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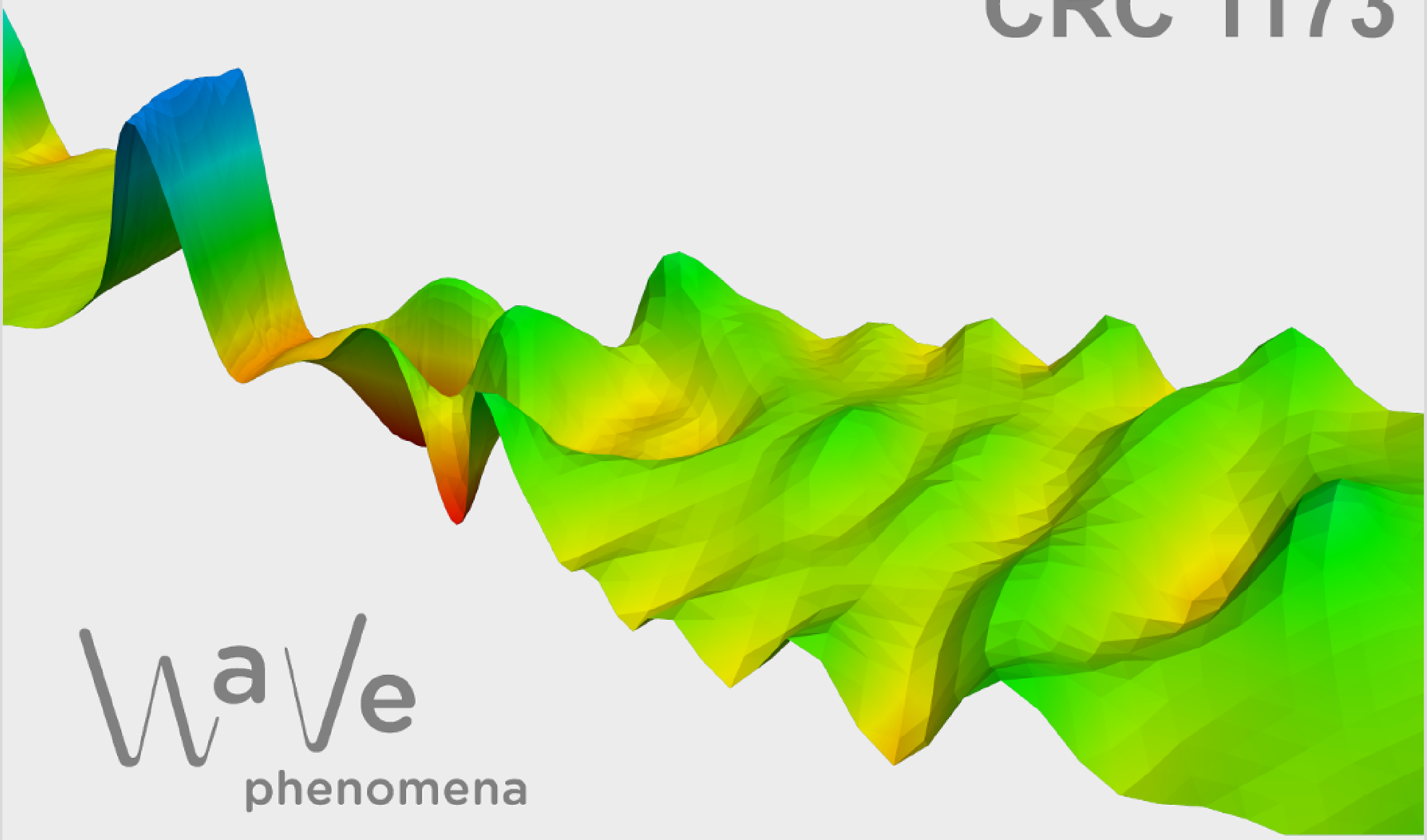

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THE CALDERON OPERATOR FOR THE MAXWELL SYSTEM IN THE EXTERIOR OF AN INFINITE CYLINDER IN $\mathbb{R}^3$

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Abstract: We study the Calderon operator for the time-harmonic Maxwell system in the "exterior" Ω^+ of an infinite cylinder in x_3-direction. The Calderon is the analogue of the Dirichlet-to-Neumann operator for the scalar Helmholtz equation. In the first part we study the case where the Calderon operator corresponds to solutions u on Ω^+ which, together with their curls, decay along x_3. In the second part we consider the case where the solution u is assumed to be quasi-periodic with respect to x_3. In both cases we derive properties of the Calderon operator with respect to coercivity and compactness. These properties are useful for the investigation of problems in all of $\mathbb{R}^3$ if one uses the Calderon operator to reduce the problem to the "interior" of the cylinder. The proofs rely heavily on properties of the Hankel functions which are studied in detail in the appendix.

MSC: 35Q61

1. Introduction

We fix some $R > 0$ and define the infinite cylinder $\Gamma = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = R^2\}$ and the exterior region $\Omega^+ = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 > R^2\}$. It is the aim to study the homogeneous Maxwell system

$$\operatorname{curl} E = i\omega\mu_0\, H\,, \quad \operatorname{curl} H = -i\omega\varepsilon_0\, E \quad \text{in } \Omega^+ \tag{1.1}$$

with the boundary condition $\nu \times E = h$ on Γ and a suitable radiation condition discussed below. Here, $\nu = \nu(x)$, $x \in \Gamma$, denotes the unit normal vector directed into Ω^+. Eliminating H from the system and renaming $u = E$ yields

$$\operatorname{curl}^2 u - k^2 u = 0 \text{ in } \Omega^+\,, \quad \nu \times u = h \text{ on } \Gamma\,, \tag{1.2}$$

where the wavenumber $k > 0$ is given by $k = \omega\sqrt{\varepsilon_0\mu_0}$.

In this paper we consider two cases. In the first case we search for fields u which decay as along the cylinder, i.e. as $|x_3| \to \infty$. More precisely, we look for solutions in the space

$$H_*(\operatorname{curl}, \Omega^+) \;:=\; \big\{u : \Omega^+ \to \mathbb{C}^3 : u|_{\Omega_\rho} \in H(\operatorname{curl}, \Omega_\rho) \text{ for all } \rho > R\big\}\,, \tag{1.3}$$

where $\Omega_\rho := \{x \in \mathbb{R}^3 : R^2 < x_1^2 + x_2^2 < \rho^2\}$, and $H(\operatorname{curl}, D)$ denotes the usual space of L^2-vector fields such that also their curl is in L^2. As we will see in Section 2 the boundary data have to be in suitable subspace of $H^{-1/2}(\operatorname{Div}, \Gamma)$ defined below.

In the second case we look for quasi-periodic solutions u, i.e. the solutions satisfy $u(x_1, x_2, x_3 + 2\pi) = e^{i\alpha 2\pi} u(x_1, x_2, x_3)$ for all $x = (x_1, x_2, x_3) \in \Omega^+$ for some parameter

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$\alpha \in \mathbb{R}$. It is clear that a α-quasi-periodic solution can only be expected if also the boundary data h are α-quasi-periodic.

For both cases we will study the corresponding Calderon operator which is defined as the operator which maps the tangential field h on Γ to the trace $\nu \times \operatorname{curl} u$ on Γ where u satisfies (1.2) and an appropriate radiation condition.

In this paper we use cylindrical coordinates (r, ϕ, x_3) and denote by $\hat{r} = (\cos\phi, \sin\phi, 0)^\top$, $\hat{\phi} = (-\sin\phi, \cos\phi, 0)^\top$, and $\hat{z} = (0,0,1)^\top$ the coordinate unit vectors. We note that $\nu = \hat{r}$ on Γ. Furthermore, we observe that the differential equation $\operatorname{curl}^2 u - k^2 u = 0$ is equivalent to the pair of equations $\Delta u + k^2 u = 0$ and $\operatorname{div} u = 0$.

Studying the exterior problem (1.2) and the corresponding Calderon operator is interesting in itself because, in contrast to the case of the exterior of a ball (see, e.g. [7]), particular emphasis has to be put on the treatment of the so-called cut-off values. Often (as, e.g., in [7]), the Calderon operator is used to reduce the problem in a unbounded domain to a problem in a bounded domain with non-local boundary conditions. For the geometry studied in this paper the quasi-periodic problem (with respect to x_3) in $\mathbb{R}^3$ is reduced to a problem in the bounded domain $\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2,\ 0 < x_3 < 2\pi\}$ with non-local boundary conditions for $x_1^2 + x_2^2 = R^2$ and quasi-periodic boundary conditions for $x_3 \in \{0, 2\pi\}$. The present paper is a necessary preparation of a forthcoming paper where the scattering problem will treated for coefficients ε and μ which are periodic with respect to x_3 in the interior $\mathbb{R}^3 \setminus \Omega^+$ of the cylinder and constant in Ω^+.

We want to mention some of the related literature. For scalar problems, i.e. the scalar Helmholtz equation $\Delta u + k^2 u = 0$ in Ω^+, the Calderon operator corresonds to the Dirichlet to Neumann operator and has been studied (for this geometry) in, e.g., [2] and [4]. The problem with the cut-off values does not occur for this case. For the Maxwell system in the half space $\{x \in \mathbb{R}^3 : x_3 > 0\}$ the Calderon operator on the plane $x_3 = 0$ has been studied intensively in [8], and it is shown that for a proper treatment weighted Sobolev spaces have to be used. Ritterbusch's approach has been applied to a different situation in [6].

2. The $H_*(\mathrm{curl})$ Case

Since we expect a solution of (1.2) in $H_*(\mathrm{curl}, \Omega^+)$ we can take the Fourier transform

$$(2.1) \qquad \hat{u}(x_1, x_2, \xi) \ := \ \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x_1, x_2, x_3)\, e^{-i\xi x_3}\, dx_3\,,\ \ \xi \in \mathbb{R}\,,$$

with respect to x_3 which has to satisfy (for every component) the two-dimensional Helmholtz equation

$$(2.2) \qquad \Delta_2 \hat{u}(\tilde{x}, \xi) + k(\xi)^2 \hat{u}(\tilde{x}, \xi) = 0 \quad \text{for } x_1^2 + x_2^2 > R^2$$

where $\tilde{x} = (x_1, x_2)$ and $k(\xi) = \sqrt{k^2 - \xi^2}$. In addition $\operatorname{div} u = 0$ translates into $\partial_1 \hat{u}^{(1)}(\tilde{x}, \xi) + \partial_2 \hat{u}^{(2)}(\tilde{x}, \xi) + i\xi\, \hat{u}^{(3)}(\tilde{x}, \xi) = 0$.

Definition 2.1. *A solution $u \in H_*(\mathrm{curl}, \Omega^+)$ of (1.2) satisfies the radiation condition if the Fourier transform $\hat{u}(\cdot, \xi)$ satisfies the two-dimensional Sommerfeld radiation condition*

$$(2.3) \qquad \frac{\partial \hat{u}(\tilde{x}, \xi)}{\partial r} \ - \ ik(\xi)\, \hat{u}(\tilde{x}, \xi) \ = \ \mathcal{O}\big(1/|\tilde{x}|^{3/2}\big)\,, \quad r = |\tilde{x}| \to \infty\,,$$

for almost all $\xi \in \mathbb{R}$.

For the definitions of the correct spaces of boundary data the cylindrical Fourier transform plays an essential role. For $g \in L^2(\Gamma) \cap C_0^\infty(\Gamma)$ we define

$$(2.4) \qquad g_m(\xi) \;:=\; \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} g(\phi, x_3)\, e^{-im\phi - i\xi x_3}\, dx_3\, d\phi\,, \quad m \in \mathbb{Z}\,,\ \xi \in \mathbb{R}\,.$$

Then this transform has an extension to $L^2(\Gamma)$, and the inverse is given by

$$g(\phi, x_3) \;=\; \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} g_m(\xi)\, e^{i\xi x_3}\, d\xi\, e^{im\phi}\,.$$

Definition 2.2. *We define the space $H^{\pm 1/2}(\Gamma)$ of scalar functions and the spaces $H^{-1/2}(\mathrm{Div}, \Gamma)$ and $H^{-1/2}(\mathrm{Curl}, \Gamma)$ of tangential vector fields by*

$$H^{\pm 1/2}(\Gamma) := \left\{ \begin{array}{l} p : \Gamma \to \mathbb{C} \;: \\ \displaystyle\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} |p_m(\xi)|^2 [1 + m^2 + \xi^2]^{\pm 1/2}\, d\xi < \infty \end{array} \right\},$$

$$H^{-1/2}(\mathrm{Div}, \Gamma) := \left\{ \begin{array}{l} h = h^\phi \hat{\phi} + h^z \hat{z} : \Gamma \to \mathbb{C}^3 \;: h^\phi := h \cdot \hat{\phi},\ h^z := h \cdot \hat{z} \textit{ satisfy} \\ \displaystyle\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{|h_m^z(\xi)|^2 + |h_m^\phi(\xi)|^2 + |\xi h_m^z(\xi) + \frac{m}{R} h_m^\phi(\xi)|^2}{\sqrt{1 + m^2 + \xi^2}} d\xi < \infty \end{array} \right\},$$

$$H^{-1/2}(\mathrm{Curl}, \Gamma) := \left\{ \begin{array}{l} h = h^\phi \hat{\phi} + h^z \hat{z} : \Gamma \to \mathbb{C}^3 \;: h^\phi := h \cdot \hat{\phi},\ h^z := h \cdot \hat{z} \textit{ satisfy} \\ \displaystyle\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{|h_m^z(\xi)|^2 + |h_m^\phi(\xi)|^2 + |\xi h_m^\phi(\xi) - \frac{m}{R} h_m^z(\xi)|^2}{\sqrt{1 + m^2 + \xi^2}} d\xi < \infty \end{array} \right\}.$$

It can be shown as in [5] that $\langle H^{-1/2}(\mathrm{Div}, \Gamma), H^{-1/2}(\mathrm{Curl}, \Gamma) \rangle$ and $\langle H^{-1/2}(\Gamma), H^{1/2}(\Gamma) \rangle$ are dual pairs with duality forms

$$\begin{aligned} \langle h, f \rangle \;&=\; 4\pi^2 R \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \left[h_m^\phi(\xi) \overline{f_m^\phi(\xi)} + h_m^z(\xi) \overline{f_m^z(\xi)} \right] d\xi\,, \\ &\qquad h \in H^{-1/2}(\mathrm{Div}, \Gamma)\,,\ f \in H^{-1/2}(\mathrm{Curl}, \Gamma)\,, \\ \langle p, q \rangle \;&=\; 4\pi^2 R \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} p_m(\xi)\, \overline{q_m(\xi)}\, d\xi\,, \quad p \in H^{-1/2}(\Gamma)\,,\ q \in H^{1/2}(\Gamma)\,. \end{aligned}$$

For the proof the following identity is essential (for $m \neq 0$)

$$\begin{aligned} & h_m^\phi(\xi) \overline{f_m^\phi(\xi)} + h_m^z(\xi) \overline{f_m^z(\xi)} \\ =\; & \frac{\frac{m}{R} \overline{f_m^\phi(\xi)} + \xi \overline{f_m^z(\xi)}}{\frac{m^2}{R^2} + \xi^2} \left[\frac{m}{R} h_m^\phi(\xi) + \xi h_m^z(\xi) \right] + \frac{\frac{m}{R} h_m^z(\xi) - \xi h_m^\phi(\xi)}{\frac{m^2}{R^2} + \xi^2} \left[\frac{m}{R} \overline{f_m^z(\xi)} - \xi \overline{f_m^\phi(\xi)} \right] \end{aligned}$$

which yields the estimate (using the Cauchy-Schwarz inequality in the numerators of the fractions)

$$
\begin{aligned}
\left|h_m^\phi(\xi)\overline{f_m^\phi(\xi)}+h_m^z(\xi)\overline{f_m^z(\xi)}\right| \;&\le\; \frac{\sqrt{|f_m^\phi(\xi)|^2+|f_m^z(\xi)|^2}}{[\frac{m^2}{R^2}+\xi^2]^{1/4}}\,\frac{|\frac{m}{R}h_m^\phi(\xi)+\xi h_m^z(\xi)|}{[\frac{m^2}{R^2}+\xi^2]^{1/4}} \\
&\quad+\frac{\sqrt{|h_m^z(\xi)|^2+|h_m^\phi(\xi)|^2}}{[\frac{m^2}{R^2}+\xi^2]^{1/4}}\,\frac{|\frac{m}{R}f_m^z(\xi)-\xi f_m^\phi(\xi)|}{[\frac{m^2}{R^2}+\xi^2]^{1/4}}\,,
\end{aligned}
\tag{2.5}
$$

and thus, using again the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\sum_{m\neq 0}\int_{-\infty}^{\infty}\left|h_m^\phi(\xi)\overline{f_m^\phi(\xi)}+h_m^z(\xi)\overline{f_m^z(\xi)}\right|d\xi \\
\le\;&\left[\sum_{m\neq 0}\int_{-\infty}^{\infty}\frac{|f_m^\phi(\xi)|^2+|f_m^z(\xi)|^2}{\sqrt{\frac{m^2}{R^2}+\xi^2}}\,d\xi\right]^{1/2}\left[\sum_{m\neq 0}\int_{-\infty}^{\infty}\frac{|\frac{m}{R}h_m^\phi(\xi)+\xi h_m^z(\xi)|^2}{\sqrt{\frac{m^2}{R^2}+\xi^2}}\,d\xi\right]^{1/2} \\
&+\left[\sum_{m\neq 0}\int_{-\infty}^{\infty}\frac{|h_m^z(\xi)|^2+|h_m^\phi(\xi)|^2}{\sqrt{\frac{m^2}{R^2}+\xi^2}}\,d\xi\right]^{1/2}\left[\sum_{m\neq 0}\int_{-\infty}^{\infty}\frac{|\frac{m}{R}f_m^z(\xi)-\xi f_m^\phi(\xi)|^2}{\sqrt{\frac{m^2}{R^2}+\xi^2}}\,d\xi\right]^{1/2}.
\end{aligned}
$$

For $m=0$ one argues analogously. This proves boundedness of $\langle\cdot,\cdot\rangle$.
Furthermore, the trace operators $u\mapsto\nu\times u$ and $u\mapsto\nu\times(u\times\nu)$ are bounded and surjective from $H(\mathrm{curl},\Omega_\rho)$ into $H^{-1/2}(\mathrm{Div},\Gamma)$ and $H^{-1/2}(\mathrm{Curl},\Gamma)$, respectively, for every $\rho>R$ which can be shown as, e.g., in [5], Section 5.1. Finally, we note that $\langle\mathrm{Div}\,u,p\rangle=-\langle u,\mathrm{Grad}\,p\rangle$ for all $u\in H^{-1/2}(\mathrm{Div},\Gamma)$ and $p\in H^{1/2}(\Gamma)$ where $\mathrm{Div}\,u$ and $\mathrm{Grad}\,u$ denote the surface divergence and surface gradient, respectively, defined as

$$
\begin{aligned}
(\mathrm{Div}\,h)(\phi,x_3)&=i\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\big[\frac{m}{R}h_m^\phi(\xi)+\xi h_m^z(\xi)\big]e^{im\phi+i\xi x_3}d\xi\,,\\
(\mathrm{Grad}\,p)(\phi,x_3)&=i\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\big[\frac{m}{R}p_m(\xi)\,\hat\phi+\xi p_m(\xi)\,\hat z\big]e^{im\phi+i\xi x_3}d\xi\,.
\end{aligned}
$$

We will see shortly that the space $H^{-1/2}(\mathrm{Div},\Gamma)$ is not quite appropriate for the bounday data h. Before we define the correct space we formally derive the form of the Calderon operator by solving the pair of equations

$$
\begin{aligned}
&\Delta\hat u(\tilde x,\xi)+k(\xi)^2\hat u(\tilde x,\xi)=0 \text{ and}\\
&\partial_{x_1}\hat u^{(1)}(\tilde x,\xi)+\partial_{x_2}\hat u^{(2)}(\tilde x,\xi)+i\xi\,\hat u^{(3)}(\tilde x,\xi)=0 \text{ for } |\tilde x|>R\,,
\end{aligned}
$$

(where again $\tilde x=(x_1,x_2)$) for the Fourier transform of u. We assume that $\xi\in\mathbb{R}$ is kept fixed with $k(\xi)\neq 0$, i.e. $|\xi|\neq k$, and require also the Sommerfeld radiation condition (2.3).

To solve this boundary value problem we make an ansatz for $\hat u$ in the cartesian form as

$$
\hat u(\tilde x,\xi)\;=\;i\xi\,\nabla_3 w(\tilde x)\;+\;k(\xi)^2\,w(\tilde x)\,\hat z\;+\;\nabla_3 v(\tilde x)\times\hat z \tag{2.6}
$$

where ∇_3 denotes the three dimensional gradient. We dropped ξ in v and w. The scalar functions $w,v:\{\tilde x\in\mathbb{R}^2:|\tilde x|>R\}\to\mathbb{C}$ are assumed to satisfy the two dimensional Helmholtz equation $\Delta_2 w + k(\xi)^2 w = 0$ and $\Delta_2 v + k(\xi)^2 v = 0$ and the Sommerfeld radiation condition (2.3). Using the product rule for the gradient and $\operatorname{curl}(fA) = f \operatorname{curl} A + \nabla f \times A$ for vector fields A and scalar functions f we can write $\hat{u}$ in the equivalent form

$$\hat{u}(\tilde{x},\xi)e^{i\xi x_3} \;=\; i\xi\, \nabla_3\big(w(\tilde{x})e^{i\xi x_3}\big) \;+\; k^2 w(\tilde{x})e^{i\xi x_3}\,\hat{z} \;+\; \operatorname{curl}\big(v(\tilde{x})e^{i\xi x_3}\,\hat{z}\big)$$

from which we observe that $\Delta\big(\hat{u}(\tilde{x},\xi)e^{i\xi x_3}\big) + k^2\big(\hat{u}(\tilde{x},\xi)e^{i\xi x_3}\big) = 0$ and $\operatorname{div}\big(\hat{u}(\tilde{x},\xi)e^{i\xi x_3}\big) = 0$. Furthermore, we compute the curl as

$$\operatorname{curl}\big[\hat{u}(\tilde{x},\xi)e^{i\xi x_3}\big] \;=\; \big[i\xi\,\nabla_3 v(\tilde{x}) + k(\xi)^2\, v(\tilde{x})\,\hat{z} + k^2\,\nabla_3 w(\tilde{x}) \times \hat{z}\big]\, e^{i\xi x_3} \tag{2.7}$$

where we used $\operatorname{curl}^2 = -\Delta + \nabla \operatorname{div}$. Since $w = w(\tilde{x},\xi)$ and $v = v(\tilde{x},\xi)$ satisfy Sommerfeld's radiation condition we can expand them into series of Hankel functions in the form

$$w(r,\phi,\xi) = \sum_{m\in\mathbb{Z}} w_m(\xi)\,\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)}\,e^{im\phi}\,, \quad v(r,\phi,\xi) = \sum_{m\in\mathbb{Z}} v_m(\xi)\,\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)}\,e^{im\phi}$$

where H_m denotes the Hankel function of type one and order m. We obtain, using $\hat{r}\times\hat{\phi} = \hat{z}$ and $\hat{r}\times\hat{z} = -\hat{\phi}$ and $\hat{\phi}\times\hat{z} = \hat{r}$,

$$\begin{aligned}\hat{u}(r,\phi,\xi) \;=\; \sum_{m\in\mathbb{Z}} \Bigg\{ & \left(w_m(\xi)\, i\xi\, \frac{k(\xi)H_m'(k(\xi)r)}{H_m(k(\xi)R)} + v_m(\xi)\,\frac{im}{r}\,\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)} \right)\hat{r} \\ & + w_m(\xi)\,k(\xi)^2 \frac{H_m(k(\xi)r)}{H_m(k(\xi)R)}\,\hat{z} \\ & - \left(w_m(\xi)\,\frac{m\xi}{r}\,\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)} + v_m(\xi)\,\frac{k(\xi)H_m'(k(\xi)r)}{H_m(k(\xi)R)} \right)\hat{\phi} \Bigg\}\, e^{im\phi}.\end{aligned} \tag{2.8}$$

Let $h \in H^{-1/2}(\operatorname{Div},\Gamma)$ be given by

$$h(\phi,x_3) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \big[h_m^\phi(\xi)\,\hat{\phi} + h_m^z(\xi)\,\hat{z}\big]\, e^{im\phi+\xi x_3} d\xi\,, \quad \phi, x_3 \in [0,2\pi]\,. \tag{2.9}$$

We determine $w_m(\xi)$ and $v_m(\xi)$ by the boundary condition $\hat{r}\times u = h$ for $r = R$. This gives

$$w_m(\xi) = -\frac{1}{k(\xi)^2}\,h_m^\phi(\xi) \quad\text{and}\quad v_m(\xi) = \frac{H_m(k(\xi)R)}{k(\xi)H_m'(k(\xi)R)}\left[\frac{m\xi}{Rk(\xi)^2}\,h_m^\phi(\xi) - h_m^z(\xi)\right] \tag{2.10}$$

provided $k(\xi) \neq 0$. Substituting this into (2.8) and re-aranging the terms (using also $k^2 = k(\xi)^2 + \xi^2$) yields

$$u(r,\phi,x_3) \;=\; \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \big\{u_m^r(r,\xi)\,\hat{r} \;+\; u_m^\phi(r,\xi)\,\hat{\phi} \;+\; u_m^z(r,\xi)\,\hat{z}\big\} e^{im\phi+i\xi x_3} d\xi \tag{2.11}$$

for $r > R$, $\phi \in (0, 2\pi)$, and $x_3 \in \mathbb{R}$ where

$$\begin{aligned}
u_m^r(r,\xi) &= h_m^\phi(\xi)\,\frac{i\xi}{rk(\xi)^2}\left(\frac{m^2 H_m(k(\xi)r)}{Rk(\xi)H_m'(k(\xi)R)} - \frac{rk(\xi)H_m'(k(\xi)r)}{H_m(k(\xi)R)}\right)\\
&\quad - h_m^z(\xi)\frac{imR}{r}\,\frac{H_m(k(\xi)r)}{Rk(\xi)H_m'(k(\xi)R)}\,,\\
u_m^\phi(r,\xi) &= h_m^\phi(\xi)\frac{m\xi}{rk(\xi)^2}\left(\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)} - \frac{rH_m'(k(\xi)r)}{RH_m'(k(\xi)R)}\right) + h_m^z(\xi)\,\frac{H_m'(k(\xi)r)}{H_m'(k(\xi)R)}\,,\\
u_m^z(r,\xi) &= -h_m^\phi(\xi)\,\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)}\,.
\end{aligned}$$

We need also the following form of $u_m^\phi(r,\xi)$:

$$\begin{aligned}
u_m^\phi(r,\xi) &= \frac{\xi R}{rk(\xi)^2}\left(\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)} - \frac{rH_m'(k(\xi)r)}{RH_m'(k(\xi)R)}\right)\big(\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big)\\
&\quad + h_m^z(\xi)\left[\frac{H_m'(k(\xi)r)}{H_m'(k(\xi)R)} + \frac{R}{r}\left(1 - \frac{k^2}{k(\xi)^2}\right)\left(\frac{H_m(k(\xi)r)}{H_m(k(\xi)R)} - \frac{rH_m'(k(\xi)r)}{RH_m'(k(\xi)R)}\right)\right].
\end{aligned}$$

Now we express $\operatorname{curl} u$. Comparing the forms of $\hat{u}$ and $\operatorname{curl} u$ of (2.6) and (2.7) we observe that they have the same form for (almost) v_m and w_n interchanged. We obtain by a similar calculation

$$\text{(2.12)}\quad \operatorname{curl} u(r,\phi,x_3) = \sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\big\{v_m^r(r.\xi)\,\hat{r} + v_m^\phi(r,\xi)\,\hat{\phi} + v_m^z(r,\xi)\,\hat{z}\big\}e^{im\phi+i\xi x_3}d\xi$$

where

$$\begin{aligned}
v_m^r(r,\xi) &= h_m^\phi(\xi)\,\frac{imk^2}{rk(\xi)^2}\left(\frac{rH_m'(k(\xi)r)}{RH_m'(k(\xi)R)} - \frac{H_m(k(\xi)r)}{H_m(k(\xi)R)}\right)\\
&\quad - i\big(\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big)\,\frac{H_m'(k(\xi)r)}{H_m'(k(\xi)R)}\,,\\
v_m^\phi(r,\xi) &= h_m^\phi(\xi)\frac{k^2}{rk(\xi)^2}\left(\frac{rk(\xi)H_m'(k(\xi)r)}{H_m(k(\xi)R)} - \frac{m^2H_m(k(\xi)r)}{Rk(\xi)H_m'(k(\xi)R)}\right)\\
&\quad + \frac{m}{r}\,\frac{H_m(k(\xi)r)}{k(\xi)H_m'(k(\xi)R)}\big(\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big)\,,\\
v_m^z(r,\xi) &= \left[\big(\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big)\,\xi - k^2 h_m^z(\xi)\right]\frac{H_m(k(\xi)r)}{k(\xi)H_m'(k(\xi)R)}\,.
\end{aligned}$$

For notational reasons we define the functions $G_m^{(j)}(r,\xi)$, $j = 1,\ldots,5$, as

$$\text{(2.13a)}\qquad G_m^{(1)}(r,\xi) := \frac{1}{k(\xi)^2}\left[\frac{m^2H_m(rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))} - \frac{rk(\xi)H_m'(rk(\xi))}{H_m(Rk(\xi))}\right],$$

$$\text{(2.13b)}\qquad G_m^{(2)}(r,\xi) := \frac{H_m(rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))}\,,$$

$$\text{(2.13c)}\qquad G_m^{(3)}(r,\xi) := \frac{H_m(rk(\xi))}{H_m(Rk(\xi))}\,,\qquad G_m^{(4)}(r,\xi) := \frac{H_m'(rk(\xi))}{H_m'(Rk(\xi))}\,,$$

$$\text{(2.13d)}\qquad G_m^{(5)}(r,\xi) := \frac{1}{k(\xi)^2}\left[\frac{H_m(rk(\xi))}{H_m(Rk(\xi))} - \frac{rH_m'(rk(\xi))}{RH_m'(Rk(\xi))}\right].$$

Then the coefficients of u and $\operatorname{curl} u$ from (2.11) and (2.12), respectively, are written as

$$\begin{aligned}
u^r_m(r,\xi) &= h^\phi_m(\xi)\,\frac{i\xi}{r}\,G^{(1)}_m(r,\xi) \;-\; h^z_m(\xi)\,\frac{imR}{r}G^{(2)}_m(r,\xi)\,,\\
u^\phi_m(r,\xi) &= h^\phi_m(\xi)\,\frac{m\xi}{r}\,G^{(5)}_m(r,\xi) \;+\; h^z_m(\xi)\,G^{(4)}_m(r,\xi)\\
&= d_m(\xi)\,\frac{\xi R}{r}\,G^{(5)}_m(r,\xi) \;+\; h^z_m(\xi)\,\left[G^{(4)}_m(r,\xi) + \frac{R[k(\xi)^2-k^2]}{r}\,G^{(5)}_m(r,\xi)\right],\\
u^z_m(r,\xi) &= -h^\phi_m(\xi)\,G^{(3)}_m(r,\xi)\,,\\
v^r_m(r,\xi) &= -h^\phi_m(\xi)\,\frac{imk^2}{r}\,G^{(5)}_m(r,\xi) \;-\; i\,d_m(\xi)\,G^{(4)}_m(r,\xi)\,,\\
v^\phi_m(r,\xi) &= -h^\phi_m(\xi)\,\frac{k^2}{r}\,G^{(1)}_m(r,\xi) \;+\; d_m(\xi)\,\frac{mR}{r}\,G^{(2)}_m(r,\xi)\,,\\
v^z_m(r,\xi) &= \left[\xi\,d_m(\xi) - k^2 h^z_m(\xi)\right] R\,G^{(2)}_m(r,\xi)\,,
\end{aligned}$$

where $d_m(\xi) := \frac{m}{R}h^\phi_m(\xi) + \xi h^z_m(\xi)$. The terms involving $G^{(1)}_m$ and $G^{(5)}_m$ are obviously singular for ξ with $k(\xi)=0$. The precise type of the singularity and the behavior for large values of $|m|+|\xi|$ are investigated in Lemma A.1 of the appendix. For $|m|\geq 2$ or $|k(\xi)|\geq 1/2$ and $r\in[R,\tilde R]$ for any fixed $\tilde R > R$ we obtain that all 6 coefficients are estimated by[1] $c\,a_m(\xi)\,\mu_m(r,\xi)$ where c in independent of r, m, and ξ, and

$$a_m(\xi) \;:=\; |h^\phi_m(\xi)| \;+\; |h^z_m(\xi)| \;+\; \Big|\frac{m}{R}h^\phi_m(\xi) \;+\; \xi h^z_m(\xi)\Big|\,, \quad |m|\geq 2\,, \tag{2.14}$$

$$\mu_m(r,\xi) \;:=\; \begin{cases} \exp\left[-\frac{|k(\xi)|^2 R\,(r-R)}{12\sqrt{m^2+|k(\xi)|^2R^2}}\right]\left(\frac{R}{r}\right)^{|m|} & \text{if } |\xi|>k\,,\\ \left(\frac{R}{r}\right)^{|m|} & \text{if } |\xi|<k\,,\end{cases} \quad m\in\mathbb{Z}\,. \tag{2.15}$$

For $|m|\leq 1$ and $|k(\xi)|\leq 1/2$ we recall from Lemma A.1 that the following functions are continuous (for every fixed $\tilde R > R$): $G^{(3)}_m$, $G^{(4)}_m$, $(r,\xi)\mapsto \frac{1}{\ln k(\xi)}G^{(2)}_0(r,\xi)$, $(r,\xi)\mapsto \frac{1}{\ln k(\xi)}G^{(j)}_{\pm1}(r,\xi)$, and $(r,\xi)\mapsto k(\xi)^2\ln k(\xi)\,G^{(j)}_0(r,\xi)$ for $j\in\{1,5\}$ and $r\in[R,\tilde R)]$ and $|k(\xi)|\leq 1/2$. Therefore, we define the weight functions

$$\rho_1(\xi) = |\ln k(\xi)| \quad\text{and}\quad \rho_0(\xi) = \frac{1}{|k(\xi)|^2\,|\ln k(\xi)|} \quad\text{for}\quad 0<|k(\xi)|\leq 1/2\,,$$

and estimate

$$|u^r_{\pm1}(r,\xi)| \;\leq\; c\left[\rho_1(\xi)|h^\phi_{\pm1}(\xi)| + |h^z_{\pm1}(\xi)|\right]\mu_1(r,\xi)\,,\quad |u^r_0(r,\xi)| \;\leq\; c\,\rho_0(\xi)|h^\phi_0(\xi)|\mu_0(r,\xi)$$

for $|k(\xi)|\leq 1/2$ and $r\in[R,\tilde R)]$. The other coefficients involving $G^{(1)}_m$, $G^{(2)}_m$, and $G^{(5)}_m$ for $|m|\leq 1$ are estimated in the same way. Therefore, we set

$$a_m(\xi) \;:=\; |h^\phi_m(\xi)| + |h^z_m(\xi)| + \rho_{|m|}(\xi)\,|h^\phi_m(\xi)|\,,\quad |m|\leq 1\,,\ |k(\xi)|\leq 1/2\,, \tag{2.16}$$

and observe that all of the 6 coefficients are bounded by $c\,|a_m(\xi)|\mu_m(r,\xi)$ for $|m|\leq 1$ and $|k(\xi)|\leq 1/2$.

These observations motivate the following space for the boundary data h:

$$H^{-1/2}_*(\mathrm{Div},\Gamma) := \left\{ h\in H^{-1/2}(\mathrm{Div},\Gamma) : \int\limits_\Xi \rho_{|m|}(\xi)\,|h^\phi_m(\xi)|^2\,d\xi < \infty \text{ for } |m|\leq 1\right\}$$

[1]for $u^\phi_m(r,\xi)$ we take the second form

where $\Xi = \{\xi : |k(\xi)| \leq 1/2\}$. We equip $H_*^{-1/2}(\mathrm{Div},\Gamma)$ with its canonical norm

$$\|h\|^2_{H_*^{-1/2}(\mathrm{Div},\Gamma)} \;:=\; \|h\|^2_{H^{-1/2}(\mathrm{Div},\Gamma)} \;+\; \sum_{|m|\leq 1} \int_{\Xi} \rho_{|m|}(\xi)\, |h_m^\phi(\xi)|^2\, d\xi\,. \tag{2.17}$$

Theorem 2.3. *For every $h \in H_*^{-1/2}(\mathrm{Div},\Gamma)$ the function u given in (2.11) is the unique solution $u \in H_*(\mathrm{curl},\Omega^+)$ of (1.2) satisfying the radiation condition of Definition 2.1. Furthermore, for every $\tilde R > R$ there exists $c = c(\tilde R) > 0$ with $\|u\|_{H(\mathrm{curl},\Omega_{\tilde R})} \leq c\|h\|_{H_*^{-1/2}(\mathrm{Div},\Gamma)}$.*

Proof. It suffices to show that $u, \mathrm{curl}\, u \in L^2(\Omega_{\tilde R},\mathbb{C}^3)$ for every $\tilde R > R$. Let $a_m(\xi)$ be defined as above for $|m| \geq 2$ and for $|m| \leq 1$ by (2.14) and (2.16), respectively. The estimate

$$|u_m^r(r,\xi)|+|u_m^\phi(r,\xi)|+|u_m^z(r,\xi)|+|v_m^r(r,\xi)|+|v_m^\phi(r,\xi)|+|v_m^z(r,\xi)| \leq c\, a_m(\xi)\, \mu_m(r,\xi)$$

for all $m \in \mathbb{Z}$, $\xi \in \mathbb{R}$, $r \in [R,\tilde R]$, and (A.5) yields

$$\int_R^{\tilde R}\int_{-\infty}^{\infty}\int_0^{2\pi} \big[|u(r,\phi,x_3)|^2 + |\,\mathrm{curl}\, u(r,\phi,x_3)|^2\big] r\, d\phi\, dx_3\, dr$$

$$\leq\; c\sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} a_m(\xi)^2 \int_R^{\tilde R} \mu_m(r,\xi)^2\, dr\, d\xi \;\leq\; c'\sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{a_m(\xi)^2}{\sqrt{1+m^2+\xi^2}}\, d\xi\,.$$

□

From the form of $\mathrm{curl}\, u$ we obtain, using again $\hat r \times \hat\phi = \hat z$ and $\hat r \times \hat z = -\hat\phi$, that $\Lambda h = \mathrm{curl}\, w \times \hat r$ is given by

$$(\Lambda h)(\phi,x_3) \;=\; \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \big[\lambda_m^z(\xi)\,\hat z + \lambda_m^\phi(\xi)\,\hat\phi\big]\, e^{im\phi+i\xi x_3} d\xi \tag{2.18}$$

where

$$\begin{aligned}
\lambda_m^z(\xi) \;&=\; -h_m^\phi(\xi)\frac{k^2}{R}\, G_m^{(1)}(R,\xi) \;+\; m\, d_m(\xi)\, G_m^{(2)}(R,\xi) &(2.19)\\
&=\; h_m^\phi(\xi)\frac{k^2}{Rk(\xi)^2}\left[\frac{Rk(\xi)H_m'(k(\xi)R)}{H_m(k(\xi)R)} - \frac{m^2 H_m(k(\xi)R)}{Rk(\xi)H_m'(k(\xi)R)}\right]\\
&\quad+ m\,\big[\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big]\frac{H_m(k(\xi)R)}{Rk(\xi)H_m'(k(\xi)R)}\\
\lambda_m^\phi(\xi) \;&=\; \big[k^2 h_m^z(\xi) - \xi\, d_m(\xi)\big]\, R\, G_m^{(2)}(R,\xi) &(2.20)\\
&=\; \big\{k^2 h_m^z(\xi) \;-\; \xi\,\big[\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big]\big\}\,\frac{H_m(k(\xi)R)}{k(\xi)H_m'(k(\xi)R)}\\
&=\; \big[k(\xi)^2\, h_m^z(\xi) - \frac{\xi m}{R}\, h_m^\phi(\xi)\big]\,\frac{H_m(k(\xi)R)}{k(\xi)H_m'(k(\xi)R)}\,.
\end{aligned}$$

Theorem 2.4. *Λ, defined by (2.18) (with (2.19), (2.20)), has the following properties.*

(a) *Λ is well-defined and bounded from $H_*^{-1/2}(\mathrm{Div},\Gamma)$ into itself.*
(b) *$\mathrm{Im}\langle\Lambda h, \hat r \times h\rangle \geq 0$ for all $h \in H_*^{-1/2}(\mathrm{Div},\Gamma)$.*

(c) *The Calderon operator* Λ_α *can be decomposed as* $\Lambda = \Lambda^D + \Lambda^C$ *where* Λ^D *and* Λ^C *are bounded from* $H_*^{-1/2}(\mathrm{Div},\Gamma)$ *into* $H^{-1/2}(\mathrm{Div},\Gamma)$ *and* $\mathrm{Div}\,\Lambda^D h = 0$ *and* $\mathrm{Curl}\,\Lambda^C h = 0$ *for all* $h \in H_*^{-1/2}(\mathrm{Div},\Gamma)$.

(d) *The operator* $\mathrm{Div}\,\Lambda$ *is bounded from* $H_*^{-1/2}(\mathrm{Div},\Gamma)$ *into* $H^{-1/2}(\Gamma)$ *and*

$$\|\Lambda^C h\|_{H^{-1/2}(\mathrm{Div},\Gamma)} \;\leq\; c\,\|\mathrm{Div}\,\Lambda h\|_{H^{-1/2}(\Gamma)} \quad \text{for all } h \in H_*^{-1/2}(\mathrm{Div},\Gamma)\,. \tag{2.21}$$

Proof. (a) For $|m| \geq 2$ or $|k(\xi)| \geq 1$ we have, using again the estimates (A.3a), that $|\lambda_m^z(\xi)|$ and $|\lambda_m^\phi(\xi)|$ are bounded by $c\,|a_m(\xi)|$ with a_m from (2.14). For $|m| \leq 1$ and $|k(\xi)| \leq 1$ we use that $\frac{1}{\rho_{|m|}(\xi)}\,G_m^{(1)}(R,\xi)$ for $|m| \leq 1$ and $\frac{1}{\rho_1(\xi)}\,G_0^{(2)}(R,\xi)$ are bounded and thus

$$\begin{aligned}
|\lambda_m^z(\xi)| &\leq c\left[|h_m^\phi(\xi)|\,\rho_{|m|}(\xi) + |d_m(\xi)|\right], \quad |m| \leq 1\,,\\
\rho_1(\xi)\,|\lambda_{\pm1}^\phi(\xi)| &\leq c\left[\rho_1(\xi)\,|k(\xi)|^2|h_{\pm1}^z(\xi)| + \rho_1(\xi)\,|h_{\pm1}^\phi(\xi)|\right]\frac{|H_1(k(\xi)R)|}{|k(\xi)||H_1'(k(\xi)R)|}\\
&\leq c\,|a_{\pm1}(\xi)|\,,\\
\rho_0(\xi)\,|\lambda_0^\phi(\xi)| &\leq c\,\rho_0(\xi)\,|k(\xi)|^2\,|h_0^z(\xi)|\,\rho_1(\xi) \;= c\,|h_0^z(\xi)|\,,
\end{aligned}$$

thus $|\lambda_m^z(\xi)| + \rho_{|m|}(\xi)|\lambda_m^\phi(\xi)| \leq c\,a_m(\xi)$ for $|m| \leq 1$ with a_m from (2.16). Finally, $\frac{m}{R}\,\lambda_m^\phi(\xi) + \xi\lambda_m^z(\xi) = -h_m^\phi(\xi)\,\frac{k^2\xi}{R}\,G_m^{(1)}(R,\xi) + mk^2h_m^z(\xi)\,G_m^{(1)}(R,\xi)$, and this is estimated by $c\,a_m(\xi)$ as before.

(b) Let h as in (2.9). Then $\hat{r} \times h(\phi,x_3) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty}\left[-h_m^z(\xi)\,\hat{\phi} + h_m^\phi(\xi)\,\hat{z}\right]e^{im\phi+\xi x_3}d\xi$ and thus

$$\langle \Lambda h, \hat{r}\times h\rangle \;=\; 4\pi^2 R\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\left\{\lambda_m^z(\xi)\,\overline{h_m^\phi(\xi)} - \lambda_m^\phi(\xi)\,\overline{h_m^z(\xi)}\right\}d\xi\,.$$

For $k(\xi) \neq 0$ we write the term in the bracket as, using the definitions of $G_m^{(1)}(R,\xi) = \frac{m^2}{k(\xi)^2}G_m^{(2)}(R,\xi) - \frac{1}{k(\xi)^2G_m^{(2)}(R,\xi)}$,

$$\begin{aligned}
&\lambda_m^z(\xi)\,\overline{h_m^\phi(\xi)} - \lambda_m^\phi(\xi)\,\overline{h_m^z(\xi)}\\
&= \left[-h_m^\phi(\xi)\frac{k^2}{R}\,G_m^{(1)}(R,\xi) + m\big(\frac{m}{R}h_m^\phi(\xi) + \xi h_m^z(\xi)\big)G_m^{(2)}(R,\xi)\right]\overline{h_m^\phi(\xi)}\\
&\quad - \left[Rk(\xi)^2h_m^z(\xi) - m\xi h_m^\phi(\xi)\right]G_m^{(2)}(R,\xi)\,\overline{h_m^z(\xi)}\\
&= |h_m^\phi(\xi)|^2\frac{k^2}{Rk(\xi)^2G_m^{(2)}(R,\xi)} \;-\; G_m^{(2)}(R,\xi)\left[\frac{k^2m^2}{Rk(\xi)^2}\,|h_m^\phi(\xi)|^2\right.\\
&\quad \left. + Rk(\xi)^2|h_m^z(\xi)|^2 \;-\; m\xi\left[h_m^z(\xi)\overline{h_m^\phi(\xi)} + (h_m^\phi(\xi)\overline{h_m^z(\xi)}\right] \;-\; \frac{m^2}{R}|h_m^\phi(\xi)|^2\right]\\
&= \frac{k^2\,|h_m^\phi(\xi)|^2}{Rk(\xi)^2G_m^{(2)}(R,\xi)} \;-\; \frac{G_m^{(2)}(R,\xi)}{Rk(\xi)^2}\left|m\xi\,h_m^\phi(\xi) \;-\; Rk(\xi)^2\,h_m^z(\xi)\right|^2
\end{aligned} \tag{2.22}$$

where we used $k^2 - k(\xi)^2 = \xi^2$.

Now we consider two cases. Let first ξ with $|\xi| > k$. Then $k(\xi) = i|k(\xi)|$ and (A.6a) and (A.6b) imply that $G_m^{(2)}(R,\xi) = \frac{H_m(k(\xi)R)}{Rk(\xi)H_m'(k(\xi)R)} = \frac{K_m(R|k(\xi)|)}{R|k(\xi)|K_m'(R|k(\xi)|)}$ which is real valued. Next we consider ξ with $|\xi| < k$. Then $k(\xi)^2 > 0$ and $k(\xi) \in \mathbb{R}_{>0}$ and thus

$$\begin{aligned}\operatorname{Im} \frac{H_m'(k(\xi)R)}{H_m(k(\xi)R)} &= \frac{Y_m'(k(\xi)R)J_m(k(\xi)R) - J_m'(k(\xi)R)Y_m(k(\xi)R)}{|H_m(k(\xi)R)|^2} \\ &= \frac{2}{\pi k(\xi)R\,|H_m(k(\xi)R)|^2} \;>\; 0\end{aligned}$$

where we used the Wronskian of J_m and Y_m. Therefore, also $\operatorname{Im} \frac{H_m(k(\xi)R)}{k(\xi)H_m'(k(\xi)R)} < 0$, and the proof of part (b) is finished.

(c) We recall the form of Λh with the coefficients $\lambda_m^z(\xi)$ and $\lambda_m^\phi(\xi)$ from (2.19) and (2.20), respectively. Since $\Lambda h \in H_*^{-1/2}(\mathrm{Div}, \Gamma)$ we have

$$\sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{|\lambda_m^z(\xi)|^2 + |\lambda_m^\phi(\xi)|^2 + |\xi\lambda_m^z(\xi) + m\lambda_m^\phi(\xi)/R|^2}{\sqrt{1+m^2+\xi^2}}\, d\xi \;<\; \infty\,,$$

and, in addition, $\int_\Xi \rho_{|m|}(\xi)\, |\lambda_m^\phi(\xi)|^2\, d\xi < \infty$ for $|m| \le 1$ where again $\Xi = \{\xi : |k(\xi)| \le 1/2\}$. We define

$$\begin{aligned}(\Lambda^D h)(\phi, x_3) &:= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{m\lambda_m^z(\xi)/R - \xi\lambda_m^\phi(\xi)}{\xi^2 + (m/R)^2} \big[\frac{m}{R}\, \hat{z} - \xi\, \hat{\phi}\big]\, e^{im\phi + i\xi x_3} d\xi\,, \\ (\Lambda^C h)(\phi, x_3) &:= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\xi\lambda_m^z(\xi) + m\lambda_m^\phi(\xi)/R}{\xi^2 + (m/R)^2} \big[\xi\, \hat{z} + \frac{m}{R}\, \hat{\phi}\big]\, e^{im\phi + i\xi x_3} d\xi\,.\end{aligned}$$

Note that for $m = 0$ the terms in the integrands reduce to $\lambda_0^\phi(\xi)\, e^{i\xi x_3} \hat{\phi}$ and $\lambda_0^z(\xi)\, e^{i\xi x_3} \hat{z}$, respectively.

Direct validation shows that $\Lambda^D h + \Lambda^C h = \Lambda h$ and $\mathrm{Div}\, \Lambda^D h = 0$ and $\mathrm{Curl}\, \Lambda^C h = 0$ for all h. Furthermore, these operators are bounded because

$$\begin{aligned}\left|\frac{m\lambda_m^z(\xi)/R - \xi\lambda_m^\phi(\xi)}{\xi^2 + (m/R)^2}\right|^2 \left[\frac{m^2}{R^2} + |\xi|^2\right] &\le c\left[|\lambda_m^z(\xi)|^2 + |\lambda_m^\phi(\xi)|^2\right], \\ \left|\frac{\xi\lambda_m^z(\xi) + m\lambda_m^\phi(\xi)/R}{\xi^2 + (m/R)^2}\right|^2 \left[|\xi|^2 + \frac{m^2}{R^2} + \left|\xi^2 + \frac{m^2}{R^2}\right|^2\right] &\le c\left|\xi\lambda_m^z(\xi) + m\lambda_m^\phi(\xi)/R\right|^2.\end{aligned}$$

(d) From the previous estimate we conclude that

$$\|\Lambda^C h\|^2_{H^{-1/2}(\mathrm{Div},\Gamma)} \;\le\; c \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\left|\xi\lambda_m^z(\xi) + m\lambda_m^\phi(\xi)/R\right|^2}{\sqrt{1+m^2+\xi^2}}\, d\xi\,.$$

Furthermore,

$$(\mathrm{Div}\, \Lambda h)(\phi, x_3) \;=\; i \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \big[\xi\, \lambda_m^z(\xi) + m\, \lambda_m^\phi(\xi)/R\big]\, e^{im\phi + i\xi x_3} d\xi$$

and thus

$$\|\,\mathrm{Div}\, \Lambda h\|^2_{H^{-1/2}(\Gamma)} \;=\; 4\pi^2 R \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{|\xi\, \lambda_m^z(\xi) + m\lambda_m^\phi(\xi)/R|^2}{\sqrt{1+m^2+\xi^2}}\, d\xi$$

which proves the estimate. □

3. The Regular Quasi-Periodic Case

Every α-quasi-periodic solution of (1.2), i.e. $\Delta u + k^2 u = 0$ and $\operatorname{div} u = 0$, has a Fourier expansion in the form $u(x) = \sum_{n\in\mathbb{Z}} \hat{u}_n(x_1, x_2) e^{i(n+\alpha)x_3}$ with

$$\hat{u}_n(x_1,x_2) \ := \ \frac{1}{2\pi}\int\limits_0^{2\pi} u(x_1,x_2,x_3)\, e^{-i(n+\alpha)x_3}\, dx_3\,,\ n\in\mathbb{Z}\,, \tag{3.1}$$

where $\hat{u}_n$ satisfies the two dimensional Helmholtz equation

$$\Delta_2 \hat{u}_n(\tilde{x}) + k_n(\alpha)^2 \hat{u}_n(\tilde{x}) = 0 \quad \text{for } x_1^2 + x_2^2 > R^2 \tag{3.2}$$

where $k_n(\alpha) = \sqrt{k^2 - (n+\alpha)^2}$ and, in addition, $\partial_{x_1}\hat{u}_n^{(1)} + \partial_{x_2}\hat{u}_n^{(2)} + i(n+\alpha)\hat{u}_n^{(3)} = 0$. The solution space $H_*(\mathrm{curl}, \Omega^+)$ from (1.3) is replaced by

$$H_{\alpha,*}(\mathrm{curl}, W^+) \ := \ \left\{ u|_{W^+} : \begin{array}{l} u\in H_{loc}(\mathrm{curl},\Omega^+),\ x_3\mapsto u(x) \text{ is } \alpha\text{-quasi-periodic,} \\ u|_{W_\rho} \in H(\mathrm{curl}, W_\rho) \text{ for all } \rho > 0 \end{array} \right\}, \tag{3.3}$$

where $W^+ = \{x\in\Omega^+ : 0 < x_3 < 2\pi\}$ and $W_\rho = \{x\in\Omega_\rho : 0 < x_3 < 2\pi\}$. Instead of the cylindrical Fourier transform (2.4) we consider the classical Fourier expansion with respect to ϕ and x_3, i.e. we define

$$\hat{g}_{n,m} \ := \ \frac{1}{4\pi^2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} g(\phi,x_3)\, e^{-im\phi - i(n+\alpha)x_3}\, dx_3\, d\phi\,, \quad m,n\in\mathbb{Z}\,,$$

where we do not indicate the dependence on α because it is fixed. The spaces $H^{\pm1/2}(\Gamma)$, $H^{-1/2}(\mathrm{Div},\Gamma)$, and $H^{-1/2}(\mathrm{Curl},\Gamma)$ from Definition 2.2 are replaced by

(3.4)

$$H_\alpha^{\pm1/2}(\Gamma) := \left\{ p:\Gamma\to\mathbb{C} \ : \sum_{n,m\in\mathbb{Z}} |\hat{p}_{n,m}|^2 [1+m^2+n^2]^{\pm1/2} < \infty \right\},$$

$$H_\alpha^{-1/2}(\mathrm{Div},\Gamma) := \left\{ \begin{array}{c} h = h^\phi\hat{\phi} + h^z\hat{z} : \Gamma\to\mathbb{C}^3 \ : h^\phi := h\cdot\hat{\phi},\ h^z := h\cdot\hat{z} \text{ satisfy} \\ \displaystyle\sum_{n,m\in\mathbb{Z}} \frac{|h^z_{n,m}|^2 + |h^\phi_{n,m}|^2 + |(n+\alpha)h^z_{n,m} + \frac{m}{R}h^\phi_{n,m}|^2}{\sqrt{1+m^2+n^2}} < \infty \end{array} \right\},$$

$$H^{-1/2}(\mathrm{Curl},\Gamma) := \left\{ \begin{array}{c} h = h^\phi\hat{\phi} + h^z\hat{z} : \Gamma\to\mathbb{C}^3 \ : h^\phi := h\cdot\hat{\phi},\ h^z := h\cdot\hat{z} \text{ satisfy} \\ \displaystyle\sum_{n,m\in\mathbb{Z}} \frac{|h^z_{n,m}|^2 + |h^\phi_{n,m}|^2 + |(n+\alpha)h^\phi_{n,m} - \frac{m}{R}h^z_{n,m}|^2}{\sqrt{1+m^2+n^2}} < \infty \end{array} \right\},$$

respectively.

In this section we consider the **regular case**, i.e. the case where $k_n(\alpha) \neq 0$ for all $n\in\mathbb{Z}$, i.e. $|n+\alpha| \neq k$ for all $n\in\mathbb{Z}$. Using the (not only formal) similarity between (3.1), (3.2) and (2.1), (2.2), we have the following quasi-periodic analog of Theorem 2.3 which we state without proof.

Theorem 3.1. *Let $\alpha\in[-1/2,1/2]$ such that $|n+\alpha|\neq k$ for all $n\in\mathbb{Z}$. For every $h\in H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ the function u, given by*

$$u(r,\phi,x_3) \ = \ \sum_{n,m\in\mathbb{Z}} \left\{ u^r_{n,m}(r)\,\hat{r} \ + \ u^\phi_{n,m}(r)\,\hat{\phi} \ + \ u^z_{n,m}(r)\,\hat{z} \right\} e^{im\phi + i(n+\alpha)x_3} \tag{3.5}$$

with coefficients

$$u^r_{n,m}(r) \;=\; h^\phi_{n,m}\,\frac{i(n+\alpha)}{r}\,G^{(1)}_{n,m}(r) \;-\; h^z_{n,m}\,\frac{imR}{r}G^{(2)}_{n,m}(r)\,, \tag{3.6a}$$

$$u^\phi_{n,m}(r) \;=\; h^\phi_{n,m}\,\frac{m(n+\alpha)}{r}\,G^{(5)}_{n,m}(r) \;+\; h^z_{n,m}\,G^{(4)}_{n,m}(r)\,, \tag{3.6b}$$

$$u^z_{n,m}(r) \;=\; -h^\phi_{n,m}\,G^{(3)}_{n,m}(r)\,, \tag{3.6c}$$

is the unique solution $u \in H_{\alpha,*}(\mathrm{curl}, W^+)$ *of* (1.2) *such that Fourier coefficients* $\hat{u}_n$ *from* (3.1) *satisfy the Sommerfeld radiation condition*

$$\frac{\partial \hat{u}_n(\tilde{x})}{\partial r} \;-\; ik_n(\alpha)\,\hat{u}_n(\tilde{x}) \;=\; \mathcal{O}\big(1/|\tilde{x}|^{3/2}\big)\,, \quad r=|\tilde{x}|\to\infty\,, \tag{3.7}$$

for every $n \in \mathbb{Z}$. *The functions* $G^{(j)}_{n,m}(r) := G^{(j)}_m(r, n+\alpha)$ *had been defined in* (2.13a)–(2.13d). *Note that these functions depend also on* α, *i.e.* $G^{(j)}_{n,m}(r) = G^{(j)}_{n,m}(r,\alpha)$. *They are well defined because* $k_n(\alpha) \neq 0$ *for all* $n \in \mathbb{Z}$.
Furthermore, for every $\tilde{R} > R$ *there exists* $c = c(\tilde{R}) > 0$ *with* $\|u\|_{H(\mathrm{curl}, W_{\tilde{R}})} \le c\|h\|_{H^{-1/2}_\alpha(\mathrm{Div},\Gamma)}$ *where again* $W_{\tilde{R}} := \{x \in \mathbb{R}^3 : R^2 < x_1^2 + x_2^2 < \tilde{R}^2,\ 0 < x_3 < 2\pi\}$.

The Calderon operator Λ_α which maps $h \in H^{-1/2}_\alpha(\mathrm{Div},\Gamma)$ into $\hat{r} \times \mathrm{curl}\, u$ on Γ has the form

$$(\Lambda_\alpha h)(\phi, x_3) \;=\; \sum_{n,m\in\mathbb{Z}} \big[\lambda^z_{n,m}\,\hat{z} + \lambda^\phi_{n,m}\,\hat{\phi}\big]\,e^{im\phi + i(n+\alpha)x_3} \tag{3.8}$$

where

$$\begin{aligned} \lambda^z_{n,m} \;&=\; -h^\phi_{n,m}\frac{k^2}{R}\,G^{(1)}_{n,m}(R) \;+\; m\,d_{n,m}\,G^{(2)}_{n,m}(R) \\ &=\; h^\phi_{n,m}\frac{k^2}{Rk_n^2}\left[\frac{Rk_nH'_m(k_nR)}{H_m(k_nR)} - \frac{m^2H_m(k_nR)}{Rk_nH'_m(k_nR)}\right] \\ &\quad + m\,\big[\frac{m}{R}h^\phi_{n,m} + (n+\alpha)h^z_{n,m}\big]\frac{H_m(k_nR)}{Rk_nH'_m(k_nR)} \end{aligned} \tag{3.9a}$$

$$\begin{aligned} \lambda^\phi_{n,m} \;&=\; \big[k^2h^z_{n,m} - (n+\alpha)\,d_{n,m}\big]\,R\,G^{(2)}_{n,m}(R) \\ &=\; \big\{k^2h^z_{n,m} \;-\; (n+\alpha)\,\big[\frac{m}{R}h^\phi_{n,m} + (n+\alpha)h^z_{n,m}\big]\big\}\,\frac{H_m(k_nR)}{k_nH'_m(k_nR)} \\ &=\; \big[k_n^2\,h^z_{n,m} - \frac{(n+\alpha)m}{R}\,h^\phi_{n,m}\big]\,\frac{H_m(k_nR)}{k_nH'_m(k_nR)}\,, \end{aligned} \tag{3.9b}$$

and $d_{n,m} := \frac{m}{R}h^\phi_{n,m} + (n+\alpha)h^z_{n,m}$. We collect a number of properties which in part corresponds to the properties of Λ from Theorem 2.4.

Theorem 3.2. *Let* α *be no cut-off value, i.e.* $k_n = k_n(\alpha)\sqrt{k^2-\xi^2} \neq 0$ *for all* $n \in \mathbb{Z}$. *Then* Λ_α, *defined by* (3.8) *(with* (3.9a), (3.9b)*), has the following properties.*

(a) Λ_α, *is well-defined and bounded from* $H^{-1/2}_\alpha(\mathrm{Div},\Gamma)$ *into itself.*
(b) $\mathrm{Im}\langle \Lambda_\alpha h, \hat{r}\times h\rangle \ge 0$ *for all* $h \in H^{-1/2}_\alpha(\mathrm{Div},\Gamma)$.
(c) $\mathrm{Im}\langle \Lambda_\alpha h, \hat{r}\times h\rangle = 0$ *implies that the corresponding solution* u, *given by* (3.5), *is decaying exponentially, i.e. there exist* $\delta, c > 0$ *with*

$$\max_{\phi,x_3\in(0,2\pi)} |u(r,\phi,x_3)| \;\le\; c\,e^{-\delta r} \quad \text{for } r \ge R+1\,. \tag{3.10}$$

Furthermore, $h^\phi_{n,m}$ *and* $h^z_{n,m}$ *vanish for all* $n, m \in \mathbb{Z}$ *with* $|n+\alpha| < k$.

(d) *The Calderon operator* Λ_α *can be decomposed as* $\Lambda_\alpha = \Lambda_\alpha^D + \Lambda_\alpha^C$ *where* Λ_α^D *and* Λ_α^C *are bounded from* $H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ *into itself and* $\mathrm{Div}\,\Lambda_\alpha^D h = 0$ *and* $\mathrm{Curl}\,\Lambda_\alpha^C h = 0$ *for all* $h \in H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$.

(e) *The operator* $\mathrm{Div}\,\Lambda_\alpha$ *is bounded from* $H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ *into* $H_\alpha^{-1/2}(\Gamma)$ *and*

$$\|\Lambda_\alpha^C h\|_{H_\alpha^{-1/2}(\mathrm{Div},\Gamma)} \;\leq\; c\,\|\,\mathrm{Div}\,\Lambda_\alpha h\|_{H_\alpha^{-1/2}(\Gamma)} \quad \text{for all } h \in H_\alpha^{-1/2}(\mathrm{Div},\Gamma)\,. \tag{3.11}$$

(f) *The operator* Λ_α^D *has a decomposition into* $\Lambda_\alpha^D = \hat\Lambda_\alpha^D + \Lambda_\alpha^K$ *where* $(h,\psi) \mapsto \langle \hat\Lambda_\alpha^D h, \psi \times \hat r\rangle$ *is hermitian on* $H_\alpha^{-1/2}(\mathrm{Div},\Gamma) \times H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ *and non-negative, i.e.* $\langle \hat\Lambda_\alpha^D h, h \times \hat r\rangle \geq 0$ *for all* h*, and* Λ_α^K *is compact.*

(g) Λ_α *depends holomorphically on* α *in the following sense. Let* $\hat\alpha \in [-1/2, 1/2]$ *not be a cut-off value. Then there exists* $\delta > 0$ *such that the mapping* $\alpha \mapsto \hat\Lambda_\alpha$ *is (strongly) holomorphic*[2] *from* $\{\alpha \in \mathbb{C} : |\alpha - \hat\alpha| < \delta\}$ *into* $\mathcal{B}\big(H_{per}^{-1/2}(\mathrm{Div},\Gamma)\big)$ *where* $H_{per}^{-1/2}(\mathrm{Div},\Gamma)$ *denotes the space* $H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ *for* $\alpha = 0$ *and* $\hat\Lambda_\alpha v = e^{-i\alpha x_3}\Lambda_\alpha\big(v\, e^{i\alpha x_3}\big)$ *for* $v \in H_{per}^{-1/2}(\mathrm{Div},\Gamma)$.

Proof. We omit the proof of parts (a) and (b) because they follows very much the proofs of the corresponding parts of Theorem 2.4. We have to set $\xi = n+\alpha$ and note that $k_n(\alpha) \neq 0$ for all $n \in \mathbb{Z}$ implies the existence of $\delta > 0$ such that $|k_n(\alpha)| \geq \delta$ for all $n \in \mathbb{Z}$.

(c) The analog of (2.22) for the quasi-periodic case yields the following form of $\langle \Lambda_\alpha h, \hat r \times h\rangle$.

$$\begin{aligned}\langle \Lambda_\alpha h, \hat r \times h\rangle \;=\; 4\pi^2 R \sum_{n,m\in\mathbb{Z}} \Bigg\{ & \frac{k^2\,|h^\phi_{n,m}|^2}{R k_n^2 G^{(2)}_{n,m}(R)} \\ & - \frac{G^{(2)}_{n,m}(R)}{R k_n^2} \big| m(n+\alpha)\, h^\phi_{n,m} \;-\; R k_n^2\, h^z_{n,m}\big|^2 \Bigg\}.\end{aligned}$$

Furthermore, the imaginary parts of each of the two terms in $\{\cdots\}$ are non-negative for all n,m and positive for $|n+\alpha| < k$. Therefore, $\mathrm{Im}\langle \Lambda_\alpha h, \hat r \times h\rangle = 0$ implies that $h^\phi_{n,m} = h^z_{n,m} = 0$ for all $m,n \in \mathbb{Z}$ with $|n+\alpha| < k$. Therefore, the Fourier series for u is written as $u(\tilde x, x_3) = \sum_{n:|n+\alpha|>k} \hat u_n(\tilde x) e^{i(n+\alpha)x_3}$ with functions $\hat u_n : \{\tilde x \in \mathbb{R}^2 : |\tilde x| > R\} \to \mathbb{C}^3$ which satisfy the Helmholtz equation (3.2) for every component. Therefore, $\hat u_n(\tilde x)$ can be expanded into a series of Hankel functions and thus, for any fixed $R_1 > R$,

$$u(r,\phi,x_3) \;=\; \sum_{n:|n+\alpha|>k}\sum_{m\in\mathbb{Z}} \hat u_{n,m}\,\frac{H_m(k_n r)}{H_m(k_n R_1)}\, e^{im\phi + i(n+\alpha)x_3} \quad \text{for } r \geq R_1\,.$$

Note that $\hat u_{n,m} \in \mathbb{C}^3$. By interior regularity arguments the field u is smooth and thus $\sum_{n:|n+\alpha|>k}\sum_{m\in\mathbb{Z}}(|n|+|m|)\,|\hat u_{n,m}| < \infty$. Now we use the estimate

$$\left|\frac{H_m(k_n r)}{H_m(k_n R_1)}\right| \;\leq\; e^{-\,\mathrm{Im}\,k_n (r-R_1)} \quad \text{for } m \in \mathbb{Z} \text{ and } r \geq R_1\,,$$

(see part (a) of Lemma 5.2 in [4]) which shows that

$$|u(r,\phi,x_3)| \;\leq\; e^{-\delta(r-R_1)} \sum_{n:|n+\alpha|>k}\sum_{m\in\mathbb{Z}} |\hat u_{n,m}| \quad \text{for } r \geq R_1$$

[2] in the sense of, e.g., [3], Section 8.5,

where $\delta = \min\{\sqrt{(n+\alpha)^2 - k^2} : |n+\alpha| > k\} > 0$. Estimates of the derivatives with respect to ϕ and x_3 follow the same lines, for the derivative with respect to r we use the estimate

$$\left|\frac{k_n H'_m(k_n r)}{H_m(k_n R_1)}\right| \;\leq\; c\,(|k_n| + |m|)\, e^{-\operatorname{Im} k_n (r-R_1)} \quad \text{for } m \in \mathbb{Z} \text{ and } r \geq R_1$$

(see part (b) of Lemma 5.2 in [4]).

Parts (d) and (e) correspond to parts (c) and (d), respectively, of Theorem 2.4 and are proven in the same way. We just state the form of Λ_α^D as

$$(\Lambda_\alpha^D h)(\phi, x_3) \;:=\; \sum_{n,m\in\mathbb{Z}} \frac{m\lambda^z_{n,m}/R - (n+\alpha)\lambda^\phi_{n,m}}{(n+\alpha)^2 + (m/R)^2} \big[\frac{m}{R}\,\hat{z} - (n+\alpha)\,\hat{\phi}\big]\, e^{im\phi + i(n+\alpha)x_3}\,,$$

where $\lambda^z_{n,m}$ and $\lambda^\phi_{n,m}$ are given in (3.9a) and (3.9b), respectively.

(f) We define $\hat{\Lambda}_\alpha^D$ by

$$(\hat{\Lambda}_\alpha^D h)(\phi, x_3) \;:=\; \sum_{m,n\in\mathbb{Z}} \frac{\hat{\lambda}_{n,m}}{(n+\alpha)^2 + (m/R)^2} \big[\frac{m}{R}\,\hat{z} - (n+\alpha)\,\hat{\phi}\big]\, e^{im\phi + i(n+\alpha)x_3}$$

where

$$\hat{\lambda}_{n,m} \;=\; \left[|k_n|^2 + \frac{m^2}{R^2}\right] \frac{K_m(R|k_n|)}{|k_n| K'_m(R|k_n|)} \left[\frac{m}{R} h^\phi_{n,m} + (n+\alpha) h^z_{n,m}\right] \text{ if } |n+\alpha| > k\,,$$

$$\hat{\lambda}_{n,m} \;=\; -\frac{|m|}{R}\left[\frac{m}{R} h^\phi_{n,m} + (n+\alpha) h^z_{n,m}\right] \text{ if } |n+\alpha| < k\,.$$

First we note that

$$\langle \hat{\lambda}_\alpha^D h, \psi \times \hat{r}\rangle \;=\; -\sum_{m,n\in\mathbb{Z}} \frac{\hat{\lambda}_{n,m}}{(n+\alpha)^2 + (m/R)^2} \big[\frac{m}{R}\,\overline{\psi^\phi_{n,m}} + (n+\alpha)\,\overline{\psi^z_{n,m}}\big]$$

which shows that the form is hermetian. The positivity follows because $\frac{K_m(R|k_n|)}{|k_n| K'_m(R|k_n|)} \leq 0$. It remains to show that $\Lambda_\alpha^D - \hat{\Lambda}_\alpha^D$ is compact. We look at the numerator of the form of Λ_α^D. Using the definitions of $\lambda^z_{n,m}$ and $\lambda^\phi_{n,m}$ from (3.9a) and (3.9b), respectively, and $R(n+\alpha)h^z_{n,m} = R d_{n,m} - m h^\phi_{n,m}$ we write

$$\begin{aligned}
&\frac{m}{R}\lambda^z_{n,m} - (n+\alpha)\lambda^\phi_{n,m} \\
=\;& -\frac{k^2 m}{R^2}\, G^{(1)}_{n,m}(R)\, h^\phi_{n,m} \;+\; \frac{m^2}{R} G^{(2)}_{n,m}(R)\, d_{n,m} \\
&+ R(n+\alpha)\big[(n+\alpha)d_{n,m} - k^2 h^z_{n,m}\big] G^{(2)}_{n,m}(R) \\
=\;& \frac{mk^2}{R^2}\, h^\phi_{n,m}\big[R^2 G^{(2)}_{n,m}(R) - G^{(1)}_{n,m}(R)\big] \;+\; R\, d_{n,m}\,\big[\frac{m^2}{R^2} + (n+\alpha)^2 - k^2\big]\, G^{(2)}_{n,m}(R)\,.
\end{aligned}$$

Using (A.3d) we estimate the first term on the right hand side by $c/\sqrt{1+m^2+n^2}|h^\phi_{n,m}|$. Therefore, this part in the representation of Λ_α^D results in a compact operator. For the second term in the previous equation we consider first the case $|n+\alpha| > k$. Then $R G^{(2)}_{n,m}(R) = \frac{K_m(|k_n|R)}{|k_n| K'_m(|k_n|R)}$, and the term coincides with the definition of $\hat{\lambda}_{n,m}$ (note that $k^2 - (n+\alpha)^2 = k_n^2 = -|k_n|^2$).

Finally, we consider the case $|n+\alpha| < k$. Then we use (A.12b) and (A.12c) and have

$$G^{(2)}_{n,m}(R) \;=\; \frac{H_m(k_n R)}{R k_n H'_m(k_n R)} \;=\; -\frac{1}{|m|}\,\big[1 + \mathcal{O}(1/|m|)\big]$$

and thus

$$R\, d_{n,m} \Big[\frac{m^2}{R^2} + (n+\alpha)^2 - k^2\Big]\, G^{(2)}_{n,m}(R) \;=\; -\frac{|m|}{R}\, d_{n,m}\big[1 + \mathcal{O}(1/|m|)\big]$$

which again coincides with the definition of $\hat{\lambda}_{n,m}$ up to $\mathcal{O}(1/|m|)$.

(g) For given $h \in H^{-1/2}_{per}(\mathrm{Div}, \Gamma)$ and $f \in H^{-1/2}_{per}(\mathrm{Curl}, \Gamma)$ we observe that $\langle \hat{\Lambda}_\alpha h, f\rangle$ is given by the series

$$\langle \hat{\Lambda}_\alpha h, f\rangle \;=\; 4\pi^2 R \sum_{n,m\in\mathbb{Z}} \big[\lambda^z_{n,m}(\alpha)\overline{f^z_{n,m}} + \lambda^\phi_{n,m}(\alpha)\overline{f^\phi_{n,m}}\big]$$

with $\lambda^z_{n,m}(\alpha)$ and $\lambda^\phi_{n,m}(\alpha)$ from (3.9a) and (3.9b), respectively, (indicating the dependence on α) where $h^\phi_{n,m}, h^\phi_{n,m}$ and $f^\phi_{n,m}, f^\phi_{n,m}$ are the coefficients of h and f, respectively, with respect to $\{e^{im\phi+inx_3} : n, m \in \mathbb{Z}\}$ (which are independent of α). The coefficients $\lambda^z_{n,m}(\alpha)$ and $\lambda^\phi_{n,m}(\alpha)$ of $\hat{\Lambda}_\alpha$ depend on α through $G^{(1)}_{n,m}(\alpha)$ and $G^{(2)}_{n,m}(\alpha)$ only. From part (a) of Lemma A.1 we conclude that $G^{(1)}_{n,m}(\alpha)$ and $G^{(2)}_{n,m}(\alpha)$ depend holomorphically on α in a neighborhood of $\hat{\alpha}$. Using the estimate (2.5) (for $h^\phi_m(\xi)$, $h^z_m(\xi)$, $f^\phi_m(\xi)$, and $f^z_m(\xi)$ replaced by $\lambda^\phi_{n,m}(\alpha)$, $\lambda^z_{n,m}(\alpha)$, $f^\phi_{n,m}$, and $f^z_{n,m}$, respectively) and the forms (3.9a) and (3.9b) of $\lambda^z_{n,m}(\alpha)$ and $\lambda^\phi_{n,m}(\alpha)$ and the estimate (A.3a) we conclude that the series is uniformly (with respect to α) convergent. Therefore, $\langle \hat{\Lambda}_\alpha h, f\rangle$ is weakly holomorphic with respect to α which implies (Theorem 8.22 of [3]) that $\alpha \mapsto \hat{\Lambda}_\alpha$ is holomorphic. □

Closely related to Λ_α is the scalar operator D_α.

Theorem 3.3. *Let α be no cut-off value. Define the operator D_α from $H^{1/2}_\alpha(\Gamma)$ into $H^{-1/2}_\alpha(\Gamma)$ by $D_\alpha p := \mathrm{Div}\,\Lambda_\alpha(\hat{r} \times \mathrm{Grad}\, p)$. Then $\mathrm{Im}\langle D_\alpha p, p\rangle \geq 0$ for all $p \in H^{1/2}_\alpha(\Gamma)$, and $\mathrm{Im}\langle D_\alpha p, p\rangle = 0$ implies that $\mathrm{Re}\langle D_\alpha p, p\rangle \geq 0$.*
Furthermore, there is an operator $\tilde{D}_\alpha$ from $H^{1/2}_\alpha(\Gamma)$ into $H^{-1/2}_\alpha(\Gamma)$ which is hermetian and non-negative, i.e. $\langle \tilde{D}_\alpha p, p\rangle \geq 0$ for all $p \in H^{1/2}_\alpha(\Gamma)$, and $D_\alpha - \tilde{D}_\alpha$ is compact.

Proof. Let $p \in H^{1/2}_\alpha(\Gamma)$ have the expansion $p(\phi, x_3) = \sum_{m,n\in\mathbb{Z}} p_{n,m}\, e^{im\phi+i(n+\alpha)x_3}$. From the definitions of Λ_α and D_α and (3.9a), (3.9b) we obtain (note that $h := \hat{r} \times \mathrm{Grad}\, p = i\sum_{m,m\in\mathbb{Z}}\big[m/R\,\hat{z} - (n+\alpha)\hat{\phi}\,\big]\, p_{n,m}\, e^{im\phi+i(n+\alpha)x_3}$),

$$(D_\alpha p)(\phi, x_3) \;=\; i \sum_{n,m\in\mathbb{Z}} \left[(n+\alpha)\lambda^z_{n,m} + \frac{m}{R}\lambda^\phi_{n,m}\right] e^{im\phi+i(n+\alpha)x_3}$$

where

$$\lambda^z_{n,m} \;=\; i(n+\alpha)\,\frac{k^2}{R}\, G^{(1)}_{n,m}(R)\, p_{n,m} \quad \text{and} \quad \lambda^\phi_{n,m} \;=\; ik^2 m\, G^{(2)}_{n,m}(R)\, p_{n,m}\,,$$

and thus

$$(D_\alpha p)(\phi, x_3) \;=\; -\frac{k^2}{R} \sum_{n,m\in\mathbb{Z}} \big[(n+\alpha)^2\, G^{(1)}_{n,m}(R) + m^2 G^{(2)}_{n,m}(R)\big]\, p_{n,m}\, e^{im\phi+i(n+\alpha)x_3}.$$

Set $h := \hat{r} \times \mathrm{Grad}\, p$. Then

$$\langle D_\alpha p, p\rangle \;=\; \langle \mathrm{Div}\,\Lambda_\alpha h, p\rangle \;=\; -\langle \Lambda_\alpha h, \mathrm{Grad}\, p\rangle \;=\; \langle \Lambda_\alpha h, \hat{r} \times h\rangle$$

and thus $\text{Im}\langle D_\alpha p, p\rangle \geq 0$ by part (b) of Theorem 3.2. Furthermore, $\text{Im}\langle D_\alpha p, p\rangle = 0$ implies $p_{n,m} = 0$ for all n with $|n+\alpha| < k$. Therefore, for p with $\text{Im}\langle D_\alpha p, p\rangle = 0$ the term $\langle D_\alpha p, p\rangle$ coincides with $\langle \tilde{D}_\alpha p, p\rangle$ where $\tilde{D}_\alpha$ is defined as

$$\begin{aligned}(\tilde{D}_\alpha p)(\phi, x_3) \;&:=\; -\frac{k^2}{R} \sum_{|n+\alpha|>k} \sum_{m\in\mathbb{Z}} \big[(n+\alpha)^2\, G^{(1)}_{n,m}(R) + m^2 G^{(2)}_{n,m}(R)\big]\, p_{n,m}\, e^{im\phi + i(n+\alpha)x_3} \\ &\quad + \frac{k^2}{R} \sum_{|n+\alpha|<k} \sum_{m\in\mathbb{Z}} |m|\, p_{n,m}\, e^{im\phi + i(n+\alpha)x_3}.\end{aligned}$$

For $|n+\alpha| > k$ we have $k_n^2 = -|k_n|^2$ and $Rk_n H'_m(Rk_n)/H_m(Rk_n) = tK'_m(t)/K_m(t)$ where we have set $t := |k_n|R$ for abbreviation. Therefore,

$$\begin{aligned} G^{(2)}_{n,m}(R) \;&=\; \frac{K_m(t)}{tK'_m(t)} \;<\; 0 \quad \text{and} \\ k_n^2 G^{(1)}_{n,m}(R) \;&=\; \frac{m^2 K_m(t)}{tK'_m(t)} - \frac{tK'_m(t)}{K_m(t)} \;=\; \frac{m^2 K_m(t)^2 - t^2 K'_m(t)^2}{tK'_m(t) K_m(t)} \\ &=\; \frac{mK_m(t) - tK'_m(t)}{tK'_m(t)K_m(t)} \big[mK_m(t) + tK'_m(t)\big] \\ &=\; -\frac{mK_m(t) - tK'_m(t)}{tK'_m(t)K_m(t)}\, t\, K_{m-1}(t) \;>\; 0 \end{aligned}$$

because $K_m(t)$ is real and positive and $K'_m(t)$ negative. Here we used the recursion formula $tK'_m(t) = -tK_{m-1}(t) - mK_m(t)$, see Formula 9.6.26 of [1]. Since $k_n^2 < 0$ we conclude that also $G^{(1)}_{n,m}(R) < 0$. Therefore, we have shown that $(n+\alpha)^2\, G^{(1)}_{n,m} + m^2 G^{(2)}_{n,m}$ is real and negative for $|n+\alpha| > k$, i.e. $\tilde{D}_\alpha$ is hermitean and positive and $\langle D_\alpha p, p\rangle = \langle \tilde{D}_\alpha p, p\rangle \geq 0$ if $\text{Im}\langle D_\alpha p, p\rangle = 0$.

Finally, we show that $D_\alpha - \tilde{D}_\alpha$ is compact from $H^{1/2}_\alpha(\Gamma)$ into $H^{-1/2}_\alpha(\Gamma)$. The difference has the form

$$\begin{aligned} &(D_\alpha - \tilde{D}_\alpha)p(\phi, x_3) \\ =\; &\frac{k^2}{R} \sum_{|n+\alpha|<k} \sum_{m\in\mathbb{Z}} \big[|m| - (n+\alpha)^2\, G^{(1)}_{n,m}(R) - m^2 G^{(2)}_{n,m}(R)\big]\, p_{n,m}\, e^{im\phi + i(n+\alpha)x_3}\,. \end{aligned}$$

Let $n \in \mathbb{Z}$ with $|n+\alpha| < k$. The difference $(D_\alpha - \tilde{D}_\alpha)p$ contains only finitely many n with this property. Then $k_n^2 > 0$, and we determine the asymptotic form of $(n+\alpha)^2\, G^{(1)}_{n,m}(R) + m^2 G^{(2)}_{n,m}(R)$, for $m \to \infty$ (for fixed n). From (A.3a) we conclude that $G^{(1)}_{n,m}(R)$ behaves as $\mathcal{O}(1/m)$ as $m \to \infty$ and (see (A.12b) and (A.12c)) $G^{(2)}_{n,m}(R) = -\frac{1}{m}\big[1 + \mathcal{O}(1/m)\big]$ as $m \to \infty$ and thus

$$(n+\alpha)^2\, G^{(1)}_{n,m}(R) + m^2 G^{(2)}_{n,m}(R) \;=\; -m\,[1 + \mathcal{O}(1/m)] \quad \text{as } m \to \infty\,.$$

This shows compactness of $D_\alpha - \tilde{D}_\alpha$. □

4. The Case of a Cut-off Value

In this section we consider the case of a cut-off value. Therefore, let $\alpha \in (-1/2, 1/2]$ and $\hat{n} \in \mathbb{Z}$ with $|\alpha + \hat{n}| = k$. Then $k_{\hat{n}}(\alpha) = 0$, and (3.2) takes the form (for $n = \hat{n}$)

(4.1) $$\Delta_2 \hat{u}_{\hat{n}}(x_1, x_2) = 0 \quad \text{for } x_1^2 + x_2^2 > R^2\,.$$

In addition we require that $\text{div}\big(\hat{u}_{\hat{n}}(x_1, x_2) e^{i(\hat{n}+\alpha)x_3}\big) = 0$ which gives an expression of $u^z := \hat{u}_{\hat{n}} \cdot \hat{z}$ in terms of (the derivatives of) the r - component $u^r := \hat{u}_{\hat{n}} \cdot \hat{r}$ and ϕ -

component $u^\phi := \hat{u}_{\hat{n}} \cdot \hat{\phi}$ of $\hat{u}_{\hat{n}}$ where we use again polar coordinates (r, ϕ). In polar coordinates the system (4.1) for (u^r, u^ϕ) takes the form

$$\Delta_2 u^r - \frac{1}{r^2} u^r - \frac{2}{r^2} \partial_\phi u^\phi = 0 \,,$$
$$\Delta_2 u^\phi - \frac{1}{r^2} u^\phi + \frac{2}{r^2} \partial_\phi u^r = 0 \,.$$

Expanding u^r and u^ϕ into Fourier series with respect to ϕ yields

$$u^r(r,\phi) \;=\; \sum_{m\in\mathbb{Z}} u^r_m(r)\, e^{im\phi}\,, \qquad u^\phi(r,\phi) \;=\; \sum_{m\in\mathbb{Z}} u^r_m\phi(r)\, e^{im\phi}$$

where u^r_m and u^ϕ_m solve the system

$$r\big[r(u^r_m)'(r)\big]' - (1+m^2)\, u^r_m(r) - 2im\, u^\phi_m = 0 \,,$$
$$r\big[r(u^\phi_m)'(r))\big]' - (1+m^2)\, u^\phi_m(r) + 2im\, u^r_m = 0 \,,$$

of ordinary differential equations. This system is easiliy solved by searching for solutions with $u^\phi_m = \pm i u^r_m$. The general solution which decays as $r \to \infty$ is given by

$$u^r_m(r) \;=\; a_m \left(\frac{R}{r}\right)^{|m|-1} + b_m \left(\frac{R}{r}\right)^{|m|+1}, \quad |m| \geq 1\,,$$
$$u^\phi_m(r) \;=\; i\,(\operatorname{sign} m) \left[a_m \left(\frac{R}{r}\right)^{|m|-1} - b_m \left(\frac{R}{r}\right)^{|m|+1}\right], \quad |m| \geq 1\,,$$
$$u^r_0(r) \;=\; a_0\, \frac{R}{r}\,,$$
$$u^\phi_0(r) \;=\; b_0\, \frac{R}{r}\,,$$

where $a_m, b_m \in \mathbb{C}$ are arbitrary with $a_{\pm 1} = 0$. The x_3-component $u^z := \hat{u}_{\hat{n}} \cdot \hat{z}$ is given by $u^z = \frac{i}{\hat{n}+\alpha} \operatorname{div}(u^r \hat{r} + u^\phi \hat{\phi}) = \frac{i}{(\hat{n}+\alpha)\, r} \big[\partial_r(r\partial_r u^r) + \partial_\phi u^\phi\big]$, i.e.

$$u^z_m(r) \;=\; a_m\, \frac{2i(1-|m|)}{R(\hat{n}+\alpha)} \left(\frac{R}{r}\right)^{|m|}, \quad |m| \geq 2\,.$$

The boundary value problem $\Delta_2 \hat{u}_{\hat{n}}(x_1,x_2) = 0$ for $x_1^2 + x_2^2 > R^2$ and $\hat{r} \times \hat{u}_{\hat{n}} = h$ for $x_1^2 + x_2^2 = R^2$ leads to $u^\phi_m(R) = h^z_m$ and $u^z_m(R) = -h^\phi_m$. Therefore, if $h^\phi_m = 0$ for $|m| \leq 1$ the coefficients a_m and b_m are given by

$$a_m \;=\; i\frac{R(\hat{n}+\alpha)}{2(1-|m|)} h^\phi_m\,, \quad b_m \;=\; i(\operatorname{sign} m) h^z_m + i\frac{R(\hat{n}+\alpha)}{2(1-|m|)} h^\phi_m \quad \text{for } |m| \geq 2\,,$$

and $b_{\pm 1} = \pm i h^z_{\pm 1}$ and $b_0 = h^z_0$. The solution is not unique because $v(r,\phi) = \frac{1}{r}\hat{r}$ solves $\Delta_2 v = 0$ and $\operatorname{div} v = 0$ and $\hat{r} \times v = 0$ for $r = R$. Therefore, we have shown part (a) of the following theorem.

Theorem 4.1. *Let $\alpha \in (-1/2, 1/2]$ be a cut-off value, i.e. $\hat{N} := \{\hat{n} \in \mathbb{Z} : |\alpha + \hat{n}| = k\} \neq \emptyset$.*

(a) For every $h \in H^{-1/2}_\alpha(\operatorname{Div}, \Gamma)$ with $h^\phi_{\hat{n},m} = 0$ for $\hat{n} \in \hat{N}$ and $|m| \leq 1$ there exists a solution $u \in H_{\alpha,*}(\operatorname{curl}, W^+)$ of (1.2) and $u(x) = \mathcal{O}(1/r)$ as $r = \sqrt{x_1^2 + x_2^2} \to \infty$ The solutions are given by (3.5) with coefficients $u^r_{n,m}(r)$, $u^\phi_{n,m}(r)$, $u^z_{n,m}(r)$ from

(3.6a), (3.6b), (3.6c), *respectively, for* $n \notin \hat{N}$ *and*

$$
\begin{aligned}
(4.2a)\quad u^r_{\hat{n},m}(r) &= i\,\frac{R(\hat{n}+\alpha)}{2(1-|m|)}h^\phi_{\hat{n},m}\left(\frac{R}{r}\right)^{|m|-1} \\
&\quad + i\left[s_m h^z_{\hat{n},m} + \frac{R(\hat{n}+\alpha)}{2(1-|m|)}h^\phi_{\hat{n},m}\right]\left(\frac{R}{r}\right)^{|m|+1}, \quad |m|\geq 2\,, \\
u^r_{\hat{n},\pm 1}(r) &= \pm i h^z_{\hat{n},\pm 1}\left(\frac{R}{r}\right)^2, \qquad u^r_{\hat{n},0}(r) \;=\; a_{\hat{n}}\frac{R}{r}\,, \\
(4.2b)\quad u^\phi_{\hat{n},m}(r) &= -s_m\,\frac{R(\hat{n}+\alpha)}{2(1-|m|)}h^\phi_{\hat{n},m}\left(\frac{R}{r}\right)^{|m|-1} \\
&\quad + \left[h^z_{\hat{n},m} + s_m\,\frac{R(\hat{n}+\alpha)}{2(1-|m|)}h^\phi_{\hat{n},m}\right]\left(\frac{R}{r}\right)^{|m|+1}, \quad |m|\geq 2\,, \\
u^\phi_{\hat{n},m}(r) &= h^z_{\hat{n},m}\left(\frac{R}{r}\right)^{|m|+1}, \quad |m|\leq 1\,, \\
(4.2c)\quad u^z_{\hat{n},m}(r) &= -h^\phi_{\hat{n},m}\left(\frac{R}{r}\right)^{|m|}, \quad |m|\geq 2\,, \qquad u^z_{\hat{n},m}(r) \;=\; 0\,, \quad |m|\leq 1\,,
\end{aligned}
$$

for $\hat{n}\in\hat{N}$ *where* $s_m = \operatorname{sign} m$ *and* $a_{\hat{n}}\in\mathbb{C}$ *is arbitrary. The solution is unique if one poses the extra condition* $\int_\gamma u(x)\cdot\hat{r}\,e^{-i(\hat{n}+\alpha)x_3}ds = 0$ *for* $\hat{n}\in\hat{N}$ *where* $\gamma := \{x\in\Gamma : 0 < x_3 < 2\pi\}$.

(b) The corresponding Calderon operator Λ_α *from* $\big\{h \in H^{-1/2}_\alpha(\mathrm{Div},\Gamma) : h^\phi_{\hat{n},m} = 0$ *for* $\hat{n}\in\hat{N}$ *and* $|m|\leq 1\big\}$ *into itself is given by* (3.8) *where* $\lambda^z_{n,m}$ *and* $\lambda^\phi_{n,m}$ *are defined by* (3.9a) *and* (3.9b), *respectively, for* $n\notin\hat{N}$ *and*

$$
\begin{aligned}
(4.3a)\qquad \lambda^z_{\hat{n},m} &= h^\phi_{\hat{n},m}\left(\frac{Rk^2}{|m|-1} - \frac{|m|}{R}\right) - s_m(\hat{n}+\alpha)\,h^z_{\hat{n},m}\,, \quad |m|\geq 2\,, \\
\lambda^z_{\hat{n},m} &= -s_m(\hat{n}+\alpha)\,h^z_{\hat{n},m}\,,\ |m| = 1\,, \quad \lambda^z_{\hat{n},0} \;=\; 0\,, \\
(4.3b)\qquad \lambda^\phi_{\hat{n},m} &= s_m(\hat{n}+\alpha)\,h^\phi_{\hat{n},m}\,,\ |m|\geq 2\,, \quad \lambda^\phi_{\hat{n},m} \;=\; 0\,,\ |m|\leq 1\,.
\end{aligned}
$$

for $\hat{n}\in\hat{N}$ *where again* $s_m = \operatorname{sign} m$.

Proof. For part (b) we compute $\operatorname{curl}\big([u^r_{\hat{n},m}(r)\hat{r} + u^\phi_{\hat{n},m}(r)\hat{\phi} + u^z_{\hat{n},m}(r)\hat{z}]e^{im\phi+i(\hat{n}+\alpha)x_3}\big)$ in cylindrical coordinates and evaluate its tangential components to obtain (4.3a) and (4.3b). We omit the details. □

Comparing (4.2a), (4.2b), (4.2c) with (3.6a), (3.6b), (3.6c), respectively, for $n = \hat{n}$ and $|m|\geq 2$ and the definitions of $G^{(j)}_{\hat{n},m}(r,\alpha) := G^{(j)}_m(r,\hat{n}+\alpha)$ of (A.4a), (A.4d), (A.4f), (A.4g), and (A.4h), respectively, we obtain the continuous dependence of the solution $u = u_\alpha$ on α at cut-off values.

Corollary 4.2. *Let* $\hat{\alpha}\in(-1/2,1/2]$ *be a cut-off value with corresponding set* $\hat{N} := \{\hat{n}\in\mathbb{Z} : |\alpha+\hat{n}| = k\}\neq\emptyset$. *For* $\alpha\in[-1/2,1/2]$ *let* $h(\alpha)\in H^{-1/2}_\alpha(\mathrm{Div},\Gamma)$ *such that* $\lim_{\alpha\to\hat{\alpha}} h^z_{n,m}(\alpha) = h^z_{n,m}(\hat{\alpha})$ *for all* $(n,m)\in\mathbb{Z}^2$ *and* $\lim_{\alpha\to\hat{\alpha}} h^\phi_{n,m}(\alpha) = h^\phi_{n,m}(\hat{\alpha})$ *for all* $(n,m)\in\mathbb{Z}^2\setminus\{(\hat{n},m) : \hat{n}\in\hat{N},\ |m|\leq 1)\}$, *and* $\lim_{\alpha\to\hat{\alpha}}\big[\frac{1}{k_{\hat{n}}(\alpha)^2\ln k_{\hat{n}}(\alpha)}h^\phi_{\hat{n},0}(\alpha)\big] = 0$ *and* $\lim_{\alpha\to\hat{\alpha}}\big[\ln k_{\hat{n}}(\alpha)h^\phi_{\hat{n},\pm 1}(\alpha)\big] = 0$ *for all* $\hat{n}\in\hat{N}$ *where again* $k_{\hat{n}}(\alpha) = \sqrt{k^2-(\hat{n}+\alpha)^2}$. *Then, for every* $\tilde{R} > R$ *the unique solution* $u_\alpha\in H_{\alpha,*}(\mathrm{curl},W^+)$ *of* (1.2), (3.7) *for* $\alpha\neq\hat{\alpha}$ *(assured by Theorem 3.1) converges in* $H(\mathrm{curl},W_{\tilde{R}})$ *to the unique solution*

$u_{\hat\alpha} \in H_{\hat\alpha,*}(\mathrm{curl}, W^+)$ *of* (1.2) *with* $u_{\hat\alpha}(x) = \mathcal{O}(1/r)$ *as* $r = \sqrt{x_1^2+x_2^2} \to \infty$ *and* $\int_\gamma u_{\hat\alpha}(x)\cdot \hat r\, e^{-i(\hat n+\hat\alpha)x_3} ds = 0$ *for* $\hat n \in \hat N$ *(assured by the previous theorem).*
Furthermore, if we define the operator T_α *from* $H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ *into* $H_0^{-1/2}(\mathrm{Div},\Gamma)$ *by* $(Th)(x) = e^{-i\alpha x_3}h(x)$ *(here* $H_0^{-1/2}(\mathrm{Div},\Gamma)$ *denotes the space* $H_\alpha^{-1/2}(\mathrm{Div},\Gamma)$ *for* $\alpha = 0$*), then* $T_\alpha\Lambda_\alpha h(\alpha)$ *converges to* $T_{\hat\alpha}\Lambda_{\hat\alpha}h(\hat\alpha)$ *in* $H_0^{-1/2}(\mathrm{Div},\Gamma)$ *as* $\alpha\to\hat\alpha$.

Proof. First we note again that (4.2a), (4.2b), and (4.2c) coincide with form of the coefficients $u^r_{n,m}$, $u^\phi_{n,m}$, and $u^z_{n,m}$ from (3.6a), (3.6b), and (3.6c), respectively, for $|m|\ge 2$ and $n=\hat n\in\hat N$, if one uses the definitions $G^{(j)}_{\hat n,m}(r,\hat\alpha) := G^{(j)}_m(r,\hat n+\hat\alpha) = G^{(j)}_m(r,\pm k)$ for $|m|\ge 2$ and $j=1,\dots,5$ from (A.4a), (A.4d), (A.4f), (A.4g), and (A.4h), respectively.

Second, for large values of n^2+m^2, i.e. for $n^2+m^2\ge k^2+4$, the series for u from (3.5) converges uniformly with respect to $(r,\alpha)\in[R,\tilde R]\times[-1/2,1/2]$ as seen from the proof of Theorem 2.3. Furthermore, part (c) of Lemma A.1, $u^r_{n,m}(r,\alpha)$, $u^\phi_{n,m}(r,\alpha)$, and $u^z_{n,m}(r,\alpha)$ converge to $u^r_{n,m}(r,\hat\alpha)$, $u^\phi_{n,m}(r,\hat\alpha)$, and $u^z_{n,m}(r,\hat\alpha)$, respectively, as $\alpha\to\hat\alpha$ for every $n,m\in\mathbb{Z}$, uniformly with respect to $r\in[R,\tilde R]$. We recall the formulas only for $n=\hat n$ and $|m|\le 1$.

$$\begin{aligned}
u^r_{\hat n,0}(r,\alpha) &= h^\phi_{\hat n,0}(\alpha)\,\frac{i(\hat n+\alpha)}{r}\,G^{(1)}_0(r,\hat n+\alpha)\,,\\
u^r_{\hat n,\pm1}(r,\alpha) &= h^\phi_{\hat n,\pm1}(\alpha)\,\frac{i(\hat n+\alpha)}{r}\,G^{(1)}_{\pm1}(r,\hat n+\alpha) \;\mp\; h^z_{\hat n,\pm1}(\alpha)\,\frac{iR}{r}G^{(2)}_{\pm1}(r,\hat n+\alpha)\,,\\
u^\phi_{\hat n,0}(r) &= h^z_{\hat n,0}(\alpha)\,G^{(4)}_0(r,\hat n+\alpha)\,,\\
u^\phi_{\hat n,\pm1}(r) &= \pm h^\phi_{\hat n,\pm1}(\alpha)\,\frac{\hat n+\alpha}{r}\,G^{(5)}_{\pm1}(r,\hat n+\alpha) \;+\; h^z_{\hat n,\pm1}(\alpha)\,G^{(4)}_{\pm1}(r,\hat n+\alpha)\,,\\
u^z_{\hat n,m}(r) &= -h^\phi_{\hat n,m}(\alpha)\,G^{(3)}_m(r,\hat n+\alpha)\,,\quad |m|\le 1\,,
\end{aligned}$$

and these converge to the corresponding coefficients for $\alpha=\hat\alpha$ by the assumptions on $h^\phi_{\hat n,m}(\alpha)$ for $|m|\le1$ and the singularity of $G^{(j)}_m$ (part (c) of Lemma A.1). The same holds for $\mathrm{curl}\,u_\alpha$.
From these two properties of the parameter-dependent coefficients of the series elementary arguments show convergence of u_α to $u_{\hat\alpha}$ in $H(\mathrm{curl}, W_{\tilde R})$.
We omit the proof for the Calderon operators because it follows the same arguments. □

Appendix A. Properties of Hankel functions

Let $k>0$ and $\delta\in(0,k/2)$ be fixed and $k(\xi)=\sqrt{k^2-\xi^2}$ for $\xi\in\mathbb{C}$ with $\xi^2\notin k^2+i\mathbb{R}$. Here we take the square function $z\mapsto\sqrt z$ to be holomorhic in $\mathbb{C}\backslash(i\mathbb{R}_{\le0})$, i.e. $\arg z\in(-\pi/2,3\pi/2)$. Then $k(\xi)\notin\mathbb{R}_{\le0}$. We note that for this choice of the branch we have $\sqrt{\overline w}=\overline{\sqrt w}$ provided $\mathrm{Re}\,w>0$. Furthermore, $|\,\mathrm{Im}\sqrt w|\le\mathrm{Re}\sqrt w$ if $\mathrm{Re}\,w>0$ because $0<\mathrm{Re}\,w=\mathrm{Re}\big([\sqrt w]^2\big)=[\mathrm{Re}\sqrt w]^2-[\mathrm{Im}\sqrt w]^2$.

Define $G_m^{(j)}(r,\xi)$, $j=1,\ldots,5$, and $\mu_m(r,\xi)$ as in (2.13a)–(2.13d) and (2.15), respectively, i.e.

$$\text{(A.1a)}\quad G_m^{(1)}(r,\xi) := \frac{1}{k(\xi)^2}\left[\frac{m^2 H_m(rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))} - \frac{rk(\xi)H_m'(rk(\xi))}{H_m(Rk(\xi))}\right],$$

$$\text{(A.1b)}\quad G_m^{(2)}(r,\xi) := \frac{H_m(rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))},$$

$$\text{(A.1c)}\quad G_m^{(3)}(r,\xi) := \frac{H_m(rk(\xi))}{H_m(Rk(\xi))}, \qquad G_m^{(4)}(r,\xi) := \frac{H_m'(rk(\xi))}{H_m'(Rk(\xi))},$$

$$\text{(A.1d)}\quad G_m^{(5)}(r,\xi) := \frac{1}{k(\xi)^2}\left[\frac{H_m(rk(\xi))}{H_m(Rk(\xi))} - \frac{rH_m'(rk(\xi))}{RH_m'(Rk(\xi))}\right],$$

$$\text{(A.1e)}\quad \mu_m(r,\xi) := \begin{cases} \exp\left[-\frac{|k(\xi)|^2 R\,(r-R)}{12\sqrt{m^2+|k(\xi)|^2R^2}}\right]\left(\frac{R}{r}\right)^{|m|} & \text{if } \operatorname{Re}(\xi^2) > k^2\,, \\ \left(\frac{R}{r}\right)^{|m|} & \text{if } \operatorname{Re}(\xi^2) < k^2\,, \end{cases}$$

respectively, for $\xi^2 \notin k^2 + i\mathbb{R}$ and $r \geq R$ and $m \in \mathbb{Z}$. Here, $H_m(z)$ denotes the Hankel function of the first kind and order $m \in \mathbb{Z}$ which is holomorphic in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$. For some $\delta > 0$ define the sets $Z_\delta^\pm$ by

$$\text{(A.2)}\quad \begin{aligned} Z_\delta^+ &:= \left\{\xi \in \mathbb{C} : |\operatorname{Re}\xi| > k + 2\delta\,,\ |\operatorname{Im}\xi| < \delta/3\right\}, \\ Z_\delta^- &:= \left\{\xi \in \mathbb{C} : |\operatorname{Re}\xi| < k - 2\delta\,,\ |\operatorname{Im}\xi| < \delta/3\right\}. \end{aligned}$$

Then $\xi^2 \notin k^2 + i\mathbb{R}$ for $\xi \in Z_\delta^+ \cup Z_\delta^-$, and the functions $G_m^{(j)}(r,\cdot)$ are obviously well defined and holomorphic in $Z_\delta^+ \cup Z_\delta^-$.

Lemma A.1. (a) *For every $\tilde{R} > R$ there exists $c = c(\delta,\tilde{R})$ with*

$$\text{(A.3a)}\quad |G_m^{(1)}(r,\xi)| + |G_m^{(2)}(r,\xi)| \leq \frac{c}{\sqrt{1+m^2+|\xi|^2}}\,\mu_m(r,\xi)\,,$$

$$\text{(A.3b)}\quad |G_m^{(3)}(r,\xi)| + |G_m^{(4)}(r,\xi)| \leq c\,\mu_m(r,\xi)\,,$$

$$\text{(A.3c)}\quad |G_m^{(5)}(r,\xi)| \leq \frac{c}{1+m^2+|\xi|^2}\,\mu_m(r,\xi)\,,$$

for all $\xi \in Z_\delta^+ \cup Z_\delta^-$ and $R \leq r \leq \tilde{R}$ and $m \in \mathbb{Z}$. Furthermore

$$\text{(A.3d)}\quad \left|R^2\,G_m^{(2)}(R,\xi) - G_m^{(1)}(R,\xi)\right| \leq \frac{c}{1+m^2+|\xi|^2}$$

for all $\xi \in Z_\delta^+ \cup Z_\delta^-$ and $m \in \mathbb{Z}$.

(b) *Let $\xi \in \mathbb{R}$ with $0 < |k(\xi)| \leq \delta$. Then the estimates* (A.3a)–(A.3d) *hold for all $R \leq r \leq \tilde{R}$ and $|m| \geq 2$ where c depends only on $\tilde{R}$.*

(c) *The functions $G_m^{(3)}$ and $G_m^{(4)}$, restricted to $(R,\tilde{R})\times(\mathbb{R}\setminus\{\pm k\})$, have extensions to continuous functions from $(R,\tilde{R})\times\mathbb{R}\to\mathbb{C}$ for all $m \in \mathbb{Z}$. The functions $G_m^{(1)}$ and $G_m^{(5)}$ are continuous for $|m| \geq 2$, and $G_m^{(2)}$ is continuous for $|m| \geq 1$. Furthermore, the following functions are continuous from $(R,\tilde{R})\times\mathbb{R}$ to $\mathbb{C}$: $(r,\xi)\mapsto \frac{1}{\ln k(\xi)}G_0^{(2)}(r,\xi)$, $(r,\xi)\mapsto\frac{1}{\ln k(\xi)}G_{\pm 1}^{(j)}(r,\xi)$ for $j \in \{1,5\}$, and $(r,\xi)\mapsto k(\xi)^2\ln k(\xi)\,G_0^{(j)}(r,\xi)$ for $j\in\{1,5\}$. The limits as $k(\xi)\to 0$, i.e. $\xi\to\pm k$,* *are given by the following expressions.*

(A.4a) $$\lim_{\xi\to\pm k} G^{(1)}_m(r,\xi) \;=\; G^{(1)}_m(r,\pm k) \;:=\; -\frac{R^2+r^2}{2\,(|m|-1)}\left(\frac{R}{r}\right)^{|m|}, \quad |m|\geq 2\,,$$

(A.4b) $$\lim_{\xi\to\pm k}\big[k(\xi^2)\ln k(\xi) G^{(1)}_0(r,\xi)\big] \;=\; -1\,,$$

(A.4c) $$\lim_{\xi\to\pm k}\frac{G^{(1)}_{\pm 1}(r,\xi)}{\ln k(\xi)} \;=\; -R^2\left(\frac{R}{r}+\frac{r}{R}\right),$$

(A.4d) $$\lim_{\xi\to\pm k} G^{(2)}_m(r,\xi) \;=\; G^{(2)}_m(r,\pm k) \;:=\; -\frac{1}{|m|}\left(\frac{R}{r}\right)^{|m|}, \quad |m|\geq 1\,,$$

(A.4e) $$\lim_{\xi\to\pm k}\frac{G^{(2)}_0(r,\xi)}{\ln k(\xi)} \;=\; 1\,,$$

(A.4f) $$\lim_{\xi\to\pm k} G^{(3)}_m(r,\xi) \;=\; G^{(3)}_m(r,\pm k) \;:=\; \left(\frac{R}{r}\right)^{|m|}, \quad m\in\mathbb{Z}\,,$$

(A.4g) $$\lim_{\xi\to\pm k} G^{(4)}_m(r,\xi) \;=\; G^{(4)}_m(r,\pm k) \;:=\; \left(\frac{R}{r}\right)^{|m|+1}, \quad m\in\mathbb{Z}\,,$$

(A.4h) $$\lim_{\xi\to\pm k} G^{(5)}_m(r,\xi) \;=\; G^{(5)}_m(r,\pm k) \;:=\; \frac{r^2-R^2}{2|m|(|m|-1)}\left(\frac{R}{r}\right)^{|m|}, \quad |m|\geq 2\,,$$

(A.4i) $$\lim_{\xi\to\pm k}\big[k(\xi^2)\ln k(\xi) G^{(5)}_0(r,\xi)\big] \;=\; \ln\frac{r}{R}\,,$$

(A.4j) $$\lim_{\xi\to\pm k}\frac{G^{(5)}_{\pm 1}(r,\xi)}{\ln k(\xi)} \;=\; R^2\left(\frac{r}{R}-\frac{R}{r}\right).$$

The limits are uniform with respect to $r\in[R,\tilde R]$ for every $\tilde R > R$.

(d) *For all $\tilde R > R$ and $(\xi, m)\in\mathbb{R}\times\mathbb{Z}$ with $|k(\xi)|\geq\delta$ there exists $c' = c'(\tilde R,\delta)$ with*

(A.5) $$\int_R^{\tilde R}\mu_m(r,\xi)^2\,dr \;\leq\; \frac{c'}{\sqrt{1+m^2+\xi^2}}\,.$$

Proof. Without loss of generality we assume that $m\geq 0$.
(a): We consider three cases: $\xi\in Z^+_\delta$ and $m\geq 1$, $\xi\in Z^+_\delta$ and $m=0$, and $\xi\in Z^-_\delta$.
Case (a1): $\xi\in Z^+_\delta$ and $m\geq 1$.
Set $z := -ik(\xi)$, thus $z^2=\xi^2-k^2$ and $k(\xi)=iz$. First we show

(i) $|z| = |k(\xi)| \geq c|\xi|$ for some c independent of ξ,
(ii) $|k(\xi)|^2 = |z^2| \geq \operatorname{Re}(z^2) = \operatorname{Re}(\xi^2)-k^2 > 3\delta^2$,
(iii) $\operatorname{Re}(z^2) > 2|\operatorname{Im}(z^2)| = 2|\operatorname{Im}(\xi^2)|$ and thus $|z^2| \leq \operatorname{Re}(z^2)+|\operatorname{Im}(z^2)| \leq \frac{3}{2}\operatorname{Re}(z^2)$.

Proofs:
(i) $|k(\xi)|^2 = |k^2-\xi^2| \geq |\xi|^2-k^2 = |\xi|^2 - \frac{k^2}{(k+2\delta)^2}(k+2\delta)^2 \geq \big(1-\frac{k^2}{(k+2\delta)^2}\big)|\xi|^2$ and $c := 1-\frac{k^2}{(k+2\delta)^2} = \frac{(k+2\delta)^2-k^2}{(k+2\delta)^2} > 0$.
(ii) $\operatorname{Re}(\xi^2)-k^2 = (\operatorname{Re}\xi)^2-(\operatorname{Im}\xi)^2-k^2 \geq (k+2\delta)^2-\delta^2/9-k^2 > 3\delta^2$.
(iii) $\operatorname{Re}(z^2)-2|\operatorname{Im}(z^2)| = (\operatorname{Re}\xi)^2-(\operatorname{Im}\xi)^2-k^2-4|\operatorname{Re}\xi||\operatorname{Im}\xi| = |\operatorname{Re}\xi|\big[|\operatorname{Re}\xi|-4|\operatorname{Im}\xi|\big]-(\operatorname{Im}\xi)^2-k^2 > (k+2\delta)[k+2\delta-4\delta/3]-\delta^2/9-k^2 > 0$.

Define $\quad\eta(z) \;:=\; \sqrt{1+z^2} \;+\; \ln\dfrac{z}{1+\sqrt{1+z^2}} \quad$ and $\quad z_r \;=\; z_{r,\xi,m} \;:=\; \dfrac{r}{m}z\,.$

We write $H_m(rk(\xi)) = H_m(imz_r) = \frac{2}{\pi}(-i)^{m+1}K_m(mz_r)$ where K_m is the modified Bessel function. We note that $\operatorname{Re} z_r > 0$. Therefore, we can use the asymptotic behavior of K_m and K'_m as m tends to infinity. Formulas 9.7.8 and 9.7.10 of [1] yield

$$\text{(A.6a)} \qquad \begin{aligned} H_m(rk(\xi)) &= \frac{2}{\pi}(-i)^{m+1}K_m(mz_r) \\ &= \frac{2}{\pi}(-i)^{m+1}\sqrt{\frac{\pi}{2m}}\,\frac{e^{-m\eta(z_r)}}{(1+z_r^2)^{1/4}}\,[1+a_{r,\xi,m}]\,, \end{aligned}$$

$$\text{(A.6b)} \qquad \begin{aligned} rk(\xi)H'_m(rk(\xi)) &= i\,\frac{2m}{\pi}(-i)^{m+2}\,z_rK'_m(mz_r) \;=\; \frac{2m}{\pi}(-i)^{m+1}\,z_rK'_m(mz_r) \\ &= -\frac{2m}{\pi}(-i)^{m+1}\sqrt{\frac{\pi}{2m}}\,(1+z_r^2)^{1/4}\,e^{-m\eta(z_r)}\,[1+c_{r,\xi,m}]\,, \end{aligned}$$

with

$$|a_{r,\xi,m}| + |c_{r,\xi,m}| \;\le\; \frac{c_1}{m\sqrt{1+|z_r|^2}} \;=\; \frac{c_1}{\sqrt{m^2+r^2|k(\xi)|^2}} \;\le\; \frac{c_2}{\sqrt{m^2+\xi^2}}$$

for all $r \ge R$, $m \in \mathbb{N}$, and $\xi \in Z_\delta^+$ where c_2 is indepedent of r, m, and ξ.
To estimate $|G_m^{(3)}(r,\xi)|$ and $|G_m^{(4)}(r,\xi)|$ we use the representations and write

$$\text{(A.7a)} \qquad \frac{H_m(rk(\xi))}{H_m(Rk(\xi))} \;=\; \left(\frac{1+z_R^2}{1+z_r^2}\right)^{1/4} e^{-m(\eta(z_r)-\eta(z_R))}\left[\frac{1+a_{r,\xi,m}}{1+a_{R,\xi,m}}\right]$$

and

$$\text{(A.7b)} \qquad \frac{rH'_m(rk(\xi))}{RH'_m(Rk(\xi))} \;=\; \left(\frac{1+z_r^2}{1+z_R^2}\right)^{1/4} e^{-m(\eta(z_r)-\eta(z_R))}\,,\left[\frac{1+c_{r,\xi,m}}{1+c_{R,\xi,m}}\right].$$

With $|1+z_r^2|^2 = \frac{1}{m^4}\big([m^2+r^2\operatorname{Re}(z^2)]^2 + r^4[\operatorname{Im}(z^2)]^2\big)$ we write

$$\frac{|1+z_R^2|^2}{|1+z_r^2|^2} = \frac{[m^2+R^2\operatorname{Re}(z^2)]^2+R^4[\operatorname{Im}(z^2)]^2}{[m^2+r^2\operatorname{Re}(z^2)]^2+r^4[\operatorname{Im}(z^2)]^2} = \frac{R^4}{r^4}\,\frac{[(m/R)^2+\operatorname{Re}(z^2)]^2+[\operatorname{Im}(z^2)]^2}{[(m/r)^2+\operatorname{Re}(z^2)]^2+[\operatorname{Im}(z^2)]^2}$$

which is less than 1 and larger than R^4/r^4 for $r \ge R$. This yields an estimate of the first factor on the right hand sides of (A.7a), (A.7b). For further use, we estimate

$$\text{(A.8)} \qquad m^4|1+z_R^2||1+z_r^2| \;=\; \frac{|1+z_r^2|}{|1+z_R^2|}\,|m^2+R^2z^2|^2\cdot\begin{cases} \le\; c\,\frac{r^2}{R^2}\,[m^2+|\xi|^2]^2\,, \\ \ge\; c\,[m^2+|\xi|^2]^2\,, \end{cases}$$

where we used $|m^2+R^2z^2| \le m^2+R^2|z|^2 \le c[m^2+|\xi|^2]$ and $|m^2+R^2z^2| \ge m^2+R^2\operatorname{Re}(z^2) \ge c[m^2+|\xi|^2]$.
To bound the exponential term in (A.7a), (A.7b) we define (compare the definition of $\eta(z)$)

$$f(r) \;:=\; \sqrt{m^2+r^2z^2} \;-\; \sqrt{m^2+R^2z^2} \;-\; m\ln\frac{m+\sqrt{m^2+r^2z^2}}{m+\sqrt{m^2+R^2z^2}}$$

and compute $f(R)=0$ and

$$f'(r) \;=\; \frac{rz^2}{\sqrt{m^2+r^2z^2}} - \frac{m}{m+\sqrt{m^2+r^2z^2}}\,\frac{rz^2}{\sqrt{m+r^2z^2}} \;=\; \frac{rz^2}{m+\sqrt{m^2+r^2z^2}}$$

for $r \geq R$ and thus

$$\begin{aligned}\operatorname{Re} f'(r) &= r\,\frac{\operatorname{Re}\big[z^2(m+\sqrt{m^2+r^2\bar{z}^2})\big]}{|m+\sqrt{m^2+r^2z^2}|^2}\\ &= r\,\frac{\operatorname{Re}(z^2)(m+\operatorname{Re}\sqrt{m^2+r^2\bar{z}^2})-\operatorname{Im}(z^2)\operatorname{Im}\sqrt{m^2+r^2\bar{z}^2}}{|m+\sqrt{m^2+r^2z^2}|^2}\end{aligned}$$

Since $\operatorname{Re}(m^2+r^2z^2)>0$ we can use the estimate $|\operatorname{Im}\sqrt{m^2+r^2\bar{z}^2}| \leq \operatorname{Re}\sqrt{m^2+r^2z^2}$ in the numerator and also in the denominator to obtain

$$\begin{aligned}\operatorname{Re} f'(r) &\geq r\,\frac{\operatorname{Re}(z^2)[m+\operatorname{Re}\sqrt{m^2+r^2z^2}]-\frac{1}{2}\operatorname{Re}(z^2)\,[m+\operatorname{Re}\sqrt{m^2+r^2z^2}]}{2[m+\operatorname{Re}\sqrt{m^2+r^2z^2}]^2}\\ &= \frac{r\ \operatorname{Re}(z^2)}{4\,[m+\operatorname{Re}\sqrt{m^2+r^2z^2}]} \geq \frac{r\ \operatorname{Re}(z^2)}{4\,[m+\sqrt{m^2+r^2|z|^2}]}\\ &\geq \frac{r\,|z|^2}{6\,[m+\sqrt{m^2+r^2|z|^2}]} \geq \frac{R\,|z|^2}{12\,\sqrt{m^2+R^2|z|^2}}\end{aligned}$$

where we used also (iii). Therefore,

$$\begin{aligned}m\operatorname{Re}[\eta(z_r)-\eta(z_R)] &= \operatorname{Re} f(r)-\operatorname{Re} f(R)+m\ln\frac{r}{R}\\ &\geq \frac{R|z|^2\,(r-R)}{12\sqrt{m^2+R^2|z|^2}} + m\ln\frac{r}{R}\end{aligned}$$

and thus

$$e^{-m\operatorname{Re}(\eta(z_r)-\eta(z_R))} \leq \exp\!\left[-\frac{|k(\xi)|^2R\,(r-R)}{12\sqrt{m^2+|k(\xi)|^2R^2}}\right]\left(\frac{R}{r}\right)^m = \mu_m(r,\xi)\,. \tag{A.9}$$

Finally, the representation $\frac{1+a_{r,\xi,m}}{1+a_{R,\xi,m}} = 1+b_{r,\xi,m}$ with $|b_{r,\xi,m}| \leq \frac{c'}{\sqrt{m^2+\xi^2}}$ implies the estimates for $G_m^{(3)}$ and $G_m^{(4)}$.

The estimate of $|G_m^{(2)}(r,\xi)|$ is proven analogously by using (A.8).

To estimate $|G_m^{(1)}(r,\xi)|$ we write

$$\begin{aligned}&k(\xi)^2\,G_m^{(1)}(r,\xi) \qquad\qquad \text{(A.10)}\\ &= \left[-\frac{m}{(1+z_r^2)^{1/4}(1+z_R^2)^{1/4}}+m(1+z_r^2)^{1/4}(1+z_R^2)^{1/4}\right]e^{-m(\eta(z_r)-\eta(z_R))}\,[1+b_{r,\xi,m}]\\ &= \frac{\sqrt{m^2+r^2z^2}\sqrt{m^2+R^2z^2}-m^2}{(m^2+r^2z^2)^{1/4}(m^2+R^2z^2)^{1/4}}\,e^{-m(\eta(z_r)-\eta(z_R))}\,[1+b_{r,\xi,m}]\end{aligned}$$

and estimate, using (A.8),

$$\begin{aligned}|\sqrt{m^2+r^2z^2}\sqrt{m^2+R^2z^2}-m^2| &= |z|^2r^2\frac{|(1+(R/r)^2)m^2+R^2z^2|}{m^2\sqrt{|1+z_r^2|}\sqrt{|1+z_R^2|}+m^2}\\ &\leq |z|^2r^2\frac{2m^2+R^2|z|^2}{m^2+|\xi|^2+m^2} \leq c|z|^2r^2\end{aligned}$$

for some constant c which is independent of m, ξ, and r. Therefore,

$$\big|G_m^{(1)}(r,\xi)\big| \leq c\,\frac{r^2}{\sqrt{m^2+|\xi|^2}}\,e^{-m\operatorname{Re}(\eta(z_r)-\eta(z_R))}$$

for some constant c which is independent of m, ξ, and r. Using (A.9) implies the estimate of $|G_m^{(1)}(r,\xi)|$.

The estimate of $|G_m^{(5)}(r,\xi)|$ follows the same way. Indeed,

$$\begin{aligned}
&\frac{H_m(rk(\xi))}{H_m(Rk(\xi))} - \frac{rH_m'(rk(\xi))}{RH_m'(Rk(\xi))} \\
=\ &\left[\frac{(1+z_R^2)^{1/4}}{(1+z_r^2)^{1/4}} - \frac{(1+z_r^2)^{1/4}}{(1+z_R^2)^{1/4}}\right] e^{-m\,\mathrm{Re}(\eta(z_r)-\eta(z_R))}[1+b_{r,\xi,m}]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(1+z_R^2)^{1/4}}{(1+z_r^2)^{1/4}} - \frac{(1+z_r^2)^{1/4}}{(1+z_R^2)^{1/4}} \\
=\ &\frac{\sqrt{1+z_R^2}-\sqrt{1+z_r^2}}{(1+z_r^2)^{1/4}(1+z_R^2)^{1/4}} = \frac{z_R^2 - z_r^2}{(1+z_r^2)^{1/4}(1+z_R^2)^{1/4}[\sqrt{1+z_R^2}+\sqrt{1+z_r^2}]} \\
=\ &\frac{(R^2-r^2)z^2}{(m^2+r^2z^2)^{1/4}(m^2+R^2z^2)^{1/4}[\sqrt{m^2+R^2z^2}+\sqrt{m^2+r^2z^2}]}
\end{aligned}$$

which is estimated by $\frac{c|z|^2}{m^2+|z|^2} \le \frac{c'|k(\xi)|^2}{m^2+|\xi|^2}$ and yields (A.3c).

Finally, for showing (A.3d) we use (A.10) for $r = R$ and $z^2 = -k(\xi)^2$ to obtain

$$k(\xi)^2 G_m^{(1)}(R,\xi) = \frac{R^2 z^2}{\sqrt{m^2+R^2z^2}} = -\frac{R^2 k(\xi)^2}{\sqrt{m^2+R^2z^2}}\,.$$

With (A.6a) and (A.6b) for $r = R$ we obtain

$$\begin{aligned}
&R^2\, G_m^{(2)}(R,\xi) - G_m^{(1)}(R,\xi) \\
=\ &-\frac{R^2}{m}\,\frac{1}{\sqrt{1+z_R^2}}\,[1+\mathcal{O}(1/\sqrt{m^2+\xi^2})] + \frac{R^2}{\sqrt{m^2+R^2z^2}}\,[1+\mathcal{O}(1/\sqrt{m^2+\xi^2})] \\
=\ &\mathcal{O}\big(1/(m^2+\xi^2)\big)\,.
\end{aligned}$$

Case (a2): $\xi \in Z_\delta^+$ and $m = 0$.
Now we use the asymptotical form of the modified Hankel functions $K_0(t)$ and $K_1(t)$ for large arguments (see 9.7.2 and 9.7.4 of [1]), i.e.

$$\text{(A.11a)}\quad H_0(rk(\xi)) = -\frac{2i}{\pi}\,K_0(rz) = -i\sqrt{\frac{2}{\pi}}\,\frac{e^{-rz}}{\sqrt{rz}}\big[1+\mathcal{O}(1/|z|\big]\,,$$

$$\text{(A.11b)}\quad H_0'(Rk(\xi)) = -H_1(Rk(\xi)) = \frac{2}{\pi}\,K_1(Rz) = -\sqrt{\frac{2}{\pi}}\,\frac{e^{-Rz}}{\sqrt{Rz}}\big[1+\mathcal{O}(1/|z|\big]\,,$$

where again $z = -ik(\xi) = \sqrt{\xi^2-k^2}$. Therefore, $|G_0^{(3)}(r,\xi)|$ and $|G_0^{(4)}(r,\xi)|$ are written as

$$\begin{aligned}
\frac{H_0(rk(\xi))}{H_0(Rk(\xi))} &= \sqrt{\frac{R}{r}}\,e^{-(r-R)z}\big[1+\mathcal{O}(1/|k(\xi)|\big]\,, \\
\frac{H_0'(rk(\xi))}{H_0'(Rk(\xi))} &= \frac{H_1(rk(\xi))}{H_1(Rk(\xi))} = \sqrt{\frac{R}{r}}\,e^{-(r-R)z}\big[1+\mathcal{O}(1/|k(\xi)|\big]\,.
\end{aligned}$$

Now we use (iii) in the estimate $|k(\xi)|^2 = |z^2| \le \mathrm{Re}(z^2) + |\,\mathrm{Im}(z^2)| \le \frac{3}{2}\,\mathrm{Re}(z^2) \le \frac{3}{2}(\mathrm{Re}\,z)^2$ and thus $e^{-(r-R)\,\mathrm{Re}\,z} \le e^{-|k(\xi)|(r-R)/12} = \mu_0(r,\xi)$.

The estimate of $|G_0^{(5)}(r,\xi)|$ is now obvious because $|G_0^{(5)}(r,\xi)| \le \frac{c}{|k(\xi)|^2}\,e^{-(r-R)\,\mathrm{Re}\,z}$.

The estimates of $|G_0^{(1)}(r,\xi)|$ and $|G_0^{(2)}(r,\xi)|$ and also (A.3d) are proven analogously by using (A.11a) and (A.11b).

Case (a3): $\xi \in Z_\delta^-$. Then $|k(\xi)|^2 \le k^2 + (k-2\delta)^2 + \delta^2/9$, i.e. the arguments in the Hankel functions are bounded. For this case we use the asymptotics of $H_m(z)$ and $H_m'(z)$ for z from compact subsets C of $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\}$, namely

$$\text{(A.12a)} \qquad H_0(z) = \frac{2i}{\pi}\left[\ln\frac{z}{2} + \gamma\right][1 + a_0(z)]\,,$$

$$\text{(A.12b)} \qquad H_m(z) = \frac{(m-1)!}{\pi i}\left(\frac{2}{z}\right)^m \big[1 + a_m(z)\big]\,, \quad m \ge 1\,,$$

$$\text{(A.12c)} \qquad z\,H_m'(z) = -\frac{m!}{\pi i}\left(\frac{2}{z}\right)^m \big[1 + b_m(z)\big]\,, \quad m \ge 1\,,$$

for $z \in C \setminus \{0\}$ where $|a_0(z)| + |a_m(z)| + |b_m(z)| \le \frac{c}{1+m}\,|z|$ for $m \ge 1$ and $z \in C$ where $c = c(C)$ is independent of m. Furthermore, a_m and b_m are continuous and vanish for $z = 0$, and $\gamma := \lim_{n\to\infty}\big[\sum_{p=1}^n \frac{1}{p} - \ln n\big]$ denotes Euler's constant.
The estimates of $|G_m^{(3)}(r,\xi)|$ and $|G_m^{(4)}(r,\xi)|$ (for all $m \ge 0$) and of $|G_m^{(2)}(r,\xi)|$ for $m \ge 1$ follow easily from these formulas. For $m = 0$ we have $|G_0^{(2)}(r,\xi)| \le c|\ln(rk(\xi))|$ which is bounded because $|k(\xi)|^2 \ge k^2 - |\xi|^2 \ge k^2 - (k-2\delta)^2 = 4\delta(k-\delta) > 0$.

To estimate $|G_m^{(1)}(r,\xi)|$ we represent $k(\xi)^2 G_m^{(1)}(r,\xi)$ as

$$\begin{aligned}
& k(\xi)^2 G_m^{(1)}(r,\xi) \\
= \; & \frac{m^2 H_m(rk(\xi))\, H_m(Rk(\xi)) - rR\,k(\xi)^2 H_m'(rk(\xi))\, H_m'(Rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))} \\
= \; & \frac{1}{2}\,\frac{[mH_m(rk(\xi)) - rk(\xi)H_m'(rk(\xi)][mH_m(Rk(\xi)) + Rk(\xi)H_m(Rk(\xi))]}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))} \\
& + \frac{1}{2}\,\frac{[mH_m(rk(\xi)) + rk(\xi)H_m'(rk(\xi)][mH_m(Rk(\xi)) - Rk(\xi)H_m(Rk(\xi))]}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))} \\
\text{(A.13)} \quad = \; & \frac{1}{2}\,\frac{rk(\xi)H_{m+1}(rk(\xi))\, Rk(\xi)H_{m-1}(Rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))} \\
& + \frac{1}{2}\,\frac{rk(\xi)H_{m-1}(rk(\xi))\, Rk(\xi)H_{m+1}(Rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))}
\end{aligned}$$

where we used the recurrence formulas $zH_m'(z) = zH_{m-1}(z) - mH_m(z) = -zH_{m+1}(z) + mH_m(z)$. Using the asymptotic forms (A.12b) and (A.12c) for $H_m(z)$ and $zH_m'(z)$, respectively, we obtain for $m \ge 2$ easily that $|k(\xi)^2 G_m^{(1)}(r,\xi)| \le \frac{c}{m-1}|k(\xi)|^2\big(\frac{R}{r}\big)^m$ for some constant which is independent of $\xi \in Z_\delta^-$, $r \ge R$, and $m \ge 2$. This yields the desired estimate for $m \ge 2$. For $m = 0$ or $m = 1$[3] we use that $|k(\xi)|$ is bounded below.

To estimate $|G_m^{(5)}(r,\xi)|$ we write

$$\begin{aligned}
& k(\xi)^2 G_m^{(5)}(r,\xi) \\
= \; & \frac{Rk(\xi)H_m'(Rk(\xi))\, H_m(rk(\xi)) - rk(\xi)H_m'(rk(\xi))\, H_m(Rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))} \\
\text{(A.14)} \quad = \; & \frac{Rk(\xi)H_{m-1}(Rk(\xi))\, H_m(rk(\xi)) - rk(\xi)H_{m-1}(rk(\xi))\, H_m(Rk(\xi))}{Rk(\xi)H_m'(Rk(\xi))\, H_m(Rk(\xi))}
\end{aligned}$$

[3] and only for these two cases

where we used again the recurrence formula $zH'_m(z) = zH_{m-1}(z) - mH_m(z)$. This expression is estimated as before by $\frac{c}{m(m-1)}|k(\xi)|^2\big(\frac{R}{r}\big)^m$ for $m \geq 2$. For $m = 0$ or $m = 1$ we use that $|k(\xi)|$ is bounded below.

For (A.3d) we write $G_m^{(1)}(R,\xi) - R^2G_m^{(2)}(R,\xi)$, using (A.13) for $r = R$ and (A.12b), (A.12c), as

$$\begin{aligned} G_m^{(1)}(R,\xi) - R^2G_m^{(2)}(R,\xi) &= \frac{R^2H_{m+1}(Rk(\xi))\,H_{m-1}(Rk(\xi))}{Rk(\xi)H'_m(Rk(\xi))\,H_m(Rk(\xi))} - \frac{R^2H_m(Rk(\xi))}{Rk(\xi)H'_m(Rk(\xi))} \\ &= -R^2\left(\frac{1}{m-1} - \frac{1}{m}\right)(1+\mathcal{O}(1/m)) \;=\; \mathcal{O}(1/m^2) \end{aligned}$$

(b) We observe that $|k(\xi)|$ is again bounded from above. Therefore, the assertion has been proven already in the previous part because a lower bound on $|k(\xi)|$ was only needed for the estimates of $|G_m^{(1)}(r,\xi)|$ and $|G_m^{(5)}(r,\xi)|$ for $m \in \{0,1\}$ and for $|G_0^{(2)}(r,\xi)|$.

(c) Continuity of $G_m^{(j)}$ for $j = 2,3,4$ and $m \geq 0$ ($m \geq 1$ for $j = 2$) and of $\xi \mapsto \frac{1}{\ln k(\xi)}G_0^{(2)}(r,\xi)$ and the form of the limits in (A.4d), (A.4e), (A.4f), (A.4g) follows directly from (A.12a), (A.12b), and (A.12c). Continuity of $G_m^{(1)}$ and $G_m^{(5)}$ for $m \geq 2$ and the form of the limits in (A.4a), (A.4h) follows from the representation (A.13) and (A.14), respectively, and (A.12b), (A.12c).
For $m = 1$ the representation (A.13) and (A.12a) – (A.12c) yields

$$\begin{aligned} &k(\xi)^2G_1^{(1)}(r,\xi) \\ =\;& \frac{\frac{rk(\xi)}{2}\frac{1}{\pi i}\big(\frac{2}{rk(\xi)}\big)^2Rk(\xi)\frac{2i}{\pi}\,\ln(Rk(\xi)) + \frac{Rk(\xi)}{2}\frac{1}{\pi i}\big(\frac{2}{Rk(\xi)}\big)^2rk(\xi)\frac{2i}{\pi}\,\ln(rk(\xi))}{\frac{1}{\pi^2}\big(\frac{2}{Rk(\xi)}\big)^2}\big[1+\mathcal{O}(k(\xi))\big] \\ =\;& R^2\left[\frac{R}{r} + \frac{r}{R}\right]k(\xi)^2\ln k(\xi)\,\big[1+\mathcal{O}(k(\xi))\big] \end{aligned}$$

which shows continuity of $\xi \mapsto \frac{1}{\ln k(\xi)}\,G_1^{(1)}(r,\xi)$ and the form (A.4c).

For $m = 0$ we obtain directly from the definition that $\xi \mapsto k(\xi)^2\ln k(\xi)\,G_0^{(1)}(r,\xi)$ is continuous with (A.4b).

For $G_0^{(5)}$ and $G_1^{(5)}$ we argue analogously using (A.14).

(d) For $|k(\xi)| \leq m$ (then $m \geq 1$) we estimate the first factor in the definition of $\mu_m(r,\xi)$ by 1 and obtain

$$\int_R^{\tilde R}\mu_m(r,\xi)^2\,dr \;\leq\; \int_R^{\tilde R}\left(\frac{R}{r}\right)^{2m}dr \;\leq\; \frac{c}{\sqrt{1+m^2+|k(\xi)|^2}}$$

where we used $m \geq |k(\xi)|$ in the last estimate. For $m \leq |k(\xi)|$ we estimate the second factor by 1 and obtain

$$\begin{aligned} \int_R^{\tilde R}\mu_m(r,\xi)^2\,dr &\leq \int_R^{\tilde R}\exp\left[-\frac{2|k(\xi)|^2R(r-R)}{10\sqrt{m^2+|k(\xi)|^2R^2}}\right]dr \\ &\leq \int_R^{\infty}e^{-c|k(\xi)|(r-R)}\,dr \;=\; \frac{1}{c|k(\xi)|} \;\leq\; \frac{c'}{\sqrt{1+m^2+|k(\xi)|^2}} \end{aligned}$$

where we used $|k(\xi)| \geq m$. $\square$

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