eta}|^{\gamma_1 - 1} + c_2^*|\tfrac{s}{\eta}|^{\gamma_2 - 1}. \end{aligned}$$

where $c_1^* = \frac{\gamma_1}{c}c_1 + \frac{\gamma_2}{c}\tilde{c}_2$ and $c_2^* = \frac{\gamma_1}{c}\tilde{c}_1 + \frac{\gamma_2}{c}c_2$. For sake of simplicity we will often omit absolute values for the estimates.

Lemma 3.20 (3 mode forcing estimate). *Let $u(s) = S(s)r(s)$ be a solution of (3.25) on $[\tilde{s}_0, s^*]$, such that $|u(s)| \leq S^*(s)C(s)$, then we estimate*

$$R_1[F_{3mode}] \leq 20c^2c_1 + (20 + c^4(\tfrac{s\wedge\tilde{s}_0}{\eta})^{-\gamma})\tilde{c}_1 + (20c^2 + c^4(\tfrac{s\wedge\tilde{s}_0}{\eta})^{-\gamma})c_2 + 20c^4\tilde{c}_2$$
$$R_2[F_{3mode}] \leq 20(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma}(c_1 + 2c^2\tilde{c}_2) + 20(\tilde{c}_1 + c^2c_2).$$

Proof. The forcing term is given by

$$F_{3mode} = (\tfrac{c}{\eta} - a_1)e_1u_2 - 2c\eta\tfrac{1}{s^2(s^2+1)}e_2u_1.$$

Therefore, we estimate

$$\begin{aligned}
R_1[e_2 2c\eta\tfrac{1}{s^2(1+s^2)}u_1] &\leq 2c^2\eta^{\gamma_1}\int_{\tilde{s}_0}^{s}\tau^{\gamma_2-4}((c_1+\tilde{c}_2)(\tfrac{\tau}{\eta})^{\gamma_1} + (\tilde{c}_1+c_2)(\tfrac{\tau}{\eta})^{\gamma_2})\\
&\leq c^4(c_1+\tilde{c}_2) + c^4(\tfrac{s\wedge\tilde{s}_0}{\eta})^{-\gamma}(\tilde{c}_1+c_2)\\
R_2[e_2 2c\eta\tfrac{1}{s^2(1+s^2)}u_1] &= 2c^2\eta^{\gamma_2}\int_{\tilde{s}_0}^{s}\tau^{\gamma_1-4}((c_1+\tilde{c}_2)(\tfrac{\tau}{\eta})^{\gamma_1} + (\tilde{c}_1+c_2)(\tfrac{\tau}{\eta})^{\gamma_2})\\
&\leq 2c^3\eta^{-\gamma}(c_1+\tilde{c}_2) + c^4(\tilde{c}_1+c_2).
\end{aligned}$$

By Taylor formula we obtain $|c\frac{1}{\eta} - a_1| \leq 18c\frac{|s|}{\eta^2}$ and so

$$\begin{aligned}
R_1[(c\tfrac{1}{\eta} - a_1)u_2e_1] &\leq (1+10c^2)c^2\eta^{1-\gamma_2}\int\tau^{\gamma_2-1}18c\tfrac{\tau}{\eta^2}((\tfrac{\tau}{\eta})^{\gamma_1-1}c_1^* + (\tfrac{\tau}{\eta})^{\gamma_2-1}c_2^*)\\
&\leq 20c^3c_1^* + 20cc_2^*\\
&\leq 20c^2c_1 + 20\tilde{c}_1 + 20c^2c_2 + 20c^4\tilde{c}_2,\\
R_2[(c\tfrac{1}{\eta} - a_1)u_2e_1] &= (1+10c^2)\eta^{1-\gamma_1}\int\tau^{\gamma_1-1}18c\tfrac{\tau}{\eta^2}((\tfrac{\tau}{\eta})^{\gamma_1-1}c_1^* + c(\tfrac{\tau}{\eta})^{\gamma_2-1}c_2^*))\\
&\leq 20c|\tfrac{s\vee\tilde{s}_0}{\eta}|^{\gamma}c_1^* + 20cc_2^*\\
&\leq 20|\tfrac{s\vee\tilde{s}_0}{\eta}|^{\gamma}(c_1 + 2c^2\tilde{c}_2) + 20\tilde{c}_1 + 20c^2c_2.
\end{aligned}$$

□

Lemma 3.21 (k_0-th current estimate). *Let $u(s) = S(s)r(s)$ be a solution of (3.25) on $[\tilde{s}_0, s^*]$ such that $|u(s)| \leq S^*(s)C(s)$, then we estimate*

$$R_1[F_j] \leq \tfrac{c^3}{\beta}(c_1+\tilde{c}_2) + \tfrac{c^3}{\beta}(\tfrac{s\wedge s_0}{\eta})^{-\gamma}(\tilde{c}_1+c_2) + \begin{cases} \frac{4c^{2+\gamma_1}}{\kappa_{k_0}\eta^{1+\gamma_2}}j(k_0,\tilde{s}_0) & on\ I_1\\ \eta^{\gamma_1}\frac{c^{4+\gamma_1}}{\kappa_{k_0}}j(k_0,\tilde{s}_0) & on\ I_3\end{cases}$$
$$R_2[F_j] \leq \min(\tfrac{1}{\beta}\tfrac{1}{2c^{1+\gamma}}\eta^{-\gamma}, \tfrac{c}{\beta}(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma})(c_1+\tilde{c}_2) + \tfrac{1}{\beta}c(c_2+\tilde{c}_1)$$
$$+ \tfrac{1}{\beta}c(c_2+\tilde{c}_1) + \begin{cases} 4\frac{c^{\gamma_2}}{\kappa_{k_0}\eta^{1+\gamma_1}}j(k_0,\tilde{s}_0) & on\ I_1\\ \eta^{\gamma_2}\frac{c^{2+\gamma_2}}{\kappa_{k_0}}j(k_0,\tilde{s}_0) & on\ I_3\end{cases}.$$

Furthermore, on I_1 we estimate

$$|j(k_0,\tilde{s}_1)| \leq \tfrac{2d^2}{\eta^2}\exp(-\tfrac{\kappa}{2^5}\xi\eta^2)j(k_0,\tilde{s}_0) \\ + \tfrac{c^2}{\beta}\big((c_1+\tilde{c}_2)(\tfrac{d}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{d}{\eta})^{\gamma_2}\big)$$

and on I_3

$$|j(k_0,\tilde{s}_1)| \leq c^2\eta^2\exp(-\kappa_{k_0}\eta^3)j(k_0,\tilde{s}_0) \\ + 2\tfrac{16^2}{\beta}\tfrac{1}{\eta^2}(c_1+c_2+\tilde{c}_1+\tilde{c}_2)$$

Proof. The equation

$$\partial_s j(k_0) = (\tfrac{2s}{1+s^2}-\kappa_{k_0}(1+s^2))j(k_0)+u_1$$

leads to

$$\begin{aligned} j(k_0) &= \tfrac{1+s^2}{1+s_0^2}\exp(-\kappa_{k_0}(s-s_0+\tfrac{1}{3}(s^3-s_0^3)))j(k_0,\tilde{s}_0) \\ &+ \tfrac{\kappa_k}{\beta}\int_{s_0}^{s} d\tau_2\ \tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))u_1(\tau_2) \\ &= j_1+j_2. \end{aligned}$$

Therefore, we estimate

$$\begin{aligned} R_1[F_{j_2}] &= \tfrac{\kappa_{k_0}}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_1\int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{\gamma_2-1}\tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))u_1(\tau_2) \\ &= \tfrac{\kappa_{k_0}}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_1\int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{\gamma_2-1}\tfrac{1+\tau_1^2}{1+\tau_2^2} \\ &\quad\cdot\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2}) \\ &\leq \tfrac{1}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_2\ \frac{((c_1+\tilde{c}_2)(\frac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\frac{\tau_2}{\eta})^{\gamma_2})\tau_2^{-\gamma_1}}{1+\tau_2^2} \\ &\quad\cdot\int_{\tau_2}^{s} d\tau_1\kappa_{k_0}(1+\tau_1^2)\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3))) \\ &= \tfrac{1}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_2\ ((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2})\tfrac{\tau_2^{-\gamma_1}}{1+\tau_2^2} \\ &\quad\cdot\big[-\exp(-\kappa_k(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))\big]_{\tau_1=\tau_2}^{\tau_1=s} \\ &\leq \tfrac{1}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_2\ (c_1+\tilde{c}_2)\eta^{-\gamma_1}\tau_2^{-2}+(c_2+\tilde{c}_1)\eta^{-\gamma_2}\tau_2^{-\gamma-2} \\ &\leq (c_1+\tilde{c}_2)\tfrac{1}{\beta}c^2\eta^{1-\gamma_2-\gamma_1}[-\tau^{-1}]_{s_0}^{s}+(c_2+\tilde{c}_1)\tfrac{1}{\beta}c^2\eta^{\gamma}[-\tau^{-\gamma-1}]_{s_0}^{s} \\ &\leq \tfrac{c^3}{\beta}(c_1+\tilde{c}_2)+\tfrac{c^3}{\beta}(\tfrac{s\wedge s_0}{\eta})^{-\gamma}(\tilde{c}_1+c_2) \end{aligned}$$

and

$$
\begin{aligned}
R_2[F_{j_2}] &= \tfrac{\kappa_{k_0}}{\beta}\eta^{1-\gamma_1}\int_{s_0}^{s} d\tau_1 \int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{\gamma_1-1}\tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))u_1(\tau_2)\\
&= \tfrac{\kappa_{k_0}}{\beta}\eta^{1-\gamma_1}\int_{s_0}^{s} d\tau_1 \int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{-\gamma_2}\tfrac{1+\tau_1^2}{1+\tau_2^2}\\
&\qquad \cdot\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2})\\
&\le \tfrac{1}{\beta}\eta^{\gamma_2}\int_{s_0}^{s} d\tau_2\ \tfrac{((c_1+\tilde{c}_2)(\frac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\frac{\tau_2}{\eta})^{\gamma_2})\tau_2^{-\gamma_2}}{1+\tau_2^2}\\
&\qquad \cdot\int_{s_0}^{\tau_2} d\tau_1\ \kappa_k(1+\tau_1^2)\exp(-\kappa_k(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))\\
&= \tfrac{1}{\beta}\int_{s_0}^{s} d\tau_2\ ((c_1+\tilde{c}_2)\tau_2^{\gamma-2}\eta^{-\gamma}+(c_2+\tilde{c}_1)\tau_2^{-2}\eta^{-\gamma_2})\\
&\qquad \cdot[\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))]_{\tau_1=\tau_2}^{\tau_1=s}\\
&\le \tfrac{1}{\beta}\int_{s_0}^{s} d\tau_2\ ((c_1+\tilde{c}_2)\tau_2^{\gamma-2}\eta^{-\gamma}+(c_2+\tilde{c}_1)\tau_2^{-2}\eta^{-\gamma_2}).
\end{aligned}
$$

We note that for the first term we obtain

$$
\tfrac{1}{\beta}\int_{s_0}^{s} d\tau_2\ \tau_2^{\gamma-2}\eta^{-\gamma} \le \min(\tfrac{c}{\beta}(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma}, \tfrac{1}{\beta c}(c\eta)^{-\gamma}),
$$

since we can either integrate it directly or first pull out s^γ and then integrate. Finally, we obtain the following estimate

$$
R_1[F_{j_2}] \le \min(\tfrac{1}{\beta}\tfrac{1}{2c^{1+\gamma}}\eta^{-\gamma}, \tfrac{c}{\beta}(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma})(c_1+\tilde{c}_2)+\tfrac{1}{\beta}c(c_2+\tilde{c}_1).
$$

On I_1 we estimate the $j(k_0)$ influence by

$$
\begin{aligned}
R_1[F_{j_1}] &= c^2\eta^{1-\gamma_2}j(s_0)\int \tau^{\gamma_2-1}\tfrac{1+\tau^2}{1+s_0^2}\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))\\
&\le 4c^{2+\gamma_1}\eta^{-1-\gamma_2}j(s_0)\int(1+\tau^2)\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))\\
&\le \tfrac{4c^{2+\gamma_1}}{\kappa_{k_0}\eta^{1+\gamma_2}}j(s_0)
\end{aligned}
$$

and

$$
\begin{aligned}
R_2[F_{j_2}] &= \eta^{1-\gamma_1}j(s_0)\int \tau^{\gamma_1-1}\tfrac{1+\tau^2}{1+s_0^2}\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))\\
&= 4\eta^{-1-\gamma_1}c^{\gamma_2}j(s_0)\int(1+\tau^2)\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))\\
&= 4\tfrac{c^{\gamma_2}}{\kappa_{k_0}\eta^{1+\gamma_1}}j(s_0).
\end{aligned}
$$

We estimate $j(k_0)$ by

$$\begin{aligned}
j(k_0,s) &= \tfrac{1+s^2}{1+\tilde{s}_0^2}\exp(-\kappa_{k_0}(s-\tilde{s}_0+\tfrac{1}{3}(s^3-\tilde{s}_0^3)))j(k_0.\tilde{s}_0)\\
&+\tfrac{\kappa_{k_0}}{\beta}\int_{\tilde{s}_0}^{s} d\tau\ \tfrac{1+s^2}{1+\tau^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))u_1(\tau)\\
&\le \tfrac{4d^2}{\eta^2}\exp(-\tfrac{\kappa}{2^5}\xi\eta^2)j(k_0,\tilde{s}_0)+\tfrac{c^2}{\beta}\big((c_1+\tilde{c}_2)(\tfrac{d}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{d}{\eta})^{\gamma_2}\big).
\end{aligned}$$

On I_3 we estimate the $j(k_0)$ influence by

$$\begin{aligned}
R_1[F_{j_1}] &= c^2\eta^{1-\gamma_2}\int \tau^{\gamma_2-1}\tfrac{1+\tau^2}{1+s_0^2}\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))j(s_0)\\
&\le c^{4+\gamma_1}\eta^{\gamma_1}\int(1+\tau^2)\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))j(s_0)\\
&\le \eta^{\gamma_1}\tfrac{c^{4+\gamma_1}}{\kappa_{k_0}}j(s_0)
\end{aligned}$$

and

$$\begin{aligned}
R_2[F_{j_2}] &= \eta^{1-\gamma_1}\int \tau^{\gamma_1-1}\tfrac{1+\tau^2}{1+s_0^2}\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))j(s_0)\\
&= \eta^{\gamma_2}c^{2+\gamma_2}\int(1+\tau^2)\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))j(s_0)\\
&= \eta^{\gamma_2}\tfrac{c^{2+\gamma_2}}{\kappa_{k_0}}j(s_0).
\end{aligned}$$

Next we want to estimate the evolution of $j(k_0)$

$$\begin{aligned}
j(k_0,\tilde{s}_1) &= \tfrac{1+\tilde{s}_1^2}{1+d^2}\exp(-\kappa_{k_0}(\tilde{s}_1-d+\tfrac{1}{3}(\tilde{s}_1^3-d^3)))j(k_0,\tilde{s}_0)\\
&+\tfrac{\kappa_{k_0}}{\beta}\int_d^{\tilde{s}_1} d\tau_2\ \tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa_k(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2}).
\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}
&\tfrac{\kappa_{k_0}}{\beta}\int_d^{\tilde{s}_1} d\tau_2\ \tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))(\tfrac{\tau_2}{\eta})^{\gamma_i}\\
&\le \tfrac{\kappa_{k_0}}{\beta}\left(\int_d^{\frac{1}{2}\tilde{s}_1}+\int_{\frac{1}{2}\tilde{s}_1}^{\tilde{s}_1}\right)\tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))(\tfrac{\tau_2}{\eta})^{\gamma_i}\\
&\le \tfrac{\kappa_{k_0}}{\beta}\eta^{-\gamma_i}\tfrac{c^{1-\gamma_i}}{\gamma_i}(1+\eta^2)\exp(-\tfrac{\kappa_{k_0}}{25}\eta^3)+\tfrac{\kappa_{k_0}}{\beta}\tfrac{2^4}{\kappa_{k_0}\eta^2}\\
&\le \tfrac{2^5}{\beta}\tfrac{1}{\eta^2},
\end{aligned}$$

which leads to

$$\begin{aligned}
|j(k_0,\tilde{s}_1)| &\le c^2\kappa_{k_0}\eta^2\exp(-\kappa_{k_0}\eta^3)j(k_0,\tilde{s}_0)\\
&+\tfrac{2^5}{\beta}\tfrac{1}{\eta^2}(c_1+c_2+\tilde{c}_1+\tilde{c}_2).
\end{aligned}$$

□

Lemma 3.22 (Forcing estimate). *Let $u(s) = S(s)r(s)$ be a solution of* (3.25) *on $[\tilde{s}_0, s^*]$ such that $|u(s)| \le S^*(s)C(s)$. We define for $|n| \ge 2$*

$$\begin{aligned}\tilde{w}(n) = 2 &\sum_{|m|\ge 2} (2c)^{|m-n|+\chi}(w + \tfrac{4}{\kappa\xi} j)(k_m, \tilde{s}_0) \\ &+ (2c)^{||m|-2|} c(2c_1^* + \tfrac{1}{c^2} c_2^*) \\ &+ (2c)^{|m|-1}(u_3(\tilde{s}_0) + \tfrac{2}{\kappa\xi\eta}(j(k_{\pm 1}, \tilde{s}_0))\end{aligned}$$

where $\chi = \chi(m,n) = -|sgn(m) - sgn(n)|$. Then we estimate

$$\begin{aligned}R_1[F_{\tilde{w}}] &= 2c^2(\tilde{w}(2) + \tilde{w}(-2)) \\ R_2[F_{\tilde{w}}] &= c^2(\tilde{w}(2) + \tilde{w}(-2))(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}\end{aligned}$$

and

$$\begin{aligned}R_1[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0) + \tfrac{2c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*) \\ R_2[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0)(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1} + \tfrac{c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*)(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}\end{aligned}$$

and

$$\begin{aligned}R_1[F_{u_3}] &= 2c\tilde{w}(1) \\ R_2[F_{u_3}] &= 2c\tilde{w}(1)(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}.\end{aligned}$$

Furthermore, we estimate

$$\begin{aligned}|w(k_n, s)| &\le \tilde{w}(n) \qquad\qquad |n| \ge 2 \\ |u_3| &\le \tilde{w}(1) = \tilde{w}(-1) \\ |j(k_n)| &\le 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_n, \tilde{s}_0) + 4\tfrac{1}{\beta\eta^2}\tilde{w}(n).\end{aligned}$$

Proof. To estimate $w(k_n, s)$ we without loss of generality assume that $n \ge 2$. We begin with the case $n \ge 3$, where we deduce that

$$\begin{aligned}\partial_s w(k_n) &= a(k_{n+1})w(k_{n+1}) - a(k_{n-1})w(k_{n-1}) - j(k_n) \\ &\le \tfrac{c}{\eta}(\tilde{w}(n-1) + \tilde{w}(n+1)) + 2e^{-\kappa\xi\eta(s-\tilde{s}_0)} j(k_0 + n, \tilde{s}_0) + 4\tfrac{1}{\beta\eta^2}\tilde{w}(n).\end{aligned}$$

We estimate

$$2\int e^{-\frac{1}{2}\kappa\xi\eta(\tau-\tilde{s}_0)} j(k_0 + n, \tilde{s}_0) \le 4\tfrac{1}{\kappa\xi\eta} j(k_n, \tilde{s}_0).$$

Thus integrating $\partial_s w(k_0 + n)$ over time yields

$$\begin{aligned}w(k_n) &\le w(k_n, \tilde{s}_0) + c(\tilde{w}(n-1) + \tilde{w}(n+1)) + 4\tfrac{1}{\kappa\xi\eta} j(k_0 + n, \tilde{s}_0) + \tfrac{4}{\beta\kappa\xi}\tilde{w}(n) \\ &< \tilde{w}(n).\end{aligned}$$

For the case $n=2$ we deduce

$$\begin{aligned}
\partial_s w(k_2) &= a(k_3)w(k_3) - a(k_1)\tfrac{1}{2}(u_3+u_2) - j(k_2)\\
&\leq \tfrac{c}{\eta}(\tilde{w}(3) + 2\tilde{w}(1) + 2c_1^*(\tfrac{s}{\eta})^{\gamma_1-1} + 2c_2^*(\tfrac{s}{\eta})^{\gamma_2-1}) + 2e^{-\frac{1}{2}\kappa\eta\xi(s-\tilde{s}_0)} j(k_2,\tilde{s}_0) + 4\tfrac{1}{\beta\kappa\xi\eta}\tilde{w}(2)\\
w(k_2) &\leq w(k_2,\tilde{s}_0) + \tfrac{4}{\kappa\xi\eta} j(k_2,\tilde{s}_0) + c(\tilde{w}(3) + 2\tilde{w}(1) + 2c_1^* + \tfrac{1}{c^2}c_2^*) + 4\tfrac{1}{\beta\kappa\xi}\tilde{w}(2)\\
&< \tilde{w}(2).
\end{aligned}$$

We estimate u_3 by

$$\begin{aligned}
\partial_s u_3 &= a(k+2)w(k_2) - a(k-2)w(k_{-2}) - j(k_1) + j(k_{-1})\\
&\leq \tfrac{2c}{\eta}(w(2)+w(-2)) + 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_{\pm1},\tilde{s}_0) + \tfrac{4}{\beta\kappa\xi\eta}(\tilde{w}(1) + c_1^* + c_2^*)\\
|u_3| &\leq |u_3(\tilde{s}_0)| + c(\tilde{w}(2) + \tilde{w}(-2)) + \tfrac{4}{\kappa\xi\eta}(j(k_{\pm1},\tilde{s}_0) + \tfrac{1}{\beta}\tilde{w}(1) + \tfrac{1}{\beta}c_1^* + \tfrac{1}{\beta}c_2^*)\\
&\leq \tilde{w}(1).
\end{aligned}$$

Non-resonant j will often be estimated similarly. Therefore we will use the following notation frequently. We estimate $j(k_n)$ for $n \geq 2$ by writing $\hat{s} = s - \frac{k_0(k_0-k)}{k+1}\eta$ and $\hat{\tau} = \tau - \frac{k_0(k_0-k)}{k+1}\eta$

$$\partial_s j(k_n) = -\kappa_{k_n}(1+\hat{s}^2) j(k_n) + 2\tfrac{\hat{s}}{1+\hat{s}^2} j(k_n) + \tfrac{1}{\beta}\kappa_{k_n} w(k_n)$$

which gives

$$\begin{aligned}
j(k_n) \leq{}& \tfrac{1+\hat{s}^2}{1+\hat{\tilde{s}}_0^2} e^{-\kappa_{k_n}((\hat{s}-\hat{\tilde{s}}_0 + \frac{1}{3}(\hat{s}^3 - \hat{\tilde{s}}_0^3))} j(k_n,\tilde{s}_0)\\
&+ \tfrac{1}{\beta}\kappa_{k_n}\int d\tau\ \tfrac{1+\hat{s}^2}{1+\hat{\tau}^2} e^{-\kappa_{k_n}((\hat{s}-\hat{\tau}+\frac{1}{3}(\hat{s}^3-\hat{\tau}^3))}\tilde{w}(n)
\end{aligned}$$

For $\tilde{s}_0 \leq \tau \leq s \leq \tilde{s}_1$ we obtain

$$\begin{aligned}
\kappa_{k_n}(\hat{s} - \hat{\tau} + \tfrac{1}{3}(\hat{s}^3 - \hat{\tau}^3)) &= \kappa_{k_n}\tfrac{1}{3}(s-\tau)(\hat{s}^2 + \hat{s}\hat{\tau} + \hat{\tau}^2 + 1)\\
&\geq \tfrac{1}{2}\kappa\max(k_n^2, k_0^2)\eta^2(s-\tau)\\
\tfrac{1+\hat{s}^2}{1+\hat{\tau}} &\leq 2.
\end{aligned}$$

So we infer

$$\begin{aligned}
j(k_n) &\leq 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_n,\tilde{s}_0)\\
&\quad + 2\kappa_{k_n}\tfrac{1}{\beta}\int d\tau\ e^{-\frac{1}{2}\kappa\xi\eta(s-\tau)}\tilde{w}(n)\\
&\leq 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_n,\tilde{s}_0) + \tfrac{4}{\beta\eta^2}\tilde{w}(n).
\end{aligned}$$

We next turn to the estimate of $j(k_{\pm1})$, where we without loss of generality consider $j(k_1)$. With the equation

$$\partial_s j(k_1) = (2\tfrac{\hat{s}}{1+\hat{s}^2} - \kappa_{k_1}(1+\hat{s}^2)) j(k_1) + \tfrac{\kappa_{k_1}}{2\beta}\kappa_{k_1}(u_3+u_2)$$

we estimate

$$\begin{aligned} j(k_1) &\le 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_1, \tilde{s}_0) \\ &\quad + \tfrac{\kappa_{k_1}}{2\beta} \int d\tau \; e^{-\frac{1}{2}\kappa\xi\eta(s-\tau)} (\tilde{w}(1) + c_1^* (\tfrac{\tau}{\eta})^{\gamma_1 - 1} + c_2^* (\tfrac{\tau}{\eta})^{\gamma_2 - 1}) \\ &\le 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_1, \tilde{s}_0) + \tfrac{2}{\beta\eta^2} (\tilde{w}(1) + c_1^* + c_2^*). \end{aligned}$$

Given these estimates, we next consider the effects on $R[\cdot]$ by forcing:

$$\begin{aligned} F_{\tilde{w}} &= e_1 (a(k_2) w(k_2) + a(k_{-2}) w(k_{-2}) \\ &\le e_1 \tfrac{c}{\eta} \tilde{w}(2) + e_1 \tfrac{c}{\eta} \tilde{w}(-2). \end{aligned}$$

For constant e_2 functions we estimate

$$\begin{aligned} R_1[e_2] &\le \tfrac{c}{2}\eta \\ R_2[e_2] &\le \tfrac{c}{3}\eta (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

Therefore, we can control $F_{\tilde{w}}$ by

$$\begin{aligned} R_1[F_{\tilde{w}}] &= c^2 (\tilde{w}(2) + \tilde{w}(-2)) \\ R_2[F_{\tilde{w}}] &= c^2 (\tilde{w}(2) + \tilde{w}(-2)) (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

For $F_{j(k_{\pm 1})}$ we use

$$\begin{aligned} F_{j(k_{\pm 1})} &= -e_2 j(k_{\pm 1}) \\ &\le e_2 (2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_{\pm 1}, \tilde{s}_0) + \tfrac{2}{\beta\kappa\xi\eta} (\tilde{w}(1) + c_1^* + c_2^*)), \end{aligned}$$

to estimate

$$\begin{aligned} R_1[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0) + \tfrac{2c}{\beta\kappa\xi} (\tilde{w}(1) + c_1^* + c_2^*) \\ R_2[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0) (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma_1} + \tfrac{c}{\beta\kappa\xi} (\tilde{w}(1) + c_1^* + c_2^*) (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

Furthermore, for F_{u_2} we estimate

$$\begin{aligned} F_{u_3} &= e_1 a_2 u_3 \\ &\le e_1 \tfrac{c}{\eta} \tilde{w}(1) \end{aligned}$$

and

$$\begin{aligned} R_1[e_1] &\le \eta \\ R_2[e_1] &\le \eta (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

to deduce

$$\begin{aligned} R_1[F_{u_3}] &\le c\tilde{w}(1) \\ R_2[F_{u_3}] &\le c\tilde{w}(1) (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

□

Proof of Proposition 3.17. For the interval I_1 we have $\tilde{s}_0 = s_0$, $\tilde{s}_1 = -d$. The initial data of r can be calculated by $r(\tilde{s}_0) = S^{-1}(s_0)u(s_0)$ and so

$$\begin{aligned} r_1(\tilde{s}_0) &= -\tfrac{\gamma_2}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_2-1}u_1(\tilde{s}_0) + \tfrac{c}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_2}u_2(\tilde{s}_0) \\ &\approx -4c^2u_1(\tilde{s}_0) + cu_2(\tilde{s}_0), \\ r_2(\tilde{s}_0) &= \tfrac{\gamma_1}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_1-1}u_1(\tilde{s}_0) - \tfrac{c}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_1}u_2(\tilde{s}_0) \\ &\approx u_1(\tilde{s}_0) - \tfrac{c}{2}u_2(\tilde{s}_0). \end{aligned}$$

For other initial data we define

$$\begin{aligned} N = &\sum_{|m|\geq 2}(2c)^{|m|}(w + \tfrac{8}{\kappa\eta\xi}j)(k_m, \tilde{s}_0) \\ &+ 2c(u_3(\tilde{s}_0) + \tfrac{8}{\kappa\eta\xi}j(k_{\pm 1}, \tilde{s}_0)) \\ &+ \tfrac{2c}{\xi\kappa}j(k_0, s_0), \end{aligned}$$

to bound the impact of the less important terms in the following bootstrap. Let $C(s)$ be defined by the terms

$$\begin{aligned} c_1 &= 45c^2u_1(s_0) + 2cu_2(s_0) + 2N, \\ \tilde{c}_1 &= 2\tfrac{c^3}{\beta\vee 1}c_2, \\ c_2 &= 2u_1(s_0) + 45cu_2(s_0) + 2N, \\ \tilde{c}_2 &= 0. \end{aligned}$$

AS $c_1 > r_1(\tilde{s}_0)$ and $c_2 > r_2(\tilde{s}_0)$ and we have a smooth solution, the estimate $|u| \leq S^*(s)C(s)$ holds at least for a small time. Let s^* be the maximal time such that $|u| \leq S^*(s)C(s)$. We then aim to show that necessarily $s^* \geq -d$, since otherwise the estimate improves, which contradicts the maximality. By Lemma 3.20, Lemma 3.21 and Lemma 3.22 we estimate

$$\begin{aligned} R_1[F_{all}] &= R_1[F_{3mode}] + R_1[F_j] + R_1[F_{\tilde{w}}] + R_1[F_{j(k_0\pm 1)}] + R_1[F_{u_3}] \\ &= 20c^2c_1 + (20 + c^4(\tfrac{s}{\eta})^{-\gamma})\tilde{c}_1 + (20c^2 + c^4(\tfrac{s}{\eta})^{-\gamma})c_2 \\ &+ \tfrac{c^3}{\beta}c_1 + \tfrac{c^3}{\beta}(\tfrac{s}{\eta})^{-\gamma}(\tilde{c}_1 + c_2) + \tfrac{4c^{2+\gamma_1}}{\kappa_{k_0}\eta^{1+\gamma_2}}j(k_0, s_0) \\ &+ 2c^2(\tilde{w}(2) + \tilde{w}(-2)) \\ &+ \tfrac{2c}{\kappa\xi\eta}j(k_{\pm 1}, \tilde{s}_0) + \tfrac{2c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*) \\ &+ 2c\tilde{w}(1) \\ &< 21c^2c_1 + \tilde{c}_1(21 + \tfrac{c^3}{\beta}(\tfrac{s}{\eta})^{-\gamma}) + c_2(21c^2 + \tfrac{c^3}{\beta}(\tfrac{s}{\eta})^{-\gamma}) + N \end{aligned}$$

and

$$
\begin{aligned}
R_2[F_{all}] &= R_2[F_{3mode}] + R_2[F_j] + R_2[F_{\tilde{w}}] + R_2[F_{j(k_0\pm 1)}] + R_2[F_{u_3}] \\
&= 20c_1 + 20\tilde{c}_1 + 20c^2 c_2 \\
&+ \tfrac{c}{\beta}c_1 + \tfrac{c}{\beta}(c_2+\tilde{c}_1) + 4\tfrac{c^{\gamma_2}}{\kappa_{k_0}\eta^{1+\gamma_1}} j(k_0, s_0) \\
&+ c^2(\tilde{w}(2)+\tilde{w}(-2)) \\
&+ \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0) + \tfrac{c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*) \\
&+ 2c\tilde{w}(1) \\
&< 21c_1 + 21\tilde{c}_1 + \tfrac{c}{\beta\wedge 1}c_2 + N.
\end{aligned}
$$

We split R_1 as

$$
R_1[all] = R_1[all][1] + R_1[all][(\tfrac{s}{\eta})^\gamma],
$$

into the part with and without a $(\frac{s}{\eta})^\gamma$ term, respectively. We then estimate

$$
\begin{aligned}
r_1(\tilde{s}_0) + R_1[all][1] &< r_1(\tilde{s}_0) + 21c^2 c_1 + 21\tilde{c}_1 + 21c^2 c_2 + N, \\
R_1[all][(\tfrac{s}{\eta})^\gamma] &< \tfrac{c^3}{1\wedge\beta}\tilde{c}_1 + \tfrac{c^3}{1\wedge\beta}c_2, \\
r_2(\tilde{s}_0) + R_2[all][1] &< r_2(\tilde{s}_0) + 21c_1 + 21\tilde{c}_1 + 2\tfrac{c}{1\wedge\beta}c_2 + N,
\end{aligned}
$$

and thus we conclude the bootstrap that

$$
\begin{aligned}
r_1(\tilde{s}_0) + R_1[all][1] &< c_1 \\
R_1[all][(\tfrac{s}{\eta})^\gamma] &< \tilde{c}_1 \\
r_2(\tilde{s}_0) + R_2[all][1] &< c_2.
\end{aligned}
$$

We can therefore extend the estimates past the time s^*, which contradicts the maximally. Therefore, we obtain that for all times $s \leq -d$ it holds that

$$
|u(s)| \leq S^*(s)C(s),
$$

which yields the upper bound

$$
\begin{aligned}
|u(-d)| &\leq \begin{pmatrix} (c\eta)^{-\gamma_1}c_1 + (\tilde{c}_1 + c_2)(c\eta)^{-\gamma_2} \\ \frac{1}{c}(c\eta)^{1-\gamma_1}c_1 + (\frac{1}{c}\tilde{c}_1 + cc_2)(c\eta)^{1-\gamma_2} \end{pmatrix} \\
&\leq 2M \begin{pmatrix} (c\eta)^{-\gamma_2} \\ (c\eta)^{1-\gamma_2} \end{pmatrix}.
\end{aligned}
$$

We next aim to establish an estimate on $\tilde{w}(n)$. For this purpose we note that

$$
\begin{aligned}
c(2c_1^* + \tfrac{1}{c^2}c_2^*) &\approx 2c_1 + \tfrac{1}{c^2}\tilde{c}_1 + 2c_2 \\
&\approx 2c_1 + 2c_2 \\
&\leq 4u_1(s_0) + 100cu_2(s_0) + 3N
\end{aligned}
$$

and

$$\tilde{w}(n) \leq 2 \sum_{|m|\geq 2} (2c)^{|m-n|+\chi}(w + \tfrac{4}{\kappa\xi\eta} j)(k_m, \tilde{s}_0)$$
$$+ (2c)^{||n|-2|} c(2c_1^* + \tfrac{1}{c^2} c_2^*)$$
$$+ (2c)^{|n|-1}(u_3(\tilde{s}_0) + \tfrac{2}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0)).$$

We hence deduce that

$$\begin{aligned}\tilde{w}(n) &\leq 2 \sum_{|m|\geq 2} (2c)^{|m-n|+\chi}(w + \tfrac{4}{\kappa\xi\eta} j)(k_m, \tilde{s}_0) \\ &+ (2c)^{||n|-2|}(4u_1(s_0) + 100cu_2(s_0) + 3N) \\ &+ (2c)^{|n|-1}(u_3(\tilde{s}_0) + \tfrac{2}{\kappa\xi\eta} j(\tilde{s}_0, k_{\pm 1})) \\ &\leq 2M_n,\end{aligned}$$

when $\chi = -|\mathrm{sgn}(m) - \mathrm{sgn}(n)|$. To prove (3.27) under the condition (3.26) we estimate

$$\begin{aligned}|u(-d) - \tilde{u}(-d)| &\leq S(-d)R[all] \\ &\leq u_1(\tilde{s}_0) \begin{pmatrix} (c\eta)^{-\gamma_2} \\ 5c(c\eta)^{\gamma_1} \end{pmatrix}.\end{aligned}$$

Furthermore, we use

$$\begin{aligned}\tilde{u}(-d) &= \begin{pmatrix} (c\eta)^{-\gamma_1} & (c\eta)^{-\gamma_2} \\ -\frac{\gamma_1}{2c}(c\eta)^{1-\gamma_1} & -\frac{\gamma_2}{2c}(c\eta)^{1-\gamma_2} \end{pmatrix} \begin{pmatrix} 4c^2 u_1(\tilde{s}_0) - 2cu_2(\tilde{s}_0) \\ u_1(\tilde{s}_0) + cu_2(\tilde{s}_0) \end{pmatrix} \\ &\approx u_1(\tilde{s}_0) \begin{pmatrix} (c\eta)^{-\gamma_2} \\ O(c)(c\eta)^{\gamma_1} \end{pmatrix}\end{aligned}$$

and thus

$$\begin{aligned}|u_1(-d) - (c\eta)^{-\gamma_2} u_1(\tilde{s}_0)| &= 10cu_1(\tilde{s}_0)(c\eta)^{-\gamma_2} \\ |u_2(-d)| &\leq 10cu_1(\tilde{s}_0)(c\eta)^{\gamma_1}.\end{aligned}$$

The remaining terms can be estimated by

$$\begin{aligned}M &\leq \tfrac{1}{1-10^{-1}} u_1(\tilde{s}_0), \\ M_n &\leq \tfrac{4}{1-10^{-1}} u_1(\tilde{s}_0).\end{aligned}$$

□

3.4.3 The Resonance and Upper Bounds in I_2

The bounds on the evolution of (3.25) on the interval $I_2 = [-d, d]$ are summarized in the following proposition:

Proposition 3.23. *Let* $c \leq \min((8\pi)^{-\frac{4}{3}}\beta^{\frac{16}{3}}, 10^{-4})$. *Consider a solution of* (3.25) *on the interval* $I = [s_0, d]$*, then it holds that*

$$\begin{aligned}
|u_1(d)| &\leq 3(c\eta)^{-\gamma_2} LM, \\
|u_2(d)| &\leq 7\pi(c\eta)^{\gamma_1} LM, \\
|u_3(d)| &\leq 7\pi(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1} LM + 2M_1, \\
|w(k_n, d)| &\leq 7\pi(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1} LM + 2M_n, \\
|j(k_n, d)| &\leq \tfrac{4}{\beta\eta^2}(7\pi(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1} LM + 2M_n), \\
|j(k_0, d)| &\leq \tfrac{4}{\beta}\min(\kappa_{k_0}\pi d^2, 1)(c\eta)^{-\gamma_2} LM.
\end{aligned}$$

For interval I_2 we are mostly concerned with the interaction between $j(k_0)$ and u_1 and in particular the growth this induces for u_2. Therefore, consider the ODE system

$$\begin{aligned}
\partial_s u_1 &= -j(k_0) + F \\
\partial_s j(k_0) &= \tfrac{\kappa_{k_0}}{\beta} u_1 + (\tfrac{2s}{1+s^2} - \kappa_{k_0}(1+s^2))j(k_0),
\end{aligned} \tag{3.30}$$

our aim is to bound the growth of $j(k_0)$ and u_1 by a factor. Let $U(\tau, s)$ be the solution of (3.30) with initial data $u_1(\tau) = 1$ and $j(\tau) = 0$ and L as the constant which satisfies

$$|U(\tau, s)| \leq L = L(\beta, \kappa, k). \tag{3.31}$$

With the restriction

$$c \leq (8\pi)^{-\frac{4}{3}}\beta^{\frac{16}{3}},$$

L is estimated by the following two cases, if $\beta \geq \pi$ we obtain $L = 1$ and if $\beta < \pi$ we obtain a $L = L(\alpha, \kappa, k) \leq \sqrt{c}$. A proof and more specific bounds can be found in 3.5 and for simplicity of presentation we here only consider two cases.

Lemma 3.24. *Let* u_1 *be a solution of* (3.30) *on* $[-d, d]$*such that* (3.31) *holds, then we estimate*

$$\tfrac{1}{L}|u_1| \leq u_1(-d) + \int_{-d}^{s} |F(\tau)|\, d\tau + |j(-d)| \int \tfrac{1+\tau^2}{1+d^2}\exp(-\kappa_k(\tau + d + \tfrac{1}{3}(s^3 + d^3))).$$

Proof. We may without loss of generality restrict to the case $j(-d) = 0$, since we can choose $\tilde{F} = F + \frac{1+s^2}{1+d^2}\exp(-\kappa_k(\tau + d + \frac{1}{3}(s^3 + d^3)))j(-d)$. By Duhamel' principle the equation (3.30) is solved by

$$u_1(s) = U(-d, s)u_1(-d) + \int U(\tau, s)F(\tau)$$

which yields the desired bound. □

Proof of Proposition 3.23. With Proposition 3.17 we estimate until time $-d$

$$
\begin{aligned}
|u_1|(-d) &\le 2M(c\eta)^{-\gamma_2} \\
|u_2|(-d) &\le 2M(c\eta)^{\gamma_1}, \\
|u_3|(-d) &\le 2M_1, \\
|w(k_n,-d)| &\le 2M_n, \\
|j|(k_0,-d) &\le \tfrac{c}{\beta}M(c\eta)^{-\gamma_2}\min(\kappa_{k_0}c^{-2},1), \\
|j|(k_{\pm1},-d) &\le \tfrac{4}{\beta\eta^2}M, \\
|j|(k_n,-d) &\le \tfrac{4}{\beta\eta^2}M_n.
\end{aligned}
$$

We next aim to prove by a bootstrap that

$$
\begin{aligned}
|u_1| &\le 3L(c\eta)^{-\gamma_2}M, \\
|u_2| &\le 7\pi L(c\eta)^{\gamma_1}M, \\
|u_3| &\le 15\pi L(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1}M + 2M_1, \\
|w(k_n)| &\le 7\pi L(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}M + 2M_n, \\
\int j(k_{\pm1}) &\le 7\pi L\tfrac{3d}{\beta\eta^2}(c\eta)^{\gamma_1}M, \\
\int j(k_n) &< \tfrac{2}{\beta}\tfrac{d}{\eta^2}(7\pi L\tfrac{5}{\eta}(c\eta)^{\gamma_1}M + 2M_n).
\end{aligned}
$$

To estimate u_1 we use Lemma 3.24 to deduce

$$
\begin{aligned}
|u_1| &\le L\left(u_1(-d) + \int a_1u_2 + a_2u_3 + \tfrac{1+s^2}{1+d^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))j(-d)\right) \\
&\le 2L(c\eta)^{-\gamma_2}M + L\tfrac{1}{\eta}(1+\eta^{-2})(7\pi L(c\eta)^{\gamma_1}M + 2M_1) \\
&+ \tfrac{L}{1+d^2}\min(\tfrac{1}{\kappa_{k_0}}, d^3)j(k_0,-d) \\
&\le 2L(c\eta)^{-\gamma_2}M + L\tfrac{1}{\eta}(1+\eta^{-2})(7\pi L(c\eta)^{\gamma_1}M + 2M_1) + L\tfrac{c}{\beta}(c\eta)^{-\gamma_2}M \\
&< 3L(c\eta)^{-\gamma_2}M,
\end{aligned}
$$

where we used that $\frac{40}{\eta}M_1 \le \frac{1}{10}M(c\eta)^{-\gamma_2}$ since $\eta \ge \frac{1}{10c}$ and that $7\pi Lc < \frac{1}{2}$. We estimate u_2 by

$$
\begin{aligned}
|u_2| &\le 2(c\eta)^{\gamma_1}M + \int 2c\eta\tfrac{1}{1+s^2}u_1 + a(k_{\pm2})w(k_{\pm2}) + j(k_{\pm1}) \\
&\le 2(c\eta)^{\gamma_1}M + 2\pi c\eta|u_1|_{L^\infty_s} + \tfrac{4}{\eta}|w(k_{\pm2})|_{L^\infty_s} + \int j(k_{\pm2}) \\
&< 7\pi L(c\eta)^{\gamma_1}M.
\end{aligned}
$$

In order to control u_3, we integrate $\partial_s u_3$ in time, which yields

$$\begin{aligned}|u_3| &\le |u_3|(-d) + \int a(k_{\pm 2})w(k_{\pm 2}) + j(k_{\pm 1}) \\ &\le 2M_1 + \tfrac{5}{\eta}(7\pi L\tfrac{4}{\eta}(c\eta)^{\gamma_1}M + 2M_2) + 7\pi L\tfrac{3d}{\beta\eta^2}(c\eta)^{\gamma_1}M \\ &< 8\pi L(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1}M + 2M_1.\end{aligned}$$

For $w(k_n)$ we first consider $|n| \ge 3$. By integrating $\partial_s w(k_n)$ we deduce

$$\begin{aligned}|w(k_n)| &\le |w(k_n, -d)| + \tfrac{2}{\eta}(|w(k_{n+1})|_{L_s^\infty} + |w(k_{n-1})|_{L_s^\infty}) + \int j(k_n) \\ &< 7\pi L(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}M + 2M_n.\end{aligned}$$

For the cases $n = \pm 2$ we similarly conclude that

$$\begin{aligned}|w(k_{\pm 2})| &\le |w(k_{\pm 2}, -d)| + \tfrac{2}{\eta}(|u_2|_{L_s^\infty} + |u_3|_{L_s^\infty} + |w(k_{\pm 3})|_{L_s^\infty}) \\ &< 7\pi L\tfrac{5}{\eta}(c\eta)^{\gamma_1}M + 2M_{\pm 2}.\end{aligned}$$

For the estimates on the current j we argue similarly as on the interval I_1 and introduce $\hat{s}$ as the shifted time coordinates. To estimate $j(k_{\pm 1})$ we integrate $\partial_s j(k_{\pm 1})$ in time:

$$\begin{aligned}j(k_{\pm 1}) &= 2\exp(-\tfrac{1}{2}\kappa\xi\eta(s+d))j(k_{\pm 1}, -d) \\ &\quad + \tfrac{\kappa_{k_{\pm 1}}}{\beta}\int \tfrac{1+\hat{s}^2}{1+\hat{\tau}^2}\exp(-\kappa_{k_{\pm 1}}(\hat{s} - \hat{\tau} + \tfrac{1}{3}(\hat{s}^3 - \hat{\tau}^3)))(u_2 \pm u_3).\end{aligned}$$

The impact of $j(k_{\pm 1})$ is bounded by

$$\begin{aligned}\int j(k_{\pm 1}) &\le \tfrac{4}{\kappa\xi\eta}j(k_{\pm 1}, -d) + \tfrac{2d}{\beta\eta^2}(|u_2|_{L_s^\infty} + |u_3|_{L_s^\infty}) \\ &\le 7\pi L\tfrac{3d}{\beta\eta^2}(c\eta)^{\gamma_1}M\end{aligned}$$

and hence yields the estimate

$$\begin{aligned}j(k_{\pm 1}) &\le 2\exp(-\tfrac{1}{2}d\kappa\xi\eta)j(k_{\pm 1}, -d) + \tfrac{4}{\beta\eta^2}(|u_3|_{L_s^\infty} + |u_2|_{L_s^\infty}) \\ &< \tfrac{4}{\beta\eta^2}(7\pi L(c\eta)^{\gamma_1}M + 2M_1).\end{aligned}$$

By integrating we thus obtain the following estimate for $j(k_n)$:

$$\begin{aligned}j(k_n) &= 2\exp(-\kappa\xi\eta(s+d))j(k_n, -d) \\ &\quad + \tfrac{\kappa_{k_n}}{\beta}\int \tfrac{1+\hat{s}^2}{1+\hat{\tau}^2}\exp(-\kappa_{k_n}(\hat{s} - \hat{\tau} + \tfrac{1}{3}(\hat{s}^3 - \hat{\tau}^3)))w(k_n),\end{aligned}$$

which leads to

$$\begin{aligned}\int j(k_n) &= \tfrac{1}{\kappa\xi\eta}j(k_n, -d) + \tfrac{1}{\beta}\tfrac{d}{\eta^2}|w(k_n)|_{L_s^\infty} \\ &\le \tfrac{2}{\beta}\tfrac{d}{\eta^2}(7\pi L\tfrac{4}{\eta}(c\eta)^{\gamma_1}M + 2M_n)\end{aligned}$$

and

$$
\begin{aligned}
j(k_n) &\leq 2\exp(-\kappa\xi d\eta) j(k_n,-d) + \tfrac{4}{\beta\eta^2}|w(k_n)|_{L^\infty_s} \\
&< \tfrac{4}{\beta\eta^2}(7\pi L(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1} M + 2M_n).
\end{aligned}
$$

We estimate $j(k_0)$ by integrating

$$
\begin{aligned}
j(k_0) &= \tfrac{1+s^2}{1+d^2}\exp(-\kappa_{k_0}(s+d+\tfrac{1}{3}(s^3+d^3))j(k_0,-d) \\
&+ \tfrac{\kappa_{k_0}}{\beta}\int \tfrac{1+s^2}{1+\tau^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))u_1(\tau).
\end{aligned}
$$

The second term can be estimated by

$$
\begin{aligned}
&\tfrac{\kappa_{k_0}}{\beta}\int \tfrac{1+s^2}{1+\tau^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))u_1(\tau) \\
&\quad\leq \tfrac{1}{\beta}\min(\kappa_{k_0}\pi d^2, 1)|u_1|_{L^\infty_s}
\end{aligned}
$$

and thus

$$
\begin{aligned}
j(k_0,d) &= \exp(-\kappa_{k_0}(2d+\tfrac{2}{3}d^3)j(k_0,-d) \\
&+ \tfrac{1}{\beta}\min(\kappa_{k_0}\pi d^2,1)|u_1|_{L^\infty_s} \\
&\leq \exp(-\kappa_{k_0}(2d+\tfrac{2}{3}d^3)\tfrac{c^2}{\beta}M(c\eta)^{-\gamma_2}\min(\kappa_{k_0},1) \\
&+ \tfrac{3LM}{\beta}\min(\kappa_{k_0}\pi d^2,1)(c\eta)^{-\gamma_2} \\
&< \tfrac{4LM}{\beta}\min(\kappa_{k_0}\pi d^2,1)(c\eta)^{-\gamma_2}.
\end{aligned}
$$

□

3.4.4 The Echo and Lower Bounds in the Interval I_2

In this section we establish the echo mechanism on the interval I_2, i.e. our aim is to show that the mode u_1 induces growth of the u_2 mode. For this echo mechanism we need the additional assumption

$$\kappa k_0^2 \min(\beta, 1) > \tfrac{1}{c}. \tag{3.32}$$

As shown in Subsection 3.2.3, this is not only a technical assumption. When k_0 is too small, the u_1 term can become negative due to the action of j and hence negate the growth of u_2 and we could even obtain $u_2(d) \approx 0$. We will use initial data of the form

$$
\begin{aligned}
u_1(-d) &= 1, \\
u_2(-d) &\leq 50c^2\eta, \\
|j|(k_0,-d) &\leq \tfrac{2c^2}{\beta}, \\
|w|(k,-d), |u_3|(-d) &\leq 5(c\eta)^{\gamma_2}, \\
|j|(k,-d) &\leq \tfrac{20}{\eta}(c\eta)^{\gamma_2}.
\end{aligned} \tag{3.33}
$$

Which corresponds to the echoes on I_1 normalized in terms of $u(-d)$. We will prove that u closely matches the following asymptotics:

$$\tilde{u}_1 = \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d))),$$
$$\tilde{u}_2 = u_2(-d) + 2c\eta\beta(1 - \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d)))).$$

Proposition 3.25. *Consider a solution of* (3.25) *with initial data* (3.33)*, then the following estimates hold:*

$$\begin{aligned} |u_1(d) - \tilde{u}_1(d)| &= 12\pi c, \\ |u_2(d) - \tilde{u}_2(d)| &\leq 24\pi c^2\eta, \\ w(k_n, d), u_3(d) &\leq 6(c\eta)^{\gamma_2}, \\ j(k_n, d) &\leq \tfrac{25}{\beta\eta^2}(c\eta)^{\gamma_2}, \\ j(k_0, d) &\leq \tfrac{2}{\beta}. \end{aligned} \tag{3.34}$$

In the following it is convenient to introduce the *good unknown*:

$$g(s) = (1 + s^2)j - \tfrac{u_1}{\beta},$$

In terms of g our equations then read

$$\partial_s u_1 = -\tfrac{1}{\beta}\tfrac{1}{1+s^2}u_1 - a_1 u_2 + a_2 u_3 - \tfrac{1}{1+s^2}g$$

and

$$\begin{aligned} \partial_s g &= 2sj(k_0) + (1 + s^2)\partial_s j(k_0) - \tfrac{1}{\beta}\partial_s u_1 \\ &= \tfrac{2s}{1+s^2}(g + \tfrac{1}{\beta}u_1) \\ &- \kappa_{k_0}(1 + s^2)g + \tfrac{2s}{1+s^2}(g + \tfrac{1}{\beta}u_1) \\ &+ \tfrac{1}{\beta^2}\tfrac{1}{1+s^2}u_1 + \tfrac{1}{\beta}a_1 u_2 - \tfrac{1}{\beta}a_2 u_3 + \tfrac{1}{\beta}\tfrac{1}{1+s^2}g \\ &= (\tfrac{4s+\frac{1}{\beta}}{1+s^2} - \kappa_{k_0}(1 + s^2))g \\ &+ \tfrac{1}{\beta}\tfrac{4s+\frac{1}{\beta}}{1+s^2}u_1 + \tfrac{1}{\beta}a_1 u_2 - \tfrac{1}{\beta}a_2 u_3. \end{aligned}$$

Therefore, (3.25) can be equivalently expressed as

$$\begin{aligned} \partial_s \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ g \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\beta}\frac{1}{1+s^2} & -a_1 & a_2 & -\frac{1}{1+s^2} \\ 2c\eta\frac{1}{1+s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\beta}\frac{4s+\frac{1}{\beta}}{1+s^2} & \frac{1}{\beta}a_1 & \frac{1}{\beta}a_2 & \frac{4s+\frac{1}{\beta}}{1+s^2} - \kappa_k(1+s^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ g \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ a(k \pm 2)w(k \pm 2) - j(k \pm 1) \\ \pm a(k \pm 2)w(k \pm 2) \mp j(k \pm 1) \\ 0 \end{pmatrix}. \end{aligned} \tag{3.35}$$

The homogeneous system with respect to (3.35) is given by

$$\partial_s \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\beta}\frac{1}{1+s^2} & 0 \\ 2c\eta\frac{1}{1+s^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} \tag{3.36}$$

with the explicit solution

$$\begin{aligned} \tilde{u}_1 &= \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d)))u_1(-d) \\ \tilde{u}_2 &= u_2(-d) + 2c\eta\beta(1 - \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d))))u_1(-d). \end{aligned} \tag{3.37}$$

In the following, we prove that the solution of (3.35) can be treated as a perturbation of (3.37). Note that we can approximate

$$\beta(1 - \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d)))) \approx \min(\beta, (\tan^{-1}(s) + \tan^{-1}(d))),$$

where "≈" in this case corresponds to the explicit bounds

$$\tfrac{1}{2}\min(\beta, \cdot) \leq \beta(1 - \exp(-\tfrac{1}{\beta}\cdot)) \leq \min(\beta, \cdot).$$

Proof of Proposition 3.25. We want to show by a bootstrap that

$$\begin{aligned} |u_1 - \tilde{u}_1| &\leq c_1 = 12\pi c \\ |u_2 - \tilde{u}_2| &\leq c_2 = (2\pi + 1)c\eta c_1 \\ |u_3|, |w(k_n)| &\leq 6(c\eta)^{\gamma_2} \\ \int j(k_n) &\leq \tfrac{13d}{\beta\eta^2}(c\eta)^{\gamma_2} \\ \int j(s, k_{\pm 1}) &\leq \tfrac{10\pi}{\beta\eta}. \end{aligned} \tag{3.38}$$

Let s^* be the maximal time such that (3.38) holds. We assume that $s^* \leq d$ and show that this leads to a contradiction by improving (3.38). The estimates of $j(k_n)$ for $n \neq 0$ are done similarly as in Proposition 3.23 and we hence omit them here. First, we estimate g:

$$\begin{aligned} g_0(s) &= \tfrac{(1+s^2)^2}{(1+d^2)^2} \exp(\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d)) - \kappa_{k_0}(s + d + \tfrac{1}{3}(s^3 + d^3)))g(-d), \\ g(s) - g_0(s) &= \tfrac{1}{\beta}\int \tfrac{(1+s^2)^2}{(1+\tau^2)^2} \exp\left(\tfrac{1}{\beta}(\tan^{-1}(s) - \tan^{-1}(\tau)) - \kappa_{k_0}(s - \tau + \tfrac{1}{3}(s^3 - \tau^3))\right) \\ &\quad \left(\tfrac{4\tau + \frac{1}{\beta}}{1+\tau^2}u_1(\tau) + \tfrac{1}{\beta}a_2u_2(\tau) - \tfrac{1}{\beta}a_3u_3(\tau)\right)\, d\tau. \end{aligned}$$

Next we estimate the size of the perturbations by

$$\int d\tau_2 \, \frac{\exp(-\frac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_2)))}{1+\tau_2^2}(g-g_0)(\tau_2)$$
$$= \tfrac{1}{\beta}\int d\tau_2 \int d\tau_1 \, \exp\Big(-\tfrac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_1)) - \kappa_{k_0}(\tau_2-\tau_1+\tfrac{1}{3}(\tau_2^3-\tau_1^3))\Big)$$
$$\cdot \tfrac{1+\tau_2^2}{(1+\tau_1^2)^2}\Big(\tfrac{4\tau_1+\frac{1}{\beta}}{1+\tau_1^2}u_1(\tau_1)+\tfrac{1}{\beta}a_2u_2(\tau_1)-\tfrac{1}{\beta}a_3u_3(\tau_1)\Big)$$
$$\le \tfrac{2}{\beta\kappa_{k_0}}\int d\tau_1 \, \exp(-\tfrac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_1)))$$
$$\cdot \tfrac{1}{(1+\tau_1^2)^2}\Big(\tfrac{4\tau_1+\frac{1}{\beta}}{1+\tau_1^2}u_1(\tau_1)+\tfrac{1}{\beta}a_2u_2(\tau_1)-\tfrac{1}{\beta}a_3u_3(\tau_1)\Big)$$
$$\le \tfrac{2}{\kappa_{k_0}\beta}(2|u_1|_{L_s^\infty}+\tfrac{4c}{\eta}(|u_2|_{L_s^\infty}+|u_3|_{L_s^\infty}))$$

and

$$\int \frac{\exp(-\frac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_2)))}{1+\tau_2^2}g_0(\tau_2)\, d\tau_2$$
$$= c^4\exp(-\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d)))g(-d)$$
$$\cdot\int(1+\tau_2^2)\exp(-\kappa_{k_0}(\tau_2+d+\tfrac{1}{3}(\tau_2^3+d^3)))g_0(-d)$$
$$= \tfrac{c^4}{\kappa_{k_0}}\exp(-\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d)))\tfrac{3}{\beta}g(-d) \le \tfrac{3}{\beta}\tfrac{c^4}{\kappa_{k_0}},$$

where we used (3.33) to obtain $g(-d) \le \frac{3}{\beta}$. To estimate u_1 we look at the difference to the homogeneous system,

$$\partial_s(u_1-\tilde{u}_1) = -\tfrac{1}{\beta}\tfrac{1}{1+s^2}(u_1-\tilde{u}_1) - a_1u_2 + a_3u_3 - \tfrac{1}{1+s^2}g$$

which leads after integrating to

$$|u_1-\tilde{u}_1| \le \tfrac{4}{\eta}(|u_2|_{L_s^\infty}+|u_3|_{L_s^\infty}) + \tfrac{1}{\kappa_{k_0}}(\tfrac{2}{\beta}|u_1|_{L_s^\infty}+\tfrac{4c}{\eta\beta}(|u_2|_{L_s^\infty}+|u_3|_{L_s^\infty})) + \tfrac{3}{\beta}\tfrac{c^4}{\kappa_{k_0}}$$
$$\le 8c\min(\beta,\pi) + 4(2\pi+1)cc_1 + \tfrac{4}{\kappa_{k_0}\beta}(1+c_1) + \tfrac{5}{\eta}6(c\eta)^{\gamma_2} < c_1$$

since $\kappa k_0^2 \ge \frac{1}{\beta c}$. We estimate $u_2-\tilde{u}_2$ by

$$\partial_s(u_2-\tilde{u}_2) = 2c\eta\tfrac{1}{1+s^2}(u_1-\tilde{u}_1) + a(k_{\pm 2})w(k_{\pm 2}) - j(k_{\pm 1})$$

which implies by integrating in s, that

$$|u_2-\tilde{u}_2| \le 2\pi c\eta c_1 + \tfrac{2}{\eta}|w(k_{\pm 2})|_{L_s^\infty} + \int j(k_{\pm 1})$$
$$\le 2\pi c\eta c_1 + (12+10\pi\tfrac{1}{\beta})c^{\gamma_2}\eta^{-\gamma_1}$$
$$< (2\pi+1)c\eta c_1.$$

Next we estimate $w(k_n)$ for $|n| \geq 3$. We remark that the estimates for u_3 and $w_{k_{\pm 2}}$ are similar and hence we omit them. By integrating over the derivative we deduce

$$
\begin{aligned}
w(k_n, -d) &\leq w(k_n, d) + \tfrac{2}{\eta}(|w(k_{n+1})|_{L_s^\infty} + |w(k_{n-1})|_{L_s^\infty}) + \int j(k_n) \\
&< 6(c\eta)^{\gamma_2}.
\end{aligned}
$$

So the bootstrap is concluded. It is left to estimate $j(k_0)$. We write

$$
\begin{aligned}
\partial_s j(k_0) &= \tfrac{\kappa_{k_0}}{\beta} u_1 + (\tfrac{2s}{1+s^2} - \kappa_{k_0}(1+s^2)) j(k_0) \\
&\leq \tfrac{\kappa_{k_0}}{\beta} u_1 - \tfrac{8}{9}\kappa_{k_0} j(k_0)
\end{aligned}
$$

where in the second line we used (3.32). By integrating, we obtain

$$
\begin{aligned}
j(k_0, s) \leq &\exp(-\tfrac{8}{9}\kappa_{k_0}(s+d)) j(k_0, -d) \\
&+ \tfrac{\kappa_{k_0}}{\beta} \int_{-d}^{s} d\tau \exp(-\tfrac{8}{9}\kappa_{k_0}(s-\tau)) u_1(\tau)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
j(k_0, d) &\leq \exp(-2d\tfrac{8}{9}\kappa_{k_0}) j(k_0, -d) \\
&\quad + \tfrac{9}{8}\tfrac{1}{\beta}|u_1|_{L_s^\infty} \\
&\quad \leq \tfrac{2}{\beta}.
\end{aligned}
$$

□

3.4.5 Proof of Theorem 3.14

In Subsections 3.4.2, 3.4.3 and 3.4.4 we proved lower and upper bounds until the time $s = d$. Furthermore, in Subsection 3.4.2 we already showed the asymptotic behavior on the interval I_3. In this subsection we need to combine the results of these subsections to obtain the final lower and upper bounds for the complete interval I^k. This will be achieved in two steps: first we conclude the bootstrap on I_3, afterwards we show that all terms result in the desired estimates.

Proof of Theorem 3.14. Following we proceed similarly as in the proof of Proposition 3.17, just for I_3. In particular we use the tools from Subsection 3.4.2. We thus need to prove the missing estimate on $[d, s_1]$. Let $r_i(d)$ be the initial data of $r(s)$. We define the c_i terms by

$$
\begin{aligned}
c_1 &= 2(r_1(d) + (21c^2 + 2\tfrac{c^3}{\beta}(c\eta)^\gamma) r_2(s_0) + N + N_j) \\
\tilde{c}_1 &= 0 \\
c_2 &= \tfrac{1}{1-2\frac{c}{\beta}}(r_2(d) + N + N_j) \\
\tilde{c}_2 &= 22c_1 + \tfrac{c}{\beta}\tilde{c}_2.
\end{aligned}
\tag{3.39}
$$

and

$$
\begin{aligned}
N &= 2c\tfrac{1}{\kappa\xi\eta^{\gamma_2}} j(k_0 \pm 1, d) + 2cu_3(d) \\
&\quad + 2\sum_{|m|\geq 2} (2c)^{|m|}(w + \tfrac{4}{\kappa\eta\xi} j)(k_m, d) \\
N_j &= 4(c\eta)^{\gamma_2} \tfrac{c^2}{\kappa_{k_0}} j(k_0, d).
\end{aligned}
$$

We prove by bootstrap that

$$
|u|(s) \leq S^*(s)C(s). \tag{3.40}
$$

Since $c_i \geq r_i(d)$ this estimate holds locally, and we again let s^* be the maximal time such that (3.40) holds. We assume that $s^* \leq s_1$ and improve the estimate, which gives a contradiction and thus proves that (3.40) holds on $[d, s_1]$. For the R_i we obtain with the Lemmas 3.20, 3.21, 3.22 that

$$
\begin{aligned}
R_1[F_{all}] &= R_1[F_{3mode}] + R_1[F_j] + R_1[F_{\tilde{w}}] + R_1[F_{j(k_0\pm 1)}] + R_1[F_{u_3}] \\
&\leq 20c^2 c_1 + 20c^4 \tilde{c}_2 + (20c^2 + c^4(c\eta)^\gamma)c_2 \\
&+ \tfrac{c^3}{\beta}(c_1 + \tilde{c}_2) + \tfrac{c^3}{\beta}(c\eta)^\gamma c_2 + (c\eta)^{\gamma_1} \tfrac{4c^4}{\kappa_{k_0}} j(k_0, d) \\
&+ 2c^2(\tilde{w}(2) + \tilde{w}(-2)) \\
&+ \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, d) + \tfrac{2c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*) \\
&+ 2c\tilde{w}(1) \\
&\leq 21c^2 c_1 + 2\tfrac{c^3}{\beta}\tilde{c}_2 + (21c^2 + \tfrac{c^3}{\beta}(c\eta)^\gamma)c_2 + N + c^2(c\eta)^\gamma N_j,
\end{aligned}
$$

and

$$
\begin{aligned}
R_2[F_{all}] &= R_2[F_{3mode}] + R_2[F_j] + R_2[F_{\tilde{w}}] + R_2[F_{j(k_0\pm 1)}] + R_2[F_{u_3}] \\
&\leq 20(\tfrac{s}{\eta})^\gamma (c_1 + 2c^2\tilde{c}_2) + 20c^2 c_2 \\
&+ \tfrac{c}{\beta}(\tfrac{s}{\eta})^\gamma (c_1 + \tilde{c}_2) + \tfrac{c}{\beta} c_2 + (c\eta)^{\gamma_2} \tfrac{c^2}{\kappa_{k_0}} j(k_0, d)) \\
&+ c^2(\tilde{w}(2) + \tilde{w}(-2))(\tfrac{s}{\eta})^{\gamma_1} \\
&+ \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, d)(\tfrac{s}{\eta})^{\gamma_1} + \tfrac{c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*)(\tfrac{s}{\eta})^{\gamma_1} \\
&+ 2c\tilde{w}(1)(\tfrac{s}{\eta})^{\gamma_1} \\
&\leq 21(\tfrac{s}{\eta})^\gamma c_1 + 2\tfrac{c}{\beta} c_2 + \tfrac{c}{\beta}(\tfrac{s}{\eta})^\gamma \tilde{c}_2 + N(\tfrac{s}{\eta})^\gamma + N_j.
\end{aligned}
$$

Therefore, we deduce that

$$
\begin{aligned}
r_1(s_0) + R_1[all][1] &\leq r_1(d) + 21c^2(c_1 + \tilde{c}_2) + (21c^2 + \tfrac{c^3}{\beta}(c\eta)^\gamma)c_2 \\
&\quad + N + c^2(c\eta)^\gamma N_j < c_1, \\
r_2(s_0) + R_2[all][1] &\leq r_2(d) + 2\tfrac{c}{\beta} c_2 + N + N_j < c_2, \\
R_2[all][(\tfrac{s}{\eta})^\gamma] &< 20c_1 + 2c^2\tilde{c}_2 < \tilde{c}_2.
\end{aligned}
$$

This concludes the bootstrap and we estimated $|u|(s) \leq S^*(s)C(s)$ for $s \leq s_1$. To finish the proof of the theorem we need to establish the norm estimate at the final time. With Proposition 3.23 we obtain the folloiwng bounds:

$$
\begin{aligned}
|u_1|(d) &\leq 3(c\eta)^{-\gamma_2} LM, \\
|u_2|(d) &\leq 7\pi (c\eta)^{\gamma_1} LM, \\
|u_3|(d) &\leq 7\pi (\tfrac{5}{\eta})^2 (c\eta)^{\gamma_1} LM + 2M_1, \\
|w(k_n, d)| &\leq 7\pi (\tfrac{5}{\eta})^{|n|-1} (c\eta)^{\gamma_1} LM + 2M_n, \\
j(k_n, d) &\leq \tfrac{4}{\eta^2 \beta} (7\pi (\tfrac{4}{\eta})^{|n|-1} (c\eta)^{\gamma_1} LM + 2M_n), \\
j(k_0, d) &\leq \tfrac{4LM}{\beta} \min(\kappa_{k_0} \pi d^2, 1)(c\eta)^{-\gamma_2}.
\end{aligned}
$$

This in turn yields

$$
\begin{aligned}
N &= 2c \tfrac{1}{\kappa \xi \eta^{\gamma_2}} j(k_0 \pm 1, d) + 2c u_3(d) \\
&\quad + 2 \sum_{|m| \geq 2} (2c)^{|m-k_0|} (w + \tfrac{8}{\kappa \eta \xi n^2} j)(k_m, d) \\
&\leq c(c\eta)^{\gamma} LM, \\
N_j &\leq 4(c\eta)^{\gamma_2} \tfrac{c^2}{\kappa_{k_0}} \tfrac{4LM}{\beta} \min(\kappa_{k_0} \pi d^2, 1)(c\eta)^{-\gamma_2} \\
&\leq \tfrac{16}{\beta} \min(\pi, \tfrac{c^2}{\kappa_{k_0}}) LM.
\end{aligned}
$$

Using these bounds, we consider Afterwards, we estimate

$$
\begin{aligned}
r(d) &= S^{-1}(d) u(d) \\
&= -2c\gamma^{-1} \begin{pmatrix} -\frac{\gamma_2}{2c} |c\eta|^{-\gamma_2+1} & -|c\eta|^{-\gamma_2} \\ \frac{\gamma_1}{2c} |c\eta|^{-\gamma_1+1} & |c\eta|^{-\gamma_1} \end{pmatrix} u(d) \\
\begin{pmatrix} |r_1| \\ |r_2| \end{pmatrix} (d) &\leq LM \begin{pmatrix} 15\pi c (c\eta)^{\gamma} \\ 4 \end{pmatrix}
\end{aligned}
$$

and hence deduce that

$$
\begin{aligned}
c_1 &= (c\eta)^{\gamma} (30\pi c + 30 \tfrac{c^2}{\beta} + c) LM \\
&\leq 31\pi c (c\eta)^{\gamma} LM \\
c_2 &= (5 + \tfrac{16\pi}{\beta}) LM.
\end{aligned}
$$

This implies the estimate

$$
\begin{aligned}
u(s_1) &\leq S^*(s_1) C(s_1) \\
&\leq LM \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2c} & 2c \end{pmatrix} \begin{pmatrix} 31\pi c (c\eta)^{\gamma} \\ (5 + \frac{16\pi}{\beta}) \end{pmatrix} \\
&\leq LM (c\eta)^{\gamma} \begin{pmatrix} 16c + (5 + \frac{16\pi}{\beta})(c\eta)^{-\gamma} \\ 16\pi \end{pmatrix},
\end{aligned}
$$

where we used that $(c\eta)^{-\gamma}\frac{1}{\kappa_{k_0}} = (\frac{k_0^2}{c\xi})^\gamma \frac{1}{\kappa k_0^2} = \frac{1}{(c\xi)^\gamma \kappa k_0^{2\gamma_2}} \ll \beta c$. For $\tilde{w}(n)$ we obtain

$$
\begin{aligned}
\tilde{w}(n) &= 2\sum_{|m|\geq 2}(2c)^{|m-n|+\chi}(w+\tfrac{4}{\kappa\eta\xi}j)(k_m,d)\\
&\quad+(2c)^{||n|-2|}c(c_1^*+\tfrac{1}{c^2}c_2^*)\\
&\quad+(2c)^{|c|-1}u_3(d)\\
&\quad\leq L(2c)^{|n|}M+M_n\\
u_3(n) &\leq L(2c)^{|n|+2}M+M_1.
\end{aligned}
$$

Furthermore, by integrating over $\partial_s j(k_n)$ we obtain

$$
\begin{aligned}
j(k_n,s_1) &\leq L\tfrac{5}{\kappa\xi\eta}((2c)^{|n|}M+M_n),\\
j(k_{\pm 1},s_1) &\leq L\tfrac{5}{\kappa\xi\eta}((2c)^{|n|+2}M+M_1).
\end{aligned}
$$

In order to estimate $j(k_0)$ we use Lemma 3.21:

$$
\begin{aligned}
|j(k_0,s_1)| &\leq Lc^2\eta^2\exp(-\kappa_{k_0}\eta^3)j(k_0,d)\\
&\quad+2\tfrac{16^2}{\beta}\tfrac{1}{\eta^2}(c_1+c_2+\tilde{c}_1+\tilde{c}_2)\\
&\leq 3\pi\tfrac{\kappa_{k_0}}{\beta}\eta^2\exp(-\kappa_{k_0}\eta^3)(\tfrac{d}{\eta})^{\gamma_2}M\\
&\quad+L4\tfrac{16^2}{\beta}\tfrac{1}{\eta^2}2M(c\eta)^{-\gamma_2}\\
&\leq \tfrac{2^{11}}{\beta}\tfrac{1}{\eta^2}M(c\eta)^{-\gamma_2}.
\end{aligned}
$$

We further estimate

$$
\begin{aligned}
M^2 &= \sum_{m,n\geq 1}10^{-m-n}(w+\tfrac{1}{\alpha_{k_n}}j)(k_n)(w+\tfrac{1}{\alpha_{k_m}}j)(k_m)\\
&\leq \tfrac{2}{1-10^{-1}}\sum_{n\geq 1}10^{-n}(w^2+\tfrac{1}{\alpha_{k_n}^2}j^2)(k_n)\\
&\leq \tfrac{2}{1-10^{-1}}\tfrac{1}{\lambda_{k_0}}\sum_{n\geq 1}(10^{-n}\tfrac{\lambda_{k_0}}{\lambda_{k_n}})\lambda_{k_n}(w^2+\tfrac{1}{\alpha_{k_n}^2}j^2)(k_n)\\
&\leq \tfrac{2}{1-10^{-1}}\tfrac{1}{\lambda_{k_0}}\|w,j\|_X(s_0)^2\\
\sum_{|n|\geq 1}\lambda_{k_n}M_n^2 &= \sum_n \lambda_{k_n}\sum_{m,l}10^{-|m-l|-|l-n|-\chi_l-\chi_m}(w+\tfrac{1}{\alpha_{k_m}}j)^2(k_m)\\
&\leq \tfrac{2}{1-10^{-1}}\sum_n\lambda_{k_n}\sum_m 10^{-|m-n|-\chi_m}(w^2+\tfrac{1}{\alpha_{k_m}^2}j^2)(k_m)\\
&\leq \tfrac{2}{1-10^{-1}}\sum_m\lambda_{k_m}(w+\tfrac{1}{\alpha_{k_m}}j)^2(k_m)\sum_n 10^{-|m-n|-\chi_m}\tfrac{\lambda_{k_n}}{\lambda_{k_m}}\\
&\leq \tfrac{2\hat{\lambda}^2}{1-10^{-1}}\|w,j\|_X^2.
\end{aligned}
$$

Combining these bounds we infer the norm estimate

$$\begin{aligned}\|w,j\|_X^2(s_1) &\leq 16\pi L^2 M^2 (c\eta)^{2\gamma}(\lambda_{k_{\pm1}} + \lambda_{k_0}(16\pi + 5(c\eta)^{-2\gamma})^2)\\ &+ \sum_{|n|\geq 1} L^2 \lambda_{k_n}(10^{-|n|}M + M_n)^2\\ &= M^2(c\eta)^{-2\gamma}(\lambda_{k_{\pm1}^2}(2c)^2 + \lambda_{k_0}^2 + 2\sum_{|n|\geq 1}\lambda_{k_n}^2 10^{-2|n|} + L^2\sum_{|n|\geq 1}\lambda_{k_n}M_n^2\\ &\leq L^2(\hat\lambda(16\pi)^2 + 3\hat\lambda^2)(c\eta)^{2\gamma}\|w,j\|_X^2(s_0).\end{aligned}$$

This finally allows us to complete the proof of the upper bound and obtain that

$$\|w,j\|_X(s_1) \leq 18\pi L\hat\lambda(c\eta)^\gamma \|w,j\|_X(s_0).$$

To prove the lower bound we use Proposition 3.17 and Proposition 3.25 and obtain that at time $s = d$ it holds that

$$\begin{aligned}|u_1(d) - \exp(-\tfrac{\pi}{\beta})(c\eta)^{\gamma_2}| &= O(c)\\ |u_2(d) - 2\beta(1-\exp(-\tfrac{\pi}{\beta}))(c\eta)^{\gamma_1}| &\leq O(c)\\ w(k_n,d), u_3(d) &\leq 6\\ j(k_n,d) &\leq \tfrac{10\pi}{\beta\eta}\tfrac{1}{\eta}\\ j(k_0,d) &\leq \tfrac{2}{\beta}.\end{aligned}$$

We calculate $\tilde u_2$ by

$$\begin{aligned}\tilde u_2(s_1) &= (0\ 1)S(s_1)S^{-1}(d)u(d)\\ &\approx (\tfrac{1}{2c}\ 2c)2c\begin{pmatrix} -c(c\eta)^{\gamma_1} & -(c\eta)^{-\gamma_2}\\ \tfrac{1}{2c}(c\eta)^{\gamma_2} & (c\eta)^{-\gamma_1}\end{pmatrix}u(d)\\ &\approx (-c(c\eta)^{\gamma_1} + 2c(c\eta)^{\gamma_2})u_1(d) + (-(c\eta)^{-\gamma_2} + 4c^2(c\eta)^{-\gamma_1})u_2(d)\\ &\approx c(c\eta)^{\gamma_1}u_1(d) + (c\eta)^{\gamma_2}u_2(d)\\ &\approx 2(c\eta)^{\gamma_1}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(-d)\\ &\approx 2(c\eta)^{\gamma}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(s_0).\end{aligned}$$

The difference $u_2 - \tilde u_2$ is estimated by

$$\begin{aligned}|u_2 - \tilde u_2| &\leq (0\ 1)S^*(s_1)R[F]\\ &\leq \tfrac{1}{2c}R_1[F] + 2cR_2[F]\\ &\leq (c\eta)^{\gamma_2}u_2(d) + O(c)\\ &= 2(c\eta)^{\gamma}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(s_0) + O(c).\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}M &\leq \tfrac{1}{1-10^{-1}}u_1(\tilde s_0)\\ M_n &\leq \tfrac{4}{1-10^{-1}}u_1(\tilde s_0).\end{aligned}$$

So we finally obtain since $\beta \geq \frac{1}{5}$

$$\begin{aligned} w(k_{-1}, t_{k_{-1}}) &\approx 2(c\eta)^{\gamma} \beta (1 - \exp(-\tfrac{\pi}{\beta})) u_1(s_0) \\ &\geq \tfrac{1}{2} \max_l (w(k_l, t_{k_l}), \end{aligned}$$

which gives

$$w(k_{-1}, t_{k_{-1}}) \geq (c\eta)^{\gamma} \min(\beta, \pi) w(k_{-1}, t_{k_{-1}}).$$

□

In this article we have studied the asymptotic (in)stability of the magneto-hydrodynamic equations with a shear, a constant magnetic field and magnetic dissipation. Here multiple effects compete to determine the long time behavior of solutions:

- Echoes in the inviscid fluid equations may lead to large norm inflation.
- The underlying magnetic field leads to an exchange between kinetic and magnetic energy. In particular, for large magnetic fields oscillation my diminish norm inflation.
- Magnetic dissipation may stabilize the flow. Hence, a priori, it is not clear whether stability requires Gevrey regularity (as for the Euler equations) or Sobolev regularity (as for the fully dissipative problem) and how the evolution depends on the size of the magnetic field α and on the resistivity κ.

As the main result of this article we show that the balance between these effects is parametrized by the parameter $\beta = \frac{\kappa}{\alpha^2} > 0$ and that the behavior for finite, positive β strongly differs from both the fully non-dissipative case and the large dissipation limit (which reduces to the Euler equations). In particular, we show that in this regime the magnetic dissipation is not strong enough to stabilize the evolution in Sobolev regularity and establish Gevrey regularity as optimal both in terms of upper and lower bounds. It remains an interesting problem for future research to determine the optimal stability classes for other partial dissipation regimes and to study the inviscid limit $\kappa \downarrow 0$.

3.5 Estimating the Growth Factor

In Section 3.4.3 we observe the evolution of (3.25) on the interval $I_2 = [-d, d]$. Here we observe the interaction between j and u_1

$$\begin{aligned} \partial_s u_1 &= -j \\ \partial_s j &= \tfrac{K}{\beta} u_1 + (\tfrac{2s}{1+s^2} - K(1+s^2)) j, \end{aligned} \tag{3.41}$$

with κ_k replaced by K for simplicity In particular we bound the growth of u_1 by a factor. Let $U(\tau, s)$ be the solution of (3.41) with initial data $u_1(\tau) = 1$ and $j(\tau) = 0$. We show that

- $|U(\tau, s)| \leq 1$ for $\beta \geq \frac{\pi}{2}$
- $|U(\tau, s)| \leq L = L(\beta, K)$ for $\beta < \frac{\pi}{2}$.

With the restriction

$$c \leq (8\pi)^{\frac{4}{3}} \beta^{\frac{16}{3}}. \tag{3.42}$$

we obtain

$$L(\beta, K) = \begin{cases} 1 & 1 \leq K \\ \sqrt{d} & \frac{1}{2}c^{\frac{3}{4}} \leq K \leq 1 \\ 2(1 + \frac{\pi}{\beta}) & \frac{2\pi}{\beta}c^3 \leq K \leq \frac{1}{2}c^{\frac{3}{4}} \\ 1 & K \leq \frac{2\pi}{\beta}c^3 \end{cases} \tag{3.43}$$

We note that (3.42) is not optimal, in the sense that Section 3.4.3 we need $Lc << 1$ and we could optimize the $\frac{1}{2}c^{\frac{3}{4}}$ term to obtain a larger L but better (3.42). However, this would yield a lot dependencies which would make the final theorem more technical to state. The most important part of this estimates is to verify that β can be very small if c is chosen small enough. First we do an energy estimate, let

$$E = u_1^2 + \frac{\beta}{K} j$$

which leads to

$$\frac{1}{2}\partial_s E \leq \left(\frac{2s}{1+s^2} - K(1+s^2)\right)_+ E.$$

Therefore, we obtain for $K \geq 1$ that $\partial_s E \leq 0$, which proves our first estimate. Furthermore, we infer for $K \leq 1$

$$E(s) \leq E(\tau) \begin{cases} 1 & s \leq 0 \\ (1+s^2)^2 & 0 \leq s \leq (\frac{K}{2})^{\frac{-}{1}3} \\ 4(K)^{-\frac{4}{3}} & (\frac{K}{2})^{-\frac{1}{3}} \leq s, \end{cases}$$

We conclude

$$u_1(s) \leq \begin{cases} 1 & s \leq 0 \\ 1 + s^2 & 0 \leq s \leq (\frac{K}{2})^{\frac{1}{3}} \\ 2(K)^{-\frac{2}{3}} & (\frac{K}{2})^{\frac{1}{3}} \leq s \end{cases}$$

which proves (3.43) for $\frac{1}{2}c^{\frac{3}{4}} \leq K \leq 1$. For small K we need to make a different ansatz. We write j as,

$$j(s) = \frac{K}{\beta} \int_{-d}^{s} \frac{1+s^2}{1+\tau^2} \exp(-K(s - \tau + \tfrac{1}{3}(s^3 - \tau^3)))u(\tau)\, d\tau$$

and so

$$u(s)-1 = -\tfrac{K}{\beta}\iint_{-d\le\tau_1\le\tau_2\le s} d(\tau_1,\tau_2)\tfrac{1+\tau_2^2}{1+\tau_1^2}\exp(-K(\tau_2-\tau_1+\tfrac{1}{3}(\tau_2^3-\tau_1^3)))u(\tau_1)$$
$$= -\tfrac{1}{\beta}\int_{-d\le\tau_1\le s} d\tau_1\ u(\tau_1)\tfrac{1}{1+\tau_1^2}[\exp(-K(\tau_2-\tau_1+\tfrac{1}{3}(\tau_2^3-\tau_1^3)))]_{\tau_2=\tau_1}^{\tau_2=s}$$
$$= -\tfrac{1}{\beta}\int_{-d\le\tau_1\le s} d\tau_1\ u(\tau_1)\tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3)))).$$

Now we exploit that u is decreasing till the smallest time such that $u(s)=0$. This holds, since if u is positive, then j is positive and so $\partial_s u = -j \le 0$. Therefore, we bound

$$\tfrac{1}{\beta}\int_{-d\le\tau_1\le s} d\tau_1\ u(\tau_1)\tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3))))$$

by 1 to deduce $0\le u(s)\le 1$. Let s be positive, then we estimate

$$\tfrac{1}{\beta}\int_{-d\le\tau_1\le s} d\tau_1\tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3))))$$
$$= \tfrac{1}{\beta}\int_{-d\le\tau_1\le -s} d\tau_1\tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3))))$$
$$+ \tfrac{1}{\beta}\int_{-s\le\tau_1\le s} d\tau_1\tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3))))$$
$$\le \tfrac{1}{\beta s}+\tfrac{\pi}{\beta}(1-\exp(-K(2s+\tfrac{2}{3}s^3)))$$
$$\le \tfrac{1}{\beta s}+\tfrac{\pi}{\beta}K(2s+\tfrac{2}{3}s^3)$$
$$\le \tfrac{1}{\beta s}+\tfrac{\pi}{\beta}Ks^3.$$

This term is less than zero if $\frac{2}{\beta}\le s\le(\frac{\beta}{2\pi K})^{\frac{1}{3}}$. We choose $s=\min((\frac{\beta}{2\pi K})^{\frac{1}{3}},d)$ maximal. When $s=d$, then $(\frac{\beta}{2\pi K})^{\frac{1}{3}}\ge d$ which is satisfied if $K\le\frac{2\pi}{\beta}c^3$ and so we obtain the last estimate of (3.43). Now we need to prove the case if $\frac{2\pi}{\beta}c^3\le K\le\frac{1}{2}c^{\frac{3}{4}}$, with the previous calculation we obtain for $s_1=(\frac{\beta}{2\pi K})^{\frac{1}{3}}$, that $0\le u(s_1)\le 1$. Then for $s\ge s_1$ we have

$$u(s)-1 = \tfrac{1}{\beta}\int_{-d\le\tau_1\le s} d\tau\ u(\tau)\tfrac{1}{1+\tau^2}(1-\exp(-K(s-\tau+\tfrac{1}{3}(s^3-\tau^3))))$$
$$|u(s)-1| \le \tfrac{1}{\beta}\int_{-d\le\tau\le s_1} d\tau\ \tfrac{1}{1+\tau^2}+\tfrac{1}{\beta}\int_{t_1\le\tau_1\le s} d\tau\ u(\tau)\tfrac{1}{1+\tau^2}$$
$$\le \tfrac{\pi}{\beta}+\tfrac{1}{\beta s_1}|u|_{L_s^\infty}.$$

Due to $K\le\frac{1}{2}c^{\frac{4}{3}}$ and (3.42) we obtain $s_1\beta=(\frac{\beta^4}{2\pi K})^{\frac{1}{3}}\ge 2$ and so

$$|u(s)| \le \tfrac{1}{1-\frac{1}{\beta s_1}}(1+\tfrac{\pi}{\beta})$$
$$\le 2(1+\tfrac{\pi}{\beta}).$$

3.6 Nonlinear Instability of Waves

In this appendix we consider the nonlinear instability of the traveling waves.

$$\begin{aligned}\partial_t w + (v\nabla w)_{\neq} &= \alpha\partial_x j + (b\nabla j)_{\neq} - (2c\sin(x)\partial_y\Delta_t^{-1}w)_{\neq} \\ \partial_t j + (v\nabla j)_{\neq} &= \kappa\Delta_t j + \alpha\partial_x w + (b\nabla w)_{\neq} - 2\partial_x\partial_y^t\Delta_t^{-1}j - (2(\partial_i v\nabla)\partial_i\Delta^{-1}j)_{\neq},\end{aligned} \tag{3.44}$$

For brevity of notation let us denote the Gevrey 2 norm with constant C by

$$\|(w,j)\|_{\mathcal{G}_C}^2 = \int\sum_k \exp(C\sqrt{|\xi|})|\mathcal{F}(w,j)|^2 d\xi.$$

Then the norm inflation result of Theorem 3.3 further implies the nonlinear instability of any non-trivial traveling wave for C sufficiently small.

Corollary 3.26. *Let $0 < c < \min(10^{-4}, 10^{-3}\frac{\kappa}{\alpha^2})$ be given and consider a traveling wave as in Lemma 3.2 and let $0 < C_2 < C_*$ where $C_* = C_*(c)$ is as in Theorem 3.3. Then the nonlinear evolution equations around the traveling wave are unstable for small initial data in $\mathcal{G}_{C_2}$ in the sense that for any $0 < C_1 < C_2$, $\epsilon > 0$ and $N > 1$ there exists initial data with*

$$\|(w_0, j_0)\|_{\mathcal{G}_{C_2}} < \epsilon$$

but such that for some time $T > 0$ it holds that

$$\|(w,j)|_{t=T}\|_{\mathcal{G}_{C_2}} \geq N\|(w_0,j_0)\|_{\mathcal{G}_{C_1}}.$$

We stress that this results considers the instability of the traveling waves and that the space with respect to which instability is established depends on the size c of the wave. A nonlinear instability result for the underlying stationary state (3.2) in the spirit of [DM23, Bed20, DZ21] further requires that the size c of the traveling is comparable to ϵ.

Proof of Corollary 3.26. We argue by contradiction. Thus suppose that the nonlinear solution is uniformly controlled in $\mathcal{G}_{C_1}$ for all times:

$$\sup_{t>0}\|(w,j)\|_{\mathcal{G}_{C_1}} \leq D\epsilon.$$

for some constant $D > 0$. Given this a priori control of regularity we may consider the nonlinear equations as a forced linear problem

$$\partial_t(w,j) + L(w,j) = F$$

where L is the linear operator considered throughout this article and F is the quadratic nonlinearity. If we denote by $S(t,\tau)$ the solution operator associated to L it then follows that for any $T > 0$

$$(w,j)_{t=T} = S(T,0)(w_0,j_0) + \int_0^T S(T,\tau)F(\tau)d\tau.$$

By the norm inflation results of Theorem 3.3 for any $C_2 < C_*$ there exists initial data and a time $T > 0$ such that

$$\|S(T,0)(w_0, j_0)\|_{L^2} \geq N\|(w_0, j_0)\|_{\mathcal{G}_{C_2}}. \tag{3.45}$$

Since this estimate is linear after multiplication with a factor we may assume that this initial data also has size smaller than ϵ. On the other hand, by the results of Section 3.3 and of Theorem 3.3 for any fixed time T, $S(T,\tau)$ is uniformly bounded as a map from L^2 to L^2. More precisely, we recall that $S(T,\tau)$ decouples with respect to the frequency ξ in y.

- For ξ with $|\xi| \gg T^2$ by the results of Section 3.3 the time interval $(0,T)$ is considered "small time" and hence $S(T,\tau)$ is bounded uniformly.
- If instead $|\xi| \leq T^2$ then Theorem 3.3 provides an upper bound of the operator norm by $\exp(C\sqrt{\xi}) \leq \exp(CT)$.

Thus there exists an extremely large constant E (depending on T) such that

$$\| \int_0^T S(T,\tau)F(\tau)d\tau\|_{L^2} \leq E \int_0^T \|F(\tau)\|_{L^2} d\tau.$$

Finally, we note that by assumption

$$\|F(\tau)\|_{L^2} \leq D^2\epsilon^2.$$

Hence, choosing $\epsilon \ll \frac{1}{ED^2NT}$ the Duhamel integral can be treated as a perturbation of (3.45), which concludes the proof. □

Chapter 4

On the Sobolev Stability Threshold for the 2D MHD Equations with Horizontal Magnetic Dissipation

This chapter is the preprint [KZ2] and is a joint work with Christian Zillinger.

NIKLAS KNOBEL AND CHRISTIAN ZILLINGER

Abstract. In this chapter we consider the stability threshold of the 2D magnetohydrodynamics (MHD) equations near a combination of Couette flow and large constant magnetic field. We study the partial dissipation regime with full viscous and only horizontal magnetic dissipation. In particular, we show that this regime behaves qualitatively differently than both the fully dissipative and the non-resistive setting.

4.1 Introduction

The equations of magnetohydrodynamics (MHD)

$$\begin{aligned}\partial_t V + V \cdot \nabla V + \nabla \Pi &= (\nu_x \partial_x^2 + \nu_y \partial_y^2) V + B \cdot \nabla B,\\ \partial_t B + V \cdot \nabla B &= (\kappa_x \partial_x^2 + \kappa_y \partial_y^2) B + B \cdot \nabla V,\\ \nabla \cdot v = \nabla \cdot b &= 0,\\ (t, x, y) &\in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R},\end{aligned} \tag{4.1}$$

model the evolution of the velocity V of conducting, non-magnetic fluids interacting with a magnetic field B. The MHD equations are commonly used in applications ranging from astrophysics and the description of plasmas to control problems for liquid metals in industrial applications [Dav16]. Similarly to the Navier-Stokes and Euler equations, questions of hydrodynamic stability and the behavior for high Reynolds numbers (that is, for ν, κ tending to zero) are a very active area of research both inner-mathematically and in view of applications.

Motivated by stability results for the isotropic full-dissipation case ($\nu_x = \nu_y = \kappa_x = \kappa_y > 0$) and instability results for the non-resistive case ($\kappa_x = \kappa_y = 0$), we are interested in the behavior of the two-dimensional magnetohydrodynamic (MHD) equations with partial dissipation, where some of the dissipation coefficients

$$\kappa_y, \kappa_x, \nu_x, \nu_y \geq 0,$$

are allowed to vanish. More specifically, we study the behavior near the stationary solution given by the combination of Couette flow and a (large) constant magnetic field

$$V_s = ye_1, \quad B_s = \alpha e_1, \tag{4.2}$$

for the case of vanishing vertical resistivity, $\kappa_y = 0$. By the symmetry $B \mapsto -B$, we consider the case $\alpha \geq 0$. For the related case of the Navier-Stokes equations (that is, without any magnetic field) the (in)stability of Couette flow at high Reynolds number is known as the Sommerfeld paradox [MB01] and is related to nonlinear instability of the Euler equations [BM15a, DM23, DZ21].

However, for the case of sufficiently small data it was proven in [BVW18] that (mixing enhanced) dissipation can counteract this instability in the Navier-Stokes equations and that (long time asymptotic) stability holds in Sobolev spaces for initial data with

$$\|\omega\|_{H^N} \leq \epsilon \ll \nu^\gamma$$

with $\gamma \geq \frac{1}{2}$. Later in [MZ22] this has been improved to $\gamma = \frac{1}{3}$. This is an example of a stability threshold result, which establishes stability for small data and determines suitable (optimal) exponents γ for given norms.

Since the addition of the magnetic field is known to possibly destabilize the dynamics (see the following discussion), our main questions concern the MHD equations (4.1) in terms of perturbations moving with the underlying shear flow:

$$\begin{aligned} v(x, y, t) &= V(x - yt, y, t) - V_s, \\ b(x, y, t) &= B(x - yt, y, t) - B_s. \end{aligned}$$

The corresponding perturbed equations in these new variables read

$$\begin{aligned} \partial_t v + v_2 e_1 - 2\partial_x \Delta_t^{-1} \nabla_t v_2 &= \nu \cdot \Delta_t v + \alpha \partial_x b + b \nabla_t b - v \nabla_t v - \nabla_t \pi, \\ \partial_t b - b_2 e_1 &= \kappa \cdot \Delta_t b + \alpha \partial_x v + b \nabla_t v - v \nabla_t b, \\ \nabla_t \cdot v = \nabla_t \cdot b &= 0. \end{aligned} \tag{4.3}$$

Here, we introduce the time-dependent derivatives $\partial_y^t = \partial_y - t\partial_x$, $\nabla_t = (\partial_x, \partial_y^t)$ and $\Delta_t = \partial_x^2 + (\partial_y^t)^2$. Furthermore, we use the following short notation for the dissipation operator:

$$\nu \cdot \Delta_t = \nu_x \partial_x^2 + \nu_y (\partial_y^t)^2,$$
$$\kappa \cdot \Delta_t = \kappa_x \partial_x^2 + \kappa_y (\partial_y^t)^2.$$

In this article we aim to establish a Sobolev stability threshold for (4.3) for the specific anisotropic, partial dissipation case

$$\kappa_y = 0, \ \kappa_x = \nu_x = \nu_y > 0.$$

In particular, we show that this setting exhibits qualitatively different behavior than the fully dissipative and the non-resistive case.

Following a similar notation as [Lis20] we make the following definition.

Definition 4.1 (Stability threshold)**.** *Consider the MHD equations* (4.1) *with anisotropic dissipation* $0 < \nu_x = \nu_y = \kappa_x =: \mu \ll 1$ *and* $\kappa_y = 0$ *and let* X *be a Banach space with norm* $\|(v, b)\|_X$*. We then say that the exponent* $\gamma = \gamma(X)$ *is a stability threshold for the space* X *if for initial data with*

$$\|(v_{in}, b_{in})\|_X \leq \epsilon \ll \mu^\gamma,$$

the corresponding solution of (4.3) *remains uniformly bounded for all future times with a quantitative control*

$$\sup_{t>0} \|(v, b)\|_X \lesssim \epsilon.$$

We remark that this definition does not require optimality (that is, instability for smaller choices of γ). Optimal stability thresholds quantify the appearance of instability in the large Reynolds number limit and are an active area of research for many fluid systems. In view of the large literature, the interested reader is referred to the following articles for the Navier-Stokes equations [BVW18, BGM17] and the Boussinesq equations [ZZ23, LWX$^+$21, TWZZ20] for a discussion and further references.

For the (isotropic) MHD equations ($\nu := \nu_x = \nu_y$ and $\kappa := \kappa_x = \kappa_y$), there exists several results for non-vanishing magnetic dissipation.

- When considering full isotropic dissipation $\nu = \kappa > 0$, Liss [Lis20] established a Sobolev threshold in the 3D case. Under a Diophantine condition on the magnetic field, he establishes stability for $\|(v, b)\|_{H^N}$ with $\gamma = 1$. For the 2D case an improvement to $\gamma = \frac{2}{3}$ is expected due to the lack of lift-up instability. Indeed, in a very recent paper, [Dol24], Dolce establishes such a threshold for the regime $0 < C\kappa^3 \leq \nu \leq \kappa$.
- In the 2D inviscid case with isotropic magnetic dissipation, $\nu = 0$ and $\kappa > 0$, in [KZ1] the authors established linear instability of nearby (in analytic regularity) so-called traveling wave type solutions in Gevrey 2 regularity. As an (almost) matching nonlinear result, [ZZ24] established a stability threshold $\gamma \geq 1$ for Gevrey $2 - \delta$ regularity for any $0 < \delta < 1$.

- The setting with only an underlying magnetic field but without shear flow exhibits qualitatively different behavior and was studied for the case of the whole space in [BSS88, RWXZ14] in the full dissipation case and in [CRW13, JLWY19] for the partially dissipative case.

To the authors' knowledge there are no such results in the literature for the non-resistive case $\kappa = 0$ with Couette flow, both for the viscous or inviscid regime $\nu = 0$ or $\nu > 0$, and neither for partial dissipation regimes. In view of linear instability results [HHKL18] (see also Proposition 4.2), for these equations any stability threshold results would need to consider unknowns different from (v, b).

As a step towards understanding this non-resistive regime, in this article we consider the 2D MHD equations with isotropic viscosity but only horizontal resistivity (while [Lis20, Dol24] consider full dissipation). In particular, we ask to which extent, as quantified by Sobolev stability thresholds, this partial dissipation regime behaves or does not behave like these extremal cases.

In the (ideal) MHD equations ($\nu = \kappa = 0$) the interaction of shear flows and the magnetic field has been shown to possibly cause instabilities, with arguments both on physical [CM91, HTY05] and mathematical grounds [HT01, ZZZ21].

As our first result, we show that this instability also persists in the viscous but non-resistive MHD. These equations exhibit norm inflation in H^N for all choices of $\nu > 0$.

Proposition 4.2 (Instability for the non-resistive MHD equations)**.** *Consider the isotropic equation with $1 \geq \nu > 0$, $\kappa = 0$, $\alpha > \frac{1}{2}$ and $N \geq 3$, then the stationary solution (4.2) is linearly unstable in H^N. More precisely, there exists initial data $(v, b)_{in} \in H^N$ such that the solution to the linearized problem satisfies*

$$\|(v, b)\|_{H^N} \approx \langle \nu t \rangle \|(v, b)_{in}\|_{H^N}$$

as $t \to \infty$. This is optimal in the sense that for all initial data $(v, b)_{in} \in H^N$ the solution to the linearized problem satisfies

$$\|(v, b)\|_{H^N} \lesssim \langle \nu t \rangle \|(v, b)_{in}\|_{H^N}.$$

The implicit constants here may depend on α. As a consequence, the nonlinear equations also exhibit arbitrarily large norm inflation in H^N. That is, for any $C = C(\nu) > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there exists initial data $(v, b)_{in}$ and a time T such that

$$\begin{aligned} \|(v, b)_{in}\|_{H^N} &= \varepsilon, \\ \|(v, b)|_{t=T}\|_{H^N} &\geq C \|(v, b)_{in}\|_{H^N}. \end{aligned}$$

In particular, there cannot exist a Sobolev threshold for $\|(v, b)\|_{H^N}$.

We remark that following the same argument also instability in suitable Gevrey spaces can be established.

As mentioned above, the isotropic fully dissipative case is known to be stable in Sobolev regularity [Lis20, Dol24]. For the associated partial dissipation regimes, in view of the underlying shear dynamics the associated vertical dissipation case is expected to behave similarly as the full dissipation case. The effects of partial dissipation are a very actively studied field of research in other fluid systems, such as the Boussinesq equations [DWZ21, CW13, ABSPW22]), but, to the authors' knowledge, is largely open in the MHD equations near Couette flow.

In the present case of horizontal resistivity, $\kappa_y = 0$ and $\nu_x = \nu_y = \kappa_x$, the lack of vertical dissipation leads to stronger instabilities, requiring finer control and use of the coupling by a strong magnetic field. Our main results are summarized in the following theorem.

Theorem 4.3. *Consider the MHD equations with horizontal resistivity, $\mu := \nu_x = \nu_y = \kappa_x > 0$ and $\kappa_y = 0$, near the stationary solution (4.2) with $\alpha > \frac{1}{2}$ and let $N \geq 6$ be given.*

Then there exist constants $c_0 = c(\alpha) > 0$, such that for all initial data $(v, b)_{in}$ which satisfy

$$\|(v, b)_{in}\|_{H^N} = \varepsilon \leq c_0 \mu^{\frac{3}{2}}$$

the corresponding solution (v, b) of (4.4) satisfies the estimates

$$\|v\|_{L^\infty H^N} + \mu^{\frac{1}{2}} \|\nabla_t v\|_{L^2 H^N} \lesssim \varepsilon,$$
$$\|b\|_{L^\infty H^N} + \mu^{\frac{1}{2}} \|\partial_x b\|_{L^2 H^N} \lesssim \varepsilon.$$

Let us comment on these results:

- Proposition 4.2 shows instability in terms of (v, b) for the non-resistive case. Hence, the (horizontal) magnetic dissipation is shown to be necessary for long-time stability results for (v, b).

  However, similarly as in the Boussinesq equations [BBZD23, Zil23], in principle stability results in terms of other unknowns such as the magnetic potential $\phi = (-\Delta_t)^{-1} \nabla_t^\perp b$ could hold for longer or even infinite times, which remains an exciting question for future research.

- Theorem 4.3 establishes a stability threshold $\gamma = \frac{3}{2}$. In particular, we stress that the lack of vertical magnetic dissipation not only poses a key challenge of our analysis but results in a different threshold value than the fully dissipative setting [Lis20, Dol24].

  Indeed, the main constraint on our stability threshold is given by the control of the nonlinearity $v \cdot \nabla_t b$ and the reduced decay rates already at the linearized level (see Section 4.2). As we show in Section 4.3.3, our estimates of the so-called reaction terms (4.23) and (4.27) require a lower bound on the threshold by $\frac{3}{2}$ and are expected to be optimal for this partial dissipation case.

- Theorem 4.3 considers the case $\mu := \nu_x = \nu_y = \kappa_x$. As we discuss in Sections 4.2 and 4.3, we expect that instead of equality it suffices to require that $\frac{1}{2\alpha}\nu_y \leq \kappa_x \leq C\nu_y^{\frac{1}{3}}$, similarly as in the full dissipation case studied in [Dol24]. These constraints naturally arise in the linearized problem studied in Proposition 4.4. Furthermore, we expect that results can be extended to the case of purely vertical viscous dissipation with additional technical effort.

- Due to missing vertical dissipation, we obtain no decay of the x-averaged magnetic field $b_=$ which is forced by the nonlinearity.

To prove our results, it is convenient to work with the unknowns

$$p_1 = \Lambda_t^{-1}\nabla_t^\perp \cdot v,\ p_2 = \Lambda_t^{-1}\nabla_t^\perp \cdot b; \quad \Lambda_t := \sqrt{-\Delta_t}.$$

Similarly to the vorticity and current, the curl operator $\nabla_t^\perp$ eliminates the pressure and yields a scalar quantity, while the operator $\Lambda_t^{-1}\nabla_t^\perp \cdot$ is of order 0. Moreover, since v and b are divergence-free, similarly to viscosity formulations of the 2D Navier-Stokes equations, it can be shown by integration by parts that

$$\begin{aligned}\|Av\|_{L^2} &= \|Ap_1\|_{L^2},\\ \|Ab\|_{L^2} &= \|Ap_2\|_{L^2},\end{aligned}$$

for all Fourier multiplier A which commute with ∇_t and Λ_t. This, in particular, includes $\langle\nabla\rangle^N$ which corresponds to the Sobolev norm $\|\cdot\|_{H^N}$.

In terms of these unknowns our equations read

$$\begin{aligned}\partial_t p_1 - \partial_x\partial_x^t\Delta_t^{-1}p_1 - \alpha\partial_x p_2 &= \mu\Delta_t p_1 + \Lambda_t^{-1}\nabla_t^\perp(b\nabla_t b - v\nabla_t v),\\ \partial_t p_2 + \partial_x\partial_x^t\Delta_t^{-1}p_2 - \alpha\partial_x p_1 &= \mu\partial_x^2 p_2 + \Lambda_t^{-1}\nabla_t^\perp(b\nabla_t v - v\nabla_t b),\end{aligned} \tag{4.4}$$

The remainder of the article is structured as follows:

- In Section 4.2, as a first step we establish linear stability of the equations (4.4). In view of the lack of vertical resistivity we here crucially rely on the interaction of p_1 and p_2 due the the underlying constant magnetic field. Moreover, we discuss the effects of partial dissipation and the resulting limited (optimal) decay rates in time.

- In Section 4.3, we introduce a bootstrap method for the proof of Theorem 4.3. Decomposing into low and high frequency contributions here yields several error terms, which are handled in different subsections. In particular, we need to distinguish between the evolution of the x-average (which does not experience enhanced dissipation due to the shear) and its L^2-orthogonal complement, as well as different frequency decompositions of the nonlinear terms (called reaction and transport terms in the literature).

- More precisely, in Subsection 4.3.2 we collect all nonlinear terms which can be estimated in a straightforward way. In view of partial magnetic dissipation a main challenge is given by the effect of $v\nabla_t b$ on p_2 at high frequencies. Here, we distinguish between terms without x-average in Subsection 4.3.3 and with average in Subsection 4.3.4 and perform a decomposition into a transport and a reaction term. The low frequency regime is discussed in Subsection 4.3.5 and does not require a very precise analysis.

- As a complementary result, in Section 4.4 we establish instability of the non-resistive, viscous MHD equations and prove Proposition 4.2. Here we first prove linear algebraic instability and then deduce a nonlinear norm inflation result as a corollary.

4.1.1 Notations and conventions

For two real numbers $a, b \in \mathbb{R}$ we denote the minimum and maximum as

$$\min(a, b) = a \wedge b,$$
$$\max(a, b) = a \vee b.$$

We use the notation $f \lesssim g$ if there exists a constant C independent of all relevant parameters such that $|f| \leq C|g|$. Furthermore, we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

Moreover, for any vector or scalar v we define

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}.$$

For a function $f(x, y) \in L^2(\mathbb{T}\times\mathbb{R})$ we denote the x-average and its L^2-orthogonal complement as

$$f_{=}(y) = \int_{\mathbb{T}} f(x, y)dx,$$
$$f_{\neq} = f - f_0.$$

Throughout this text, unless noted otherwise, the spatial variables $(x, y) \in \mathbb{T}\times\mathbb{R}$ are periodic in the horizontal direction and the respective Fourier variables are denoted as

$$(k, \xi) \in (\mathbb{Z}, \mathbb{R})$$

or (l, η). The norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^N}$ refer to the standard Lebesgue and Sobolev norms for functions on $\mathbb{T} \times \mathbb{R}$. For time-dependent functions we denote $L^p H^s = L^p_t H^s$ as the space with the norm

$$\|f\|_{L^p H^s} = \left\| \|f\|_{H^s(\mathbb{T}\times\mathbb{R})} \right\|_{L^p(0,T)}.$$

We define the weight A^N and $A_\mu^{N'}$ by the Fourier multipliers

$$
\begin{aligned}
A^N &= M\langle\nabla\rangle^N, \\
A_\mu^{N'} &= M\langle\nabla\rangle^{N'} e^{c\mu t\mathbf{1}_{k\neq 0}},
\end{aligned}
$$

for $3 < N' \leq N-2$ and $0 < c < \frac{1}{2}(1-\sqrt{\frac{2}{3}})$. With slight abuse of notation we identify the multiplier operators with their Fourier symbols. The operator M is a time dependent Fourier multiplier, introduced in [BVW18], and is defined to satisfy the following equation:

$$
\begin{aligned}
-\tfrac{\dot{M}}{M} &= \tfrac{|k|}{k^2+|\xi-kt|^2}, \\
M(0,k,\xi) &= 1.
\end{aligned}
$$

That is, M is given as

$$
M(t,k,\xi) = \exp\left(-\int_0^t d\tau \tfrac{|k|}{k^2+(\xi-k\tau)^2}\right).
$$

In particular, the operator M is comparable to the identity in the sense that

$$
1 \geq M(t,k,\xi) \geq c
$$

for some constant c and all $k \neq 0$ (and $M(t,0,\xi) := 1$ for $k=0$).

The operators A thus define energies comparable to Sobolev (semi)norms:

$$
\begin{aligned}
\|A^N \cdot\|_{L^2} &\approx \|\cdot\|_{H^N}, \\
\|A_\mu^{N'} \cdot\|_{L^2} &\approx \|e^{c\mu t\mathbf{1}_{k\neq 0}} \cdot\|_{H^N}.
\end{aligned}
$$

In particular, since N is sufficiently large, the norm defined by A^N satisfies an algebra property.

4.2 Linear Stability

In this section we study the stability of the linearized version of the equations (4.4):

$$
\begin{aligned}
\partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \cdot \Delta_t p_1, \\
\partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \cdot \Delta_t p_2, \\
\nu = (\mu,\mu),\ \kappa &= (\mu, 0).
\end{aligned}
\tag{4.5}
$$

The ode tools to establish stability of such systems are well-known in related systems such as the Boussinesq equations [LWX+21, BBZD23, BZD20, MZZ23, Zil21b].

Our main results are summarized in the following proposition.

Proposition 4.4 (Linear stability)**.** *Let $\mu > 0$, $\alpha > \frac{1}{2}$ and $N \geq 6$ be as in Theorem 4.3. Then the equations* (4.5) *are stable in H^N in the sense that for any choice of initial data $p_{in} \in H^N$ the corresponding solution satisfies*

$$\|p\|_{L^\infty H^N} + \mu^{1/2}\|\nabla_t p_1\|_{L^2 H^N} + \mu^{1/2}\|\partial_x p_2\|_{L^2 H^N} \lesssim e^{-C\mu t}\|p_{in}\|_{H^N}.$$

As we discuss after the proof, in the case $\frac{1}{2\alpha}\nu \leq \kappa \leq \nu^{1/3}$ the optimal decay rate for large times is given by $\mu = \min(\nu^{1/3}, \kappa)$. In particular, the coupling induced by the underlying magnetic field cannot yield enhanced dissipation rates for both components once the viscous dissipation becomes too large.

Proof of Proposition 4.4. We note that in this linear evolution equation (4.5) all coefficient functions are independent of both x and y. Therefore the equations decouple after a Fourier transform and we may equivalently consider the ode system

$$\begin{aligned} &\partial_t \hat{p}_1 - \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\hat{p}_1 - \alpha i k \hat{p}_2 = -\nu(k^2+(\xi-kt)^2)\hat{p}_1, \\ &\partial_t \hat{p}_2 + \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\hat{p}_2 - \alpha i k \hat{p}_1 = -\kappa k^2 \hat{p}_2, \end{aligned} \tag{4.6}$$

for an arbitrary but fixed frequency $(k, \eta) \in \mathbb{Z} \times \mathbb{R}$. Since the equations are trivial for $k = 0$, in the following we further without loss of generality may assume that $k \neq 0$. Furthermore, after shifting t by $\frac{\xi}{k}$, we may assume that $\xi = 0$ and thus obtain a system of the form

$$\partial_t \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\frac{t}{1+t^2} - \nu k^2(1+t^2) & i\alpha k \\ i\alpha k & \frac{t}{1+t^2} - \kappa k^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{4.7}$$

where we dropped the hats for simplicity of notation.

In a first naive estimate, we can test this equations with (p_1, p_2) and obtain that

$$\partial_t(|p_1|^2 + |p_2|^2) \leq (\frac{|t|}{1+t^2} - \mu k^2)(|p_1|^2 + |p_2|^2),$$

which already yields the desired decay for times $|t| \gg (\mu k^2)^{-1}$. However, a Gronwall-type estimate on the remaining interval would only yield a very rough upper bound on the possible growth by

$$\exp\left(\int_{|t| \lesssim (\mu k^2)^{-1}} \frac{|t|}{1+t^2} dt\right) \lesssim (1 + (\mu k^2)^{-1})^2.$$

In order to improve this estimate, a common trick is to make use of the fact that $|\alpha|$ is relatively large and to consider

$$E = |p_1|^2 + |p_2|^2 - \frac{t}{1+t^2}\Re\left(\frac{1}{i\alpha k} p_1 \overline{p_2}\right).$$

Since $|\alpha| > \frac{1}{2}$ this energy is positive definite and comparable to $|p_1|^2 + |p_2|^2$, with a constant which degenerates as $|\alpha| \downarrow \frac{1}{2}$.

Computing the time derivative of the last term, we note that

$$\begin{aligned}
&\frac{t}{1+t^2}\partial_t \Re\left(\frac{1}{i\alpha k} p_1 \overline{p_2}\right) \\
&\leq \frac{t}{1+t^2}(|p_1|^2 - |p_2|^2) \\
&\quad + \frac{|t|}{1+t^2}\frac{1}{|\alpha|}\nu k(1+t^2)|p_1||p_2| \\
&\quad + \frac{|t|}{1+t^2}\frac{1}{|\alpha|}\kappa k|p_1||p_2| \\
&\quad + \mathcal{O}(t^{-2})|p_1||p_2|.
\end{aligned}$$

The first term exactly cancels out the possibly large contribution in $\partial_t(|p_1|^2 + |p_2|^2)$. For the second and third term we use the fact that $\frac{1}{\alpha} < 2$ and that these terms can hence be absorbed into the dissipation terms at the cost of a slight loss of constants, provided that

$$\frac{1}{2\alpha}\nu \leq \kappa.$$

We estimate

$$\begin{aligned}
|t|\frac{1}{|\alpha|}\nu k|p_1||p_2| &\leq \tfrac{2}{3}\nu k^2(1+t^2)|p_2|^2 + \tfrac{3}{8}\tfrac{1}{|\alpha|k}\nu k^2|p_2| \\
&\leq \tfrac{2}{3}\nu k^2(1+t^2)|p_2|^2 + \tfrac{3}{4}\kappa k^2|p_2|
\end{aligned}$$

and

$$\frac{|t|}{1+t^2}\frac{1}{|\alpha|}\kappa k|p_1||p_2| \leq \frac{8}{1+t^2}\frac{1}{|\alpha|^2}|p_1|^2 + \tfrac{1}{8}\kappa k^2|p_2|^2$$

Noting that $\partial_t \frac{|t|}{1+t^2} = \mathcal{O}(t^{-2})$ is integrable in time, we thus arrive at

$$\partial_t E \lesssim \mathcal{O}(t^{-2})E - \nu k^2(1+t^2)|p_1|^2 - \kappa k^2|p_2|^2.$$

Further defining

$$\tilde{E} = E\exp\left(\int^t \mathcal{O}(\tau^{-2})d\tau\right),$$

it follows that $\tilde{E} \approx E$ decays exponentially in time and that the damping terms are integrable in time, which yields the desired result. □

We further remark that for t (corresponding to times $t + \frac{\xi}{k}$) such that $|t| \lesssim |\alpha|(\mu k^2)^{-1/2}$ the system (4.7) exhibits fast damping in both components (due to the coupling by α). However, for times much larger than this (that is, far away from $\frac{\xi}{k}$) the decay is limited. Indeed, after relabeling $p_1 \mapsto ip_1$ and introducing the energy E to control contributions by $\frac{t}{1+t^2}$, this follows from the fact that the eigenvalues of the matrix

$$\begin{pmatrix} -\mu k^2(1+t^2) & -\alpha \\ \alpha & -\mu k^2 \end{pmatrix}$$

are given by

$$\lambda_{1,2} = -\frac{\mu k^2(2+t^2)}{2} \pm \sqrt{\frac{1}{4}(\mu k^2 t^2)^2 - \alpha^2}.$$

In the first case, the square root is purely imaginary and hence $\Re(\lambda_1) = \Re(\lambda_2)$ is comparable to the stronger dissipation term

$$-\mu k^2(1+t^2).$$

For large times, instead the same eigenvalue computation yields

$$\lambda_1 \approx -\mu k^2 \langle t \rangle^2, \ \lambda_2 \approx -\mu k^2$$

and hence the decay estimates of Proposition 4.4 are not improved to $\mu^{1/3}$. This linear result thus highlights the effects of the coupling induced by the underlying constant magnetic field and shows which optimal decay estimates can be expected. In particular, it clearly illustrates that the loss of vertical magnetic dissipation incurs a change of decay rate compared to the fully dissipative case.

4.3 Bootstrap Hypotheses and Outline of Proof

We next turn to the full nonlinear problem (4.4), where we intend to treat the nonlinear contributions as errors and make use of the smallness of our initial data.

Our approach here follows a bootstrap argument, which is by now standard in the field (see, for instance, [BVW18]). In the notation of Section 4.1.1 we assume that at the initial time

$$\|A^N p\|_{L^2}^2 + \|A_\mu^{N'} p\|_{L^2}^2 \leq c_0 \epsilon^2 \tag{4.8}$$

for $3 < N' \leq N-2$. The constant $c_0 = c_0(\alpha) > 0$ will later be chosen small enough and tends to 0 as $\alpha \to \frac{1}{2}$. Given this estimate at the initial time, our aim in the remainder of this section is to establish the following estimates for the corresponding solution:

- **High frequency estimates**

$$\begin{aligned} &\|A^N p_1\|_{L^\infty L^2}^2 + \mu \|A^N \nabla_t p_1\|_{L^2 L^2}^2 + \|\sqrt{-\tfrac{\dot M}{M}} A^N p_1\|_{L^2 L^2}^2 < \varepsilon^2, \\ &\|A^N p_2\|_{L^\infty L^2}^2 + \mu \|A^N \partial_x p_2\|_{L^2 L^2}^2 + \|\sqrt{-\tfrac{\dot M}{M}} A^N p_2\|_{L^2 L^2}^2 < \varepsilon^2. \end{aligned} \tag{4.9}$$

- **Low frequency estimates**

$$\begin{aligned}
&\|A_\mu^{N'} p_1\|_{L^\infty L^2}^2 + \mu \|A_\mu^{N'} \nabla_t p_1\|_{L^2L^2}^2 + \|\sqrt{-\tfrac{\dot M}{M}} A_\mu^{N'} p_1\|_{L^2L^2}^2 < \varepsilon^2, \\
&\|A_\mu^{N'} p_2\|_{L^\infty L^2}^2 + \mu \|A_\mu^{N'} \partial_x p_2\|_{L^2L^2}^2 + \|\sqrt{-\tfrac{\dot M}{M}} A_\mu^{N'} p_2\|_{L^2L^2}^2 < \varepsilon^2.
\end{aligned} \tag{4.10}$$

By local well-posedness and our assumptions on the initial data, these estimates are satisfied at least on some (small) time interval $(0, T)$. In our bootstrap approach we assume for the sake of contradiction that the maximal time T with this property is finite. We then show that on that same time interval all estimates hold with improved bounds instead, which however would imply that the estimates could be extended for a small additional time, contradicting the maximality of T.

With this understanding, we suppress T in our notation (see Section 4.1.1) and all L^p norms in time are understood to be norms on $L^p(0, T)$.

The splitting into high and low frequencies is essential to close the estimates in Subsection 4.3.3 and Subsection 4.3.4. In particular, we need the $e^{-c\mu t}$ decay to bound the so-called reaction error. Moreover, we require strong control of commutators involving A in order to control the so-called transport error. Both error terms impose strong restrictions on the energies and do not allow to close estimates in an easy way. We overcome this difficulty by linking separate energy estimates in the high frequency part and the low frequency part. On the one hand, we can use the additional $e^{-c\mu t}$ in the low frequency part to our benefit in the analysis of the high frequency part. On the other hand, the difference in regularity allows us to control derivatives in the low frequency estimate by the using high frequency estimate.

Given a solution (p_1, p_2) of (4.4) and letting $A = A^N, A_\mu^{N'}$, computing time derivatives we need to control

$$\begin{aligned}
&\partial_t \|Ap_1\|_{L^2}^2 + 2(1-c)\mu \|A\nabla_t p_1\|_{L^2}^2 + 2\|\sqrt{-\tfrac{\dot M}{M}} Ap_1\|_{L^2}^2 \\
&\leq 2\langle A^2 p_1, \partial_x \partial_x^t \Delta_t^{-1} p_1 + \Lambda_t^{-1} \nabla_t^\perp (b\nabla_t b - v\nabla_t v)\rangle =: L[p_1] + NL[p_1], \\
&\partial_t \|Ap_2\|_{L^2}^2 + 2(1-c)\mu \|A\partial_x p_2\|_{L^2}^2 + 2\|\sqrt{-\tfrac{\dot M}{M}} Ap_2\|_{L^2}^2 \\
&\leq 2\langle A^2 p_2, -\partial_x \partial_x^t \Delta_t^{-1} p_2 + \Lambda_t^{-1} \nabla_t^\perp (b\nabla_t v - v\nabla_t b)\rangle =: L[p_2] + NL[p_2].
\end{aligned}$$

Here we have split contributions into linear (that is, quadratic integrals) and nonlinear terms (that is, trilinear integrals). Note that the choice of $0 < c < \frac{1}{2}(1 - \sqrt{\frac{2}{3}})$ is made such that $1 - c$ is not too small to absorb linear effects for α close to $\frac{1}{2}$. For later reference, we note that the bootstrap assumptions (4.9) and (4.10) yield the following estimates:

$$\|\partial_x^2 \Lambda_t^{-1} \Lambda^{-1} p\|_{H^N} \lesssim \tfrac{1}{t} \|p_{\neq}\|_{H^N} \tag{4.11}$$

and for $A = A^N, A_\mu^{N'}$

$$\begin{aligned} \|Ap_{1,\neq}\|_{L^2L^2} &\lesssim \mu^{-\frac{1}{2}}\varepsilon, \\ \|Ap_{2,\neq}\|_{L^2L^2} &\lesssim \mu^{-\frac{1}{2}}\varepsilon. \end{aligned} \tag{4.12}$$

Furthermore, for the nonlinear terms we will often use the equality

$$\begin{aligned} \|Av\|_{L^2} &= \|Ap_1\|_{L^2}, \\ \|Ab\|_{L^2} &= \|Ap_2\|_{L^2}. \end{aligned}$$

Throughout the following sections, we aim to establish smallness of the contributions by the linear terms $L[\cdot]$ and nonlinear terms $NL[\cdot]$. More precisely, we establish the following proposition.

Proposition 4.5 (Control of errors)**.** *Under the assumptions of Theorem 4.3 suppose that the initial data satisfies the smallness condition* (4.8) *and let $T > 0$ be such the high and low frequency estimates* (4.9), (4.10) *are satisfied. Then on the same time interval it holds that*

$$\begin{aligned} \int_0^T L[p_1] + L[p_2] dt &\leq \tfrac{1}{2\alpha}(c_0 + 1)\varepsilon^2 + O(\mu^{-1}\varepsilon^3), \\ \int_0^T NL[p_1] + NL[p_2] dt &\leq \mu^{-\frac{3}{2}}\varepsilon^3. \end{aligned}$$

As a consequence, supposing that $\alpha > \frac{1}{2}$ and $\epsilon \ll \mu^{3/2}$, this implies that both the high frequency and low frequency estimates (4.9), (4.10) improve and thus T can only have been maximal if $T = \infty$, which proves Theorem 4.3. Thus proving Proposition 4.5 is our main concern in this section and our proof is split over the following subsections. The most important estimates, highlighting the effects of partial dissipation, are established in Subsections 4.3.1, 4.3.3 and 4.3.4.

We note that the nonlinear terms

$$\begin{aligned} \langle Ap_1, \Lambda_t^{-1}\nabla_t^\perp A(b\nabla_t b - v\nabla_t v)\rangle &= -\langle Av, A(b\nabla_t b - v\nabla_t v)\rangle, \\ \langle Ap_2, \Lambda_t^{-1}\nabla_t^\perp A(b\nabla_t v - v\nabla_t b)\rangle &= -\langle Ab, A(b\nabla_t v - v\nabla_t b)\rangle, \end{aligned}$$

for $A = A^N, A_\mu^{N'}$ are all trilinear products involving

$$a^1a^2a^3 \in \{vvv, vbb, bbv, bvb\}$$

and we will use this notation to refer to the specific terms. Since the x-averages do not experience fast (mixing enhanced) decay under the dissipation, we split these products as

$$\begin{aligned} \langle Aa^1, A(a^2\nabla_t a^3)\rangle = &\langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{\neq})_{\neq}\rangle \\ &+ \langle Aa^1_{\neq}, A(a^2_{=}\nabla_t a^3_{\neq})\rangle \\ &+ \langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{=})\rangle \\ &+ \langle Aa^1_{=}, A(a^2_{\neq}\nabla_t a^3_{\neq})_{=}, \end{aligned}$$

where the full splitting is only used for the bvb term.

4.3.1 Estimate of the linear error

In this subsection we establish the estimate of the linear terms in Proposition 4.5. Here, we use some of the same techniques as in the proof of linear stability in Section 4.2, but instead focus on establishing quantitative bounds on the time integral.

Taking a Fourier transform of (4.4) yields

$$\begin{aligned} &\partial_t \hat{p}_1 - \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\hat{p}_1 - \alpha i k \hat{p}_2 = -\mu(k^2+(\xi-kt)^2)\hat{p}_1 + \mathcal{F}[NL[p_1]], \\ &\partial_t \hat{p}_2 + \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\hat{p}_2 - \alpha i k \hat{p}_1 = -\mu k^2 \hat{p}_2 + \mathcal{F}[NL[p_2]]. \end{aligned} \tag{4.13}$$

Recalling the various contributions, we aim to estimate

$$\begin{aligned} &\langle A^2 p_2, -\partial_x \partial_y^t \Delta_t^{-1} p_2\rangle + \langle A^2 p_1, \partial_x \partial_y^t \Delta_t^{-1} p_1\rangle \\ &= \sum_k \int d\xi A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}(|\hat{p}_1|^2 - |\hat{p}_2|^2). \end{aligned}$$

In the following, with slight abuse of notation, we omit the hat denoting the Fourier transform and only consider $k \neq 0$, since for $k=0$ this term vanishes.

Similarly as in the linear stability results of Section 4.2, we note that the Fourier multiplier a priori is not integrable in time and cannot easily be estimated by the partial dissipation. Hence, we rely on the coupling induced by the underlying magnetic field to eliminate some of this contribution and to provide better decay. More precisely, multiplying the equations (4.13) with $\hat{p}_2, \hat{p}_1$ and omitting the hats for simplicity of notation, we obtain the following identity:

$$\begin{aligned} &|p_1(k)|^2 - |p_2(k)|^2 \\ &= -\tfrac{1}{i\alpha k}(p_1 \overline{i\alpha k p_1} + i\alpha k p_2 \overline{p_2}) \\ &= -\tfrac{1}{i\alpha k} p_1 (\partial_t \overline{p}_2 + \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\ \overline{p}_2 + \mu k^2 \overline{p}_2 - \overline{\mathcal{F}}[NL[p_2]])) \\ &\quad - \tfrac{1}{i\alpha k}\overline{p}_2 (\partial_t p_1 - \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} p_1 + (\mu k^2 + \mu(\xi-kt)^2)p_1 - \mathcal{F}[NL[p_1]] \\ &= \tfrac{-1}{i\alpha k}(\partial_t(p_1\overline{p}_2) + \mu(k^2+(\xi-kt)^2)p_1\overline{p}_2 + \mu k^2 p_1 \overline{p}_2) \\ &\quad - \tfrac{1}{\alpha i k}(p_1,p_2)\cdot \mathcal{F}[\Lambda_t^{-1}\nabla_t^{\perp}(b\nabla_t b - v\nabla_t v, b\nabla_t v - v\nabla_t b)]. \end{aligned}$$

Thus we split L into two linear terms and one nonlinear term:

$$\begin{aligned} L &= 2\sum_k \int d\xi A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\tfrac{-1}{i\alpha k}\partial_t(p_1\overline{p}_2) \\ &\quad + 2\sum_k \int d\xi A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2}\tfrac{-1}{i\alpha k}(2\mu k^2 + \mu(\xi-kt)^2)p_1\overline{p}_2 \\ &\quad - \tfrac{2}{\alpha}\langle A\partial_y^t \Delta_t^{-1}(p_1,p_2)_{\neq}, A\Lambda_t^{-1}\nabla_t^{\perp}(b\nabla_t b - v\nabla_t v, b\nabla_t v - v\nabla_t b)_{\neq}\rangle \\ &= L_1 + L_\mu + ONL. \end{aligned} \tag{4.14}$$

We estimate L_μ by

$$
\begin{aligned}
L_\mu &= \tfrac{2}{\alpha}\mu \sum_{k\neq 0} \int d\xi A^2 \tfrac{(2k^2+(\xi-kt)^2)(\xi-kt)}{k^2+(\xi-kt)^2} p_1 \overline{p}_2 \\
&= \tfrac{2}{\alpha}\mu \sum_{k\neq 0} \int d\xi A^2 \tfrac{(2k^2+(\xi-kt)^2)(\xi-kt)}{(k^2+(\xi-kt)^2)^{\frac{3}{2}}} p_1 (k^2+(\xi-kt)^2)^{\frac{1}{2}} \overline{p}_2 \\
&\leq \tfrac{2}{\alpha}\mu \sup_s \left(\tfrac{(2+s^2)s}{(1+s^2)^{\frac{3}{2}}} \right) \|A\partial_x p_2\|_{L^2} \|A\nabla_t p_1\|_{L^2} \\
&\leq \sqrt{\tfrac{2}{3}} \tfrac{1}{\alpha}\mu (\|A\partial_x p_2\|_{L^2}^2 + \|A\nabla_t p_1\|_{L^2}^2),
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\left| \tfrac{(2k^2+(\xi-kt)^2)(\xi-kt)}{(k^2+(\xi-kt)^2)^{\frac{3}{2}}} \right| &= \left| \tfrac{(2+(\frac{\xi}{k}-t)^2)(\frac{\xi}{k}-t)}{(1+(\frac{\xi}{k}-t)^2)^{\frac{3}{2}}} \right| \\
&\leq \sup_s \left(\tfrac{(2+s^2)s}{(1+s^2)^{\frac{3}{2}}} \right) \\
&\leq \sqrt{\tfrac{2}{3}}.
\end{aligned}
$$

To estimate L_1, we integrate by parts in time to deduce that

$$
\begin{aligned}
&\int d\tau \sum_k \int d\xi A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} \tfrac{-1}{i\alpha k} \partial_t (p_1 \overline{p}_2) \\
&= \left[\tfrac{-1}{i\alpha} \sum_k \int d\xi A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} p_1 \overline{p}_2 \right]_0^t \\
&\quad + \int d\tau \tfrac{1}{i\alpha} \sum_k \int d\xi p_1 \overline{p}_2 \partial_t (A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2}) \\
&= \left[\tfrac{-1}{i\alpha} \sum_k \int d\xi A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} p_1 \overline{p}_2 \right]_0^t \\
&\quad + \int d\tau \tfrac{2}{i\alpha} \sum_k \int d\xi p_1 \overline{p}_2 \tfrac{\dot{M}}{M} A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} \\
&\quad + c\mu \mathbf{1}_{A=A_\mu^{N'}} \int d\tau \tfrac{2}{i\alpha} \sum_k \int d\xi p_1 \overline{p}_2 A^2 \tfrac{(\xi-kt)}{k^2+(\xi-kt)^2} \\
&\quad + \int d\tau \tfrac{1}{i\alpha} \sum_k \int d\xi p_1 \overline{p}_2 A^2 \tfrac{k(k^2-(kt-\xi)^2)}{(k^2+(\xi-kt)^2)^2}.
\end{aligned}
$$

So we infer by Hölder's inequality that

$$\begin{aligned}
&\int d\tau \sum_k \int d\xi A^2 \tfrac{k(\xi-kt)}{k^2+(\xi-kt)^2} \tfrac{-1}{i\alpha k} \partial_t (p_1 \overline{p}_2) \\
&\leq \tfrac{1}{\alpha}(\|Ap_1(0)\|_{L^2}\|Ap_2(0)\|_{L^2} + \|Ap_1(t)\|_{L^2}\|Ap_2(t)\|_{L^2}) \\
&\quad + \mu\|A\partial_x p_1\|_{L^2L^2}\|A\sqrt{-\tfrac{\dot{M}}{M}}p_2\|_{L^2L^2} \\
&\quad + \tfrac{1}{\alpha}\|A\sqrt{-\tfrac{\dot{M}}{M}}p_1\|_{L^2L^2}\|A\sqrt{-\tfrac{\dot{M}}{M}}p_2\|_{L^2L^2}
\end{aligned}$$

and thus

$$\begin{aligned}
&\int L d\tau - \int ONL d\tau \\
&\leq \tfrac{1}{2\alpha}(\|Ap_1(0)\|_{L^2}^2 + \|Ap_2(0)\|_{L^2}^2) \\
&\quad + \tfrac{1}{2\alpha}(\|Ap_1\|_{L^\infty L^2}^2 + \|Ap_2(t)\|_{L^\infty L^2}^2) \\
&\quad + \tfrac{1}{\alpha}(\mu(1-c)\|\partial_x Ap\|_{L^2}^2 + \mu(1-c)\|\partial_y^t Ap\|_{L^2}^2 + \|\sqrt{-\tfrac{\dot{M}}{M}}Ap\|_{L^2}^2).
\end{aligned}$$

Using the dissipation estimates (4.12), we therefore obtain

$$\int L d\tau \leq \tfrac{1}{2\alpha}(c_0+1)\varepsilon^2 + \int ONL d\tau, \tag{4.15}$$

where the ONL part will be estimated at the beginning of the next subsection.

4.3.2 Immediate nonlinear estimates for A^N

In this subsection, we collect some estimates which can be obtained in a straight forward approach using standard techniques (e.g. see [BVW18]). In particular, for these terms we are not constrained by the lack of vertical resistivity. For most estimates we do not aim to establish optimal (mixing enhanced) bounds, since these bounds are in any case better than the ones involving horizontal resistivity and hence do not affect the over all stability threshold. In the following we write $A = A^N$.

ONL estimate: Using integration by parts in space and Hölder's inequality, the nonlinear contribution in (4.14) can be estimated by

$$\begin{aligned}
ONL &= \tfrac{2}{\alpha}\langle A\partial_y^t \Delta_t^{-1} v_{\neq}, A(b\nabla_t b - v\nabla_t v)_{\neq}\rangle \\
&\quad + \tfrac{2}{\alpha}\langle A\partial_y^t \Delta_t^{-1} b_{\neq}, A(b\nabla_t v - v\nabla_t b)_{\neq}\rangle \\
&= \tfrac{2}{\alpha}\langle A\partial_y^t \Delta_t^{-1}(\nabla_t^\perp \otimes v_{\neq}), A(b\otimes b - v\otimes v)_{\neq}\rangle \\
&\quad + \tfrac{2}{\alpha}\langle A\partial_y^t \Delta_t^{-1}(\nabla_t^\perp \otimes b_{\neq}), A(b\otimes v - v\otimes b)_{\neq}\rangle \\
&\lesssim \tfrac{2}{\alpha}\|A(v,b)_{\neq}\|_{L^2}^2 \|A(v,b)\|_{L^2}.
\end{aligned}$$

Recalling the bounds (4.12) and integrating in time we thus obtain that

$$\int ONL d\tau \lesssim \mu^{-1}\varepsilon^3. \tag{4.16}$$

Estimates with an x-average in the second component: Let $a^1a^2a^3 \in \{vvv, vbb, bbv, bvb\}$, then we need to estimate the trilinear products

$$\begin{aligned}\langle Aa^1_{\neq}, A(a^2_{=}\nabla_t a^3_{\neq})\rangle &= \langle Aa^1_{\neq}, A(a^2_{=,1}\partial_x a^3_{\neq})\rangle \\ &\lesssim \|Aa^1_{\neq}\|_{L^2}\|Aa^2_{=,1}\|_{L^2}\|A\partial_x a^3_{\neq}\|_{L^2}.\end{aligned}$$

Integrating in time and again using the bound (4.12) yields a control by

$$\int d\tau \langle Aa^1, A(a^2_{=}\nabla_t a^3)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{4.17}$$

The influence of the underlying x-averaged velocity and magnetic field on the average-less parts can thus be easily controlled by the dissipation, provided $\epsilon \ll \mu$. In the following we focus on terms involving $a^2_{\neq}$.

vvv estimate: We first discuss the velocity non-linearity and use the algebra property of H^N and the bounds on A to estimate

$$\langle Av, Av_{\neq}\nabla_t v\rangle \leq \|Av\|_{L^2}\|Av_{\neq}\|_{L^2}\|A\nabla_t v\|_{L^2}.$$

Here, the contribution by $\|A\nabla_t v\|_{L^2}$ is square integrable in time due to the dissipation (4.12), while $\|Av_{\neq}\|_{L^2}$ is square integrable in time by the inviscid damping estimates (4.11). Integrating in time thus yields a bound by

$$\int d\tau \langle Av, A(v_{\neq}\nabla_t v)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{4.18}$$

vbb estimate: For the contributions by the vbb nonlinearity we intend to argue similarly, but have to account for the lack of vertical magnetic dissipation (which we compensate for by using the full fluid dissipation). We may split the integral as

$$\begin{aligned}\langle Av, A(b_{\neq}\nabla_t b)\rangle &= \int Av_1 A(b_{1,\neq}\partial_x + b_{2,\neq}\partial_y^t)b_1 \\ &\quad + \int Av_2 A(b_{1,\neq}\partial_x + b_{2,\neq}\partial_y^t)b_2.\end{aligned}$$

For the second term we integrate by parts to obtain

$$\int Av_1 A(b_{2,\neq}\partial_y^t b_1) = -\int A\partial_y^t v_1 A(b_{2,\neq}b_1) - \int Av_1 A(\partial_y^t b_{2,\neq}b_1).$$

Furthermore, since b is divergence-free, it holds that $\partial_y^t b_2 = -\partial_x b_1$ and hence

$$\langle Av, A(b_{\neq}\nabla_t b)\rangle \leq \|Av\|_{L^2}\|Ab_{\neq}\|_{L^2}\|A\partial_x b\|_{L^2} + \|\partial_y^t v\|_{L^2}\|Ab_{\neq}\|_{L^2}\|Ab_2\|_{L^2}.$$

We may therefore estimate this term using the full fluid and horizontal magnetic dissipation (4.12) and integrating in time yields a bound by

$$\int d\tau \langle Av, A(b_{\neq}\nabla_t b)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{4.19}$$

bbv estimate: Finally, for the *bbv* contribution, we may again use the full fluid dissipation and the algebra property of A (and H^N) to obtain a bound

$$\langle Ab, A(b_{\neq}\nabla_t v)\rangle \lesssim \|Ab\|_{L^2}\|Ab_{\neq}\|_{L^2}\|A\nabla_t v\|_{L^2}.$$

Integrating in time and using (4.12) we thus obtain a bound by

$$\int d\tau \langle Ab, A(b_{\neq}\nabla_t v)\rangle \lesssim \mu^{-1}\varepsilon^3. \tag{4.20}$$

4.3.3 High frequency *bvb* term without x-average

Having established several straightforward estimates using the full fluid dissipation, in this and the following subsections we establish bounds for the high frequency (that is, A^N terms as in (4.9)) terms involving *bvb*. For simplicity, we write $A = A^N$ and aim to establish the estimate

$$\langle Ab, A(v_{\neq}\nabla_t b)\rangle \lesssim \mu^{-\frac{3}{2}}\varepsilon^3.$$

We split the *bvb* term according to (non)vanishing x-averages:

$$\begin{aligned}\langle Ab, A(v_{\neq}\nabla_t b)\rangle &= \langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{\neq})_{\neq}\rangle \\ &\quad + \langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{=})_{\neq}\rangle \\ &\quad + \langle Ab_{=}, A(v_{\neq}\nabla_t b_{\neq})_{=}\rangle.\end{aligned}$$

Let us first consider the term without any x-averages, which can be written as

$$\begin{aligned}\langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{\neq})\rangle &= \int Ab_{1,\neq}A((v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{1,\neq}) \\ &\quad + \int Ab_{2,\neq}A((v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{2,\neq}).\end{aligned}$$

We estimate the second contribution using the algebra property of H^N and that $\partial_y^t b_2 = -\partial_x b_1$, since b is divergence-free:

$$\begin{aligned}&\int d\tau \int Ab_{2,\neq}A(v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{2,\neq} \\ &\quad \leq \int d\tau \|Ab_{2,\neq}\|_{L^2}(\|Av_{1,\neq}\|_{L^2}\|A\partial_x b_{2,\neq}\|_{L^2} + \|Av_{2,\neq}\|_{L^2}\|A\partial_y^t b_{2,\neq}\|_{L^2}) \\ &\quad \leq \int d\tau \|Ab_{2,\neq}\|_{L^2}(\|Av_{1,\neq}\|_{L^2}\|\partial_x b_{2,\neq}\|_{L^2} + \|Av_{2,\neq}\|_{L^2}\|A\partial_x b_{1,\neq}\|_{L^2}).\end{aligned}$$

Employing Hölder's inequality this contribution can thus be estimated as

$$\begin{aligned}&\int d\tau \int Ab_{2,\neq}A((v_{1,\neq}\partial_x + v_{2,\neq}\partial_y^t)b_{2,\neq}) \\ &\quad \leq \int d\tau \|Ab_{2,\neq}\|_{L^2}\|Av_{\neq}\|_{L^2}\|A\partial_x b_{\neq}\|_{L^2} \\ &\quad \leq \|Ab_{2,\neq}\|_{L^2L^2}\|Av_{\neq}\|_{L^\infty L^2}\|A\partial_x b_{\neq}\|_{L^2L^2} \\ &\quad \lesssim \mu^{-1}\varepsilon^3.\end{aligned} \tag{4.21}$$

It remains to control the contribution by $b_{1,\neq}$, which in view to the lack of vertical resistivity is the hardest term to control. Since the velocity field v is divergence-free, we observe that

$$\int Ab_{1,\neq}(v_{\neq}\nabla_t Ab_{1,\neq}) = 0.$$

Therefore, we obtain the following cancellations and introduce a splitting in Fourier space:

$$\begin{aligned}
\int Ab_{1,\neq}A(v_{\neq}\nabla_t b_{1,\neq}) &= \int Ab_{1,\neq}(A(v\nabla_t b_{1,\neq}) - (v_{\neq}\nabla_t Ab_{1,\neq})) \\
&= \sum_{k,l,k-l\neq 0} \iint d(\xi,\eta)A(k,\xi)b_1(k,\xi)\tfrac{(A(k,\xi)-A(l,\eta))(\xi l-\eta k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}} \\
&\qquad p_1(k-l,\xi-\eta)b_1(l,\eta) \\
&= T + R + \mathcal{R}.
\end{aligned}$$

Here, the Fourier regions

$$\begin{aligned}
\Omega_T &= \{|k-l,\xi-\eta| \le \tfrac{1}{8}|l,\eta|\}, \\
\Omega_R &= \{|l,\eta| \le \tfrac{1}{8}|k-l,\xi-\eta|\}, \\
\Omega_{\mathcal{R}} &= \{\tfrac{1}{8}|l,\eta| \le |k-l,\xi-\eta| \le 8|l,\eta|\},
\end{aligned}$$

correspond to the the *transport* (T) or low-high term, *reaction* (R) or high-low term and the *remainder* ($\mathcal{R}$) or high-high term. In the following we omit the $\neq$ subscripts.

Transport term: Since $|k-l,\xi-\eta| \le \frac{1}{8}|l,\eta|$ we obtain that $|l,\eta| \approx |k,\xi|$. Without loss of generality we assume that $\xi \le \eta$, since we can use either of the following splittings

$$\begin{aligned}
\xi l - k\eta &= (\xi-\eta)l - (k-l)\eta \\
&= (\xi-\eta)k - \xi(k-l).
\end{aligned}$$

Thus using the second equality we estimate

$$\begin{aligned}
T \le\ & \|\partial_y \Lambda_t^{-1} p_1\|_{L^\infty}\|Ab_1\|_{L^2}\|\partial_x Ab_1\|_{L^2} \\
&+ \sum_{k,l\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_T}(\mathbf{1}_{2\langle t\rangle(k\vee l)\ge\xi} + \mathbf{1}_{2\langle t\rangle(k\vee l)\le\xi}) \\
&\qquad \cdot A(k,\xi)b_1(k,\xi)\tfrac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}p_1(k-l,\xi-\eta)b_1(l,\eta),
\end{aligned}$$

where we distinguished between $2\langle t\rangle(k\vee l) \ge \xi$ and $2\langle t\rangle(k\vee l) \le \xi$.

The first case is estimated by using the dissipation and (4.11):

$$\begin{aligned}\sum_{k,l\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{\xi\leq 2(k\vee l)\langle t\rangle}A(k,\xi)b_1(k,\xi)&\\ \cdot\tfrac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}&p_1(k-l,\xi-\eta)b_1(l,\eta)\\ &\lesssim \langle t\rangle\|Ab_1\|_{L^2}\|\Lambda_t^{-1}\partial_x p_1\|_{L^\infty}\|\partial_x Ab_1\|_{L^2}\\ &\lesssim \|Ab_1\|_{L^2}\|\Lambda\partial_x p_1\|_{L^\infty}\|\partial_x Ab_1\|_{L^2}\\ &\lesssim \|Ab_1\|_{L^2}\|Ap_1\|_{L^2}\|\partial_x Ab_1\|_{L^2}.\end{aligned}$$

For the second case, $2\langle t\rangle(k\vee l)\leq \xi$, we need to estimate

$$\begin{aligned}(A^N(k,\xi)-A^N(l,\eta)) &= (M(k,\xi)|k,\xi|^N - M(l,\eta)|l,\eta|^N)\\ &= M(k,\xi)(|k,\xi|^N-|l,\eta|^N)\\ &\quad + M(l,\eta)(\tfrac{M(k,\xi)}{M(l,\eta)}-1)|l,\eta|^N.\end{aligned}$$

By the mean value theorem, we obtain

$$\begin{aligned}|k,\xi|^N-|l,\eta|^N &\leq N|k-\theta l,\xi-\theta\eta|^{N-1}|k-l,\xi-\eta|\\ &\lesssim |k-l,\xi-\eta|(|l,\eta|^{N-1}+|k-l,\xi-\eta|^{N-1})\\ &\lesssim |k-l,\xi-\eta||l,\eta|^{N-1}.\end{aligned}$$

For the differences in M we use that for $a,b>0$ it holds that $|e^{a-b}-1|\leq e^{a+b}-1$ and hence

$$\begin{aligned}|\tfrac{M_1(k,\xi)}{M_1(l,\eta)}-1| &= |\exp\left(\int_0^t \tfrac{|l|}{l^2+(\eta-ls)^2}-\tfrac{|k|}{k^2+(\xi-ks)^2}ds\right)-1|\\ &\leq |\exp\left(\int_0^t \tfrac{|l|}{l^2+(\eta-ls)^2}+\tfrac{|k|}{k^2+(\xi-ks)^2}ds\right)-1|.\end{aligned}$$

Thus for $\eta\geq\xi\geq 2t(k\vee l)$ by integrating we obtain that

$$\begin{aligned}|\tfrac{M_1(k,\xi)}{M_1(l,\eta)}-1| &\leq \exp\left(\tfrac{1}{|l|}\int_0^t \tfrac{1}{1+(\frac{\eta}{l}-s)^2}ds+\tfrac{1}{|k|}\int_0^t\tfrac{1}{1+(\frac{\xi}{k}-s)^2}ds\right)-1\\ &\leq \exp(\tfrac{1}{\eta}+\tfrac{1}{\xi})-1\\ &\lesssim \tfrac{1}{\eta}+\tfrac{1}{\xi}.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\sum_{k,l,k-l\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{\xi\geq 2(k\vee l)t}A(k,\xi)b_1(k,\xi)&\tfrac{(A(k,\xi)-A(l,\eta))\xi(l-k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}\\ p_1(k-l,\xi-\eta)b_1(l,\eta)&\\ &\lesssim \|Ab_1\|_{L^2}\|\Lambda_t^{-1}\partial_x p_1\|_{L^\infty}\|Ab_1\|_{L^2}\\ &\lesssim \langle t\rangle^{-1}\|Ab_1\|_{L^2}\|\Lambda\partial_x p_1\|_{L^\infty}\|Ab_1\|_{L^2}\\ &\lesssim \langle t\rangle^{-1}\|Ab_1\|_{L^2}\|Ap_1\|_{L^2}\|Ab_1\|_{L^2},\end{aligned}$$

where we used the estimate (4.11). Combining all estimates, we have derived the following estimate of the transport term:

$$\begin{aligned}\int T d\tau &\lesssim \|Ab_1\|_{L^\infty L^2}\|Ap_1\|_{L^\infty L^2}\|Ab_1\|_{L^2L^2} \\ &\lesssim \mu^{-\frac{1}{2}}\varepsilon^3.\end{aligned} \tag{4.22}$$

Reaction term: Since $|l,\eta| \leq \frac{1}{8}|k-l,\xi-\eta|$ we obtain $|k-l,\xi-\eta| \approx |k,\xi|$. With the identity

$$\xi l - k\eta = l(\xi - \eta - (k-l)t) - (k-l)(\eta - lt)$$

and $A(k,\xi) - A(l,\eta) \lesssim A(k-l,\xi-\eta)$ we infer

$$\begin{aligned}R = &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta)\mathbf{1}_{\Omega_R} A(k,\xi)b_1(k,\xi)\tfrac{(A(k,\xi)-A(l,\eta))(l(\xi-\eta-(k-l)t)-(k-l)(\eta-lt))}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}} \\ &\qquad \cdot p_1(k-l,\xi-\eta)b_1(l,\eta) \\ &\leq \|Ab_1\|_{L^2}\|A\partial_y^t\Lambda_t^{-1}p_1\|_{L^2}\|\partial_x b_1\|_{L^\infty} \\ &\quad + \|Ab_1\|_{L^2}\|A\Lambda_t^{-1}p_1\|_{L^2}\|\partial_y^t\partial_x^2 b_1\|_{L^\infty} \\ &\quad + \|\partial_x Ab_1\|_{L^2}\|A\Lambda_t^{-1}p_1\|_{L^2}\|\partial_y^t\partial_x b_1\|_{L^\infty}.\end{aligned}$$

We split $\partial_y^t = \partial_y - t\partial_x$ and use the definition of the low-frequency multiplier $A_\mu^{N'}$ to estimate

$$\begin{aligned}\|\langle\partial_x\rangle^2\partial_y^t b_1\|_{L^\infty} &\leq \|\langle\partial_x\rangle^2\partial_y b_1\|_{L^\infty} + \|\langle\partial_x\rangle^2 t\partial_x b_1\|_{L^\infty} \\ &\leq t\|\Lambda^{N'} b_1\|_{L^2} \\ &\lesssim te^{-c\mu t}\|A_\mu^{N'} b_1\|_{L^2} \\ &\lesssim \mu^{-1}\|A_\mu^{N'} b_1\|_{L^2}.\end{aligned}$$

Therefore, integrating in time yields the estimate

$$\begin{aligned}\int R d\tau \lesssim &\|Ab_1\|_{L^2L^2}\left(\|A\partial_y^t\Lambda_t^{-1}p_1\|_{L^2L^2}\|Ab_1\|_{L^\infty L^2}\right) \\ &\quad + \mu^{-1}\|A\partial_x b_1\|_{L^2L^2}\|A\Lambda_t^{-1}p_1\|_{L^2L^2}\|A_\mu^{N'} b_1\|_{L^\infty L^2} \\ \lesssim &\varepsilon^3\mu^{-\frac{3}{2}}.\end{aligned} \tag{4.23}$$

$\mathcal{R}$ term: We consider the Fourier region where $\frac{1}{8}|l,\eta| \leq |k-l,\xi-\eta| \leq 8|l,\eta|$. Thus, we have the bounds $|k,\xi| \lesssim |l,\eta|$ and $A(k,\xi) \lesssim A(l,\eta) \approx A(k-l,\xi-\eta)$. Furthermore, we note that

$$\xi l - \eta k \leq |l,\eta|^2,$$

and thus estimate the remainder terms as

$$\begin{aligned}\mathcal{R} = &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) 1_{\Omega_{\mathcal{R}}} A(k,\xi) b_1(k,\xi) \\ &\qquad \tfrac{(A(k,\xi)-A(l,\eta))(\xi l-\eta k)}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}} p_1(k-l,\xi-\eta) b_1(l,\eta) \\ \lesssim\; &\|Ab_1\|_{L^2} \|A\Lambda_t^{-1} p_1\|_{L^2} \|\Lambda^2 b_1\|_{L^\infty} \\ \lesssim\; &\|Ab_1\|_{L^2} \|A\Lambda_t^{-1} p_1\|_{L^2} \|Ab_1\|_{L^2}.\end{aligned}$$

Hence after integrating in time, we deduce that

$$\int \mathcal{R} \lesssim \; \|Ab_1\|_{L^2 L^2} \|\sqrt{-\tfrac{\dot{M}}{M}} Ap_1\|_{L^2L^2} \|Ab_1\|_{L^\infty L^2} \lesssim \mu^{-\frac{1}{2}} \varepsilon^3. \tag{4.24}$$

Combining the estimates (4.21), (4.22), (4.23) and (4.24), we finally conclude that

$$\langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{\neq})_{\neq}\rangle \lesssim \mu^{-\frac{3}{2}} \varepsilon^3. \tag{4.25}$$

4.3.4 High frequency estimates for bvb terms with x-averages

In this subsection we aim to estimate the remaining terms in the bvb integrals, which involve x-averages. We consider the two terms

$$\begin{aligned}&\langle Ab_{\neq}, A(v_{\neq}\nabla_t b_{=})_{\neq}\rangle + \langle Ab_{=}, A(v_{\neq}\nabla_t b_{\neq})_{=}\rangle \\ &\quad = \langle Ab_{1,\neq}, A(v_{\neq}\nabla_t b_{1,=})_{\neq}\rangle + \langle Ab_{1,=}, A(v_{\neq}\nabla_t b_{1,\neq})_{=}\rangle,\end{aligned}$$

where we used that $b_{2,=}=0$, since b is divergence-free. Using integration by parts and the fact that v is divergence-free, we obtain that

$$\begin{aligned}&\langle Ab_{1,\neq}, v_{\neq}\nabla_t Ab_{1,=}\rangle + \langle Ab_{1,=}, v_{\neq}\nabla_t Ab_{1,\neq}\rangle \\ &\qquad\qquad = \langle v_{\neq}, \nabla_t (Ab_{1,=}Ab_{1,\neq})\rangle = 0,\end{aligned}$$

and thus

$$\begin{aligned}&\langle Ab_{1,\neq}, A(v_{\neq}\nabla_t b_{1,=})\rangle + \langle Ab_{1,=}, A(v_{\neq}\nabla_t b_{1,\neq})\rangle \\ &\quad = \langle Ab_{1,\neq}, A(v_{\neq}\nabla_t b_{1,=}) - v_{\neq}\nabla_t Ab_{1,=}\rangle + \langle Ab_{1,=}, A(v_{\neq}\nabla_t b_{1,\neq}) - v_{\neq}\nabla_t Ab_{1,\neq}\rangle \\ &\quad = \sum_{k\neq 0} \iint d(\xi,\eta) A(k,\xi) b_1(k,\xi) \tfrac{(A(k,\xi)-A(0,\eta))(-k\eta)}{\sqrt{k^2+(\xi-\eta-kt)^2}} p_1(k,\xi-\eta) b_1(0,\eta) \\ &\qquad + \sum_{k\neq 0} \iint d(\xi,\eta) A(0,\xi) b_1(0,\xi) \tfrac{(A(0,\xi)-A(k,\eta))(-k\xi)}{\sqrt{k^2+(\xi-\eta-kt)^2}} p_1(k,\xi-\eta) b_1(-k,\eta).\end{aligned}$$

Again we split this integrals into the transport T, reaction R and remainder terms $\mathcal{R}$ with the same definition of sets in Fourier space:

$$\begin{aligned}\Omega_T &= \{|\xi-\eta| \leq \tfrac{1}{8}|\eta|\}, \\ \Omega_R &= \{|\eta| \leq \tfrac{1}{8}|\xi-\eta|\}, \\ \Omega_{\mathcal{R}} &= \{\tfrac{1}{8}|\eta| \leq |\xi-\eta| \leq 8|\eta|\}.\end{aligned}$$

Transport term: Since $|\xi-\eta|\leq \frac{1}{8}|\eta|$ we obtain that $|\eta|\approx|\xi|$.

In our estimates, we distinguish the cases $\xi\vee\eta\leq 2k\langle t\rangle$ and $\xi\vee\eta\geq 2k\langle t\rangle$. In the first case, $\xi\vee\eta\leq 2k\langle t\rangle$ we obtain a bound by

$$\begin{aligned}
&\sum_{k\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{\xi\vee\eta\leq k\langle t\rangle}A(k,\xi)b_1(k,\xi)\tfrac{(A(k,\xi)-A(0,\eta))k\eta}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(0,\eta)\\
&+\sum_{k\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{\xi\vee\eta\leq k\langle t\rangle}A(0,\xi)b_1(0,\xi)\tfrac{(A(0,\xi)-A(k,\eta))k\xi}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(-k,\eta)\\
&\qquad\leq t\|Ab_=\|_{L^2}\|\partial_x^2\Lambda_t^{-1}p_{1,\neq}\|_{L^\infty}\|Ab_{1,\neq}\|_{L^2}\\
&\qquad\lesssim \|Ab_=\|_{L^2}\|Ap_{1,\neq}\|_{L^2}\|Ab_{1,\neq}\|_{L^2}.
\end{aligned}$$

In the case $\xi\vee\eta\geq 2k\langle t\rangle$, we instead estimate

$$\begin{aligned}
A(k,\xi)-A(0,\eta)&\leq M(k,\xi)(\xi^2+k^2)^{\frac{N}{2}}-\eta^N\\
&=(M(k,\xi)-1)(\xi^2+k^2)^{\frac{N}{2}}+((\xi^2+k^2)^{\frac{N}{2}}-\eta^N).
\end{aligned}$$

Since $\xi\geq 2k\langle t\rangle$, in the first summand we may bound

$$\begin{aligned}
M(k,\xi)-1&=\exp\left(-\int_0^t \tfrac{|k|}{k^2+(\xi-ks)^2}ds\right)-1\\
&\lesssim \tfrac{1}{\xi}\lesssim\tfrac{1}{\eta}.
\end{aligned}$$

By the mean value theorem we further infer

$$(\xi^2+k^2)^{\frac{N}{2}}-\eta^N\leq((\xi-\theta\eta)^2+k^2)^{\frac{N-1}{2}}|k,\xi-\eta|\lesssim|k,\xi-\eta|(\xi^2+k^2)^{\frac{N-1}{2}}.$$

Thus, using that $k\leq\xi\lesssim\eta$, we deduce that

$$\begin{aligned}
A(k,\xi)-A(0,\eta)&\lesssim|k,\xi-\eta|\eta^{N-1},\\
A(k,\eta)-A(0,\xi)&\lesssim|k,\xi-\eta|\eta^{N-1},
\end{aligned}$$

where the proof for $A(k,\eta)-A(0,\xi)$ is analogous. Finally, we obtain

$$\begin{aligned}
&\langle Ab_{\neq},\mathbf{1}_{\Omega_T}\mathbf{1}_{\eta\geq kt}A(v_{\neq}\nabla_t b_=)\rangle+\langle Ab_=,\mathbf{1}_{\Omega_T}1_{\eta\geq kt}A(v_{\neq}\nabla_t b_{\neq})\rangle\\
&\quad\lesssim\sum_{k\neq 0}\iint d(\xi,\eta)A(k,\xi)b_1(k,\xi)\tfrac{|k,\xi-\eta|\eta^{N-1}}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(0,\eta)\\
&\qquad+\sum_{k\neq 0}\iint d(\xi,\eta)A(0,\xi)b_1(0,\xi)\tfrac{|k,\xi-\eta|\eta^{N-1}}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_1(k,\xi-\eta)b_1(k,\eta)\\
&\quad\lesssim\|Ab_=\|_{L^\infty}\|A\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|Ab_{1,\neq}\|_{L^2}\\
&\quad\lesssim\|Ab_=\|_{L^\infty}\|A\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|Ab_{1,\neq}\|_{L^2},
\end{aligned}$$

and integrating in time yields the desired bound:

$$\begin{aligned}
\int\langle Ab_{\neq},\mathbf{1}_{\Omega_T}A(v_{\neq}\nabla_t b_=)\rangle+\langle Ab_=,\mathbf{1}_{\Omega_T}A(v_{\neq}\nabla_t b_{\neq})\rangle d\tau&\\
\lesssim\mu^{-1}\varepsilon^3.&
\end{aligned}\tag{4.26}$$

Reaction term: Since $|\eta| \leq \frac{1}{8}|\xi-\eta|$ we obtain $|\xi-\eta| \approx |\xi|$ and thus

$$\begin{aligned}
R &= \langle Ab_{\neq}, \mathbf{1}_{\Omega_R} A((v_{\neq}\nabla_t b_{=}) - v_{\neq}\nabla_t Ab_{=})\rangle + \langle Ab_{=}, \mathbf{1}_{\Omega_R}(A(v_{\neq}\nabla_t b_{\neq}) - v_{\neq}\nabla_t Ab_{\neq})_{=}\rangle \\
&\leq \sum_{k\neq 0} \iint d(\xi,\eta) \mathbf{1}_{\Omega_R} A(k,\xi) b_1(k,\xi) \tfrac{(A(k,\xi)-A(0,\eta))k\eta}{\sqrt{k^2+(\xi-\eta-kt)^2}} p_1(k,\xi-\eta) b_1(0,\eta) \\
&\quad + \sum_{k\neq 0} \iint d(\xi,\eta) \mathbf{1}_{\Omega_R} A(0,\xi) b_1(0,\xi) \tfrac{(A(0,\xi)-A(-k,\eta))k\xi}{\sqrt{k^2+(\xi-\eta-kt)^2}} p_1(k,\xi-\eta) b_1(-k,\eta) \\
&\lesssim \|Ab_{1,\neq}\|_{L^2} \|A\partial_x \Lambda_t^{-1} p_{1,\neq}\|_{L^2} \|\partial_y b_{1,=}\|_{L^\infty} \\
&\quad + \|Ab_{1,=}\|_{L^2} \|A\partial_y \partial_x^{-1} \Lambda_t^{-1} p_{1,\neq}\|_{L^2} \|\partial_x^2 b_{1,\neq}\|_{L^\infty}.
\end{aligned}$$

Expressing $\partial_y = \partial_y^t + t\partial_x$ in terms of the time-dependent derivatives, at this point we require the splitting into high and low frequency estimates. More precisely, using the additional time decay of the low-frequency part, we estimate

$$\begin{aligned}
\|A\partial_y \partial_x^{-1} \Lambda_t^{-1} p_{1,\neq}\|_{L^2} &\leq \|A\partial_y^t \partial_x^{-1} \Lambda_t^{-1} p_{1,\neq}\|_{L^2} + t\|A\Lambda_t^{-1} p_{1,\neq}\|_{L^2} \\
&\lesssim \|Ap_{1,\neq}\|_{L^2} + t\|A\Lambda_t^{-1} p_{1,\neq}\|_{L^2}
\end{aligned}$$

and using the definition of $A_\mu^{N'}$ we can absorb the growth of the factor t at the cost of a power of μ:

$$\begin{aligned}
\|\partial_x^2 b_{1,\neq}\|_{L^\infty} &\leq \|\Lambda^{N'} b_{1,\neq}\|_{L^2} \\
&\lesssim e^{-c\mu t} \|A_\mu^{N'} b_{1,\neq}\|_{L^2} \\
&\lesssim \mu^{-1} \langle t\rangle^{-1} \|A_\mu^{N'} b_{1,\neq}\|_{L^2}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
R &\lesssim \|A^N p_{1,\neq}\|_{L^2} \|A^N b_{1,=}\|_{L^2} \|A^N b_{1,\neq}\|_{L^2} \\
&\quad + \mu^{-1} \|A^N b_{1,=}\|_{L^2} \|A\Lambda_t^{-1} p_{1,\neq}\|_{L^2} \|A_\mu^{N'} b_{1,\neq}\|_{L^2}.
\end{aligned}$$

Integrating in time then yields the estimate

$$\int R d\tau \lesssim \mu^{-\frac{3}{2}} \varepsilon^3. \tag{4.27}$$

$\mathcal{R}$ term: The remainder term $\mathcal{R}$ can be estimated by the same argument as in the case without x-averages in Subsection 4.3.3.

Combining the estimates (4.26), (4.27) and (4.25), we conclude that the *bvb* term can be controlled as

$$\langle Ab, A(v_{\neq}\nabla_t b)\rangle \lesssim \mu^{-\frac{3}{2}} \varepsilon^3. \tag{4.28}$$

4.3.5 Low frequency estimates

In this subsection we establish the estimates on the low frequency errors. For simplicity of presentation we present the proof of these estimates for the *bvb* nonlinearity. The estimates with an x-average in the second component are analogous to the ones in Subsection 4.3.2. The arguments for the vvv, vbb, bbv or ONL trilinear terms are also analogous.

We aim to establish the bound

$$\langle A_\mu^{N'} b, A_\mu^{N'}(v_{\neq}\nabla_t b)\rangle \lesssim \mu^{-\frac{1}{2}}\varepsilon^3,$$

and, as in the previous section, separately discuss the transport, reaction and remainder term.

For the transport term, we note that

$$\begin{aligned} v_{\neq}\nabla_t &= \nabla_t^{\perp}\Lambda_t^{-1}p_1\nabla_t \\ &= \nabla^{\perp}\Lambda_t^{-1}p_1\nabla. \end{aligned}$$

Hence, we may rewrite

$$\langle A_\mu^{N'} b, A_\mu^{N'}(v_{\neq}\nabla_t b)\rangle = \langle A_\mu^{N'} b, A_\mu^{N'}(\nabla^{\perp}\Lambda_t^{-1}p_{1,\neq}\nabla b)\rangle.$$

In a first step, we estimate the $b_{\neq}$ term by using the algebra property of $A^{N'}$:

$$\begin{aligned} &\langle A_\mu^{N'} b, A_\mu^{N'}(\nabla^{\perp}\Lambda_t^{-1}p_{1,\neq}\nabla b_{\neq})\rangle \\ &\qquad \leq \|A_\mu^{N'} b\|_{L^2} e^{c\mu_x t}\big(\|A^{N'}\nabla^{\perp}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|\nabla b_{\neq}\|_{L^\infty} + \\ &\qquad\qquad \|\nabla^{\perp}\Lambda_t^{-1}p_{1,\neq}\|_{L^\infty}\|A^{N'}\nabla b_{\neq}\|_{L^2}\big) \\ &\qquad \leq \|A_\mu^{N'} b\|_{L^2}\big(\|A^{N}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|A_\mu^{N'} b_{\neq}\|_{L^2} + \|A_\mu^{N'}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|A^{N} b_{\neq}\|_{L^2}\big). \end{aligned}$$

Integrating in time then yields the estimate

$$\int d\tau \langle A_\mu^{N'} b, A_\mu^{N'}(v_{\neq}\nabla_t b_{\neq})\rangle \lesssim \mu^{-\frac{1}{2}}\varepsilon^3. \tag{4.29}$$

Furthermore, we estimate the $b_{=}$ term by partial integration and the algebra property of $A^{N'}$

$$\begin{aligned} &\langle A_\mu^{N'} b, A_\mu^{N'}(\nabla^{\perp}\Lambda_t^{-1}p_{1,\neq}\nabla b_{=})\rangle \\ &= -\langle A_\mu^{N'} b_{1,\neq}, A_\mu^{N'}(\partial_x\Lambda_t^{-1}p_{1,\neq}\partial_y b_{1,=})\rangle \\ &= \langle \partial_x A_\mu^{N'} b_{1,\neq}, A_\mu^{N'}(\Lambda_t^{-1}p_{1,\neq}\partial_y b_{1,=})\rangle \\ &\leq \|\partial_x A_\mu^{N'} b_{1,\neq}\|_{L^2} e^{c\mu t}\big(\|A^{N'}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|\partial_y b_{1,=}\|_{L^\infty} \\ &\qquad + \|\Lambda_t^{-1}p_{1,\neq}\|_{L^\infty}\|\partial_y^{N'+1} b_{1,=}\|_{L^2}\big) \\ &\lesssim \|\partial_x A_\mu^{N'} b_{1,\neq}\|_{L^2}\big(\|A_\mu^{N'}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|A^{N'} b_{1,=}\|_{L^2} \\ &\qquad + \|A_\mu^{N'}\Lambda_t^{-1}p_{1,\neq}\|_{L^2}\|A^{N} b_{1,=}\|_{L^2}\big). \end{aligned}$$

Integrating in time then yields that

$$\int d\tau \langle A_\mu^{N'} b_{\neq}, A_\mu^{N'}(v_{\neq}\nabla_t b_{=})\rangle \lesssim \mu^{-\frac{1}{2}}\varepsilon^3. \tag{4.30}$$

This concludes our proof of Proposition 4.5 and hence of Theorem 4.3. More precisely, the claimed estimates for both A^N and $A_\mu^{N'}$ are obtained by combining the respective linear estimate (4.15), the high frequency nonlinear estimates (4.16), (4.17), (4.18), (4.19), (4.20), (4.28), and the low frequency estimates given in (4.29) and (4.30).

We emphasize that the stability threshold of $\frac{3}{2}$ is determined by the estimates for the action of the $v \cdot \nabla_t b$ nonlinearity in the estimate (4.28) and, in particular, by the estimates of the reaction terms (4.23) and (4.27). These estimates are expected to be optimal and together with the linear estimates of Section 4.2 highlight the effects of the lack of vertical resistivity.

The partial dissipation case considered in this article

$$\kappa_y = 0,\ \nu_x = \nu_y = \kappa_x > 0,$$

shows the large impact of (partial) magnetic resistivity on the behavior of the MHD equations and the (de)stabilizing role of the magnetic field. As mentioned following Theorem 4.3, more generally our methods of proof extend to the case where κ_x is bounded below in terms of ν:

$$\nu_y^{1/3} \geq \kappa_x \geq \frac{1}{2\alpha}\nu_y.$$

The complementary regime, where κ_x tends to zero quicker than ν_y remains an interesting topic for future work. The limiting case, $\kappa_x = 0$, and the associated instability is discussed in the following section.

4.4 Instability of the Non-Resistive MHD System

As a complementary result, in this section we consider the non-resistive MHD equations and establish the instability estimates of Proposition 4.2.

4.4.1 Linear instability

We begin by studying the linearized MHD equations with isotropic viscosity and vanishing resistivity:

$$\begin{aligned} &\partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 = \nu \Delta_t p_1, \\ &\partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 = 0. \end{aligned} \tag{4.31}$$

Lemma 4.6 (Quantitative linear instability of the non-resistive MHD equations)**.** *Under the assumptions of Proposition 4.2, for the linearized equations* (4.31) *there exists initial data* p_{in} *such that*

$$\begin{aligned} \|p(t)\|_{H^N} &\geq t^{\frac{\nu}{8\alpha^2}} \|p_{in}\|_{H^N}, \\ \|p(t)\|_{H^{N-1}} &\geq t^{\frac{\nu^2}{32\alpha^4}} \|p_{in}\|_{H^N}. \end{aligned} \tag{4.32}$$

Furthermore, for all solutions such that at time τ it holds $p(\tau) \in H^N$, then we obtain

$$\|p(t)\|_{H^N} \lesssim \langle \nu(t-\tau)\rangle \|p(\tau)\|_{H^N} \tag{4.33}$$

for all $t > \tau$.

Proof of Lemma 4.6. After a Fourier transform (4.31) yields

$$\begin{aligned} \partial_t p_1(k) &= -\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} p_1(k) + \alpha k p_2(k) - \nu(k^2 + (\xi - kt)^2) p_1(k), \\ \partial_t p_2(k) &= \tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} p_2(k) - \alpha k p_1(k). \end{aligned} \tag{4.34}$$

Here, in order to simplify notation we have relabeled $p_2 \mapsto ip_2$ so that we obtain only real-valued coefficient functions.

For the lower bound we fix $k = -1$ and $\xi \geq 3\frac{\alpha^2}{\nu}$ and choose $p_1(0, k, \xi) = 0$, $p_2(0, k, \xi) = 1$. In this case, the Duhamel integral formula yields that

$$p_1 = -\alpha \int_0^t d\tau_1 \sqrt{\tfrac{1+(\tau_1+\xi)^2}{1+(t+\xi)^2}} \exp(-\nu(t - \tau + \tfrac{1}{3}((t+\xi)^3 - (\tau_1+\xi)^3))) p_2(\tau_1),$$

and that

$$\begin{aligned} &p_2 - \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} \\ &= -\alpha k \int_0^t d\tau_2 \sqrt{\tfrac{1+(t+\xi)^2}{1+(\tau_2+\xi)^2}} p_1(\tau_2) \\ &= -\alpha^2 \int_0^t d\tau_2 \int_0^{\tau_1} d\tau_1 \tfrac{\sqrt{1+(t+\xi)^2}\sqrt{1+(\tau_1+\xi)^2}}{1+(\tau_2+\xi)^2} p_2(\tau_1) \\ &\quad \cdot \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2+\xi)^3 - (\tau_1+\xi)^3))) \end{aligned}$$

Denoting $|p_2|_\infty(t) = \sup_{0\leq\tau\leq t} |p_2(\tau)|$, the double integral term can be bounded by

$$\alpha^2 |p_2|_\infty \int_0^t d\tau_1 \int_{\tau_2}^t d\tau_2 \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2+\xi)^3 - (\tau_1+\xi)^3))).$$

Furthermore, we may estimate

$$\begin{aligned} &\int_0^t d\tau_1 \int_{\tau_2}^t d\tau_2 \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2+\xi)^3 - (\tau_1+\xi)^3))) \\ &\quad = \int_0^t d\tau_1 \int_{\tau_2}^t d\tau_2 \tfrac{1+(\tau_2+\xi)^2}{1+(\tau_2+\xi)^2} \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2+\xi)^3 - (\tau_1+\xi)^3))) \\ &\quad \leq \int_0^t d\tau_1 \int_{\tau_2}^t d\tau_2 \tfrac{1+(\tau_2+\xi)^2}{1+(\tau_1+\xi)^2} \exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2+\xi)^3 - (\tau_1+\xi)^3))) \\ &\quad \leq \tfrac{1}{\nu} \int_0^t d\tau_1 \tfrac{1}{1+(\tau_1+\xi)^2} [\exp(-\nu(\tau_2 - \tau_1 + \tfrac{1}{3}((\tau_2+\xi)^3 - (\tau_1+\xi)^3)))]_{\tau_2=\tau_1}^{\tau_2=t} \\ &\quad \leq \tfrac{1}{\nu\xi}. \end{aligned}$$

Hence, we obtain that

$$|p_2 - \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} p_{2,in}| \leq \tfrac{\alpha^2}{\nu \xi} |p_2|_\infty =: c|p_2|_\infty,$$

For later reference we note that $c = \frac{\alpha^2}{\nu\xi}$ satisfies $0 < c \leq \frac{1}{3}$ and hence $0 < \frac{c}{1-c} \leq \frac{1}{2}$.

Then it follows that

$$|p_2| \leq c|p_2|_\infty + \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}|,$$

and, since $\xi \geq 0$, the function $\sqrt{\frac{1+(t+\xi)^2}{1+\xi^2}}$ is monotonly increasing in time. This implies that

$$|p_2|_\infty \leq \frac{1}{1-c} \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}|$$

Hence we infer

$$\begin{aligned} |p_2 - \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} p_{2,in}| &\leq \frac{c}{1-c} \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}| \\ &\leq \frac{1}{2} \sqrt{\tfrac{1+(t+\xi)^2}{1+\xi^2}} |p_{2,in}|. \end{aligned}$$

Since $0 < \frac{c}{1-c} \leq \frac{1}{2}$, p_2 is comparable to $\sqrt{\frac{1+(t+\xi)^2}{1+\xi^2}} p_{2,in}$.

We next keep $k = -1$ fixed but combine this construction for different ξ. More precisely, let $a(\xi)$ be such that $\text{supp}_\xi(a(\xi)) \subset [3\frac{\alpha^2}{\nu}, 4\frac{\alpha^2}{\nu}]$ and $\int (2+\xi^2)^{\frac{N}{2}} a^2(\xi) = 1$. Then for the initial data

$$p_{in}(k, \xi) = \mathbf{1}_{k=-1} a(\xi)$$

it holds that

$$\begin{aligned} \|p_{in}\|_{H^N} &= 1, \\ \|p(t)\|_{H^N} &\geq t \tfrac{\nu}{8\alpha^2}, \\ \|p(t)\|_{H^{N-1}} &\geq t \tfrac{\nu^2}{32\alpha^4}, \end{aligned}$$

which proves (4.32).

We prove the upper bound in three steps

1. Let $t \geq \tau \geq \nu^{-1} + \frac{\xi}{k}$, then we estimate $|p|(t) \lesssim \langle \nu(t-\tau) \rangle |p|(\tau)$.
2. Let $\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau \geq -\nu^{-1} + \frac{\xi}{k}$, then we estimate $|p|(t) \lesssim |p|(\tau)$.
3. Let $-\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau$, then we estimate $|p|(t) \lesssim |p|(\tau)$.

From (1-3) estimate (4.33) follows directly. In the following, we prove (1-3).

1. Let $t \geq \tau \geq \nu^{-1} + \frac{\xi}{k}$. Then we obtain

$$\begin{aligned}\partial_t |p|^2 &\leq 2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}|p_2|^2 \\ &\quad + (-2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} - 2\nu k^2(1+(t-\tfrac{\xi}{k})^2))|p_1|^2 \\ &\leq 2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}|p_2|^2.\end{aligned}$$

Thus we obtain by Gronwall's Lemma

$$\begin{aligned}|p|^2(t) &\leq \tfrac{1+(t-\frac{\xi}{k})^2}{1+(\tau-\frac{\xi}{k})^2}|p|^2(\tau) \\ &\leq \tfrac{1+(\tau-\frac{\xi}{k})^2+(t-\frac{\xi}{k})^2-(\tau-\frac{\xi}{k})^2}{1+(\tau-\frac{\xi}{k})^2}|p|^2(\tau) \\ &\leq (1+\tfrac{(t-\frac{\xi}{k})^2-(\tau-\frac{\xi}{k})^2}{1+(\tau-\frac{\xi}{k})^2})|p|^2(\tau) \\ &\leq 2\langle \nu(t-\tau)\rangle^2 |p|^2(\tau).\end{aligned}$$

2. Let $\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau \geq -\nu^{-1} + \frac{\xi}{k}$. We define the energy

$$E = |p|^2 + \tfrac{2}{\alpha k}\tfrac{s}{1+s^2}p_1 p_2.$$

As $\alpha > \frac{1}{2}$, E is positive definite with

$$(1-\tfrac{1}{2\alpha})|p|^2 \leq E \leq (1+\tfrac{1}{2\alpha})|p|^2.$$

Then we derive in time and infer

$$\begin{aligned}\partial_t E + \nu k^2(1+s^2))p_1^2 &= \tfrac{2}{\alpha k}\tfrac{1-s^2}{(1+s^2)^2}p_1 p_2 \\ &\quad + \tfrac{2}{\alpha}\nu k s p_1 p_2 \\ &\leq \tfrac{2}{\alpha k}\tfrac{1}{1+s^2}p_1 p_2 \\ &\quad + \tfrac{2}{\alpha}\nu p_2^2 + \nu k(1+s^2)p_1^2.\end{aligned}$$

This further implies that

$$\partial_t E \lesssim (\tfrac{1}{1+s^2} + \nu)E.$$

By Gronwall's lemma we infer

$$\begin{aligned}E(t) &\leq \exp(C\int_\tau^t \tfrac{1}{1+\tau_1^2} + \nu d\tau_1)E(\tau) \\ &\leq \exp(C(\pi + 2\nu\nu^{-1})E(\tau) \\ &\lesssim E(\tau).\end{aligned}$$

Therefore, we obtain that

$$|p|(t) \lesssim |p|(\tau).$$

3. Let $-\nu^{-1} + \frac{\xi}{k} \geq t \geq \tau$. Then we obtain that

$$\begin{aligned}\partial_t |p|^2 &\leq 2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}|p_2|^2 \\ &+ (-2\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} - 2\nu k^2(1+(t-\tfrac{\xi}{k})^2))|p_1|^2 \\ &\leq 0.\end{aligned}$$

Thus we arrive at the desired estimate

$$|p|(t) \lesssim |p|(\tau).$$

□

4.4.2 Nonlinear norm inflation

We next consider the nonlinear non-resistive MHD equations in their perturbative form around the stationary solution (4.2):

$$\begin{aligned}\partial_t p_1 - \partial_x \partial_x^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1 + \nabla_t^{\perp} \Lambda_t^{-1}(b\nabla_t b - v\nabla_t v), \\ \partial_t p_2 + \partial_x \partial_x^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \nabla_t^{\perp} \Lambda_t^{-1}(b\nabla_t v - v\nabla_t b).\end{aligned} \tag{4.35}$$

The following lemma establishes the norm inflation result of Proposition 4.2.

Lemma 4.7 (Nonlinear norm inflation for the non-resistive MHD equations)**.** *Under the same assumptions of Proposition 4.2, we consider the non-resistive nonlinear MHD equations* (4.35)*. Then for all $C = C(\nu) > 1$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists initial data p_{in} such that*

$$\|p_{in}\|_{H^N} = \varepsilon$$

and

$$\|p\|_{L^\infty H^N} \geq \varepsilon C.$$

Proof. For the sake of contradiction we assume that there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and for any choice of initial data with $\|p_{in}\|_{H^N} = \epsilon$ it holds that

$$\|p\|_{L^\infty H^N} \leq \varepsilon C.$$

Our plan is to choose initial data such that for a choice of ε and t we obtain a contradiction to this bound. In particular, we choose p_{in} as the data of the linear instability result, Lemma 4.6, such that the associated linear solution p_{lin} satisfies

$$\begin{aligned}\|p_{in}\|_{H^N} &= \varepsilon, \\ \|p_{lin}(t)\|_{H^{N-1}} &\geq t\tfrac{\nu^2}{32\alpha^4}.\end{aligned}$$

Let $S(\tau,t)$ be the solution operator for the linearized system. Then in view of (4.33) we have the estimate

$$\|S(\tau,t)\|_{H^N\to H^N} \lesssim \langle \nu(t-\tau)\rangle \leq \langle \nu t\rangle. \tag{4.36}$$

Thus, since

$$\partial_t(p-p_{lin}) \leq L(p-p_{lin}) + NL[p],$$

we deduce that

$$\begin{aligned}
\|p-p_{lin}\|_{H^{N-1}}^2 &\leq \int_0^t \|S(\tau,t)\|_{H^N\to H^N}\|p-p_{lin}\|_{H^{N-1}}\|p\|_{H^{N-1}}\|\nabla_t p\|_{H^{N-1}} \\
&\lesssim \|p-p_{lin}\|_{L^\infty H^{N-1}}\|p\|_{L^\infty H^{N-1}}\|p\|_{L^\infty H^N} 2\int_0^t \tau\langle \nu\tau\rangle \\
&\lesssim t^2\langle \nu t\rangle \varepsilon^2 C^2 \|p-p_{lin}\|_{L^\infty H^{N-1}}.
\end{aligned}$$

This yields the estimate

$$\|p-p_{lin}\|_{L^\infty H^{N-1}} \leq \tilde{C} t^2 \langle \nu t\rangle \varepsilon^2 C^2,$$

for some $\tilde{C}$. Finally, we obtain

$$\begin{aligned}
\|p\|_{H^{N-1}} &\geq \|p_{lin}\|_{H^{N-1}} - \|p-p_{lin}\|_{L^\infty H^{N-1}} \\
&\geq \|p_{lin}\|_{H^{N-1}} - t^2\langle \nu t\rangle \varepsilon^2 \tilde{C} C^2 \\
&\geq t\varepsilon\left(\tfrac{\nu^2}{32\alpha^4} - t^2\langle \nu t\rangle \varepsilon C^2\right).
\end{aligned}$$

This completes our proof by contradiction provided this term is large enough for a given small ε and suitable time. Indeed for the choice $\varepsilon \leq \frac{1}{8}\frac{\nu^6}{32^3 C^4 \tilde{C}\alpha^{10}}$ at the time $t = 2C\frac{32\alpha^4}{\nu^2}$ it holds that

$$\|p\|_{H^{N-1}} \geq \tfrac{1}{2} t \tfrac{\nu^2}{32\alpha^4}\varepsilon \geq C\varepsilon.$$

This concludes our proof of the nonlinear norm inflation and hence completes our proof of Proposition 4.2. □

The behavior of the MHD equations and, in particular, the interaction of shear flows, the magnetic field and dissipation are an area of current active research [Lis20, Dol24, ZZ24, KZ1]. However, prior works have focused on cases where the resistivity is at least as strong as the fluid viscosity and where thus the behavior is closely related to that of the Navier-Stokes equations. In contrast, the non-resistive MHD equations exhibit additional instability, as for instance shown in Proposition 4.2.

Motivated by this dichotomy, in this article we have studied the anisotropic, partial dissipation regime

$$\kappa_y = 0,\ \kappa_x = \nu_x = \nu_y$$

and the associated stability threshold in the inviscid limit. As shown in Theorem 4.3 and highlighted in the estimates of Sections 4.2, 4.3.4 and 4.3.3, this partial dissipation regime behaves qualitatively differently than both the fully dissipative case and the non-resistive case. Moreover, our analysis crucially used the coupling of the velocity field and magnetic field induced by the underlying magnetic field, which allowed us to obtain improved estimates for the magnetic field despite the lack of the symmetry of the dissipation.

Partial, anisotropic dissipation in the MHD equations is thus shown to give rise to distinct regimes with different (in)stability properties and demonstrates an intricate interplay of shear dynamics, magnetic interaction and anisotropic dissipation. A more complete understanding of all these regimes, the case of resistivity vanishing faster than viscosity and a characterization of the (in)stability properties of the ideal MHD equations remain exciting questions for future research.

Chapter 5

Sobolev Stability for the 2D MHD Equations in the Non-Resistive Limit

This chapter is the preprint [K].

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Abstract. This chapter considers the stability of the 2D magnetohydrodynamics (MHD) equations close to a combination of Couette flow and a constant magnetic field. We consider the ideal conductor limit for the case when viscosity ν is larger than resistivity κ, $\nu \geq \kappa > 0$. For this regime, we establish a bound on the Sobolev stability threshold. Furthermore, for $\kappa \leq \nu^3$ this system exhibits instability, which leads to norm inflation of size $\nu\kappa^{-\frac{1}{3}}$.

5.1 Introduction

The equations of magnetohydrodynamics (MHD)

$$\begin{aligned} \partial_t V + V \cdot \nabla V + \nabla \Pi &= \nu \Delta V + B \cdot \nabla B, \\ \partial_t B + V \cdot \nabla B &= \kappa \Delta B + B \cdot \nabla V, \\ \nabla \cdot V = \nabla \cdot B &= 0, \\ (t, x, y) &\in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R} =: \Omega, \end{aligned} \tag{5.1}$$

model the evolution of a magnetic field $B : \Omega \to \mathbb{R}^2$ interacting with the velocity $V : \Omega \to \mathbb{R}^2$ of a conducting fluid. The MHD equations are a common model used in astrophysics, planetary magnetism and controlled nuclear fusion [Dav16]. The quantities $\nu, \kappa \geq 0$ correspond to fluid viscosity and magnetic resistivity.

The pressure $\Pi : \Omega \to \mathbb{R}$ ensures that the velocity remains divergence-free. A fundamental problem of fluid dynamics and plasma physics is the stability and long-time behavior of solutions to equation (5.1) and in particular stability of specific solutions. We consider the combination of an affine shear flow, called Couette flow, and a constant magnetic field:

$$\begin{aligned} V_s &= ye_1, \\ B_s &= \alpha e_1. \end{aligned}$$

In particular, the solution combines the effects of mixing due to shear and coupling by the magnetic field. The Couette flow mixes any perturbation, which leads to increased dissipation rates, called enhanced dissipation, and stabilizes the equation. The coupling with a constant magnetic field propagates this mixing to magnetic perturbations. However, the magnetic field weakens the mixing, especially if viscosity is larger than resistivity, inviscid damping gets counteracted by algebraic growth for specific time regimes.

In the related case of the Navier-Stokes equation, that is when no magnetic field is present, one observes turbulent solutions as viscosity reaches small values. In contrast, the linearized problem around Couette flow is stable for all values of the viscosity. These phenomena are known as the Sommerfeld paradox [LL11] and highlight instability due to nonlinear effects. In [BM15a, DM23, DZ21, BM14, IJ13] various authors show sharp stability in Gevrey 2 spaces (spaces between C^∞ and analytic) for the inviscid case $\nu = 0$. The nonlinear instability can be suppressed by the viscosity for initial data sufficiently small in Sobolev spaces, ensuring stability [BVW18, MZ22, BGM17].

When considering the MHD equations without Couette flow, the constant magnetic field stabilizes the equation. The dynamics of small initial perturbations of the ideal MHD equation around a strong enough magnetic field is close to the linearized system [BSS88]. For stability in several dissipation regimes we refer to [WZ17, RWXZ14, HXY18, RWXZ14, Sch88, CF23, Koz89] and references therein. However, global in time wellposedness for the non-resistive case is still open (see the discussion in [CF23]). Furthermore, a shear flow leads to qualitatively different behavior and instabilities [HT01, HHKL18].

Recently, the MHD equation around Couette flow has gathered significant interest [Lis20, KZ1, ZZ24, Dol24, KZ2]. Already on a linear level, the behavior of the MHD changes for different values of ν and κ. In [Lis20] Liss proved the first stability threshold for the MHD equations. He considered the full dissipative regime of $\kappa = \nu > 0$ and proved the stability of the three-dimensional MHD equation for initial data which is sufficiently small in Sobolev spaces. For the analogous two-dimensional problem, Dolce [Dol24] proved stability in the more general setting of $0 < \kappa^3 \lesssim \nu \leq \kappa$. In [KZ2] Zillinger and the author considered the case of only horizontal resistivity and full viscosity and established stability for small data in Sobolev spaces. For the regime of vanishing viscosity $\nu = 0$ and non-vanishing resistivity $\kappa > 0$, in [KZ1] we constructed a linear stability and instability mechanism around nearby traveling waves in Gevrey 2 spaces. In a corresponding nonlinear stability result, Zhao and Zi [ZZ24] proved the almost matching nonlinear result of Gevrey σ stability for $1 \le \sigma < 2$ and for sufficiently small perturbations.

The results mentioned above on stability around Couette flow focus on the setting when resistivity is larger than viscosity $\nu \le \kappa$. Indeed in the setting $\nu > 0$ and $\kappa = 0$, the magnetic effects dominate leading to a linear instability mechanism and thus a growth of the magnetic field by νt for specific initial data [KZ2].

In this paper, we consider the setting $0 < \kappa \le \nu$. In particular, this also includes the non-resistive limit $\kappa \downarrow 0$ independent of ν. To the author's knowledge the stability of the regime $\kappa < \nu$ has not previously been studied for the MHD equation around Couette flow. To state the main result, we define the perturbative unknowns

$$\begin{aligned} v(x,y,t) &= V(x+yt,y,t) - V_s, \\ b(x,y,t) &= B(x+yt,y,t) - B_s, \end{aligned}$$

where the change of variables $x \mapsto x + yt$ follows the characteristics of the Couette flow. For these unknowns, equation (5.1) becomes

$$\begin{aligned} \partial_t v + v_2 e_1 - 2\partial_x \Delta_t^{-1} \nabla_t v_2 &= \nu \Delta_t v + \alpha \partial_x b + b \nabla_t b - v \nabla_t v - \nabla_t \pi, \\ \partial_t b - b_2 e_1 &= \kappa \Delta_t b + \alpha \partial_x v + b \nabla_t v - v \nabla_t b, \\ \nabla_t \cdot v = \nabla_t \cdot b &= 0. \end{aligned} \tag{5.2}$$

Due to the change of variables the spatial derivatives become time-dependent, i.e. $\partial_y^t = \partial_y - t\partial_x$, $\nabla_t = (\partial_x, \partial_y^t)^T$ and $\Delta_t = \partial_x^2 + (\partial_y^t)^2$.

For equation (5.2) we establish Lipschitz stability for initial data which is sufficiently small in Sobolev spaces, in the sense that there exists a bound on the initial data $\varepsilon_0 = \varepsilon_0(\nu, \kappa)$ and a Lipschitz constant $L = L(\nu, \kappa)$ such that for initial data which satisfies

$$\|(v,b)_{in}\|_{H^N} = \varepsilon \le \varepsilon_0,$$

the corresponding solution is globally bounded in time by

$$\|(v,b)(t)\|_{H^N} \le L\varepsilon.$$

For the non-resistive case, $\kappa = 0$, global wellposedness is an open problem and so Lipschitz stability in Sobolev spaces is unclear. Thus, naturally the question arises, which ε_0 and L are optimal and how they behave in the limit $\nu, \kappa \downarrow 0$. We denote a Sobolev stability threshold as $\gamma_1, \gamma_2 \in \mathbb{R}$, such that for $\varepsilon_0 = c_0 \nu^{\gamma_1} \kappa^{\gamma_2}$ with small $c_0 > 0$ we obtain

$$\begin{aligned} \|(v,b)_{in}\|_{H^N} &\le c_0 \nu^{\gamma_1} \kappa^{\gamma_2} \to \text{ stability}, \\ \|(v,b)_{in}\|_{H^N} &\gg c_0 \nu^{\gamma_1} \kappa^{\gamma_2} \to \text{possible instability}. \end{aligned}$$

This extends the common convention in the field (eg. see [BVW18]) to allow for two independent parameters ν and κ. In particular, it agrees with the common convention when restricting to the case $\nu \approx \kappa$. It allows us to discuss cases where κ tends to zero much quicker than ν. Establishing a possible instability is highly nontrivial since for the nonlinear setting it is difficult to construct solutions that exhibit norm inflation. To the author's knowledge, there does not exist any nonlinear instability result for the MHD equation around Couette flow in Sobolev spaces.

For accessibility and simplicity of notation, we state our main result as the following theorem (see Theorem 1 for a detailed description).

Theorem 5.1. *Consider $\alpha > \frac{1}{2}$, $N \geq 5$ and a small enough constant $c_0 = c_0(\alpha) > 0$. Let $0 < \kappa \leq \nu \leq \frac{1}{40}(1 - \frac{1}{2\alpha})^{\frac{6}{5}}$, then we obtain Sobolev stability for initial data which is sufficiently small in Sobolev spaces, where the estimates qualitatively differ for the regimes $\kappa \gtrsim \nu^3$ and $\kappa \lesssim \nu^3$. More precisely:*

- *In the regime of $\nu^3 \lesssim \kappa$, for all initial data which satisfy*
$$\|(v, b)_{in}\|_{H^N} = \varepsilon \leq c_0 \nu^{\frac{1}{12}} \kappa^{\frac{1}{2}},$$
the global in time solution (v, b) of (5.2) satisfies the Lipschitz bound
$$\sup_{t>0} \|(v, b)(t)\|_{H^N} \lesssim \varepsilon.$$

- *In the regime of $\nu^3 \gtrsim \kappa$, for all initial data which satisfy*
$$\|(v, b)_{in}\|_{H^N} = \varepsilon \leq c_0 \nu^{-\frac{11}{12}} \kappa^{\frac{5}{6}},$$
the global in time solution (v, b) of (5.2) satisfies the Lipschitz bound
$$\sup_{t>0} \|(v, b)(t)\|_{H^N} \lesssim \nu \kappa^{-\frac{1}{3}} \varepsilon.$$

In particular, we obtain Lipschitz stability for the Lipschitz constant $L \approx \max(1, \nu\kappa^{-\frac{1}{3}})$ for the smallness parameter $\varepsilon_0 \approx \min(\nu^{\frac{1}{12}} \kappa^{\frac{1}{2}}, \nu^{-\frac{11}{12}} \kappa^{\frac{5}{6}})$.

In the proof, we employ an energy method similar to [BBZD23, MZZ23, Zil21b, Dol24, KZ2]. In the following, we outline the main challenges and novelties of the proof:

- The imbalance of resistivity κ and viscosity ν yields two cases $\nu^3 \lesssim \kappa$ and $\nu^3 \gtrsim \kappa$ (or equivalently $1 \lesssim \nu\kappa^{-\frac{1}{3}}$ or $1 \gtrsim \nu\kappa^{-\frac{1}{3}}$). These cases give different values for L, namely 1 and $\nu\kappa^{-\frac{1}{3}}$. By Proposition 5.3, the norm inflation of $\nu\kappa^{-\frac{1}{3}}$ appears in the linear dynamics and thus is sharp.

- We consider the case $\nu^3 \gtrsim \kappa$. On certain time scales the viscosity is so strong that fluid effects get suppressed while the effects of the magnetic field dominate. Thus, the term $\partial_t b = e_1 b_2$ in (5.1) generates algebraic growth in specific regimes (see Subsection 5.2). Estimating this linear effect yields the norm inflation by $L = \nu\kappa^{-\frac{1}{3}}$. The algebraic growth appears on different time scales depending on the frequency, a precise estimate of the nonlinear terms is necessary.

- For the case $\nu^3 \lesssim \kappa$ the algebraic growth is bounded by a finite constant. In the subcase $\nu = \kappa$ the sum of the threshold parameters is $\gamma_1 + \gamma_2 = \frac{7}{12}$ which is a slight improvement over $\frac{2}{3}$ in [Dol24]. This improvement is attained by the choice of our adapted unknowns which changes the structure of the nonlinearity.

- In the proof of Theorem 5.1 we perform a low and high frequency decomposition $a = a_{hi} + a_{low}$. For high frequencies, the nonlinear term consist of $a_{low}\nabla_t a_{hi}$, called transport term and $a_{hi}\nabla_t a$, called reaction term (including $hi - hi$ interactions). Compared to the Navier-Stokes equation, in the case of the MHD equation, it is vital to bound the transport term precisely. In particular, for $\kappa \lesssim \nu^3$ the previously mentioned algebraic growth affects the estimate of the transport term strongly.

- The threshold is determined by the nonlinear term $v\nabla_t b = \Lambda_t^{-1}\nabla^\perp p_1 \nabla b$ acting on b in (5.2), for the natural unknown $p_1 = \Lambda_t^{-1}\nabla^\perp v$ (which we discuss later in more detail). In our estimates we rely on two stabilizing effects, the strong viscosity of v and the Λ_t^{-1} in front of p_1. For the nonlinear term $v\nabla_t b$ both effects fall onto v. Due to the weaker integrability of the b this term determines the threshold after integrating in time.

With the main challenges in mind, let us comment on the results:

- The size of the constant magnetic field $\alpha > \frac{1}{2}$ results in a strong interaction between v and b. Due to this interaction, the decay in v and growth in b are in balance (see Lemma 5.2). Constants may depend on α and degenerate as $\alpha \downarrow \frac{1}{2}$. For example we obtain $\lim_{\alpha \downarrow \frac{1}{2}} c_0(\alpha) = 0$.

- Figure 5.1 shows which areas stability has been proven. The graphic shows only qualitative behavior and after rescaling we obtain the same graphic. The resistivity κ is on the vertical axis and the viscosity ν is on the horizontal axis. We prove stability for the regime $0 < \kappa \leq \nu$, which we divide into two segments: $\nu^3 \lesssim \kappa$ in orange and $\nu^3 \gtrsim \kappa$ in red. In [Dol24] Dolce considered the regime of $0 < (\frac{16}{\alpha}\kappa)^3 \leq \nu \leq \kappa$, which is in blue. The authors of [ZZ24] considered the line $\nu = 0$ which is in purple. The black line corresponds to $\nu = \kappa > 0$ of [Lis20].

  Stability for the regimes $0 < \nu \leq (\frac{16}{\alpha}\kappa)^3$, $\kappa = 0 < \nu$ and $\kappa = \nu = 0$ remain open. For the set $0 < \nu \ll \kappa^3$, we expect that an adjusted application of the methods used in this article yield stability. We expect stability for the case $\kappa = 0$ and $0 < \nu$ to be very difficult since we obtain linear growth for the p_2 variable. For $\Lambda_t^{-1}p_2$ we obtain linear stability but then there is no time decay in the magnetic field and so we lack an important stabilizing effect. In the inviscid case, $\kappa = \nu = 0$ the linearized system is stable in the p variables. However, due to the lack of dissipation, it is very challenging to bound the nonlinear terms.

- Our threshold consists of parameters γ_1 and γ_2. An alternative notation is to impose the relation $\nu \approx \kappa^\delta$ for some $0 \leq \delta \leq 1$. With that convention

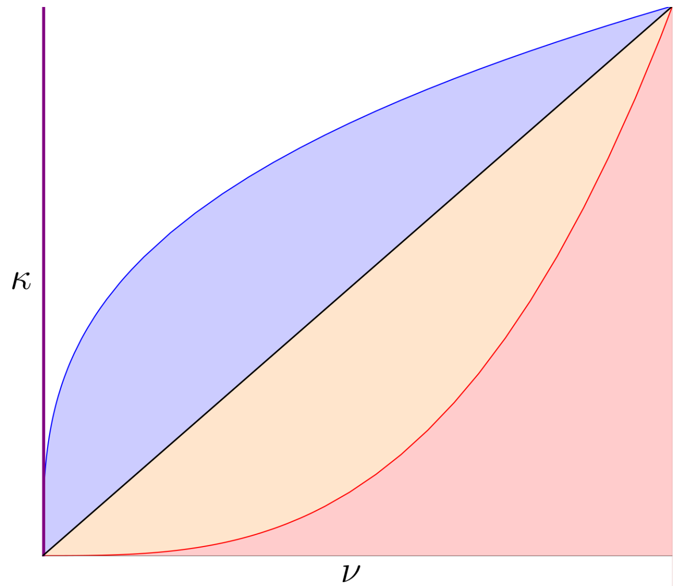

Figure 5.1: Sketch of areas with results for stability.

we obtain stability if $\varepsilon \leq c_0 \kappa^{\gamma(\delta)}$ for

$$\gamma = \begin{cases} \frac{1}{2} + \frac{\delta}{12} & \delta \geq \frac{1}{3} \\ \frac{5}{6} - \frac{11}{12}\delta & \text{otherwise.} \end{cases}$$

The remainder of this article is structured as follows:

- In Section 5.2 we discuss the linearized system. We identify two different time regions where "circular movement" or "strong viscosity" determine the linearized behavior. We estimate both effects separately and then establish the estimates for the linearized system.
- In Section 5.3 we prove the main theorem. We employ a bootstrap approach, where we control errors in Proposition 5.4. The main difficulty is to bound the linear growth and the nonlinear effect of $v\nabla_t b$ acting on b.

Notations and Conventions

For $a, b \in \mathbb{R}$ we denote their minimum and maximum as

$$\min(a, b) = a \wedge b,$$
$$\max(a, b) = a \vee b.$$

We write $f \lesssim g$ if $f \leq Cg$ for a constant C independent of ν and κ. Furthermore, we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. We denote the Lebesgue spaces $L^p =$

$L^p(\mathbb{T}\times\mathbb{R})$ and the Sobolev spaces $H^N = H^N(\mathbb{T}\times\mathbb{R})$ for some $N \in \mathbb{N}$. For time-dependent functions, we denote $L^pH^s = L^p_tH^s$ as the space with the norm

$$\|f\|_{L^pH^s} = \left\| \|f\|_{H^s(\mathbb{T}\times\mathbb{R})} \right\|_{L^p(0,T)}, \tag{5.3}$$

where omit writing the T. We write the time-dependent spatial derivatives

$$\begin{aligned} \partial_y^t &= \partial_y - t\partial_x, \\ \nabla_t &= (\partial_x, \partial_y^t)^T, \\ \Delta_t &= \partial_x^2 + (\partial_y^t)^2, \end{aligned}$$

and the half Laplacians as

$$\begin{aligned} \Lambda &= (-\Delta)^{\frac{1}{2}}, \\ \Lambda_t &= (-\Delta_t)^{\frac{1}{2}}. \end{aligned}$$

The function $f \in H^N$ is decomposed into its x average and the orthogonal complement

$$\begin{aligned} f_=(y) &= \int f(x,y)dx, \\ f_{\neq} &= f - f_=. \end{aligned}$$

The adapted unknowns

For the following, it is useful to change to the unknowns $p_{1,\neq} = \Lambda_t^{-1}\nabla_t^{\perp} v_{\neq}$ and $p_{2,\neq} = \Lambda_t^{-1}\nabla_t^{\perp} b_{\neq}$. However, since $\Lambda_t^{-1}\nabla_t^{\perp}$ is not a bounded operator on the x average, we define

$$\begin{aligned} p_{1,\neq} &= \Lambda_t^{-1}\nabla_t^{\perp} v_{\neq}, \\ p_{1,=} &= v_{1,=}, \\ p_{2,\neq} &= \Lambda_t^{-1}\nabla_t^{\perp} b_{\neq}, \\ p_{2,=} &= b_{1,=}. \end{aligned}$$

Thus (5.2) can be equivalently expressed as

$$\begin{aligned} \partial_t p_1 - \partial_x\partial_y^t\Delta_t^{-1}p_1 - \alpha\partial_x p_2 &= \nu\Delta_t p_1 + \Lambda_t^{-1}\nabla_t^{\perp}(b\nabla_t b - v\nabla_t v), \\ \partial_t p_2 + \partial_x\partial_y^t\Delta_t^{-1}p_2 - \alpha\partial_x p_1 &= \kappa\Delta_t p_2 + \Lambda_t^{-1}\nabla_t^{\perp}(b\nabla_t v - v\nabla_t b), \\ p|_{t=0} &= p_{in}. \end{aligned} \tag{5.4}$$

These unknowns are particularly useful since

$$\begin{aligned} \|Ap_1\|_{L^2} &= \|Av\|_{L^2}, \\ \|Ap_2\|_{L^2} &= \|Ab\|_{L^2}, \end{aligned}$$

for all Fourier multipliers A such that one side is finite. We note that we can recover $v_{\neq}$ and $b_{\neq}$ from $p_{\neq}$ by

$$\begin{aligned} v_{\neq} &= -\Lambda_t^{-1}\nabla_t^{\perp} p_{1,\neq} \\ b_{\neq} &= -\Lambda_t^{-1}\nabla_t^{\perp} p_{2,\neq}. \end{aligned}$$

The operator $\Lambda_t^{-1}\nabla_t^{\perp}$ can be seen as the perpendicular Riesz transform shifted in time on frequency space applied to either a vector or a scalar. It satisfies $(\Lambda_t^{-1}\nabla_t^{\perp}) \circ (\Lambda_t^{-1}\nabla_t^{\perp}) = -\operatorname{Id}$ in H^N.

5.2 Linear Stability and Norm Inflation

In this section, we consider the behavior of the linearized version of (5.4):

$$\begin{aligned} \partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1, \\ \partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \Delta_t p_2. \end{aligned} \tag{5.5}$$

For this equation, we establish the following proposition:

Proposition 5.2 (Linear energy estimate)**.** *Consider $\alpha > \frac{1}{2}$, $N \geq 0$ and $0 < \kappa \leq \nu$. Let $p_{in} \in H^N$ with $p_{in,=} = \int p_{in} dx = 0$, then the solution p of* (5.5) *satisfies the bound*

$$\|p(t)\|_{H^N} \lesssim e^{-c\kappa^{\frac{1}{3}} t}(1 + \nu \kappa^{-\frac{1}{3}}) \|p_{in}\|_{H^N}. \tag{5.6}$$

Furthermore, we obtain for specific initial data norm inflation of $\nu\kappa^{-\frac{1}{3}}$. The proof uses Lemma 3 from [KZ2].

Proposition 5.3 (Linear norm inflation)**.** *Consider $\alpha > \frac{1}{2}$, $N \geq 1$ and $\nu \geq \max(2\kappa, \kappa^{\frac{1}{3}})$, then there exist initial data p_{in} and such that at the time $T = \kappa^{-\frac{1}{3}}$ the solution p of* (5.5) *satisfies*

$$\|p(T)\|_{H^N} \gtrsim \nu \kappa^{-\frac{1}{3}} \|p_{in}\|_{H^N}. \tag{5.7}$$

For the proof of the propositions, we perform a Fourier transform $(x, y) \mapsto (k, \xi)$ and replace p_1 by ip_1. Hence the system (5.5) can be equivalently (for $k \neq 0$) written as

$$\begin{aligned} \partial_t p_1 &= -\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} p_1 - \alpha k p_2 - \nu k^2 (1 + (t - \tfrac{\xi}{k})^2) p_1, \\ \partial_t p_2 &= \tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} p_2 + \alpha k p_1 - \kappa k^2 (1 + (t - \tfrac{\xi}{k})^2) p_2. \end{aligned} \tag{5.8}$$

Here, with slight abuse of notation, we omit writing the Fourier transformation. This equation has several effects that appear in different regimes of $t - \frac{\xi}{k}$, which we discuss in the following. The effect of circular movement appears for $|t - \frac{\xi}{k}| \lesssim \nu^{-1}$ and the effect of strong viscosity for $\nu^{-1} \lesssim |t - \frac{\xi}{k}| \lesssim \kappa^{-\frac{1}{3}}$. Then before proceeding to the proof of the propositions, we briefly sketch these effects.

Circular movement

To highlight the effect of the constant magnetic field α in (5.5) we consider the toy model

$$\begin{aligned} \partial_t p_1 &= -\alpha k p_2, \\ \partial_t p_2 &= \alpha k p_1. \end{aligned} \tag{5.9}$$

This is solved by

$$p(t) = \begin{pmatrix} \cos(\alpha k t) & -\sin(\alpha k t) \\ \sin(\alpha k t) & \cos(\alpha k t) \end{pmatrix} p_{in}.$$

We call this effect of the constant magnetic field (5.9) circular movement, which leads to a transfer between p_1 and p_2. This circular movement is counteracted by viscosity for times away from $\frac{\xi}{k}$.

Effect of strong viscosity

Let us consider the case when $0 < \kappa \ll \nu$ and for simplicity of notation let $k = 1$ and $\xi = 0$. Due to the viscosity, we obtain $p_1 \approx 0$ for large times $t \geq t_0 \gg 1$. Then from (5.8) we deduce the toy model

$$\partial_t p_2 = (\tfrac{t}{1+t^2} - \kappa(1+t^2))p_2. \tag{5.10}$$

The first term in (5.10) leads to linear growth until the resistivity is strong enough for the second term to take over. This is seen in the explicit solution of (5.10)

$$p_2(t) = \tfrac{\langle t \rangle}{\langle t_0 \rangle} \exp\left(-\kappa \int_{t_0}^{t} 1 + \tau^2 \, d\tau\right) p_2(t_0).$$

This is estimated by

$$p_2(t) \lesssim t_0^{-1} \kappa^{-\frac{1}{3}} e^{-c\kappa^{\frac{1}{3}}(t-t_0)} p_2(t_0),$$

which corresponds to the maximal growth which we obtain. In the following, we will see that $t_0 \approx \nu^{-1}$ is the time after which viscosity dominates. The reader may expect that the enhanced dissipation timescale $\nu^{-\frac{1}{3}}$ would be the relevant timescale, but the combination of circular movement and the viscosity gives enough decay for p_2 such that the linear growth gets suppressed until the time ν^{-1}.

Proof of Proposition 5.2

Proof. For simplicity of notation, we introduce the new variable $s = t - \frac{\xi}{k}$ and initial time $s_{in} = -\frac{\xi}{k}$. Then equation (5.8) reads

$$\begin{aligned} \partial_s p_1 &= -\tfrac{s}{1+s^2} p_1 - \alpha k p_2 - \nu k^2 (1+s^2) p_1, \\ \partial_s p_2 &= \tfrac{s}{1+s^2} p_2 + \alpha k p_1 - \kappa k^2 (1+s^2) p_2. \end{aligned}$$

Further we change the unknown to $\tilde{p} = \exp(\frac{\kappa}{2}k^2(s - s_{in} + \frac{1}{3}(s^3 - s_{in}^3)))p$. For $\tilde{\kappa} = \frac{\kappa}{2}$ and $\tilde{\nu} = \nu - \frac{\kappa}{2}$, this yield the equation

$$\begin{aligned}\partial_s \tilde{p}_1 &= -\tfrac{s}{1+s^2}\tilde{p}_1 - \alpha k \tilde{p}_2 - \tilde{\nu} k^2(1+s^2)\tilde{p}_1,\\ \partial_s \tilde{p}_2 &= \tfrac{s}{1+s^2}\tilde{p}_2 + \alpha k \tilde{p}_1 - \tilde{\kappa} k^2(1+s^2)\tilde{p}_1.\end{aligned}$$

Let us denote $s_0 := \nu^{-1}$ and in the following, we distinguish between times $|s| \leq s_0$ and $|s| \geq s_0$. We first consider the case $s_{in} \leq -s_0$. For $|s| \leq s_0$, the circular movement is not suppressed by the viscosity.

We define the energy $E = |\tilde{p}|^2 + \frac{1}{\alpha k}\frac{2s}{1+s^2}\tilde{p}_1\tilde{p}_2$, then E is a positive quadratic form due to our assumption $\alpha > \frac{1}{2}$ and satisfies

$$(1 - \tfrac{1}{2\alpha k})|\tilde{p}|^2 \leq E \leq (1 + \tfrac{1}{2\alpha k})|\tilde{p}|^2.$$

We calculate the time derivative

$$\begin{aligned}&\partial_s E + \tilde{\nu} k^2(1+s^2)\tilde{p}_1^2 + \tilde{\kappa} k^2(1+s^2)\tilde{p}_2^2\\ &\quad = \tfrac{1}{\alpha k}\partial_s(\tfrac{2s}{1+s^2})\tilde{p}_1\tilde{p}_2 - 2s\tfrac{(\tilde{\nu}-\tilde{\kappa})k}{\alpha}\tilde{p}_1\tilde{p}_2\\ &\quad \leq \tfrac{1}{\alpha k}\partial_s(\tfrac{2s}{1+s^2})\tilde{p}_1\tilde{p}_2 + \tfrac{1}{2}\tilde{\nu} k^2(1+s^2)\tilde{p}_1^2 + \tfrac{2\tilde{\nu}}{\alpha^2}\tilde{p}_2^2\end{aligned}$$

and so with $|\tilde{p}|^2 \leq \frac{2\alpha}{2\alpha-1}E$ we infer

$$|\partial_s E| \leq \tfrac{\alpha}{\alpha-\frac{1}{2}}(\tfrac{1}{1+s^2} + 2\tfrac{\tilde{\nu}}{\alpha^2})E.$$

Gronwall's lemma implies

$$E(s_0) \leq \exp\left(\tfrac{\alpha}{\alpha-\frac{1}{2}}(\pi + 2\tfrac{\tilde{\nu}}{\alpha^2}|s_0|)r\right)E(-s_0).$$

Since $\nu s_0 = 1$, we deduce

$$E(s_0) \lesssim E(-s_0)$$

and thus

$$|\tilde{p}(s_0)| \lesssim |\tilde{p}(-s_0)|. \tag{5.11}$$

Consider the case $|s| \geq s_0$, we calculate

$$\begin{aligned}\tfrac{1}{2}\partial_s|\tilde{p}|^2 \leq{}& (-\tfrac{s}{1+s^2} - \tilde{\nu} k^2(1+s^2))\tilde{p}_1^2\\ &+ (\tfrac{s}{1+s^2} - \tilde{\kappa} k^2(1+s^2))\tilde{p}_2^2,\end{aligned}$$

and since $(-\frac{s}{1+s^2} - \tilde{\nu} k^2(1+s^2)) \leq 0$ for all $|s| \geq s_0$ we conclude

$$\partial_s|\tilde{p}|^2 \leq (\tfrac{2s}{1+s^2} - \tilde{\kappa} k^2(1+s^2))_+\tilde{p}_2^2.$$

Thus we obtain the estimate

$$|\tilde{p}(s)|^2 \leq \begin{cases} |\tilde{p}(s_{in})|^2 & s \leq -s_0, \\ \frac{1+s^2}{1+s_0^2}|\tilde{p}(s_0)|^2 & s_0 \leq s \leq 2(\kappa k^2)^{-\frac{1}{3}}, \\ (1+4\nu^2\kappa^{-\frac{2}{3}}k^{-\frac{4}{3}})|\tilde{p}(s_0)|^2 & s_0 \vee 2(\kappa k^2)^{-\frac{1}{3}} \leq s. \end{cases}$$

Combining this with (5.11) we infer

$$|\tilde{p}(s)| \lesssim (1+\nu\kappa^{-\frac{1}{3}}k^{-\frac{2}{3}})|p(s_{in})|.$$

The case $s_{in} \geq -s_0$ is established similarly since we only bound the growth. With

$$\exp(-\tfrac{\kappa}{2}k^2(s-s_{in}+\tfrac{1}{3}(s^3-s_{in}^3))) \lesssim e^{-c\kappa^{\frac{1}{3}}t}$$

we deduce

$$\begin{aligned} |p(s)| &\lesssim e^{-c\kappa^{\frac{1}{3}}t}|\tilde{p}|(s) \\ &\lesssim (1+\nu\kappa^{-\frac{1}{3}}k^{-\frac{2}{3}})e^{-c\kappa^{\frac{1}{3}}t}|p|(s_{in}). \end{aligned}$$

Equation (5.8) decouples in ξ and k, so we infer the proposition with this estimate.

□

Proof of Proposition 5.3

Proof. We introduce the notations $\tilde{p}(t,k,\xi) = \exp(\kappa k^2 \int_0^t (1+(\tau-\frac{\xi}{k})^2) d\tau) p(t,k,\xi)$ and $\tilde{\nu} := \nu - \kappa$. Then the equations (5.8) read

$$\begin{aligned} \partial_t \tilde{p}_1 &= -\frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}\tilde{p}_1 - \alpha k \tilde{p}_2 - \tilde{\nu}k^2(1+(t-\tfrac{\xi}{k})^2)\tilde{p}_1, \\ \partial_t \tilde{p}_2 &= \frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}\tilde{p}_2 + \alpha k \tilde{p}_1, \\ \tilde{p}_{in} &= p_{in}. \end{aligned} \tag{5.12}$$

We point out that this exactly agrees with the linearized equation with the non-resistive case. Specifically in Lemma 3 of [KZ2] it was shown that there exists frequency localized initial data such that for all $t > 0$ and frequencies $\xi \in [2\frac{\alpha^2}{\tilde{\nu}}, 4\frac{\alpha^2}{\tilde{\nu}}]$ it holds that

$$\tilde{p}_2(t,-1,\xi) \geq \tfrac{t}{2\xi} p_{2,in}(-1,\xi) \gtrsim \tilde{\nu} t p_{2,in}(-1,\xi). \tag{5.13}$$

From this lower bound, we deduce the norm inflation for the non-resistive limit. For times $\tau \in [0, \kappa^{-\frac{1}{3}}]$ and frequencies $\xi \in [2\frac{\alpha^2}{\tilde{\nu}}, 4\frac{\alpha^2}{\tilde{\nu}}]$ we obtain

$$|\tau + \xi| \lesssim \nu^{-1} + \kappa^{-\frac{1}{3}}.$$

Then for $T = \kappa^{-\frac{1}{3}}$ there exist a $C = C(\alpha)$, such that

$$1 \geq \exp(-\kappa \int_0^T (1 + (t+\xi)^2)dt) \gtrsim \exp(-C\kappa(\nu^{-1} + \kappa^{-\frac{1}{3}})^2 \kappa^{-\frac{1}{3}}) \geq \exp(-2C) \gtrsim 1.$$

Where we used in the last estimate, that $\kappa \leq \nu^3$ yields $\kappa(\nu^{-1} + \kappa^{-\frac{1}{3}})^2\kappa^{-\frac{1}{3}} \leq 2$. From (5.13) and $\tilde{\nu} = \nu - \kappa \geq \frac{1}{2}\nu$, since $\nu \geq 2\kappa$, we deduce, that

$$p_2(T, -1, \xi) = \exp(-\kappa \int_0^T (1 + (t+\xi)^2)dt)\tilde{p}_2(T, -1, \xi) \gtrsim \nu\kappa^{-\frac{1}{3}} p_{2,in}(-1, \xi),$$

From this, we infer the norm inflation

$$\|p(T)\|_{H^N} \gtrsim \nu\kappa^{-\frac{1}{3}} \|p_{in}\|_{H^N}.$$

□

5.3 Sobolev Stability for the Nonlinear System

The following theorem is a more general statement of Theorem 5.1. We dedicate the remainder of the section to the proof.

Theorem 1. *Let $\alpha > \frac{1}{2}$ and $N \geq 5$, then there exist $c_0, c > 0$, such that for all $0 < \kappa \leq \nu \leq \frac{1}{40}(1 - \frac{1}{2\alpha})^{\frac{6}{5}}$ there exist $L = \max(1, \nu\kappa^{-\frac{1}{3}})$, such that for all initial data, which satisfy*

$$\begin{aligned} \|(v,b)_{in,\neq}\|_{H^N} &= \varepsilon \leq c_0 L^{-1} \nu^{\frac{1}{12}} \kappa^{\frac{1}{2}}, \\ \|(v,b)_{in,=}\|_{H^N} &\leq \tilde{\varepsilon}, \qquad \text{with } \varepsilon \leq \tilde{\varepsilon} \leq \nu^{-\frac{1}{12}}\varepsilon \end{aligned} \tag{5.14}$$

the corresponding solution of (5.2) *satisfies the bound*

$$\begin{aligned} \|(v,b)_{\neq}(t)\|_{L^\infty H^N} + \|\nabla_t(\nu v, \kappa b)_{\neq}\|_{L^2 H^N} &\lesssim L e^{-c\kappa^{\frac{1}{3}} t}\varepsilon, \\ \|(v,b)_{=}(t)\|_{L^\infty H^N} + \|\partial_y(\nu v, \kappa b)_{=}\|_{L^2 H^N} &\lesssim \tilde{\varepsilon}. \end{aligned} \tag{5.15}$$

Furthermore, we obtain the following enhanced dissipation estimates

$$\|v_{\neq}\|_{L^2 H^N} \lesssim L\nu^{-\frac{1}{6}} e^{-c\kappa^{\frac{1}{3}} t}\varepsilon,$$
$$\|b_{\neq}\|_{L^2 H^N} \lesssim L\kappa^{-\frac{1}{6}} e^{-c\kappa^{\frac{1}{3}} t}\varepsilon.$$

This theorem implies Theorem 5.1. With slight abuse of notation, we write L as the ν and κ dependent part of the Lipschitz constant. We prove this theorem by using a bootstrap method. Let A be the Fourier weight

$$A := M|\nabla|^N e^{c\kappa^{\frac{1}{3}} t \mathbf{1}_{\neq}},$$

where $M = M_L M_1 M_\kappa M_\nu M_{\nu^3}$ are defined as

$$\begin{aligned}
\frac{-\dot M_L}{M_L} &= \frac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2}\mathbf{1}_{\{\nu^{-1}\le t-\frac{\xi}{k}\le (c_1\kappa k^2)^{-\frac13}\}} && k\neq 0,\\
\frac{-\dot M_1}{M_1} &= C_\alpha \frac{|k|+\nu^{\frac{1}{12}}|k|^2}{k^2+(\xi-kt)^2} && k\neq 0,\\
\frac{-\dot M_\nu}{M_\nu} &= \frac{\nu^{\frac13}}{1+\nu^{\frac23}(t-\frac{\xi}{k})^2} && k\neq 0,\\
\frac{-\dot M_\kappa}{M_\kappa} &= \frac{\kappa^{\frac13}}{1+\kappa^{\frac23}(t-\frac{\xi}{k})^2} && k\neq 0,\\
\frac{-\dot M_{\nu 3}}{M_{\nu 3}} &= \frac{C_\alpha \nu}{1+\nu^2(t-\frac{\xi}{k})^2} && k\neq 0,\\
M.(t=0) &= M.(k=0)=1.
\end{aligned}$$

The weight M_L is an adaption of the weight $m^{\frac12}$ in [Lis20] to our setting. The method of using time-dependent Fourier weights is common when working with solutions around Couette flow and the other weights are modifications of previously used weights (cf. [BVW18, MZ22, Lis20, ZZ24]). For simplicity, here we only state their main properties and refer to Appendix 5.4 for a detailed description. The constants $C_\alpha = \frac{2}{\min(1,\alpha-\frac12)}$, $c = \frac{1}{200}(1-\frac{1}{2\alpha})^2$ and $c_1 = \frac{1}{20}(1-\frac{1}{2\alpha})$ are determined through the linear estimates. For the weights we obtain

$$\begin{aligned}
L^{-1} \le \min(1, \nu^{-1}\kappa^{\frac13}k^{\frac23}) &\lesssim M_L \le 1,\\
M_1 \approx M_\kappa \approx M_\nu &\approx M_{\nu^3} \approx 1.
\end{aligned} \tag{5.16}$$

We note that the weight M_L is distinct from the others due to its lower bound L^{-1}, which depends on ν and κ. The weight M_L is necessary to bound the linear growth in the region $\nu^{-1} \lesssim t - \frac{\xi}{k} \lesssim (\kappa k^2)^{-\frac13}$. Controlling the effects of M_L is one of the main challenges in the proof of Theorem 1. We recall the unknowns p and equation (5.4)

$$\begin{aligned}
\partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha \partial_x p_2 &= \nu \Delta_t p_1 + \Lambda_t^{-1}\nabla_t^\perp (b\nabla_t b - v\nabla_t v),\\
\partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha \partial_x p_1 &= \kappa \Delta_t p_2 + \Lambda_t^{-1}\nabla_t^\perp (b\nabla_t v - v\nabla_t b),\\
p|_{t=0} &= p_{in}.
\end{aligned} \tag{5.17}$$

Let $\chi \in C^\infty(\mathbb{R}_+ \times \mathbb{Z} \times \mathbb{R})$ be a Fourier multiplier defined by

$$\chi = \chi(k,\xi) = \begin{cases} 1 & |t-\frac{\xi}{k}| \le \nu^{-1} \\ 0 & |t-\frac{\xi}{k}| \ge 2\nu^{-1} \end{cases} \tag{5.18}$$

$$\partial_t \chi \le 2\nu. \tag{5.19}$$

We define the main energy

$$E := \|Ap_{\neq}\|_{L^2}^2 + \tfrac{2}{\alpha}\Re\langle \partial_y^t \Delta_t^{-1}\chi A p_{1,\neq}, A p_{2,\neq}\rangle.$$

Here $\Re$ denotes the real part, in the following, we omit writing the symbol $\Re$ since we derive an upper bound. As $\alpha > \frac12$, this energy is positive definite and satisfies

$$(1-\tfrac{1}{2\alpha})\|Ap\|_{L^2}\leq E\leq(1+\tfrac{1}{2\alpha})\|Ap\|_{L^2}. \tag{5.20}$$

In the following, we assume initial data as in Theorem 1, i.e. (5.14). We use a bootstrap approach to prove the following two estimates globally in time:
The **energy estimate without x-average**

$$\begin{aligned}&\|E\|_{L^\infty}+\|A\nabla_t\otimes(\nu p_{1,\neq},\kappa p_{2,\neq})\|^2_{L^2L^2}\\ &\qquad+\sum_{j=1,\nu,\kappa,\nu^3}\|\sqrt{\tfrac{-\dot M_j}{M_j}}Ap_{\neq}\|^2_{L^2L^2}\leq(C\varepsilon)^2\end{aligned} \tag{5.21}$$

The **energy estimate with x-average**

$$\|p_=\|^2_{L^\infty H^N}+\|\partial_y(\nu p_{1,=},\kappa p_{2,=})\|^2_{L^2H^N}\leq(C\tilde\varepsilon)^2. \tag{5.22}$$

We then prove that the equality in the estimates is not attained at time T. By local wellposedness, the estimates thus remain valid at least for a short additional time. This contradicts the maximality and thus T has to be infinite. We note that we suppress in our notation the T in the estimates (see (5.3)). With $1\leq\kappa^{-\frac13}(\frac{-\dot M_\kappa}{M_\kappa}+\kappa k^2(1+(t-\frac{\xi}{k})^2)$ and $1\leq\nu^{-\frac13}(\frac{-\dot M_\nu}{M_\nu}+\nu k^2(1+(t-\frac{\xi}{k})^2))$ we infer from (5.20) and (5.21) the enhanced dissipation estimates

$$\|Ap_{1,\neq}\|_{L^2L^2}\leq 2(1-\tfrac{1}{2\alpha})^{-1}\nu^{-\frac16}C\varepsilon, \tag{5.23}$$

$$\|Ap_{2,\neq}\|_{L^2L^2}\leq 2(1-\tfrac{1}{2\alpha})^{-1}\kappa^{-\frac16}C\varepsilon. \tag{5.24}$$

By the construction of M_1 we obtain

$$\|\partial_x\Lambda_t^{-1}Ap\|_{L^2}\lesssim\nu^{-\frac{1}{12}}\varepsilon.$$

We obtain the energy estimate by deriving the energy E

$$\begin{aligned}\partial_tE&+2\|A\nabla_t\otimes(\nu p_{1,\neq},\kappa p_{2,\neq})\|^2_{L^2}+2\|\sqrt{\tfrac{-\dot M}{M}}Ap_{\neq}\|^2_{L^2}\\ =&2c\kappa^{\frac13}\|Ap_{\neq}\|^2_{L^2}\\ &-2\langle A(1-\chi)p_{1,\neq},\partial_x\partial_x^t\Delta_t^{-1}Ap_{1,\neq}\rangle\\ &+2\langle A(1-\chi)p_{2,\neq},\partial_x\partial_x^t\Delta_t^{-1}Ap_{2,\neq}\rangle\\ &+\tfrac{4}{\alpha}\langle\chi\partial_y^t\Delta_t^{-1}Ap_{1,\neq},\dot Ap_{2,\neq}\rangle\\ &+\tfrac{2}{\alpha}\langle\chi\partial_t(\partial_y^t\Delta_t^{-1})Ap_{1,\neq},Ap_{2,\neq}\rangle\\ &+\tfrac{2}{\alpha}\langle\partial_t(\chi)\partial_y^t\Delta_t^{-1}Ap_{1,\neq},Ap_{2,\neq}\rangle\\ &+\tfrac{2|\nu+\kappa|}{\alpha}\langle\chi\partial_y^tAp_{1,\neq},Ap_{2,\neq}\rangle\\ &+2\langle Av_{\neq},A(b\nabla_tb-v\nabla_tv)\rangle\\ &+2\langle Ab_{\neq},A(b\nabla_tv-v\nabla_tb)\rangle\\ &+\tfrac{2}{\alpha}\langle\chi A\partial_y^t\Delta_t^{-1}b_{\neq},A(b\nabla_tb-v\nabla_tv)\rangle\\ &+\tfrac{2}{\alpha}\langle\chi A\partial_y^t\Delta_t^{-1}v_{\neq},A(b\nabla_tv-v\nabla_tb)\rangle\\ =&L_1+L_{NR}+L_R+NL_{\neq}+ONL.\end{aligned} \tag{5.25}$$

Where we denoted by ONL all the terms which include the operator $\partial_y^t \Delta_t^{-1}$ and NL the one which does not. Furthermore, for the energy of x-averages, we obtain

$$\begin{aligned}\partial_t \|p_=\|_{H^N} &+ \|\partial_y(\nu p_{1,=}, \kappa p_{2,=})\|_{H^N} \\ &\leq \langle \langle \partial_y \rangle^N v_{1,=}, \langle \partial_y \rangle^N (b\nabla_t b - v\nabla_t v)_= \rangle \\ &\quad + \langle \langle \partial_y \rangle^N b_{1,=}, \langle \partial_y \rangle^N (b\nabla_t v - b\nabla_t v)_= \rangle \\ &= NL_=.\end{aligned} \tag{5.26}$$

In the following subsections, we establish the following proposition:

Proposition 5.4 (Control of errors)**.** *Under the assumptions of Theorem 1, there exists a constant $C = C(\alpha) > 0$ such that if* (5.21) *and* (5.22) *are satisfied for $T > 0$, then the following estimate holds*

$$\begin{aligned}\int_0^T L_1 + L_R + L_{NR}\, dt &\leq \tfrac{17+\frac{3}{2}\alpha}{10}(C\varepsilon)^2 + 2\|\sqrt{\tfrac{-\dot M_L}{M_L}} Ap_2\|^2_{L^2L^2}, \\ \int_0^T NL_{\neq} + ONL\, dt &\lesssim L\nu^{-\frac{1}{12}}\kappa^{-\frac{1}{2}}\varepsilon^3 + (L\kappa^{-\frac{1}{3}} + \kappa^{-\frac{1}{2}})\tilde\varepsilon\varepsilon^2, \\ \int_0^T NL_=\, dt &\lesssim L\nu^{-\frac{1}{12}}\kappa^{-\frac{1}{2}}\tilde\varepsilon\varepsilon^2.\end{aligned} \tag{5.27}$$

With this proposition we deduce Theorem 1:

Proof of Theorem 1. By a standard application of the Banach fixed-point theorem we obtain local well-posedness, see Appendix 5.5. Thus for all initial data, there exists a time interval $[0, T]$ such that (5.21) and (5.22) hold. Let T^* be the maximal time such that (5.21) and (5.22) hold. Let c_0 be a given, small constant and suppose for the sake of contradiction that $T^* < \infty$. With the estimates (5.25), (5.26) and (5.27) and since c_0 is small we obtain that the estimates (5.21) and (5.22) do not attain equality. Thus by local existence, T^* is not the maximal time and thus we obtain a contradiction. Therefore, for small enough c_0, (5.21) and (5.22) hold global in time and so we infer Theorem 1. ☐

The remainder of the section is dedicated to the proof of Proposition 5.4. We rearrange and use partial integration to infer that

$$\begin{aligned}&\langle Av_{\neq}, b\nabla_t Ab_{\neq} - v\nabla_t Av_{\neq}\rangle + \langle Ab_{\neq}, b\nabla_t Av_{\neq} - v\nabla_t Ab_{\neq}\rangle \\ &= \langle b, \nabla_t(Av_{\neq}Ab_{\neq})\rangle - \tfrac{1}{2}\langle v, \nabla_t(Av_{\neq}Av_{\neq}) + \nabla_t(Ab_{\neq}Ab_{\neq})\rangle \\ &= 0.\end{aligned} \tag{5.28}$$

The NL term consists of trilinear products with the unknowns

$$a^1a^2a^3 \in \{vvv, vbb, bbv, bvb\}. \tag{5.29}$$

Thus, we denote the nonlinear terms

$$\begin{aligned}NL_{\neq}[a^1a^2a^3] =&\langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{\neq})_{\neq} - a^2_{\neq}\nabla_t Aa^3_{\neq}\rangle,\\ &+\langle Aa^1_{\neq}, A(a^2_{=}\nabla_t a^3_{\neq}) - a^2_{=}\nabla_t a^3_{\neq}\rangle,\\ &+\langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{=})\rangle,\\ NL_{=}[a^1a^2a^3] =&\langle\langle\partial_y\rangle^N a^1_{=}, \langle\partial_y\rangle^N (a^2_{\neq}\nabla_t a^3_{\neq})_{=}\rangle.\end{aligned}$$

If we do not use specific choices for $a^1a^2a^3$ we write just NL. Similarly, we use $a^1a^2a^3 \in \{bvv, bbb, vbv, vvb\}$ for ONL. Furthermore, we always use i such that $p_i = \Lambda_t^{-1}\nabla_t^{\perp}a^2$ in the sense that $i = 1$ if $a^2 = v$ and $i = 2$ if $a^2 = b$. We perform the energy estimates in the next subsections:

- In Subsection 5.3.1 we estimate the linear error terms. In this subsection, the split with χ into resonant and non-resonant regions depending on ν is vital.
- In Subsection 5.3.2 we conclude the energy estimate for the nonlinear term without x average. Here it is necessary to perform a low and high frequency decomposition. This gives us a reaction and a transport term. In particular, for $\kappa \downarrow 0$ bounding the transport term is very challenging due to the linear growth.
- In Subsections 5.3.3, 5.3.4 and 5.3.5 we estimate nonlinear terms with an x-average component.
- In Subsection 5.3.6 we estimate nonlinear term which arise due χ in the resonant regions. For these terms, we obtain an additional Λ_t^{-1}, which has a stabilizing effect. This stabilizing effect is necessary due to a nonlinear term consisting of only magnetic components.

5.3.1 Linear estimates

In this section, we establish estimates of the linear errors L_1, L_R and L_{NR} of (5.25). In order to estimate L_1, we use (5.23) and (5.24) to deduce

$$\int L_1 d\tau = 2c\kappa^{\frac{1}{3}}\|Ap_{\neq}\|^2_{L^2L^2} \leq 8(1-\tfrac{1}{2\alpha})^{-1}c(C\varepsilon)^2.$$

For the L_{NR} terms in (5.25), we infer

$$\begin{aligned}\langle A(1-\chi)p_{1,\neq}, A\partial_x\partial_x^t\Delta_t^{-1}p_{1,\neq}\rangle &= \sum_{k\neq 0}\int d\xi (1-\chi)\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}A^2p_1^2\\ &\leq \nu^3\|A\nabla_t p_{1,\neq}\|^2_{L^2},\end{aligned}$$

since $(1-\chi)\frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} \le (1-\chi)(\nu)^3(1+(t-\frac{\xi}{k})^2)$ due to $\chi = 1$ for $|t-\frac{\xi}{k}| \le \nu^{-1}$. Furthermore, using (5.34) we estimate

$$\begin{aligned}\langle A(1-\chi)p_{2,\neq}, A\partial_x\partial_x^t\Delta_t^{-1}p_2\rangle &= \sum_{k\neq 0}\int d\xi(1-\chi)\tfrac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2}A^2p_2^2 \\ &\le \sum_{k\neq 0}\int d\xi(1-\chi)(\tfrac{-\dot{M}_L}{M_L}+\kappa c_1(1+(t-\tfrac{\xi}{k})^2))A^2p_2^2 \\ &\le \|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|_{L^2}^2 + \kappa c_1\|\nabla_t Ap_{2,\neq}\|_{L^2}^2.\end{aligned}$$

Thus with (5.21) we deduce

$$\int L_{NR}\ d\tau \le (2\nu^2+2c_1)(C\varepsilon)^2 + 2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|_{L^2L^2}^2.$$

For L_R, we estimate in frequency space

$$\begin{aligned}|(\mathbf{1}_{\neq}\partial_y^t\Delta_t^{-1})^\wedge| &= |(\tfrac{\xi-kt}{k^2+(\xi-kt)^2})_{k\neq 0}| \le \tfrac{1}{2}, \\ \tfrac{-\dot{M}_L}{M_L}|(\partial_y^t\Delta_t)^\wedge| &\le \left|\left(\tfrac{(t-\frac{\xi}{k})^2}{k(1+(t-\frac{\xi}{k})^2)^2}\right)_{k\neq 0}\right| \le C_\alpha^{-1}\tfrac{-\dot{M}_1}{M_1}.\end{aligned}$$

So it follows that

$$\begin{aligned}&\tfrac{4}{\alpha}\langle\chi\partial_y^t\Delta_t^{-1}Ap_{1,\neq}, \dot{A}p_{2,\neq}\rangle \\ &= \tfrac{4}{\alpha}\langle\chi Ap_{1,\neq}, (c\kappa^{\frac{1}{3}} + \tfrac{\dot{M}_1}{M_1} + \tfrac{\dot{M}_L}{M_L} + \tfrac{\dot{M}_\kappa}{M_\kappa} + \tfrac{\dot{M}_\nu}{M_\nu} + \tfrac{\dot{M}_{\nu 3}}{M_{\nu 3}})\partial_y^t\Delta_t^{-1}Ap_{2,\neq}\rangle \\ &\le \tfrac{2c}{\alpha}\kappa^{\frac{1}{3}}\|Ap_{\neq}\|_{L^2}^2 + (1+C_\alpha^{-1})\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{\neq}\|_{L^2}^2 \\ &\quad + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\kappa}{M_\kappa}}Ap_{\neq}\|_{L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\nu}{M_\nu}}Ap_{\neq}\|_{L^2}^2 + \tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_{\nu 3}}{M_{\nu 3}}}Ap_{\neq}\|_{L^2}^2.\end{aligned}$$

We use the estimate in frequency space

$$|(((\partial_x^2-(\partial_y^t)^2)\Delta_t^{-2})_{\neq})^\wedge| = \left|\left(\tfrac{1-(t-\frac{\xi}{k})^2}{k^2(1+(t-\frac{\xi}{k})^2)^2}\right)_{k\neq 0}\right| \le C_\alpha^{-1}\tfrac{-\dot{M}_1}{M_1},$$

to infer that

$$\begin{aligned}\tfrac{1}{\alpha}\langle\chi Ap_{1,\neq}, A\partial_x^{-1}(\partial_x^2-(\partial_y^t)^2)\Delta_t^{-2}p_{2,\neq}\rangle &\le C_\alpha^{-1}\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{1,\neq}\|_{L^2}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{2,\neq}\|_{L^2} \\ &\le C_\alpha^{-1}\tfrac{1}{2\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}Ap_{\neq}\|_{L^2}^2.\end{aligned}$$

With (5.19) we deduce

$$\langle\partial_y^t\Delta_t^{-1}\partial_t(\chi)Ap_{1,\neq}, Ap_{2,\neq}\rangle \le \nu\|Ap_{1,\neq}\|_{L^2}\|A\Lambda_t^{-1}p_{2,\neq}\|_{L^2}.$$

By the Fourier support of χ (see (5.18)) and the definition of M_{ν^3} we obtain $\chi \leq 2C_\alpha^{-1}\nu^{-1}\frac{-\dot{M}_{\nu^3}}{M_{\nu^3}}\chi$, which yields

$$\tfrac{|\nu+\kappa|}{\alpha}\langle \chi A\partial_y^t p_{1,\neq}, Ap_{2,\neq}\rangle \leq 2C_\alpha^{-1}\tfrac{\nu^{\frac{1}{2}}}{\alpha}\|A\partial_y^t p_{1,\neq}\|_{L^2}\|\sqrt{\tfrac{-M_{\nu^3}}{M_{\nu^3}}}Ap_{2,\neq}\|_{L^2}.$$

Thus for the linear error L_R we infer

$$\begin{aligned}\int L_R d\tau \leq{}& (\tfrac{2c}{\alpha}+\nu^{\frac{5}{6}})(C\varepsilon)^2\\ &+\tfrac{1+C_\alpha^{-1}}{\alpha}\|\sqrt{\tfrac{-\dot{M}_1}{M_1}}p_{\neq}\|_{L^2L^2}^2+\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\kappa}{M_\kappa}}p_{\neq}\|_{L^2L^2}^2+\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\nu}{M_\nu}}p_{\neq}\|_{L^2L^2}^2\\ &+\tfrac{1+C_\alpha^{-1}}{2\alpha}\|\sqrt{\tfrac{-\dot{M}_{\nu^3}}{M_{\nu^3}}}Ap_{2,\neq}\|_{L^2L^2}+C_\alpha^{-1}\tfrac{1}{2\alpha}\nu\|A\partial_y^t p_{1,\neq}\|_{L^2L^2}^2.\end{aligned}$$

Combining the estimates for all linear terms, we obtain

$$\begin{aligned}&\int L+L_R+L_{NR}\,d\tau\\ &\leq ((8+\tfrac{2}{\alpha})(1-\tfrac{1}{2\alpha})^{-1}c+2\nu^2+2c_1+\nu^{\frac{5}{6}})(C\varepsilon)^2\\ &\quad+2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_2\|_{L^2L^2}^2\\ &\quad+(1+\tfrac{3}{2}C_\alpha^{-1})\tfrac{1}{\alpha}\|\chi\sqrt{\tfrac{-\dot{M}_1}{M_1}}p_{\neq}\|_{L^2L^2}^2+\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\kappa}{M_\kappa}}p_{\neq}\|_{L^2L^2}^2+\tfrac{1}{\alpha}\|\sqrt{\tfrac{-\dot{M}_\nu}{M_\nu}}p_{\neq}\|_{L^2L^2}^2\\ &\quad+\tfrac{1+C_\alpha^{-1}}{2\alpha}\|\sqrt{\tfrac{-M_{\nu^3}}{M_{\nu^3}}}Ap_{2,\neq}\|_{L^2L^2}+C_\alpha^{-1}\tfrac{1}{2\alpha}\nu\|A\partial_y^t p_{1,\neq}\|_{L^2L^2}^2\\ &\leq (12(1-\tfrac{1}{2\alpha})^{-1}c+2\nu^2+2c_1+\nu^{\frac{5}{6}}+\tfrac{1+2C_\alpha^{-1}}{2\alpha})(C\varepsilon)^2\\ &\quad+2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|_{L^2L^2}^2.\end{aligned}$$

Since $\alpha>\frac{1}{2}$ we deduce $\frac{1+2C_\alpha^{-1}}{\alpha}<1+\frac{1}{2\alpha}$. Choosing the constants such that

$$\begin{aligned}c&=\tfrac{1}{200}(1-\tfrac{1}{2\alpha})^2,\\ c_1&=\tfrac{1}{20}(1-\tfrac{1}{2\alpha}),\end{aligned}$$

and recalling that

$$\nu\leq\tfrac{1}{40}(1-\tfrac{1}{2\alpha})^{\frac{6}{5}},$$

we conclude that $(12(1-\frac{1}{2\alpha})^{-1}c+2\nu^2+2c_1+\nu^{\frac{5}{6}}+\frac{1+2C_\alpha^{-1}}{2\alpha}<\frac{17+\frac{3}{2\alpha}}{20}$. Thus we obtain the estimate

$$\int L_1+L_R+L_{NR}d\tau\leq\tfrac{17+\frac{3}{2\alpha}}{20}(C\varepsilon)^2+2\|\sqrt{\tfrac{-\dot{M}_L}{M_L}}Ap_{2,\neq}\|_{L^2L^2}^2.$$

This yields the first estimate of Proposition 5.4.

5.3.2 Nonlinear terms without an x-average

We apply the notation of (5.29) and aim to estimate terms of the form

$$\begin{aligned}
&\langle Aa^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{\neq}) - a^2_{\neq}\nabla_t Aa^3_{\neq}\rangle \\
&= \sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) \tfrac{A(k,\xi)-A(l,\eta)}{A(k-l,\xi-\eta)A(l,\eta)} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)t)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&= T + R.
\end{aligned}$$

Here, we split the integral into the *reaction* R and the *transport* T terms which correspond to the sets

$$\begin{aligned}
\Omega_R &= \{|k-l,\xi-\eta| \geq \tfrac{1}{8}|l,\eta|\}, \\
\Omega_T &= \{|k-l,\xi-\eta| < \tfrac{1}{8}|l,\eta|\},
\end{aligned}$$

in Fourier space. We split the weights

$$\begin{aligned}
A(k,\xi) - A(l,\eta) = e^{ct\kappa^{\frac{1}{3}}}&(M_L(k,\xi) - M_L(l,\eta))M_1(k,\xi)M_\kappa(k,\xi)M_\nu(k,\xi)M_{\nu^3}(k,\xi)|k,\xi|^N \\
+ e^{ct\kappa^{\frac{1}{3}}}&(|k,\xi|^N - |l,\eta|^N)M_L(l,\eta)M_1(k,\xi)M_\kappa(k,\xi)M_\nu(k,\xi)M_{\nu^3}(k,\xi) \\
+ e^{ct\kappa^{\frac{1}{3}}}&(M_1(k,\xi) - M_1(l,\eta))M_L(l,\eta)M_\kappa(k,\xi)M_\nu(k,\xi)M_{\nu^3}(k,\xi)|l,\eta|^N \\
+ e^{ct\kappa^{\frac{1}{3}}}&(M_\kappa(k,\xi) - M_\kappa(l,\eta))M_1(l,\eta)M_L(l,\eta)M_\nu(k,\xi)M_{\nu^3}(k,\xi)|l,\eta|^N \\
+ e^{ct\kappa^{\frac{1}{3}}}&(M_\nu(k,\xi) - M_\nu(l,\eta))M_1(l,\eta)M_L(l,\eta)M_\kappa(l,\eta)M_{\nu^3}(k,\xi)|l,\eta|^N \\
+ e^{ct\kappa^{\frac{1}{3}}}&(M_{\nu^3}(k,\xi) - M_{\nu^3}(l,\eta))M_1(l,\eta)M_L(l,\eta)M_\kappa(l,\eta)M_\nu(l,\eta)|l,\eta|^N
\end{aligned}$$

and thus by (5.16) we estimate

$$\begin{aligned}
\tfrac{A(k,\xi)-A(l,\eta)}{A(k-l,\xi-\eta)A(l,\eta)} \lesssim\, & e^{-ct\kappa^{\frac{1}{3}}} \tfrac{|M_L(k,\xi)-M_L(l,\eta)|}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \tfrac{|\xi,\eta|^N}{|l,\eta|^N|k-l,\xi-\eta|^N} \\
&+ e^{-ct\kappa^{\frac{1}{3}}} \tfrac{\left||k,\xi|^N-|l,\eta|^N\right|}{|l,\eta|^N|k-l,\xi-\eta|^N} \tfrac{1}{M_L(k-l,\xi-\eta)} \\
&+ e^{-ct\kappa^{\frac{1}{3}}} \sum_{j=1,\kappa,\nu,\nu^3} |M_j(k,\xi) - M_j(l,\eta)| \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{1}{M_L(k-l,\xi-\eta)}.
\end{aligned} \tag{5.30}$$

Reaction term: On the set Ω_R it holds that $|k-l,\xi-\eta| \geq \frac{1}{8}|l,\eta|$, thus $|k,\xi|, |l,\eta| \lesssim |k-l,\xi-\eta|$. From (5.16), (5.30) and $\frac{|k,\xi|^N}{|l,\eta|^N|k-l,\xi-\eta|^N}, \frac{||k,\xi|^N-|l,\eta|^N|}{|l,\eta|^N|k-l,\xi-\eta|^N} \lesssim \frac{1}{|l,\eta|^N}$ we infer

$$\tfrac{A(k,\xi)-A(l,\eta)}{A(k-l,\xi-\eta)A(l,\eta)} \lesssim \tfrac{1}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \tfrac{1}{|l,\eta|^N}.$$

With $\xi l - k\eta = (\xi - \eta - (k-l)t)l - (k-l)(\eta - lt)$ and Hölder's inequality we deduce

$$\begin{aligned}
R &= e^{-ct\kappa^{\frac{1}{3}}} \sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_R} \tfrac{A(k,\xi)-A(l,\eta)}{A(k-l,\xi-\eta)A(l,\eta)} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&\lesssim e^{-ct\kappa^{\frac{1}{3}}} \sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_R} \tfrac{1}{|l,\eta|^N} \tfrac{1}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \tfrac{|(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&\lesssim \|Aa^1_{\neq}\|_{L^2} \|\tfrac{1}{M_L} Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} Aa^3_{\neq}\|_{L^2} \\
&\quad + \|Aa^1_{\neq}\|_{L^2} \|\partial_x \tfrac{1}{M_L} \Lambda_t^{-1} Ap_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} A\partial_y^t a^3_{\neq}\|_{L^2}.
\end{aligned}$$

We use (5.36) and (5.37) to infer

$$\begin{aligned}
R &\lesssim \|Aa^1_{\neq}\|_{L^2} (\|Ap_{i,\neq}\|_{L^2} + \nu \|(\Lambda_t \wedge \kappa^{-\frac{1}{3}}) Ap_{i,\neq}\|_{L^2})(\|Aa^3_{\neq}\|_{L^2} + \nu\|A(\Lambda_t \wedge \kappa^{-\frac{1}{3}}) a^3_{\neq}\|_{L^2}) \\
&\quad + L\|Aa^1_{\neq}\|_{L^2} (\|A\partial_x \Lambda_t^{-1} p_{i,\neq}\|_{L^2} + \nu \|Ap_{i,\neq}\|_{L^2}) \|\partial_y^t Aa^3_{\neq}\|_{L^2}.
\end{aligned}$$

Integrating in time yields

$$\int R d\tau \lesssim L \nu^{-\frac{1}{12}} \kappa^{-\frac{1}{2}} \varepsilon^3.$$

Transport term: On the set Ω_T it holds that $|k-l,\xi-\eta| < \frac{1}{8}|l,\eta|$ and thus it follows that $|k,\xi| \approx |l,\eta|$. By the mean value theorem, there exists $\theta \in [0,1]$ such that

$$\begin{aligned}
\big| |k,\xi|^N - |l,\eta|^N \big| &\leq N|k-l,\xi-\eta||k-\theta l, \xi - \theta\eta|^{N-1} \\
&\lesssim |k-l,\xi-\eta||l,\eta|^{N-1}.
\end{aligned}$$

Thus with (5.30) and Lemma 5.7 we conclude, that

$$\begin{aligned}
\tfrac{A(k,\xi)-A(l,\eta)}{A(k-l,\xi-\eta)A(l,\eta)} &\lesssim \tfrac{1}{M_L(l,\eta)} (\tfrac{1}{|l|} + \nu^{\frac{1}{12}}) \tfrac{1}{|k-l,\xi-\eta|^{N-1}} && (5.31) \\
&\quad + \sum_{j=\kappa,\nu,\nu^3} |M_j(k,\xi) - M_j(l,\eta)| \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} && (5.32) \\
&\quad + \tfrac{M_L(k,\xi)-M_L(l,\eta)}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N}. && (5.33)
\end{aligned}$$

Based on this estimate, in the following we distinguish between different regimes in frequency,

$$
\begin{aligned}
T = &\sum_{k,l,k-l\neq 0} \int d(\xi,\eta)\mathbf{1}_{\Omega_T} \tfrac{A(k,\xi)-A(l,\eta)}{A(k-l,\xi-\eta)A(l,\eta)} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
\lesssim &\sum_{k,l,k-l\neq 0} \int d(\xi,\eta)\mathbf{1}_{\Omega_T} \tfrac{1}{M_L(l,\eta)}(\tfrac{1}{|l|}+\nu^{\frac{1}{12}}) \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&+ \sum_{k,l,k-l\neq 0} \int d(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{|\frac{\eta}{l}-t|\geq|\frac{\xi-\eta}{k-l}-t|} \tfrac{\sum_{j=\kappa,\nu,\nu^3}|M_j(k,\xi)-M_j(l,\eta)|}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^{N}} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&+ \sum_{k,l,k-l\neq 0} \int d(\xi,\eta)\mathbf{1}_{\Omega_T}\mathbf{1}_{|\frac{\eta}{l}-t|\leq|\frac{\xi-\eta}{k-l}-t|} \tfrac{\sum_{j=\kappa,\nu,\nu^3}|M_j(k,\xi)-M_j(l,\eta)|}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^{N}} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&+ \sum_{k,l,k-l\neq 0} \int d(\xi,\eta)\mathbf{1}_{\Omega_T} \tfrac{M_L(k,\xi)-M_L(l,\eta)}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^{N}} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
= &\, T_{1,1}+T_{1,2}+T_{1,3}+T_2.
\end{aligned}
$$

Here, the $T_{1,1}$ term is due to estimate (5.31). For (5.32) we distinguish between the frequencies $|\frac{\eta}{l}-t|\geq|\frac{\xi-\eta}{k-l}-t|$ in $T_{1,2}$ and $|\frac{\eta}{l}-t|\leq|\frac{\xi-\eta}{k-l}-t|$ in $T_{1,3}$. The M_L commutator (5.33) is T_2, which requires further splitting. For $T_{1,1}$ we use $\xi l - k\eta = (\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)$, (5.16) and (5.37) to estimate

$$
\begin{aligned}
T_{1,1} = &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta)\mathbf{1}_{\Omega_T} \tfrac{1}{M_L(l,\eta)}(\tfrac{1}{|l|}+\nu^{\frac{1}{12}}) \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
&\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
&\lesssim \|Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2} + \|Aa^1_{\neq}\|_{L^2}\|A\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\tfrac{1}{M_L}A\partial_y^t a^3_{\neq}\|_{L^2} \\
&\quad + \nu^{\frac{1}{12}}\|Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|\tfrac{1}{M_L}A\partial_x a^3_{\neq}\|_{L^2} \\
&\leq L\|Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|Aa^3_{\neq}\|_{L^2} + L\|Aa^1_{\neq}\|_{L^2}\|A\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|A\partial_y^t a^3_{\neq}\|_{L^2} \\
&\quad + \nu^{\frac{1}{12}}\|Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|\Lambda_t a^3_{\neq}\|_{L^2}.
\end{aligned}
$$

After integrating in time we deduce that

$$
\int T_{1,1}d\tau \lesssim (L+\nu^{-\frac{1}{12}})\kappa^{-\frac{1}{2}}\varepsilon^3.
$$

For $T_{1,2}$ we use $|\frac{\eta}{l}-t|\geq|\frac{\xi-\eta}{k-l}-t|$ to infer that $|\xi l-k\eta| = |(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)|\leq 2|(k-l)(\eta-lt)|$. Furthermore, with $\sum_{i=\nu,\kappa,\nu^3}|M_i(k,\xi)-M_i(l,\eta)|\approx$

1 and (5.35) we conclude that

$$\begin{aligned}
T_{1,2} \lesssim & \sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \geq |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{|(k-l)(\eta-lt)|}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
& \qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
\lesssim & \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} A\partial_y^t a^3_{\neq}\|_{L^2} \\
\lesssim & L \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2}.
\end{aligned}$$

So after integrating in time, we obtain

$$\int T_{1,2} d\tau \lesssim L\kappa^{-\frac{1}{2}} \varepsilon^3.$$

For $T_{1,3}$, we use $|\frac{\eta}{l}-t| \leq |\frac{\xi-\eta}{k-l}-t|$ to infer $\xi l - k\eta \leq 2(\xi-\eta-(k-l)t)l$. Furthermore, with (5.43) we deduce

$$\sum_{j=\kappa,\nu,\nu^3} |M_j(k,\xi) - M_j(l,\eta)| \lesssim \nu^{\frac{1}{3}} \tfrac{|\xi l - k\eta|}{|kl|}.$$

Combining these two estimates by Hölder's inequality and (5.35) it follows, that

$$\begin{aligned}
T_{1,3} \lesssim & \sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \leq |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{\nu^{\frac{1}{3}}(\xi l - k\eta)^2}{kl((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
& \qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
\lesssim & \sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \leq |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{\nu^{\frac{1}{3}}(\xi-\eta-(k-l)t)^2 l^2}{kl((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\
& \qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
\lesssim & \nu^{\frac{1}{3}} \sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_T} \mathbf{1}_{|\frac{\eta}{l}-t| \leq |\frac{\xi-\eta}{k-l}-t|} \tfrac{1}{M_L(l,\eta)} \tfrac{|\xi-\eta-(k-l)t|}{|k-l,\xi-\eta|^{N-1}} \\
& \qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\
\lesssim & \nu^{\frac{1}{3}} \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t p_{i,\neq}\|_{L^2} \|\tfrac{1}{M_L} Aa^3_{\neq}\|_{L^2} \\
\lesssim & L\nu^{\frac{1}{3}} \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t p_{i,\neq}\|_{L^2} \|Aa^3_{\neq}\|_{L^2}.
\end{aligned}$$

Thus integrating in time yields

$$\int T_{1,3} d\tau \leq L\nu^{\frac{1}{6}} \kappa^{-\frac{1}{2}} \varepsilon^3.$$

To estimate the T_2 term, we split the integral into the sets

$$\begin{aligned}
\Omega_1 &= \{\min(t - \tfrac{\eta}{l}, t - \tfrac{\xi-\eta}{k-l}) \geq \nu^{-1}\}, \\
\Omega_2 &= \{t - \tfrac{\eta}{l} \geq \nu^{-1} \geq t - \tfrac{\xi-\eta}{k-l}\}, \\
\Omega_3 &= \{t - \tfrac{\xi-\eta}{k-l} \geq \nu^{-1} \geq t - \tfrac{\eta}{l}\}, \\
\Omega_4 &= \{t - \tfrac{\xi}{k} \geq \nu^{-1} \geq \max(t - \tfrac{\eta}{l}, t - \tfrac{\xi-\eta}{k-l})\}.
\end{aligned}$$

For frequencies such that $\nu^{-1} \geq \max(t - \frac{\eta}{l}, t - \frac{\xi}{k})$, then $M_L(k,\xi) - M_L(l,\eta) = 0$ and hence the commutator vanishes. Thus the sets Ω_j covers all regions of the support. The sets Ω_1, Ω_2 and Ω_3 are chosen to distinguish between $\frac{1}{M_L} = 1$ and $\frac{1}{M_L} > 1$ for different frequencies and on set Ω_4 we use strong dissipation in the first component. We split the set T_2 into

$$\begin{aligned} T_2 = &\sum_{k,l,k-l\neq 0} \int d(\xi,\eta) \mathbf{1}_{\Omega_T} \tfrac{M_L(k,\xi)-M_L(l,\eta)}{M_L(k-l,\xi-\eta)M_L(l,\eta)} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \tfrac{1}{|k-l,\xi-\eta|^N} \\ &\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta)(\mathbf{1}_{\Omega_1} + \mathbf{1}_{\Omega_2} + \mathbf{1}_{\Omega_3} + \mathbf{1}_{\Omega_4}) \\ = &\, T_{2,1} + T_{2,2} + T_{2,3} + T_{2,4}. \end{aligned}$$

For $T_{2,1}$ we use (5.35) to deduce

$$\begin{aligned} T_{2,1} \leq \nu^2 &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) 1_{\Omega_1} \langle t - \tfrac{\eta}{l} \wedge \kappa^{-\frac{1}{3}} \rangle \langle t - \tfrac{\xi-\eta}{k-l} \wedge \kappa^{-\frac{1}{3}} \rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \\ &\qquad \tfrac{(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\ \lesssim &\, \nu^2 \|Aa^1_{\neq}\|_{L^2} \|(\Lambda_t \wedge \kappa^{-\frac{1}{3}}) Ap_{i,\neq}\|_{L^2} \|A\Lambda_t a^3_{\neq}\|_{L^2} \\ &+ L\nu \|Aa^1_{\neq}\|_{L^2} \|(\Lambda_t \wedge \kappa^{-\frac{1}{3}}) \Lambda_t^{-1} \Lambda^{-1} Ap_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2} \end{aligned}$$

and so

$$\int T_{2,1} d\tau \lesssim L\nu^{\frac{5}{6}} \kappa^{-\frac{1}{2}} \varepsilon^3.$$

Now we consider $T_{2,2}$. By (5.35) we infer

$$\begin{aligned} T_{2,2} \leq &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) \mathbf{1}_{\Omega_2} \nu \langle t - \tfrac{\eta}{l} \wedge \kappa^{-\frac{1}{3}} \rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\ &\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\ \lesssim &\, \nu \|Aa^1_{\neq}\|_{L^2} \|Ap_{i,\neq}\|_{L^2} \|A\Lambda_t a^3_{\neq}\|_{L^2} \\ &+ L \|Aa^1_{\neq}\|_{L^2} \|A\Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2}. \end{aligned}$$

Integrating in time yields

$$\int T_{2,2} d\tau \lesssim L\kappa^{-\frac{1}{2}} \varepsilon^3.$$

To estimate $T_{2,3}$, we need to distinguish between different choices of a. Using (5.35) we estimate

$$\begin{aligned} T_{2,3} = &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) \mathbf{1}_{\Omega_3} \nu \langle t - \tfrac{\xi-\eta}{k-l} \wedge \kappa^{-\frac{1}{3}} \rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{(\xi-\eta-(k-l)t)l-(k-l)(\eta-lt)}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\ &\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\ \lesssim &\, \nu \|Aa^1_{\neq}\|_{L^2} \|(\Lambda_t \wedge \kappa^{-\frac{1}{3}}) Ap_{i,\neq}\|_{L^2} \|A\partial_x a^3_{\neq}\|_{L^2} \\ &+ \nu \|Aa^1_{\neq}\|_{L^2} \|A(\Lambda_t \wedge \kappa^{-\frac{1}{3}}) \Lambda^{-1} \Lambda_t^{-1} p_{i,\neq}\|_{L^2} \|A\partial_y^t a^3_{\neq}\|_{L^2} \end{aligned}$$

and thus after integrating in time

$$\int T_{2,3}[vvv]d\tau \lesssim \nu^{-\frac{1}{2}}\varepsilon^3,$$
$$\int T_{2,3}[bvb]d\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^3,$$
$$\int T_{2,3}[bbv]d\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^3.$$

In the case of vbb, we use (5.35) to estimate

$$\begin{aligned} T_{2,3}[vbb] = &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta)\nu\langle t - \tfrac{\xi-\eta}{k-l}\rangle \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{(\xi-\eta-(k-l)t)k-(k-l)(\xi-kt)}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\ &\qquad (Av)(k,\xi)(Ap_2)(k-l,\xi-\eta)(Ab)(l,\eta) \\ &\lesssim \nu\|\partial_x Av_{\neq}\|_{L^2}\|(\Lambda_t \wedge \kappa^{-\frac{1}{3}})Ap_{2,\neq}\|_{L^2}\|Ab_{\neq}\|_{L^2} \\ &\quad + \nu\|\partial_y^t Av_{\neq}\|_{L^2}\|(\Lambda_t \wedge \kappa^{-\frac{1}{3}})\Lambda_t^{-1}Ap_{2,\neq}\|_{L^2}\|Ab_{\neq}\|_{L^2} \\ &\leq \nu\|Ab_{\neq}\|_{L^2}\|A\nabla_t v_{\neq}\|_{L^2}\|A\Lambda_t p_{2,\neq}\|_{L^2}. \end{aligned}$$

Thus after integrating in time, we obtain

$$\int T_{2,3}[vbb]d\tau \lesssim \nu^{\frac{1}{2}}\kappa^{-\frac{1}{2}}\varepsilon^3.$$

For $T_{2,4}$ we obtain that $M(l,\eta) = M(k-l,\xi-\eta) = 1$. We use $t - \frac{\xi}{k} \geq \nu \geq \max(t-\frac{\eta}{l}, t - \frac{\xi-\eta}{k-l})$ to deduce that

$$1 = \tfrac{kt-\xi}{kt-\xi} = \tfrac{k}{k}\tfrac{t-\frac{\xi}{k}}{t-\frac{\xi}{k}} \leq \nu\tfrac{|\xi-kt|}{|k|}.$$

With $\xi l - k\eta = (\xi-\eta-(k-l)t)l - (k-l)(\eta - lt)$ we infer that

$$\begin{aligned} T_{2,4} = &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) \tfrac{1}{|k-l,\xi-\eta|^N} \tfrac{\xi l - k\eta}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\ &\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\ = \nu &\sum_{k,l,k-l\neq 0} \iint d(\xi,\eta)|\xi-kt| \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{(\xi-\eta-(k-l)t)l}{|l|((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\ &\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\ &+ \sum_{k,l,k-l\neq 0} \iint d(\xi,\eta) \tfrac{1}{|k-l,\xi-\eta|^{N-1}} \tfrac{(k-l)(\eta-lt)}{((k-l)^2+(\xi-\eta-(k-l)\tau)^2)^{\frac{1}{2}}} \\ &\qquad (Aa^1)(k,\xi)(Ap_i)(k-l,\xi-\eta)(Aa^3)(l,\eta) \\ \leq \nu &\|A\partial_y^t a^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|Aa^3_{\neq}\|_{L^2} \\ &+ \|Aa^1_{\neq}\|_{L^2}\|\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2}\|A\partial_y^t a^3_{\neq}\|_{L^2}. \end{aligned}$$

Thus integrating in time yields

$$\int T_{2,4} d\tau \lesssim \kappa^{-\frac{1}{2}} \varepsilon^3.$$

5.3.3 Nonlinear terms with an x-average in the second component

We apply the notation of (5.29)

$$\begin{aligned}
&\langle A a^1_{\neq}, A(a^2_{1,=} \partial_x a^3_{\neq}) - a^2_{1,=} \partial_x A a^3_{\neq} \rangle \\
&= \sum_{k \neq 0} \iint d(\xi,\eta)(Aa^1)(k,\xi)(A(k,\xi) - A(k,\eta)) k a^2_1(0,\xi-\eta) a^3(k,\eta) \\
&= R + T.
\end{aligned}$$

Here we split into reaction and transport terms according to the sets

$$\begin{aligned}
\Omega_R &= \{|\xi - \eta| \geq \tfrac{1}{8}|k,\eta|\}, \\
\Omega_T &= \{|\xi - \eta| < \tfrac{1}{8}|k,\eta|\}.
\end{aligned}$$

Reaction term On the set Ω_R it holds that $|\xi - \eta| \geq \frac{1}{8}|k,\eta|$, then we obtain $|A(k,\xi) - A(k,\eta)| \lesssim |\xi - \eta|^N$ and thus with (5.35), it follows that

$$\begin{aligned}
\sum_{k \neq 0} \iint d(\xi,\eta)(Aa^1)(k,\xi)(A(k,\xi) - A(k,\eta)) k a^2_1(0,\xi-\eta) a^3(k,\eta) \\
&\lesssim \|Aa^1_{\neq}\|_{L^2} \|a^2_=\|_{H^N} \|\partial_x a^3_{\neq}\|_{L^\infty} \\
&\lesssim \|Aa^1_{\neq}\|_{L^2} \|a^2_=\|_{H^N} \|\tfrac{1}{M_L} A a^3_{\neq}\|_{L^2} \\
&\lesssim L \|Aa^1_{\neq}\|_{L^2} \|a^2_=\|_{H^N} \|A a^3_{\neq}\|_{L^2}.
\end{aligned}$$

Integrating in time yields a bound

$$\int R \, d\tau \lesssim L \kappa^{-\frac{1}{3}} \varepsilon^2 \tilde{\varepsilon}.$$

Transport term On the set Ω_L it holds that $|k,\eta| \geq \frac{1}{8}|\xi - \eta|$. By the mean value theorem there exists a $\theta \in [0,1]$

$$\big| |k,\eta|^N - |k,\xi|^N \big| \lesssim |\xi - \eta| |k, \eta - \theta\xi|^{N-1} \lesssim |\xi - \eta| |k,\eta|^{N-1}.$$

Thus, we can estimate the difference in A by

$$\begin{aligned}
|A(k,\xi) - A(k,\eta)| &\lesssim (M_L(k,\xi) - M_L(k,\eta))|k,\xi|^N \\
&\quad + M_L(k,\eta)|k,\xi|^N \sum_{j=1,\kappa,\nu,\nu^3} |M_j(k,\xi) - M_j(k,\eta)| \\
&\quad + M_L(k,\eta)(|k,\eta|^N - |k,\xi|^N) \\
&\lesssim \tfrac{1}{k} |\xi - \eta| |k,\xi|^N
\end{aligned}$$

where we used (5.40),(5.42) and (5.43) to estimate the differences in M_j. So we infer, that

$$T \leq \sum_{k\neq 0} \iint d(\xi,\eta)\mathbf{1}_{\Omega_T}|Aa^1|(k,\xi)|a_1^2|(0,\xi-\eta)\tfrac{1}{M_L(k,\eta)}|Aa^3|(k,\eta).$$

and thus integrating in time yields

$$\int T d\tau \lesssim L\|Aa^1_{\neq}\|_{L^2L^2}\|a^2_{=}\|_{L^\infty H^N}\|Aa^3_{\neq}\|_{L^2L^2} \lesssim L\kappa^{-\frac{1}{3}}\varepsilon^2\tilde{\varepsilon}.$$

5.3.4 Nonlinear terms with an x-average in the third component

We aim to estimate

$$\begin{aligned}
&\langle Aa^1_{1,\neq}, A(a^2_{2,\neq}\partial_y a^3_{1,=})\rangle \\
&= \sum_{k\neq 0}\iint d(\xi,\eta)(Aa_1^1)(k,\xi)A(k,\xi)\tfrac{k\eta}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_i(k,\xi-\eta)a_1^3(0,\eta) \\
&= R+T
\end{aligned}$$

where we split into the reaction and transport terms according to the sets

$$\begin{aligned}
\Omega_R &= \{|k,\xi-\eta| \geq \tfrac{1}{8}|\eta|\}, \\
\Omega_T &= \{|k,\xi-\eta| < \tfrac{1}{8}|\eta|\}.
\end{aligned}$$

Reaction term On the set Ω_R it holds that $|k,\xi-\eta| \geq \frac{1}{8}|\eta|$. With (5.35) we infer

$$\begin{aligned}
R &= \sum_{k\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_R}(Aa_1^1)(k,\xi)A(k,\xi)\tfrac{k\eta}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_i(k,\xi-\eta)a_1^3(0,\eta) \\
&\lesssim \|Aa^1_{1,\neq}\|_{L^2}\|A\tfrac{1}{M_L}\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\partial_y a^3_{1,=}\|_{L^\infty} \\
&\lesssim \|Aa^1_{1,\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|a^3_{1,=}\|_{H^N}.
\end{aligned}$$

Integrating in time then yields

$$\int R d\tau \lesssim \kappa^{-\frac{1}{3}}\varepsilon^2\tilde{\varepsilon}.$$

Transport term On the set Ω_T it holds that $|k,\xi-\eta| \leq \frac{1}{8}|\eta|$, then with (5.35) we estimate

$$\begin{aligned}
T &= \sum_{k\neq 0}\iint d(\xi,\eta)\mathbf{1}_{\Omega_T}(Aa_1^1)(k,\xi)A(k,\xi)\tfrac{k\eta}{\sqrt{k^2+(\xi-\eta-kt)^2}}p_i(k,\xi-\eta)a_1^3(0,\eta) \\
&\lesssim \|Aa^1_{\neq}\|_{L^2}\|\partial_x\Lambda_t^{-1}p_{i,\neq}\|_{L^\infty}\|\partial_y a^3_{=}\|_{H^N} \\
&\lesssim \|Aa^1_{\neq}\|_{L^2}\|\tfrac{1}{M_L}A\Lambda_t^{-1}p_{i,\neq}\|_{L^2}\|\partial_y a^3_{=}\|_{H^N} \\
&\lesssim \|Aa^1_{\neq}\|_{L^2}(\|\Lambda_t^{-1}Ap_i\|_{L^2}+\nu\|Ap_{i,\neq}\|_{L^2})\|\partial_y a^3_{=}\|_{H^N}.
\end{aligned}$$

Integrating in time yields

$$\int T d\tau \lesssim \kappa^{-\frac{1}{2}} \varepsilon^2 \tilde{\varepsilon}.$$

5.3.5 Nonlinear terms with an x-average in first component

Now we turn to

$$\begin{aligned}
&\langle \langle \partial_y \rangle^N a^1_{=,1}, \langle \partial_y \rangle^N (a^2_{\neq} \nabla_t a^3_{\neq,1})_= \rangle \\
&= -\sum_{k\neq 0} \iint d(\xi,\eta) \langle \xi \rangle^{2N} a^1_1(0,\xi) \tfrac{k\xi}{\sqrt{k^2+(\xi-\eta+kt)^2}} p_i(-k,\xi-\eta) a^3_1(k,\eta).
\end{aligned}$$

Applying Hölder's inequality, the Sobolev embedding and the definition of A yields

$$\begin{aligned}
&\langle \langle \partial_y \rangle^N a^1_{1,=}, \langle \partial_y \rangle^N (a^2_{\neq} \nabla_t a^3_{\neq,1})_= \rangle \\
&\leq \| \partial_y \langle \partial_y \rangle^N a^1_= \|_{L^2} (\| \partial_x \Lambda_t^{-1} p_{i,\neq} \|_{L^\infty} \| \langle \partial_y \rangle^N a^3_{\neq} \|_{L^2} + \| \langle \partial_y \rangle^N \Lambda_t^{-1} p_{i,\neq} \|_{L^2} \| \partial_x a^3_{\neq} \|_{L^\infty} \\
&\leq \| \partial_y a^1_= \|_{H^N} \| A \tfrac{1}{M_L} \Lambda_t^{-1} p_{i,\neq} \|_{L^2} \| A \tfrac{1}{M_L} a^3_{\neq} \|_{L^2}
\end{aligned}$$

With (5.35) we infer

$$\begin{aligned}
&\langle \langle \partial_y \rangle^N a^1_{1,=}, \langle \partial_y \rangle^N (a^2_{\neq} \nabla_t a^3_{\neq,1})_= \rangle \\
&\lesssim \| \partial_y a^1_= \|_{H^N} (\| A \Lambda_t^{-1} p_{i,\neq} \|_{L^2} + \nu \| A p_{i,\neq} \|_{L^2}) (\| A a^3_{\neq} \|_{L^2} + \nu \| A (\Lambda_t \wedge \kappa^{-\frac{1}{3}}) a^3_{\neq} \|_{L^2}) \\
&\lesssim \| \partial_y a^1_= \|_{H^N} \| A \Lambda_t^{-1} p_{i,\neq} \|_{L^2} (\| A a^3_{\neq} \|_{L^2} + \nu \| (\Lambda_t \wedge \kappa^{-\frac{1}{3}}) a^3_{\neq} \|_{L^2}) \\
&\quad + \nu \| \partial_y a^1_= \|_{H^N} \| A p_{i,\neq} \|_{L^2} (\| A a^3_{\neq} \|_{L^2} + \nu \| (\Lambda_t \wedge \kappa^{-\frac{1}{3}}) a^3_{\neq} \|_{L^2}).
\end{aligned}$$

Integrating in time yields

$$\int \langle \langle \partial_y \rangle^N a^1_=, \langle \partial_y \rangle^N (a^2_{\neq} \nabla_t a^3_{\neq,1})_= \rangle d\tau \lesssim L \kappa^{-\frac{1}{2}} \varepsilon^2 \tilde{\varepsilon}.$$

5.3.6 Other nonlinear terms

In this subsection, we aim to estimate

$$\begin{aligned}
\langle \chi A \partial_y^t \Delta_t^{-1} a^1_{\neq}, A(a^2 \nabla_t a^3) \rangle &= \langle \chi A \partial_y^t \Delta_t^{-1} a^1_{\neq}, A(a^2_{\neq} \nabla_t a^3_{\neq}) \rangle \\
&\quad + \langle \chi A \partial_y^t \Delta_t^{-1} a^1_{\neq}, A(a^2_{\neq} \nabla_t a^3_=) \rangle \\
&\quad + \langle \chi A \partial_y^t \Delta_t^{-1} a^1_{\neq}, A(a^2_= \nabla_t a^3_{\neq}) \rangle.
\end{aligned}$$

with the choices $a^1 a^2 a^3 \in \{bvv, bbb, vbv, vvb\}$. We start with the case of no x-averages and use

$$\xi l - k\eta = (\xi - kt)(l-k) + k(\xi - \eta - (k-l)t)$$

and (5.35) to infer

$$
\begin{aligned}
&\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{\neq})\rangle \\
&= \sum_{k,l,k-l\neq 0}\iint d(\xi,\eta)\tfrac{\xi-kt}{k^2+(\xi-kt)^2}\tfrac{\xi l-k\eta}{\sqrt{(k-l)^2+(\xi-\eta-(k-l)t)^2}}A^2(k,\xi)a^1(k,\xi)p_i(k-l,\xi-\eta)a^3(l,\eta) \\
&\lesssim \|Aa^1_{\neq}\|_{L^2}\|\tfrac{1}{M_L}\partial_x\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2} \\
&\quad + \|\partial_x\Lambda_t^{-1}Aa^1_{\neq}\|_{L^2}\|\tfrac{1}{M_L}Ap_{i,\neq}\|_{L^2}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2} \\
&\lesssim L\|Aa^1_{\neq}\|_{L^2}(\|\partial_x\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2}+\nu\|Ap_{i,\neq}\|_{L^2})\|Aa^3_{\neq}\|_{L^2} \\
&\quad + L(1+\nu\kappa^{-\frac{1}{3}})\|\partial_x\Lambda_t^{-1}Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|Aa^3_{\neq}\|_{L^2}.
\end{aligned}
$$

Thus integrating in time yields

$$\int\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{\neq}\nabla_t a^3_{\neq})\rangle d\tau \lesssim L\kappa^{-\frac{1}{2}}\varepsilon^3.$$

For the case, when the average is in the second component, we use partial integration to estimate

$$
\begin{aligned}
&\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{1,=}\partial_x a^3_{\neq})\rangle \\
&= -\langle \chi A\partial_x\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{1,=}a^3_{\neq})\rangle \\
&\lesssim \|\partial_x\Lambda_t^{-1}a^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|\tfrac{1}{M_L}Aa^3_{\neq}\|_{L^2} \\
&\lesssim L\|\partial_x\Lambda_t^{-1}a^1_{\neq}\|_{L^2}\|a^2_{=}\|_{H^N}\|Aa^2_{\neq}\|_{L^2}
\end{aligned}
$$

and thus integrating in time and using $L=\max(1,\nu\kappa^{-\frac{1}{3}})$ yields

$$\int\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{1,=}\partial_x a^3_{\neq})\rangle d\tau \lesssim \kappa^{-\frac{1}{2}}\varepsilon^2\tilde{\varepsilon}.$$

For the case when the average is in the third component, we obtain

$$
\begin{aligned}
&\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{2,\neq}\partial_y a^3_{=})\rangle \\
&= \sum_{k\neq 0}\iint d(\xi,\eta)\chi\tfrac{\xi-kt}{k^2+(\xi-kt)^2}\tfrac{k\eta}{\sqrt{k^2+(\xi-\eta-t)^2}}\tfrac{A(\xi,k)}{A(\xi-\eta,k)}(Aa^1)(k,\xi)(Ap_i)(k,\xi-\eta)a^3(0,\eta).
\end{aligned}
$$

Thus by $\eta=\xi-kt-(\xi-\eta-kt)$ we estimate

$$
\begin{aligned}
&\langle \chi A\partial_y^t\Delta_t^{-1}a^1_{\neq}, A(a^2_{2,\neq}\partial_y a^3_{=})\rangle \\
&\leq \|(\partial_y^t)^2\Delta_t^{-1}Aa^1_{\neq}\|_{L^2}\|\partial_x\Lambda_t^{-1}\tfrac{1}{M_L}Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N} \\
&+ \|\partial_x\partial_y^t\Delta_t^{-1}Aa^1_{\neq}\|_{L^2}\|\tfrac{1}{M_L}Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N} \\
&\leq L\|Aa^1_{\neq}\|_{L^2}\|\partial_x\Lambda_t^{-1}Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N} \\
&+ L\|\partial_x\Lambda_t^{-1}Aa^1_{\neq}\|_{L^2}\|Ap_{i,\neq}\|_{L^2}\|a^3_{=}\|_{H^N}
\end{aligned}
$$

Integrating in time and using $L = \max(1, \nu\kappa^{-\frac{1}{3}})$ yields

$$\int \langle \chi A \partial_y^t \Delta_t^{-1} a^1_{\neq}, A(a^2_{2,\neq} \partial_y a^3_{=}) \rangle d\tau \lesssim \kappa^{-\frac{1}{2}} \varepsilon^2 \tilde{\varepsilon}.$$

Which concludes the estimate

$$\int ONL d\tau \lesssim L\kappa^{-\frac{1}{2}} \tilde{\varepsilon}\varepsilon^2.$$

Combining the estimates of Subsection 5.3.2 to 5.3.6 completes the proof of Proposition 5.4and thus Theorem 1. □

In this article, we have shown that the MHD equations around Couette flow with magnetic resistivity smaller than fluid viscosity $\nu \geq \kappa > 0$ are stable for initial data which is small enough in Sobolev spaces. If the resistivity is much smaller than the viscosity, $\nu\kappa^{-\frac{1}{3}} > 0$, large viscosity destabilizes the equation, leading to norm inflation of size $\nu\kappa^{-\frac{1}{3}}$. Controlling this norm inflation is a major new challenge compared to other dissipation regimes.

5.4 Construction of the Weights

Let A be the Fourier weight

$$A := M \langle \nabla \rangle e^{c\kappa^{\frac{1}{3}} t \mathbf{1}_{\neq}},$$

with $M = M_1 M_L M_\kappa M_\nu M_{\nu^3}$ defined as

$$\begin{aligned}
\frac{-\dot{M}_L}{M_L} &= \frac{t - \frac{\xi}{k}}{1 + (\frac{\xi}{k} - t)^2} \mathbf{1}_{\{\nu^{-1} \leq t - \frac{\xi}{k} \leq (c_1 \kappa k^2)^{-\frac{1}{3}}\}} && k \neq 0, \\
\frac{-\dot{M}_1}{M_1} &= C_\alpha \frac{|k| + \nu^{\frac{1}{12}} |k|^2}{k^2 + (\xi - kt)^2} && k \neq 0, \\
\frac{-\dot{M}_\nu}{M_\nu} &= \frac{\nu^{\frac{1}{3}}}{1 + \nu^{\frac{2}{3}} (t - \frac{\xi}{k})^2} && k \neq 0, \\
\frac{-\dot{M}_\kappa}{M_\kappa} &= \frac{\kappa^{\frac{1}{3}}}{1 + \kappa^{\frac{2}{3}} (t - \frac{\xi}{k})^2} && k \neq 0, \\
\frac{-\dot{M}_{\nu^3}}{M_{\nu^3}} &= \frac{C_\alpha \nu}{1 + \nu^2 (t - \frac{\xi}{k})^2} && k \neq 0, \\
M.(t = 0) &= M.(k = 0) = 1.
\end{aligned}$$

The weight M_L is an adaption of the weight $m^{\frac{1}{2}}$ in [Lis20] to the present setting and M_{ν^3} we use to differentiate between resonant and non-resonant regions. The method of using time-dependent Fourier weights is common when working at solutions around Couette flow and the other weights are modifications of previously used weights (cf. [BVW18, MZ22, Lis20, ZZ24] for shear related systems such as Navier-Stokes). The constants $C_\alpha = \frac{2}{\min(1, \alpha - \frac{1}{2})}$, $c = \frac{1}{20}(1 - \frac{1}{2\alpha})^2$

and $c_1 = \frac{1}{20}(1-\frac{1}{2\alpha})$ are determined through the linear estimates. For the weights we obtain that for all times $t > 0$, it holds that

$$M_1 \approx M_\kappa \approx M_\nu \approx M_{\nu^3} \approx 1,$$
$$L^{-1} \leq \min(1, \nu^{-1}\kappa^{\frac{1}{3}}k^{\frac{2}{3}}) \lesssim M_L \leq 1.$$

Lemma 5.5 (M_L properties)**.** *The weight M_L satisfies the following bounds*

$$\mathbf{1}_{|t-\frac{\xi}{k}|\geq\nu^{-1}} \frac{t-\frac{\xi}{k}}{1+(t-\frac{\xi}{k})^2} \leq \frac{-\dot{M}_L}{M_L} + \kappa k^2 c_1(1+(t-\tfrac{\xi}{k})^2), \tag{5.34}$$
$$\frac{1}{M_L(k,\xi)} \leq 1 + \nu^{\frac{1}{2}}\langle t - \tfrac{\xi}{k}\rangle \wedge \kappa^{-\frac{1}{3}}. \tag{5.35}$$

Furthermore, it follows for $a \in H^1$, that

$$\|\tfrac{1}{M_L}a_{\neq}\|_{L^2} \leq \|a_{\neq}\|_{L^2} + \|(\Lambda_t^{-1} \wedge \kappa^{-\frac{1}{3}})a_{\neq}\|_{L^2}, \tag{5.36}$$
$$\|\tfrac{1}{M_L}\partial_x a_{\neq}\|_{L^2} \leq \|\Lambda_t a_{\neq}\|_{L^2}. \tag{5.37}$$

Proof. This follows immediately, from the definition of M_L. □

Lemma 5.6 (Enhanced dissipation estimates)**.** *The weights M_ν and M_κ satisfy the following bounds*

$$\tfrac{1}{2}\nu^{\frac{1}{3}} \leq \frac{-\dot{M}_\kappa}{M_\kappa} + \nu(k^2 + (\xi - kt)^2), \tag{5.38}$$
$$\tfrac{1}{2}\kappa^{\frac{1}{3}} \leq \frac{-\dot{M}_\nu}{M_\nu} + \kappa(k^2 + (\xi - kt)^2). \tag{5.39}$$

Proof. This follows immediately, from the definition of M_ν and M_κ. □

Lemma 5.7 (Difference estimates)**.** *Let $k, l \in \mathbb{Z} \setminus \{0\}$ and $\xi, \eta \in \mathbb{R}$, then there hold the following bounds on differences*

$$1 - \frac{M_1(k,\xi)}{M_1(k,\eta)} \lesssim \frac{|\xi-\eta|}{|k|}, \tag{5.40}$$
$$1 - \frac{M_1(k,\xi)}{M_1(l,\eta)} \lesssim \frac{|k-l|}{|l|} + \nu^{\frac{1}{12}}, \tag{5.41}$$
$$M_L(k,\eta) - M_L(k,\xi) \leq 2\frac{|\xi-\eta|}{k}, \tag{5.42}$$
$$1 - \frac{M_j(k,\xi)}{M_j(l,\eta)} \leq 2j^{\frac{1}{3}}\frac{|\xi l - k\eta|}{|kl|}, \qquad j \in \{\kappa, \nu, \nu^3\} \tag{5.43}$$

Proof. We start with the M_1 estimate (5.40) and consider $M_1(k,\xi) \leq M_1(k,\eta)$

$$\begin{aligned} 1 - \frac{M_1(k,\xi)}{M_1(k,\eta)} &= 1 - \exp\left(-\left|\int_0^t \frac{|k|+|k|^2\nu^{\frac{1}{12}}}{k^2+(\xi-k\tau)^2} - \frac{|k|+|k|^2\nu^{\frac{1}{12}}}{k^2+(\eta-kt)^2}d\tau\right|\right), \\ &\leq 1 - \exp\left(-\int_{[\frac{\xi}{k},\frac{\eta}{k}]\cup t-[\frac{\xi}{k},\frac{\eta}{k}]} 1\ d\tau\right) \lesssim \frac{|\xi-\eta|}{|k|}. \end{aligned}$$

The case $M_1(k,\xi) \geq M_1(k,\eta)$ follows by the same argument and $M_1(k,\xi) \approx M_1(k,\eta) \approx 1$. For (5.41) we consider the case $M_1(k,\xi) \leq M_1(l,\eta)$ and infer that

$$\begin{aligned} 1 - \tfrac{M_1(k,\xi)}{M_1(l,\eta)} &= 1 - \exp\left(-\left|\int_0^t \tfrac{|k|+|k|^2\nu^{\frac{1}{12}}}{k^2+(\xi-kt)^2} - \tfrac{|l|+|l|^2\nu^{\frac{1}{12}}}{l^2+(\eta-lt)^2}\right|\right), \\ &= 1 - \exp(-2\pi(\tfrac{1}{l\wedge k} + \nu^{\frac{1}{12}})) \lesssim \tfrac{1}{l\wedge k} + \nu^{\frac{1}{12}} \lesssim \tfrac{|k-l|}{|l|} + \nu^{\frac{1}{12}}. \end{aligned}$$

The case $M_1(k,\xi) \geq M_1(l,\eta)$ follows by the same argument and $M_1(k,\xi) \approx M_1(l,\eta) \approx 1$. For (5.42) we consider the case $M_L(k,\xi) \leq M_L(k,\eta)$ and thus

$$M_L(k,\eta) - M_L(k,\xi) = M_L(k,\eta)(1 - \tfrac{M_L(k,\xi)}{M_L(k,\eta)}) \lesssim 1 - \tfrac{M_L(k,\xi)}{M_L(k,\eta)}.$$

We infer

$$\begin{aligned} 1 - \tfrac{M_L(k,\xi)}{M_L(k,\eta)} &= 1 - \exp\left(-\left|\int_0^t \tfrac{\tau-\frac{\xi}{k}}{1+(\tau-\frac{\xi}{k})^2} d\tau - \int_0^t \tfrac{\tau-\frac{\eta}{k}}{1+(\tau-\frac{\eta}{k})^2} d\tau\right|\right), \\ &= 1 - \exp\left(-\int_{-[\frac{\xi}{k},\frac{\eta}{k}]\cup t-[\frac{\xi}{k},\frac{\eta}{k}]} 1 d\tau\right), \\ &\leq 2\tfrac{|\xi-\eta|}{k}. \end{aligned}$$

The case $M_L(k,\xi) \geq M_L(k,\eta)$ follows by the same argument.

For (5.43) we estimate the M_κ difference, since the M_ν and M_{ν^3} differences are done similar. Let $M_\kappa(k,\xi) \geq M_\kappa(l,\eta)$, then it follows

$$\begin{aligned} 1 - \tfrac{M_\kappa(k,\xi)}{M_\kappa(l,\eta)} &= 1 - \exp\left(-\kappa^{\frac{1}{3}}\left|\int_0^t \tfrac{1}{1+\kappa^{\frac{2}{3}}(t-\frac{\xi}{k})^2} - \tfrac{1}{1+\kappa^{\frac{2}{3}}(t-\frac{\eta}{l})^2}\right|\right), \\ &\leq 1 - \exp\left(-\kappa^{\frac{1}{3}}\left|\int_0^t \mathbf{1}_{-[\frac{\xi}{k},\frac{\eta}{l}]\cup t-[\frac{\xi}{k},\frac{\eta}{l}]}(\tau) d\tau\right|\right), \\ &\leq 1 - \exp\left(-2\kappa^{\frac{1}{3}}\left|\tfrac{\xi}{k} - \tfrac{\eta}{l}\right|\right), \\ &\lesssim \kappa^{\frac{1}{3}}\tfrac{|\xi l - k\eta|}{|kl|}. \end{aligned}$$

The case $M_\kappa(k,\xi) \leq M_\kappa(l,\eta)$ follows from the same steps and $M_\kappa(k,\xi) \approx M_\kappa(l,\eta)$.

□

5.5 Local Wellposedness

We expect the local wellposedness result to be well-known, but were not able to find it stated in the literature. In the following, we prove the local wellposedness by a standard application of the Banach fixed-point theorem.

Proposition 5.8. *Consider equation* (5.17) *with initial data* $p_{in} \in H^N$ *for* $N \geq 5$. *Then there exists a time* T *such that there exists a unique solution* $p(t) \in H^N$ *to* (5.17) *for all* $t \in [0,T]$ *.*

Proof. We prove existence with the Banach fixed-point theorem. Let $T = 1 + 2\|p_{in}\|_{H^N}(1+\frac{8}{\kappa})$ and let X be the space

$$X = \{p \in L^\infty H^N \cap CH^{N-2} :\ p(t=0) = p_{in},\ \|p\|^2_{L^\infty H^N} + \tfrac{\kappa}{2}\|\nabla_t p\|^2_{L^\infty H^N} \leq 2\|p_{in}\|^2_{H^N}\}$$

with the norm

$$\|p\|^2_X := \|p\|^2_{L^\infty H^N} + \tfrac{\kappa}{2}\|\nabla_t p\|^2_{L^\infty H^N}.$$

We define $F :\ X \mapsto X$ as a mapping $q \mapsto p = F(q)$ such that p solves

$$\begin{aligned}
\partial_t p_1 - \partial_x \partial_y^t \Delta_t^{-1} p_1 - \alpha\partial_x p_2 &= \nu\Delta_t p_1 + \Lambda_t^{-1}\nabla_t^\perp(\nabla_t^\perp \Lambda_t^{-1} q_2 \nabla_t b - \nabla_t^\perp \Lambda_t^{-1} q_1 \nabla_t v),\\
\partial_t p_2 + \partial_x \partial_y^t \Delta_t^{-1} p_2 - \alpha\partial_x p_1 &= \kappa\Delta_t p_2 + \Lambda_t^{-1}\nabla_t^\perp(\nabla_t^\perp \Lambda_t^{-1} q_2 \nabla_t v - \nabla_t^\perp \Lambda_t^{-1} q_1 \nabla_t b),\\
p|_{t=0} &= p_{in}.
\end{aligned}$$

Then the mapping F satisfies:

1. The mapping $F :\ X \to X$ is well defined on X.
2. The mapping F is a contraction, i.e. $\|F(p)-F(\tilde{p})\|_{L^\infty H^N} \leq \frac{1}{2}\|p-\tilde{p}\|_{L^\infty H^N}$.

Since X is a complete metric space, if we prove (1) and (2), then it follows that F has a unique fixpoint by the Banach fixed-point theorem.

1. Let $q \in X$, then we obtain for $p = F(q)$

$$\begin{aligned}
\partial_t\|p\|^2_{H^N} + \kappa\|\nabla_t p\|^2_{H^N} &\leq \|p\|^2_{H^N} + \langle \Lambda^N v, \Lambda^N(\nabla_t^\perp \Lambda_t^{-1} q_2 \nabla_t b - \nabla_t^\perp \Lambda_t^{-1} q_1 \nabla_t v)\rangle\\
&\quad + \langle \Lambda^N b, \Lambda^N(\nabla_t^\perp \Lambda_t^{-1} q_2 \nabla_t v - \nabla_t^\perp \Lambda_t^{-1} q_1 \nabla_t b)\rangle\\
&\leq \|p\|^2_{H^N} + \|p\|_{H^N}\|q\|_{H^N}\|\nabla_t p^{n+1}\|_{H^N}\\
&\leq \|p\|^2_{H^N} + \tfrac{2}{\kappa}\|p\|^2_{H^N}\|q\|^2_{H^N} + \tfrac{\kappa}{2}\|\nabla_t p\|^2_{H^N}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\|p\|^2_X &\leq \|p_{in}\|^2_{H^N} + T(1+\tfrac{1}{\kappa}\|q\|^2_{L^\infty H^N})\|p\|^2_{L^\infty H^N}\\
&\leq \|p_{in}\|^2_{H^N} + T(1+\tfrac{2}{\kappa}\|p_{in}\|^2_{H^N})\|p\|^2_{L^\infty H^N},
\end{aligned}$$

Since

$$T(1+\tfrac{4}{\kappa}\|p_{in}\|^2_{H^N}) < \tfrac{1}{2}$$

we infer the bound

$$\|p\|^2_X \leq 2\|p_{in}\|^2_{H^N}.$$

As $\partial_t p \in H^{N-2}$, it follows that $p \in CH^{N-2}$ and thus $p \in X$.

2. We show that F is a contraction. Let $q, \tilde{q} \in X$ we denote $p = F(q)$ and $\tilde{p} = F(\tilde{q})$. We need to show that

$$\|p - \tilde{p}\|_X < \tfrac{1}{2}\|q - \tilde{q}\|_X,$$

by time estimate we obtain

$$\begin{aligned}
\partial_t \|p - \tilde{p}\|_{H^N}^2 + \kappa\|\nabla_t(p - \tilde{p})\|_{H^N}^2 \leq{}& \|p - \tilde{p}\|_{H^N}^2 \\
&+ \langle \Lambda^N(v - \tilde{v}), \Lambda^N(\nabla_t^\perp \Lambda_t^{-1} q_2 \nabla_t b - \nabla_t^\perp \Lambda_t^{-1} q_1 \nabla_t v)\rangle \\
&+ \langle \Lambda^N(b - \tilde{b}), \Lambda^N(\nabla_t^\perp \Lambda_t^{-1} q_2 \nabla_t v - \nabla_t^\perp \Lambda_t^{-1} q_1 \nabla_t b)\rangle \\
&- \langle \Lambda^N(v - \tilde{v}), \Lambda^N(\nabla_t^\perp \Lambda_t^{-1} \tilde{q}_2 \nabla_t \tilde{b} - \nabla_t^\perp \Lambda_t^{-1} \tilde{q}_1 \nabla_t v)\rangle \\
&- \langle \Lambda^N(b - \tilde{b}), \Lambda^N(\nabla_t^\perp \Lambda_t^{-1} \tilde{q}_2 \nabla_t \tilde{v} - \nabla_t^\perp \Lambda_t^{-1} \tilde{q}_1 \nabla_t \tilde{b})\rangle \\
\leq{}& \|p - \tilde{p}\|_{H^N}^2 + \|p - \tilde{p}\|_{H^N} \left(\|q - \tilde{q}\|_{H^N}\|\nabla_t \tilde{p}\|_{H^N} + \|q\|_{H^N}\|\nabla_t(p - \tilde{p})\|_{H^N}\right) \\
\leq{}& \|p - \tilde{p}\|_{H^N}^2 (1 + \tfrac{2}{\kappa}\|q\|_{H^N}^2) \\
&+ \|p - \tilde{p}\|_{H^N}\|q - \tilde{q}\|_{H^N}\|\nabla_t \tilde{p}\|_{H^N} + \tfrac{\kappa}{2}\|\nabla_t(p - \tilde{p})\|_{H^N}^2.
\end{aligned}$$

Integrating in time yields

$$\begin{aligned}
\|p - \tilde{p}\|_{L^\infty H^N}^2 &+ \tfrac{\kappa}{2}\|\nabla_t(p - \tilde{p})\|_{L^2 H^N}^2 \\
&\leq \|p - \tilde{p}\|_{L^\infty H^N}^2 T(1 + \tfrac{2}{\kappa}\|q\|_{L^\infty H^N}^2) \\
&\quad + \sqrt{T}\|p - \tilde{p}\|_{L^\infty H^N}\|q - \tilde{q}\|_{L^\infty H^N}\|\nabla_t \tilde{p}\|_{L^2 H^N} \\
&\leq \|p - \tilde{p}\|_{L^\infty H^N}^2 T(1 + \tfrac{2}{\kappa}\|q\|_{L^\infty H^N}^2 + 4\|\nabla_t \tilde{p}\|_{L^2 H^N}^2) \\
&\quad + \tfrac{T}{4}\|q - \tilde{q}\|_{L^\infty H^N}^2.
\end{aligned}$$

Choosing T such that

$$T(1 + \tfrac{2}{\kappa}\|q\|_{H^N}^2 + \|\nabla_t \tilde{p}\|_{L^2 H^N}^2) \leq T + 2T\|p_{in}\|_{H^N}(1 + \tfrac{8}{\kappa}) < \tfrac{1}{2},$$

it follows, that

$$\|p - \tilde{p}\|_X^2 \leq \tfrac{1}{2}\|p - \tilde{p}\|_{H^N} + \tfrac{T}{4}\|q - \tilde{q}\|_{L^\infty H^N}^2.$$

We hence conclude, that

$$\|p - \tilde{p}\|_X^2 \leq \tfrac{1}{2}\|q - \tilde{q}\|_X^2.$$

□

Appendix A

Mathematical Background and Notation

Fourier transformation

We consider the Fourier transformation on the space $\mathbb{T} \times \mathbb{R}$. Due to linearity, all the properties can be deduced by the Fourier transformation on $\mathbb{R}$ and the Fourier series on $\mathbb{T}$. In the following, we list the main properties (see [Gra14]).

Let $f \in L^2(\mathbb{T} \times \mathbb{R})$, we define for $(k, \xi) \in \mathbb{Z} \times \mathbb{R}$ it's Fourier transform as

$$\mathcal{F}f(k, \xi) := \hat{f}(k, \xi) : = \tfrac{1}{2\pi} \int_0^{2\pi} dx \int dy \; e^{-i(kx+\xi y)} f(x, y)$$

For a function $g \in L^2(\mathbb{Z} \times \mathbb{R})$ we define the inverse Fourier transform

$$\mathcal{F}^{-1}g(x, y) := \check{g}(x, y) : = \tfrac{1}{2\pi} \sum_k \int d\xi \; e^{i(kx+\xi y)} g(k, \xi).$$

Then, it holds that

$$\mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{Id}_{L^2} \qquad \mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{Id}_{L^2}\,.$$

The Fourier transformation acts as an isometry on L^2, i.e. for all $f \in L^2$

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}.$$

For functions f such that $\partial_x f \in L^2$ and $\partial_y f \in L^2$ it holds that

$$\begin{aligned} (\partial_x f)^\wedge &= ik\hat{f}, \\ (\partial_y f)^\wedge &= i\xi\hat{f}. \end{aligned}$$

Sobolev and Gevrey spaces

For $1 \leq p \leq \infty$ and $s \geq 0$ the spaces L^p and H^s corresponds to the classical Lebesgue and Sobolev spaces for functions on the set $\mathbb{T} \times \mathbb{R}$. Sobolev spaces satisfy the Sobolev embedding

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^s}$$

for $s > 1$.

For time dependent functions we define $L^p H^s = L^p(0, T; H^s)$ as the space with the norm

$$\| \cdot \|_{L^p H^s} = \|\| \cdot \|_{H^s}\|_{L^p(0,T)}.$$

There are different definitions of Gevrey classes and spaces. The main concept is that functions in Gevrey spaces decay exponentially in the Fourier variables on the L^2 norm. In the following, we will provide two definitions, the one we use in Chapter 3 and a more general version.

Let $1 \leq \sigma$, then a function $f \in L^2(\mathbb{T} \times \mathbb{R})$ belongs to the Gevrey-σ spaces (in sense of Chapter 3) if

$$\sum_k \int \exp(\lambda |\xi|^{\frac{1}{\sigma}}) |\hat{f}(k, \xi)|^2 \, d\xi$$

for some constant $\lambda > 0$.
For the more general version, let $1 \leq \sigma$, we define the Gevrey-σ class as the Gevrey spaces $G_\sigma^{\lambda,N}$ for $\lambda > 0$ and $N \geq 0$ with

$$G_\sigma^{\lambda,N} := \{ \ f \in L^2 \ : \ \|f\|_{G_\sigma^{\lambda,N}} < \infty \ \} \tag{A.1}$$

with

$$\|f\|^2_{G_\sigma^{\lambda,N}} = \sum_k \int d\xi \ \langle k, \xi \rangle^N \exp(\lambda |k, \xi|^{\frac{1}{\sigma}}) |\hat{f}|^2(k, \xi). \tag{A.2}$$

The λ is called the radius of convergence.

Notations

For two real numbers $a, b \in \mathbb{R}$, we denote the minimum and maximum as

$$\min(a, b) = a \wedge b,$$
$$\max(a, b) = a \vee b.$$

We write $f \lesssim g$ if there exists a constant C independent of all relevant parameters such that $|f| \leq C|g|$. Furthermore, we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

Moreover, for any vector or scalar v we define

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}.$$

For a function $f \in L^2(\mathbb{T} \times \mathbb{R})$ we denote the x-average and its L^2-orthogonal complement as

$$f_{=}(y) = \int_{\mathbb{T}} f(x, y) dx,$$
$$f_{\neq} = f - f_{=}.$$

Appendix B

Derivation of the MHD Equations from the Navier-Stokes and Maxwell Equations

In this appendix, we derive the MHD equations from the Navier-Stokes and Maxwell Equations. We follow the book of Davidson [Dav16]. The MHD equations model an electrical conduction and non-magnetic fluid and are derived from the Navier-Stokes and Maxwell equations under the vanishing charge density assumption. The MHD equations are derived in three steps. First, we write down the governing equations, then we explain the simplifications and finally we derive the MHD equations.

Governing Equations

A (conduction) fluid satisfies the Navier-Stokes equation

$$\begin{aligned} \partial_t V + V \cdot \nabla V + \nabla \Pi_v &= \nu \Delta V + F, \\ \operatorname{div}(V) &= 0, \end{aligned} \tag{B.1}$$

Here V is the fluid velocity, Π_v is the fluid pressure, ν is the fluid viscosity and F is the applied forces.

The conduction fluid generates an electric and magnetic field which satisfies

Maxwell's equations

$$\begin{aligned}\nabla \cdot E &= \tfrac{\rho}{\varepsilon}, \\ \nabla \cdot B &= 0, \\ \nabla \times E &= -\partial_t B, \\ \nabla \times B &= \mu J + \mu\varepsilon\partial_t E.\end{aligned} \tag{B.2}$$

Here E is the electric field, B is the magnetic field, J is the current density, ρ is the charge density, ε is the permittivity of free space and μ is the magnetic constant. In a conducting fluid, the current density is proportional to the Lorentz force

$$f = q(E + V \times B) \tag{B.3}$$

on free charges, with q as the charge.

In a stationary conductor, the current density J is proportional to the force applied to free charges, and thus, the Ohmic law $J = \sigma\tilde{E}$ applies there. For a conducting fluid, we consider the electric field measured in a moving frame with the velocity of the conducting fluid

$$J = \sigma(E + V \times B). \tag{B.4}$$

With the electrical conductivity σ.

The Lorentz force (B.3) affects the moving particles, but we are more interested in the volumetric version. Thus we sum over the charge q of a unit volume of the conductor. Then the sum over charges is the density $\sum q = \rho$ and the sum over moving charges is the current $\sum qV = J$ and so we obtain the force

$$F = \rho E + J \times B. \tag{B.5}$$

Simplification for the MHD Equations

The charge density ρ in a conducting fluid is small and negligible compared to other effects. Thus for the MHD equations, we assume that

$$\rho = 0. \tag{B.6}$$

Due to the charge conservation, we obtain that the current is divergent free

$$\nabla \cdot J = -\partial_t \rho = 0.$$

The second simplification is, for the last term of (B.2) the $\mu\varepsilon\partial_t E$ only is relevant on relativistic scales. Therefore it can be neglected in the model and we obtain

$$\nabla \times B = \mu J. \tag{B.7}$$

Derivation of the MHD equations

Then with (B.6) and (B.7) Maxwell's equations (B.2) change to the pre Maxwell equations

$$\nabla \cdot E = 0, \tag{B.8}$$
$$\nabla \cdot B = 0, \tag{B.9}$$
$$\nabla \times E = -\partial_t B, \tag{B.10}$$
$$\nabla \times B = \mu J. \tag{B.11}$$

In the following, we establish a closed formula for the magnetic and velocity field. We apply (B.10), (B.4) and (B.11) to infer

$$\begin{aligned}\partial_t B &= -\nabla \times E \\ &= -\nabla \times (\tfrac{1}{\sigma} J - V \times B) \\ &= -\tfrac{1}{\sigma\mu} \nabla \times \nabla \times B + \nabla \times (V \times B).\end{aligned}$$

Then we define the resistivity $\kappa := \frac{1}{\sigma\mu}$ and use $\nabla \cdot V = \nabla \cdot B = 0$ to infer

$$\begin{aligned}-\tfrac{1}{\sigma\mu} \nabla \times \nabla \times B &= \kappa \Delta B, \\ \nabla \times (V \times B) &= B \cdot \nabla V - V \cdot \nabla B.\end{aligned}$$

Therefore, for B we obtain the equation

$$\begin{aligned}\partial_t B + V \cdot \nabla B &= \kappa \Delta B + B \cdot \nabla V \\ \operatorname{div}(B) &= 0.\end{aligned} \tag{B.12}$$

From (B.5), (B.6) and (B.11) we infer

$$F = J \times B = \operatorname{curl}(B) \times B = B \cdot \nabla B - \tfrac{1}{2} \nabla |B|^2.$$

We define the pressure $\Pi = \Pi_v + \frac{1}{2} \nabla |B|^2$ and thus we obtain for (B.1)

$$\begin{aligned}\partial_t V + V \cdot \nabla V + \nabla \Pi &= \nu \Delta V + B \cdot \nabla B, \\ \operatorname{div}(V) &= 0.\end{aligned} \tag{B.13}$$

Combining the equations (B.12) and (B.13) we infer the MHD equations

$$\begin{aligned}\partial_t V + V \cdot \nabla V + \nabla \Pi &= \nu \Delta V + B \cdot \nabla B, \\ \partial_t B + V \cdot \nabla B &= \kappa \Delta B + B \cdot \nabla V, \\ \operatorname{div}(B) &= \operatorname{div}(V) = 0.\end{aligned}$$

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