Odd-frequency superfluidity from a particle-number-conserving perspective

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We investigate odd-in-time—or *odd-frequency*—pairing of fermions in equilibrium systems within the particle-number-conserving framework of Penrose, Onsager, and Yang, where superfluid order is defined by macroscopic eigenvalues of reduced density matrices. We show that odd-frequency pair correlations are synonymous with even fermion-exchange symmetry in a time-dependent correlation function that generalises the two-body reduced density matrix. Macroscopic even-under-fermion-exchange pairing is found to emerge from conventional Penrose-Onsager-Yang condensation in two-body or higher-order reduced density matrices through the symmetry-mixing properties of the Hamiltonian. We identify and characterize a *transformer* matrix responsible for producing macroscopic even fermion-exchange correlations that coexist with a conventional Cooper-pair condensate, while a *generator* matrix is shown to be responsible for creating macroscopic even fermion-exchange correlations from hidden orders such as a multiparticle condensate. The transformer scenario is illustrated using the spin-balanced s-wave superfluid with Zeeman splitting as an example. The generator scenario is demonstrated by the composite-boson condensate arising for itinerant electrons coupled to magnetic excitations. Structural analysis of the transformer and generator matrices is shown to provide general conditions for odd-frequency pairing order to arise in a given system. Our formalism facilitates a fully general derivation of the Meissner effect for odd-frequency superconductors that holds also beyond the regime of validity for mean-field theory.

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I. INTRODUCTION

Superconductivity is a striking phenomenon whose macroscopic consequences include zero electric resistivity, the Meissner and Josephson effects, and magnetic-flux quantization [1]. The mechanism of conventional superconductivity is well understood [2]: bosonic Cooper pairs with s-wave symmetry are formed by phononmediated attraction between electrons, and these bosons condense at low temperatures. Unconventional superconductors, which do not conform to the same pattern, have been the subject of active research for decades [3,4]. Known unconventional superconductors involve Cooper pairs with different orbital symmetry (e.g., p-wave or d-wave), or alternative pairing mechanisms like repulsive electron-electron interactions or spin fluctuations. Odd-frequency superconductivity is an exotic hypothesized form of unconventional superconductivity [5]. Going back to an attempt by Berezinskii to explain superfluidity in ^{3}He [6], odd-frequency superconductivity is based on the mathematical possibility that natural systems might be described by a finite anomalous pair correlation function [7,8]

$$F_{\mathbf{ij}}(t_1, t_2) = \langle T\, c_{\mathbf{i}}^\dagger(t_1)\, c_{\mathbf{j}}^\dagger(t_2)\rangle \tag{1}$$

with odd symmetry under exchange of the time arguments, which also implies an even symmetry under exchange of the fermion indices. This is in contrast to the standard theories of superconductivity (both conventional and unconventional), which are based on equal-time pair correlation functions with odd pair-exchange symmetry. Here $\mathbf{i}$ and $\mathbf{j}$ denote sets of indices used for labeling single-fermion quantum numbers, and T is the time-ordering operator. The name "odd-frequency" refers to the fact that an odd symmetry under exchange of the time arguments [i.e., $F_{\mathbf{ij}}(t_1, t_2)$ satisfying $F_{\mathbf{ij}}(t_1, t_2) = -F_{\mathbf{ij}}(t_2, t_1)$] implies an odd symmetry in the frequency domain.

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Even though some theoretical models suggest that odd-frequency superconductivity may exist in bulk materials and in the absence of conventional (even-frequency) superconductivity [6,9–21], and despite an intense search for such phases, no conclusive evidence has been found to date [5]. In a separate scenario, odd-frequency pair correlations have been proposed to occur in the presence of a conventional even-frequency Cooper-pair condensate when translational and/or spin-rotational symmetries are broken [22–30]. This alternative is supported by indirect experimental evidence for odd-frequency correlations in heterostructures and near defects, through spectroscopic measurements and observations of the density of states [31–33]. Some theoretical understanding of the apparent absence of bulk odd-frequency superconductors has been obtained through a discussion about the thermodynamic stability of odd-frequency superconducting states [34–37] and the recent development of *no-go* theorems within the framework of Eliashberg theory [38–40]. Thus, whether odd-frequency superconductivity is realized in nature still remains an open question.

Existing theories of odd-frequency superconductivity are deeply rooted in the formalism of anomalous Green's functions [8], which violates particle-number conservation for electrons. This is potentially problematic, as the electron number is fundamentally expected to be conserved [41]. While number-nonconserving theories of superconductivity and superfluidity have been very successful in describing and predicting many important phenomena [42], they are at odds with nature [43] and thus not fully satisfactory. Moreover, with the recent development of ultracold-atom experiments in optical lattices and microtraps, it has become possible to study superfluidity in systems where the particle number is fixed [44–47] and small, even down to single digits [48], such that the consequences of number fluctuations—or their absence—matter. It is thus important to ask whether odd-frequency superconductivity can be described using a particle-number conserving formalism, and what this would look like. It is the purpose of this paper to take the first step in answering this question. To this end, we develop a number-conserving formalism of superfluid and superconducting states based on a time-dependent two-particle correlation function. This work generalizes the Penrose-Onsager-Yang criterion for fermionic superfluidity [49,50], which is based on the presence of a macroscopic eigenvalue of the two-particle reduced density matrix, i.e., one that scales linearly with the particle number N [51]. Our extension includes a relative-time dependence of pairing between two particles. It thus provides a natural framework for the study of odd-frequency superfluidity, while it reduces to the usual criterion in the case of even-frequency superfluidity [52]. The relative time in this formalism refers to a time delay for probing the fermion-pair correlations and not the timescale of nonequilibrium dynamics of the host system.

A particle-number-conserving description also pertains to the important concept of off-diagonal long-range order [50], which not only demonstrates that a superconductor is a state of macroscopic quantum coherence, but also allows one to draw conclusions about phenomena such as flux quantization, the Meissner effect, and related phenomena, even if conventional mean-field approaches do not apply. This aspect is particularly important for states with exotic order, such as odd-frequency pairing states. In fact, an important conclusion drawn from our formalism is the existence of a diamagnetic Meissner effect for (even- and) odd-frequency superconductors. Generalizing well-known arguments for conventional superconductors [53], we find that the presence of off-diagonal long-range order at any value of the relative time implies the absence of a (near-) homogeneous magnetic field. In particular, this means that odd-frequency superconductors exhibit a diamagnetic Meissner effect even if no even-frequency superconducting order is concurrently present.

There are additional fundamental open questions in the context of odd-frequency pairing that require a new perspective and where our approach could offer new insights. For example, the relation of the anomalous Green's functions of Eq. (1) that describes the creation of a Cooper pair, and

$$\overline{F}_{\mathbf{ij}}(t_1, t_2) = \langle T\, c_{\mathbf{i}}(t_1)\, c_{\mathbf{j}}(t_2)\rangle \tag{2}$$

that stands for the annihilation of a pair, is nontrivial [34,35]. Using the Lehmann or spectral representation of correlation functions, adapted to anomalous correlation functions [7,54,55], it follows after Fourier transformation that $F_{\mathbf{ij}}(\omega) = \overline{F}^*_{\mathbf{ji}}(-\omega)$. On the other hand, for states with well-defined symmetry under time exchange, i.e., for either even- or odd-frequency superconductors, it was shown in Ref. [35] that $F_{\mathbf{ij}}(\omega) = \overline{F}^*_{\mathbf{ji}}(\omega)$ yields consistent solutions. Only if the second relation holds is the thermodynamic stability of odd-frequency pairing at a continuous transition guaranteed, and the superfluid stiffness possesses the correct sign. While both expressions agree for even-frequency pairing, they are inconsistent for odd-frequency pairing. Even though a consistency argument was made in Ref. [35], there exists no proof that the second relation is correct for a given microscopic Hamiltonian. If it is, one might have to revise the spectral representation of the Gor'kov function [7,54,55], even for systems of infinite size. Addressing these crucial issues ultimately requires a particle-number-conserving description as presented in this paper.

Finally, having a particle-conserving description of macroscopic pairing order is extremely important if one wants to employ numerical approaches, such as density matrix renormalization group or Monte Carlo simulations, which are performed for a fixed particle number [56–62]. With our approach, we offer a first direct and unbiased way to probe odd-frequency superconductivity using these numerical methods.

To provide the reader with a general outline and motivation for the main part of this article, the fundamentals of our theoretical formalism and the obtained results are briefly summarized in Sec. II. This is followed by a thorough development of the particle-conserving theory describing time-dependent pair correlations, presented in Sec. III. The rigorous formalism discussed in Sec. III rests on minimal assumptions about basic properties of the physical system and is therefore widely applicable. As a first demonstration of its utility, a fully general derivation of the Meissner effect for odd-frequency superconductors is presented in Sec. IV. Section V is devoted to further illustrating the formalism by its application to two generic systems exhibiting symmetric-pairing order. The spin-balanced s-wave

Fermi superfluid subject to Zeeman splitting considered in Sec. V A serves as an example for the case where symmetric pair correlations arise in conjunction with conventional (antisymmetric-pairing, even-frequency) superfluidity. We refer to this situation as the *transformer scenario*. In contrast, the emergence of symmetric-pairing correlations in the absence of ordinary superfluidity is referred to as the *generator scenario*. We elucidate this alternative situation using the composite-boson condensate as an example (Sec. V B). Our conclusions are presented in Sec. VI. Details of some mathematical derivations are provided in Appendixes.

II. OUTLINE OF THEORETICAL APPROACH AND OVERVIEW OF MAIN RESULTS

A system of N fermions realizes a Cooper-pair condensate when the two-body reduced density matrix $\rho_{\mathbf{ij},\mathbf{kl}} = \langle c_\mathbf{i}^\dagger c_\mathbf{j}^\dagger c_\mathbf{l} c_\mathbf{k}\rangle$ has an eigenvalue of order N [50], which is generally referred to as a *macroscopic* eigenvalue [41]. In this case, the density matrix factorizes to leading order in N:

$$\rho_{\mathbf{ij},\mathbf{kl}} = \phi_\mathbf{ij}\, \phi_\mathbf{kl}^* + \tilde{\rho}_{\mathbf{ij},\mathbf{kl}}, \tag{3}$$

where the dominant eigenvector $\phi_\mathbf{ij}$ is the pair-condensate order parameter with eigenvalue $n_0 = \sum_\mathbf{ij} |\phi_\mathbf{ij}|^2 \sim O(N)$. The remaining part $\tilde{\rho}_{\mathbf{ij},\mathbf{kl}}$ of the two-body reduced density matrix has no macroscopic eigenvalue. The pair-condensate order parameter $\phi_\mathbf{ij}$ is the fixed-N analog of the anomalous pair correlator $F_\mathbf{ij}(0,0)$ of Eq. (1) at equal time. Both $\phi_\mathbf{ij}$ and $F_\mathbf{ij}$ may equally serve as a starting point for the approximate description of superfluidity and superconductivity [41], but only $\phi_\mathbf{ij}$ is well defined (and finite) when the number of particles is fixed.

In this work, we study the properties of the time-dependent two-body correlation matrix (T2bCM)

$$\rho_{\mathbf{ij},\mathbf{kl}}(t_1,t_2) = \langle c_\mathbf{i}^\dagger(t_1)\, c_\mathbf{j}^\dagger(t_2)\, c_\mathbf{l}(t_2)\, c_\mathbf{k}(t_1)\rangle. \tag{4}$$

This two-particle correlation function with a specific choice of the time arguments serves as a useful generalization of the two-body reduced density matrix, to which it reduces when $t_1 = t_2 = 0$. Because the T2bCM is a Hermitian and positive-semidefinite matrix in the fermion-pair index space (see Sec. III C), we know that its eigenvalues are real and nonnegative, and its eigenvectors are orthogonal. The T2bCM is a natural quantity for introducing time dependence into the description of superfluidity and superconductivity in a number-conserving formalism. We show that the T2bCM provides a general framework for the study of odd-frequency pairing of fermions.

Assuming systems governed by a time-independent and Hermitian Hamiltonian, the T2bCM depends only on the relative time $t = t_1 - t_2$. A macroscopic pairing order is signified by the presence of a macroscopic eigenvalue of the T2bCM. In this case, we find a time-dependent factorization to leading order in N,

$$\rho_{\mathbf{ij},\mathbf{kl}}(t,0) = \phi_\mathbf{ij}(t)\, \phi_\mathbf{kl}^*(t) + \tilde{\rho}_{\mathbf{ij},\mathbf{kl}}(t), \tag{5}$$

which is analogous to the time-independent case of Eq. (3). The time-dependent dominant eigenvalue is $n_0(t) = \sum_\mathbf{ij} |\phi_\mathbf{ij}(t)|^2 \sim O(N)$, and the remaining part $\tilde{\rho}_{\mathbf{ij},\mathbf{kl}}(t,0)$ has nonmacroscopic eigenvalues $\sim O(N^0)$ or smaller. The time-dependent pair-condensate order parameter $\phi_\mathbf{ij}(t)$ is the fixed-N analog of the anomalous pair correlator $F_\mathbf{ij}(t,0)$ at relative time t. Equation (5) is the starting point for our study of odd-frequency pairing of fermions. It is reminiscent of the factorization of the two-body correlation function postulated by Gorkov [7] in the context of superconductivity, but it is rigorous and applies to any finite and number-conserving system.

In order to study odd-frequency superfluidity, we introduce the symmetric and antisymmetric parts of the time-dependent pair-condensate order parameter

$$\phi_\mathbf{ij}(t) = \phi_\mathbf{ij}^{(a)}(t) + \phi_\mathbf{ij}^{(s)}(t), \tag{6}$$

where $\phi_\mathbf{ij}^{(a)}(t) = -\phi_\mathbf{ji}^{(a)}(t)$ is antisymmetric under index exchange $\mathbf{i} \leftrightarrow \mathbf{j}$, and $\phi_\mathbf{ij}^{(s)}(t) = \phi_\mathbf{ji}^{(s)}(t)$ is symmetric. For the time-ordered correlation function $F_\mathbf{ij}(t_1,t_2)$ of Eq. (1), even symmetry under the exchange of fermion indices is equivalent to odd symmetry in the relative time due to the properties of the time-ordering operator. We thus take the emergence of a macroscopic symmetric part $\phi_\mathbf{ij}^{(s)}(t) \sim O(\sqrt{N})$ as synonymous with the existence of odd-frequency pairing order. If a macroscopic antisymmetric part $\phi_\mathbf{ij}^{(a)}(t)$ exists, it indicates the presence of (conventional) even-frequency pairing order.

The time dependence of the pair-condensate order parameter is also important. Specifically, at relative time $t = 0$, it is purely antisymmetric, $\phi_\mathbf{ij}(0) = -\phi_\mathbf{ji}(0)$. A nonzero value of the symmetric part $\phi_\mathbf{ij}^{(s)}(t)$ can develop only at $t \neq 0$. Thus, we can distinguish two distinct scenarios for the presence of macroscopic symmetric-pairing (i.e., odd-frequency) order, depending on whether a nonzero and macroscopic $\phi_\mathbf{ij}^{(a)}(0)$ exists.

Coexistence of symmetric and antisymmetric pairing—Transformer scenario. A finite $\phi_\mathbf{ij}^{(a)}(0) \equiv \phi_\mathbf{ij}$ signals the presence of an ordinary Cooper-pair condensate. At finite relative time t, a macroscopic symmetric-pairing order can coexist with the ordinary Cooper-pair condensate. A small-t expansion reveals that the symmetric part of the pair-condensate order parameter is related to the conventional Cooper-pair order parameter $\phi_\mathbf{ij}$ by a transformation matrix,

$$\phi_\mathbf{ij}^{(s)}(t) = -i\,\frac{t}{\hbar}\,\frac{1}{n_0}\sum_\mathbf{kl} \tau_{\mathbf{ij},\mathbf{kl}}\, \phi_\mathbf{kl} + O(t^2), \tag{7a}$$

$$\tau_{\mathbf{ij},\mathbf{kl}} = \frac{1}{2}\,\langle (c_\mathbf{i}^\dagger\, H\, c_\mathbf{j}^\dagger + c_\mathbf{j}^\dagger\, H\, c_\mathbf{i}^\dagger)\, c_\mathbf{l} c_\mathbf{k}\rangle. \tag{7b}$$

This transformation matrix evidently depends not only on the properties of the underlying quantum state, but also on the Hamiltonian H of the system. We refer to this situation as the *transformer scenario*. We derive more general expressions for arbitrary time dependence in Sec. III E 1 and illustrate the transformer scenario using the spin-balanced s-wave Fermi superfluid with finite Zeeman splitting as an example in Sec. V A.

Emergence of symmetric pairing from hidden orders—Generator scenario. In the absence of a conventional Cooper-pair condensate, i.e., when $\phi_\mathbf{ij}(0) \equiv \phi_\mathbf{ij} = 0$, macroscopic pairing order can emerge only at $t \neq 0$ and is embodied in an order parameter whose leading time dependence is linear in t,

$$\phi_{\mathbf{ij}}(t) = \frac{t}{\sqrt{2}} \sqrt{n_0''(0)}\, \chi_{0,\mathbf{ij}} + O(t^2), \tag{8}$$

where $n_0''(0)$ is the second derivative of the dominant eigenvalue with respect to t, and $\chi_{0,\mathbf{ij}}$ are the elements of a normalized eigenvector of the T2bCM at $t = 0$. Symmetric-pairing order is present when the order parameter (8) has a nonvanishing symmetric part $\phi_{\mathbf{ij}}^{(s)}(t) \neq 0$. We refer to this situation as the *generator scenario*. In the special case when the order parameter is fully symmetric, i.e., $\phi_{\mathbf{ij}}(t) = \phi_{\mathbf{ij}}^{(s)}(t)$, the eigenvalue equation determining $n_0''(0)$ and $\chi_{0,\mathbf{ij}}$ simplifies to $\sum_{\mathbf{kl}} \gamma_{\mathbf{ij},\mathbf{kl}}\, \chi_{0,\mathbf{kl}} = [\hbar^2\, n_0''(0)/2]\, \chi_{0,\mathbf{ij}}$, with generator matrix elements

$$\gamma_{\mathbf{ij},\mathbf{kl}} = \tfrac{1}{4} \langle (c_{\mathbf{i}}^\dagger\, H\, c_{\mathbf{j}}^\dagger + c_{\mathbf{j}}^\dagger\, H\, c_{\mathbf{i}}^\dagger)(c_{\mathbf{l}}\, H\, c_{\mathbf{k}} + c_{\mathbf{k}}\, H\, c_{\mathbf{l}}) \rangle + O(N^0). \tag{9}$$

These results are derived in a more general setting in Sec. III E 2. Normally, a macroscopic $n_0''(0)$ can arise if the underlying quantum state has a composite, multiparticle condensate. Such a hidden order is not visible in the conventional two-body reduced density matrix. We illustrate the generator scenario in Sec. V B, using the composite-boson condensate formed by Cooper pairs coupled to magnons as an example.

The transformer and generator scenarios exhaust all possibilities for symmetric-pairing order with leading linear-in-t dependence to emerge in any system. Thus, analyzing the structure of the transformer and generator matrices for a particular physical situation provides the means to identify necessary and sufficient conditions under which odd-frequency superfluidity may be realized. Our theory could be extended to discuss the potential for unconventional pairing orders to manifest via a higher-order t dependence.

Using the generalized Penrose-Onsager-Yang formalism, we show that macroscopic quantum coherence in the T2bCM implies the conventional (diamagnetic) Meissner effect in both the transformer and generator scenarios.

III. PENROSE-ONSAGER-TYPE FORMALISM FOR TIME-DEPENDENT PAIR CORRELATIONS

A. Basic definitions and assumptions

Our starting point is the time-dependent two-particle correlation function (four-point function) with a specific choice for the time arguments. Equation (4) defines the matrix elements of the time-dependent two-body correlation matrix (T2bCM) $\underline{\underline{\rho}}(t_1, t_2)$. In this article, we generally denote a matrix in two-particle index space that has matrix elements $M_{\mathbf{ij},\mathbf{kl}}$ by the symbol $\underline{\underline{M}}$. Each individual index $\mathbf{i}$ labels a single-particle state that is created (annihilated) at time t by its corresponding fermion operator $c_{\mathbf{i}}^\dagger(t)$ $[c_{\mathbf{i}}(t)]$. An index pair $\mathbf{ij}$ refers to the fermionic two-particle state where one of the fermions is in state $\mathbf{i}$ and the other in state $\mathbf{j}$.

The expectation value in Eq. (4) is to be taken with respect to the system's many-body ground state (if we consider the zero-temperature limit) or a thermal mixture of many-body states (when considering a system at finite temperature). Since we are interested in describing superfluidity from a particle-conserving perspective, we furthermore restrict ourselves, in the zero-temperature case, to situations where the ground state is an eigenstate of the fermion-number operator $\hat{N} = \sum_{\mathbf{i}} c_{\mathbf{i}}^\dagger c_{\mathbf{i}}$ with eigenvalue N. [Here and in the following, we use the shorthand notation where $c_{\mathbf{i}} \equiv c_{\mathbf{i}}(0)$.] In the finite-temperature case, we work over statistical ensembles in which the microstates all have the same particle number N. We also assume that the Hamiltonian H itself commutes with $\hat{N}$. These assumptions underlie the majority of the discussion in this article, and it will be explicitly stated whenever we deviate from them.

Vectors in two-particle index space are denoted as $\underline{f}$ and have components $f_{\mathbf{ij}}$. Vectors that are invariant (invariant up to a minus sign) under the particle-index exchange $\mathbf{i} \leftrightarrow \mathbf{j}$ are called symmetric (antisymmetric). An arbitrary vector can be projected onto its symmetric and antisymmetric parts by applying the projectors $\underline{\underline{S}}$ and $\underline{\underline{A}}$ respectively, which have the matrix elements

$$S_{\mathbf{ij},\mathbf{kl}} = \tfrac{1}{2}(\delta_{\mathbf{i},\mathbf{k}}\, \delta_{\mathbf{j},\mathbf{l}} + \delta_{\mathbf{i},\mathbf{l}}\, \delta_{\mathbf{j},\mathbf{k}}), \tag{10a}$$

$$A_{\mathbf{ij},\mathbf{kl}} = \tfrac{1}{2}(\delta_{\mathbf{i},\mathbf{k}}\, \delta_{\mathbf{j},\mathbf{l}} - \delta_{\mathbf{i},\mathbf{l}}\, \delta_{\mathbf{j},\mathbf{k}}). \tag{10b}$$

Here $\delta_{\mathbf{a},\mathbf{b}}$ is a multidimensional Kronecker delta function with discrete vector arguments $\mathbf{a}$ and $\mathbf{b}$. The definitions in Eq. (10) yield the following properties of the projector matrices:

$$\underline{\underline{A}} + \underline{\underline{S}} = \underline{\underline{1}}, \quad \underline{\underline{A}}^2 = \underline{\underline{A}}, \quad \underline{\underline{S}}^2 = \underline{\underline{S}}, \quad \underline{\underline{A}}\,\underline{\underline{S}} = \underline{\underline{S}}\,\underline{\underline{A}} = \underline{\underline{0}}, \tag{11}$$

where $\underline{\underline{0}}$ and $\underline{\underline{1}}$ are the zero and unit matrices in two-particle index space, respectively.

B. The two-body reduced density matrix

Setting all time arguments in Eq. (4) to zero shows that $\underline{\underline{\rho}}(0,0) \equiv \underline{\underline{\rho}}$, with its matrix elements

$$\rho_{\mathbf{ij},\mathbf{kl}}(0,0) \equiv \rho_{\mathbf{ij},\mathbf{kl}} = \langle c_{\mathbf{i}}^\dagger c_{\mathbf{j}}^\dagger c_{\mathbf{l}} c_{\mathbf{k}} \rangle, \tag{12}$$

corresponds to the two-body reduced density matrix (2bRDM), which is central to the standard discussion of Cooper-pair condensation [41,50].

The 2bRDM is Hermitian and positive-semidefinite. For systems where the total particle number N is fixed, the trace evaluates to $\mathrm{Tr}\, \underline{\underline{\rho}} = N(N-1)$. As a consequence, the eigenvalues of the 2bRDM are non-negative and sum to the constant $N(N-1)$; hence, they can be interpreted as occupation numbers of fermion-pair states that relate to the eigenvectors of $\underline{\underline{\rho}}$.

Yang [50] further showed that the eigenvalues are bounded from above by N. Following Leggett [41], and based on the ideas of Penrose, Onsager, and Yang [49,50], we define pseudo-Bose-Einstein condensation (pseudo-BEC) of fermion (Cooper) pairs by the presence of at least one eigenvalue that is macroscopic, i.e., of order N. This becomes a rigorous definition in the thermodynamic limit, i.e., when the number of particles N is taken to infinity while keeping the fermion number density constant.

We will usually be concerned with the situation where only a single eigenvalue of $\underline{\underline{\rho}}$ is macroscopic, realizing a simple pseudo-BEC [63]. The fact that the sum over all eigenvalues is $O(N^2)$ implies that there are typically many remaining eigenvalues of order unity. The pseudo-BEC has to be contrasted with the situation where all the eigenvalues are of order unity, which would represent an uncondensed system configuration with no macroscopic order [49,50].

C. General properties of the time-dependent two-body correlation matrix

Evidently, the Hermiticity, positive-semidefiniteness and trace properties of the 2bRDM $\underline{\underline{\rho}} \equiv \underline{\underline{\rho}}(0,0)$ are instrumental in drawing conclusions about the macroscopicity of its eigenvalues and, thus, the possibility of time-independent conventional macroscopic order. It is now demonstrated that the T2bCM $\underline{\underline{\rho}}(t_1,t_2)$ defined in Eq. (4) satisfies these same three properties, even for nonzero and generally distinct time arguments t_1 and t_2.

(i) Hermiticity: We have

$$\begin{aligned}\rho^*_{\mathbf{kl},\mathbf{ij}}(t_1,t_2) &= \langle c^\dagger_{\mathbf{i}}(t_1)\, c^\dagger_{\mathbf{j}}(t_2)\, c_{\mathbf{l}}(t_2)\, c_{\mathbf{k}}(t_1)\rangle \\ &= \rho_{\mathbf{ij},\mathbf{kl}}(t_1,t_2), \end{aligned} \tag{13}$$

so that $\underline{\underline{\rho}}^\dagger(t_1,t_2) = \underline{\underline{\rho}}(t_1,t_2)$ for all t_1 and t_2. Thus, the T2bCM (4) is Hermitian.

(ii) Positive semidefiniteness: For an arbitrary vector $\underline{f}$ in two-particle index space, we have

$$\begin{aligned}\underline{f}^\dagger\, \underline{\underline{\rho}}(t_1,t_2)\, \underline{f} &= \sum_{\mathbf{ij},\mathbf{kl},\alpha} f^*_{\mathbf{ij}}\, \langle c^\dagger_{\mathbf{i}}(t_1)\, c^\dagger_{\mathbf{j}}(t_2)|\alpha\rangle\langle\alpha|c_{\mathbf{l}}(t_2)\, c_{\mathbf{k}}(t_1)\rangle\, f_{\mathbf{kl}} \\ &= \sum_\alpha \left|\sum_{\mathbf{ij}} f^*_{\mathbf{ij}}\, \langle c^\dagger_{\mathbf{i}}(t_1)\, c^\dagger_{\mathbf{j}}(t_2)|\alpha\rangle\right|^2 \geqslant 0, \end{aligned} \tag{14}$$

where a resolution of the identity in the $(N-2)$-particle Hilbert space was inserted. Thus, $\underline{\underline{\rho}}(t_1,t_2)$ is positive-semidefinite, and its eigenvalues are always non-negative.

(iii) Time-independent trace: Lastly, for a system with fixed particle number N, and a number-conserving Hamiltonian H, we have $\hat{N}(t) = \sum_{\mathbf{i}} c^\dagger_{\mathbf{i}}(t)\, c_{\mathbf{i}}(t) = \hat{N}(0) \equiv \hat{N}$. It follows that the trace of the T2bCM is equal to the time-independent constant $N(N-1)$:

$$\begin{aligned}\mathrm{Tr}\underline{\underline{\rho}}(t_1,t_2) &= \sum_{\mathbf{ij}} \big\langle c^\dagger_{\mathbf{i}}(t_1)\, c^\dagger_{\mathbf{j}}(t_2)\, c_{\mathbf{j}}(t_2)\, c_{\mathbf{i}}(t_1)\big\rangle \\ &= \sum_{\mathbf{i}} \big\langle c^\dagger_{\mathbf{i}}(t_1)\, \hat{N}\, c_{\mathbf{i}}(t_1)\big\rangle = N(N-1). \end{aligned} \tag{15}$$

Given points (i), (ii), and (iii) above, it is assured that the T2bCM $\underline{\underline{\rho}}(t_1,t_2)$ has non-negative eigenvalues that sum to $N(N-1)$. Thus, the interpretation of the eigenvalues as occupation numbers of generalized fermion-pair orbitals is also valid for the T2bCM.

In the following, we assume that the system's time evolution is determined by a Hermitian time-independent Hamiltonian H that commutes with the fermion-number operator $\hat{N}$, and that the system is in an equilibrium state with fixed particle number N. This may be the ground state of H or a thermal mixture. In this case, the T2bCM depends only on the relative time $t = t_1 - t_2$, and we will denote it by $\underline{\underline{\rho}}(t) \equiv \underline{\underline{\rho}}(t,0)$. The matrix elements of this quantity are then given by

$$\rho_{\mathbf{ij},\mathbf{kl}}(t) = \big\langle c^\dagger_{\mathbf{i}}\, \exp\left(\tfrac{-it}{\hbar}\, H\right) c^\dagger_{\mathbf{j}}\, c_{\mathbf{l}}\, \exp\left(\tfrac{it}{\hbar}\, H\right) c_{\mathbf{k}}\big\rangle, \tag{16}$$

specializing to the 2bRDM (12) when $t = 0$; $\underline{\underline{\rho}}(0) \equiv \underline{\underline{\rho}}$. We note that expressions that we derive for the short-time expansion of this correlation function might prove very useful for numerical approaches that want to probe whether a given many-body state displays odd-frequency pairing correlations.

D. Index-exchange symmetry of the time-dependent two-body correlation matrix

The number-conserving formalism leads us to reconsider the question of what odd-frequency pairing correlations mean. Let us first establish that permuting the time arguments in the pair correlation function of Eq. (1) is equivalent to permuting the fermion operators with a change of sign, i.e.,

$$F_{\mathbf{ij}}(t_1,t_2) = -F_{\mathbf{ji}}(t_2,t_1). \tag{17}$$

This follows directly from the definition of the time-ordering operator T [5,6,19]. As a consequence, any odd-in-time component of the pair correlation function (equivalent to odd-frequency in the frequency domain) is even under exchange of the fermion indices. The search for odd-in-time (or odd-frequency) pairing correlations is thus equivalent to the search for even-under-fermion-exchange pairing correlation. We argue that the latter is a more natural way to think about this type of unconventional pairing than the symmetry of the pairing correlations in time or frequency, and we will use this formulation in the following. Moreover, the exchange-symmetry concept can be meaningfully applied also to correlation functions that are defined without time ordering, such as the T2bCM introduced in Eq. (4). As shown in our analysis presented below, the T2bCM exhibits its own particular correspondence between index-exchange symmetry and time dependence. To be specific, we focus our discussion on the t dependence of $\underline{\underline{\rho}}(t)$ defined in Eq. (16), reminding the reader that t denotes the relative time $t \equiv t_1 - t_2$.

The resolution of the identity in Eq. (11) can be used to decompose $\underline{\underline{\rho}}(t)$ into its parts that are either fully symmetric, fully antisymmetric, or have mixed symmetry under index exchange,

$$\underline{\underline{\rho}}(t) = \underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}} + \underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} + \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}} + \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}. \tag{18}$$

It is easy to show (see the previous section) that both matrices $\underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}}$ and $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}$ are Hermitian and positive-semidefinite, and thus have real and non-negative eigenvalues. Furthermore, $\mathrm{Tr}\underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}} + \mathrm{Tr}\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} = \mathrm{Tr}\underline{\underline{\rho}}(t) = N(N-1)$. At $t = 0$, all terms in Eq. (18) containing the projector $\underline{\underline{S}}$ vanish because $\underline{\underline{S}}\,\underline{\underline{\rho}}(0) = \underline{\underline{\rho}}(0)\,\underline{\underline{S}} = \underline{\underline{0}}$ due to the anticommutation relations satisfied by fermion operators. Thus, the only nonzero contribution to the 2bRDM $\underline{\underline{\rho}}(0)$ is the purely antisymmetric block:

$$\underline{\underline{\rho}}(0) = \underline{\underline{A}}\,\underline{\underline{\rho}}(0)\,\underline{\underline{A}}. \tag{19}$$

The part in Eq. (18) that is fully symmetric under index exchange, as well as the parts that have mixed symmetry, emerge only at $t \neq 0$. To illustrate this more clearly, it is useful to consider the small-t expansion

$$\underline{\underline{\rho}}(t) = \underline{\underline{\rho}}(0) + t\,\underline{\underline{\rho}}'(0) + \frac{t^2}{2}\,\underline{\underline{\rho}}''(0) + O(t^3), \tag{20}$$

where the terms $\underline{\underline{\rho}}'(0)$ and $\underline{\underline{\rho}}''(0)$ have matrix elements given by

$$\begin{aligned}\rho'_{\mathbf{ij},\mathbf{kl}}(0) &= \partial_t\,\rho_{\mathbf{ij},\mathbf{kl}}(t)|_{t=0}\\ &= \frac{i}{\hbar}(\langle c_{\mathbf{i}}^\dagger c_{\mathbf{j}}^\dagger c_{\mathbf{l}} H c_{\mathbf{k}}\rangle - \langle c_{\mathbf{i}}^\dagger H c_{\mathbf{j}}^\dagger c_{\mathbf{l}} c_{\mathbf{k}}\rangle), \qquad (21a)\\ \rho''_{\mathbf{ij},\mathbf{kl}}(0) &= \partial_t^2\,\rho_{\mathbf{ij},\mathbf{kl}}(t)|_{t=0}\\ &= \frac{1}{\hbar^2}(\langle c_{\mathbf{i}}^\dagger H c_{\mathbf{j}}^\dagger c_{\mathbf{l}} H c_{\mathbf{k}}\rangle - \langle c_{\mathbf{i}}^\dagger H^2 c_{\mathbf{j}}^\dagger c_{\mathbf{l}} c_{\mathbf{k}}\rangle\\ &\quad - \langle c_{\mathbf{i}}^\dagger c_{\mathbf{j}}^\dagger c_{\mathbf{l}} H^2 c_{\mathbf{k}}\rangle). \qquad (21b)\end{aligned}$$

Inspection of Eqs. (21) reveals that $\underline{\underline{\rho}}'(0)$ and $\underline{\underline{\rho}}''(0)$ obey the symmetry constraints

$$\underline{\underline{\rho}}'(0) = \underline{\underline{A}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}} + \underline{\underline{A}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{S}} + \underline{\underline{S}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}}, \tag{22a}$$

$$\underline{\underline{\rho}}''(0) = \underline{\underline{A}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{A}} + \underline{\underline{A}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}} + \underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{A}} + \underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}}. \tag{22b}$$

Therefore, the small-t limit for the fully symmetric block of the T2bCM is $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} = t^2\underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}}/2 + O(t^3)$, while the mixed-symmetry blocks $\underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}$ and $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}}$ are $O(t)$. See Fig. 1 for an illustration of this general structure. The relationship between index-exchange symmetry and leading-order t dependence signals the relevance of certain blocks of the T2bCM for the description of odd-frequency pairing correlations.

E. Index-exchange symmetry of eigenvectors

Our discussion in the previous Sec. III D elucidated the connection between index-exchange symmetry and t dependence of the T2bCM $\underline{\underline{\rho}}(t)$. In particular, it was shown that $\underline{\underline{\rho}}(0)$ is purely antisymmetric, as required for a 2bRDM, and that the part $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}$ is $O(t^2)$ in the limit $t \to 0$. We now explore ramifications of this structure for the eigenvectors of the T2bCM, which are essential for describing pairing correlations present in the system [41]. Insights obtained here about general properties of T2bCM eigenvectors underpin the description of macroscopic symmetric-pairing order in terms of a single eigenvector with macroscopic eigenvalue, developed in the subsequent Sec. III F.

$$\underline{\underline{\rho}}(t) = \left(\begin{array}{c|c} \underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}} = \mathcal{O}(t^0) & \underline{\underline{A}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} = \mathcal{O}(t^1) \\ \hline \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}} = \mathcal{O}(t^1) & \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} = \mathcal{O}(t^2) \end{array}\right) \begin{array}{l} \}\,\underline{\underline{A}} \\ \}\,\underline{\underline{S}} \end{array}$$

FIG. 1. General structure of the time-dependent two-body correlation matrix (T2bCM) $\underline{\underline{\rho}}(t)$. Here $\underline{\underline{A}}$ and $\underline{\underline{S}}$ denote the projectors onto subspaces spanned by two-particle basis states that are antisymmetric and symmetric, respectively, under particle exchange. See Eq. (10).

We start with the spectral decomposition

$$\underline{\underline{\rho}}(t) = \sum_\alpha n_\alpha(t)\,\underline{\chi}_\alpha(t)\,\underline{\chi}_\alpha^\dagger(t), \tag{23}$$

where the $n_\alpha(t)$ are eigenvalues of the T2bCM, and $\underline{\chi}_\alpha(t)$ the corresponding normalized eigenvectors. The eigenvalues $n_\alpha(0)$ of the 2bRDM $\underline{\underline{\rho}}(0)$ are usually interpreted as occupation numbers for two-fermion states described by wave functions $\underline{\chi}_\alpha(0)$ [41]. In fact, as shown in more detail in Appendix B, the vectors $\underline{\chi}_\alpha(0)$ with nonzero eigenvalue $n_\alpha(0)$ are fully antisymmetric; $\underline{\chi}_\alpha(0) \equiv \underline{\underline{A}}\,\underline{\chi}_\alpha(0)$, and so possess the crucial property of two-fermion wavefunctions to be antisymmetric under index exchange $\mathbf{i} \leftrightarrow \mathbf{j}$, required by Fermi statistics. As was demonstrated in Sec. III C, the eigenvalues $n_\alpha(t)$ of the T2bCM defined in Eq. (16) are also non-negative and can therefore be interpreted as occupation numbers for the generalized natural pair orbitals $\underline{\chi}_\alpha(t)$. However, due to the form of Eq. (18), the eigenvectors $\underline{\chi}_\alpha(t)$ for $t \neq 0$ are not constrained to have an identically zero symmetric part and, therefore, can encode symmetric pair correlations that do not correspond to any ordinary two-fermion bound state. One of the important observations resulting from the number-conserving formalism is that there are two distinct mechanisms for symmetric components $\underline{\underline{S}}\,\underline{\chi}_\alpha(t)$ of T2bCM eigenvectors $\underline{\chi}_\alpha(t)$ to exist. We elucidate these two mechanisms in turn below.

1. Transformer mechanism

The transformer mechanism applies to T2bCM eigenvectors $\underline{\chi}_\alpha(t)$ with a nonzero eigenvalue $n_\alpha(0) > 0$ at $t = 0$. In this case, the eigenvector belongs to the $\underline{\underline{A}}\,\underline{\underline{\rho}}\,\underline{\underline{A}}$ block and satisfies the eigenvalue equation of the 2bRDM

$$\underline{\underline{\rho}}\,\underline{\chi}_\alpha(0) = n_\alpha(0)\,\underline{\chi}_\alpha(0). \tag{24}$$

While being antisymmetric at $t = 0$, $\underline{\chi}_\alpha(0) \equiv \underline{\underline{A}}\,\underline{\chi}_\alpha(0)$, such eigenvectors generally develop a finite symmetric part for $t \neq 0$. A Taylor expansion of the eigenvalue equation for $\underline{\chi}_\alpha(t)$ in small t reveals that a symmetric component appears in first order (see Appendix B 1 for details),

$$\underline{\chi}_\alpha(t) = \left(\underline{\underline{1}} - \frac{i\,t}{\hbar\, n_\alpha(0)}\,\underline{\underline{\tau}}\right)\underline{\underline{A}}\,\underline{\chi}_\alpha(t) + O(t^2). \tag{25}$$

The leading-order *transformer* matrix $\underline{\underline{\tau}}$ transforms the antisymmetric eigenvector into the symmetric subspace. It is defined by

$$\underline{\underline{\tau}} = i\,\hbar\,\underline{\underline{S}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}}, \tag{26}$$

and its matrix elements are given by Eq. (7b). As is evident from Eq. (25), the part $\underline{\underline{S}}\,\underline{\chi}_\alpha(t)$ of the eigenvector $\underline{\chi}_\alpha(t)$ that is symmetric under index exchange has a leading $O(t)$ dependence on relative time t. Thus, symmetric-pairing correlations that are necessarily odd-in-t to leading order are a feature of any system for which the transformer $\underline{\underline{\tau}}$ with matrix elements given in Eq. (7b) is finite.

As shown in Appendix A, the transformer scenario can be generalized to all orders in t. The symmetric part of the eigenvector $\underline{\chi}_\alpha(t)$ may be written as

$$\underline{\underline{S}}\,\underline{\chi}_\alpha(t) = \frac{-i\,t}{\hbar\, n_\alpha(t)}\,\underline{\underline{\tau}}_\alpha(t)\,\underline{\underline{A}}\,\underline{\chi}_\alpha(t), \tag{27}$$

with the generalized transformer $\underline{\underline{\tau}}_\alpha(t)$ for the αth eigenstate that is given formally by

$$\underline{\underline{\tau}}_\alpha(t) = \left[\underline{\underline{1}} - \frac{1}{n_\alpha(t)}\,\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}\right]^{-1}\frac{i\,\hbar}{t}\,\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}}. \tag{28}$$

Expanding these equations to leading order in t readily yields Eqs. (25) and (26).

2. Generator mechanism

The transformer mechanism discussed above is not the only way in which symmetric pair correlations can arise. In addition, we have to consider eigenvectors $\underline{\chi}_\beta(t)$ with vanishing eigenvalues at zero time; $n_\beta(0) = 0$. To make this explicit, we rewrite the spectral decomposition (23) of the T2bCM as follows:

$$\underline{\underline{\rho}}(t) = \sum_\alpha{}' n_\alpha(t)\,\underline{\chi}_\alpha(t)\,\underline{\chi}_\alpha^\dagger(t) + \sum_\beta{}'' n_\beta(t)\,\underline{\chi}_\beta(t)\,\underline{\chi}_\beta^\dagger(t). \tag{29}$$

Here $\sum_\alpha'$ is the restricted sum over eigenvectors that have nonzero eigenvalues $n_\alpha(0) > 0$. They are antisymmetric at $t = 0$ and may or may not develop symmetric parts for $t \neq 0$, depending on the structure of the transformer $\underline{\underline{\tau}}_\alpha(t)$. In contrast, $\sum_\beta''$ contains only eigenvectors whose associated eigenvalues vanish at $t = 0$. Those eigenvalues grow quadratically as a function of t; $n_\beta(t) \sim O(t^2)$, because $n_\beta(t) \geqslant 0$ for all times. The eigenvectors $\underline{\chi}_\beta(t)$ are of unknown symmetry but may have a symmetric component $\underline{\underline{S}}\,\underline{\chi}_\beta(t) \neq \underline{0}$. To identify these specific contributions, we consider the symmetric-pair-correlation *generator*

$$\underline{\underline{\gamma}}(t) = \hbar^2 \sum_\beta{}'' \frac{n_\beta(t)}{t^2}\,\underline{\underline{S}}\,\underline{\chi}_\beta(t)\,[\underline{\underline{S}}\,\underline{\chi}_\beta(t)]^\dagger \tag{30a}$$

$$= \frac{\hbar^2}{t^2}\,\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} - \sum_\alpha{}' \underline{\underline{\tau}}_\alpha(t) \times \frac{\underline{\underline{A}}\,\underline{\chi}_\alpha(t)[\underline{\underline{A}}\,\underline{\chi}_\alpha(t)]^\dagger}{n_\alpha(t)}\,\underline{\underline{\tau}}_\alpha^\dagger(t), \tag{30b}$$

where the form given in (30b) follows using Eqs. (27) and (29). By construction, $\underline{\underline{\gamma}}(t)$ can only be finite if the fully symmetric part $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}$ of the T2bCM is not entirely accounted for by the transformer mechanism, i.e., the generator embodies any and all symmetric-pairing correlations arising by complementary mechanisms. Its $t = 0$ limit is given by

$$\underline{\underline{\gamma}} = \frac{\hbar^2}{2}\sum_\beta{}'' n_\beta''(0)\,\underline{\underline{S}}\,\underline{\chi}_\beta(0)[\underline{\underline{S}}\,\underline{\chi}_\beta(0)]^\dagger \tag{31a}$$

$$= \frac{\hbar^2}{2}\,\underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}} - \underline{\underline{\tau}}\,\underline{\underline{Q}}[\underline{\underline{\rho}}(0)]^{-1}\underline{\underline{Q}}\,\underline{\underline{\tau}}^\dagger. \tag{31b}$$

Here we introduced the projector $\underline{\underline{Q}} = \sum_\alpha' \underline{\chi}_\alpha(0)\,\underline{\chi}_\alpha^\dagger(0)$ onto the subspace where the 2bRDM has no zero modes. See Fig. 2 for a refinement of the T2bCM structure presented earlier in Fig. 1. The expression given in Eq. (31b) enables straightforward calculation of $\gamma_{\mathbf{ij},\mathbf{kl}}$ after diagonalizing the 2bRDM, using also the transformer matrix elements from Eq. (7b), and

$$\frac{\hbar^2}{2}[\underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}}]_{\mathbf{ij},\mathbf{kl}} = \frac{1}{4}\langle(c_\mathbf{i}^\dagger\,H\,c_\mathbf{j}^\dagger + c_\mathbf{j}^\dagger\,H\,c_\mathbf{i}^\dagger)(c_\mathbf{l}\,H\,c_\mathbf{k} + c_\mathbf{k}\,H\,c_\mathbf{l})\rangle. \tag{32}$$

Inspecting $\underline{\underline{\gamma}}$ for a given system of interest provides basic insights into the nature of its generated symmetric-pairing correlations, i.e., those arising beyond the transformer mechanism.

It can be shown that the $\underline{\chi}_\beta(0)$ having $n_\beta(0) = 0$ are the solutions of the eigenvalue equation

$$\{\underline{\underline{P}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{P}} - 2\,\underline{\underline{P}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{Q}}[\underline{\underline{\rho}}(0)]^{-1}\underline{\underline{Q}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{P}}\}\underline{\chi}_\beta(0) = n_\beta''(0)\,\underline{\chi}_\beta(0), \tag{33}$$

where $\underline{\underline{P}} \equiv \sum_\beta'' \underline{\chi}_\beta(0)\,\underline{\chi}_\beta^\dagger(0) = \underline{\underline{1}} - \underline{\underline{Q}}$. (See Appendix B 2 for details, and Fig. 2 for an illustration of the subspaces projected on by $\underline{\underline{P}}$ and $\underline{\underline{Q}}$.) As per Eq. (31a), finiteness of the generator matrix $\underline{\underline{\gamma}}$ is an indicator for one or more of the eigenvectors $\underline{\chi}_\beta(0)$ to have a symmetric part $\underline{\underline{S}}\,\underline{\chi}_\beta(0) \neq \underline{0}$. In the particular case where $\underline{\chi}_\beta(0)$ is fully symmetric, i.e., $\underline{\chi}_\beta(0) = \underline{\underline{S}}\,\underline{\chi}_\beta(0)$, Eq. (33) becomes an eigenvalue equation for $\underline{\underline{\gamma}}$ (see further discussion in Appendix B 2),

$$\underline{\underline{\gamma}}\,\underline{\chi}_\beta(0) = \frac{\hbar^2}{2}\,n_\beta''(0)\,\underline{\chi}_\beta(0). \tag{34}$$

Thus, T2bCM eigenvectors that are fully symmetric in the $t = 0$ limit become also eigenvectors of the generator matrix $\underline{\underline{\gamma}}$ in that same limit.

F. Macroscopic symmetric-pairing order

Generalizing the conventional Penrose-Onsager-Yang approach [49,50], we associate pair condensation into a superfluid state with having a single term in the T2bCM's spectral decomposition (23) with a macroscopic eigenvalue, $n_0(t) \sim O(N)$, signifying that the pair orbital $\underline{\chi}_0(t)$ has a macroscopic occupation number. The physical properties of the system are

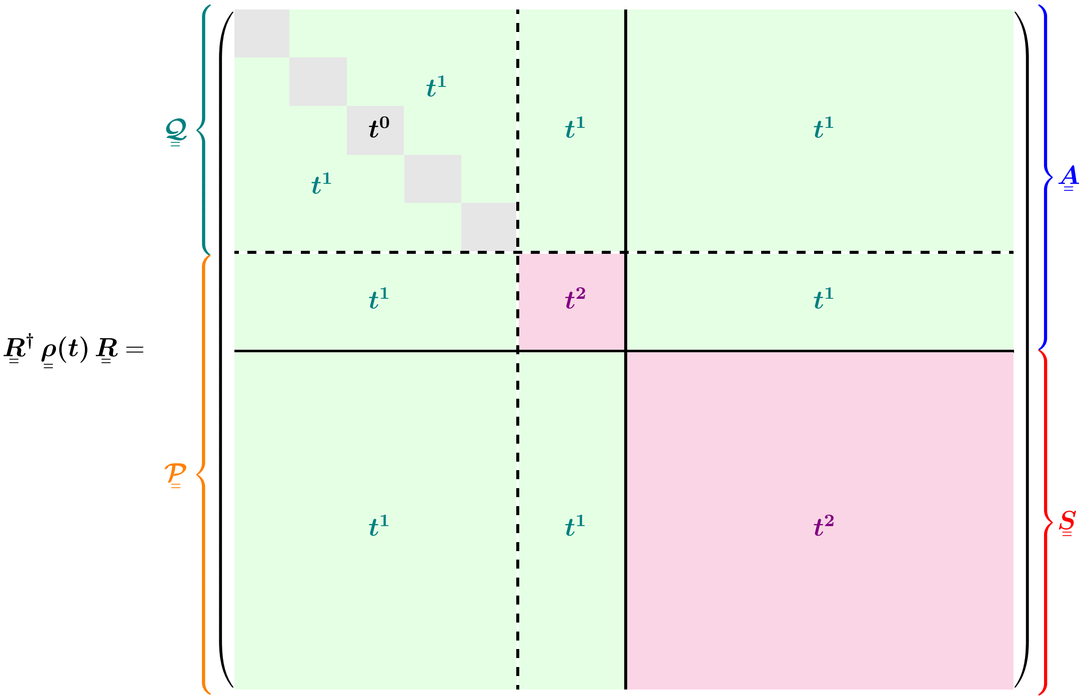

FIG. 2. Structure of the time-dependent two-body correlation matrix (T2bCM) $\underline{\underline{\rho}}(t)$ in the representation where $\underline{\underline{\rho}}(0)$ is diagonal. The form shown here is related to the form shown in Fig. 1 by a unitary transformation $\underline{\underline{R}}$, where the columns of $\underline{\underline{R}}$ are the normalized eigenvectors of $\underline{\underline{\rho}}(0)$. Labels t^n in the various blocks indicate their leading-order small-t dependence. The operators $\underline{\underline{A}}$ and $\underline{\underline{S}}$ project onto subspaces of two-particle states that are antisymmetric and symmetric, respectively, under particle exchange. The subspace where $\underline{\underline{\rho}}(0)$ has positive eigenvalues is singled out by the projector $\underline{\underline{\mathcal{Q}}}$, and $\underline{\underline{\mathcal{P}}} = \underline{\underline{1}} - \underline{\underline{\mathcal{Q}}}$ projects onto the nullspace of $\underline{\underline{\rho}}(0)$.

then dominated by the condensate order parameter [41]

$$\underline{\phi}(t) = \sqrt{n_0(t)}\,\underline{\chi}_0(t), \tag{35}$$

as the quantity $\underline{\phi}(t)/\sqrt{N}$ remains finite in the thermodynamic limit $N \to \infty$. Thus, to leading order in large N, the T2bCM factorizes as expressed in Eq. (5), in generalization of the analogous factorization in Gorkov's time-ordered four-point function [7].

In the $t = 0$ limit, our criterion for superfluid order to exist, and the definition (35) of the associated order parameter, recover the case of conventional even-frequency pair condensation. Clearly, finiteness of $\underline{\phi}(0)$ requires $n_0(0) > 0$. In this case, $\underline{\chi}_0(0)$ is a macroscopic eigenvector of the 2bRDM and, thus, necessarily antisymmetric. The order parameter then specializes to the form $\underline{\phi}(0) = \sqrt{n_0(0)}\,\underline{\chi}_0(0)$ that is familiar from the particle-number-conserving description of ordinary Cooper-pair condensates [41].

For $t \neq 0$, $\underline{\phi}(t)$ as defined in Eq. (35) will have a finite symmetric part $\underline{\underline{S}}\,\underline{\phi}(t)$ whenever $\underline{\underline{S}}\,\underline{\chi}_0(t) \neq \underline{0}$. However, to be properly part of the macroscopic pairing order parameter, finiteness of $\underline{\underline{S}}\,\underline{\phi}(t)/\sqrt{N}$ as $N \to \infty$ is required. As we now show, the two mechanisms identified in Sec. III E lead to two possible scenarios for a system to develop a macroscopic $\underline{\underline{S}}\,\underline{\phi}(t)$, indicating the presence of symmetric-pairing order.

Finite and macroscopic $n_0(0)$: Transformer scenario. As discussed above, in the case where $n_0(0)$ is finite, the macroscopic eigenvector $\underline{\chi}_0(t)$ is antisymmetric at $t = 0$, and the system exhibits conventional antisymmetric-pairing (even-frequency) superfluid order. By virtue of the transformer mechanism, the order parameter (35) generally has a symmetric part $\underline{\underline{S}}\,\underline{\phi}(t)$ whose leading-order small-t dependence is linear. The conditions under which $\underline{\underline{S}}\,\underline{\phi}(t)/\sqrt{N}$ will also be finite in the thermodynamic limit depend on specifics of the physical system. Assuming this to be the case, we define $\underline{\phi}^{(\mathrm{s})}(t) \equiv \underline{\underline{S}}\,\underline{\phi}(t)$ and find

$$\underline{\phi}^{(\mathrm{s})}(t) = -i\,\frac{t}{\hbar}\,\frac{1}{\sqrt{n_0(0)}}\,\underline{\underline{\tau}}\,\underline{\underline{A}}\,\underline{\chi}_0(0) + O(t^2). \tag{36}$$

Thus, in the presence of conventional antisymmetric-pair condensation signaled by a macroscopic eigenvalue $n_0(0) \sim O(N)$ with associated antisymmetric eigenvector $\underline{\chi}_0(0) \equiv \underline{\underline{A}}\,\underline{\chi}_0(0)$, the symmetric-pairing order parameter $\underline{\phi}^{(\mathrm{s})}(t)$ defined according to (36) emerges alongside the conventional superfluid order. The transformer scenario covers the existing proposals for odd-frequency superconductivity to occur in conjunction with conventional even-frequency order [22–30]. We present an illustrative example in Sec. V A.

Vanishing $n_0(0)$, macroscopic $n_0''(0)$: Generator scenario. In the absence of conventional macroscopic pair condensation at $t = 0$, the generator mechanism of Sec. III E provides an avenue for a macroscopic $n_0(t) \sim O(N)$ to emerge at $t \neq 0$ and to cause symmetric-pairing order. As a prerequisite, Eq. (33) must have a single macroscopic eigenvalue $n_0''(0) \sim O(N)$. The small-t limit for the macroscopic eigenvalue of the T2bCM is then of the form $n_0(t) = t^2\, n_0''(0)/2 + O(t^3)$. In this situation, the order parameter (35) has a leading-order small-t

dependence $\sim t$, and its symmetric component

$$\underline{\phi}^{(\mathrm{s})}(t) = \frac{t}{\sqrt{2}}\sqrt{n_0''(0)}\,\underline{\underline{S}}\,\underline{\chi}_0(0) + O(t^2) \tag{37}$$

represents symmetric-pairing order. Note that the symmetric order parameter $\underline{\phi}^{(\mathrm{s})}(t)$ can be obtained from the dominant eigenpair of the generator matrix $\underline{\underline{\gamma}}(t)$ in the special case where the eigenvector $\underline{\chi}_0(t)$ is fully symmetric. More generally, a macroscopic eigenvalue of the generator matrix implies a macroscopic symmetric order parameter $\underline{\phi}^{(\mathrm{s})}(t)$ under weak additional assumptions. For a detailed discussion and proof, see Appendix C.

The generator scenario applies to instances of bulk odd-frequency order where fermion-pair correlations exist only at $t \neq 0$ [6,9–21]. An example of such a case is discussed in Sec. V B. Having a macroscopic eigenvalue $n_0''(0)$ arising from Eq. (33) [or, if applicable, Eq. (34)] generally implies the existence of some type of hidden order [64] in the system of interest, and the generator scenario describes how symmetric-pairing order arises as its manifestation. The order parameter $\underline{\phi}(t) \propto t\sqrt{n_0''(0)/2}$ in this case and, thus, the fermion-pair condensate fraction scales as $(t/t^*)^2$ for $t < t^* \equiv \sqrt{2/n_0''(0)}$. While this reduction may create practical difficulties for revealing symmetric-pairing order [65] for too small values of t/t^*, the T2bCM is nonetheless dominated by the single macroscopic contribution as per Eq. (5) and, thus, exhibits the hallmarks of macroscopic quantum coherence [41,50]. We leave a more detailed discussion of physical consequences associated with the timescale t^* and a condensate fraction $\propto (t/t^*)^2$ for future research. In this context, it is important to keep in mind that the time $t = t_1 - t_2$ in Eq. (4) describes the internal dynamics of the Cooper pair, akin to the time argument in the dynamical Eliashberg formalism [66,67] of superconductivity, and not the out-of-equilibrium dynamics that enters, e.g., a formulation in terms of a time-dependent Ginzburg-Landau theory. The former is the relative time t of the two fermions forming the pair, while the latter is the total time $(t_1 + t_2)/2$ [68].

Higher-order scenario. In the transformer and generator scenarios described so far, macroscopic pairing order manifests already in the leading-order-in-t contribution to $n_0(t)$. For the transformer scenario, this is $n_0(0)$, the dominant eigenvalue of the 2bRDM as per Eq. (24). In the generator scenario where $n_0(0) = 0$, the leading term $n_0''(0)t^2/2$ in the small-t limit is macroscopic because $n_0''(0)$ is the single macroscopic eigenvalue of the generator matrix $\underline{\underline{\gamma}}$ [or, more generally, Eq. (33)]. Other scenarios are possible, where the macroscopic nature (i.e., scaling with particle number $\sim N$) develops solely at finite t and becomes apparent only at higher orders in the small-t expansion of $n_0(t)$.

Our formalism describes odd-frequency superfluidity in terms of the symmetric-pairing order parameter $\underline{\phi}^{(\mathrm{s})}(t)$, as defined in Eqs. (36) and (37) for the transformer scenario and the generator scenario, respectively. How $\underline{\phi}^{(\mathrm{s})}(t)$ can be related to order parameters from number-nonconserving theories is discussed in Appendix D.

IV. MEISSNER EFFECT

One of the most striking applications of the Penrose-Onsager-Yang formalism for conventional superfluids is the possibility to derive the Meissner effect and flux quantization solely from macroscopic quantum coherence in the 2bRDM, without strong assumptions about system properties [53,69–72]. Here we show how the analogous description for the T2bCM developed in Sec. III similarly lends itself to discussing the Meissner effect for odd-frequency superconductors, which has been a hotly debated issue [5,18,34–37,73].

The general approach is based on the observation that a spatial translation in a uniform and time-independent magnetic field $\mathbf{B}$ is equivalent to a gauge transformation of the magnetic vector potential $\mathbf{A}(\mathbf{r})$ [74]. Specifically, for the symmetric gauge $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}$, it is straightforward to show the relation

$$\mathbf{A}(\mathbf{r}+\mathbf{a}) = \mathbf{A}(\mathbf{r}) + \nabla\Lambda_{\mathbf{a}}(\mathbf{r}), \tag{38a}$$

$$\Lambda_{\mathbf{a}}(\mathbf{r}) = -\mathbf{a}\cdot\mathbf{A}(\mathbf{r}). \tag{38b}$$

Gauge invariance of the overall system dynamics then implies

$$c_{\mathbf{r}+\mathbf{a}} = \exp\left[i\,\frac{q}{\hbar}\,\Lambda_{\mathbf{a}}(\mathbf{r})\right]c_{\mathbf{r}}, \tag{39}$$

where $c_{\mathbf{r}}$ denotes the annihilation operator for a fermion with charge q in real-space representation. The relation (39) can be used to infer the transformational behavior of the 2bRDM

$$\rho_{\mathbf{r}_i\mathbf{r}_j,\,\mathbf{r}_k\mathbf{r}_l} = \left\langle c^\dagger_{\mathbf{r}_i}\,c^\dagger_{\mathbf{r}_j}\,c_{\mathbf{r}_l}\,c_{\mathbf{r}_k}\right\rangle \tag{40}$$

under translations. Together with the approximate factorization (3) that represents off-diagonal long-range order (ODLRO) in an ordinary Cooper-pair condensate [50], this leads to the condition $\mathbf{B} = \mathbf{0}$, embodying the familiar diamagnetic Meissner effect [53,69–72].

Here we are interested in discussing the implications of macroscopic coherence in the real-space representation of the T2bCM

$$\rho_{\mathbf{r}_i\mathbf{r}_j,\,\mathbf{r}_k\mathbf{r}_l}(t) = \left\langle c^\dagger_{\mathbf{r}_i}(t)\,c^\dagger_{\mathbf{r}_j}\,c_{\mathbf{r}_l}\,c_{\mathbf{r}_k}(t)\right\rangle \tag{41}$$

instead of the 2bRDM $\rho_{\mathbf{r}_i\mathbf{r}_j,\,\mathbf{r}_k\mathbf{r}_l} \equiv \rho_{\mathbf{r}_i\mathbf{r}_j,\,\mathbf{r}_k\mathbf{r}_l}(0)$. Due to the gauge invariance of the system Hamiltonian H and, therefore, of the time-evolution operator $\exp(\frac{-it}{\hbar}H)$, the relation (39) generalizes to

$$c_{\mathbf{r}+\mathbf{a}}(t) = \exp\left[i\,\frac{q}{\hbar}\,\Lambda_{\mathbf{a}}(\mathbf{r})\right]c_{\mathbf{r}}(t). \tag{42}$$

As a result, the T2bCM satisfies

$$\rho_{\mathbf{r}_i+\mathbf{a}\,\mathbf{r}_j+\mathbf{a},\,\mathbf{r}_k+\mathbf{a}\,\mathbf{r}_l+\mathbf{a}}(t) = \exp\left\{-i\,\frac{q}{\hbar}[\Lambda_{\mathbf{a}}(\mathbf{r}_i) + \Lambda_{\mathbf{a}}(\mathbf{r}_j) - \Lambda_{\mathbf{a}}(\mathbf{r}_k) - \Lambda_{\mathbf{a}}(\mathbf{r}_l)]\right\}\rho_{\mathbf{r}_i\mathbf{r}_j,\,\mathbf{r}_k\mathbf{r}_l}(t). \tag{43}$$

The approximate factorization (5) of the T2bCM due to the existence of a macroscopic eigenvalue implies that the T2bCM is dominated by the macroscopically coherent contribution at large length scales. Specifically,

$$\rho_{\mathbf{r}_i\,\mathbf{r}_j,\,\mathbf{r}_k\,\mathbf{r}_l}(t) \to \phi_{\mathbf{r}_i\,\mathbf{r}_j}(t)\,\phi^*_{\mathbf{r}_k\,\mathbf{r}_l}(t) \tag{44}$$

is valid for

$$|\mathbf{r}_i + \mathbf{r}_j - \mathbf{r}_k - \mathbf{r}_l|/2 \geqslant L \gg |\mathbf{r}_i - \mathbf{r}_j| \approx |\mathbf{r}_k - \mathbf{r}_l|. \tag{45}$$

This is the generalization of the concept of ODLRO [50] to the situation where $t \neq 0$, with L denoting the length scale beyond which the part $\tilde{\rho}_{\mathbf{ij},\mathbf{kl}}(t)$ in Eq. (5) has decayed and only the contribution from the macroscopic eigenvector of the T2bCM remains. Let us now assume that the factorization (44) is valid at a particular value of the relative time t. Consistency of (44) with (43) then requires that the order parameter transforms under translations as

$$\phi_{\mathbf{r}_i+\mathbf{a}\,\mathbf{r}_j+\mathbf{a}}(t) = f_{\mathbf{a}}\,\exp\left\{-\,i\,\frac{q}{\hbar}\,[\Lambda_{\mathbf{a}}(\mathbf{r}_i) + \Lambda_{\mathbf{a}}(\mathbf{r}_j)]\right\}\phi_{\mathbf{r}_i\,\mathbf{r}_j}(t), \tag{46}$$

with a displacement-dependent phase factor $f_{\mathbf{a}}$. Performing two consecutive translations, first along vector **a** then along vector **b**, leads to the order-parameter transformation

$$\phi_{\mathbf{r}_i+\mathbf{a}+\mathbf{b}\,\mathbf{r}_j+\mathbf{a}+\mathbf{b}}(t) = f_{\mathbf{b}}\,f_{\mathbf{a}}\,\exp\left\{-i\,\frac{q}{\hbar}\,[\Lambda_{\mathbf{b}}(\mathbf{r}_i+\mathbf{a}) + \Lambda_{\mathbf{b}}(\mathbf{r}_j+\mathbf{a}) + \Lambda_{\mathbf{a}}(\mathbf{r}_i) + \Lambda_{\mathbf{a}}(\mathbf{r}_j)]\right\}\phi_{\mathbf{r}_i\,\mathbf{r}_j}(t), \tag{47a}$$

$$= \exp\left[i\,\frac{q}{\hbar}\,\mathbf{B}\cdot(\mathbf{a}\times\mathbf{b})\right] f_{\mathbf{a}+\mathbf{b}}\,\exp\left\{-i\,\frac{q}{\hbar}\,[\Lambda_{\mathbf{a}+\mathbf{b}}(\mathbf{r}_i) + \Lambda_{\mathbf{a}+\mathbf{b}}(\mathbf{r}_j)]\right\}\phi_{\mathbf{r}_i\,\mathbf{r}_j}(t). \tag{47b}$$

To obtain (47b), we made use of the identity $\Lambda_{\mathbf{b}}(\mathbf{r}+\mathbf{a}) + \Lambda_{\mathbf{a}}(\mathbf{r}) = \Lambda_{\mathbf{a}+\mathbf{b}}(\mathbf{r}) - \frac{1}{2}\,\mathbf{B}\cdot(\mathbf{a}\times\mathbf{b})$. Performing the two translations in opposite order yields, however,

$$\phi_{\mathbf{r}_i+\mathbf{a}+\mathbf{b}\,\mathbf{r}_j+\mathbf{a}+\mathbf{b}}(t) = f_{\mathbf{a}}\,f_{\mathbf{b}}\,\exp\left\{-i\,\frac{q}{\hbar}\,[\Lambda_{\mathbf{a}}(\mathbf{r}_i+\mathbf{b}) + \Lambda_{\mathbf{a}}(\mathbf{r}_j+\mathbf{b}) + \Lambda_{\mathbf{b}}(\mathbf{r}_i) + \Lambda_{\mathbf{b}}(\mathbf{r}_j)]\right\}\phi_{\mathbf{r}_i\,\mathbf{r}_j}(t), \tag{48a}$$

$$= \exp\left[-i\,\frac{q}{\hbar}\,\mathbf{B}\cdot(\mathbf{a}\times\mathbf{b})\right] f_{\mathbf{a}+\mathbf{b}}\,\exp\left\{-i\,\frac{q}{\hbar}\,[\Lambda_{\mathbf{a}+\mathbf{b}}(\mathbf{r}_i) + \Lambda_{\mathbf{a}+\mathbf{b}}(\mathbf{r}_j)]\right\}\phi_{\mathbf{r}_i\,\mathbf{r}_j}(t). \tag{48b}$$

The required consistency of the results (47b) and (48b) leads to the condition

$$\frac{2\,q}{\hbar}\,\mathbf{B}\cdot(\mathbf{a}\times\mathbf{b}) = 2\pi s, \tag{49}$$

with integer s. Arbitrariness of the displacements **a** and **b** appearing on the left-hand side of Eq. (49) makes it impossible for the requirement to be generally satisfied, except for $\mathbf{B} = \mathbf{0}$. For systems with discrete translation invariance, the smallest field allowed would amount to placing a flux quantum in the unit cell, often corresponding to extremely large magnetic-field values.

In the derivation of the Meissner effect from ODLRO for conventional superconductors [53,69–72], the factorization (44) is used at $t = 0$. However, as our arguments show, it is sufficient if this factorization is valid at any particular value of t, as the conclusion of Eq. (49) is independent of time. Remarkably, the diamagnetic Meissner effect then follows even for purely odd-frequency superconductors in the generator scenario.

We have explicitly demonstrated the incompatibility of a translationally invariant quantum state exhibiting macroscopic pairing order in the T2bCM with a homogeneous magnetic field, generalizing the line of reasoning originally advanced by Sewell [53,71]. However, our analysis including the dependence on relative time extends also to related applications of Sewell's arguments, e.g., to discuss flux quantization in multiply connected geometries [69], the incompatibility of magnetic fields that vary slowly in space [70], and the existence of vortex lattices [72].

The above derivation of the Meissner effect applies to the system as a whole, described by the full order parameter $\phi_{\mathbf{r}_i\,\mathbf{r}_j}(t)$. Within our formalism, it is thus impossible to discuss individual contributions to the Meissner response arising from the antisymmetric-pairing and symmetric-pairing parts in the transformer scenario [73].

Our formalism establishes on very general grounds the connection between macroscopic pairing order in the T2bCM and the conventional diamagnetic Meissner effect, extending the previous understanding [53,69–72] about conventional antisymmetric-pairing orders to symmetric-pairing order, even in the case where no ODLRO is present in the 2bRDM. These arguments extend beyond the range of validity for particle-nonconserving approaches and are valid even in cases that cannot be described by approximate theories, or by Gorkov's anomalous pair-correlation function (1). On the other hand, the formalism cannot yield direct insight about whether the macroscopic order constitutes the energetically stable phase of the system [72] and is thus unable to resolve on its own the ongoing stability debate [34–37]. But, as outlined briefly in Sec. I, the particle-conserving theory is still crucial for testing the validity of basic arguments aimed at establishing thermodynamic stability of odd-frequency paring order.

V. APPLICATION TO SPECIFIC INSTANCES OF SYMMETRIC-PAIRING ORDER

We illustrate the strength of the particle-conserving formalism developed in the previous section by applying it to particular physical realizations of symmetric-pairing (i.e., odd-in-time; odd-frequency) order. In Sec. V A the transformer scenario underlying odd-frequency pair correlations emerging in a spin-balanced s-wave Fermi superfluid with Zeeman splitting is discussed. Section V B examines the generator scenario realized for a composite-boson condensate that has been proposed [12–19] as a generic type of system exhibiting odd-frequency superconductivity in the absence of ordinary (i.e., antisymmetric-pairing) superfluid order.

A. Zeeman-spin-split Fermi superfluid: Example of transformed antisymmetric-to-symmetric pairing

Motivated by the emergence of odd-frequency superconductivity in superconductor-ferromagnet hybrid structures [22], the s-wave Fermi superfluid subject to Zeeman spin splitting has been studied as a model system for bulk odd-frequency order [30]. This system also constitutes a particular realization of a generic multiband superconductor where odd-frequency pairs are expected to exist [27–29]. Its many-particle Hamiltonian is given by

$$H = \sum_{\mathbf{q}\sigma} \epsilon_{\mathbf{q}\sigma}\, c^\dagger_{\mathbf{q}\sigma} c_{\mathbf{q}\sigma} - U \sum_{\mathbf{q}\mathbf{q}'} c^\dagger_{\mathbf{q}\uparrow} c^\dagger_{-\mathbf{q}\downarrow} c_{-\mathbf{q}'\downarrow} c_{\mathbf{q}'\uparrow}, \tag{50}$$

where $\mathbf{q}$ and $\sigma \in \{\uparrow, \downarrow\}$ denote quantum numbers of a fermion's linear momentum and spin-1/2 degrees of freedom, respectively, and $U > 0$ is the strength of the spin-singlet orbital-s-wave pairing interaction. The single-particle energy dispersion is given by

$$\epsilon_{\mathbf{q}\uparrow(\downarrow)} = \epsilon_{|\mathbf{q}|} \overset{-}{{}_{(+)}} h, \tag{51}$$

where h is the Zeeman spin-splitting energy, and the $\mathbf{q}$-dependence is assumed to be isotropic.

Explicit calculation of the transformer matrix elements defined by Eq. (7b), using the Hamiltonian H from Eq. (50) and identifying general fermion-state indices $\mathbf{i}$ with the combined momentum and spin quantum numbers $\mathbf{q}_i$ and σ_i, yields

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}^{(\mathrm{sp})} + \underline{\underline{\tau}}^{(\mathrm{int})}, \tag{52}$$

with the single-particle and interaction-related contributions to the transformer given by

$$\tau^{(\mathrm{sp})}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j,\,\mathbf{q}_k\sigma_k\,\mathbf{q}_l\sigma_l} = \frac{1}{2}\left(\epsilon_{\mathbf{q}_j\sigma_j} - \epsilon_{\mathbf{q}_i\sigma_i}\right) \rho_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j,\,\mathbf{q}_k\sigma_k\,\mathbf{q}_l\sigma_l}(0), \tag{53a}$$

$$\tau^{(\mathrm{int})}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j,\,\mathbf{q}_k\sigma_k\,\mathbf{q}_l\sigma_l} = -\frac{U}{2}\sum_{\mathbf{q}} \left\langle c^\dagger_{\mathbf{q}\uparrow}\, c^\dagger_{-\mathbf{q}\downarrow} \left(\varsigma_j\, c^\dagger_{\mathbf{q}_i\sigma_i}\, c_{-\mathbf{q}_j\bar{\sigma}_j} + \varsigma_i\, c^\dagger_{\mathbf{q}_j\sigma_j}\, c_{-\mathbf{q}_i\bar{\sigma}_i}\right) c_{\mathbf{q}_l\sigma_l}\, c_{\mathbf{q}_k\sigma_k}\right\rangle. \tag{53b}$$

For the compact notation of Eq. (53b), we use $\bar{\sigma}$ to denote the opposite of σ; i.e., $\bar{\sigma} = \downarrow (\uparrow)$ if $\sigma = \uparrow (\downarrow)$, and ς takes the values $+1\,(-1)$ when $\sigma = \uparrow (\downarrow)$. The structure of the expressions from Eqs. (53a) and (53b) suggests that the transformer is generally finite for a system described by the Hamiltonian (50), implying that symmetric-pairing correlations exist. However, for actual symmetric-pairing *order* to emerge, these correlations must become a macroscopic property of the system so that $\underline{\phi}^{(\mathrm{s})}(t)$ given by Eq. (36) satisfies $\underline{\phi}^{(\mathrm{s})}(t) \sim O(\sqrt{N})$ in the large-N limit. As we now show, the macroscopicity of symmetric-pairing correlations derives from the antisymmetric-paring order in the superfluid.

In the absence of a Zeeman term (i.e., $h = 0$), the ground state of a fermion system with Hamiltonian (50) is known to be a condensate of s-wave spin-singlet Cooper pairs. For finite values of h, this may remain the case, as long as the Zeeman splitting h is smaller than a critical value (the Chandrasekhar-Clogston limit [75,76]), or at arbitrary values of h in the absence of spin-relaxation processes where the populations of spin-$\uparrow$ and spin-$\downarrow$ particles are separately conserved and adjusted to be equal, as is typical for ultracold-atom experiments [77–80]. In either case, the zero-temperature ground state of the Zeeman-spin-split Fermi gas remains unpolarized with all macroscopic properties unchanged from the superfluid phase at $h = 0$ [81–84]. This superfluid state thus has a conventional Cooper pair condensate [50] with a 2bRDM with a single macroscopic eigenvector $\underline{\chi}_0$ corresponding to spin-singlet s-wave pairing. Specifically, the 2bRDM factorizes to leading order in N [see Eq. (3)],

$$\rho_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j,\,\mathbf{q}_k\sigma_k\,\mathbf{q}_l\sigma_l} = n_0\, \chi_{0,\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j}\, \chi^*_{0,\mathbf{q}_k\sigma_k\,\mathbf{q}_l\sigma_l} + \tilde{\rho}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j,\,\mathbf{q}_k\sigma_k\,\mathbf{q}_l\sigma_l}, \tag{54a}$$

with $n_0 \sim O(N)$, and

$$\chi_{0,\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j} = \delta_{\mathbf{q}_j,-\mathbf{q}_i}\left(\chi_{0,\mathbf{q}_i}\, \delta_{\sigma_i,\uparrow}\, \delta_{\sigma_j,\downarrow} - \chi_{0,-\mathbf{q}_i}\, \delta_{\sigma_i,\downarrow}\, \delta_{\sigma_j,\uparrow}\right). \tag{54b}$$

Substituting the expression (53a) of $\underline{\underline{\tau}}^{(\mathrm{sp})}$ for $\underline{\underline{\tau}}$ in the formula (36) of the symmetric-pairing order parameter, using also (54b) for the macroscopic eigenvector $\underline{\chi}_0$, we find to leading order in small t

$$\phi^{(\mathrm{s})}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j}(t) = -\,i\,\frac{t}{\hbar}\,h\,\sqrt{n_0}\;\delta_{\mathbf{q}_j,-\mathbf{q}_i} \times \left(\chi_{0,\mathbf{q}_i}\, \delta_{\sigma_i,\uparrow}\, \delta_{\sigma_j,\downarrow} + \chi_{0,-\mathbf{q}_i}\, \delta_{\sigma_i,\downarrow}\, \delta_{\sigma_j,\uparrow}\right). \tag{55}$$

Thus, for finite Zeeman splitting h, odd-frequency spin-triplet s-wave order emerges alongside even-frequency spin-singlet s-wave superfluidity in a Fermi gas [30].

In the derivation of (55), only the single-particle contribution $\underline{\underline{\tau}}^{(\mathrm{sp})}$ to the transformer was included, and this yielded the form of symmetric-pairing order consistent with previous studies of the Zeeman-spin-split Fermi superfluid [30]. Intriguingly, this result emerged as a direct consequence of the Zeeman splitting in the single-particle term of the Hamiltonian (50), while the interaction term was only implicitly relevant as the source of conventional antisymmetric-pairing order in the Fermi superfluid. In principle, an explicitly interaction-dependent contribution to symmetric-pairing order may arise based on the part $\underline{\underline{\tau}}^{(\mathrm{int})}$ of the transformer, adding to the Zeeman-splitting-facilitated portion (55). While this possibility cannot be ruled out completely, we expect it to be rarely relevant. See Appendix E for a more detailed discussion.

B. Composite-boson condensate: Example of generated symmetric-pairing order

Early studies of odd-frequency superfluidity envisioned situations where it appears on its own, i.e., not alongside ordinary antisymmetric (even-frequency) Cooper pairing of fermions as in the transformer scenario discussed in the previous section. Nevertheless, the possibility of another type of even-frequency order being the fundamental origin of odd-frequency pair correlations has tantalized ongoing research efforts [5]. In particular, a relationship between symmetric-pairing (odd-frequency) order and composite-boson condensation has been actively investigated [12–19].

Here we consider a one-dimensional-lattice realization of the model proposed in Refs. [16–19]. A system of itinerant electrons coupled to bosonic spin excitations (magnons) is described by the Hamiltonian

$$H = -K \sum_{r\sigma}(c^\dagger_{r+1\,\sigma} c_{r\sigma} + c^\dagger_{r\sigma} c_{r+1\,\sigma}) + J \sum_{r}(c^\dagger_{r\uparrow} c_{r\downarrow}\, b_r + c^\dagger_{r\downarrow} c_{r\uparrow}\, b^\dagger_r), \qquad (56)$$

where $c^\dagger_{r\sigma}$ creates an electron with spin σ at lattice position r, and $K > 0$ is the nearest-neighbor electron-hopping energy. Creation and annihilation of a magnon at site r [described by boson operators $b^\dagger_r$ and b_r, respectively] incurs a spin flip of electrons at the same site, with an associated (exchange-) energy scale J. In the following, we assume that the ground state used to calculate the expectation values determining the 2bRDM [Eq. (12)], the transformer $\underline{\underline{\tau}}$ [Eq. (7b)], and the generator $\underline{\underline{\gamma}}$ [Eq. (31b) with (32)] exhibits no independent electron-pair or magnon condensates. This implies that the 2bRDM has no macroscopic eigenvalue.

Without a macroscopic eigenvalue of the 2bRDM, there is no possibility for symmetric-pairing order to arise via the transformer scenario. *Generated* symmetric-pairing order, on the other hand, emerges when the generator $\underline{\underline{\gamma}}$ has an eigenvector with macroscopic eigenvalue. See Eq. (34) [more generally, Eq. (33) and further discussion in Appendix C]. Calculation of the generator, defined in Eq. (31b), still requires knowledge of the system's transformer matrix $\underline{\underline{\tau}}$. We present details of how $\underline{\underline{\tau}}$ and $\underline{\underline{\gamma}}$ are obtained in Appendix F 1. With the assumption that the system ground state is an eigenstate of both the electron number operator $\hat{N} \equiv \sum_{r\sigma} c^\dagger_{r\sigma} c_{r\sigma}$ and the magnon number operator $\hat{N}_\mathrm{b} \equiv \sum_r b^\dagger_r b_r$, the generator matrix for the system described by the Hamiltonian (56) is found to only have terms dependent on the exchange-coupling strength J;

$$\begin{aligned}\gamma_{r_i\sigma_i\, r_j\sigma_j,\, r_k\sigma_k\, r_l\sigma_l} = \frac{J^2}{4}\Big[&\delta_{\sigma_i,\uparrow}\,\delta_{\sigma_j,\uparrow}\,\delta_{\sigma_k,\uparrow}\,\delta_{\sigma_l,\uparrow}\,\big\langle\big(c^\dagger_{r_i\uparrow}\, c^\dagger_{r_j\downarrow}\, b^\dagger_{r_j} - c^\dagger_{r_i\downarrow}\, c^\dagger_{r_j\uparrow}\, b^\dagger_{r_i}\big)\big(b_{r_l}\, c_{r_l\downarrow}\, c_{r_k\uparrow} - b_{r_k}\, c_{r_l\uparrow}\, c_{r_k\downarrow}\big)\big\rangle\\ &+ \delta_{\sigma_i,\downarrow}\,\delta_{\sigma_j,\downarrow}\,\delta_{\sigma_k,\downarrow}\,\delta_{\sigma_l,\downarrow}\,\big\langle\big(c^\dagger_{r_i\uparrow}\, c^\dagger_{r_j\downarrow}\, b_{r_i} - c^\dagger_{r_i\downarrow}\, c^\dagger_{r_j\uparrow}\, b_{r_j}\big)\big(b^\dagger_{r_k}\, c_{r_l\downarrow}\, c_{r_k\uparrow} - b^\dagger_{r_l}\, c_{r_l\uparrow}\, c_{r_k\downarrow}\big)\big\rangle\\ &+ \big\langle c^\dagger_{r_i\uparrow}\, c^\dagger_{r_j\uparrow}\big(\delta_{\sigma_i,\uparrow}\,\delta_{\sigma_j,\downarrow}\, b_{r_j} - \delta_{\sigma_i,\downarrow}\,\delta_{\sigma_j,\uparrow}\, b_{r_i}\big)\big(\delta_{\sigma_k,\uparrow}\,\delta_{\sigma_l,\downarrow}\, b^\dagger_{r_l} - \delta_{\sigma_k,\downarrow}\,\delta_{\sigma_l,\uparrow}\, b^\dagger_{r_k}\big)\, c_{r_l\uparrow}\, c_{r_k\uparrow}\big\rangle\\ &+ \big\langle c^\dagger_{r_i\downarrow}\, c^\dagger_{r_j\downarrow}\big(\delta_{\sigma_i,\uparrow}\,\delta_{\sigma_j,\downarrow}\, b^\dagger_{r_i} - \delta_{\sigma_i,\downarrow}\,\delta_{\sigma_j,\uparrow}\, b^\dagger_{r_j}\big)\big(\delta_{\sigma_k,\uparrow}\,\delta_{\sigma_l,\downarrow}\, b_{r_k} - \delta_{\sigma_k,\downarrow}\,\delta_{\sigma_l,\uparrow}\, b_{r_l}\big)\, c_{r_l\downarrow}\, c_{r_k\downarrow}\big\rangle\Big].\end{aligned} \qquad (57)$$

The right-hand side of Eq. (57) contains various generalized three-body reduced density matrices describing correlations between an itinerant-electron pair and a magnon. A macroscopic eigenvalue of $\underline{\underline{\gamma}}$ would have to arise from a hidden order [5,64] involving such combinations of electronic and magnetic degrees of freedom. Our particle-number-conserving formalism enables a detailed discussion of the possibility that condensation of bosonic fermion pairs coupled with bosonic spin excitations underpin symmetric-pairing order in the system under consideration.

The generator's connection with composite-boson condensation is made particularly apparent by focusing on its matrix elements satisfying $r_i = r_j = r$ and $r_k = r_l = r'$,

$$\gamma_{r\sigma_i\, r\sigma_j,\, r'\sigma_k\, r'\sigma_l} = J^2\,\big[\delta_{\sigma_i,\uparrow}\,\delta_{\sigma_j,\uparrow}\,\delta_{\sigma_k,\uparrow}\,\delta_{\sigma_l,\uparrow}\langle c^\dagger_{r\uparrow}\, c^\dagger_{r\downarrow}\, b^\dagger_r\, b_{r'}\, c_{r'\downarrow}\, c_{r'\uparrow}\rangle + \delta_{\sigma_i,\downarrow}\,\delta_{\sigma_j,\downarrow}\,\delta_{\sigma_k,\downarrow}\,\delta_{\sigma_l,\downarrow}\langle c^\dagger_{r\uparrow}\, c^\dagger_{r\downarrow}\, b_r\, b^\dagger_{r'}\, c_{r'\downarrow}\, c_{r'\uparrow}\rangle\big]. \qquad (58)$$

The combinations of electron-pair operators with spin excitations appearing on the right-hand-side of Eq. (58) correspond to the order-parameter structure of the composite-boson condensate proposed, e.g., in Refs. [18,19]. Hypothesizing a form of the fixed-particle-number ground state that maximizes the composite-boson condensate (generalizing an approach pioneered by Yang [50]), we find that the three-body reduced density matrix with elements

$$\rho^{(3b)}_{r_i\sigma_i\, r_j\sigma_j\, r_o,\, r_k\sigma_k\, r_l\sigma_l\, r_p} = \langle c^\dagger_{r_i\sigma_i}\, c^\dagger_{r_j\sigma_j}\, b^\dagger_{r_o}\, b_{r_p}\, c_{r_l\sigma_l}\, c_{r_k\sigma_k}\rangle \tag{59}$$

factorizes to leading order in N (see Appendix F 2)

$$\rho^{(3b)}_{r_i\sigma_i\, r_j\sigma_j\, r_o,\, r_k\sigma_k\, r_l\sigma_l\, r_p} = n_0^{(3b)}\, \chi^{(3b)}_{0,r_i\sigma_i\, r_j\sigma_j\, r_o}\, \chi^{(3b)*}_{0,r_k\sigma_k\, r_l\sigma_l\, r_p} + \tilde{\rho}^{(3b)}_{r_i\sigma_i\, r_j\sigma_j\, r_o\, r_k\sigma_k\, r_l\sigma_l\, r_p}. \tag{60a}$$

Here $n_0^{(3b)} \sim O(N)$, the residual matrix $\underline{\underline{\tilde{\rho}}}^{(3b)}$ has no macroscopic contribution, and the macroscopic eigenvector of the three-body reduced density matrix has the form

$$\chi^{(3b)}_{0,r_i\sigma_i\, r_j\sigma_j\, r_o} = \chi_0^{(3b)}\, \delta_{r_j,r_i}\, \delta_{r_o,r_i}\big(\delta_{\sigma_i,\uparrow}\, \delta_{\sigma_j,\downarrow} - \delta_{\sigma_i,\downarrow}\, \delta_{\sigma_j,\uparrow}\big). \tag{60b}$$

As a result, the generator (57) satisfies the eigenvalue equation (34) with a symmetric eigenvector $\underline{\chi}_0(0)$;

$$\chi_{0,r_i\sigma_i\, r_j\sigma_j}(0) = \chi_0^{(3b)}\, \delta_{r_j,r_i}\, \delta_{\sigma_i,\uparrow}\, \delta_{\sigma_j,\uparrow}, \tag{61a}$$

$$n_0''(0) = \frac{2J^2}{\hbar^2}\, n_0^{(3b)}. \tag{61b}$$

The generated symmetric-pairing order parameter defined in Eq. (37) then takes the form

$$\phi^{(s)}_{r_i\sigma_i\, r_j\sigma_j}(t) = \frac{t}{\hbar}\, J\, \sqrt{n_0^{(3b)}}\, \chi_0^{(3b)}\, \delta_{r_j,r_i}\, \delta_{\sigma_i,\uparrow}\, \delta_{\sigma_j,\uparrow} \tag{62}$$

to leading order in small t. Thus, odd-frequency spin-polarized-triplet pairing of electrons emerges from the condensation of s-wave singlet electron pairs coupled to magnons [16–18].

The formalism illustrated here provides a general recipe for systematically identifying avenues toward generating symmetric-pairing order. Given a microscopic model, the reduced density matrices making up the generator designate the channels for equal-time many-particle condensation that underpins symmetric-pairing order. In the above consideration, we assumed condensation of singlet-fermion pairs coupled to magnons and found symmetric-pairing correlations in the spin-polarized triplet channel. In this scenario, the relevant terms of the generator (57) are those from the first two lines. Alternatively, the structure of terms in the last two lines of Eq. (57) implies that composite-boson condensation involving triplet fermion pairs would generate symmetric-singlet pairing order.

Here we have established a direct causal link between the presence of a composite-boson condensate and macroscopic symmetric-pairing correlations. This example demonstrates how generated symmetric-pairing order is generally a consequence of some type of hidden multi-particle condensation ensuring a macroscopic eigenvalue of the generator matrix. The general form of the generator as given in Eq. (31b), in conjunction with the matrix elements (32), should enable a comprehensive classification of system Hamiltonians that can give rise to generated odd-frequency superfluidity.

VI. CONCLUSIONS

We present a formalism to describe odd-in-time (also called *odd-frequency*) pairing, i.e., pair correlations that are occurring only between two fermions present at different times t_1 and t_2. Our approach is based on a thorough study of the time-dependent two-body correlation matrix (T2bCM) $\underline{\underline{\rho}}(t_1, t_2)$ [Eq. (4)]. The T2bCM is well defined in real physical systems that conserve particle number and could therefore, in principle, be probed directly in experiments similar to those that have recently been performed [48] or proposed [85] to measure even-in-time ($t_1 = t_2$) pair correlations. In addition, the T2bCM has the required properties for being a suitable generalization of the two-body reduced density matrix (2bRDM) $\underline{\underline{\rho}} \equiv \underline{\underline{\rho}}(0, 0)$ utilized in particle-conserving descriptions of conventional fermion superfluidity [41,50]. While the formalism applies more generally, we have focused on equilibrium situations described by a time-independent Hermitian Hamiltonian H conserving fermion number N. In that case, the T2bCM depends only on the relative time $t = t_1 - t_2$; $\underline{\underline{\rho}}(t_1, t_2) = \underline{\underline{\rho}}(t, 0) \equiv \underline{\underline{\rho}}(t)$, and $\underline{\underline{\rho}}(0)$ corresponds to the 2bRDM. Generalizing the particle-number-conserving description of fermion-pair condensation [41,49,50], the existence of a pair condensate is signaled by $\underline{\underline{\rho}}(t)$ having an eigenvector $\underline{\chi}_0(t)$ with macroscopic eigenvalue $n_0(t) \sim O(N)$. The condensate order parameter $\underline{\phi}(t)$ [Eq. (35)] then satisfies $\underline{\phi}(t) \sim O(\sqrt{N})$ in the large-N (i.e., the thermodynamic) limit. We establish odd-in-t pairing order to be associated with the symmetric part $\underline{\phi}^{(s)}(t)$ of the order parameter, i.e., the part that does not change sign under fermion exchange. Our derivation of the diamagnetic Meissner effect from off-diagonal long-range order at any value of t suggests that the presence of a macroscopic eigenvalue of the T2bCM implies superfluid phenomena, even if no macroscopic pairing is present in the 2bRDM. Two scenarios are identified for symmetric-pairing order embodied by $\underline{\phi}^{(s)}(t)$ to exist.

The *transformer* scenario can occur when $n_0(0) \neq 0$ is macroscopic and, thus, $\underline{\chi}_0(0)$ is an antisymmetric eigenvector of the 2bRDM. In such a case, $\underline{\phi}^{(s)}(t)$ arises from the transformation of even-in-t order into odd-in-t order, facilitated by the transformer matrix $\underline{\underline{\tau}}$ [Eq. (36) with Eq. (26)]. We illustrate the transformer scenario using the spin-polarized Fermi superfluid as an example [Sec. V A]. The transformer matrix $\underline{\underline{\tau}}$ provides a natural way to quantify the propensity for symmetric-pairing (i.e., odd-frequency) order to emerge in the presence of an even-frequency Cooper-pair condensate. It is thus similar to the recently proposed superconducting-fitness measure [86,87] whose connection with odd-frequency superconductivity has been established within Bogoliubov–de Gennes mean-field theory [29]. Our formalism could be utilized for further detailed investigation of the superconducting-fitness concept, especially its generalization beyond mean-field theory.

The alternative to the transformer scenario is the *generator* scenario, where symmetric-pairing order emerges without an ordinary Cooper-pair condensate present. The order parameter $\underline{\phi}^{(s)}(t)$ is then associated with an eigenvector $\underline{\chi}_0(t)$ of the T2bCM $\underline{\underline{\rho}}(t)$ that has a symmetric part at $t = 0$ [Eq. (37)] and satisfies the eigenvector equation (33) with a macroscopic eigenvalue $n_0''(0) \sim O(N)$. The propensity of a system to host generated symmetric-pairing order is embodied in the generator matrix $\underline{\underline{\gamma}}$ [Eq. (31)]. We elucidate how the generator scenario transpires in a model system of itinerant electrons coupled to magnons due to the presence of a composite-boson (fermion pair + magnon) condensate [Sec. V B]. Our formalism lends itself to establishing the direct link between a system's hidden order [64] and odd-frequency superconductivity. Future research could perform a systematic search for realizations of symmetric-pairing order in other many-particle model systems [88] based on consideration of their generator matrices. Exotic multiparticle condensates [89,90] are particularly promising candidates for unveiling generated odd-frequency order.

Extensions of the present theoretical description could consider symmetric-pairing order associated with a higher-order t dependence. Further investigation of the transformer and generator matrices would also inform efforts to design systems with odd-frequency superfluid order present. A search for measurable quantities whose response functions are related to a system's generator matrix may yield indirect experimental probes of generated symmetric-pairing order. More generally, due to its focus on physically realistic particle-number-conserving quantities, we expect our work to boost the overall development of direct detection schemes for odd-in-t pairing. In addition, the fixed-N theory developed here is ideally suited to study fermion pairing occurring in cold-atom gases [48,85] and nuclear matter [91–93], which are new platforms for realizing and investigating symmetric-pairing order [5,94,95].

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APPENDIX A: DERIVATION OF GENERAL TRANSFORMER

We start by applying the symmetrizer $\underline{\underline{S}}$ on both sides of the eigenvalue equation for $\underline{\chi}_\alpha(t)$, which yields

$$\begin{aligned} n_\alpha(t)\,\underline{\underline{S}}\,\underline{\chi}_\alpha(t) &= \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\chi}_\alpha(t) \\ &= \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,[\underline{\underline{S}}\,\underline{\chi}_\alpha(t) + \underline{\underline{A}}\,\underline{\chi}_\alpha(t)]. \end{aligned} \tag{A1}$$

Assuming a nonvanishing antisymmetric contribution $\underline{\underline{A}}\,\underline{\chi}_\alpha(t)$, the implicit relation (A1) between an eigenvector's symmetric and antisymmetric parts is formally resolved as

$$\underline{\underline{S}}\,\underline{\chi}_\alpha(t) = [n_\alpha(t)\,\underline{\underline{1}} - \underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}\,]^{-1}\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{A}}\,\underline{\chi}_\alpha(t). \tag{A2}$$

Here the invertibility of the expression between the square brackets requires $n_\alpha(t)$ to be distinct from any eigenvalues of $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}$. This can be guaranteed for any $n_\alpha(0) > 0$ and small enough t because all eigenvalues of $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}$ are $\sim O(t^2)$ and, thus, can become degenerate with $n_\alpha(t)$ only at strictly finite t. In the physically relevant transformer scenario where $n_\alpha(t)$ is the only macroscopic eigenvalue both at zero and finite t, the invertibility of the bracketed expression is guaranteed.

The derived expression (A2) resolves to Eq. (27), with Eq. (28) defining the general form of the transformer matrix $\underline{\underline{\tau}}_\alpha(t)$ under the additional assumption that the eigenvalue $n_\alpha(t)$ does not vanish. The definition of the transformer matrix $\underline{\underline{\tau}}_\alpha(t)$ ensures that it has an α-independent zero-t limit $\underline{\underline{\tau}}_\alpha(0) = \underline{\underline{\tau}}$ as given in Eq. (26), with matrix elements given by Eq. (7b).

APPENDIX B: SMALL-t EXPANSION FOR T2BCM EIGENVECTORS AND EIGENVALUES

We consider a general eigenvalue equation $\underline{\underline{\rho}}(t)\,\underline{\chi}_\alpha(t) = n_\alpha(t)\,\underline{\chi}_\alpha(t)$ for the T2bCM $\underline{\underline{\rho}}(t)$. Inserting the expansion of $\underline{\underline{\rho}}(t)$ up to $O(t^2)$ given in Eq. (20), as well as analogous expansions for the eigenvector $\underline{\chi}_\alpha(t)$ and eigenvalue $n_\alpha(t)$,

$$\underline{\chi}_\alpha(t) = \underline{\chi}_\alpha(0) + t\,\underline{\chi}_\alpha'(0) + \frac{t^2}{2}\,\underline{\chi}_\alpha''(0) + O(t^3), \tag{B1a}$$

$$n_\alpha(t) = n_\alpha(0) + t\,n_\alpha'(0) + \frac{t^2}{2}\,n_\alpha''(0) + O(t^3), \tag{B1b}$$

and equating coefficients of powers t^0, t^1, and t^2, yields the relations

$$[n_\alpha(0) - \underline{\underline{\rho}}(0)]\underline{\chi}_\alpha(0) = \underline{0}, \tag{B2a}$$

$$[n_\alpha(0) - \underline{\underline{\rho}}(0)]\underline{\chi}_\alpha'(0) = [\underline{\underline{\rho}}'(0) - n_\alpha'(0)]\underline{\chi}_\alpha(0), \tag{B2b}$$

$$\begin{aligned} [n_\alpha(0) - \underline{\underline{\rho}}(0)]\underline{\chi}_\alpha''(0) &= [\underline{\underline{\rho}}''(0) - n_\alpha''(0)]\underline{\chi}_\alpha(0) \\ &\quad + 2[\underline{\underline{\rho}}'(0) - n_\alpha'(0)]\underline{\chi}_\alpha'(0). \end{aligned} \tag{B2c}$$

Relations involving higher-order t-derivatives can be straightforwardly obtained by expanding each quantity up to a higher power of t, but any features we are interested in as part of the present work can be readily illustrated based on Eqs. (B2a), (B2b), and (B2c). The formal structure of these relations is analogous to that emerging in the context of time-independent perturbation theory in quantum mechanics [96]. However, in the situation focused on here, the perturbation is controlled by the small parameter t. A further twist on the familiar perturbation-theory approach is that we are interested in separating symmetric and antisymmetric contributions to the eigenvector $\underline{\chi}_\alpha(t)$.

Multiplication of Eq. (B2a) from the left with $\underline{\underline{S}}$ and using $\underline{\underline{S}}\,\underline{\underline{\rho}}(0) = \underline{\underline{0}}$ yields

$$n_\alpha(0)\,\underline{\underline{S}}\,\underline{\chi}_\alpha(0) = \underline{0}. \tag{B3}$$

Thus, in the $t = 0$ limit, eigenvectors are purely antisymmetric if the eigenvalue is nonzero, because $n_\alpha(0) \neq 0$ implies $\underline{\underline{S}}\,\underline{\chi}_\alpha(0) = \underline{0}$ based on Eq. (B3). As finite-t corrections can contain both antisymmetric and symmetric contributions, $\underline{\chi}_\alpha(t)$ could remain purely antisymmetric or become a mixture of antisymmetric and symmetric parts. We discuss this case in Appendix B 1 below.

The case of $n_\alpha(0) = 0$ has to be treated carefully because of the potential for degeneracy between symmetric and antisymmetric subspaces. The degeneracy may be lifted at finite t, where mixing between the two sectors may occur, which will affect the $t \to 0$ limit. We use degenerate perturbation theory to treat this case in Appendix B 2.

1. Eigenvectors with finite eigenvalues at $t = 0$

Equation (B2a) constitutes the eigenvalue equation for the 2bRDM $\underline{\underline{\rho}}(0)$. We assume that the eigenvalue problem for $\underline{\underline{\rho}}(0)$ has been solved for a particular system of interest. Here we focus on the set of nonzero eigenvalues, $n_\alpha(0) > 0$, and their associated $\underline{\chi}_\alpha(0)$. As discussed above, these are fully antisymmetric; $\underline{\chi}_\alpha(0) \equiv \underline{\underline{A}}\,\underline{\chi}_\alpha(0)$.

Multiplying Eq. (B2b) with $\underline{\underline{S}}$ from the left and remembering $\underline{\underline{S}}\,\underline{\underline{\rho}}(0) = \underline{\underline{0}}$, as well as $\underline{\underline{S}}\,\underline{\chi}_\alpha(0) = \underline{0}$, we find

$$n_\alpha(0)\,\underline{\underline{S}}\,\underline{\chi}'_\alpha(0) = \underline{\underline{S}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}}\;\underline{\chi}_\alpha(0). \tag{B4}$$

Multiplying instead with $\underline{\underline{A}}$ and remembering $\underline{\underline{\rho}}(0) = \underline{\underline{A}}\,\underline{\underline{\rho}}(0)\,\underline{\underline{A}}$, from which follows $\underline{\underline{A}}\,\underline{\underline{\rho}}(0) = \underline{\underline{\rho}}(0)\,\underline{\underline{A}}$, and using also $\underline{\underline{A}}\,\underline{\chi}_\alpha(0) = \underline{\chi}_\alpha(0)$, one obtains

$$[n_\alpha(0) - \underline{\underline{\rho}}(0)]\underline{\underline{A}}\,\underline{\chi}'_\alpha(0) = [\underline{\underline{A}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}} - n'_\alpha(0)]\underline{\chi}_\alpha(0). \tag{B5}$$

Lastly, multiplication of (B2b) with $\underline{\chi}^\dagger_\alpha(0)$ from the left, using the zero-mode property of $\underline{\chi}_\alpha(0)$ [Eq. (B2a)], yields

$$n'_\alpha(0) = \underline{\chi}^\dagger_\alpha(0)\,\underline{\underline{\rho}}'(0)\,\underline{\chi}_\alpha(0). \tag{B6}$$

Thus, we obtain implicit determining relations for $\underline{\underline{S}}\,\underline{\chi}'_\alpha(0)$, $\underline{\underline{A}}\,\underline{\chi}'_\alpha(0)$ and $n'_\alpha(0)$ in terms of the eigenvector $\underline{\chi}_\alpha(0)$ and eigenvalue $n_\alpha(0) > 0$ of the 2bRDM.

Rearranging Eq. (B4) straightforwardly yields

$$\underline{\underline{S}}\,\underline{\chi}'_\alpha(0) = -\frac{i}{\hbar\, n_\alpha(0)}\,\underline{\underline{\tau}}\,\underline{\chi}_\alpha(0) \tag{B7}$$

in terms of the universal transformer from Eq. (26). Equation (B7) is indeed the $t \to 0$ limit of the more general relation (27). The explicit expression for $\underline{\underline{A}}\,\underline{\chi}'_\alpha(0)$ is found from Eq. (B5), but we omit this here.

We continue by analyzing Eq. (B2c). Multiplying from the left with $\underline{\underline{S}}$ and using the relations $\underline{\underline{S}}\,\underline{\underline{\rho}}(0) = \underline{\underline{0}}$, $\underline{\underline{S}}\,\underline{\chi}_\alpha(0) = \underline{0}$, and $\underline{\underline{S}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{S}} = \underline{\underline{0}}$, we obtain

$$\begin{aligned} n_\alpha(0)\,\underline{\underline{S}}\,\underline{\chi}''_\alpha(0) &= \underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{A}}\,\underline{\chi}_\alpha(0) + 2\,\underline{\underline{S}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}}\,\underline{\chi}'_\alpha(0) \\ &\quad - 2\,n'_\alpha(0)\,\underline{\underline{S}}\,\underline{\chi}'_\alpha(0). \end{aligned} \tag{B8}$$

Multiplying instead with $\underline{\underline{A}}$, we find

$$\begin{aligned} &[n_\alpha(0) - \underline{\underline{\rho}}(0)]\underline{\underline{A}}\,\underline{\chi}''_\alpha(0) \\ &\quad = [\underline{\underline{A}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{A}} - n''_\alpha(0)]\underline{\chi}_\alpha(0) \\ &\qquad + 2\,\underline{\underline{A}}\,\underline{\underline{\rho}}'(0)\,\underline{\chi}'_\alpha(0) - 2\,n'_\alpha(0)\,\underline{\underline{A}}\,\underline{\chi}'_\alpha(0). \end{aligned} \tag{B9}$$

Finally, multiplying (B2c) with $\underline{\chi}^\dagger_\alpha(0)$ from the left, using also Eq. (B2a), we find

$$n''_\alpha(0) = \underline{\chi}^\dagger_\alpha(0)\,\underline{\underline{\rho}}''(0)\,\underline{\chi}_\alpha(0) + 2\,\underline{\chi}^\dagger_\alpha(0)[\underline{\underline{\rho}}'(0) - n'_\alpha(0)]\underline{\chi}'_\alpha(0). \tag{B10}$$

Thus, we have obtained implicit relations for all relevant quantities to order t^2. In particular, Eq. (B8) yields

$$\begin{aligned} \underline{\underline{S}}\,\underline{\chi}''_\alpha(0) = \frac{1}{n_\alpha(0)}\,&[\underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{A}}\,\underline{\chi}_\alpha(0) + 2\,\underline{\underline{S}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{A}}\,\underline{\chi}'_\alpha(0) \\ &- 2\,n'_\alpha(0)\,\underline{\underline{S}}\,\underline{\chi}'_\alpha(0)], \end{aligned} \tag{B11}$$

consistent with the general relation (27).

The perturbative scheme with small parameter t for obtaining eigenvectors and eigenvalues of the T2bCM that have $n_\alpha(0) > 0$ sketched above can be straightforwardly extended to higher orders. The general structure of the obtained perturbative expressions is analogous to those found for the eigenvalues and eigenvectors of the Hamiltonian in time-independent nondegenerate perturbation theory of quantum mechanics [96]. Results found for the symmetric contribution to the eigenvector accord with its general t dependence discussed in Appendix A.

2. Eigenvectors with vanishing eigenvalues at $t = 0$

We now consider eigenvectors of the T2bCM whose eigenvalue vanishes in the $t = 0$ limit; $n_\alpha(0) = 0$. Due to the positive-semidefiniteness of T2bCM eigenvalues, i.e., $n_\alpha(t) \geqslant 0$, $n'_\alpha(0) = 0$ must also hold in this case, and Eq. (B1b) specializes to

$$n_\alpha(t) = \frac{t^2}{2}\,n''_\alpha(0) + O(t^3). \tag{B12}$$

The fact that $\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}} = t^2\underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}}/2 + O(t^3)$ implies that eigenvectors that are entirely in the fully symmetric sector of the T2bCM must have vanishing eigenvalues in the $t = 0$ limit. However, eigenvectors with $n_\alpha(0) = 0$ may also exist in the antisymmetric sector, which would be signaled by having zero modes in the 2bRDM $\underline{\underline{\rho}}(0)$. We now present a careful treatment of the most general case where there exists such a degeneracy between symmetric and antisymmetric subspaces.

We start by defining projectors onto the degenerate and nondegenerate subspaces. Specifically, we introduce

$$\underline{\underline{Q}} = \sum_\alpha{}' \,\underline{\chi}_\alpha(0)\,\underline{\chi}^\dagger_\alpha(0), \tag{B13}$$

where $\sum'_\alpha$ indicates the restricted sum over states having $n_\alpha(0) > 0$. Its complement is

$$\underline{\underline{P}} \equiv \underline{\underline{1}} - \underline{\underline{Q}} = \sum_\beta{}'' \,\underline{\chi}_\beta(0)\,\underline{\chi}^\dagger_\beta(0), \tag{B14}$$

with $\sum''_\beta$ being the restricted sum over states having $n_\beta(0) = 0$. Eigenvectors from the degenerate subspace satisfy $\underline{\chi}_\alpha(0) = \underline{\underline{P}}\,\underline{\chi}_\alpha(0)$. Also, by construction, we have $\underline{\underline{\rho}}(0) \equiv \underline{\underline{Q}}\,\underline{\underline{\rho}}(0)\,\underline{\underline{Q}}$ and

$$\underline{\underline{P}}\,\underline{\underline{\rho}}(0) = \underline{\underline{\rho}}(0)\,\underline{\underline{P}} = \underline{\underline{0}}. \tag{B15}$$

We now analyze Eq. (B2) using the projectors $\underline{\underline{P}}$ and $\underline{\underline{Q}}$, having set $n_\alpha(0) = 0$ and $n'_\alpha(0) = 0$.

Multiplying Eq. (B2b) from the left by $\underline{\underline{P}}$, using also the relations (B15) and remembering that both $n_\alpha(0)$ and $n'_\alpha(0)$

vanish, we find $\underline{\underline{\mathcal{P}}}\,\underline{\underline{\rho}}'(0)\,\underline{\chi}_\alpha(0) = \underline{0}$ for any eigenvector from the degenerate subspace, implying

$$\underline{\underline{\mathcal{P}}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{\mathcal{P}}} = \underline{\underline{0}}. \tag{B16}$$

Muliplying (B2b) with $\underline{\underline{\mathcal{Q}}}$ instead and rearranging yields

$$\underline{\underline{\mathcal{Q}}}\,\underline{\chi}'_\alpha(0) = -\underline{\underline{\mathcal{Q}}}\,[\underline{\underline{\rho}}(0)]^{-1}\,\underline{\underline{\mathcal{Q}}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{\mathcal{P}}}\,\underline{\chi}_\alpha(0), \tag{B17}$$

where the inverse on the r.h.s. is well defined because $\underline{\underline{\mathcal{Q}}}$ projects onto the subspace where $\underline{\underline{\rho}}(0)$ has no zero modes and is thus invertible.

Inserting $n_\alpha(0) = 0$ and $n'_\alpha(0) = 0$ into Eq. (B2c), multiplying from the left with $\underline{\underline{\mathcal{P}}}$ and using the relation (B16), we find

$$n''_\alpha(0)\,\underline{\chi}_\alpha(0) = \underline{\underline{\mathcal{P}}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{\mathcal{P}}}\,\underline{\chi}_\alpha(0) + 2\,\underline{\underline{\mathcal{P}}}\,\underline{\underline{\rho}}'(0)\,\underline{\underline{\mathcal{Q}}}\,\underline{\chi}'_\alpha(0). \tag{B18}$$

Utilising the explicit expression for $\underline{\underline{\mathcal{Q}}}\,\underline{\chi}'_\alpha(0)$ given in Eq. (B17) yields the eigenvalue equation (33) whose solution determines all the $\underline{\chi}_\alpha(0)$ and their associated $n''_\alpha(0)$.

In principle, the eigenvector $\underline{\chi}_\alpha(t)$ can be constructed order by order in t by continuing the perturbative treatment. However, for our purposes, knowledge of $\underline{\chi}_\alpha(0)$ suffices. In particular, if a single eigenvalue $n''_0(0)$ among those obtained via Eq. (33) turns out to be macroscopic, then pairing order exists in the system at finite t, even though no order is present for $t = 0$. Most generally, the macroscopic eigenvector $\underline{\chi}_0(0)$ can be an arbitrary superposition of symmetric and antisymmetric parts. The system exhibits symmetric-pairing (i.e., odd-frequency) order when $\underline{\underline{S}}\,\underline{\chi}_0(0)$ is finite.

Further simplifications arise for eigenvectors that are fully symmetric in the $t = 0$ limit. Left-multiplying Eq. (33) by $\underline{\underline{S}}$, assuming $\underline{\chi}_\beta(0) = \underline{\underline{S}}\,\underline{\chi}_\beta(0)$, and using $\underline{\underline{S}}\,\underline{\underline{\mathcal{P}}} = \underline{\underline{\mathcal{P}}}\,\underline{\underline{S}} = \underline{\underline{S}}$ gives

$$\left\{\underline{\underline{S}}\,\underline{\underline{\rho}}''(0)\,\underline{\underline{S}} - \frac{2}{\hbar^2}\,\underline{\underline{\tau}}\,\underline{\underline{\mathcal{Q}}}[\underline{\underline{\rho}}(0)]^{-1}\,\underline{\underline{\mathcal{Q}}}\,\underline{\underline{\tau}}^\dagger\right\}\underline{\chi}_\beta(0) = n''_\beta(0)\,\underline{\chi}_\beta(0), \tag{B19}$$

which is equivalent to the eigenvalue equation for the generator matrix $\underline{\underline{\gamma}}$ [Eq. (34) from the main text].

APPENDIX C: RELATING EIGENVALUES OF THE GENERATOR MATRIX TO EIGENVALUES OF THE T2bCM

A fundamental connection can be established between the eigenvalues of the generator matrix $\underline{\underline{\gamma}}$ and the eigenvalues $n''_\beta(0)$ from Eq. (33). Considering the spectral decomposition

$$\underline{\underline{\gamma}} = \sum_\nu g_\nu\,\underline{\lambda}_\nu\,\underline{\lambda}_\nu^\dagger \tag{C1}$$

in conjunction with the expression given in Eq. (31a), one obtains the sum rule

$$g_\nu = \frac{\hbar^2}{2}\sum_\beta{}'' \, n''_\beta(0)\,|\underline{\lambda}_\nu^\dagger\,\underline{\chi}_\beta(0)|^2. \tag{C2}$$

For the special case where an eigenvector $\underline{\lambda}_\nu$ coincides with one of the $\underline{\chi}_\beta(0)$ and is therefore orthogonal to all the others, Eq. (C2) yields $g_\nu = \hbar^2 n''_\beta(0)/2$, consistent with Eq. (34). More generally, specifics of the relationship between the g_ν and the $n''_\beta(0)$ are encoded by the overlap matrix $|\underline{\lambda}_\nu^\dagger\,\underline{\chi}_\beta(0)|^2$.

Here we will prove that a macroscopic eigenvalue of $\underline{\underline{\gamma}}(t)$ implies the existence of a macroscopic eigenvalue $n_\beta(t) \sim N\,O(t^2)$ of the T2bCM, under the further assumption that the corresponding eigenvector has a finite overlap with some eigenvector of the T2bCM. More specifically, the assumption is that the overlap of the respective eigenvectors remains finite for large N. As the image of the generator matrix coincides with the entire symmetric subspace of the T2bCM, this assumption is reasonable.

We start by defining more precisely what we mean by a macroscopic eigenvalue of $\underline{\underline{\gamma}}(t)$. Let us assume that there is a systematic way to change the number N of fermions in the system. Then an eigenvalue $g_\nu(t)$ of $\underline{\underline{\gamma}}(t)$ is macroscopic if

$$\frac{g_\nu(t)}{N} \geqslant c > 0, \tag{C3}$$

for all $N > N_c$ for some constants c and N_c.

Now let $g_0(t)$ be a macroscopic eigenvalue of $\underline{\underline{\gamma}}(t)$, with constants c and N_c as explained above, satisfying

$$\underline{\underline{\gamma}}(t)\,\underline{\lambda}_0(t) = g_0(t)\,\underline{\lambda}_0(t). \tag{C4}$$

Let $\underline{\chi}_\beta(t)$ be an eigenvector of the T2bCM with a vanishing eigenvalue $n_\beta(0)$ at $t = 0$. Let us further assume that the overlap of $\underline{\chi}_\beta(t)$ with $\underline{\lambda}_0(t)$ is bounded from below,

$$|\underline{\lambda}_0^\dagger(t)\,\underline{\chi}_\beta(t)|^2 \geqslant \delta > 0, \tag{C5}$$

for all $N > N_\delta$ for some constants δ and N_δ.

We can now derive an inequality for the eigenvalue $n_\beta(t)$ of the T2bCM:

$$\begin{aligned}
n_\beta(t) &= \underline{\chi}_\beta^\dagger(t)\,\underline{\underline{\rho}}(t)\,\underline{\chi}_\beta(t)\\
&\geqslant \underline{\chi}_\beta^\dagger(t)\,\underline{\underline{S}}\,\underline{\underline{\rho}}(t)\,\underline{\underline{S}}\,\underline{\chi}_\beta(t)\\
&\geqslant \frac{t^2}{\hbar^2}\,\underline{\chi}_\beta^\dagger(t)\,\underline{\underline{\gamma}}(t)\,\underline{\chi}_\beta(t)\\
&\geqslant g_0\,\frac{t^2}{\hbar^2}\,|\underline{\lambda}_0^\dagger(t)\,\underline{\chi}_\beta(t)|^2\\
&\geqslant g_0\,\frac{t^2}{\hbar^2}\,\delta\\
&\geqslant N\,t^2\,\frac{c\,\delta}{\hbar^2},
\end{aligned} \tag{C6}$$

where the last two lines only hold for sufficiently large N. We have made use of the properties of $\underline{\underline{\rho}}(t)$ and $\underline{\underline{\gamma}}(t)$ being positive-semidefinite, as well as the fact that $\underline{\underline{S}}$ is a projector. If we have to weaken the assumptions to hold only for $t \to 0$, then, by continuity, the last line will still hold for a small interval around $t = 0$.

APPENDIX D: RELATING ORDER-PARAMETER DEFINITIONS OF NUMBER-CONSERVING AND NONCONSERVING FORMALISMS

Particle-nonconserving descriptions of pairing focus on the anomalous pair-correlation function $\underline{F}(t_1, t_2)$ defined via Eq. (1). This quantity emerges from postulating [7,97] the factorization of the time-ordered two-particle correlation function,

$$\langle T\, c_{\mathbf{i}}^\dagger(t_1)\, c_{\mathbf{j}}^\dagger(t_2)\, c_{\mathbf{l}}(t_2)\, c_{\mathbf{k}}(t_1)\rangle \to F_{\mathbf{ij}}(t_1, t_2)\, F_{\mathbf{kl}}^*(t_1, t_2), \tag{D1}$$

in the presence of a pair condensate. In contrast, the generalized Penrose-Onsager-type approach developed here considers the approximate factorization of the T2bCM $\underline{\underline{\rho}}(t, 0) \equiv \underline{\underline{\rho}}(t)$ to leading order in N [Eq. (5)] in terms of the order parameter $\underline{\phi}(t)$ [Eq. (35)].

Although $\underline{F}(t, 0)$ and $\underline{\phi}(t)$ are very different quantities, they can be linked conceptually [41]

$$\underline{F}(t, 0) \longleftrightarrow \underline{\phi}(t). \tag{D2}$$

The even-frequency and odd-frequency parts of $\underline{F}(t, 0)$ correspond to its antisymmetric and symmetric contributions [5,6],

$$\underline{F}^{(\mathrm{e})}(t, 0) \equiv \underline{\underline{A}}\, \underline{F}(t, 0) \longleftrightarrow \underline{\phi}^{(\mathrm{a})}(t), \tag{D3a}$$

$$\underline{F}^{(\mathrm{o})}(t, 0) \equiv \underline{\underline{S}}\, \underline{F}(t, 0) \longleftrightarrow \underline{\phi}^{(\mathrm{s})}(t). \tag{D3b}$$

In situations where only leading-order terms in the limit $t \to 0$ are considered relevant, the quantities

$$\Delta^{(\mathrm{e})} \equiv \underline{F}(0, 0) \longleftrightarrow \underline{\phi}^{(\mathrm{a})}(0), \tag{D4a}$$

$$\Delta^{(\mathrm{o})} \equiv \partial_t\, \underline{F}(t, 0)|_{t=0} \longleftrightarrow \lim_{t\to 0} \underline{\phi}^{(\mathrm{s})}(t)/t \tag{D4b}$$

are sometimes used as the order parameters [5,18,19].

APPENDIX E: INTERACTION-RELATED CONTRIBUTION TO TRANSFORMER-INDUCED SYMMETRIC-PAIRING ORDER

We consider the specific case of a Zeeman-spin-split Fermi superfluid discussed in Sec. V A. Using Eq. (52), the symmetric-pairing order parameter (36) for this model system can be decomposed as

$$\underline{\phi}^{(\mathrm{s})}(t) = \underline{\phi}^{(\mathrm{s,sp})}(t) + \underline{\phi}^{(\mathrm{s,int})}(t), \tag{E1}$$

where the leading-order small-t expression of $\underline{\phi}^{(\mathrm{s,sp})}(t)$ is given in Eq. (55), and

$$\underline{\phi}^{(\mathrm{s,int})}(t) = \frac{-it}{\hbar}\, \frac{1}{\sqrt{n_0(0)}}\, \underline{\underline{\tau}}^{(\mathrm{int})}\, \underline{\underline{A}}\, \underline{\chi}_0(0) \tag{E2}$$

to leading order in small t. Taking the explicit form of the interaction part of the transformer $\underline{\underline{\tau}}^{(\mathrm{int})}$ from Eq. (53b) and utilizing also Eq. (54b), we obtain

$$\phi^{(\mathrm{s,int})}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j}(t) = i\,U\, \frac{t}{\hbar}\, \frac{1}{\sqrt{n_0(0)}} \left\langle \sum_{\mathbf{q}} c_{\mathbf{q}\uparrow}^\dagger\, c_{-\mathbf{q}\downarrow}^\dagger \big(\varsigma_j\, c_{\mathbf{q}_i\sigma_i}^\dagger\, c_{-\mathbf{q}_j\bar\sigma_j} + \varsigma_i\, c_{\mathbf{q}_j\sigma_j}^\dagger\, c_{-\mathbf{q}_i\bar\sigma_i}\big) \sum_{\mathbf{q}'} \chi_{0,\mathbf{q}'}\, c_{-\mathbf{q}'\downarrow}\, c_{\mathbf{q}'\uparrow} \right\rangle. \tag{E3}$$

Here we used again our compact notation where $\bar\sigma$ denotes the opposite of σ; i.e., $\bar\sigma = \downarrow (\uparrow)$ if $\sigma = \uparrow (\downarrow)$, and ς takes the values $+1\,(-1)$ when $\sigma = \uparrow (\downarrow)$.

To gain further insight into the general form of $\underline{\phi}^{(\mathrm{s,int})}$, we adopt the Yang-model description of a Fermi superfluid [50], where the ground state is the pair-condensate state

$$|\Psi_N\rangle = \mathcal{N}_N (B^\dagger)^{\frac{N}{2}}\, |\mathrm{vac}\rangle, \tag{E4}$$

involving the pair-creation operator

$$B^\dagger = \frac{1}{\sqrt{m}} \sum_{\mathbf{q}} c_{\mathbf{q}\uparrow}^\dagger\, c_{-\mathbf{q}\downarrow}^\dagger. \tag{E5}$$

Here $m \gg N$ indicates the number of single-particle modes (excluding spin), $|\mathrm{vac}\rangle$ is the vacuum state, and the normalization factor $\mathcal{N}_N$ is in principle known but does not need to be specified here. For our purposes, we only need to employ the relation

$$B|\Psi_N\rangle = \left[\frac{N}{2} - \frac{N(N-2)}{4m}\right]^{\frac{1}{2}} |\Psi_{N-2}\rangle \tag{E6}$$

and the known form of entries in the macroscopic eigenvector of the 2bRDM [see Eq. (54b)],

$$\chi_{0,\mathbf{q}} = \frac{1}{\sqrt{2m}}. \tag{E7}$$

With the input of Eqs. (E6) and (E7), assuming also that the expectation value on the r.h.s. of Eq. (E3) is calculated in the Yang state $|\Psi_N\rangle$, the interaction contribution to the symmetric-pairing order parameter becomes

$$\phi^{(\mathrm{s,int})}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j}(t) = i\,U\, \frac{t}{\hbar}\, \frac{1}{\sqrt{n_0(0)}}\, \sqrt{2m}\, \langle \Psi_N | B^\dagger \big(\varsigma_j\, c_{\mathbf{q}_i\sigma_i}^\dagger\, c_{-\mathbf{q}_j\bar\sigma_j} + \varsigma_i\, c_{\mathbf{q}_j\sigma_j}^\dagger\, c_{-\mathbf{q}_i\bar\sigma_i}\big) B |\Psi_N\rangle, \tag{E8a}$$

$$= i\,m\,U\, \frac{t}{\hbar}\, \frac{N}{\sqrt{n_0(0)}} \left[1 - \frac{N-2}{2m}\right] \frac{1}{\sqrt{2m}}\, \langle \Psi_{N-2} | \big(\varsigma_j\, c_{\mathbf{q}_i\sigma_i}^\dagger\, c_{-\mathbf{q}_j\bar\sigma_j} + \varsigma_i\, c_{\mathbf{q}_j\sigma_j}^\dagger\, c_{-\mathbf{q}_i\bar\sigma_i}\big) |\Psi_{N-2}\rangle. \tag{E8b}$$

On the r.h.s. of Eq. (E8b), normalization factors have been distributed such that the result emerging in the limits of $m \to \infty$ and large $N \ll m$ is readily apparent. In particular, with $n_0(0) \sim O(N)$, it appears that, as far as the scaling as a function of N is concerned, $\underline{\phi}^{(\mathrm{s,int})}(t)$ could be a relevant contribution to the symmetric-pairing order parameter.

The vector entries of $\underline{\phi}^{(\mathrm{s,int})}(t)$ given in Eq. (E8b) are related to a combination of entries from the single-particle reduced density matrix for the $N-2$-particle Yang state. This again illustrates the general importance of single-particle physics for facilitating the transformation of antisymmetric pairing order into symmetric pairing order. Within the Yang-model description, exact results for the single-particle reduced density matrix yield

$$\langle \Psi_{N-2}| c^\dagger_{\mathbf{q}_i\sigma_i}\, c_{-\mathbf{q}_j\bar{\sigma}_j} |\Psi_{N-2}\rangle = \langle \Psi_{N-2}| c^\dagger_{\mathbf{q}_j\sigma_j}\, c_{-\mathbf{q}_i\bar{\sigma}_i} |\Psi_{N-2}\rangle = \frac{N-2}{2m}\, \delta_{\mathbf{q}_j,-\mathbf{q}_i}\, \delta_{\sigma_j,\bar{\sigma}_i}, \tag{E9}$$

and we find

$$\phi^{(\mathrm{s,int})}_{\mathbf{q}_i\sigma_i\,\mathbf{q}_j\sigma_j}(t) \propto -\varsigma_i \langle \Psi_{N-2}| \big(c^\dagger_{\mathbf{q}_i\sigma_i}\, c_{\mathbf{q}_i\sigma_i} - c^\dagger_{-\mathbf{q}_i\bar{\sigma}_i}\, c_{-\mathbf{q}_i\bar{\sigma}_i} \big) |\Psi_{N-2}\rangle\, \delta_{\mathbf{q}_j,-\mathbf{q}_i}\, \delta_{\sigma_j,\bar{\sigma}_i} = 0. \tag{E10}$$

Thus, within the Yang-model description, the interaction part of the transformer does not cause symmetric-pairing order, and the latter originates entirely from the single-particle portion of the transformer that gives rise to the order parameter from Eq. (55).

The fundamental reason why the interaction-related part of the transformer yields no symmetric-pairing order can be gleaned from inspecting the correlation function appearing on the r.h.s. of Eq. (E10). The two terms being subtracted are related via the time-reversal operation, which inverts both the spin and orbital momentum. This observation leads us to surmise that the vanishing of $\underline{\phi}^{(\mathrm{s,int})}(t)$ holds more generally on symmetry grounds, even beyond the strict limits of applicability for the Yang model.

APPENDIX F: GENERATED SYMMETRIC-PAIRING ORDER IN A SYSTEM OF ITINERANT ELECTRONS COUPLED TO MAGNONS

1. Form of the transformer and generator matrices

Using the definition Eq. (7b), with H given by Eq. (56), we find the expression

$$\begin{aligned}\tau_{r_i\sigma_i\, r_j\sigma_j,\, r_k\sigma_k\, r_l\sigma_l} = {}& \frac{K}{2}\left[\rho_{r_i+1\,\sigma_i\, r_j\sigma_j,\, r_k\sigma_k\, r_l\sigma_l}(0) + \rho_{r_i-1\,\sigma_i\, r_j\sigma_j,\, r_k\sigma_k\, r_l\sigma_l}(0) + \rho_{r_j+1\,\sigma_j\, r_i\sigma_i,\, r_k\sigma_k\, r_l\sigma_l}(0) + \rho_{r_j-1\,\sigma_j\, r_i\sigma_i,\, r_k\sigma_k\, r_l\sigma_l}(0)\right] \\ &+ \frac{J}{2}\Big[\delta_{\sigma_i,\uparrow} \big\langle c^\dagger_{r_j\sigma_j}\, c^\dagger_{r_i\downarrow}\, b^\dagger_{r_i}\, c_{r_l\sigma_l}\, c_{r_k\sigma_k}\big\rangle + \delta_{\sigma_i,\downarrow} \big\langle c^\dagger_{r_j\sigma_j}\, c^\dagger_{r_i\uparrow}\, b_{r_i}\, c_{r_l\sigma_l}\, c_{r_k\sigma_k}\big\rangle \\ &+ \delta_{\sigma_j,\uparrow} \big\langle c^\dagger_{r_i\sigma_i}\, c^\dagger_{r_j\downarrow}\, b^\dagger_{r_j}\, c_{r_l\sigma_l}\, c_{r_k\sigma_k}\big\rangle + \delta_{\sigma_j,\downarrow} \big\langle c^\dagger_{r_i\sigma_i}\, c^\dagger_{r_j\uparrow}\, b_{r_j}\, c_{r_l\sigma_l}\, c_{r_k\sigma_k}\big\rangle \Big]\end{aligned} \tag{F1}$$

for the transformer. The terms $\propto J$ in Eq. (F1) are expectation values involving unbalanced boson creation and annihilation operators $b^\dagger_r$ and b_r. If the state defining the expectation values in Eq. (F1) is an eigenstate of the boson number operator $\hat{N}_\mathrm{b} = \sum_r b^\dagger_r b_r$, then these terms must vanish identically.

Expectation values involving unbalanced boson operators also arise in the calculation of the part $\propto \underline{\underline{S}}\, \underline{\underline{\rho}}''(0)\, \underline{\underline{S}}$ of the generator [see the first term in Eq. (31b)], and they are neglected on the same grounds in that context. The remaining terms $\propto K^2$ arising due to the $\underline{\underline{S}}\, \underline{\underline{\rho}}''(0)\, \underline{\underline{S}}$ part are cancelled by the quadratic-in-$\underline{\underline{\tau}}$ contribution to the generator [see the second term in Eq. (31b)]. The result for the generator depends only on terms $\propto J^2$, and can be expressed as Eq. (57).

2. Reduced density matrices for the composite fermion-pair–magnon condensate

To demonstrate the typical behavior of reduced density matrices corresponding to composite-condensate systems [19], we consider a Yang-type state for composite bosons,

$$\big|\Psi^{(3\mathrm{b})}_N\big\rangle = \mathcal{N}_N\, (B^\dagger_{\mathrm{fb}})^{\frac{N}{2}} |\mathrm{vac}\rangle, \tag{F2a}$$

$$B^\dagger_{\mathrm{fb}} = \frac{1}{\sqrt{m}} \sum_r c^\dagger_{r\uparrow} c^\dagger_{r\downarrow} b^\dagger_r, \tag{F2b}$$

$$B_{\mathrm{fb}} \big|\Psi^{(3\mathrm{b})}_N\big\rangle = \left[\frac{N}{2} - \frac{N(N-2)}{4m}\right]^{\frac{1}{2}} \big|\Psi^{(3\mathrm{b})}_{N-2}\big\rangle, \tag{F2c}$$

where m is the total number of one-dimensional lattice sites. Equations (F2) constitute a particular example of a composite-boson condensate that is an eigenstate of both the fermion and boson number operators $\hat{N}$ and $\hat{N}_\mathrm{b}$.

In the limit of a large lattice, i.e., for $m \gg N$, the system's 2bRDM and the three-body reduced density matrix defined by Eq. (59), respectively, take the form

$$\big\langle \Psi^{(3\mathrm{b})}_N \big| c^\dagger_{r_i\sigma_i}\, c^\dagger_{r_j\sigma_j}\, c_{r_l\sigma_l}\, c_{r_k\sigma_k} \big| \Psi^{(3\mathrm{b})}_N \big\rangle \approx \frac{N}{2m}\, \delta_{\sigma_j,\bar{\sigma}_i}\, \delta_{r_i,r_j}\, \delta_{r_l,r_k}\, (\delta_{r_i\sigma_i,\, r_k\sigma_k}\, \delta_{r_j\sigma_j,\, r_l\sigma_l} - \delta_{r_j\sigma_j,\, r_k\sigma_k}\, \delta_{r_i\sigma_i,\, r_l\sigma_l}), \tag{F3a}$$

$$\left\langle \Psi_N^{(3b)} \middle| c^{\dagger}_{r_i\sigma_i} c^{\dagger}_{r_j\sigma_j} b^{\dagger}_{r_o} b_{r_p} c_{r_l\sigma_l} c_{r_k\sigma_k} \middle| \Psi_N^{(3b)} \right\rangle \approx \frac{N}{2m} \varsigma_i \varsigma_k \delta_{r_i,r_j} \delta_{r_l,r_k} \delta_{r_i,r_o} \delta_{r_p,r_k} \delta_{\sigma_j,\bar{\sigma}_i} \delta_{\sigma_l,\bar{\sigma}_k}$$
$$\equiv N \left[\frac{1}{\sqrt{2m}} \varsigma_i \delta_{r_j,r_i} \delta_{r_o,r_i} \delta_{\sigma_j,\bar{\sigma}_i} \right] \left[\frac{1}{\sqrt{2m}} \varsigma_k \delta_{r_l,r_k} \delta_{r_p,r_k} \delta_{\sigma_l,\bar{\sigma}_k} \right]^* . \qquad \text{(F3b)}$$

We utilized again the notation where $\bar{\sigma}$ is the opposite of σ; i.e., $\bar{\sigma} = \downarrow (\uparrow)$ if $\sigma = \uparrow (\downarrow)$, and ς takes the values $+1\,(-1)$ when $\sigma = \uparrow (\downarrow)$. The 2bRDM (F3a) is a block-diagonal matrix in two-particle index space with degenerate eigenvalues 0 and $N/m \ll 1$ that are nonmacroscopic. It can be verified that the vectors from three-body index space appearing in Eq. (F3b) are normalized. The rank-1 three-body reduced density matrix thus has eigenvalue N, which is consequently macroscopic, in the limit $m \gg N$. Identifying $1/\sqrt{2m} = \chi_0^{(3b)}$ yields Eq. (60b).

[1] J. B. Ketterson and S. N. Song, *Superconductivity* (Cambridge University Press, Cambridge, 1999).

[2] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Theory of superconductivity, Phys. Rev. **108**, 1175 (1957).

[3] M. Sigrist and K. Ueda, Phenomenological theory of unconventional superconductivity, Rev. Mod. Phys. **63**, 239 (1991).

[4] V. P. Mineev and K. V. Samokhin, *Introduction to Unconventional Superconductivity* (CRC Press, Boca Raton, FL, 1999).

[5] J. Linder and A. V. Balatsky, Odd-frequency superconductivity, Rev. Mod. Phys. **91**, 045005 (2019).

[6] V. L. Berezinskii, New model of the anisotropic phase of superfluid He^3, JETP Lett. **20**, 287 (1974).

[7] L. P. Gorkov, On the energy spectrum of superconductors, Sov. Phys. JETP **7**, 505 (1958).

[8] H. Bruus and K. Flensberg, *Many-Body Quantum Theory in Condensed Matter Physics* (Oxford University Press, Oxford, 2004).

[9] T. R. Kirkpatrick and D. Belitz, Disorder-induced triplet superconductivity, Phys. Rev. Lett. **66**, 1533 (1991).

[10] A. Balatsky and E. Abrahams, New class of singlet superconductors which break the time reversal and parity, Phys. Rev. B **45**, 13125 (1992).

[11] E. Abrahams, A. Balatsky, J. R. Schrieffer, and P. B. Allen, Interactions for odd-ω-gap singlet superconductors, Phys. Rev. B **47**, 513 (1993).

[12] V. J. Emery and S. Kivelson, Mapping of the two-channel Kondo problem to a resonant-level model, Phys. Rev. B **46**, 10812 (1992).

[13] A. V. Balatsky and J. Bonča, Even- and odd-frequency pairing correlations in the one-dimensional $t - J - h$ model: A comparative study, Phys. Rev. B **48**, 7445 (1993).

[14] P. Coleman, E. Miranda, and A. Tsvelik, Odd-frequency pairing in the Kondo lattice, Phys. Rev. B **49**, 8955 (1994).

[15] P. Coleman, E. Miranda, and A. Tsvelik, Three-body bound states and the development of odd-frequency pairing, Phys. Rev. Lett. **74**, 1653 (1995).

[16] A. V. Balatsky, E. Abrahams, D. J. Scalapino, and J. R. Schrieffer, Properties of odd gap superconductors, Phys. B: Condens. Matter **199–200**, 363 (1994).

[17] J. R. Schrieffer, A. V. Balatsky, E. Abrahams, and D. J. Scalapino, Odd frequency pairing in superconductors, J. Supercond. **7**, 501 (1994).

[18] E. Abrahams, A. Balatsky, D. J. Scalapino, and J. R. Schrieffer, Properties of odd-gap superconductors, Phys. Rev. B **52**, 1271 (1995).

[19] H. P. Dahal, E. Abrahams, D. Mozyrsky, Y. Tanaka, and A. V. Balatsky, Wave function for odd-frequency superconductors, New J. Phys. **11**, 065005 (2009).

[20] Y. Fuseya, H. Kohno, and K. Miyake, Realization of odd-frequency p-wave spin-singlet superconductivity coexisting with antiferromagnetic order near quantum critical point, J. Phys. Soc. Jpn. **72**, 2914 (2003).

[21] H. Kusunose, Y. Fuseya, and K. Miyake, Possible odd-frequency superconductivity in strong-coupling electron-phonon systems, J. Phys. Soc. Jpn. **80**, 044711 (2011).

[22] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Odd triplet superconductivity and related phenomena in superconductor-ferromagnet structures, Rev. Mod. Phys. **77**, 1321 (2005).

[23] M. Eschrig, T. Löfwander, T. Champel, J. C. Cuevas, J. Kopu, and G. Schön, Symmetries of pairing correlations in superconductor-ferromagnet nanostructures, J. Low Temp. Phys. **147**, 457 (2007).

[24] Y. Tanaka, A. A. Golubov, S. Kashiwaya, and M. Ueda, Anomalous Josephson effect between even- and odd-frequency superconductors, Phys. Rev. Lett. **99**, 037005 (2007).

[25] H. Kusunose, M. Matsumoto, and M. Koga, Strong-coupling superconductivity with mixed even- and odd-frequency pairing, Phys. Rev. B **85**, 174528 (2012).

[26] B. Sothmann, S. Weiss, M. Governale, and J. König, Unconventional superconductivity in double quantum dots, Phys. Rev. B **90**, 220501(R) (2014).

[27] A. M. Black-Schaffer and A. V. Balatsky, Odd-frequency superconducting pairing in multiband superconductors, Phys. Rev. B **88**, 104514 (2013).

[28] Y. Asano and A. Sasaki, Odd-frequency Cooper pairs in two-band superconductors and their magnetic response, Phys. Rev. B **92**, 224508 (2015).

[29] C. Triola, J. Cayao, and A. M. Black-Schaffer, The role of odd-frequency pairing in multiband superconductors, Ann. Phys. (Berlin) **532**, 1900298 (2020).

[30] D. Chakraborty and A. M. Black-Schaffer, Interplay of finite-energy and finite-momentum superconducting pairing, Phys. Rev. B **106**, 024511 (2022).

[31] A. Di Bernardo, Z. Salman, X. L. Wang, M. Amado, M. Egilmez, M. G. Flokstra, A. Suter, S. L. Lee, J. H. Zhao, T. Prokscha *et al.*, Intrinsic paramagnetic Meissner effect due to s-wave odd-frequency superconductivity, Phys. Rev. X **5**, 041021 (2015).

[32] A. Pal, J. A. Ouassou, M. Eschrig, J. Linder, and M. G. Blamire, Spectroscopic evidence of odd frequency superconducting order, Sci. Rep. **7**, 40604 (2017).

[33] V. Perrin, F. L. N. Santos, G. C. Ménard, C. Brun, T. Cren, M. Civelli, and P. Simon, Unveiling odd-frequency pairing around a magnetic impurity in a superconductor, Phys. Rev. Lett. **125**, 117003 (2020).

[34] R. Heid, On the thermodynamic stability of odd-frequency superconductors, Z. Phys. B **99**, 15 (1995).

[35] D. Solenov, I. Martin, and D. Mozyrsky, Thermodynamical stability of odd-frequency superconducting state, Phys. Rev. B **79**, 132502 (2009).

[36] H. Kusunose, Y. Fuseya, and K. Miyake, On the puzzle of odd-frequency superconductivity, J. Phys. Soc. Jpn. **80**, 054702 (2011).

[37] Y. V. Fominov, Y. Tanaka, Y. Asano, and M. Eschrig, Odd-frequency superconducting states with different types of Meissner response: Problem of coexistence, Phys. Rev. B **91**, 144514 (2015).

[38] F. Schrodi, A. Aperis, and P. M. Oppeneer, Induced odd-frequency superconducting state in vertex-corrected Eliashberg theory, Phys. Rev. B **104**, 174518 (2021).

[39] D. Pimenov and A. V. Chubukov, Odd-frequency pairing and time-reversal symmetry breaking for repulsive interactions, Phys. Rev. B **106**, 104515 (2022).

[40] E. Langmann, C. Hainzl, R. Seiringer, and A. V. Balatsky, Obstructions to odd-frequency superconductivity in Eliashberg theory, arXiv:2207.01825 (2022).

[41] A. J. Leggett, *Quantum Liquids* (Oxford University Press, Oxford, 2006).

[42] P. W. Anderson, Coherent matter field phenomena in superfluids, in *Belfer Graduate School of Science Annual Science Conference Proceedings*, edited by A. Gelbart (Yeshiva University, New York, 1969), Vol. 2, pp. 21–40.

[43] A. S. Wightman, Superselection rules; old and new, Nuovo Cim. B **110**, 751 (1995).

[44] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. **80**, 885 (2008).

[45] I. Bloch, J. Dalibard, and S. Nascimbène, Quantum simulations with ultracold quantum gases, Nat. Phys. **8**, 267 (2012).

[46] E. Altman, K. R. Brown, G. Carleo, L. D. Carr, E. Demler, C. Chin, B. DeMarco, S. E. Economou, M. A. Eriksson, K. Fu *et al.*, Quantum simulators: Architectures and opportunities, PRX Quantum **2**, 017003 (2021).

[47] X. Li, S. Wang, X. Luo, Y.-Y. Zhou, K. Xie, H.-C. Shen, Y.-Z. Nie, Q. Chen, H. Hu, Y.-A. Chen *et al.*, Observation and quantification of the pseudogap in unitary Fermi gases, Nature (London) **626**, 288 (2024).

[48] M. Holten, L. Bayha, K. Subramanian, S. Brandstetter, C. Heintze, P. Lunt, P. M. Preiss, and S. Jochim, Observation of Cooper pairs in a mesoscopic two-dimensional Fermi gas, Nature (London) **606**, 287 (2022).

[49] O. Penrose and L. Onsager, Bose-Einstein condensation and liquid helium, Phys. Rev. **104**, 576 (1956).

[50] C. N. Yang, Concept of off-diagonal long-range order and the quantum phases of liquid He and of superconductors, Rev. Mod. Phys. **34**, 694 (1962).

[51] We use the term "macroscopic" in the sense that a macroscopic quantity is expected to grow proportionally to the number of fermions N when the system size is increased. For a mathematically precise definition, see Appendix C. The actual value of N could be of the order of Avogadro's number, e.g., for solid-state systems, but it is typically much smaller (and thus not truly macroscopic in the thermodynamic sense) in ultracold-atom experiments.

[52] There exist also other particle-conserving theories for conventional superfluidity [98–101]. Here we choose to follow the Penrose-Onsager-Yang approach because of its versatility [41].

[53] G. L. Sewell, Off-diagonal long-range order and the Meissner effect, J. Stat. Phys. **61**, 415 (1990).

[54] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).

[55] Z. Wang and S.-C. Zhang, Strongly correlated topological superconductors and topological phase transitions via Green's function, Phys. Rev. B **86**, 165116 (2012).

[56] M. Frick, I. Morgenstern, and W. von der Linden, Off-diagonal long range order in an electron-phonon model for high-T_c superconductors, Int. J. Mod. Phys. C **03**, 105 (1992).

[57] M. Guerrero, G. Ortiz, and J. E. Gubernatis, Pairing correlations in the attractive Hubbard model on chains, ladders, and squares, Phys. Rev. B **62**, 600 (2000).

[58] S. Sorella, G. B. Martins, F. Becca, C. Gazza, L. Capriotti, A. Parola, and E. Dagotto, Superconductivity in the two-dimensional t–J model, Phys. Rev. Lett. **88**, 117002 (2002).

[59] P. W. Anderson, P. A. Lee, M. Randeria, T. M. Rice, N. Trivedi, and F. C. Zhang, The physics behind high-temperature superconducting cuprates: The plain vanilla version of RVB, J. Phys.: Condens. Matter **16**, R755 (2004).

[60] G. E. Astrakharchik, K. V. Krutitsky, M. Lewenstein, and F. Mazzanti, One-dimensional Bose gas in optical lattices of arbitrary strength, Phys. Rev. A **93**, 021605(R) (2016).

[61] U. Ebling, A. Alavi, and J. Brand, Signatures of the BCS-BEC crossover in the yrast spectra of Fermi quantum rings, Phys. Rev. Res. **3**, 023142 (2021).

[62] A. Wietek, Fragmented Cooper pair condensation in striped superconductors, Phys. Rev. Lett. **129**, 177001 (2022).

[63] In principle, there could be multiple macroscopic eigenvalues of the 2bRDM, corresponding to a fragmented pseudo-BEC, but this situation appears to be rare in nature [41].

[64] G. Aeppli, A. V. Balatsky, H. M. Rønnow, and N. A. Spaldin, Hidden, entangled and resonating order, Nat. Rev. Mater. **5**, 477 (2020).

[65] V. Kornich, F. Schlawin, M. A. Sentef, and B. Trauzettel, Direct detection of odd-frequency superconductivity via time- and angle-resolved photoelectron fluctuation spectroscopy, Phys. Rev. Res. **3**, L042034 (2021).

[66] G. M. Éliashberg, Interactions between electrons and lattice vibrations in a superconductor, Sov. Phys. JETP **11**, 696 (1960).

[67] G. M. Éliashberg, Temperature Green's function for electrons in a superconductor, Sov. Phys. JETP **12**, 1000 (1961).

[68] G.-A. Inkof, K. Schalm, and J. Schmalian, Quantum critical Eliashberg theory, the Sachdev-Ye-Kitaev superconductor and their holographic duals, npj Quantum Mater. **7**, 56 (2022).

[69] H. T. Nieh, G. Su, and B.-H. Zhao, Off-diagonal long-range order: Meissner effect and flux quantization, Phys. Rev. B **51**, 3760 (1995).

[70] C. Au and B.-H. Zhao, From ODLRO to the Meissner effect and flux quantization, Phys. Lett. A **209**, 235 (1995).

[71] G. L. Sewell, Off-diagonal long range order and superconductive electrodynamics, J. Math. Phys. **38**, 2053 (1997).

[72] M. A. Rampp and J. Schmalian, Integer and fractionalized vortex lattices and off-diagonal long-range order, J. Phys. Commun. **6**, 055013 (2022).

[73] F. Parhizgar and A. M. Black-Schaffer, Diamagnetic and paramagnetic Meissner effect from odd-frequency pairing in multiorbital superconductors, Phys. Rev. B **104**, 054507 (2021).

[74] Y. B. Levinson, Translational invariance in uniform fields and the equation for the density matrix in the Wigner representation, Sov. Phys. JETP **30**, 362 (1970).

[75] B. S. Chandrasekhar, A note on the maximum critical field of high-field superconductors, Appl. Phys. Lett. **1**, 7 (1962).

[76] A. M. Clogston, Upper limit for the critical field in hard superconductors, Phys. Rev. Lett. **9**, 266 (1962).

[77] C. Chin, M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, J. Hecker Denschlag, and R. Grimm, Observation of the pairing gap in a strongly interacting Fermi gas, Science **305**, 1128 (2004).

[78] M. W. Zwierlein, C. H. Schunck, A. Schirotzek, and W. Ketterle, Direct observation of the superfluid phase transition in ultracold Fermi gases, Nature (London) **442**, 54 (2006).

[79] M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Fermionic superfluidity with imbalanced spin populations, Science **311**, 492 (2006).

[80] G. Veeravalli, E. Kuhnle, P. Dyke, and C. J. Vale, Bragg spectroscopy of a strongly interacting Fermi gas, Phys. Rev. Lett. **101**, 250403 (2008).

[81] F. Chevy and C. Mora, Ultra-cold polarized Fermi gases, Rep. Prog. Phys. **73**, 112401 (2010).

[82] L. Radzihovsky and D. E. Sheehy, Imbalanced Feshbach-resonant Fermi gases, Rep. Prog. Phys. **73**, 076501 (2010).

[83] A. Cichy and R. Micnas, The spin-imbalanced attractive Hubbard model in $d = 3$: Phase diagrams and BCS-BEC crossover at low filling, Ann. Phys. (NY) **347**, 207 (2014).

[84] G. Spada, R. Rossi, F. Simkovic, R. Garioud, M. Ferrero, K. Van Houcke, and F. Werner, High-order expansion around BCS theory, arXiv:2103.12038 (2021).

[85] T. Malas-Danzé, A. Dugelay, N. Navon, and H. Kurkjian, Probing pair correlations in Fermi gases with Ramsey-Bragg interferometry, SciPost Phys. **17**, 024 (2024).

[86] A. Ramires and M. Sigrist, Identifying detrimental effects for multiorbital superconductivity: Application to Sr_2 RuO_4, Phys. Rev. B **94**, 104501 (2016).

[87] A. Ramires, D. F. Agterberg, and M. Sigrist, Tailoring T_c by symmetry principles: The concept of superconducting fitness, Phys. Rev. B **98**, 024501 (2018).

[88] P. Coleman, *Introduction to Many-Body Physics* (Cambridge University Press, Cambridge, 2015).

[89] S.-S. Zhang, W. Zhu, and C. D. Batista, Pairing from strong repulsion in triangular lattice Hubbard model, Phys. Rev. B **97**, 140507(R) (2018).

[90] K. G. Nazaryan and L. Fu, Magnonic superconductivity, arXiv:2403.14756 (2024).

[91] D. J. Dean and M. Hjorth-Jensen, Pairing in nuclear systems: From neutron stars to finite nuclei, Rev. Mod. Phys. **75**, 607 (2003).

[92] A. Sedrakian and J. W. Clark, Superfluidity in nuclear systems and neutron stars, Eur. Phys. J. A **55**, 167 (2019).

[93] Z. H. Yang, Y. L. Ye, B. Zhou, H. Baba, R. J. Chen, Y. C. Ge, B. S. Hu, H. Hua, D. X. Jiang, M. Kimura *et al.*, Observation of the exotic 0_2^+ cluster state in ^{8}He, Phys. Rev. Lett. **131**, 242501 (2023).

[94] R. M. Kalas, A. V. Balatsky, and D. Mozyrsky, Odd-frequency pairing in a binary mixture of bosonic and fermionic cold atoms, Phys. Rev. B **78**, 184513 (2008).

[95] M. Arzamasovs and B. Liu, Engineering frequency-dependent superfluidity in Bose-Fermi mixtures, Phys. Rev. A **97**, 043607 (2018).

[96] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Addison-Wesley, San Francisco, 2011).

[97] M. H. Cohen, Generalized self-consistent-field theory: Gor'kov factorization, Phys. Rev. **137**, A497 (1965).

[98] R. W. Richardson and N. Sherman, Exact eigenstates of the pairing-force Hamiltonian, Nucl. Phys. **52**, 221 (1964).

[99] J. von Delft and F. Braun, Superconductivity in ultrasmall grains: Introduction to Richardson's exact solution, in *Quantum Mesoscopic Phenomena and Mesoscopic Devices in Microelectronics*, edited by I. O. Kulik and R. Ellialtioğlu (Kluwer Academic, Dordrecht, 2000), pp. 361–370.

[100] D. Janssen and P. Schuck, Symmetry conserving generalisation of Hartree Fock Bogoliubov theory: I. Particle number, Z. Phys. A **301**, 255 (1981).

[101] H. Koizumi and A. Ishikawa, Berry connection from many-body wave functions and superconductivity: Calculations by the particle number conserving Bogoliubov-de Gennes equations, J. Supercond. Novel Magn. **34**, 2795 (2021).